Pacific Journal of Mathematics

N_∞-OPERADS AND ASSOCIAHEDRA

SCOTT BALCHIN, DAVID BARNES AND CONSTANZE ROITZHEIM

Volume 315 No. 2 December 2021

PACIFIC JOURNAL OF MATHEMATICS
Vol. 315, No. 2, 2021
https://doi.org/10.2140/pjm.2021.315.285

N_∞-OPERADS AND ASSOCIAHEDRA

SCOTT BALCHIN, DAVID BARNES AND CONSTANZE ROITZHEIM

We provide a combinatorial approach to studying the collection of N_∞-operads in G-equivariant homotopy theory for G a finite cyclic group of prime power order. In particular, we show that for $G = C_{p^n}$ the natural order on the collection of N_∞-operads is in bijection with the poset structure of the $(n+1)$-associahedron. We further provide a lower bound for the number of possible N_∞-operads for any finite cyclic group G. As such, we have reduced an intricate problem in equivariant homotopy theory to a manageable combinatorial problem.

1. Introduction

Let X be a topological space equipped with a multiplication $m : X \times X \to X$. We say that the multiplication is *homotopy commutative* if the diagram

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "\text{twist}"'] & X \\ X \times X \arrow[ur, "m"'] & \end{tikzcd}$$

commutes up to homotopy, and all higher coherences are satisfied up to homotopy. Homotopy commutativity is neatly encoded by the theory of E_∞-operads [May 1972]. These are those symmetric topological operads such that the space $\mathcal{O}_n$ is Σ_n-contractible for all $n \geq 0$. As the homotopy type of such operads is determined by the homotopy theory of the underlying spaces $\mathcal{O}_n$, it follows that all E_∞-operads are homotopy equivalent [Berger and Moerdijk 2003]. In particular, there is a unique (up to homotopy) way for a space to be homotopy commutative.

We now move to an equivariant setting. We fix a finite group G and consider topological spaces equipped with a G-action. In this setting, constructing an appropriate version of homotopy commutativity via E_∞-operads has its difficulties. For example, there are G-operads whose underlying nonequivariant operads are E_∞, but whose derived category of algebras are inequivalent.

MSC2010: primary 18D50, 55P91; secondary 06A07, 52B20, 55N91.
Keywords: operads, equivariant spectra, ring spectra, associahedra, Catalan numbers.

To correctly encode homotopy commutativity in the equivariant setting, N_∞-operads were developed. Each N_∞-operad encodes a different notion of homotopy commutativity with respect to the structure of the group. This is in stark contrast to the nonequivariant case where there was a unique way of being homotopy commutative. Understanding different N_∞-operads for a group G is, however, in general, challenging. As such, having access to a combinatorial framework in which to study them is of great value.

Recent work by Blumberg and Hill [2015] led to the conjecture, soon verified by [Bonventre and Pereira 2021; Gutiérrez and White 2018; Rubin 2021], that for a group G, the data of an N_∞-operad is equivalent to a certain "indexing system". In Section 2 we show that this again is equivalent to a subgraph of the lattice of subgroups satisfying two rules. Such a description appears under the name of *transfer systems* in [Rubin 2019]. This opens the door to a more combinatorial approach to studying these operads for a fixed group G, which sets the stage for this current paper.

We start with the case of G being a cyclic group C_{p^n} in Section 3. A constructive approach leads to our first result that there are $\mathsf{Cat}(n+1)$ many N_∞-operads for C_{p^n}, where $\mathsf{Cat}(n)$ denotes the n-th Catalan number. In particular, there are as many N_∞-operads for C_{p^n} as there are binary trees with $n+2$ leaves.

The relation does not just stop there, though. Binary trees are one way of encoding associahedra (also known as *Tamari lattices* or *Stasheff polytopes*), where a binary tree corresponds to a vertex, and two vertices are related by a directed edge if one tree can be obtained from another by moving a branch to the right. On the other side, the set of all N_∞-operads for C_{p^n} can be ordered by inclusion of the corresponding graphs. We prove that these two posets are in fact isomorphic as posets, i.e., there is an isomorphism between N_∞-operads and binary trees which is order-preserving and order-reflecting.

When moving to a general cyclic group, unfortunately, one will quickly find the combinatorics of the N_∞-operads unmanageable. This is due to the fact that in the corresponding graph diagram of an N_∞-operad for $C_{p_1^{n_1}\cdots p_k^{n_k}}$, the edges not induced from the C_{p^i} become hard to describe. We explain this phenomenon in Section 4 by developing the terms of *pure* and *mixed* N_∞-operads and give a nontrivial lower bound for the number of N_∞-operads for an arbitrary finite cyclic group G.

This new approach of N_∞-operads as graph diagrams sheds light on the mysterious nature of the theory of equivariant homotopy commutativity. In particular, we have provided a finite and therefore computable approach to an involved problem.

2. A brief tour of the theory of N_∞-operads

We shall assume that the reader is somewhat familiar with G-equivariant homotopy theory in the sense of May [1996]. We shall assume that G is a finite group throughout. Our objects of interest, N_∞-operads, are a special class of G-operad, whence we begin our exposition.

Definition 1. A *G-operad* $\mathcal{O}$ is a symmetric operad in G-spaces. That is, we have a sequence of $(G \times \Sigma_n)$-spaces $\mathcal{O}_n$ for all $n \geqslant 0$ such that

(1) there is a G-fixed identity element $1 \in \mathcal{O}_1$,

(2) there are G-equivariant composition maps

$$\mathcal{O}_k \times \mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k} \to \mathcal{O}_{n_1+\cdots+n_k}$$

which satisfy the usual compatibility conditions with each other and the symmetric group actions.

A certain subclass of G-operads, known as N_∞-operads, is used to describe different levels of homotopy commutativity in genuine G-equivariant stable homotopy theory; see Blumberg and Hill [2015]. That is, they are a generalization of E_∞-operads to the equivariant setting. Recall that for a group G, a *family* $\mathcal{F}$ is a collection of subgroups which is closed under passage to subgroups and conjugacy. A *universal space* for a family $\mathcal{F}$ is a G-space $E\mathcal{F}$ such that for all subgroups H,

$$(E\mathcal{F})^H \simeq \begin{Bmatrix} * & H \in \mathcal{F} \\ \varnothing & H \notin \mathcal{F} \end{Bmatrix}.$$

Definition 2. An *N_∞-operad* is a G-operad $\mathcal{O}$ such that

(1) the space $\mathcal{O}_0$ is G-contractible,

(2) the action of Σ_n on $\mathcal{O}_n$ is free,

(3) $\mathcal{O}_n$ is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$ for $H \leqslant G$.

We will denote by $N_\infty(G)$ the collection of all (homotopy classes of) N_∞-operads for a given group G.

Although Definition 2 is perfectly good for theoretical purposes, we need a more computationally exploitable definition of N_∞-operads. We first introduce the intermediary notion of indexing systems.

Definition 3. A *categorical coefficient system* is a contravariant functor $\underline{\mathcal{C}} : O_G^{\mathrm{op}} \to$ **Cat** from the orbit category of G to the category of small categories. Such a coefficient system is called *symmetric monoidal* if it takes values in symmetric monoidal categories and strong monoidal functors. We are particularly interested in the coefficient system $\mathcal{S}et$ with disjoint union which sends a subgroup H to the category $\mathbf{Set}^H$ of H-sets, where **Set** denotes the category of finite sets. A subsymmetric coefficient $\underline{\mathcal{C}}$ of $\mathcal{S}et$ is said to be an *indexing system* if it is closed under subobjects and self-induction (i.e., $T \in \underline{\mathcal{C}}(K)$ and $H/K \in \underline{\mathcal{C}}(H)$ implies $H \times_K T \in \underline{\mathcal{C}}(H)$).

The following result was first conjectured in [Blumberg and Hill 2015] and has subsequently been proven to hold in three independent articles by Bonventre and Pereira; Gutiérrez and White; and Rubin. The result uses the existence of a model structure on the category of N_∞-operads whose weak equivalences are those maps which (at level n) induce weak homotopy equivalences after taking Γ-fixed points for all $\Gamma \subseteq G \times \Sigma_n$.

Proposition 4 [Bonventre and Pereira 2021; Gutiérrez and White 2018; Rubin 2021]. *The homotopy category of N_∞-operads is equivalent to the poset category of indexing systems.*

An algebra for an N_∞-operad has structure above and beyond being a G-spectrum whose underlying nonequivariant spectrum is an E_∞-algebra. This additional structure was fundamental to Hill, Hopkins and Ravenel [Hill et al. 2016].

Theorem 5 [Blumberg and Hill 2018, Theorems 4.13 and 4.14]. *Let G be a finite group and $\mathcal{O}$ an N_∞-operad. If R is an $\mathcal{O}$-algebra in G-spectra, then the Mackey functor $\underline{\pi}_0(R)$ is an $\mathcal{O}$-Tambara functor.*

That is, each $\underline{\pi}_0(R)(G/H) := \pi_0^H(R)$ is a commutative ring, the restriction maps are monoidal and the transfer maps satisfy the Frobenius relations. Furthermore, if $H/K \in \underline{\mathcal{C}}(H)$, for $\underline{\mathcal{C}}$ the indexing system determined by $\mathcal{O}$, there is a multiplicative (but not usually additive) norm map

$$N_K^H : \pi_0^K(R) \longrightarrow \pi_0^H(R).$$

The norm maps satisfy Frobenius-type relations describing their interaction with addition and the restriction and transfer maps. An $\mathcal{O}$-Tambara functor is also known as an *incomplete Tambara functor*. In particular, the maps N_H^H are always present and are the identity maps. If one has a norm map for each pair of subgroups $K \leqslant H$ of G, then the homotopy groups of an $\mathcal{O}$-algebra are a Tambara functor in the original sense of [Tambara 1993].

We now compare the notion of indexing systems to *transfer systems* from [Rubin 2019]. This notion was also independently discovered by the authors.

Lemma 6 [Rubin 2019, §6]. *An indexing system determines, and is determined by, a set $\mathcal{F}_H$ for each $H \leqslant G$ consisting of subgroups K of H, written as H/K, satisfying the following axioms*:

(identity) $H/H \in \mathcal{F}_H$.

(conjugation) $H/K \in \mathcal{F}_H$ *implies* $gHg^{-1}/gKg^{-1} \in \mathcal{F}_{gHg^{-1}}$.

(restriction) $H/K \in \mathcal{F}_H$ *implies* $M/(M \cap K) \in \mathcal{F}_M$ *for all* $M \leqslant H$.

(composition) $H/K \in \mathcal{F}_H$ *and* $K/L \in \mathcal{F}_K$ *implies* $H/L \in \mathcal{F}_H$.

We call this data a ***transfer system****.*

We rewrite this definition into a form more directly useful for our purposes.

Definition 7. Given a transfer system $\mathcal{F}_H$, define a set (of abstract symbols)

$$\{N_K^H \mid H/K \in \mathcal{F}_H, \quad H \leqslant G\}$$

and call the symbol N_K^H a *norm map* from K to H.

Corollary 8. *Let G be a finite group. Up to homotopy, an N_∞-operad for G is the data of a set of norm maps $X = \{N_K^H\}_{1 \leqslant K < H \leqslant G}$ satisfying the following rules (and all conjugates thereof):*

(restriction) *If $N_K^H \in X$ and $M < H$, then $N_{K\cap M}^M \in X$.*

(composition) *If $N_L^K \in X$ and $N_K^H \in X$, then $N_L^H \in X$.*

In particular, homotopy classes of N_∞-operads can be described as certain subgraphs of the lattice of subgroups of G.

Proof. Up to homotopy, an N_∞-operad $\mathcal{O}$ uniquely specifies a transfer system $\mathcal{F}$. We know that $H/K \in \mathcal{F}_H$ implies

$$M/(M \cap K) \in \mathcal{F}_M.$$

In terms of norm maps, this is precisely the restriction rule. The second axiom of a transfer system says that $H/K \in \mathcal{F}_H$ and $K/L \in \mathcal{F}_K$ implies $H/L \in \mathcal{F}_H$. In terms of norm maps, this is precisely the composition rule.

For the converse, we construct a transfer system from a set of norm maps X satisfying the two axioms of the statement. We define $\mathcal{F}_H$ as the set of H/K such that $N_K^H \in X$ and H/H. One can check this defines a transfer system and hence a homotopy class of N_∞-operads. $\square$

Remark 9. Our notation has been chosen so that the Mackey functor of an algebra R over an N_∞-operad $\mathcal{O}$ has multiplicative norm maps $N_K^H : \pi_0^K(R) \to \pi_0^H(R)$ whenever N_K^H is in the (abstract) set of norm maps.

Remark 10. Given a set of norm maps $\{N_K^H\}$ which do not satisfy the rules of Corollary 8, then one can find a minimal set X containing $\{N_K^H\}$ which does satisfy the rules. We refer to this as the *completion* of the set $\{N_K^H\}$ to a transfer system X. See also [Rubin 2019, Theorem A.2].

This result leads to the following corollary, which motivates the results in this paper, namely, that for a finite group G, it makes sense to attempt to enumerate the number of N_∞-operad structures, and to understand the associated poset structure.

Corollary 11. *Let G be a finite group. Then the number of N_∞-operads for G is finite. Moreover, the set $N_\infty(G)$ admits a canonical poset structure given by inclusions of sets of the corresponding transfer systems.*

3. The case $G = C_{p^n}$

We will begin with the case of cyclic groups of the form C_{p^n}. We note that the choice of p here is arbitrary as the subgroup lattices of C_{p^n} and C_{q^n} are both isomorphic to the poset $\underline{n} = \{0 < 1 < \cdots < n\}$. To ease the notation we shall denote by N_i^j the norm map $N_{C_{p^i}}^{C_{p^j}}$ for $i \leqslant j$.

Before we continue to the theoretics, let us manually compute the first handful of values of $|N_\infty(C_{p^n})|$. The purpose of this is two-fold. Firstly it will give the reader an idea of how such computations are done, and second, for the avid integer sequence fan, these examples will suggest the general form for the sequence $\{|N_\infty(C_{p^n})|\}_{n\in\mathbb{N}}$. Note that we will not write the identity norm maps N_i^i and shall only consider the nontrivial norm maps.

Example 12. The case of $G = C_{p^0}$ is trivial. That is, there are no choices of nontrivial indexing systems to make, and therefore $|N_\infty(C_{p^0})| = 1$. This is exactly the fact that for nonequivariant stable homotopy theory, there is only a single notion of commutativity as one may expect. We will write the single (up to homotopy) N_∞-operad structure as $\{\varnothing\}$ to indicate that there are no nontrivial norm maps.

Example 13. The situation for $G = C_p$ is only marginally more involved than the trivial case. Here we have a subgroup lattice $\{C_{p^0} < C_{p^1}\}$. Therefore the only choice to make is if we wish to include the nontrivial norm map N_0^1 or not. Therefore there are two N_∞-operad structures (up to homotopy), namely $\{\varnothing\}$ and $\{N_0^1\}$.

Example 14. We shall now look at $G = C_{p^2}$. This is the first case where we need to take care of the rules appearing in Corollary 8. As always, we have the trivial N_∞-operad $\{\varnothing\}$ which we shall write diagrammatically as

$$\left(C_{p^0} \qquad C_{p^1} \qquad C_{p^2} \right) \quad = \quad \{\varnothing\}\,.$$

At the other extreme, we could add in all of the norm maps. One can easily check the conditions to see that this will always be a valid N_∞-operad. We shall display this N_∞-operad as

$$\left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd} \right) \quad = \quad \{N_0^1, N_1^2, N_0^2\}$$

where an arrow from C_{p^i} to C_{p^j} indicates the existence of the norm map N_i^j for $i < j$.

The technical part then, of course, is to identify what other N_∞-operads can appear between these two extremes. There are 2^3 different possibilities to try (indeed, there are three different norm maps which we must choose whether to include or not). Instead of investigating all of the remaining cases, we shall just show the failure of the ones that do not have an N_∞-operad structure. Figures 1, 2 and 3 give the invalid diagrams.

$$\text{✗} \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_0^1, N_1^2\}$$

Figure 1. This diagram is not valid as it violates the composition rule of Corollary 8. If we were to complete this diagram in the sense of Remark 10 to get a valid N_∞-operad then we would need to add in the norm map N_0^2, and we would get the previous operad.

$$\text{✗} \qquad \left(\begin{tikzcd} C_{p^0} \arrow[rr, bend left] & C_{p^1} & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_0^2\}$$

Figure 2. This diagram is not valid as it does not satisfy the restriction rules. To satisfy the rule we would need to also have the norm map N_0^1, and then all of the rules would be satisfied. The resulting operad would be different from the above two.

$$\text{✗} \qquad \left(\begin{tikzcd} C_{p^0} \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_1^2, N_0^2\}$$

Figure 3. This is the final invalid diagram, which suffers from the same deficiency as the one above; that is, it does not satisfy the restriction rules.

Consequently, we can write down the elements of $N_\infty(C_{p^2})$. Note that in particular, $|N_\infty(C_{p^2})| = 5$. We implore the reader to check these for themselves to gain confidence with the rules of Corollary 8. The valid N_∞-operad structures are as follows:

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} & C_{p^1} & C_{p^2} \end{tikzcd}\right) \quad = \quad \{\varnothing\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_0^1\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_0^1, N_0^2\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_1^2\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) \quad = \quad \{N_0^1, N_1^2, N_0^2\}$$

From our first analysis, we have obtained the integer sequence $1, 2, 5$ counting the number of N_∞-operads for C_{p^0}, C_{p^1} and C_{p^2} respectively. If one were to take the time to check the possibilities for C_{p^3}, they would see that there are 14 possibilities. Therefore the examples suggest a relation to the Catalan numbers. The next section

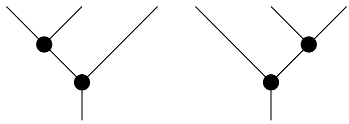


Figure 4. The two binary trees giving the second Catalan number.

will be devoted to recalling the necessary results regarding the Catalan numbers before we prove the first main result, Theorem 20, which says that $|N_\infty(C_{p^n})|$ coincides with the $(n+1)$-st Catalan number.

3.1. ***A recollection of the Catalan numbers.*** The Catalan numbers are a sequence of numbers which regularly appear in enumeration problems. The n-th Catalan number, which we denote $\mathsf{Cat}(n)$, is given as

$$\mathsf{Cat}(n) = \frac{(2n)!}{(n+1)!n!} .$$

The first few terms of the sequence are therefore $\mathsf{Cat}(0) = 1$, $\mathsf{Cat}(1) = 1$, $\mathsf{Cat}(2) = 2$, $\mathsf{Cat}(3) = 5$ and $\mathsf{Cat}(4) = 14$. There are many surprising and amusing ways to define the Catalan numbers. Let us recall a few.

- $\mathsf{Cat}(n)$ is the number of valid expressions containing n pairs of parentheses.
- $\mathsf{Cat}(n)$ is the number of triangulations of a regular $(n+2)$-gon.
- $\mathsf{Cat}(n)$ is the number of rooted binary trees with $n+1$ leaves.

This is but a few of a multitude of descriptions given in [Stanley 1999]. The last interpretation involving binary trees will be our canonical representation. Figure 4 gives the corresponding binary trees in the case of $n = 2$.

The following well-known recurrence relation will be fundamental to the proof of the main result in this section.

Lemma 15. *The Catalan numbers satisfy, and are determined by, the recurrence relation*

$$\begin{aligned} \mathsf{Cat}(0) &= 1, \\ \mathsf{Cat}(n+1) &= \sum_{i=0}^{n} \mathsf{Cat}(i)\mathsf{Cat}(n-i) \quad \text{for } n \geqslant 0. \end{aligned}$$

3.2. ***An operation on N_∞-operads.*** To facilitate the proof of Theorem 20, we first introduce a binary operation

$$\odot : N_\infty(C_{p^i}) \times N_\infty(C_{p^j}) \to N_\infty(C_{p^{i+j+2}}).$$

To be able to define this function explicitly, we need some auxiliary notation. We consider an $X \in N_\infty(C_{p^i})$ as being described by its finite set of norm maps.

Secondly, for formal reasons, we will fix the convention that $N_\infty(C_{p^{-1}})$ is defined to be the set containing only the empty set.

For $X \in N_\infty(C_{p^i})$ and $Y \in N_\infty(C_{p^j})$, we now define $X \odot Y \in N_\infty(C_{p^{i+j+2}})$ to be the N_∞-operad described by the set of norm maps

$$X \odot Y := X \coprod \Sigma^{i+2} Y \coprod \{N_{i+1}^k\}_{i+1<k\leq i+j+2}.$$

Here, Σ is the following notation. For a norm map $N_{k_1}^{k_2}$, we write

$$\Sigma^n N_{k_1}^{k_2} := N_{k_1+n}^{k_2+n}$$

and let $\Sigma^n Y$ denote the set resulting from applying Σ^n to each norm map in Y. The symbol Σ is used to invoke the idea of a suspension or shift.

Figures 5, 6 and 7 give a pictorial presentation of $X \odot Y$. We exclude the norm maps for X and Y from the diagrams for clarity. Note that, in particular, we can see that this operation is not commutative (nor is it associative).

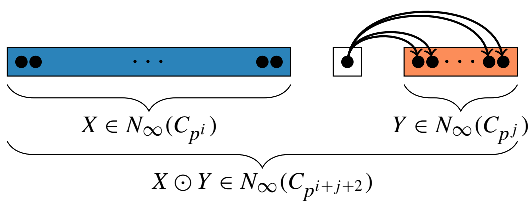


Figure 5. The general picture for the operation $X \odot Y$. We highlight that the last vertex in X occurs at position i, the pivot point is in spot $i+1$, and the first vertex of the suspension of Y occurs at position $i+2$.

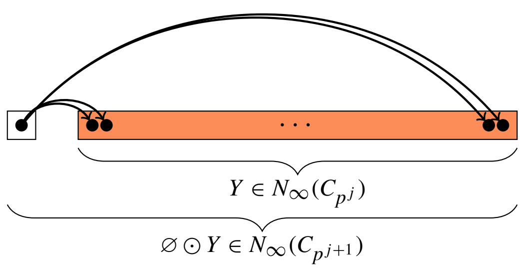


Figure 6. The general picture for the operation $\varnothing \odot Y$.

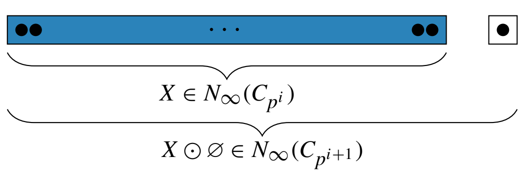


Figure 7. The general picture for the operation $X \odot \varnothing$.

Let us give some explicit examples of this construction before we prove that the resulting set of norm maps does indeed give an N_∞-operad as we have claimed.

Example 16. Let

$$X = \left(\begin{tikzcd} C_{p^0} & C_{p^1} \end{tikzcd} \right) \in N_\infty(C_{p^1}), \qquad Y = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} \end{tikzcd} \right) \in N_\infty(C_{p^1}).$$

Then

$$X \odot Y = \left(\begin{tikzcd} C_{p^0} & C_{p^1} & C_{p^2} \arrow[r] \arrow[rr, bend left] & C_{p^3} \arrow[r] & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4}),$$

and

$$Y \odot X = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \arrow[r] \arrow[rr, bend left] & C_{p^3} & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4}).$$

Example 17. Let

$$X = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \arrow[r] & C_{p^3} \end{tikzcd} \right) \in N_\infty(C_{p^3})$$

$$Y = N_\infty(C_{p^{-1}}) = \varnothing.$$

Then

$$X \odot Y = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \arrow[r] & C_{p^3} & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4})$$

and

$$Y \odot X = \left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] \arrow[rrrr, bend left] & C_{p^1} \arrow[r] & C_{p^2} & C_{p^3} \arrow[r] & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4}).$$

Proposition 18. *For $X \in N_\infty(C_{p^i})$ and $Y \in N_\infty(C_{p^j})$, $X \odot Y$ satisfies the rules of Corollary 8, and therefore is a valid object in $N_\infty(C_{p^{i+j+2}})$ for $-1 \leqslant i, j$. Moreover, the converse is true; that is, if $X \odot Y \in N_\infty(C_{p^{i+j+2}})$, then it follows that X and Y are both valid N_∞-operads for their respective groups.*

Proof. We must check that $X \odot Y$ satisfies the restriction and composition conditions. The simplest way to do this is to appeal to Figure 5. First of all, note that the norm maps coming from X are disjoint from the rest of the structure, and as we have assumed that X is a valid N_∞-operad for $G = C_{p^i}$, this part does not need further consideration.

The restriction rule for the remaining norm maps is clear. This rule is satisfied due to the addition of the norm maps $\{N_{i+1}^k\}_{i+1<k\leq i+j+2}$. The composition rule will be satisfied because Y was chosen to be in $N_\infty(C_{p^j})$, and suspending it to its new position will not affect this.

To see the converse of the statement, take two lattices X and Y of size i and j respectively such that $X \odot Y \in N_\infty(C_{p^{i+j+2}})$. We first of all note that X must be an object of $N_\infty(C_{p^i})$. Clearly if Y was not an object in $N_\infty(C_{p^j})$, then neither would its shift. Therefore it only remains to show that the addition of the norm maps $\{N^k_{i+1}\}_{i+1<k\leq i+j+2}$ has no possibility of invalidating Y. As mentioned above, adding these maps only serves to ensure the restriction rule is satisfied for the additional point, hence they cannot turn Y into a invalid diagram. □

Remark 19. The $\odot$ operation has an operadic interpretation, as explained to the authors by J. Rubin. Suppose that we have two transfer systems X and Y which realise the operads $\mathcal{O}$ and $\mathcal{P}$ which are C_{p^i} and C_{p^j} N_∞-operads, respectively.

The inclusion $C_{p^i} \hookrightarrow C_{p^{i+j+2}}$ gives rise to a left derived induction functor, which, when applied to $\mathcal{O}$, realises X as a $C_{p^{i+j+2}}$ N_∞-operad. Similarly, the quotient map $\pi : C_{p^{i+j+2}} \to C_{p^j}$ gives a left derived restriction functor, which, when applied to $\mathcal{P}$, realises $\Sigma^{i+2}Y$. Finally, there is a little disks operad $\mathcal{D}$ which realises the set of norm maps $\{N^k_{i+1}\}_{i+1<k\leq i+j+2}$. The homotopy coproduct of these three operads realises $X \odot Y$.

In particular, this result tells us that it is possible to use a homotopy colimit construction to inductively form the homotopy categories of N_∞-operads for $G = C_{p^n}$. Of course, one hopes for a general result like this for arbitrary G, but as we will see in Section 4, the situation becomes extremely complex.

3.3. ***Computing the cardinality of $N_\infty(C_{p^n})$.*** We now come to the first main result of this paper which gives the link between the set of N_∞-operads for C_{p^n} and the Catalan numbers. We shall prove that the cardinalities of these sets satisfy the defining recurrence relation for the Catalan numbers, and then we show how to construct a bijection between these N_∞-operads and binary trees.

Theorem 20. *The cardinalities $|N_\infty(C_{p^n})|$ satisfy the recurrence relation*

$$|N_\infty(C_{p^{-1}})| = 1,$$
$$|N_\infty(C_{p^n})| = \sum_{i=0}^{n} |N_\infty(C_{p^{i-1}})||N_\infty(C_{p^{n-i-1}})| \quad \text{for } n \geqslant 0.$$

In particular we have that $|N_\infty(C_{p^n})| = \mathsf{Cat}(n+1)$.

Proof. To prove this we shall show that every N_∞-operad in $Z \in N_\infty(C_{p^n})$ can be written in the form $X \odot Y$ for (unique) $X \in N_\infty(C_{p^{i-1}})$ and $Y \in N_\infty(C_{p^{n-i-1}})$. This fact, along with Proposition 18, completes the argument.

Suppose that $Z \in N_\infty(C_{p^n})$. We let $k \in \mathbb{Z}$ be the minimum integer such that the norm map N^n_k is in Z. We have three cases to deal with here, either $k = 0$, $0 < k < n$ or $k = n$ (i.e., there is no such norm map). We start with the two extreme cases before dealing with the intermediate one.

- When $k = 0$, we construct Z as $X \odot Y$ for $X = \varnothing \in N_\infty(C_{p^{-1}})$, and Y an N_∞-operad for $G = C_{p^{n-1}}$ as in Figure 6.

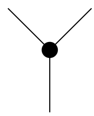
- When $k = n$, we construct Z as $X \odot Y$ for $Y = \varnothing \in N_\infty(C_{p^{-1}}))$, and X an N_∞-operad for $G = C_{p^{n-1}}$ as in Figure 7.
- When $0 < k < n$, we observe that we have two disjoint parts to Z; namely we are able to split off the subgroups C_{p^i} for $0 \leqslant i < k$. Let us denote this part as X (which lives in $N_\infty(C_{p^{k-1}})$, and the remaining part Z'. The crucial observation to make now is that Z' looks like $\varnothing \odot Y$ for some $Y \in N_\infty(C_{p^{n-k-1}})$. We therefore conclude that $Z = X \odot Y$ as required. □

Corollary 21. *Every N_∞-operad Z for $G = C_{p^n}$ can be decomposed uniquely as $Z = X \odot Y$ for some N_∞-operads X and Y.*

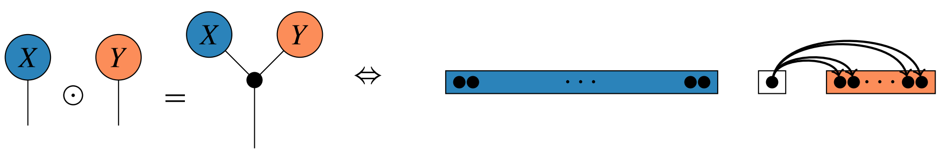
Corollary 22. *There is a bijection of sets*

$$\{N_\infty(C_{p^n})\} \Leftrightarrow \{\textit{rooted binary trees with } n{+}2 \textit{ leaves}\}.$$

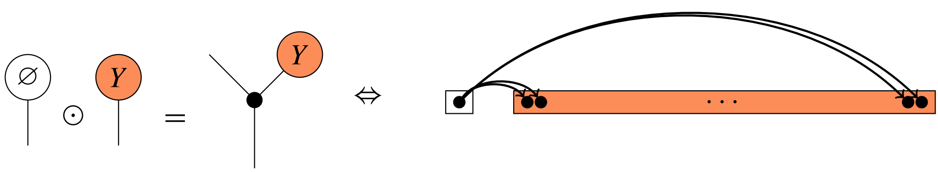
Proof. This follows immediately from Theorem 20 and the discussion in Section 3.1; however, it will be beneficial to the next section to spell out exactly how the correspondence works inductively. To the trivial N_∞-operad for $G = C_{p^0}$ we assign the following binary tree:



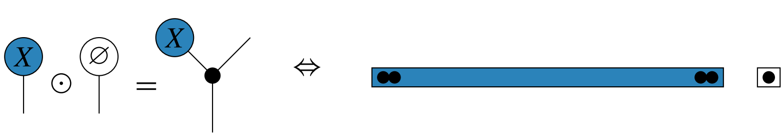
We will make the convention that $\varnothing$ is the empty tree. Assume that $n > 0$; we know from Theorem 20 that any N_∞-operad is of the form $X \odot Y$. We then have a binary tree associated to X and a binary tree associated to Y, and we can form the binary tree associated to $X \odot Y$ in the following way:



Following the convention of the empty diagram, we see that



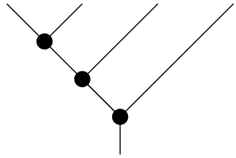
and



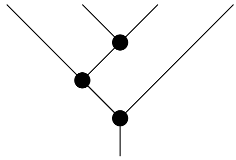
□

Example 23. One may use the above algorithm to compute the binary trees associated to the objects of $N_\infty(C_{p^2})$ as follows:

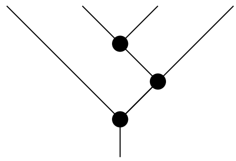
$$\left(C_{p^0} \qquad C_{p^1} \qquad C_{p^2} \right) \Leftrightarrow$$



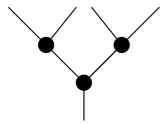
$$\left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \end{tikzcd} \right) \Leftrightarrow$$



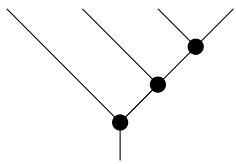
$$\left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} & C_{p^2} \end{tikzcd} \right) \Leftrightarrow$$



$$\left(\begin{tikzcd} C_{p^0} & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd} \right) \Leftrightarrow$$



$$\left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd} \right) \Leftrightarrow$$



3.4. ***The relation to the associahedron.*** We shall now see that the relationship between $N_\infty(C_{p^n})$ and the Catalan numbers runs deeper than just the result of Theorem 20. Recall that we can put an order on binary trees. Indeed, let X and Y be binary trees with $n+1$ edges. Then we say that $X < Y$ if Y can be obtained from X by a (finite sequence of) *clockwise* tree rotation operations, i.e., by moving a branch from left to right.

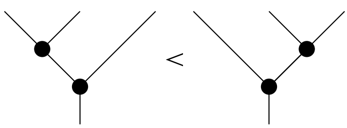


Figure 8. An example of an order relation between two binary trees.

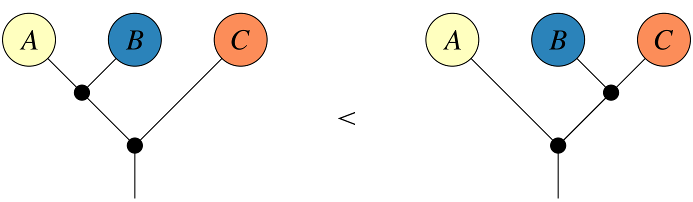


Figure 9. A general example of an order relation between two binary trees.

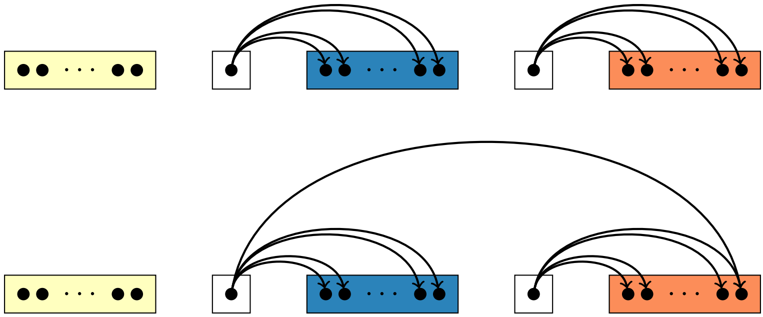


Figure 10. The diagrams corresponding to the trees in Figure 9.

A more general example is given by Figure 9. Furthermore, if a binary tree X were to contain the left-hand side as a subtree, then we could make a binary tree Y by replacing that subtree with the right-hand side. We would see that Y is obtained from X by a (finite sequence of) clockwise tree rotation operations, so $X < Y$.

The poset structure on the set of binary trees with $n+1$ edges is known as the n-associahedron; see [Stasheff 1963]. We shall denote this poset structure as $\mathcal{A}_n$.

We can also implement a poset structure on $N_\infty(C_{p^n})$ by fixing that $X < Y$ if Y can be obtained from X via the addition of norm maps. For example, we have

$$\left(C_{p^0} \qquad\qquad C_{p^1} \right) < \left(C_{p^0} \longrightarrow C_{p^1} \right)$$

Remark 24. Note that $Z \odot -$ and $- \odot Z$ preserve this ordering. That is, if $X < Y$ then it follows that $Z \odot X < Z \odot Y$ and $X \odot Z < Y \odot Z$.

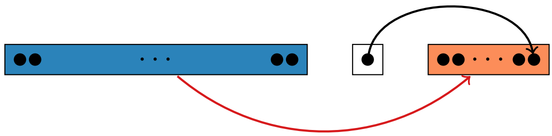
We now prove our main theorem which tells us that these poset structures actually agree.

Theorem 25. *There is an order-preserving and order-reflecting bijection of posets*

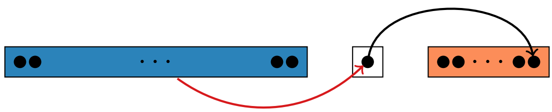
$$\{N_\infty(C_{p^n})\} \Leftrightarrow \{\textit{rooted binary trees with } n{+}2 \textit{ leaves}\} \Leftrightarrow \mathcal{A}_{n+1}.$$

Proof. Let us begin by showing that a clockwise tree rotation corresponds to the addition of an arrow in the corresponding N_∞-diagram, or more specifically, the addition of a norm map. We shall do this by appealing to the diagrammatic representations, as it provides the cleanest proof. Consider a branch move as in Figure 9. The left-hand tree in Figure 9 corresponds to the first norm diagram in Figure 10, which we may call $(A \odot B) \odot C$. The right-hand tree corresponds to the second (where restrictions of the largest arrow are omitted for clarity), which is $A \odot (B \odot C)$.

We now show that adding an arrow in a norm diagram induces a clockwise tree rotation in the corresponding binary trees. We shall do this by induction on n. Note that the base case can be easily checked, see Figure 8, and the corresponding discussion of the general case. Example 23 then illustrates the next case. Suppose that we begin with an arbitrary norm diagram $A \odot Z$ to which we have added the red arrow.



We can assume that the new arrow starts in A and ends in Z. By the composition and restriction rule, we can, without loss of generality, assume that the new arrow ends at the pivot (the added vertex in $A \odot Z$).



We can split up the left-hand block into a diagram of the form $X \odot Y$ for some smaller diagrams X and Y. This gives three different cases to consider based on where the new arrow begins. These situations are summarised in Figure 11. In particular, the new arrow could start in Y, giving Case 1, it could start in X, giving Case 3, or the final option is that the new arrow begins at the vertex arising from the $\odot$ operation in $X \odot Y$.

Case 2 has already been covered in the beginning of this proof, as adding an arrow connecting the two pivots is equivalent to adding an arrow from the first pivot to the rightmost vertex. Therefore, it is the situation described in Figures 9 and 10. In that part we illustrated the corresponding tree move to the addition of such an arrow.

For Case 1, it is important to note that adding the arrow (1) creates the arrow (2) via composition, but that adding the arrow (2) does *not* create arrow (1) by any of the rules. However, we can say that adding arrow (1) is equivalent to first adding arrow (2) and then arrow (1).

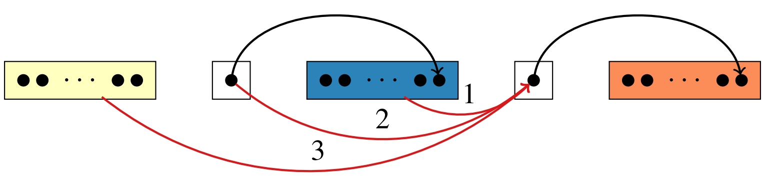
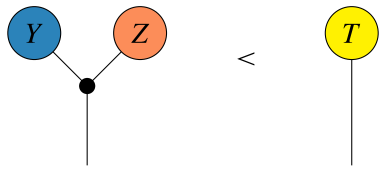
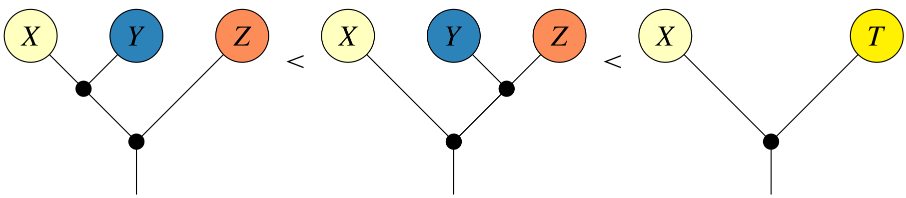


Figure 11. The three cases for adding a nontrivial norm map to $(X \odot Y) \odot Z$

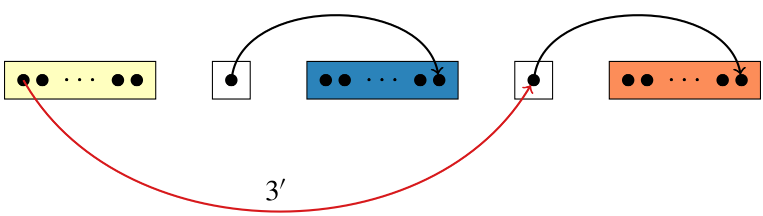
Hence, we add arrow (2) to $(X \odot Y) \odot Z$. By Case 2, this gives $X \odot (Y \odot Z)$ and in terms of trees gives Figure 9. We then add arrow (1). This only affects the term $Y \odot Z$, giving a new diagram T with $Y \odot Z < T$. Moreover, $Y \odot Z$ and T are in $\{N_\infty(C_{p^k})\}$ for $k < n$. By induction, we have that the trees corresponding to $Y \odot Z$ and T are ordered correctly.



As the order on norm maps is preserved by $X \odot -$, we have $X \odot (Y \odot Z) < X \odot T$; see Remark 24. The corresponding operation on trees also preserves the order to give the following:



For Case 3, we repeat our earlier decomposition. Either, the new arrow starts at the leftmost vertex or we can split X into $X_1 \odot X_2$ and see that we are in Case 1 or Case 2 (which are solved) or we are in Case 3 for this new decomposition. For Case 3, we can continue to split up X_1 into smaller diagrams $X_1 = X_1' \odot X_2'$, and so on. Continuing in this way, the only new case is the following, which we recognise as adding the arrow $(3')$ from the leftmost vertex to the pivot of $Y \odot Z$:



The arrow $(3')$ adds the arrow (α) in Figure 12 by restriction (but not vice versa). Therefore, we can obtain our diagram by first adding (α) and then $(3')$. The diagram $X \odot Y$ with (α) added is of the form $\varnothing \odot B$ for some other B. In particular, $X \odot Y \leq \varnothing \odot B$, which by induction induces the correct ordering on the corresponding trees. Using the order-preserving properties of $\odot$ we have $(X \odot Y) \odot Z \leq (\varnothing \odot B) \odot Z$.

If we now add the arrow $(3')$ to $((\varnothing \odot B) \odot Z$, we are adding a new arrow from a pivot to a pivot, which is Case 2. $\square$

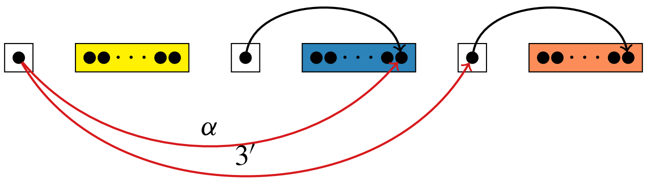


Figure 12. The arrow additions for Case 3.

4. Generalising to other cyclic groups

We would like to have a closed formula for the cardinality of $N_\infty(G)$ for all finite cyclic G. We shall explore the obstructions to obtaining such a result in this section. The main result is the construction of a lower bound of the number of such operads for $G = C_{p_1^{n_1}} \cdots C_{p_k^{n_k}}$. Let us highlight the style of norm maps that we must deal with in this situation. Figure 13 gives the 10 possible N_∞-operads for $G = C_{pq}$.

A key observation to make is that there is an "odd one out" among these diagrams. In particular, consider the following:

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q & C_{pq} \end{tikzcd}$$

This transfer system is different from the other nine because it is the only one where the diagonal is not forced by the composition and restriction rules of Corollary 8. That is, if we were to remove the norm map N_1^{pq}, then the resulting diagram is still a valid N_∞-operad. It follows that this N_∞-operad cannot be formed by just combining those for $G = C_p$ and $G = C_q$. We will call such an operad *mixed*. If it can be obtained from the component groups, then we will call it *pure*.

The main result of this section will be to give a closed expression for the number of pure N_∞-operads for $G = C_{p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}}$, which provides a nontrivial lower bound for the total number of N_∞-operads for G.

$$\begin{tikzcd} C_1 & C_p \\ C_q & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r] & C_p \\ C_q & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[d] & C_p \\ C_q & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r] & C_p \\ C_q \arrow[r] & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[d] & C_p \arrow[d] \\ C_q & C_{pq} \end{tikzcd}$$

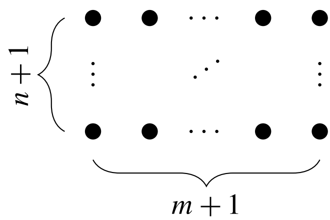
$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_p \\ C_q & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q \arrow[r] & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \arrow[d] \\ C_q & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q & C_{pq} \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \arrow[d] \\ C_q \arrow[r] & C_{pq} \end{tikzcd}$$

Figure 13. The 10 possible N_∞-operad structures for $G = C_{pq}$.

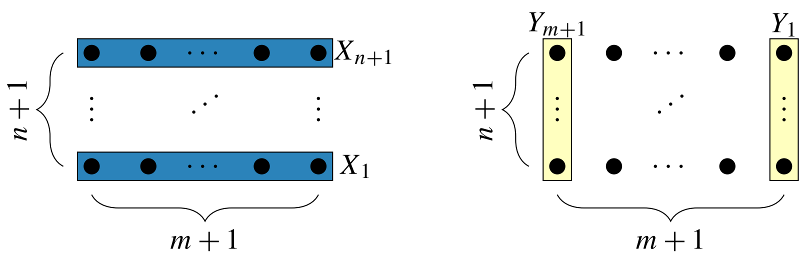
Trying to manually enumerate the norm maps for $G = C_{p^n q^m}$, $p \neq q$ or even just $C_{p^3 q}$ shows that the situation is already intangibly complicated. Indeed, we have computationally verified that there are 544 N_∞-operads for $C_{p^3 q}$.

4.1. ***Enumerating pure operads.*** We begin with a more formal definition of "pure" and "mixed".

Let Z be an N_∞-operad for $G = C_{p^n q^m}$. That is, Z is an N_∞-diagram on the lattice below:



Then we can consider the rows and columns of these diagrams to obtain a family of diagrams for $G = C_{p^m}$, namely $\{X_i\}_{1 \leqslant i \leqslant n+1}$ and a family of diagrams for $G = C_{p^n}$, namely $\{Y_i\}_{1 \leqslant i \leqslant m+1}$. Note that these are indeed valid diagrams as can be seen from observing the restriction and composition rules:



We shall say that an N_∞-operad is *pure* if it is completely determined by the systems $\{X_i\}$ and $\{Y_j\}$ in the sense of Remark 10. If an operad is not pure, then we will say that it is *mixed*. Note that an operad is mixed if and only if after removing all norm maps of the form $N_{p^i}^{p^j q}$, $j \neq i$, and completing the set of norm maps according to the rules of Corollary 8, one does *not* recover the original operad one started with.

Example 26. The following operad is pure as it has no diagonals, that is, no norm maps of the form $N_{p^i}^{p^j q}$. Therefore there is no condition to check

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_p \\ C_q & C_{pq} \end{tikzcd}$$

The following is also pure, as when we remove the diagonal (highlighted in red) then the composition rule of Corollary 8 is violated. Completing the set of norm maps according to the rules forces the diagonal, and we recover the original operad that we started with:

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q \arrow[r] & C_{pq} \end{tikzcd}$$

By using the restriction rules, we see that there is a natural ordering on the systems $\{X_i\}$ and $\{Y_j\}$. Indeed, $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_{n+1}$ and $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_{m+1}$.

Definition 27. We will denote by $\mathcal{P}(n,r)$ the number of length r paths in the n-Tamari lattice $\mathcal{A}_n$. For example, $\mathcal{P}(n,2)$ gives the sequence 1, 1, 3, 13, 68, 399, 2530, 16965, ... (starting at $n=0$). In [Châtel and Pons 2015] this is given the closed form

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Theorem 28. *The number of pure N_∞-operads for $G = C_{p^n}C_{q^m}$ is given as*

$$\mathcal{P}(n+1,m)\mathcal{P}(m+1,n).$$

In general, for $G = C_{p_1^{n_1}} \cdots C_{p_k^{n_k}}$ the number of pure operads is

$$\prod_{j,i=1}^{k} \mathcal{P}(n_i+1, n_j).$$

Proof. This is an exercise in counting using the orderings $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_{n+1}$ and $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_{m+1}$. Once we have picked X_1, we must take a (possibly stationary) path of length n through the Tamari lattice $\mathcal{A}_{m+1}$ to pick the other entries. Therefore, there are $\mathcal{P}(m+1,n)$ such options for the X_i. We then have the choices for the Y_j giving us a total of $\mathcal{P}(n+1,m)$ options via a similar argument. Combining these, we get the required total of $\mathcal{P}(n+1,m)\mathcal{P}(m+1,n)$.

The proof for the general case follows similarly. □

Example 29. One can compute the first few values for the sequence appearing in Theorem 28 (starting at $n=0$ for $m=1$) to be 1, 9, 52, 340, 2394, 17710, ... This sequence does not appear on the OEIS at the time of writing.

Acknowledgements

We are very grateful for support and hospitality from the University of Kent and from the Isaac Newton Institute for Mathematical Sciences during the programme "Homotopy harnessing higher structures", which was supported by EPSRC grant EP/R014604/1. We furthermore thank Anna Marie Bohmann and Magdalena Kędziorek for helpful discussions, as well as Jonathan Rubin for insightful comments on an earlier version of this paper.

References

[Berger and Moerdijk 2003] C. Berger and I. Moerdijk, "Axiomatic homotopy theory for operads", *Comment. Math. Helv.* **78**:4 (2003), 805–831. MR Zbl

[Blumberg and Hill 2015] A. J. Blumberg and M. A. Hill, "Operadic multiplications in equivariant spectra, norms, and transfers", *Adv. Math.* **285** (2015), 658–708. MR Zbl

[Blumberg and Hill 2018] A. J. Blumberg and M. A. Hill, "Incomplete Tambara functors", *Algebr. Geom. Topol.* **18**:2 (2018), 723–766. MR Zbl

[Bonventre and Pereira 2021] P. Bonventre and L. A. Pereira, "Genuine equivariant operads", *Adv. Math.* **381** (2021), art. id. 107502. MR Zbl

[Châtel and Pons 2015] G. Châtel and V. Pons, "Counting smaller elements in the Tamari and m-Tamari lattices", *J. Combin. Theory Ser. A* **134** (2015), 58–97. MR Zbl

[Gutiérrez and White 2018] J. J. Gutiérrez and D. White, "Encoding equivariant commutativity via operads", *Algebr. Geom. Topol.* **18**:5 (2018), 2919–2962. MR Zbl

[Hill et al. 2016] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, "On the nonexistence of elements of Kervaire invariant one", *Ann. of Math.* (2) **184**:1 (2016), 1–262. MR Zbl

[May 1972] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics **271**, Springer, 1972. MR Zbl

[May 1996] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics **91**, Amer. Math. Soc., Providence, RI, 1996. MR Zbl

[Rubin 2019] J. Rubin, "Characterizations of equivariant Steiner and linear isometries operads", 2019. arXiv

[Rubin 2021] J. Rubin, "Detecting Steiner and linear isometries operads", *Glasg. Math. J.* **63**:2 (2021), 307–342. MR

[Stanley 1999] R. P. Stanley, *Enumerative combinatorics, Vol. 2*, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1999. MR Zbl

[Stasheff 1963] J. D. Stasheff, "Homotopy associativity of H-spaces, II", *Trans. Amer. Math. Soc.* **108** (1963), 293–312. MR Zbl

[Tambara 1993] D. Tambara, "On multiplicative transfer", *Comm. Algebra* **21**:4 (1993), 1393–1420. MR Zbl

Received March 9, 2020. Revised January 10, 2021.

SCOTT BALCHIN
MAX PLANCK INSTITUTE FOR MATHEMATICS
BONN
GERMANY
balchin@mpim-bonn.mpg.de

DAVID BARNES
MATHEMATICAL SCIENCES RESEARCH CENTRE
QUEEN'S UNIVERSITY BELFAST
UNITED KINGDOM
d.barnes@qub.ac.uk

CONSTANZE ROITZHEIM
SCHOOL OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE
UNIVERSITY OF KENT
CANTERBURY
UNITED KINGDOM
csrr@kent.ac.uk

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Robert Lipshitz
Department of Mathematics
University of Oregon
Eugene, OR 97403
lipshitz@uoregon.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2021 is US $520/year for the electronic version, and $705/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2021 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 315 No. 2 December 2021