ON THE RATIONALIZATION OF THE $K(n)$-LOCAL SPHERE

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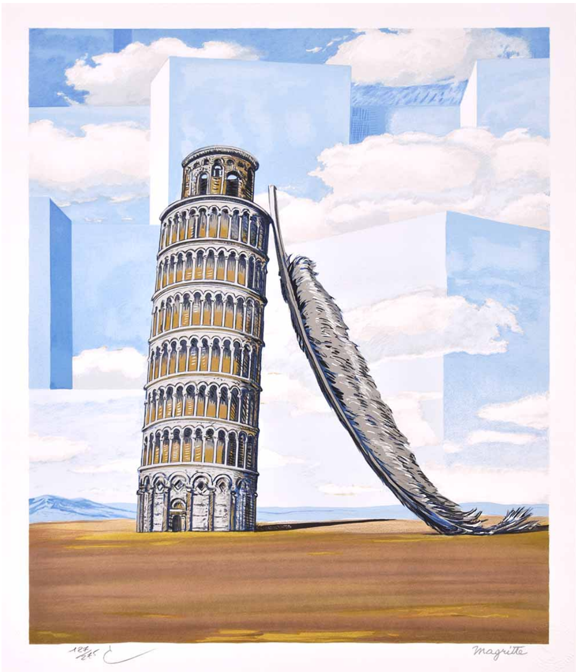
Abstract. We compute the rational homotopy groups of the $K(n)$-local sphere for all heights n and all primes p, verifying a prediction that goes back to the pioneering work of Morava in the early 1970s. More precisely, we show that the inclusion of the Witt vectors into the Lubin–Tate ring induces a split injection on continuous stabilizer cohomology with torsion cokernel of explicit bounded exponent, thereby proving Hopkins' chromatic splitting conjecture and Goerss's vanishing conjecture rationally. The key ingredients are the equivalence between the Lubin–Tate tower and the Drinfeld tower due to Faltings and Scholze–Weinstein, integral p-adic Hodge theory, and an integral refinement of a theorem of Tate on the Galois cohomology of non-archimedean fields.



René Magritte, *Memory of a Journey*, 1955.

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Contents

1. Introduction

A central problem in homotopy theory is to understand the homotopy groups of spheres $\pi_{k+d}S^k$, i.e., the group of continuous maps $S^{k+d} \to S^k$ up to homotopy. Since every space may be approximated by a CW complex built from iteratively attaching cells, the homotopy groups encode how all spaces are built up to homotopy. It follows from the Freudenthal suspension theorem that $\pi_{k+d}S^k$ stabilizes for $k > d+1$, so one may first seek to determine the stable homotopy groups of spheres, $\pi_d S^0 := \lim_{k\to\infty} \pi_{k+d}S^k$. These are abelian groups, which vanish for all $d < 0$, while $\pi_0 S^0 \cong \mathbb{Z}$ encoding the degree of a map, while Serre showed that $\pi_d S^0$ is finitely generated and torsion in all positive degree d.

However, early attempts at understanding $\pi_* S^0$ through explicit calculations in small degrees only provided limited information about the large-scale structure. Chromatic homotopy theory begins with the deep observation that the elements of $\pi_* S^0$ may be organized into certain periodic families of increasing periodicity. From a modern perspective, these periodic families are captured by a filtration on $S^0_{(p)}$, the localization of the sphere spectrum for any given prime number p, obtained from localizations $L_n S^0$ of the sphere spectrum by passing to homotopy groups. These localizations provide successive approximations to the sphere spectrum

$$S^0 \to \ldots \to L_n S^0 \to \ldots \to L_1 S^0 \to L_0 S^0 \simeq S^0_{\mathbb{Q}}$$

whose associated graded pieces are given by the $K(n)$-local spheres $L_{K(n)}S^0$, implicitly depending on p. The homotopy limit of this tower recovers $S^0_{(p)}$, and so a fundamental problem in the field is to understand the homotopy groups $\pi_* L_{K(n)}S^0$.

In the case $n = 0$, we have $L_{K(0)}S^0 \cong S^0_{\mathbb{Q}}$, the rational sphere spectrum. As an immediate consequence of the aforementioned theorem of Serre, $\pi_* S^0_{\mathbb{Q}} \cong \mathbb{Q} \otimes \pi_* S^0$ is $\mathbb{Q}$ in degree 0 and 0 otherwise. In the case $n = 1$, $\pi_* L_{K(1)}S^0$ was calculated in the 1970s by Adams–Baird (unpublished) and Ravenel [Rav84]. The case $n = 2$ took many years of work by many people and was only recently resolved (see [SY95, SW02b, SW02a, GHMR05, Beh12, Koh13, BGH22] for an incomplete list); even stating the answer is very involved. Consequently, a full computation of $\pi_* L_{K(n)}S^0$ for $h > 2$ seems to be out of reach.

In light of this, attention over the last few decades has gradually turned towards understanding structural features of $\pi_* L_{K(n)}S^0$. Since the 1970s and motivated by the work of Lazard and Morava, a guiding problem has been to determine the location of the free $\mathbb{Z}_p$-summands in $\pi_* L_{K(n)}S^0$ or, equivalently, to understand $\mathbb{Q} \otimes \pi_* L_{K(n)}S^0$. Through the full force of the computations mentioned above, this is now known for $n \leq 2$ and all primes p. In this paper, we resolve this question completely for all n and all primes p:

Theorem A. *There is an isomorphism of graded $\mathbb{Q}$-algebras*

$$\mathbb{Q} \otimes \pi_* L_{K(n)}S^0 \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \ldots, \zeta_n),$$

where the latter is the exterior $\mathbb{Q}_p$-algebra on generators ζ_i in degree $1-2i$.

In particular, this result confirms the rational part of Hopkins' chromatic splitting conjecture [Hov95] for all primes p and all heights n. Previously, this was only known for $n \leq 2$ through the explicit computation in the works listed above. As such, Theorem A constitutes the first general result in the direction of this conjecture since the construction of the class $\zeta \in \pi_{-1}L_{K(n)}S^0$ by Devinatz and Hopkins [DH04] in the early 2000s. We will give a more thorough explanation of the homotopical context for our results in Section 2.

In order to explain our approach, we recall that the homotopy groups of $L_{K(n)}S^0$ can be approached algebraically through Lubin and Tate's deformation theory of formal groups. Let Γ_n be a formal group of dimension 1 and height n over $\overline{\mathbb{F}}_p$, and let $\mathbb{G}_n = \mathrm{Aut}(\Gamma_n, \overline{\mathbb{F}}_p)$ be the so-called Morava stabilizer group, defined as the group of automorphisms of Γ_n that lie over an automorphism of $\overline{\mathbb{F}}_p$. Then $\mathbb{G}_n$ is an extension of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ by $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_n)$. Let $W = W(\overline{\mathbb{F}}_p)$ be the ring of p-typical Witt vectors. By Lubin–Tate theory, there is a complete local ring $A \cong W[\![u_1, \ldots, u_{n-1}]\!]$, corepresenting deformations of Γ_n, which admits a continuous action by $\mathbb{G}_n$. Further, the invariant differentials ω of the universal deformation of Γ_n form an invertible A-module, and the natural actions of $\mathbb{G}_n$ on A and ω extend to an action of $\mathbb{G}_n$ on the graded ring $A_* = \bigoplus_{t\in 2\mathbb{Z}} \omega^{\otimes t/2}$ (i.e., A_* is evenly concentrated). The Devinatz–Hopkins spectral sequence [DH04] takes the form

$$H^s_{\mathrm{cts}}(\mathbb{G}_n, A_t) \implies \pi_{t-s}L_{K(n)}S^0,$$

where H^s_{cts} refers to continuous cohomology. Thus understanding the bi-graded ring $H^s_{\mathrm{cts}}(\mathbb{G}_n, A_t)$ is of great importance in chromatic homotopy theory. However, the action of $\mathbb{G}_n$ on A_* is extremely difficult to describe [DH95].

Consider instead the problem of computing $H^*_{\mathrm{cts}}(\mathbb{G}_n, W)$, where $\mathbb{G}_n$ acts on W through its quotient $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. A classical theorem of Lazard [Laz65] states that the cohomology of a p-adic Lie group G with $\mathbb{Q}_p$-coefficients can be computed in terms of Lie algebra cohomology. Applied to $G = \mathcal{O}_D^\times$, Lazard's theorem provides an isomorphism of graded $\mathbb{Q}_p$-algebras:

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, W[1/p]) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \ldots, x_n) \tag{1.0.1}$$

Here, the right hand side is the exterior $\mathbb{Q}_p$-algebra on generators x_i of degree $2i-1$.

Remarkably, and verified by extensive calculations for heights $n \leq 2$ over the last 40 years, work of Morava [Mor85] from the early 1970s suggests that the natural map of $\mathbb{Z}_p$-modules

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, W) \longrightarrow H^*_{\mathrm{cts}}(\mathbb{G}_n, A) \tag{1.0.2}$$

is a rational isomorphism; i.e., it becomes an isomorphism after inverting p. The main result of this paper establishes a refinement of this conjecture:

Theorem B. *For every integer $s \geq 0$, the natural map $W \hookrightarrow A$ induces a split injection*

$$H^s_{\mathrm{cts}}(\mathbb{G}_n, W) \hookrightarrow H^s_{\mathrm{cts}}(\mathbb{G}_n, A)$$

whose complement is killed by a power of p independent of s. In particular,

$$H^s_{\mathrm{cts}}(\mathbb{G}_n, W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^s_{\mathrm{cts}}(\mathbb{G}_n, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

We have not attempted to make explicit the power of p which kills the complement of $H^s_{\mathrm{cts}}(\mathbb{G}_n, W)$ in $H^s_{\mathrm{cts}}(\mathbb{G}_n, A)$, though this should be possible by our methods. In fact, at heights $n = 1, 2$ this complement is zero, suggesting that this might be the case in general; this is known as the "chromatic vanishing conjecture".

The proof of Theorem B is summarized in Section 3.5. It relies upon recent advances in p-adic geometry. Ultimately we draw much of our power from the isomorphism, due to Faltings [Fal02a] and clarified by Scholze and Weinstein [SW13], between the Lubin–Tate and Drinfeld towers. Faltings' isomorphism may be regarded as an equivalence of stacks:

$$[\mathrm{LT}/\mathbb{G}_n] \simeq [\mathcal{H}/\mathrm{GL}_n(\mathbb{Z}_p)]. \tag{1.0.3}$$

The cohomology of the stack on the left accesses $H^*_{\mathrm{cts}}(\mathbb{G}_n, A)$. The main idea is to use this isomorphism to replace the opaque action of $\mathbb{G}_n$ on Lubin–Tate space LT with the far more transparent action of the group $\mathrm{GL}_n(\mathbb{Z}_p)$ on Drinfeld's symmetric space $\mathcal{H}$. Theorem A is then readily deduced from Theorem B via the the Devinatz–Hopkins spectral sequence.

The groups appearing in (1.0.3) are profinite, and accordingly the stacks appearing there must be construed as living on the pro-étale topology on rigid-analytic spaces. Thereby the pro-étale cohomology of rigid-analytic spaces, first considered in [Sch13a] and expanded in [BMS18], enters the picture as an indispensable tool. Much of this article is concerned with controlling the pro-étale cohomology of a rigid-analytic space X over a local field K of characteristic $(0,p)$, for example $X = \mathrm{LT}$ or $X = \mathcal{H}$.

The results we obtain are new even for the case when $X = \operatorname{Spa} K$ is a single point. In that case our result (Theorem 4.0.3) is a refinement of a classical theorem of Tate. It states a bound on the torsion part of $H^1_{\mathrm{cts}}(\mathrm{Gal}(\overline{K}/K), \mathcal{O}_C)$, where C is the completion of $\overline{K}$ and $\mathcal{O}_C$ is the ring of integers in C.

Outline of the document. Since this paper is written with an audience of both arithmetic geometers and homotopy theorists in mind, we have chosen to include the additional background material that might be familiar for one group but not necessarily the other. In that spirit, we begin in Section 2 with a rapid review of the key players in chromatic homotopy theory, focusing on stating the chromatic splitting conjecture, before constructing the desired splitting and deducing Theorem A from Theorem B. Section 3 then collect preliminary material from arithmetic geometry, including the notion of adic space, a short treatment of continuous cohomology from the condensed perspective, and the pro-étale topology. The section concludes with an outline of the proof of Theorem B, see Section 3.5, which is then fleshed out in the rest of the paper. In Section 4, we establish an integral refinement of a theorem of Tate, by determining bounds on the torsion exponents in the Galois cohomology of $\mathcal{O}_C$ and its Tate twists, where C is the completion of an algebraic closure of a local field K of mixed characteristic. We then globalize this result in Section 5 to the pro-étale cohomology of the generic fiber of a semistable formal scheme over $\mathcal{O}_K$ with coefficients in the sheaf of bounded functions. Finally, in Section 6, we put all the pieces in action to prove Theorem B, by first applying methods separately to the Drinfeld tower and the Lubin–Tate tower, and then deducing our main theorem via the isomorphism of towers.

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2. Chromatic homotopy theory

The goal of this section is to place Theorem B in the context of stable homotopy theory and deduce Theorem A from Theorem B. To this end, we begin with a rapid review of some relevant material from chromatic homotopy theory, before turning to the applications. We refer the reader interested in a more thorough introduction to the subject to the following sources: [Lur10, BB20, BGH22]. The homotopy theory experts can safely skip ahead to the new results, beginning in Section 2.5.

2.1. Chromatic characteristics.

Our starting point is the chromatic perspective on the category of spectra as envisioned by Morava [Mor85] and Ravenel [Rav84] and established by Devinatz, Hopkins, and Smith [DHS88, HS98]. This story has been told by many, and we take a revisionistic approach following [BB20], also freely using the language of higher algebra as developed by Lurie in [Lur09, Lur17].

In order to motivate the homotopical constructions, let us first recast some familiar concepts from algebra in more category-theoretic language; the resulting definitions can then be transported more easily to higher algebra. Let $\mathrm{Mod}(\mathbb{Z})$ be the symmetric monoidal abelian category of abelian groups. A non-trivial unital associative ring $A \in \mathrm{Mod}(\mathbb{Z})$ is said to be a division algebra (or skew field) if any module over A is free. Two division algebras A and B are said to be of the same characteristic if $A \otimes_{\mathbb{Z}} B \neq 0$. It is straightforward to verify that this notion induces an equivalence relation on the collection of all division algebras in $\mathrm{Mod}(\mathbb{Z})$.

It turns out that we can classify all such characteristics: Indeed, the minimal representative of the equivalence classes of characteristics of division algebras in $\mathrm{Mod}(\mathbb{Z})$ are given by the prime fields $\mathbb{F}_p$ for primes p and $\mathbb{Q}$. This fact is essentially a translation of the basic classification of prime fields in classical algebra.

Stable homotopy theory is the study of the category Sp of spectra, which forms a higher analogue of the category of abelian groups; Waldhausen and May coined the term 'brave new algebra.' The role of the integers is then played by the sphere spectrum S^0, and the tensor product is replaced by the smash product written as $\otimes$ or $\otimes_{S^0}$ for emphasis. Equipped with this structure, Sp forms a symmetric monoidal stable ∞-category; we may therefore speak of rings and their modules in this setting. Interpreted in this context, formally there is an identification $\mathrm{Sp} = \mathrm{Mod}(S^0)$. The next definition is then the natural higher algebraic counterpart to the concept of characteristic as discussed above:

Definition 2.1.1. A *division algebra* in Sp is a unital associative ring spectrum A such that every A-module M splits into a direct sum of shifts of free rank 1 modules. Two division algebras $A, B \in \mathrm{Sp}$ are of the same *(chromatic) characteristic* if and only if $A \otimes_{S^0} B \neq 0$.

We again obtain an equivalence relation on the collection of all division algebras in Sp, so we are naturally led to ask if we can understand the equivalence classes. This question has been answered completely in the aforementioned seminal work of Devinatz, Hopkins, and Smith. Stating their classification in the form we want requires a short detour. Let p be a prime, let κ be a perfect field of characteristic p, let Γ be a 1-dimensional height n commutative formal group over κ, and finally let $n \in \mathbb{N} \cup \{\infty\}$. It is an insight of Morava, based on earlier work of Quillen, that this data lifts to Sp: there exists a multiplicative cohomology theory $K(\Gamma, \kappa)^*$ with the following properties:

(1) The value of $K(\Gamma, \kappa)^*$ on a point is given by

$$K(\Gamma, \kappa)^*(\mathrm{pt}) \cong \begin{cases} \mathbb{Q} & \text{if } n = 0 \\ \kappa[v_n^{\pm 1}] & \text{if } 0 < n < \infty \\ \kappa & \text{if } n = \infty, \end{cases}$$

where v_n is a formal variable in degree $2p^n - 2$.

(2) $K(\Gamma, \kappa)^*$ is complex oriented, and the $K(\Gamma, \kappa)^*$-cohomology of complex projective space represents the formal group Γ:

$$\mathrm{Spf}(K(\Gamma, \kappa)^*(\mathbf{CP}^\infty)) \cong \Gamma.$$

(3) $K(\Gamma, \kappa)^*$ satisfies the Künneth formula for any two spectra X, Y:

$$K(\Gamma, \kappa)^*(X \otimes Y) \cong K(\Gamma, \kappa)^*(X) \otimes_{K(\Gamma,\kappa)^*} K(\Gamma, \kappa)^*(Y).$$

By Brown representability, $K(\Gamma, \kappa)^*$ is represented in the category of spectra by a (unital and associative) ring spectrum $K(\Gamma, \kappa)$, known as *Morava K-theory* (at height n and over the field κ). Since the ring of coefficients $K(\Gamma, \kappa)^*(\mathrm{pt})$ is a graded field, $K(\Gamma, \kappa)$ itself must be a division algebra in the sense of Definition 2.1.1. For example, if $\hat{\mathbb{G}}_a$ is the formal additive group, then $K(\hat{\mathbb{G}}_a, \mathbb{Q})$ and $K(\hat{\mathbb{G}}_a, \mathbb{F}_p)$ represent singular cohomology with coefficients in $\mathbb{Q}$ and $\mathbb{F}_p$, respectively. If $\hat{\mathbb{G}}_m$ is the formal multiplicative group, then $K(\hat{\mathbb{G}}_m, \mathbb{F}_p)$ is a summand of mod p complex topological K-theory. Generalizing the last example, for any prime p and any height $n \in \mathbb{N}$, there exists a formal group Γ_n of height n over $\mathbb{F}_p$. Mildly abusing notation, we set $K(n, p) \coloneqq K(\Gamma_n, \mathbb{F}_p)$, keeping the prime implicit. (After base change to an algebraically closed field, Γ_n is unique up to isomorphism by a theorem of Lazard [Laz75].) We also set $K(0, p) = K(\hat{\mathbb{G}}_a, \mathbb{Q})$ and $K(\infty, p) = K(\hat{\mathbb{G}}_a, \mathbb{F}_p)$.

Armed with a good collection of division algebras, we can now return to the classification of characteristics in Sp to state:

Theorem 2.1.2 (Devinatz–Hopkins–Smith). *The collection of Morava K-theories $K(n, p)$ for p ranging through the primes and $n \in \mathbb{N} \cup \{\infty\}$ forms a complete and pairwise distinct set of representatives for the characteristics of division algebras in* Sp*. Moreover, the $K(n)$s are minimal in the sense that any division algebra of the same characteristic as $K(n, p)$ is a module over $K(n, p)$.*

In other, more plain terms: the Morava K-theories $K(n, p)$ provide precisely the prime fields of the category of spectra. A couple of remarks are in order.

- (Non-commutativity) In contrast to the situation in classical algebra, the Morava K-theories in intermediate characteristic $0 < n < \infty$ cannot be made commutative. In fact, they do not even afford the structure of an $\mathbb{E}_2$-ring spectrum, see for instance [ACB19]. This is the main reason to work with division algebras in the definition of characteristic.
- (Interrelation) A finitely generated abelian group M with $M \otimes \mathbb{Q} \neq 0$ also satisfies $M \otimes \mathbb{F}_p \neq 0$ for all primes p. This statement has a chromatic refinement: If X is a finite spectrum, i.e., a compact object in Sp, then $K(n, p)^*(X) \neq 0$ implies $K(n+1, p)^*(X) \neq 0$. Note that both the classical algebraic as well as the chromatic statement are false in general for non-compact objects.

To access and isolate the part of Sp that is visible to a fixed Morava K-theory $K(n, p)$, we need another important tool from stable homotopy theory, namely the theory of Bousfield localization. These form a suitable generalization of localizations and (derived) completions familiar from commutative algebra.

Fix an arbitrary spectrum $M \in \mathrm{Sp}$. A spectrum X is said to be *E-acyclic* if $M_*(X) = 0$; a spectrum Y is then called *M-local* if any map $X \to Y$ from an M-acyclic spectrum X is null, i.e., factors through a zero object. Intuitively, we wish to quotient Sp by the ideal of all M-acyclic spectra to focus on those spectra which are "seen" by M. Bousfield [Bou79] rigorously proved that this works, thereby constructing a localization functor $L_M \colon \mathrm{Sp} \to \mathrm{Sp}$ with the following properties:

(1) $L_M X = 0$ if and only if X is M-acyclic;

(2) L_M is idempotent and has essential image spanned by the M-local spectra;
(3) for any $Z \in \mathrm{Sp}$, there is a natural map $Z \to L_M Z$ which exhibits $L_M Z$ as the initial map out of Z to an M-local spectrum.

These properties characterize L_M uniquely up to homotopy. Finally, we denote the full subcategory of M-local spectra by Sp_M; alternatively, this category is obtained from Sp by inverting all M-equivalences, i.e., those maps which induce isomorphisms in M_*-homology. Via localization, Sp_M inherits a structure of symmetric monoidal ∞-category from Sp, with tensor product given by the *localized* smash product.

Some examples might be illuminating. If $M = \mathbb{Q}$, then $L_{\mathbb{Q}}$ is rationalization, whose effect on homotopy groups of any spectrum is tensoring with $\mathbb{Q}$. The element $p \in \mathbb{Z} \cong \pi_0 S^0$ is represented by a map $p\colon S^0 \to S^0$, whose cofiber we denote by S^0/p, the *mod p Moore spectrum.* On the one hand, the local category $\mathrm{Sp}_{S^0/p}$ is the category of p-complete spectra, and localization at S^0/p has the effect of derived p-completion on homotopy groups. It is customary to write $X_p \coloneqq L_{S^0/p}X$ and $\mathrm{Sp}_p \coloneqq \mathrm{Sp}_{S^0/p}$. On the other hand, localizing at $M = S^0[1/p]$, the colimit over multiplication by p on S^0, has the effect of inverting p on S^0. Similarly, we can construct spectral analogues of p-localization, by inverting all primes but p, to obtain the category $\mathrm{Sp}_{(p)}$ of p-local spectra. For more information about various localizations on Sp, we refer to [Bou79].

The main example of interest to us is $\mathrm{Sp}_{K(n,p)}$, the so-called $K(n,p)$-local category. In light of Theorem 2.1.2, it forms an irreducible piece of the category of spectra, as we will explain momentarily, and it is one of the key objects of study in chromatic homotopy theory. Note that the functor $L_{K(n,p)}$ and thus $\mathrm{Sp}_{K(n,p)}$ only depend on the characteristic and are in particular independent of the choice of Γ_n.

2.2. **Chromatic divide and conquer.** From now on, we will restrict attention to the category of p-local spectra for a fixed prime p, and usually drop the prime from the notation. Intuitively speaking, the idea of the chromatic approach to stable homotopy theory is to filter the category of p-local spectra $\mathrm{Sp}_{(p)}$ by its subcategories of mixed chromatic characteristics $(0, 1, \ldots, n)$ for $n \to \infty$. Here we say that a spectrum has mixed chromatic characteristic $(0, 1, \ldots, n)$ if it is local with respect to the direct sum $K(0) \oplus K(1) \oplus \ldots \oplus K(n)$, and we write L_n for the associated Bousfield localization. There is then a sequence of Bousfield localization functors and natural transformations,

$$\ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \tag{2.2.1}$$

the so-called *chromatic tower*. Note that the bottom layer is rationalization $L_0 = L_{\mathbb{Q}}$. Also note that the infinite height has been omitted, a curiosity justified by the chromatic convergence theorem of Hopkins–Ravenel [Rav92]: If X is a p-local finite spectrum, then it can be recovered from its chromatic tower (2.2.1):

$$X \simeq \lim_n L_n X.$$

Given X, it is then sensible to ask for the graded pieces of its chromatic filtration, i.e., the difference between $L_n X$ and $L_{n-1}X$. This is captured by the *chromatic fracture square*, taking the form of a (homotopy) pullback square that exists for any height $n > 0$ and an arbitrary spectrum X:

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X. \end{array} \tag{2.2.2}$$

Geometrically, one should think of this square as being analogous to the gluing square for a sheaf over an open-closed decomposition of a space. In this picture, $L_{K(n)}X$ corresponds to the sheaf over a formal neighborhood of a point, while $L_{n-1}X$ is the restriction to the open complement of the point. The term $L_{n-1}L_{K(n)}X$ along with the maps pointing to it then control the gluing process.

The weakest form of Hopkins' *chromatic splitting conjecture* stipulates that the bottom horizontal map in (2.2.2) is split for $X = S^0$ (and hence for any finite spectrum X), so that the chromatic assembly process takes a particularly simple form. The strong form of the conjecture (Conjecture 2.4.2 below) gives a complete description of $L_{n-1}L_{K(n)}S^0$ in terms of the L_iS^0 for $i \leq n$. Assuming it, one could inductively reduce the study of the L_nS^0 to that of the $L_{K(n)}S^0$, whose homotopy groups are identified after inverting p by our Theorem A.

2.3. **Morava E-theory and the cohomology of the stabilizer group.** Just as a height n formal group over $\mathbb{F}_p$ gives rise to the spectrum $K(n)$, the Lubin–Tate ring also admits a spectral incarnation. Let Γ be a height n formal group over a perfect field κ of characteristic p. Let $A(\Gamma,\kappa)$ denote its ring of deformations, so that $A(\Gamma,\kappa) \cong W(\kappa)[\![u_1,\ldots,u_{n-1}]\!]$. An unpublished theorem of Goerss–Hopkins–Miller, revisited and extended by Lurie in [Lur18], lifts this data to a commutative algebra in $\mathrm{Sp}_{K(n)}$, called $E(\Gamma,\kappa)$, with the property that

$$\pi_*E(\Gamma,\kappa) \cong A(\Gamma,\kappa)[\beta,\beta^{-1}],\ |\beta| = 2.$$

The commutative ring spectrum $E(\Gamma,\kappa)$ is known as *Morava E-theory* or *Lubin–Tate theory.*

In fact, Goerss, Hopkins, and Miller prove something stronger. Consider the 1-category of formal groups over perfect fields FG. The objects of FG are given by pairs (Γ,κ) as above, and a morphism $(\Gamma,\kappa) \to (\Gamma',\kappa')$ in FG consists of a ring map $i\colon \kappa \to \kappa'$ together with an isomorphism of formal groups $i^*\Gamma \xrightarrow{\sim} \Gamma'$. Goerss, Hopkins, and Miller produce a fully faithful functor $E(-,-)\colon \mathrm{FG} \to \mathrm{CAlg}(\mathrm{Sp})$. It is important to note that the source is a 1-category and that the target is an ∞-category. Thus this theorem identifies a very rigid portion of $\mathrm{CAlg}(\mathrm{Sp})$, in which the mapping spaces are homotopy equivalent to sets.

The underlying spectrum of $E(\Gamma,\kappa)$ is easily constructed as a consequence of the Landweber exact functor theorem. That theorem produces cohomology theories out of the complex cobordism spectrum with specified formal groups, so long as these satisfy a certain hypothesis. The universal deformation of Γ over $A(\Gamma,\kappa)$ satisfies the hypothesis, and this gives rise to Morava E-theory. However, producing Morava E-theory as a commutative algebra in $\mathrm{Sp}_{K(n)}$ is quite a bit more difficult and requires either obstruction theory or a derived deformation theory of formal groups—this is the content of the Goerss–Hopkins–Miller theorem.

Let Γ_n be any formal group of height n over $\overline{\mathbb{F}}_p$; then Γ_n is unique up to isomorphism. We write $E_n = E(\Gamma,\overline{\mathbb{F}}_p)$. We let $\mathbb{G}_n = \mathrm{Aut}(\Gamma_n,\overline{\mathbb{F}}_p) \cong \mathrm{Aut}\, E_n$, the *Morava stabilizer group.* As a topological group, $\mathbb{G}_n$ may be identified with the profinite completion $\widehat{D^\times}$ of $D^\times$, where D is the central simple algebra of invariant $\frac{1}{n}$ over $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_D = \mathrm{End}_{\overline{\mathbb{F}}_p}(\Gamma_n)$.

It turns out that $L_{E_n} = L_n$, so E_n does not provide us with a new localization functor. Furthermore, there is a close relationship between E_n and $K(n)$, akin to the one between a local ring and its residue field. Since the ideal $I_n = (p,u_1,\ldots,u_{n-1})$ of π_0E_n is generated by a regular sequence, we may form a (not necessarily commutative) ring spectrum E_n/I_n by iterated cofibers. This will have the property that $\pi_*(E_n/I_n) \cong \pi_*(E_n)/I_n = \kappa[\beta,\beta^{-1}]$. The ring spectrum obtained in this way is equivalent as a spectrum to a finite direct sum of suspensions of $K(\Gamma_n,\kappa)$:

$$E_n/I_n \simeq \bigoplus_{0\leq i\leq p^n-2} \Sigma^{2i}K(\Gamma_n,\kappa).$$

Here, the direct sum accounts for the fact that E_n is 2-periodic, while the periodicity of $K(\Gamma_n,\kappa)$ is $2p^n-2$, an issue that could be avoided by considering the 2-periodizatiopn of $K(\Gamma_n,\kappa)$.

The unit in $\mathrm{Sp}_{K(n)}$ is the $K(n)$-local sphere, denoted $L_{K(n)}S^0$. Being the unit, $L_{K(n)}S^0$ is the initial object in $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$, so there is a canonical map of commutative algebras

$L_{K(n)}S^0 \to E_n$. Since $\mathbb{G}_n$ acts on E_n through commutative algebra automorphisms, this map is $\mathbb{G}_n$-equivariant for the trivial action on $L_{K(n)}S^0$. A result of Devinatz and Hopkins [DH04], reinterpreted in Rognes' framework of spectral Galois extensions [Rog08], says that the unit map $L_{K(n)}S^0 \to E_n$ exhibits E_n as a pro-Galois extension of $L_{K(n)}S^0$ in $\mathrm{Sp}_{K(n)}$, with Galois group $\mathbb{G}_n$. Concretely, this means that we have canonical equivalences of commutative ring spectra

$$L_{K(n)}S^0 \simeq E_n^{h\mathbb{G}_n} \quad \text{and} \quad L_{K(n)}(E_n \otimes E_n) \simeq C_{\mathrm{cts}}(\mathbb{G}_n, E_n), \tag{2.3.1}$$

where $C_{\mathrm{cts}}(\mathbb{G}_n, E_n)$ denotes the ring spectrum of continuous functions on $\mathbb{G}_n$ with coefficients in E_n. This enables us to run Galois descent along $L_{K(n)}S^0 \to E_n$. Form the associated $K(n)$-local cosimplicial Amitsur complex

$$L_{K(n)}S^0 \to E_n^{\hat{\otimes}\bullet+1} \coloneqq L_{K(n)}\left(E_n \rightrightarrows E_n \otimes E_n \mathrel{\substack{\to\\[-0.6ex]\to\\[-0.6ex]\to}} E_n^{\otimes 3} \dots\right), \tag{2.3.2}$$

where we have omitted the degeneracy maps from the display. Applying the homotopy groups to the resolution (2.3.2) and using (2.3.1) to identify the abutment and E_2-page, we obtain a Bousfield–Kan spectral sequence of signature

$$E_2^{s,t} \cong H^s_{\mathrm{cts}}(\mathbb{G}_n, \pi_t E_n) \implies \pi_{t-s}L_{K(n)}S^0, \tag{2.3.3}$$

where H^*_{cts} denotes cohomology with continuous cocycles.

Since $\mathbb{G}_n$ has finite virtual cohomological dimension, this spectral sequence provides an excellent approximation to the homotopy groups of $L_{K(n)}S^0$: it converges strongly with a finite horizontal vanishing line on some finite page, i.e., there exists $r \geq 2$ and some $N > 0$ such that $E_r^{s,t} = 0$ for all $s > N$. In fact, if p is odd and $2(p-1) \geq n^2$, then we may take $r = 2$ and $N = n^2$. The spectral sequence (2.3.3), often simply referred to as "the" descent spectral sequence in chromatic homotopy theory, provides a gateway between stable homotopy theory and p-adic geometry.

2.4. **The chromatic splitting conjecture and the vanishing conjecture.** Computational evidence at low heights $n \leq 2$ suggests that the continuous cohomology of the action of $\mathbb{G}_n$ on $W \subseteq \pi_0 E_n$ largely controls the behavior of $\pi_* L_{K(n)}S^0$, as we shall now explain. Recall the isomorphism (1.0.1)

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, W[1/p]) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \dots, x_n),$$

where the latter is the exterior $\mathbb{Q}_p$-algebra on generators x_i in degree $2i-1$.

Each of the classes x_i can be lifted[1] to a class $\tilde{x}_i$ in the integral cohomology ring $H^*_{\mathrm{cts}}(\mathbb{G}_n, W)$. Let

$$\varphi\colon H^*_{\mathrm{cts}}(\mathbb{G}_n, W) \to H^*_{\mathrm{cts}}(\mathbb{G}_n, \pi_0 E_n) \cong E_2^{*,0} \tag{2.4.1}$$

be the natural map induced from the inclusion $W \hookrightarrow A \cong \pi_0 E_n$.

Hopkins' *chromatic splitting conjecture*, recorded in [Hov95], predicts that the classes $\tilde{x}_i$ carry all the relevant information about the homotopy groups of $L_{K(n)}S^0$; more precisely:

Conjecture 2.4.2 (Chromatic splitting conjecture)**.** *For p odd and each $i = 1, \dots, n$, we have the following behavior.*

(1) For each $i = 1, \dots, n$, the class $\varphi(\tilde{x}_i)$ survives the spectral sequence (2.3.3), thereby giving rise to a homotopy class

$$e_i \in \pi_{1-2i}L_{K(n)}S^0,$$

or all the same, a map $e_i\colon S_p^{1-2i} \to L_{K(n)}S^0$.

(2) The composition $S_p^{1-2i} \xrightarrow{e_i} L_{K(n)}S^0 \to L_{n-1}L_{K(n)}S^0$ factors through a map $\overline{e}_i\colon L_{n-i}S_p^{1-2i} \to L_{n-1}L_{K(n)}S^0$.

[1] In the chromatic practice, there are certain preferred choices of lifts, but these will not matter for our purposes here. For details, we refer to [Hov95].

(3) The $\bar{e}_i$ induce an equivalence of spectra

$$L_{n-1}L_{K(n)}S^0 \cong \bigwedge_{i=1}^{n}(L_{n-i}S_p^{1-2i}) := \bigoplus_{\substack{0\le j\le n\\ 1\le i_1<\ldots<i_j\le n}} \left(\bigotimes_{k=1}^{j} L_{n-i_k}S_p^{1-2i_k}\right),$$

where the right hand side is indexed on the $\mathbb{Z}_p$-module generators of the exterior algebra $\Lambda_{\mathbb{Z}_p}(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$.

A more refined formulation of the conjecture as well as the necessary modifications for the prime 2 can be found in [BGH22]. It has been verified by explicit computation of both sides for heights $n \le 2$ and all primes p. After tensoring with $\mathbb{Q}$, the chromatic splitting conjecture predicts that

$$\mathbb{Q} \otimes \pi_* L_{K(n)}S^0 \cong \Lambda_{\mathbb{Q}_p}(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n), \ |e_i| = 1-2i. \tag{2.4.3}$$

Indeed, there are natural equivalences $\mathbb{Q} \otimes L_i X \simeq \mathbb{Q} \otimes X$ for all spectra X and all $i \ge 0$, so (3) of Conjecture 2.4.2 rationalizes to (2.4.3). Further chromatic consequences of Conjecture 2.4.2 can be found in [BB20].

A related question concerns the map φ appearing in (2.4.1). The following conjecture[2] was formulated by Beaudry, Goerss, and Henn in [BGH22, Page 3].

Conjecture 2.4.4 (Vanishing conjecture)**.** *The inclusion of coefficients $W \hookrightarrow \pi_0 E_n$ induces an isomorphism $H^*_{\mathrm{cts}}(\mathbb{G}_n, W) \cong H^*_{\mathrm{cts}}(\mathbb{G}_n, \pi_0 E_n)$.*

Conjecture 2.4.4 has been verified for all $n \le 2$ and for all primes p; moreover, in these cases, it does not need to be modified for $p = 2$, but rather accounts for the additional complications witnessed there. If correct, this conjecture would substantially simplify the task of understanding $\pi_* L_{K(n)}S^0$. Our goal in this paper is to prove (2.4.3) for all heights n and all primes p.

2.5. **Power operations and the splitting of $W \to A$.** Fix a prime p and a height $n \ge 1$, and let $E = E_n$ be Morava E-theory. Thus $A = E^0 \cong W[\![u_1, \ldots, u_{n-1}]\!]$, where $W = W(\overline{\mathbb{F}}_p)$. Our proof of Theorem A begins with the following proposition.

Proposition 2.5.1. *The inclusion $W \hookrightarrow A$ admits a continuous $\mathbb{G}_n$-equivariant (additive) splitting. In other words, there is a $\mathbb{G}_n$-equivariant decomposition of topological abelian groups*

$$A \cong W \oplus A^c.$$

Our proof of Proposition 2.5.1 uses input from homotopy theory, namely the power operations on Morava E-theory. We will briefly recall this theory here:

Before we can give the proof of the proposition, we need to show that power operations on E-theory are continuous in a suitable sense.

Power operations on Morava E-theory are a consequence of the $\mathbb{E}_\infty$-ring structure on E, which endows E with multiplication maps

$$(E^{\otimes m})_{h\Sigma_m} \to E.$$

This structure is essentially unique by a theorem of Goerss, Hopkins, and Miller [GH04]. For $m \in \mathbb{N}$, these are natural multiplicative operations

$$P^m : E^0 \to E^0(B\Sigma_m).$$

They are defined to be the composite $[S^0, E] \to [(S^0)_{h\Sigma_m}, (E^{\otimes m})_{h\Sigma_m}] \to [(S^0)_{h\Sigma_m}, E]$, where the first map is given by applying the mth tensor power (recalling that S^0 is the unit) and applying homotopy orbits for the resulting Σ_m-action and the second map makes use of the $\mathbb{E}_\infty$-ring structure on E.

[2]The name comes from the equivalent formulation that $H^*_{\mathrm{cts}}(\mathbb{G}_n, \pi_0 E_n/W) = 0$.

The operation P^0 is the constant function 1 and the operation P^1 is the identity map on E^0. Since the E-cohomology of a symmetric group is a free E^0-module, there is a Kunneth isomorphism

$$E^0(B\Sigma_i \times B\Sigma_j) \cong E^0(B\Sigma_i) \otimes_{E^0} E^0(B\Sigma_j).$$

Power operations have the property that, if $i + j = m$, then the composite

$$E^0 \xrightarrow{P^m} E^0(B\Sigma_m) \to E^0(B\Sigma_i \times B\Sigma_j) \cong E^0(B\Sigma_i) \otimes_{E^0} E^0(B\Sigma_j)$$

is $P^i \otimes P^j$.

Although P^m is not additive, by [BMMS86, Chapter VIII], the ideal $I_{\mathrm{tr}} \subseteq E^0(B\Sigma_m)$ generated by the images of the transfer maps along $\Sigma_i \times \Sigma_j \subseteq \Sigma_m$ has the property

$$P^m/I_{\mathrm{tr}} \colon E^0 \to E^0(B\Sigma_m)/I_{\mathrm{tr}}$$

is a ring map. A theorem of Strickland's [Str98] proves that $E^0(B\Sigma_m)/I_{\mathrm{tr}}$ is a finitely generated free E^0-module and that there is a canonical isomorphism of formal schemes over Lubin–Tate space

$$\mathrm{Spf}(E^0(B\Sigma_m)/I_{\mathrm{tr}}) \cong \mathrm{Sub}_m(\bar{\Gamma}),$$

where $\bar{\Gamma}$ is the universal deformation of the formal group Γ and $\mathrm{Sub}_m(\bar{\Gamma})$ is the formal scheme classifying subgroup schemes of order m in $\bar{\Gamma}$. Note that no such subgroup exists unless $m = p^k$ for some k and thus $\mathrm{Sub}_m(\bar{\Gamma}) = \emptyset$ if $m \neq p^k$. Ando, Hopkins, and Strickland [AHS04, Section 3] proved that the map P^m/I_{tr} classifies the deformation associated to the quotient of $\bar{\Gamma}$ by the universal subgroup of order m. In particular, they show that P^m/I_{tr} is a continuous ring map for all $m \in \mathbb{N}$.

Lemma 2.5.2. *The power operations on Morava E-theory are continuous with respect to the I_n-adic topology on E^0 and $E^0(B\Sigma_m)$.*

Proof. The proof makes use of the fact that P^m/I_{tr} is continuous as well as an application of Hopkins–Kuhn–Ravenel character theory [HKR00].

Assume that $m = \sum_{i=0}^{j} a_i p^i$ is the base p expansion of m. The restriction map

$$E^0(B\Sigma_m) \to \bigotimes_{i=0}^{j} E^0(B\Sigma_{p^i})^{\otimes a_i}$$

is injective as E is p-local and $\prod_{i=0}^{j} \Sigma_{p^k}^{\times a_i}$ contains the Sylow p-subgroup of Σ_m. All of the tensor products are over E^0. We are reduced to proving that P^{p^k} is continuous.

For $0 \leq i \leq k$, consider the E^0-algebra map

$$E^0(B\Sigma_{p^k}) \to (E^0(B\Sigma_{p^i})/I_{\mathrm{tr}})^{\otimes p^{k-i}}$$

given by restriction to $\Sigma_{p^i}^{\times p^{k-i}}$, applying the Kunneth isomorphism, and then taking the quotient by the ideal I_{tr}. The composite of the power operation P^{p^k} with this map is continuous since it may be identified with the map $(P^{p^i}/I_{\mathrm{tr}})^{\otimes p^{k-i}}$.

Taking the product of these maps for all $0 \leq i \leq k$, we get a map

$$E^0(B\Sigma_{p^k}) \to \prod_{i=0}^{k} (E^0(B\Sigma_{p^i})/I_{\mathrm{tr}})^{\otimes p^{k-i}}.$$

Hopkins–Kuhn–Ravenel character theory implies that this map is injective. Since the composite of the p^kth power operation with this map is continuous, the p^kth power operation is continuous. □

Proof of Proposition 2.5.1. Recall that $A = E^0$. Let $P^m\colon E^0 \to E^0(B\Sigma_m)$ be the mth power operation, determined by the $\mathbb{E}_\infty$-ring structure on E. Let

$$\beta_m\colon E^0 \xrightarrow{P_m} E^0(B\Sigma_m) \xrightarrow{\mathrm{Tr}^e_{\Sigma_m}} E^0$$

be the composite of the mth power operation with the $K(n)$-local transfer map along the surjection from Σ_m to the trivial group.

The standard relations among the power operations implies that the formal sum $\beta(x) = \sum_{m\geq 0} \beta_m x^m$, considered as map $E^0 \to E^0[\![x]\!]$, satisfies $\beta(x+y) = \beta(x)\beta(y)$. Since $\beta_0(a) = 1$ and $\beta_1(a) = a$ for all $a \in E^0$, the map β factors through a homomorphism from the additive group E^0 to the subgroup $1 + xE^0[\![x]\!]$ of $E^0[\![x]\!]^\times$. Now we may quotient the target by the maximal ideal in E^0 to obtain a map $E^0 \to \overline{\mathbb{F}}_p[\![x]\!]$ that sends addition to multiplication. The big Witt vectors $W_{\mathrm{big}}(\overline{\mathbb{F}}_p)$ may be canonically identified (additively) with the abelian group of units in $\overline{\mathbb{F}}_p[\![x]\!]$ with constant coefficient 1 under multiplication. Further, the p-typical Witt vectors $W = W(\overline{\mathbb{F}}_p)$ splits off of the big Witt vectors. The quotient map $(1 + x\overline{\mathbb{F}}_p[\![x]\!]) \cong W_{\mathrm{big}}(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p$ is given by reading off the coefficient of x, and this quotient map factors through W. The maps constructed so far fit into a diagram:

$$\begin{array}{ccccccc}
E^0 & \xrightarrow{\beta} & E^0[\![x]\!] & \longrightarrow & \overline{\mathbb{F}}_p[\![x]\!] & & \\
 & \searrow & \uparrow & & \uparrow & \searrow & \\
 & & 1 + xE^0[\![x]\!] & \longrightarrow & 1 + x\overline{\mathbb{F}}_p[\![x]\!] \cong W_{\mathrm{big}}(\overline{\mathbb{F}}_p) & \longrightarrow W \longrightarrow & \overline{\mathbb{F}}_p
\end{array}$$

Let $\gamma\colon E^0 \to W$ be the composition of the maps appearing in the diagram. Precomposing with the inclusion $W \to E^0$, we obtain an additive endomorphism f of W. This map is the identity modulo p, and therefore f is an automorphism. The map $\alpha := f^{-1} \circ \gamma$ is therefore a section of $W \to E^0$.

Further, the maps that go into the construction of α are $\mathbb{G}_n$-equivariant. For P_m, this follows from the fact that $\mathbb{G}_n$ acts on E via $\mathbb{E}_\infty$-ring maps. The transfer map is $\mathbb{G}_n$-equivariant as it is given by restriction along a map of spectra (alternatively by the formula for this transfer and the action of $\mathbb{G}_n$ on the level of characters). □

Corollary 2.5.3. *The inclusion $W \hookrightarrow A$ induces a split injection $H^*_{\mathrm{cts}}(\mathbb{G}_n, W) \to H^*_{\mathrm{cts}}(\mathbb{G}_n, A)$ with cokernel $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c)$.*

Theorem B has thus been reduced to the statement that $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c)$ is p-power torsion. This will be established in the course of the next sections.

2.6. **The proof of Theorem A assuming Theorem B.** We finish this section by explaining how to deduce Theorem B from Theorem A. The key point is that, rationally, the cohomology of the stabilizer group action on the homotopy groups of Morava E-theory simplifies dramatically in non-zero degrees:

Lemma 2.6.1. *For all $t \neq 0$ and all $s \in \mathbb{Z}$, we have $H^s_{\mathrm{cts}}(\mathbb{G}_n, \mathbb{Q} \otimes \pi_t E_n) = 0$.*

Proof. Recall that the (extended) Morava stabilizer group $\mathbb{G}_n$ can be described naturally as a semidirect product

$$1 \longrightarrow \mathcal{O}_D^\times \longrightarrow \mathbb{G}_n \simeq \mathcal{O}_D^\times \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1, \qquad (2.6.2)$$

where $\mathcal{O}_D^\times$ is isomorphic to the automorphism group of our chosen formal group law Γ_n over $\overline{\mathbb{F}}_p$. The center of $\mathcal{O}_D^\times$ is isomorphic to $\mathbb{Z}_p^\times$ and we may consider the central subgroup $\mathbb{Z}_p \subset \mathbb{Z}_p^\times \trianglelefteq \mathcal{O}_D^\times$, which we can take to be generated by the element $1 + p \in \mathbb{Z}_p^\times$. Fixing some integer t, the associated convergent Lyndon–Hochschild–Serre spectral sequence for continuous cohomology (e.g., [Ser02, Section I.2.6(b)]) has signature

$$H^p_{\mathrm{cts}}(\mathcal{O}_D^\times/\mathbb{Z}_p, H^q_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Q}\otimes\pi_t E_n)) \implies H^{p+q}_{\mathrm{cts}}(\mathcal{O}_D^\times, \mathbb{Q}\otimes\pi_t E_n).$$

It is thus enough to show that

$$H^q_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Q}\otimes\pi_t E_n) = 0$$

for $t \neq 0$. To this end, we use that the generator of $\mathbb{Z}_p$ acts by multiplication by $(1+p)^t$, see for example [BB20, Secton 3.3.2(c)]. The continuous $\mathbb{Z}_p$-cohomology of $\mathbb{Q}\otimes\pi_t E_n$ is thus computed via the complex

$$\mathbb{Q}\otimes\pi_t E_n \xrightarrow{(1+p)^t-1} \mathbb{Q}\otimes\pi_t E_n.$$

Since $\mathbb{Q}\otimes\pi_t E_n$ is a $\mathbb{Q}_p$-vector space, when $t\neq 0$ the action by $(1+p)^t-1$ is invertible, so the complex is acyclic. We then conclude by another application of the Lyndon–Hochschild–Serre spectral sequence, this time for the extension (2.6.2). □

Proposition 2.6.3. *Theorem B implies Theorem A.*

Proof. We will use the Devinatz–Hopkins spectral (2.3.3) which we recall has signature

$$E_2^{s,t} \cong H^s_{\mathrm{cts}}(\mathbb{G}_n, \pi_t E_n) \implies \pi_{t-s} L_{K(n)} S^0$$

This spectral sequence converges strongly and collapses on a finite page with a horizontal vanishing line. One the one hand, it follows that rationalization yields another strongly convergent spectral sequence

$$\mathbb{Q}\otimes E_2^{s,t} \cong H^s_{\mathrm{cts}}(\mathbb{G}_n, \mathbb{Q}\otimes\pi_t E_n) \implies \mathbb{Q}\otimes\pi_{t-s} L_{K(n)} S^0.$$

Here, the identification of the E_2-term uses that $\mathbb{G}_n$ is a compact group so that rationalization commutes with taking continuous cohomology; see for example [BP21, Corollary 12.9]. On the other hand, Lemma 2.6.1 implies that the rationalized Devinatz–Hopkins spectral sequence collapses on the E_2-page, resulting in a graded isomorphism

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, \mathbb{Q}\otimes\pi_0 E_n) \cong \mathbb{Q}\otimes\pi_{-*} L_{K(n)} S^0.$$

Since the spectral sequence is multiplicative, this is in fact an isomorphism of graded rings. Theorem B combined with Lazard's theorem [Laz65] as stated in Lemma 3.4.4 identifies the left hand side as

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, \mathbb{Q}\otimes\pi_0 E_n) \cong H^*_{\mathrm{cts}}(\mathbb{G}_n, \mathbb{Q}\otimes W) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \ldots, x_n),$$

with x_i in cohomological degree $2i-1$. This gives Theorem A. □

3. Arithmetic prerequisites

To complete the proof of Theorem B, we must show that the complement of the split injection

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, W) \to H^*_{\mathrm{cts}}(\mathbb{G}_n, A)$$

is p^N-torsion for some $N \geq 0$. Our proof lies entirely within the domain of p-adic geometry. After reviewing these topics, we offer a summary of the proof at the end of this section.

3.1. **Adic spaces.** We offer the reader a brief summary of the necessary techniques from non-archimedean analytic geometry, starting with Huber's category of *adic spaces* [Hub94]. This category contains all formal schemes, rigid-analytic varieties, and perfectoid spaces. For a more leisurely exposition, see the last named author's chapter in [BCKW19].

A topological ring A is a *Huber ring* if it contains an open subring A_0 whose topology is induced by a finitely generated ideal $I \subset A_0$. A subset S of a Huber ring A is *bounded* if for every $n \geq 0$ there exists $N \geq 0$ such that $I^N S \subset I^n$. A single element $f \in A$ is *power-bounded* if $\{f^n\}_{n\geq 1}$ is bounded. A *Huber pair* is a pair (A, A^+) consisting of a Huber ring A and an open and integrally closed subring $A^+ \subset A$ whose elements are power-bounded. A *continuous valuation* on a Huber ring A is a continuous multiplicative function $| \ | : A \to H \cup \{0\}$, where H is a totally ordered abelian group (written multiplicatively). The adic spectrum $\operatorname{Spa}(A, A^+)$ is the set of equivalence classes of continuous valuations satisfying $|A^+| \leq 1$. It is endowed with the topology generated by *rational subsets* of the form

$$U = U\left(\frac{f_1, \dots, f_r}{g}\right) = \left\{ | \ | \in \operatorname{Spa}(A, A^+) \;\middle|\; |f_i| \leq |g| \neq 0, \ i = 1, \dots, r \right\}$$

for $f_1, \dots, f_r \in A$ generating an open ideal and $g \in A$. Then $\operatorname{Spa}(A, A^+)$ is quasi-compact.

One defines presheaves of rings $\mathcal{O}_X^+ \subset \mathcal{O}_X$ on $X = \operatorname{Spa}(A, A^+)$ as follows. For the rational subset U above, we declare that $\mathcal{O}_X(U)$ is the completion of $A[f_i/g]$ with respect to the topology in which $A_0[f_i/g]$ (with its I-adic topology) is an open subring, and $\mathcal{O}_X^+(U)$ is the completion of the integral closure of $A^+[f_i/g]$ in $A[f_i/g]$. Then $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is another Huber pair. For certain classes of Huber pairs (A, A^+) (including all which are considered in this article), the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves. In such cases, the triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ is an *affinoid adic space*. A general adic space is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ which is locally isomorphic to an affinoid adic space. The sheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are the *structure sheaf* and *integral structure sheaf*, respectively.

As a basic example, if $X = \operatorname{Spa}(W, W)$, then $\mathcal{O}_X = \mathcal{O}_X^+$ and the ringed space $(X, \mathcal{O}_X)$ is isomorphic to $\operatorname{Spec} W$. In particular it has two points: a generic point lying in $\operatorname{Spa}(K, W)$ (which extends to the usual absolute value on K) and a special point (which satisfies $|p| = 0$).

A *rigid-analytic space* over K is an adic space over $\operatorname{Spa}(K, W)$ that is locally isomorphic to an affinoid adic space of the form $\operatorname{Spa}(A, A^+)$, where (A, A^+) obeys a certain finiteness condition. Namely, A is isomorphic to a ring quotient of a Tate algebra $K\langle T_1, \dots, T_d\rangle$, and A^+ is equal to the subring of power-bounded elements of A. A special case is $A = K\langle T_1, \dots, T_d\rangle$, in which case $\operatorname{Spa}(A, A^+)$ is the closed ball of radius 1. (Chronologically, Tate's theory of rigid-analytic spaces long predates Huber's theory of adic spaces; rigid-analytic spaces as Tate defined them sit inside of adic spaces over $\operatorname{Spa}(K, W)$ as a full subcategory, so there is no harm in thinking of them this way.)

Fix a continuous real-valued valuation $| \ |$ representing the sole point of $\operatorname{Spa}(K, W)$, so as to fix a value of $|p|$.

Example 3.1.1 (The rigid-analytic open ball)**.** Let $A = W[\![T_1, \dots, T_d]\!]$, and let $\mathfrak{B}^d = \operatorname{Spa}(A, A)$, the formal d-dimensional unit ball over $\operatorname{Spa}(W, W)$. Let $B^{d,\circ}$ be the fiber of $\mathfrak{B}^d$ over the generic point of $\operatorname{Spa}(W, W)$; i.e., $B^{d,\circ}$ is the locus where $|p| \neq 0$. Let us observe that $B^{d,\circ}$ is exhausted by affinoid (closed) balls over K. For each real number r of the form $r = |p|^{1/n}$ with $n = 1, 2, 3, \dots$, let B_r^d be the following rational subset of $\mathfrak{B}^d$:

$$B_r^d = U\left(\frac{T_1^n, \dots, T_d^n, p}{p}\right).$$

Then B_r^d is an affinoid rigid-analytic space over K. We have

$$B^{d,\circ} = \varinjlim_{r<1} B_r^d,$$

since for each continuous valuation on A with $|p| \neq 0$, we must have $|T_i|^n \leq |p|$ for n sufficiently large. Therefore $B^{d,\circ}$ is a rigid-analytic space over K.

There is an important distinction between the global sections of $\mathcal{O}_X$ and $\mathcal{O}_X^+[1/p]$ for a rigid-analytic space X over K. When X is quasi-compact, these agree, but in general they are quite different. In the situation of the rigid-analytic open ball, we have that

$$H^0(B^{d,\circ}, \mathcal{O}_X^+)[1/p] = W[\![T_1, \dots, T_d]\!][1/p]$$

is the ring of power series in $K[\![T_1, \dots, T_d]\!]$ which are bounded on the open unit ball. Whereas, $H^0(B^{d,\circ}, \mathcal{O}_X)$ is the much larger ring of power series in $K[\![T_1, \dots, T_d]\!]$ which converge on the open unit ball.

3.2. **Remarks on continuous cohomology.** When G is a topological group and M is a topological abelian group admitting a continuous G-action, Tate's continuous cohomology groups $H^i_{\mathrm{cts}}(G, M)$ are defined using the complex of continuous cocycles. At this level of generality, however, one has no abelian category of topological abelian groups, and so $M \mapsto H^i_{\mathrm{cts}}(G, M)$ has no interpretation as the ith derived functor of fixed points $M \mapsto M^G$.

The language of condensed mathematics [CS] is well-suited to resolve this issue in all contexts which will arise in this article. We quickly review the main points of the theory, ignoring set-theoretic issues throughout.[3] The pro-étale site of a point $*_{\text{proét}}$ is the category of profinite sets with jointly surjective continuous maps as the covers. A *condensed set* is a sheaf of sets on $*_{\text{proét}}$. Similarly there are condensed groups, rings, etc. There is a functor $X \mapsto \underline{X}$ from topological spaces/groups/rings to condensed sets/groups/rings, via $\underline{X}(S) = C_{\mathrm{cts}}(S, X)$, meaning continuous maps $S \to X$. This functor is fully faithful when restricted to compactly generated topological spaces [CS, Proposition 1.7]

Let Cond(Ab) be the category of condensed abelian groups. Then Cond(Ab) is an abelian category containing all limits and colimits [CS, Theorem 1.10]. It has a symmetric monoidal tensor product $\mathcal{M} \otimes \mathcal{N}$ and an internal Hom-functor $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})$ related by the adjunction

$$\mathrm{Hom}(\mathcal{P}, \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})) \cong \mathrm{Hom}(\mathcal{P} \otimes \mathcal{M}, \mathcal{N}).$$

The forgetful functor Cond(Ab) $\to$ Cond(Set) has a left adjoint $\mathcal{X} \mapsto \mathbb{Z}[\mathcal{X}]$, the "free condensed abelian group on $\mathcal{X}$". In the case $\mathcal{X} = \underline{S}$ for S profinite, we abuse notation and write $\mathbb{Z}[S]$ for $\mathbb{Z}[\underline{S}]$.

Cond(Ab) has enough projectives, and we can form the derived category $D(\mathrm{Cond(Ab)})$. Then $D(\mathrm{Cond(Ab)})$ admits a derived tensor product $\otimes$ and a derived internal hom functor RHom satisfying the usual adjunction relation.

In the language of condensed mathematics, the notion of completeness goes by the term "solid". For a profinite set $S = \varprojlim S_i$ with each S_i finite, the free solid abelian group on S is defined as

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim \underline{\mathbb{Z}[S_i]},$$

where $\mathbb{Z}[S_i]$ is the free abelian group on S_i, considered as a discrete topological group. A *solid abelian group* is a condensed abelian group $\mathcal{M}$ such that for all profinite sets S, any morphism $\underline{S} \to \mathcal{M}$ extends uniquely to a morphism $\mathbb{Z}[S]^{\blacksquare} \to \mathcal{M}$. Let Solid denote the category of solid abelian groups; by [CS, Theorem 5.8], Solid is closed under all limits and colimits in Cond(Ab). Then Solid is an abelian subcategory of Cond(Ab). The functor $D(\mathrm{Solid}) \to D(\mathrm{Cond(Ab)})$ is fully faithful. For an object $\mathcal{C}$ of $D(\mathrm{Cond(Ab)})$, the following are equivalent:

(1) $\mathcal{C}$ lies in its essential image of $D(\mathrm{Solid}) \to D(\mathrm{Cond(Ab)})$.
(2) $H^i(\mathcal{C})$ is a solid abelian group for all $i \in \mathbb{Z}$.

[3] They can be dealt with as in [CS, Lecture I].

(3) For all profinite sets S, the natural map
$$\mathrm{RHom}(\mathbb{Z}[S]^{\blacksquare}, \mathcal{C}) \to \mathrm{RHom}(\mathbb{Z}[S], \mathcal{C})$$
is an isomorphism.

The following lemma shows how solidity and completeness are related.

Lemma 3.2.1. *Let M be a topological abelian group which is separated and complete for a linear topology. Then $\underline{M}$ is a solid abelian group.*

Proof. The hypothesis means there is a directed system of open subgroups $M_n \subset M$ inducing the topology on M, such that the map $M \to \varprojlim M/M_n$ is an isomorphism.

Let $S = \varprojlim S_i$ with each S_i finite, and let $f\colon S \to M$ be continuous. For each n, the map $S \to M/M_n$ is locally constant, so it must factor through $S_i \to M/M_n$ for some $i = i(n)$. This map can be extended to a morphism of condensed abelian groups $\underline{\mathbb{Z}[S_i]} \to \underline{M/M_n}$. After passage to the limit in n, we obtain a morphism $\mathbb{Z}[S]^{\blacksquare} \to \underline{M}$. □

Now suppose $\mathcal{G}$ is a condensed group. A $\mathcal{G}$-action on a condensed abelian group $\mathcal{M}$ is a morphism $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ satisfying the usual axioms.

Lemma 3.2.2. *Let M be a topological abelian group which is separated and complete for a linear topology. Let G be a profinite group which acts continuously on M. Then $\underline{M}$ is a solid abelian group with $\underline{G}$-action.*

Proof. The claim in the lemma is that the group action $G \times M \to M$ can be upgraded to an action $\underline{G} \times \underline{M} \to \underline{M}$ on the level of condensed sets. That is, we need for every profinite $S = \varprojlim S_i$ an action
$$C_{\mathrm{cts}}(S, G) \times C_{\mathrm{cts}}(S, M) \to C_{\mathrm{cts}}(S, M)$$
which is functorial in S. Therefore let $f\colon S \to G$ and $h\colon S \to M$ be continuous. Since $M \cong \varprojlim M/M_n$ for a directed system of open subgroups M_n, it is enough to produce an action with values in $C_{\mathrm{cts}}(S, M/M_n)$ compatibly in n.

The continuity of the action of G on M means exactly that there exists an index N and an open subgroup $H \subset G$ such that $HM_N \subset M_n$. Since h is continuous, it is locally constant modulo M_N; that is, the composition $S \to M \to M/M_N$ factors through $h_i\colon S_i \to M/M_N$ for some i. After replacing i, we may also assume that $f\colon S \to G \to G/H$ factors through $f_i\colon S_i \to G/H$. Then the sum $\sum_{s \in S_i} f_i(s)h(s)$ is well-defined in M/M_N; this is the required action of f on h. □

For a profinite group G, let Solid_G be the category of solid abelian groups admitting an action of $\underline{G}$. Then Solid_G is an abelian category.

Lemma 3.2.3. *Let G be a profinite group. Consider the functor of fixed points $\mathrm{Solid}_G \to \mathrm{Solid}$ defined by $\mathcal{M} \mapsto \mathcal{M}^G$, i.e., the right adjoint to the trivial action functor $\mathrm{Solid} \to \mathrm{Solid}_G$. Let $\mathcal{C} \mapsto R\Gamma(G, \mathcal{C})$ be its derived functor $D(\mathrm{Solid}_G) \to D(\mathrm{Solid})$, and let $H^i(G, \mathcal{C}) = R^i(G, \mathcal{C})$.*

(1) For an object $\mathcal{C}$ of Solid_G, the object $R\Gamma(G, \mathcal{C})$ is the totalization of the double complex
$$\mathcal{C} \to \underline{\mathrm{RHom}}(\mathbb{Z}[G], \mathcal{C}) \to \underline{\mathrm{RHom}}(\mathbb{Z}[G^2], \mathcal{C}) \to \cdots .$$
The transition maps are derived from the usual formulas in group cohomology.

(2) In particular, if M is an abelian group which is separated and complete for a linear topology, and G acts continuously on M, then
$$H^i(G, \underline{M}) \cong \underline{H^i_{\mathrm{cts}}(G, M)}.$$
Here $H^i_{\mathrm{cts}}(G, M)$ is Tate's continuous cohomology.

Proof. The idea is to construct a projective resolution of the trivial module $\mathbb{Z}$ in Solid_G. Let

$$\mathbb{Z}[G]^{\blacksquare} = \varprojlim_H \underline{\mathbb{Z}[G/H]}$$

be the "solid Iwasawa algebra"; here H runs over open subgroups of G. We claim that $\mathbb{Z}[G]^{\blacksquare}$ is projective in Solid_G. Indeed, let $\mathcal{M} \to \mathbb{Z}[G]^{\blacksquare}$ be a surjection; we want to produce a section. Since $\mathcal{M}$ is solid, it is enough to produce a morphism $\underline{G} \to \mathcal{M}$ such that the composition $\underline{G} \to \mathcal{M} \to \mathbb{Z}[G]^{\blacksquare}$ is the natural map. The required morphism is $g \mapsto gm$, where $m \in \mathcal{M}(*)$ lifts the identity section $* \to \mathbb{Z}[G]^{\blacksquare}$.

Similarly, the solid Iwasawa algebras $\mathbb{Z}[G^n]^{\blacksquare}$ are projective. Thus we have the usual projective resolution of the trivial $\underline{G}$-module $\underline{\mathbb{Z}}$:

$$\underline{\mathbb{Z}} \to \mathbb{Z}[G]^{\blacksquare} \to \mathbb{Z}[G^2]^{\blacksquare} \to \dots. \tag{3.2.4}$$

If $\mathcal{C}$ is an object of $D(\mathrm{Solid}_G)$, then $R\Gamma(G, \mathcal{C})$ is the totalization of the double complex with terms $\underline{\mathrm{RHom}}(\mathbb{Z}[G^n]^{\blacksquare}, \mathcal{C})$. Since $\mathcal{C}$ is solid, the latter is isomorphic to $\underline{\mathrm{RHom}}(\mathbb{Z}[G^n], \mathcal{C})$. This is (1).

In the case $\mathcal{C} = \underline{M}$ described in (2), we have (since $\mathbb{Z}[G]$ is a free condensed abelian group)

$$\underline{\mathrm{RHom}}(\mathbb{Z}[G^n], \underline{M}) \cong \underline{\mathrm{Hom}}(G^n, \underline{M}) \cong \underline{C_{\mathrm{cts}}(G^n, M)},$$

so that $R\Gamma(G, \underline{M})$ is nothing but the condensed version of the complex which computes Tate's continuous cohomology. Therefore $H^i(G, \underline{M})$ is the condensed version of $H^i_{\mathrm{cts}}(G, M)$. □

Lemma 3.2.3 shows that as long as M is separated and complete for a linear topology, the continuous cohomology $H^i_{\mathrm{cts}}(G, M)$ really is the derived functor of G-fixed points in an abelian category. This allows us to seamlessly use the language of derived categories to compute $H^i_{\mathrm{cts}}(G, M)$.

Notation 3.2.5 (Continuous homotopy fixed points)**.** Let G be a profinite group. We write

$$\begin{array}{rcl} D(\mathrm{Solid}_G) & \to & D(\mathrm{Solid}) \\ \mathcal{C} & \mapsto & \mathcal{C}^{hG} \end{array}$$

for the functor $R\Gamma(G, \mathcal{C})$ described in Lemma 3.2.3; i.e., the derived functor of $\mathcal{M} \mapsto \mathcal{M}^G$. Similarly, if M is a topological abelian group which is separated and complete for a linear topology, and G acts continuously on M, then we write M^{hG} for $R\Gamma(G, \underline{M})$.

By Lemma 3.2.3, M^{hG} is a complex of solid abelian groups which computes $H^i_{\mathrm{cts}}(G, M)$; it would also be appropriate to use the notation $R\Gamma_{\mathrm{cts}}(G, M)$ for M^{hG}. If $H \subset G$ is a closed normal subgroup, then M^{hH} is an object of $D(\mathrm{Solid}_{G/H})$, and then the quasi-isomorphism

$$M^{hG} \cong \left(M^{hH}\right)^{h(G/H)}$$

formally implies the Hochschild-Serre spectral sequence in continuous cohomology:

$$H^i_{\mathrm{cts}}(G/H, H^j_{\mathrm{cts}}(H, M)) \implies H^{i+j}_{\mathrm{cts}}(G, M).$$

Crucially, we can apply this picture to the example where $M = A$ is the Lubin–Tate ring and $G = \mathbb{G}_n$ is the Morava stabilizer group. Another important example we will encounter occurs when K is a nonarchimedean local field, $G = \mathrm{Gal}(\overline{K}/K)$, and $M = \mathcal{O}_C$, the valuation ring of the completion C of an algebraic closure $\overline{K}$ of K.

3.3. **The pro-étale site for rigid-analytic spaces.** Let X be a rigid-analytic space over K. We swiftly recall some material from [Sch13a, §3,4] regarding the pro-étale site $X_{\text{proét}}$.

Objects in $X_{\text{proét}}$ are formal limits $U = \varprojlim U_i$, where i runs over a cofiltered index set, the U_i are rigid-analytic spaces étale over X, and each transition map $U_i \to U_j$ commutes with the maps to X. It is required that $U_i \to U_j$ is finite étale and surjective for large $i > j$. For an object $U = \varprojlim U_i$, let $|U| = \varprojlim |U_i|$ be its underlying topological space. A covering in $X_{\text{proét}}$ is a family of pro-étale morphisms $f_j \colon U_j \to U$ such that the underlying topological space $|U|$ is covered by the $f_j(U_j)$.

The integral structure sheaf $\hat{\mathcal{O}}_X^+$ on $X_{\text{proét}}$ is defined as the p-adic completion:

$$\hat{\mathcal{O}}_X^+(U) = \left(\varinjlim \mathcal{O}_{U_i}^+(U_i)\right)^\wedge_{(p)},$$

and the structure sheaf is $\hat{\mathcal{O}}_X := \hat{\mathcal{O}}_X^+[1/p]$.

The pro-étale cohomology $H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X^+)$ of a rigid-analytic space X will be of special interest to us. To investigate it, we will make crucial use of perfectoid spaces. Let us recall the relevant definitions from [SW20], adapted to the case where all structures live in characteristic 0. A topological $\mathbb{Q}_p$-algebra R is *perfectoid* if the following conditions hold:

(1) R is uniform, meaning its subring R° of power-bounded elements is bounded,
(2) R° is p-adically complete,
(3) There exists an element $\varpi \in R^\circ$ such that $\varpi^p | p$ holds in R°, and such that the pth power Frobenius map
$$R^\circ/\varpi \to R^\circ/\varpi^p$$
is an isomorphism.

In particular there exists for all $n \geq 1$ an element ϖ_n whose p^nth power is the product of ϖ by a unit in R°. A Huber pair (R, R^+) is a *perfectoid affinoid algebra* over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ if R is perfectoid. An adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ is perfectoid if it admits a cover by adic spectra of perfectoid affinoid algebras.

Suppose that X is a rigid-analytic space over a field containing $\mathbb{Q}_p$. We say that an object $U \in X_{\text{proét}}$ is *affinoid perfectoid* if $U = \varprojlim U_i$, where $U_i = \text{Spa}(R_i, R_i^+)$ is affinoid, and if R^+ is the p-adic completion of $\varinjlim R_i^+$, and $R = R^+[1/p]$, then R is a perfectoid K-algebra. More generally, an object $U \in X_{\text{proét}}$ is perfectoid if it admits an open cover by perfectoid affinoid subobjects.

Perfectoid objects are useful to the calculation of $H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X^+)$ for the following two reasons:

(1) The affinoid perfectoid objects $U \in X_{\text{proét}}$ form a basis for the topology [Sch13a, Proposition 4.8].
(2) Suppose $U \in X_{\text{proét}}$ is affinoid perfectoid. Then for all $i \geq 1$, $H^i(U_{\text{proét}}, \hat{\mathcal{O}}_X^+)$ is almost zero, in the sense that it is annihilated by ϖ_n for all n [Sch13a, Lemma 4.10]. In particular $H^i(U_{\text{proét}}, \hat{\mathcal{O}}_X) = 0$ for $i \geq 1$.

The facts above suggest a strategy for computing $H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X^+)$: By (1), there exists a pro-étale cover $f_i \colon U_i \to X$ where each U_i is perfectoid affinoid, and by (2), $R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}_X^+)$ is "almost" computed by the Čech complex $\mathcal{O}^+(X) \to \prod_i \mathcal{O}^+(U_i) \to \prod_{i,j} \mathcal{O}^+(U_i \times_X U_j) \to \cdots$. The precise consequences for the pro-étale cohomology of rigid-analytic spaces will be reviewed in Section 5.

It will be important to consider the pro-étale cohomology of a rigid-analytic space as a complex of condensed abelian groups. For a rigid-analytic space X, observe that we have a morphism of sites:

$$\lambda_X \colon X_{\text{proét}} \to *_{\text{proét}}$$

Indeed, if $S = \varprojlim S_i$ is a profinite set, then $X \times S = \varprojlim X \times S_i$ is an object of $X_{\text{proét}}$. Consequently, if $\mathcal{F}$ is a sheaf of abelian groups on $X_{\text{proét}}$, we may define:

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \mathcal{F}) = R(\lambda_X)_*\mathcal{F},$$

an object of $D(\text{Cond}(\text{Ab}))$ whose value on $*$ is $R\Gamma(X_{\text{proét}}, \mathcal{F})$.

In particular we have $R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+)$, the condensed pro-étale cohomology of the completed integral structure sheaf. On the other hand, we may consider $\hat{\mathcal{O}}^+$ to be a sheaf of *topological* groups, by endowing $\hat{\mathcal{O}}^+(U)$ with the p-adic topology for all affinoid perfectoid $U \in X_{\text{proét}}$. Let $\hat{\mathcal{O}}^+_{\text{cond}}$ be the sheaf of condensed abelian groups on $X_{\text{proét}}$ defined by

$$\hat{\mathcal{O}}^+_{\text{cond}}(U) = \underline{\hat{\mathcal{O}}^+(U)}.$$

By Lemma 3.2.1, $\hat{\mathcal{O}}^+_{\text{cond}}$ is a sheaf of solid abelian groups, since $\hat{\mathcal{O}}^+(U)$ is p-adically separated and complete for all affinoid perfectoid U. Then the cohomology $R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_{\text{cond}})$ is a complex of solid abelian groups.

Lemma 3.3.1. *We have an isomorphism in* $D(\text{Cond}(\text{Ab}))$*:*

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+) \cong R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_{\text{cond}})$$

In particular $R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+)$ *lies in* $D(\text{Solid})$.

Proof. Since $X_{\text{proét}}$ admits a basis consisting of perfectoid affinoids, this reduces to the claim that for all perfectoid affinoids $U = \text{Spa}(R, R^+) \in X_{\text{proét}}$, we have an isomorphism in Cond(Ab):

$$(\lambda_U)_*\hat{\mathcal{O}}^+ \cong \mathcal{O}^+_{\text{cond}}(U)$$

and Solid admits all limits, it is enough to see that $(\lambda_U)_*\hat{\mathcal{O}}^+$ is solid for $U = \text{Spa}(R, R^+)$ perfectoid affinoid. This in turn reduces to the following calculation, for any profinite set $S = \varprojlim S_i$:

$$\begin{aligned} H^0(U \times S, \hat{\mathcal{O}}^+) &\cong \left(\varinjlim H^0(U \times S_i, \mathcal{O}^+)\right)^\wedge_{(p)} \\ &\cong \left(\varinjlim C_{\text{cts}}(S_i, R^+)\right)^\wedge_{(p)} \\ &\cong C_{\text{cts}}(S, R^+) \\ &\cong \mathcal{O}^+_{\text{cond}}(U)(S) \end{aligned}$$

In the penultimate step we used the fact that R^+ is p-adically complete. □

In the context of the proof of Lemma 3.3.1 we have

$$C_{\text{cts}}(S, R^+) \cong \underline{\text{Hom}}(\mathbb{Z}[S], \underline{R^+}).$$

Applying this over a covering of X by perfectoid affinoids $U = \text{Spa}(R, R^+)$ in $X_{\text{proét}}$, we obtain an isomorphism in $D(\text{Cond}(\text{Ab}))$:

$$R\Gamma_{\text{cond}}((X \times S)_{\text{proét}}, \hat{\mathcal{O}}^+) \cong \underline{\text{RHom}}(\mathbb{Z}[S], R\Gamma_{\text{cond}}(X, \hat{\mathcal{O}}^+)). \tag{3.3.2}$$

Let G be a profinite group. A *pro-étale G-torsor* over X is an object $Y \to X$ admitting an action $G \times Y \to Y$ lying over the trivial action of X, such that the map $G \times Y \to Y \times_X Y$ given by $(g, y) \mapsto (y, gy)$ is an isomorphism.

Proposition 3.3.3. *Let X be a rigid-analytic space, and let $Y \to X$ be a perfectoid pro-étale G-torsor. There is an isomorphism in* $D(\text{Solid})$*:*

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+) \cong R\Gamma_{\text{cond}}(Y_{\text{proét}}, \hat{\mathcal{O}}^+)^{hG}.$$

Proof. Since $Y \to X$ is pro-étale, the pro-étale cohomology of X can be computed by means of the simplicial cover:

$$\cdots \substack{\to\\\to\\\to} Y \times_X Y \rightrightarrows Y$$

Namely, $R\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)$ is quasi-isomorphic to the totalization of the corresponding double complex

$$R\Gamma_{\mathrm{cond}}(Y, \hat{\mathcal{O}}^+) \to R\Gamma_{\mathrm{cond}}(Y \times_X Y, \hat{\mathcal{O}}^+) \to \cdots .$$

Since $Y \to X$ is a G-torsor, the nth term in the complex is quasi-isomorphic to

$$R\Gamma_{\mathrm{cond}}((G^n \times Y)_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) \cong R\underline{\mathrm{Hom}}(\mathbb{Z}[G^n], R\Gamma_{\mathrm{cond}}(Y_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+))$$

by (3.3.2). Lemma 3.2.3(1) identifies the latter as the nth term in a double complex whose totalization computes $R\Gamma_{\mathrm{cond}}(Y_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)^{hG}$. □

Example 3.3.4. Let K be a nonarchimedean field of characteristic $(0, p)$. Let $\overline{K}$ be an algebraic closure, and let C be the metric completion of $\overline{K}$. Then $\mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(K, \mathcal{O}_K)$ is a pro-étale torsor for the group $\mathrm{Gal}(\overline{K}/K)$. Furthermore, since C is algebraically closed, every pro-étale cover of $\mathrm{Spa}(C, \mathcal{O}_C)$ is split, meaning that $H^i(\mathrm{Spa}(C, \mathcal{O}_C)_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) = 0$ for $i > 0$. Therefore by Proposition 3.3.3 we have an isomorphism in $D(\mathrm{Solid})$:

$$R\Gamma_{\mathrm{cond}}(\mathrm{Spa}(K, \mathcal{O}_K)_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) \cong \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)}.$$

3.4. **Continuous cohomology of p-adic Lie groups.** As Theorem B is a statement about the continuous cohomology of the Morava stabilizer group $\mathbb{G}_n \cong \mathcal{O}_D^\times \rtimes \hat{\mathbb{Z}}$, it will be useful to collect some basic results on the continuous cohomology of p-adic Lie groups such as $\mathcal{O}_D^\times$.

The first systematic study of the continuous cohomology of p-adic Lie groups was undertaken by Lazard [Laz65]. Lazard's results include comparision theorems such as [Laz65, Théorème V.2.4.10], which we briefly summarize. If G is a $\mathbb{Q}_p$-analytic group admitting a p-valuation, Lazard defines its Lie algebra $\mathrm{Lie}\, G$ over $\mathbb{Q}_p$. Then if V is a finite-dimensional $\mathbb{Q}_p$-vector space admitting a continuous action of G, then $\mathrm{Lie}\, G$ acts on G, and we have an isomorphism $H^*_{\mathrm{cts}}(G, V) \cong H^*(\mathrm{Lie}\, G, V)$. (If $\mathbf{G}$ is an algebraic group over $\mathbb{Q}_p$, then any sufficiently small subgroup of $\mathbf{G}(\mathbb{Q}_p)$ satisfies Lazard's hypotheses, with $\mathrm{Lie}\, G$ being the usual Lie algebra of $\mathbf{G}$.)

The following is a well-known consequence of Lazard's theorem, see for example [Mor85, Remark 2.2.5].

Lemma 3.4.1. *Let G be either of the groups $\mathrm{GL}_n(\mathbb{Z}_p)$ or $\mathcal{O}_D^\times$. Consider the trivial action of G on $\mathbb{Q}_p$. There is an isomorphism of graded $\mathbb{Q}_p$-algebras:*

$$H^*_{\mathrm{cts}}(G, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \ldots, x_{2n-1}).$$

In the case of $G = \mathcal{O}_D^\times$, the outer automorphism $\mathrm{ad}\,\Pi$ acts as the identity on $H^*_{\mathrm{cts}}(G, \mathbb{Q}_p)$.

Proof. By Lazard's result, $H^*_{\mathrm{cts}}(G, \mathbb{Q}_p) \cong H^*(\mathrm{Lie}\, G, \mathbb{Q}_p)$, so we are reduced to calculating Lie algebra cohomology. In the case of $G = \mathcal{O}_D^\times$, the outer automorphism $\mathrm{ad}\,\Pi$ on G corresponds to the inner automorphism of $\mathrm{Lie}\, G$, which acts trivially on $H^*(\mathrm{Lie}\, G, \mathbb{Q}_p)$.

The Lie algebra cohomology of a reductive Lie algebra $\mathfrak{g}$ over a field k of characteristic zero is well-studied [CE48], [Kos50]; we give a brief exposition. If $k = \mathbb{R}$ and $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, then $H^*(\mathfrak{g}, \mathbb{R})$ is isomorphic to the de Rham cohomology ring $H^*_{\mathrm{dR}}(G)$. Generally, this is a graded-commutative $\mathbb{R}$-algebra whose primitive elements live in odd degree. In the special case $G = U(n)$, the sequence of fibrations $U(n-1) \to U(n) \to S^{2n-1}$ allows one to identify the rational cohomology $H^*_{\mathrm{dR}}(U(n))$ with the rational cohomology of a product of spheres $S^1 \times S^3 \times \cdots \times S^{2n-1}$. Thereby we obtain an isomorphism

$$H^*(\mathfrak{gl}_n(\mathbb{Q}), \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_1, x_3, \ldots, x_{2n-1}), \tag{3.4.2}$$

since the two sides become isomorphic after tensoring with $\mathbb{C}$. (See [Kos50] for an explicit description, due to Dynkin, of the elements $x_1, x_3, \dots, x_{2n-1}$ in terms of cocycles.) The isomorphism (3.4.2) implies the lemma for $G = \mathrm{GL}_n(\mathbb{Z}_p)$, since $\operatorname{Lie} G = \mathfrak{gl}_n(\mathbb{Q}_p)$.

Now suppose $G = \mathcal{O}_D^\times$. Then $\operatorname{Lie} G = D$ is a twist of $\mathfrak{gl}_n(\mathbb{Q}_p)$ in the sense that there is an isomorphism:

$$\operatorname{Lie} G \otimes \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathfrak{gl}_n(\overline{\mathbb{Q}}_p). \tag{3.4.3}$$

This implies that $H^*(\operatorname{Lie} G, \mathbb{Q}_p)$ is an exterior algebra as claimed, since it becomes one after tensoring with $\overline{\mathbb{Q}}_p$. □

Lemma 3.4.4. *Let $W = W(\overline{\mathbb{F}}_p)$ and $K = W[1/p]$.*

(1) The continuous cohomology $H^i_{\mathrm{cts}}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), W)$ is $\mathbb{Z}_p$ if $i = 0$, and is 0 otherwise.
(2) Let $\mathbb{G}_n$ act on K through its quotient $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. There is an isomorphism of graded $\mathbb{Q}_p$-algebras:

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, K) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})$$

Proof. (1) Since $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$, it is enough to show vanishing of cohomology in degree 1. Since W is p-adically complete, this is further reduced to showing that $H^1_{\mathrm{cts}}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \overline{\mathbb{F}}_p) = 0$; this is true because $x \mapsto x^p - x$ is surjective on $\overline{\mathbb{F}}_p$.

(2) Consider the Hochschild–Serre spectral sequence:

$$H^i_{\mathrm{cts}}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), H^j_{\mathrm{cts}}(\mathcal{O}_D^\times, W)) \implies H^{i+j}_{\mathrm{cts}}(\mathbb{G}_n, W).$$

Consider the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on $H^j_{\mathrm{cts}}(\mathcal{O}_D^\times, W) = H^j_{\mathrm{cts}}(\mathcal{O}_D^\times, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W$. The action on the first factor is trivial by Lemma 3.4.1, and the action on the second factor has no higher cohomology by (1). Therefore:

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, W) \cong H^*_{\mathrm{cts}}(\mathcal{O}_D^\times, \mathbb{Z}_p),$$

at which point we tensor with $\mathbb{Q}_p$ and apply Lemma 3.4.1. □

3.5. **Overview of the proof of Theorem B.** Let p be a prime number, and let Γ be a formal group of dimension 1 and height n over $\overline{\mathbb{F}}_p$. Let LT be the functor of deformations of Γ. Then LT is representable by a formal scheme $\operatorname{Spf} A$ whose coordinate ring is isomorphic to a power series ring over $W = W(\overline{\mathbb{F}}_p)$:

$$A \cong W[\![u_1, \dots, u_{n-1}]\!].$$

Let $\mathcal{O}_D = \operatorname{End} \Gamma$; then $\mathcal{O}_D$ is the ring of integers in a division algebra D over $\mathbb{Q}_p$ of invariant $1/n$. Explicitly, $\mathcal{O}_D$ is generated over $W(\mathbb{F}_{p^n})$ by an element Π satisfying $\Pi^n = p$ and $\Pi\alpha = \sigma(\alpha)\Pi$, where $\sigma \in \operatorname{Aut} W(\mathbb{F}_{p^n})$ is the Frobenius automorphism. The Morava stabilizer group $\mathbb{G}_n$ from Section 2 is the profinite completion of $D^\times$; it fits into an exact sequence:

$$1 \to \mathcal{O}_D^\times \to \mathbb{G}_n \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1.$$

There is a continuous action of $\mathbb{G}_n$ on A.

Our main theorem (Theorem B) concerns the continuous cohomology ring $H^*_{\mathrm{cts}}(\mathbb{G}_n, A)$. The action of $\mathbb{G}_n$ on A is rather inexplicit, and it seems difficult to compute $H^*_{\mathrm{cts}}(\mathbb{G}_n, A)$ directly in terms of cocycles. To address this problem, we pass from the formal scheme LT to its rigid-analytic generic fiber LT_K, which is isomorphic to the open unit ball over K. The use of LT_K has a precedent in chromatic homotopy theory, namely in applications of the *Gross-Hopkins period morphism* [HG94]:

$$\pi\colon \mathrm{LT}_K \to \mathbb{P}(M(\Gamma)).$$

Here $M(\Gamma) \cong K^n$ is the rational Dieudonné module of Γ, and $\mathbb{P}(M(\Gamma)) \cong \mathbb{P}^{n-1}_K$ is the corresponding projective space, considered as a rigid-analytic space over K. The morphism π is étale, surjective, equivariant for the action of $\mathcal{O}_D^\times$, and fairly explicit in terms of the variables

$u_1, \dots, u_{n-1}$. Using the period morphism π, it is possible in principle to give formulas for the action of $\mathcal{O}_D^\times$ on A, see [DH95].

For our work we will not use the period morphism but rather a different structure possessed by LT_K: *the isomorphism between the two towers*. This was discovered by Faltings [Fal02b], see also [FGL08] for more details. We review the form given in [SW13, Theorem D], which treats this phenomenon as an isomorphism between perfectoid spaces.

Let $\mathcal{H}$ be *Drinfeld's symmetric space* (also called Drinfeld's upper half-space) in dimension $n-1$. This is a rigid-analytic space over $\mathbb{Q}_p$, defined as

$$\mathcal{H} = \mathbb{P}^{n-1}_{\mathbb{Q}_p} \backslash \bigcup_H H$$

where $\mathbb{P}^{n-1}_{\mathbb{Q}_p}$ is rigid-analytic projective space, and H runs over all $\mathbb{Q}_p$-rational hyperplanes in $\mathbb{P}^{n-1}_{\mathbb{Q}_p}$. Then $\mathcal{H}$ admits an action of the group $\mathrm{GL}_n(\mathbb{Q}_p)$. Then $\mathcal{H}$ admits an action of the group $\mathrm{GL}_n(\mathbb{Q}_p)$ and in particular of its subgroup $\mathrm{GL}_n(\mathbb{Z}_p)$.

Theorem 3.5.1. *There exists a perfectoid space $\mathcal{X}$ and a diagram of adic spaces:*

$$\begin{array}{ccc} & \mathcal{X} & \\ {}^{\mathrm{GL}_n(\mathbb{Z}_p)}\swarrow & & \searrow^{\mathbb{G}_n} \\ \mathrm{LT}_K & & \mathcal{H} \end{array}$$

Here, $\mathcal{X}$ admits commuting actions of $\mathbb{G}_n$ and $\mathrm{GL}_n(\mathbb{Z}_p)$. The morphism to LT_K is a pro-étale $\mathrm{GL}_n(\mathbb{Z}_p)$-torsor, which is equivariant for the action of $\mathbb{G}_n$. The morphism to $\mathcal{H}$ is a pro-étale $\mathbb{G}_n$-torsor, which is equivariant for the action of $\mathrm{GL}_n(\mathbb{Z}_p)$.

Proof. This is an application of a general duality statement [SW13, Theorem E] between Rapoport-Zink spaces at infinite level. Both LT_K and the base change $\mathcal{H}_K$ arise as the generic fiber of a deformation problem of formal groups. We have already seen that Lubin-Tate space LT parametrizes deformations of a formal group Γ over $\overline{\mathbb{F}}_p$ of dimension 1 and height n. Whereas, $\mathcal{H}_K$ is the generic fiber of a formal scheme $\mathfrak{H}_W$ over $\mathrm{Spf}\, W$ which parametrizes deformations of a *special formal $\mathcal{O}_D$-module* X in the sense of [Dri76]. The relation between X and Γ is:

$$X = \Gamma \oplus \Gamma^{(p)} \oplus \cdots \oplus \Gamma^{(p^{n-1})},$$

where $\Gamma^{(p^k)}$ is the pullback of Γ under the p^kth power Frobenius automorphism of $\overline{\mathbb{F}}_p$.

Trivialization of the torsion in the universal deformation of Γ (resp., X) produces a pro-étale torsor over LT_K (resp., $\mathcal{H}_K$) with group $\mathrm{GL}_n(\mathbb{Z}_p)$ (resp., $\mathcal{O}_D^\times$), known as the Lubin–Tate tower (resp., the Drinfeld tower). Applied to this situation, [SW13, Theorem E] is the statement that the two towers are isomorphic in the limit to the same perfectoid space $\mathcal{X}$.

Note that $\mathcal{H}_K \to \mathcal{H}$, being a pullback of $\mathrm{Spa}\, K \to \mathrm{Spa}\, \mathbb{Q}_p$, is a pro-étale $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-torsor. It remains to be seen why the composition $\mathcal{X} \to \mathcal{H}_K \to \mathcal{H}$ is a pro-étale torsor for the group $\mathbb{G}_n$. For this, we observe that there is an $\mathcal{O}_D$-equivariant isomorphism $\Gamma \xrightarrow{\sim} \Gamma^{(p^n)}$ (for instance, the p^nth power isogeny divided by p). From this one constructs an isomorphism $i\colon X \xrightarrow{\sim} X^{(p)}$ which satisfies the relation $i \circ \alpha = (\Pi\alpha\Pi^{-1}) \circ i$ for all $\alpha \in \mathcal{O}_D^\times$. The isomorphism i induces an automorphism of $\mathcal{X}$ lying over the Frobenius automorphism of $\mathrm{Spa}\, K$ and satisfying the same relation with respect to $\mathcal{O}_D^\times$. This automorphism is exactly the necessary structure required to extend the action of $\mathcal{O}_D^\times$ on $\mathcal{X}$ to an action of $\mathbb{G}_n$. □

Theorem 3.5.1 suggests a strategy for accessing the cohomology ring $H^*_{\mathrm{cts}}(\mathbb{G}_n, A)$. The diagram in Theorem 3.5.1 witnesses an isomorphism in $D(\mathrm{Solid})$:

$$R\Gamma_{\mathrm{cond}}(\mathrm{LT}_{K,\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)^{h\mathbb{G}_n} \xrightarrow{\sim} R\Gamma_{\mathrm{cond}}(\mathcal{H}_{K,\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} \tag{3.5.2}$$

Indeed by Proposition 3.3.3, both objects are isomorphic to

$$R\Gamma_{\mathrm{cond}}(\mathcal{X}_{\text{proét}}, \hat{\mathcal{O}}^+)^{h(\mathbb{G}_n \times \mathrm{GL}_n(\mathbb{Z}_p))}.$$

The isomorphism in (3.5.2) is helpful because it translates the opaque action of $\mathcal{O}_D^\times$ on LT_K into the transparent action of $\mathrm{GL}_n(\mathbb{Z}_p)$ on $\mathcal{H}$. To completely leverage (3.5.2), we will have to say something about the $\hat{\mathcal{O}}^+$-cohomology of $X_{\text{proét}}$, where X is LT_K or $\mathcal{H}$, respectively. Let us write X_{an} for the analytic topology, to distinguish it from the pro-étale topology. The following comparison statements appear as Theorem 6.2.5 and Theorem 6.3.2.

Theorem 3.5.3. *The pro-étale cohomology of $\hat{\mathcal{O}}^+$ on LT_K and $\mathcal{H}$ can be approximated as follows.*

(1) There is a morphism of differential graded solid W-algebras, which is equivariant for the action of $\mathbb{G}_n$:

$$A[\varepsilon] \to R\Gamma_{\mathrm{cond}}(\mathrm{LT}_{K,\text{proét}}, \hat{\mathcal{O}}^+).$$

(2) There is a morphism of differential graded solid $\mathbb{Z}_p$-algebras, which is equivariant for the action of $\mathrm{GL}_n(\mathbb{Z}_p)$:

$$\mathbb{Z}_p[\varepsilon] \to R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\text{proét}}, \hat{\mathcal{O}}^+).$$

Here $R[\varepsilon]$ is shorthand for the complex $\underline{R} \xrightarrow{0} \underline{R}$ in degrees 0,1. Let A be the cofiber of either of the above morphisms in $D(\mathrm{Solid})$. Then $H^i(A) = 0$ for $i \leq 0$, and there exists a single power of p which annihilates $H^i(A)$ for every $i \geq 1$.

The proof of Theorem 3.5.3 requires the full force of the integral p-adic Hodge theory theorems of [BMS18] and [vK19]. The effect is to reduce the study of $R\Gamma_{\mathrm{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+)$ (where X is either of LT_K or $\mathcal{H}$) to the case of a point $X = \mathrm{Spa}(K, \mathcal{O}_K)$. By Example 3.3.4, the pro-étale cohomology of $\mathrm{Spa}(K, \mathcal{O}_K)$ agrees with the Galois cohomology $\mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)}$, where C is the completion of an algebraic closure of K. This cohomology was controlled by Tate [Tat67], in a way that is valid for any local field of characteristic $(0, p)$. Expressed in our language, Tate's result is that there is a morphism of differential graded solid $\mathcal{O}_K$-algebras

$$W[\varepsilon] \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K)}$$

whose cofiber has p^N-torsion cohomology groups, for some absolute constant N.

Combining (3.5.2) with Theorem 3.5.3, we obtain a diagram in $D(\mathrm{Solid})$:

$$\begin{aligned} A^{h\mathbb{G}_n} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\varepsilon] \quad &\cong \quad A[\varepsilon]^{h\mathbb{G}_n} \\ &\to \quad R\Gamma_{\mathrm{cond}}(\mathrm{LT}_{K,\text{proét}}, \hat{\mathcal{O}}^+)^{h\mathbb{G}_n} \\ &\cong \quad R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} \\ &\leftarrow \quad \mathbb{Z}_p[\varepsilon]^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} \\ &\cong \quad \mathbb{Z}_p^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\varepsilon] \end{aligned}$$

Here, each of the two arrows not labeled as an isomorphism has cofiber whose cohomology groups are annihilated by some uniform power of p.

We will briefly indicate how this is used to prove Theorem B, leaving the details for Section 6. By Proposition 2.5.1, we have a $\mathbb{G}_n$-equivariant splitting $A = W \oplus A^c$. After inverting p in the above diagram, we arrive at an isomorphism in cohomology:

$$\left(H^*_{\mathrm{cts}}(\mathbb{G}_n, K) \oplus H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right) \otimes \mathbb{Q}_p[\varepsilon] \cong H^*_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{Z}_p), \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\varepsilon] \tag{3.5.4}$$

By Lemmas 3.4.1 and 3.4.4, $H^*_{\mathrm{cts}}(\mathbb{G}_n, K)$ and $H^*_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{Z}_p), \mathbb{Q}_p)$ are isomorphic to the same exterior $\mathbb{Q}_p$-algebra. In particular $\dim_{\mathbb{Q}_p} H^i_{\mathrm{cts}}(\mathbb{G}_n, K) = \dim_{\mathbb{Q}_p} H^i_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{Z}_p), \mathbb{Q}_p)$ for all i.

Comparing dimensions of the $\mathbb{Q}_p$-vector spaces in (3.5.4) shows that $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$, which is to say that $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c)$ is torsion. A more careful analysis shows that $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c)$ is killed by a uniform power of p, which is the assertion of Theorem B.

4. The Galois cohomology of $\mathcal{O}_C$

Let us fix some definitions. A *nonarchimedean field* is a field K which is complete with respect to the topology induced from a nontrivial nonarchimedean valuation $|\,| : K \to \mathbb{R}_{\geq 0}$. (Some authors do not require K to be complete, but for our purposes it will be useful to always assume this.) Let $\mathcal{O}_K$ be its valuation ring; i.e., the subring of elements with $|\alpha| \leq 1$. Let κ be the residue field of $\mathcal{O}_K$. The characteristic of a nonarchimedean field K refers to the pair $(\operatorname{char} K, \operatorname{char} \kappa)$.

A *local field* is a nonarchimedean field satisfying the additional conditions: (a) the valuation on K is discrete, in other words $\mathcal{O}_K$ is a discrete valuation ring, and (b) the residue field κ is perfect.

Let L be a (possibly infinite) Galois extension of a nonarchimedean field K. The valuation on K extends uniquely to a valuation on L, and the completion $\hat{L}$ is a nonarchimedean field admitting a continuous action of $\mathrm{Gal}(L/K)$. (If L/K is finite then it is not necessary to complete.) It is an interesting problem to compute the continuous cohomology groups $H^i_{\mathrm{cts}}(\mathrm{Gal}(L/K), \mathcal{O}_{\hat{L}})$, or at least to approximate these as $\mathcal{O}_K$-modules. For simplicity we will assume throughout that $\operatorname{char} K = 0$.

A basic result along these lines is due to Ax, which settles the problem in degree 0.

Theorem 4.0.1 ([Ax64])**.** *Let K be a nonarchimedean field with $\operatorname{char} K = 0$. Let L/K be a Galois extension of nonarchimedean fields. Then the subfield of $\hat{L}$ fixed by $\mathrm{Gal}(L/K)$ is exactly K. Consequently $H^0(\mathrm{Gal}(L/K), \mathcal{O}_{\hat{L}}) = \mathcal{O}_K$.*

Results on higher cohomology tend to require that K be a local field. A classical result attributed to Noether states that if L/K is a finite tamely ramified Galois extension of local fields, then $\mathcal{O}_L$ is a free $\mathcal{O}_K[\mathrm{Gal}(L/K)]$-module, and therefore $H^i(\mathrm{Gal}(L/K), \mathcal{O}_L) = 0$ for $i > 0$. For arbitrary finite extensions there is the following result of Sen:

Theorem 4.0.2 ([Sen69])**.** *Let K be a local field of characteristic $(0, p)$. Let L/K be a finite Galois extension. Then $H^1(\mathrm{Gal}(L/K), \mathcal{O}_L)$ is α-torsion for any $\alpha \in \mathcal{O}_K$ with $|\alpha| \leq |p|^{1/(p-1)}$.*

The theorem applies to L/K infinite, but only in the sense that the non-continuous H^1 can be controlled.

The following is the main theorem of this section. The techniques used in its proof are due to Tate; control over the error terms is the only added value.

Theorem 4.0.3. *Let K be a local field of characteristic $(0, p)$, and let C be the completion of an algebraic closure $\overline{K}/K$. Let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. Then:*

1. *$H^0(\Gamma_K, \mathcal{O}_C) = \mathcal{O}_K$.*
2. *There exists an injective map of $\mathcal{O}_K$-modules*
$$\mathcal{O}_K \to H^1_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C)$$
whose cokernel is p^k-torsion. Here we can take $k = 4$ or $k = 5$ when p is odd or even, respectively. If $p \nmid e_K$ (where e_K is the absolute ramification index of K), these bounds can be improved to $k = 3$ and $k = 4$, respectively.
3. *$H^i_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C)$ is p-torsion for $i > 1$.*

A more detailed version of Theorem 4.0.3 appears as Theorem 4.2.5. We also obtain an explicit bound on the cohomology of the nontrivial Tate twists $\mathcal{O}_C(j)$.

Theorem 4.0.4. *With notation as in Theorem 4.0.3, let $j \neq 0$ be an integer, and let $\mathcal{O}_C(j)$ be the jth Tate twist of $\mathcal{O}_C$ as a $\mathrm{Gal}(\overline{K}/K)$-module. Then:*

(1) $H^0(\Gamma_K, \mathcal{O}_C(j)) = 0$.

(2) $H^1_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C(j))$ is $p^{M+\mathrm{ord}_p(j)}$-torsion. Here $M = M_K$ is a constant which only depends on K and which is insensitive to passage to a tamely ramified extension of K. If $p \nmid e_K$ we may take $M = 2$ if p is odd and $M = 3$ if $p = 2$.

(3) For $i \geq 2$, $H^i_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C(j))$ is p-torsion if p is odd and p^2-torsion if $p = 2$.

The theorem appears later as Theorem 4.3.3.

4.1. **Infinitely ramified $\mathbb{Z}_p$-extensions.** Suppose K is a local field of characteristic $(0, p)$. We fix some notation regarding valuations. We will have use for the (additively written) valuations v and v_K on K. These are normalized so that $v(p) = v_K(\pi) = 1$, where π is a uniformizer of K. The two valuations are related by $v_K = e_K v$, where $e_K = v_K(p)$ is the (absolute) ramification index of K, and they extend uniquely to $\mathbb{Q}$-valued valuations on $\overline{K}$.

Let K_∞/K be an infinitely ramified Galois extension with $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$. Then $\mathrm{Gal}(K_\infty/K)$ has an obvious filtration by subgroups $p^n\mathbb{Z}_p$ for $n = 0, 1, \ldots$. We write K_n/K for the finite extension corresponding to the subgroup $p^n\mathbb{Z}_p \subset \mathbb{Z}_p$.

On the other hand we have the upper numbering ramification filtration $\mathrm{Gal}(K_\infty/K)^u$ for real numbers $u \geq -1$. By the Hasse-Arf theorem, the jumps of this filtration occur at rational integers. That is, there is a sequence of integers $-1 = u_0 \leq u_1 \leq u_2 < \ldots$ such that

$$\mathrm{Gal}(K_\infty/K)^u \cong p^n\mathbb{Z}_p \text{ whenever } u_n < u \leq u_{n+1}$$

for all $n = 0, 1, \ldots$. The two filtrations on $\mathrm{Gal}(K_\infty/K)$ are "eventually compatible" in the following sense.

Lemma 4.1.1. *Let $u_1, u_2, \ldots$ be the sequence defined above.*

(1) There exists $N = N_K \geq 1$ such that $u_N \geq 0$ and $u_{n+1} = u_n + e_K$ for all $n \geq N$.

(2) If $e_K = 1$ and K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension, then the sequence $u_1, u_2, \ldots$ is $1, 2, 3, \ldots$.

(3) Suppose L/K is a finite tamely ramified Galois extension. Then $L_\infty = LK_\infty$ is an infinitely ramified $\mathbb{Z}_p$-extension of L. The sequence of jumps for L_∞/L is $e_{L/K}u_1, e_{L/K}u_2, \ldots$. In particular, if $p \nmid e_K$ and K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension, then the sequence of jumps for K_∞/K is $e_K, 2e_K, \ldots$.

(4) Let $N = N_K$ as above. Consider the $\mathbb{Z}_p$-extension K_∞/K_N. Its sequence of jumps is $e_{K_N}, 2e_{K_N}, \ldots$.

Proof. Part (1) is well-known (see [Tat67, Proposition 5]), so we only give a sketch. It is possible to reduce to the case that the residue field of K is quasi-finite, in which case local class field theory applies. The reciprocity map of local class field theory $K^\times \to \mathrm{Gal}(K_\infty/K)$ sends $1 + \mathfrak{m}_K^u$ surjectively onto $\mathrm{Gal}(K_\infty/K)^u$ for each $u \geq 1$. The statement now follows from the observations that (a) since K_∞/K is infinitely ramified, the inertia group $\mathrm{Gal}(K_\infty/K)^0$ is an open subgroup of $\mathrm{Gal}(K_\infty/K)$, and (b) $(1 + \mathfrak{m}_K^u)^p = 1 + p\mathfrak{m}_K^u = 1 + \mathfrak{m}_K^{u+e_K}$ for u large enough ($u > e_K/(p-1)$ suffices).

For part (2): The computation of the higher ramification groups of the cyclotomic $\mathbb{Z}_p$-extension over $\mathbb{Q}_p$ (or any unramified extension thereof) is standard, see for instance [Ser79, Chapter IV, §4].

For part (3), note that the the wild inertia groups $\mathrm{Gal}(\overline{K}/K)^{>0}$ and $\mathrm{Gal}(\overline{K}/L)^{>0}$ agree. Let $\psi_{L/K}(u)$ be the Herbrand function. The transitivity of the Herbrand function implies that $\mathrm{Gal}(\overline{K}/L)^{\psi_{L/K}(u)} = \mathrm{Gal}(\overline{K}/K)^u$ for all $u > 0$. On the other hand $\psi_{L/K}(u) = e_{L/K}u$ for $u > 0$ since L/K is tamely ramified. Since the ramification filtration is compatible with quotients we have $\mathrm{Gal}(L_\infty/L)^{e_{L/K}u} = \mathrm{Gal}(K_\infty/K)^u$ for all $u > 0$, and so the jumps for $\mathrm{Gal}(L_\infty/L)$ occur at $e_{L/K}u_1, e_{L/K}u_2, \dots$.

Part (4) follows once again from the transitivity of the Herbrand function. □

Let us call a $\mathbb{Z}_p$-extension K_∞/K *regular* if its sequence of jumps is $e_K, 2e_K, \dots$. Part (4) of the lemma will allow us to reduce our arguments to the case that K_∞/K is regular, in which case the ramification is tightly controlled.

For a finite extension L/K of local fields, we let $\mathcal{D}_{L/K}$ denote the relative different. Recall the formula for the valuation of the different in the case that L/K is Galois:

$$v_K(\mathcal{D}_{L/K}) = \int_{-1}^{\infty} \left(1 - \frac{1}{\#\,\mathrm{Gal}(L/K)^u}\right)\, du \tag{4.1.2}$$

Applied to K_n/K, we find:

$$v_K(\mathcal{D}_{K_n/K}) = \sum_{k=0}^{n-1} (u_{k+1} - u_k)\left(1 - \frac{1}{p^{n-k}}\right) \tag{4.1.3}$$

Lemma 4.1.4. *Assume that K_∞/K is regular. Then for all $n \geq 0$:*

$$v_{K_{n+1}}(\mathcal{D}_{K_{n+1}/K_n}) = e_K p^{n+1} - e_K + p - 1$$

Proof. Use the transitivity of differents together with (4.1.3) and $v_{K_{n+1}} = p^{n+1} v_K$. □

We need the following result on traces.

Lemma 4.1.5 ([Ser79, Chapter V Lemma 4])**.** *Let L/K be a cyclic extension of order p. Let $d = v_L(\mathcal{D}_{L/K})$. Letting $\mathfrak{m}_K$ denote the maximal ideal of $\mathcal{O}_K$, and similarly for $\mathfrak{m}_L$, we have for each $i \geq 0$:*

$$\mathrm{tr}_{L/K}(\mathfrak{m}_L^i) = \mathfrak{m}_K^j,$$

where $j = \left[\frac{i+d}{p}\right]$.

Lemma 4.1.6. *Assume K_∞/K is regular. We have the following inequalities.*

1. *For $x \in K_{n+1}$ we have:*

$$\left|\mathrm{tr}_{K_{n+1}/K_n}(x)\right| \leq |p|^{1-\frac{1}{p^{n+1}}}\, |x|$$

2. *For $x \in K_n$ we have:*

$$\left|\mathrm{tr}_{K_n/K}(x)\right| \leq |p|^{n-\frac{1}{p-1}}\, |x|$$

Proof. Applying Lemma 4.1.5 to K_{n+1}/K_n, we find $d = e_K p^{n+1} - e_K + p - 1$, and so

$$\begin{aligned} v_{K_n}(\mathrm{tr}_{K_{n+1}/K_n}(x)) &\geq \left[\frac{v_{K_{n+1}}(x) + d}{p}\right] \\ &\geq \frac{v_{K_{n+1}}(x) + d - (p-1)}{p} \\ &= v_{K_n}(x) + e_K p^n - \frac{e_K}{p} \end{aligned}$$

Translating this in terms of absolute values gives (1).

Applying (1) inductively to $x \in K_n$ gives (2). □

Keep the assumption that K_∞/K is regular. Define the normalized traces by $t = p^{-n}\operatorname{tr}_{K_n/K}$ and $t_n = p^{-1}\operatorname{tr}_{K_{n+1}/K_n}$. Then Lemma 4.1.6 translates into:

$$|t(x)| \le |p|^{-\frac{1}{p-1}} |x|, \; x \in K_\infty \tag{4.1.7}$$

and

$$|t_n(x)| \le |p|^{-\frac{1}{p^{n+1}}} |x| \tag{4.1.8}$$

Let σ be a topological generator of $\mathrm{Gal}(K_\infty/K)$.

Lemma 4.1.9. *Assume that K_∞/K is regular. For $x \in K_\infty$ we have:*

$$|x - t(x)| \le |p|^{-1-\frac{1}{p(p-1)}} |\sigma(x) - x|$$

Proof. For each $x \in K_{n+1}$ we have

$$x - t_n(x) = \sum_{i=1}^{p-1} p^{-1}(1 + \sigma^{p^n} + \cdots + \sigma^{(i-1)p^n})(1 - \sigma^{p^n})(x),$$

so that

$$|x - t_n(x)| \le |p|^{-1} \left|(1 - \sigma^{p^n})x\right| \le |p|^{-1} |\sigma(x) - x|. \tag{4.1.10}$$

We will prove by induction on $n \ge 1$ the following statement which implies the lemma: for $x \in K_n$, we have

$$|x - t(x)| \le |p|^{-1-\frac{1}{p^2}-\cdots-\frac{1}{p^n}} |\sigma(x) - x|.$$

The base case is (4.1.10). Assume the statement for n, and then for $x \in K_{n+1}$ we have:

$$|x - t(x)| \le \max\{|x - t_n(x)|, |t_n(x) - t(x)|\}$$

Treating each quantity on the right side in turn, we have

$$|x - t_n(x)| \le |p|^{-1} |\sigma(x) - x|$$

by (4.1.10), and then applying (4.1.7) and (4.1.8), we find:

$$\begin{aligned} |t_n(x) - t(x)| &= |t_n(x) - t(t_n(x))| \\ &\le |p|^{-1-\frac{1}{p^2}-\cdots-\frac{1}{p^n}} |(\sigma - 1)t_n(x)| \\ &\le |p|^{-1-\frac{1}{p^2}-\cdots-\frac{1}{p^{n+1}}} |\sigma(x) - x|. \end{aligned}$$

□

The bounds in Lemma 4.1.6 and Lemma 4.1.9 show that $t\colon K_\infty \to K$ is continuous and therefore extends uniquely to a K-linear map $t\colon \hat{K}_\infty \to K$ satisfying the bounds for all $x \in \hat{K}_\infty$:

$$|t(x)| \le p^{-\frac{1}{p-1}} |x| \tag{4.1.11}$$

$$|x - t(x)| \le |p|^{-1-\frac{1}{p(p-1)}} |\sigma(x) - x| \tag{4.1.12}$$

Lemma 4.1.13. *The natural map*

$$\mathcal{O}_K \cong H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), \mathcal{O}_K) \to H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), \mathcal{O}_{\hat{K}_\infty}) \tag{4.1.14}$$

is injective. If in addition K_∞/K is regular, the cokernel of the map is α-torsion for any $\alpha \in \mathcal{O}_K$ with $v(\alpha) > 1 + \frac{p+1}{p(p-1)}$. In particular the cokernel is p^2-torsion for $p \neq 2$ and p^3-torsion for $p = 2$.

Proof. The map in (4.1.14) is injective, for if $\alpha \in \mathcal{O}_K$ lies in the kernel, then $\alpha = (\sigma - 1)\beta$ for $\beta \in \mathcal{O}_{\hat{K}_\infty}$, but then $\alpha = t(\alpha) = 0$.

Assume now K_∞/K is regular. Let us write the superscript $t = 0$ to mean the kernel of the normalized trace t wherever this is defined. For each $n \geq 1$, the operator $\sigma - 1$ is injective on $K_n^{t=0}$ and hence an isomorphism; write $(\sigma - 1)^{-1}$ for its inverse. The inequality (4.1.12) shows that $(\sigma - 1)^{-1}$ is bounded on $K_\infty^{t=0}$ with operator norm $\leq |p|^{-1-1/p(p-1)}$, so it extends to an operator $(\sigma - 1)^{-1}$ on $\hat{K}_\infty^{t=0}$ with the same operator norm. Therefore

$$H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), \mathcal{O}^{t=0}_{\hat{K}_\infty}) \cong \frac{\mathcal{O}^{t=0}_{\hat{K}_\infty}}{(\sigma - 1)\mathcal{O}^{t=0}_{\hat{K}_\infty}}$$

is α-torsion for any $\alpha \in \mathcal{O}_K$ with $v(\alpha) \geq 1 + 1/p(p-1)$.

On the other hand, the inequality (4.1.11) shows that the inclusion

$$\mathcal{O}_K \oplus \mathcal{O}^{t=0}_{\hat{K}_\infty} \hookrightarrow \mathcal{O}_{\hat{K}_\infty}$$

is β-torsion for any $\beta \in \mathcal{O}_K$ with $v(\beta) \geq 1/(p-1)$. Combining this paragraph with the last, we find that the cokernel of (4.1.14) is killed by any product $\alpha\beta$, where $v(\alpha) \geq 1 + 1/p(p-1)$ and $v(\beta) \geq 1/(p-1)$. If K contains elements of sufficiently small positive valuation, then the conclusion of the lemma holds: the cokernel of (4.1.14) is killed by any element of valuation $> 1+(p+1)/p(p-1)$. In general, one can let L/K be a sufficiently ramified finite tame extension, and let $L_\infty = LK_\infty$. Then L_∞/L is regular by Lemma 4.1.1(3). The result descends from L to K using Noether's theorem ($H^i(\mathrm{Gal}(L/K), \mathcal{O}_L) = 0$ for $i > 0$). □

We are ready to state the main result of this subsection. Let K be a general local field of characteristic $(0, p)$, and let K_∞/K be an infinitely ramified $\mathbb{Z}_p$-extension. The inclusion $\mathcal{O}_K \hookrightarrow \mathcal{O}_{\hat{K}_\infty}$ induces a map of complexes:

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)}. \tag{4.1.15}$$

Note that $\mathrm{Gal}(K_\infty/K)$ acts trivially on $\mathcal{O}_K$. A choice of isomorphism $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ induces a quasi-isomorphism $\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \cong \mathcal{O}_K[\varepsilon]$.

Proposition 4.1.16. *Let X be the cofiber of the morphism in (4.1.15), so that we have an exact triangle*

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to X.$$

Then $H^i(X) = 0$ for all $i \neq 1$. As for $H^1(X)$, it is p^3-torsion if $p \neq 2$, and p^4-torsion if $p = 2$. If K_∞/K is regular, the bounds can be improved to p^2 and p^3, respectively.

Proof. We have $H^0(X) = 0$ because $H^0(\mathrm{Gal}(K_\infty/K), \mathcal{O}_{\hat{K}_\infty}) = \mathcal{O}_K$ and because the map in (4.1.14) is injective, by the same reasoning used in the proof of Lemma 4.1.13. Also we have $H^i(X) = 0$ for $i > 1$ because $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ has cohomological dimension 1. Thus we are reduced to studying $H^1(X)$, which is the cokernel of (4.1.14). If K_∞/K is regular, the desired statement is Lemma 4.1.13.

If K_∞/K is not regular, let $N = N_K$ be the bound from Lemma 4.1.1, so that K_∞/K_N is regular. Let X_N be the complex analogous to X. Then we have a left exact sequence:

$$0 \to H^1(\mathrm{Gal}(K_N/K), \mathcal{O}_{K_N}) \to H^1(X) \to H^1(X_N)$$

By Sen's theorem (Theorem 4.0.2), $H^1(\mathrm{Gal}(K_N/K), \mathcal{O}_{K_N})$ is p-torsion. Lemma 4.1.13, $H^1(X_N)$ is p^2- or p^3-torsion as p is odd or even. Therefore $H^1(X)$ is p^3 or p^4-torsion as p is odd or even. □

4.2. **The Galois cohomology of $\mathcal{O}_C$.** Keep the assumption that K is a local field of characteristic $(0,p)$. Let $\overline{K}$ be an algebraic closure, and let C be the metric completion of $\overline{K}$. We are interested in $\mathcal{O}_C^{h\Gamma_K}$, the complex computing the continuous cohomology of Γ_K on $\mathcal{O}_C$. The idea is to use an infinitely ramified $\mathbb{Z}_p$-extension K_∞/K as an intermediary:

$$\mathcal{O}_C^{h\Gamma_K} \cong \left(\mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K_\infty)}\right)^{h\,\mathrm{Gal}(K_\infty/K)}. \tag{4.2.1}$$

First we deal with the inner term on the right side of (4.2.1).

Lemma 4.2.2. *Define Y_0 by the exact triangle*

$$\mathcal{O}_{\hat{K}_\infty} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K_\infty)} \to Y_0,$$

where the first morphism is induced from $\mathcal{O}_{\hat{K}_\infty} \hookrightarrow \mathcal{O}_C$. Then $H^0(Y_0)=0$, and for all $i>0$, $H^i(Y_0)$ is almost zero, in the sense that it is α-torsion for any $\alpha\in\mathcal{O}_{\hat{K}_\infty}$ with $v(\alpha)>0$.

Proof. See [Tat67, §3, Corollary 1]. In modern terms, $\hat{K}_\infty$ is a perfectoid field, and for such fields there is an "almost purity" result: for $i>0$, the cohomology $H^i_{\mathrm{cts}}(\mathrm{Gal}(\overline{K}/K_\infty),\mathcal{O}_C)$ is almost zero. The vanishing of $H^0(Y_0)$ is a case of Ax's theorem (Theorem 4.0.1). □

The complex Y_0 in Lemma 4.2.2 admits a $\mathrm{Gal}(K_\infty/K)$-action, so we may define the derived invariants $Y := Y_0^{h\,\mathrm{Gal}(K_\infty/K)}$.

Lemma 4.2.3. *We have*

$$H^i(Y) = \begin{cases} 0, & i=0,\\ \pi\text{-torsion}, & i\geq 1.\end{cases}$$

Here π is uniformizer for K. In particular $H^i(Y)$ is p-torsion for $i\geq 1$.

Proof. Consider the spectral sequence

$$H^i_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), H^j(Y_0)) \implies H^{i+j}(Y).$$

The left side only has nonzero terms for $i=0,1$. Combining this with Lemma 4.2.2, we find that $H^0(Y)=0$ and $H^i(Y)$ is π^2-torsion for all $i>0$.

To improve the result as in the lemma, we let L/K be a finite Galois extension such that $e_{L/K}\geq 2$ and $p\nmid \#\,\mathrm{Gal}(L/K)$. Let Y_L be the complex defined analogously to Y, using $L_\infty = LK_\infty$. We may identify $\mathrm{Gal}(L_\infty/K_\infty)$ with $\mathrm{Gal}(L/K)$. Then $\mathrm{Gal}(L/K)$ acts on Y_L, and there is an exact triangle

$$A\to Y\to Y_L^{h\,\mathrm{Gal}(L/K)},$$

where A is the cofiber of $\mathcal{O}_{\hat{K}_\infty}\to\mathcal{O}_{\hat{L}_\infty}^{h\,\mathrm{Gal}(L/K)}$. But by Noether's theorem, this cofiber is 0, and so $Y\cong Y_L^{h\,\mathrm{Gal}(L/K)}$. Now consider the spectral sequence

$$H^i(\mathrm{Gal}(L/K), H^j(Y_L)) \implies H^{i+j}(Y_L^{h\,\mathrm{Gal}(L/K)})$$

Since $\#\,\mathrm{Gal}(L/K)$ is invertible in $\mathcal{O}_K$, the left side is only nonzero for $i=0$, and so $H^i(Y)\cong H^i(Y_L)^{\mathrm{Gal}(L/K)}$. We have just seen that this is π_L^2-torsion, where π_L is a uniformizer for L. Since $e_{L/K}\geq 2$, we find that $H^i(Y)$ is π-torsion as well. □

We have a composition of morphisms:

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\Gamma_K} \tag{4.2.4}$$

induced by $\mathcal{O}_K\hookrightarrow\mathcal{O}_{\hat{K}_\infty}\hookrightarrow\mathcal{O}_C$. Recall $\Gamma_K=\mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K.

Theorem 4.2.5. *Define Z by the exact triangle*

$$\mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C^{h\Gamma_K} \to Z,$$

where the first morphism is the composition in (4.2.4). Then:

$$H^i(Z) = \begin{cases} 0, & i=0, \\ p^k\text{-torsion}, & i=1 \\ p\text{-torsion}, & i \geq 2 \end{cases}$$

Here we can take $k=4$ or $k=5$ as p is odd or even respectively. If K_∞/K_0 is regular (for instance if $p \nmid e_K$), these bounds can be improved to 3 and 4, respectively.

Proof. Consider the three exact triangles:

$$\begin{aligned} \mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} &\to \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} \to X \\ \mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(K_\infty/K)} &\to \mathcal{O}_C^{h\Gamma_K} \to Y \\ \mathcal{O}_K^{h\,\mathrm{Gal}(K_\infty/K)} &\to \mathcal{O}_C^{h\Gamma_K} \to Z \end{aligned}$$

Here the first triangle is from Proposition 4.1.16, the second is obtained by applying $h\,\mathrm{Gal}(K_\infty/K)$ to the triangle defining Y_0, and the third is as in the theorem. By the octahedral axiom, we have an exact triangle $X \to Z \to Y$. The result now follows by combining Proposition 4.1.16 with Lemma 4.2.3. □

4.3. **Galois cohomology of characters.** Once again suppose K is a local field of characteristic $(0,p)$. Let

$$\chi\colon \mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p^\times$$

be a character; i.e., a continuous homomorphism. We assume that χ is infinitely ramified, in the sense that the image of the inertia group under χ is infinite. Let K_∞ be the fixed field of the kernel of χ. Let $\mathbb{Z}_p(\chi)$ be $\mathbb{Z}_p$ with an action of $\mathrm{Gal}(\overline{K}/K)$ through χ. Then if M is any p-adically complete $\mathrm{Gal}(\overline{K}/K)$-module, we may define a new such module by $M(\chi) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)$. The present goal is to bound the continuous cohomology of $\mathrm{Gal}(K_\infty/K)$ acting on $\mathcal{O}_{\hat{K}_\infty}(\chi)$.

Let $U \subset \mathbb{Z}_p^\times$ be the largest subgroup which is isomorphic to $\mathbb{Z}_p$. Thus $U = 1 + p\mathbb{Z}_p$ for p odd and $U = 1 + p^2\mathbb{Z}_p$ for $p=2$. Let K_0 be the fixed field of $\chi^{-1}(U)$; then K_∞/K_0 is an infinitely ramified $\mathbb{Z}_p$-extension. As usual, we let K_n/K_0 be the fixed field of $p^n\,\mathrm{Gal}(K_\infty/K_0)$. Finally, we define an integer $r \geq 1$ by

$$\chi(\mathrm{Gal}(K_\infty/K_0)) = 1 + p^r\mathbb{Z}_p.$$

Lemma 4.3.1. *Let $N \geq 0$ be large enough so that K_∞/K_N is regular. Then*

$$H^i_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{\hat{K}_\infty}(\chi)) = \begin{cases} 0, & i=0 \\ (p^k\text{-torsion}), & i=1. \end{cases}$$

Here we may take $k = N+r+1$ if p is odd and $N+r+2$ if $p=2$.

Proof. We first prove the result in the case that K_∞/K_0 is regular. Write $t\colon \hat{K}_\infty \to K_0$ for the normalized trace. As in the proof of Lemma 4.1.13, we make use of the fact that the inclusion

$$\mathcal{O}_{K_0} \oplus \mathcal{O}_{\hat{K}_\infty}^{t=0} \hookrightarrow \mathcal{O}_{\hat{K}_\infty}$$

has α-torsion cokernel for any $\alpha \in \mathcal{O}_{K_N}$ with $v(\alpha) \geq 1/(p-1)$.

We are therefore reduced to studying the continuous cohomology of $\mathrm{Gal}(K_\infty/K_0)$ acting on $\mathcal{O}_{K_0}(\chi)$ and $\mathcal{O}_{\hat{K}_\infty}^{t=0}(\chi)$, respectively. Let $\sigma \in \mathrm{Gal}(K_\infty/K_0)$ be a topological generator, and let $\lambda = \chi(\sigma)^{-1}$. Then for any p-adically complete $\mathrm{Gal}(K_\infty/K_0)$-module M, the continuous cohomology of $M(\chi)$ is computed by the complex $\sigma - \lambda\colon M \to M$. Note that $\lambda = 1 + p^r u$ for a p-adic unit $u \in \mathbb{Z}_p^\times$.

In the case $M = \mathcal{O}_{K_0}$, we have $H^0(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{K_0}(\chi)) = 0$, since σ acts on $\mathcal{O}_{K_0}(\chi)$ as the scalar $\lambda \neq 1$. On the other hand $H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{K_0}(\chi))$ is p^r-torsion, being the cokernel of multiplication by $\lambda - 1 = p^r u$.

To treat $M = \mathcal{O}^{t=0}_{\hat{K}_\infty}$, we apply Lemma 4.1.9 to the regular extension K_∞/K_0. We find that $(\sigma - 1)^{-1}$ is defined on $\hat{K}^{t=0}_\infty$ and has operator norm $\leq |p|^{-1-1/p(p-1)}$.

Let

$$\mu = (\sigma-1)^{-1}(\sigma - \lambda) = 1 - (\lambda - 1)(\sigma - 1)^{-1}.$$

In the case that $r \geq 2$, we have $|\lambda - 1|\,|\sigma - 1|^{-1} < 1$, and so μ has continuous inverse satisfying $\left|\mu^{-1}\right| \leq 1$. Therefore $(\sigma - \lambda)^{-1} = (\sigma - 1)^{-1}\mu^{-1}$ exists on $\hat{K}^{t=0}_\infty$ and has operator norm bounded by $|p|^{-1-1/p(p-1)}$. Thus $H^0(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}^{t=0}_{\hat{K}_\infty}(\chi)) = 0$ and $H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}^{t=0}_{\hat{K}_\infty}(\chi))$ is annihilated by any element $\alpha \in \mathcal{O}_{K_0}$ with $v(\alpha) \geq 1 + 1/p(p-1)$.

If $r = 1$, the idea is to apply the same argument to the regular extension K_∞/K_1, noting that $\lambda^p = \chi(\sigma^p)$ now satisfies $|\lambda^p - 1|\,|\sigma^p - 1|^{-1} < 1$. Thus $H^i_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_1), \mathcal{O}^{t=0}_{\hat{K}_\infty}(\chi))$ is 0 for $i = 0$ and is $p^{1+1/p(p-1)}$-torsion for $i = 1$. The inflation-restriction sequence allows us to deduce the same results for K_∞/K_0.

We have found that $H^0(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{\hat{K}_\infty}(\chi)) = 0$ and $H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{\hat{K}_\infty}(\chi))$ is α-torsion for any $\alpha \in \mathcal{O}_{K_0}$ with $v(\alpha) \geq \max\{r, 1 + 1/p(p-1)\} + 1/(p-1)$. The latter term is $\geq r+1$ or $\geq r+2$ as p is odd or even, respectively, which implies the lemma in the case that K_∞/K_0 is regular. (To handle the problem that $\mathcal{O}_{K_0}$ may not contain elements of sufficiently small valuation, it may be necessary to pass to a tamely ramified extension L/K_0 as in the proof of Lemma 4.1.13.) This concludes the proof of the lemma when K_∞/K_0 is regular.

In general, suppose $N \geq 0$ is large enough so that K_∞/K_N is regular. Note that $\mathrm{Gal}(K_\infty/K_N)$ is generated by σ^{p^N} and that $\chi(\sigma^{p^N}) \equiv 1 \pmod{p^{N+r}}$. Thus the preceding argument shows that $H^0_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_N), \mathcal{O}_{\hat{K}_\infty}(\chi)) = 0$ and $H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_N), \mathcal{O}_{\hat{K}_\infty}(\chi))$ is p^{N+r+1}- or p^{N+r+2}-torsion as p is odd or even, respectively. The same results for K_∞/K_0 now follow from the inflation-restriction sequence.

We conclude that $H^0(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{\hat{K}_\infty}(\chi)) = 0$ and $H^1_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{\hat{K}_\infty}(\chi))$ is α-torsion for any $\alpha \in \mathcal{O}_{K_0}$ with $v(\alpha) \geq \max\{N + r, 1 + 1/p(p-1)\} + 1/(p-1)$. The latter term is $\geq N+r+1$ or $\geq N+r+2$ as p is odd or even, respectively, which implies the lemma. □

Lemma 4.3.2. *Assume that K_∞/K_N is regular. Let $a_p = 0$ if p is odd, and $a_p = 1$ if $p = 2$. We have:*

$$H^i_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K), \mathcal{O}_{\hat{K}_\infty}(\chi)) = \begin{cases} 0, & i = 0, \\ (p^{N+r+1+a_p}\text{-torsion}), & i = 1, \\ (p^{a_p}\text{-torsion}), & i > 1. \end{cases}$$

Proof. Combine the spectral sequence

$$H^i(\mathrm{Gal}(K_0/K), H^j_{\mathrm{cts}}(\mathrm{Gal}(K_\infty/K_0), \mathcal{O}_{\hat{K}_\infty}(\chi))) \implies H^{i+j}(\mathrm{Gal}(K_\infty/K), \mathcal{O}_{\hat{K}_\infty}(\chi))$$

with Lemma 4.3.1. The only terms on the left that contribute occur when $j = 1$. If p is odd, the $i > 0$ terms on the left side vanish because $\#\,\mathrm{Gal}(K_0/K)$ is invertible in $\mathcal{O}_K$. If $p = 2$ then $\#\,\mathrm{Gal}(K_0/K) = 2$, and the best we can say is that the $i > 0$ terms on the left side are 2-torsion. □

As a special case, assume that $p \nmid e_K$, and let

$$\chi\colon \Gamma_K \to \mathbb{Z}_p^\times$$

be the p-adic cyclotomic character, so that $\tau(\zeta) = \zeta^{\chi(\tau)}$ for any $\tau \in \Gamma_K$ and any pth power root of unity ζ. In this case K_∞/K_0 is a regular $\mathbb{Z}_p$-extension. For any $j \in \mathbb{Z}$ we write $\mathbb{Z}_p(j) = \mathbb{Z}_p(\chi^j)$ for the jth Tate twist, and similarly $\mathcal{O}_C(j) = \mathcal{O}_C \otimes \mathbb{Z}_p(j)$. We have $\chi^j(\mathrm{Gal}(K_\infty/K_0)) = 1 + p^r\mathbb{Z}_p$, where $r = \mathrm{ord}_p(j) + 1$.

We conclude the section with a bound on the cohomology of $\mathcal{O}_C(j)$.

Theorem 4.3.3. *Let K be a local field of characteristic $(0, p)$. Let j be a nonzero integer. Finally, let $a_p = 0$ for p odd and $a_p = 1$ for $p = 2$. Then*

$$H^i_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C(j)) = \begin{cases} 0, & i = 0, \\ p^{N+\mathrm{ord}_p(j)+2+a_p}\text{-torsion}, & i = 1, \\ p^{1+a_p}\text{-torsion}, & i > 1. \end{cases}$$

Here N is chosen large enough so that K_∞/K_N is a regular $\mathbb{Z}_p$-extension. In particular if $p \nmid e_K$ we may take $N = 0$.

Proof. Let K_∞/K be the extension obtained by adjoining all pth power roots of unity. As in Lemma 4.2.3, define a complex Y_0 by the exact triangle

$$\mathcal{O}_{\hat{K}_\infty}^{h\,\mathrm{Gal}(\overline{K}/K_\infty)} \to \mathcal{O}_C^{h\,\mathrm{Gal}(\overline{K}/K_\infty)} \to Y_0.$$

Then $H^0(Y_0) = 0$ and $H^i(Y_0)$ is almost zero for $i > 0$. Twisting by χ and taking derived $\mathrm{Gal}(K_\infty/K)$-invariants, we obtain an exact triangle

$$\mathcal{O}_{\hat{K}_\infty}(\chi)^{h\,\mathrm{Gal}(K_\infty/K)} \to \mathcal{O}_C(\chi)^{h\Gamma_K} \to Y,$$

where $Y = Y_0(\chi)^{h\,\mathrm{Gal}(K_\infty/K)}$. The same method of proof for Lemma 4.2.3 shows that $H^0(Y) = 0$ and $H^i(Y)$ is p-torsion for all $i > 0$. Combining these bounds with those obtained in Lemma 4.3.2, we obtain the bounds appearing in the theorem. $\square$

5. Pro-étale cohomology of rigid-analytic spaces

Let K be a local field of characteristic $(0,p)$, and let X be a smooth rigid-analytic space over K. In this section we present some results on the pro-étale cohomology $H^*(X_{\text{proét}},\hat{\mathcal{O}})$ and $H^*(X_{\text{proét}},\hat{\mathcal{O}}^+)$.

5.1. **The rational comparison isomorphism.** There is a convenient basis for the topology $X_{\text{proét}}$, consisting of those $U=\varprojlim U_i$ which are affinoid perfectoid. Such covers are convenient because if U is affinoid perfectoid, then $H^i(U_{\text{proét}},\hat{\mathcal{O}}_X)=0$ for $i>0$ [Sch13a, Lemma 4.10]. As a consequence, if $f_i\colon U_i\to X$ is an affinoid perfectoid covering, then $R\Gamma(X_{\text{proét}},\hat{\mathcal{O}}_X)$ is computed by the Čech complex $\hat{\mathcal{O}}_X(X)\to\prod_i\hat{\mathcal{O}}_X(U_i)\to\prod_{i,j}\hat{\mathcal{O}}_X(U_i\times_X U_j)\to\cdots$. As a special case, if X is an affinoid rigid-analytic space, and $U\to X$ is a pro-étale torsor for a profinite group G with U affinoid perfectoid, then

$$R\Gamma(X_{\text{proét}},\hat{\mathcal{O}}_X)\cong H^0(U,\hat{\mathcal{O}}_X)^{hG}. \tag{5.1.1}$$

If C is an algebraically closed nonarchimedean field of characteristic $(0,p)$, and X is a smooth rigid-analytic space over C, the local nature of $X_{\text{proét}}$ is well-understood, and the cohomology of $\hat{\mathcal{O}}_X$ can be related to differentials:

Theorem 5.1.2 ([Sch13b, Proposition 3.23])**.** *Let X be a smooth rigid-analytic space over C. Let $\nu\colon X_{\text{proét}}\to X_{\text{ét}}$ be the projection. Then for each $j\geq 0$ there is an isomorphism of $\mathcal{O}_{X_{\text{ét}}}$-modules:*

$$\Omega^j_{X_{\text{ét}}/C}(-j)\cong R^j\nu_*\hat{\mathcal{O}}_X.$$

The essential calculation behind Theorem 5.1.2 goes back to Faltings. Étale locally, X is a finite cover of the d-dimensional torus

$$\begin{aligned}\mathbb{T}^d &= \operatorname{Spa}(R_d,R_d^+)\\ R_d^+ &= \mathcal{O}_C\left\langle T_1^{\pm},\dots,T_d^{\pm1}\right\rangle\\ R_d &= R_d^+[1/p].\end{aligned}$$

There is an affinoid perfectoid pro-étale torsor $\tilde{\mathbb{T}}^d\to\mathbb{T}^d$ for the group $\mathbb{Z}_p(1)^d$, namely

$$\begin{aligned}\tilde{\mathbb{T}}^d &= \operatorname{Spa}(\tilde{R}_d,\tilde{R}_d^+)\\ \tilde{R}_d^+ &= \mathcal{O}_C\left\langle T_1^{\pm1/p^\infty},\dots,T_d^{\pm1/p^\infty}\right\rangle\\ \tilde{R}_d &= \tilde{R}_d^+[1/p].\end{aligned}$$

By (5.1.1) we have $H^i(\mathbb{T}^d_{\text{proét}},\mathcal{O}_X)\cong H^i_{\text{cts}}(\mathbb{Z}_p(1)^d,\tilde{R}_d)$. But now an explicit calculation shows that for all i the natural map

$$H^i_{\text{cts}}(\mathbb{Z}_p(1)^d,R_d^+)\to H^i_{\text{cts}}(\mathbb{Z}_p(1)^d,\tilde{R}_d^+)$$

is injective, with cokernel killed by $(\zeta_p-1)^i$. Note that $H^i(\mathbb{Z}_p(1)^d,R_d^+)\cong\bigwedge^i_{R_d^+}(R_d^+)^d$. The upshot is that $R^j\nu_*\hat{\mathcal{O}}_X\cong\bigwedge^d R^1\nu_*\hat{\mathcal{O}}_X$, and $R^1\nu_*\hat{\mathcal{O}}_X$ is a locally free $\mathcal{O}_{X_{\text{ét}}}$-module of rank d. This already suggests that $R^j\nu_*\hat{\mathcal{O}}_X$ should be related to differentials. We refer the reader to [Sch13b, Lemma 3.24] for a functorial construction of the isomorphism in Theorem 5.1.2.

Now suppose once again that X is defined over the discretely valued field K. Once again we write $\nu\colon X_{\text{proét}}\to X_{\text{ét}}$ for the projection; let $\nu_C\colon X_{C,\text{proét}}\to X_{C,\text{ét}}$ be the corresponding projection for X_C. We have

$$R\nu_*\hat{\mathcal{O}}_X=R\nu_*(\hat{\mathcal{O}}_{X_C}^{h\Gamma_K})=(R(\nu_C)_*\hat{\mathcal{O}}_{X_C})^{h\Gamma_K}.$$

Applying Theorem 5.1.2, we find that $(R(\nu_C)_*\hat{\mathcal{O}}_{X_C})^{h\Gamma_K}$ admits a filtration with graded pieces $\Omega^j_{X_C/C}(-j)^{h\Gamma_K}$. Now for a quasi-compact object $U \in X_{\text{ét}}$, the derived global sections of the jth piece on U are

$$\begin{aligned} R\Gamma(U, \Omega^j_{X_C/C}(-j)^{h\Gamma_K}) &= (R\Gamma(U, \Omega^j_{X/K})\hat{\otimes}_K C(-j))^{h\Gamma_K} \\ &= R\Gamma(U, \Omega^j_{X/K})\hat{\otimes}_K C(-j)^{h\Gamma_K}. \end{aligned}$$

By Theorem 4.0.3, the terms with nonzero j vanish, and the $j = 0$ term is $R\Gamma(U, \mathcal{O}_{X_{\text{ét}}})[\varepsilon]$. Therefore

$$R\nu_*\mathcal{O}_X \cong \mathcal{O}_{X_{\text{ét}}}[\varepsilon].$$

We have proved:

Theorem 5.1.3. *Let K be a discretely valued nonarchimedean field of characteristic 0 with perfect residue field. Let X/K be a smooth rigid-analytic space. There is an isomorphism*

$$R\Gamma(X_{\text{ét}}, \mathcal{O}_X)[\varepsilon] \cong R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}_X).$$

(By étale descent, the "ét" on the left side of this can be replaced with "an".)

5.2. **Integral p-adic Hodge theory.** Suppose again that C is an algebraically closed nonarchimedean field of characteristic $(0,p)$, and that X is a smooth rigid-analytic space over C. When X has a sufficiently nice formal model over $\mathcal{O}_C$, the theorems of [BMS18] and [vK19] can be used to gain control over the integral pro-étale cohomology $R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_X)$. The setup is as follows. Let $\mathfrak{X}$ be a formal scheme over $\operatorname{Spf}\mathcal{O}_C$. We assume that $\mathfrak{X}$ is *semistable of dimension d* in the sense of [vK19]. This means that $\mathfrak{X}$ can be covered by affine opens $\mathfrak{U}$ which admit an étale $\mathcal{O}_C$-morphism to

$$\operatorname{Spf}\mathcal{O}_C\left\langle T_0, \dots, T_r, T^{\pm}_{r+1}, \dots, T^{\pm}_d\right\rangle/(T_0\cdots T_r = \pi) \tag{5.2.1}$$

where $\pi \in \mathcal{O}_C$ is a nonunit. We shall assume that $\log_{|p|}|\pi| \in \mathbb{Q}$. (The values of r and π may vary with $\mathfrak{U}$.) Then $\mathfrak{X}$ carries a log structure associated to the subpresheaf $\mathcal{M} = \mathcal{O}_{\mathfrak{X}_{\text{ét}}} \cap \mathcal{O}_{\mathfrak{X}_{\text{ét}}}[1/p]^\times$ of $\mathcal{O}_{\mathfrak{X}_{\text{ét}}}$. Let $\Omega^1_{\mathfrak{X}_{\text{ét}},\log}$ be the sheaf of log-differentials on $\mathfrak{X}_{\text{ét}}$; that is, the sheaf generated by Kähler differentials $\Omega^1_{\mathfrak{X}_{\text{ét}}}$ together with logarithmic differentials df/f for $f \in \mathcal{M}$. (We refer to *continuous* differentials throughout; the right way to construct $\Omega^1_{\mathfrak{X}_{\text{ét}},\log}$ is to do it over $\mathcal{O}_C/p^n$ and then take a limit over n.) The formal scheme $\mathfrak{X}$ is log-smooth, so $\Omega^1_{\mathfrak{X}_{\text{ét}},\log}$ is a locally free $\mathcal{O}_{\mathfrak{X}_{\text{ét}}}$-module of rank d. Finally, let $\Omega^j_{\mathfrak{X}_{\text{ét}},\log} = \bigwedge^j \Omega^1_{\mathfrak{X}_{\text{ét}},\log}$.

The adic generic fiber $X = \mathfrak{X}^{\text{ad}}_C$ is a smooth rigid-analytic variety. The integral version of Theorem 5.1.2 compares $\Omega^j_{\mathfrak{X}_{\text{ét}}}$ with $R^j\nu_*\hat{\mathcal{O}}^+_X$, where we have relabeled ν as the map of sites

$$\nu\colon X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}.$$

To make this work there are two additional ingredients: the Breuil-Kisin twist and the décalage functor.

The Breuil-Kisin twist $\mathcal{O}_C\{1\}$ is a free $\mathcal{O}_C$-module of rank 1 carrying a $\operatorname{Gal}(\overline{K}/K)$-action. See [BMS18, Definition 8.2] for its precise definition. There is a canonical Galois-equivariant injection $\mathcal{O}_C(1) \hookrightarrow \mathcal{O}_C\{1\}$ whose cokernel is killed by $(\zeta_p - 1)$.

The décalage functor will be reviewed in Section 5.4. In our context it appears as an endofunctor $L\eta_{(\zeta_p-1)}$ on the derived category of $\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C,\text{ét}}}$-modules. For the moment we need two facts concerning $L\eta_{(\zeta_p-1)}$: it is a lax monoidal functor, and also there is a natural map

$$a\colon L\eta_{(\zeta_p-1)}\mathcal{C} \to \mathcal{C} \tag{5.2.2}$$

whenever $\mathcal{C}$ is bounded below by 0. Let

$$\widetilde{\Omega}_{\mathfrak{X}} = L\eta_{\zeta_p-1}R\nu_*\hat{\mathcal{O}}^+_X,$$

so that $\widetilde{\Omega}_{\mathfrak{X}}$ is a complex of $\mathcal{O}_{\mathfrak{X}_{\text{ét}}}$-modules.

Theorem 5.2.3 ([BMS18, Theorem 8.3],[vK19, Theorem 4.11])**.** *For each $j \geq 0$ there is a canonical isomorphism of sheaves of $\mathcal{O}_{\mathfrak{X}_{\text{ét}}}$-modules:*

$$\Omega^j_{\mathfrak{X}_{\text{ét}},\log}\{-j\} \cong H^j(\widetilde{\Omega}_{\mathfrak{X}}).$$

We will require upgrading Theorem 5.2.3 into a statement about an isomorphism of sheaves of condensed abelian groups. For an abelian group A, let $A^{(p)}$ be the condensed abelian group associated to A, considered as a topological abelian group with its p-adic topology. Note that if A is p-adically separated and complete, then $A^{(p)}$ is a solid abelian group by Lemma 3.2.1. We continue to write $A \mapsto A^{(p)}$ for the extension of this functor to $D(\text{Ab}) \to D(\text{Cond}(\text{Ab}))$.

Lemma 5.2.4. *Let X be an affinoid rigid-analytic space over a nonarchimedean field of characteristic $(0,p)$. There is a natural isomorphism in $D(\text{Cond}(\text{Ab}))$:*

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+) \cong R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+)^{(p)}$$

Proof. By Lemma 3.3.1, $R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+) \cong R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_{\text{cond}})$.

We claim there exists a pro-étale cover of X by an affinoid perfectoid space $\tilde{X}$ which is *strictly totally disconnected* [Sch22, Definition 1.14]: this means that every étale cover of $\tilde{X}$ has a section. First note there exists a pro-étale cover $X' \to X$ which is affinoid perfectoid by [Sch13a, Proposition 4.8], and then there exists a pro-étale cover $\tilde{X} \to X'$ with $\tilde{X}$ strictly totally disconnected by [Sch22, Lemma 7.18].

Such an $\tilde{X}$ has no higher pro-étale cohomology, so that $R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_{\text{cond}})$ is computed by the Čech complex associated to the simplicial complex associated to the pro-étale cover $\tilde{X} \to X$. Explicitly, $R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_{\text{cond}})$ is quasi-isomorphic to the complex of condensed abelian groups with terms $\mathcal{A}^i = H^0(\tilde{X}^{(i)}, \hat{\mathcal{O}}^+_{\text{cond}})$, where $\tilde{X}^{(i)}$ is the i-fold fiber product of $\tilde{X}$ over X.

Since the $\tilde{X}^{(i)}$ are affinoid perfectoid, we have $\mathcal{A}^i = H^0(\tilde{X}^{(i)}, \hat{\mathcal{O}}^+)^{(p)}$. Since the complex with terms $H^0(\tilde{X}^{(i)}, \hat{\mathcal{O}}^+)$ computes $R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+)$, the result follows. □

Now let us return to the context of Theorem 5.2.3, so that X has a semistable formal model $\mathfrak{X}$ over $\mathcal{O}_C$. We may define sheaves $R\nu_*\hat{\mathcal{O}}^+_{\text{cond}}$ and $\tilde{\Omega}_{\mathfrak{X},\text{cond}} = L\eta_{\zeta_p-1}R\nu_*\hat{\mathcal{O}}^+_{\text{cond}}$ of solid abelian groups on $\mathfrak{X}_{\text{ét}}$. On the other hand, the sheaf of continuous differentials $\Omega^j_{\mathfrak{X}_{\text{ét}},\log}$ takes values in p-adically separated and complete abelian groups by definition. Therefore by Lemma 3.2.1, the sheaf $\Omega^j_{\mathfrak{X}_{\text{ét}},\log,\text{cond}}$ defined by $U \mapsto \Omega^j_{\mathfrak{X}_{\text{ét}},\log}(U)^{(p)}$ is a sheaf of solid abelian groups.

Similarly, the sheaf $\hat{\mathcal{O}}^+$ on $X_{\text{proét}}$ takes values in p-adically separated and complete abelian groups, and therefore we may upgrade it to a sheaf $\hat{\mathcal{O}}^+_{\text{cond}}$ of solid abelian groups. For any $U \in X_{\text{proét}}$ affinoid perfectoid, the value of $\hat{\mathcal{O}}^+_{\text{cond}}(U)$ on S is

$$\hat{\mathcal{O}}^+_{\text{cond}}(U)(S) = C_{\text{cts}}(S, \hat{\mathcal{O}}^+(U)) \cong \hat{\mathcal{O}}^+(U \times S),$$

and as a result the derived global sections of $\hat{\mathcal{O}}^+_{\text{cond}}$ as a condensed abelian group computes $R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+)$:

$$R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+_{\text{cond}}) \cong R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+)$$

One also has sheaves of condensed abelian groups $R\nu_*\hat{\mathcal{O}}^+_{\text{cond}}$ and $\tilde{\Omega}_{\mathfrak{X},\text{cond}} = L\eta_{\zeta_p-1}R\nu_*\hat{\mathcal{O}}^+_{\text{cond}}$.

Proposition 5.2.5. *For each $j \geq 0$ there is a canonical isomorphism of sheaves of condensed abelian groups on $\mathfrak{X}_{\text{ét}}$:*

$$\Omega^j_{\mathfrak{X}_{\text{ét}},\log,\text{cond}}\{-j\} \cong H^j(\widetilde{\Omega}_{\mathfrak{X},\text{cond}}).$$

Proof. The claim in the proposition will follow from Theorem 5.2.3 as soon as one knows that

$$\tilde{\Omega}_{\mathfrak{X},\mathrm{cond}} = \left(\tilde{\Omega}_{\mathfrak{X}}\right)^{(p)},$$

since the $\Omega^j_{\mathfrak{X}_{\text{ét}},\log,\mathrm{cond}}$ have the corresponding property. It is enough to check this after passing to derived global sections over any $U \in \mathfrak{X}_{\text{ét}}$; without loss of generality we can take $U = \mathfrak{X}$, in which case:

$$\begin{aligned} R\Gamma(\mathfrak{X}, \tilde{\Omega}_{\mathfrak{X},\mathrm{cond}}) &= L\eta_{(\zeta_p-1)} R\Gamma_{\mathrm{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+) \\ &\cong L\eta_{(\zeta_p-1)} \left(R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+)^{(p)}\right) \\ &\cong R\Gamma(\mathfrak{X}, \tilde{\Omega}_{\mathfrak{X}})^{(p)} \end{aligned}$$

In the last step we applied the easily verified fact that $L\eta_{\zeta_p-1}$ commutes with the operation $A \mapsto A^{(p)}$. □

5.3. **The integral comparison isomorphism for affine semistable formal schemes.** The idea now is to present integral versions of the comparison isomorphism in Theorem 5.1.3. The first version applies to the setting where K is a local field of characteristic $(0,p)$, and X is a rigid-analytic space over K admitting an *affine* semistable model $\mathfrak{X}$ over $\mathcal{O}_K$.

We have a diagram of sites endowed with sheaves of solid rings:

$$\begin{array}{ccc} (X_{\text{proét}}, \hat{\mathcal{O}}^+_{\mathrm{cond}}) & \longrightarrow & (\mathfrak{X}_{\text{ét}}, \mathcal{O}_{\mathrm{cond}}) \\ \downarrow & & \\ ((\operatorname{Spa} K)_{\text{proét}}, \hat{\mathcal{O}}^+_{\mathrm{cond}}) & & \end{array}$$

Each of the morphisms in the diagram induces a map of ring objects in $D(\mathrm{Solid})$, which can be tensored together to form a map:

$$H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X},\mathrm{cond}}) \otimes R\Gamma_{\mathrm{cond}}(\operatorname{Spa} K_{\text{proét}}, \hat{\mathcal{O}}^+) \to R\Gamma_{\mathrm{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+).$$

Theorem 5.3.1. *Let $\mathfrak{X}$ be an affine semistable formal scheme of dimension d over $\mathcal{O}_K$ with generic fiber X. There exist constants $M_{K,d,i}$ independent of $\mathfrak{X}$ such that the cofiber of*

$$H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X},\mathrm{cond}}) \otimes R\Gamma_{\mathrm{cond}}(\operatorname{Spa} K_{\text{proét}}, \hat{\mathcal{O}}^+) \to R\Gamma_{\mathrm{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+)$$

has ith cohomology group killed by $p^{M_{K,d,i}}$ for all $i \geq 0$.

Explicitly:

(1) $M_{K,d,0} = 0$,
(2) $M_{K,d,1} = 2d+1$
(3) *For* $2 \leq i \leq d+1$,

$$M_{K,d,i} = M_K + \mathrm{ord}_p(i-1) + (2+a_p)(i-2) + 2d + 2$$

where M_K is the bound from Theorem 4.0.4, and $a_p = 0$ for p odd and $a_p = 1$ for $p = 2$.
(4) *For* $i \geq d+2$,

$$M_{K,d,i} = (4+a_p)d + 2.$$

We record here the more convenient bound

$$M_{K,d,i} \leq M_K + 6d, \tag{5.3.2}$$

valid for any p and all cohomological degrees.

Proof. Let C be the completion of an algebraic closure of K. To ease notational burden, we will drop the "cond" from $\mathcal{O}$, $\mathcal{O}^+_{\mathrm{cond}}$, $\tilde{\Omega}$, etc., with the tacit assumption that these are to be interpreted as sheaves of solid abelian groups.

The map in the theorem factors is defined as the following composition:

$$\begin{aligned} H^0(\mathfrak{X}, \mathcal{O}) \otimes R\Gamma(\operatorname{Spa} K_{\text{proét}}, \hat{\mathcal{O}}^+) &\cong R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{O}) \otimes \mathcal{O}_C^{h\Gamma_K} \\ &\cong R\Gamma(\mathfrak{X}_{\mathcal{O}_C,\text{ét}}, \mathcal{O})^{h\Gamma_K} \\ &\xrightarrow{\varepsilon} R\Gamma(\mathfrak{X}_{\mathcal{O}_C,\text{ét}}, \tilde{\Omega})^{h\Gamma_K} \\ &\xrightarrow{\alpha} R\Gamma(\mathfrak{X}_{\mathcal{O}_C,\text{ét}}, R\nu_*\hat{\mathcal{O}}^+)^{h\Gamma_K} \\ &\cong R\Gamma(X_{C,\text{proét}}, \hat{\mathcal{O}}^+)^{h\Gamma_K} \\ &\cong R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+) \end{aligned}$$

The map ε comes from the unit morphism $\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C}} \to \tilde{\Omega}$. (Note that since $L\eta_{\zeta_p - 1}$ is lax monoidal, $\tilde{\Omega}$ is a commutative algebra object.) The map labeled α comes from the functor a appearing in (5.2.2). The cofibers of both maps are controlled in the following two subsections, and the bound $M_{K,d,i}$ obtained in the theorem is obtained by adding the bounds obtained therein. □

5.4. **Controlling the cofiber of ε.** The map ε arises from a morphism of sheaves $\mathcal{O}_{\mathrm{cond}} \to \tilde{\Omega}$ on $\mathfrak{X}_{\mathcal{O}_C,\text{ét}}$ which fits into a cofiber sequence:

$$\mathcal{O}_{\mathrm{cond}} \to \tilde{\Omega} \to \tilde{\Omega}_{\geq 1}.$$

Thus the cofiber of ε is:

$$\operatorname{cof}(\varepsilon) = R\Gamma(\mathfrak{X}, \tilde{\Omega}_{\geq 1})^{h\Gamma_K}.$$

By Theorem 5.2.3, $\tilde{\Omega}_{\geq 1}$ admits a filtration whose associated graded pieces are

$$\Omega^j_{\mathfrak{X}_{\mathcal{O}_C},\log}\{-j\} \cong \Omega^j_{\mathfrak{X}_{\mathcal{O}_C},\log} \otimes \mathcal{O}_C\{-j\}$$

Therefore the cofiber of ε has a finite filtration with associated graded pieces

$$R\Gamma(\mathfrak{X}, \Omega^j_{\mathfrak{X},\log}) \otimes \mathcal{O}_C\{-j\}^{h\Gamma_K}$$

for $j = 1, \ldots, d$. (Here the $\otimes$ means the tensor product in $D(\mathrm{Solid})$.)

Lemma 5.4.1. *For all $j \geq 0$ we have:*

(1) $H^0(\Gamma_K, \mathcal{O}_C\{-j\}) = 0$.
(2) $H^1_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C\{-j\})$ is $p^{M+\mathrm{ord}_p(j)+1}$-torsion, where $M = M_K$ is the constant in Theorem 4.0.4.
(3) For $i \geq 2$, $H^i_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C\{-j\})$ is p^2-torsion if p is odd, or p^3-torsion if $p = 2$.

Proof. Since there is an injective Γ_K-equivariant map $\mathcal{O}_C(1) \to \mathcal{O}_C\{1\}$ whose cokernel is killed by $(\zeta_p - 1)$, there is an injective map $\mathcal{O}_C\{-j\} \to \mathcal{O}_C(-j)$ whose cokernel is killed by $(\zeta_p - 1)^d$. After adjusting this map by an appropriate power of p, we may assume that the cokernel is simply killed by p. The claimed bound follows from applying the long exact sequence in cohomology together with Theorem 4.3.3. □

Proposition 5.4.2. *Considering the cohomology groups of the cofiber of ε, we have:*

(1) For $i = 0, 1$, $H^i(\operatorname{cof}(\varepsilon)) = 0$.
(2) For $2 \leq i \leq d+1$, $H^i(\operatorname{cof}(\varepsilon))$ is $p^{M+\mathrm{ord}_p(i-1)+1+(2+a_p)(i-2)}$-torsion, where $M = M_K$ is the constant from Theorem 4.3.3, and $a_p = 0$ for p odd and $a_p = 1$ for $p = 2$.
(3) For $i \geq d+2$, $H^i(\operatorname{cof}(\varepsilon))$ is $p^{(2+a_p)d}$-torsion.

Proof. Follows from Lemma 5.4.1 together with the spectral sequence

$$H^0(\mathfrak{X}, \Omega^j_{\mathfrak{X},\log}) \otimes H^i_{\mathrm{cts}}(\Gamma_K, \mathcal{O}_C\{-j\}) \implies H^{i+j}(\tilde{\Omega}_{\geq 1}),$$

noting that the left side is only nonzero for $j = 1, \ldots, d$ and $i \geq 1$. □

5.5. **Controlling the décalage functor.** We record here some lemmas regrading the $L\eta$ functor, recalling and extending slightly the results in §6 of [BMS18]. Let $(T, \mathcal{O}_T)$ be a ringed topos. Let $D(\mathcal{O}_T)$ be the derived category of $\mathcal{O}_T$-modules. Let $\mathcal{I} \subset \mathcal{O}_T$ be an invertible ideal sheaf. We use $L\eta_{\mathcal{I}}$ to denote the lax symmetric monoidal functor $D(\mathcal{O}_T) \to D(\mathcal{O}_T)$ as in [BMS18, Corollary 6.5., Proposition 6.7]. This functor has the effect of killing the $\mathcal{I}$-torsion in the cohomology of a complex. The functor $L\eta_{\mathcal{I}}$ commutes with truncations and in particular preserves the subcategories of bounded complexes $D^{\geq 0}(\mathcal{O}_T)$, $D^{\leq d}(\mathcal{O}_T)$, $D^{[0,d]}(\mathcal{O}_T)$.

Lemma 5.5.1. *Let $\mathcal{C}$ be an object in $D(\mathcal{O}_T)$. Then*

1. *Assume that $\mathcal{C} \in D^{\geq 0}(\mathcal{O}_T)$ and that $H^0(\mathcal{C})$ is $\mathcal{I}$-torsion free. We have a natural map in $D(\mathcal{O}_T)$:*
$$a \colon L\eta_{\mathcal{I}}(\mathcal{C}) \to \mathcal{C}.$$
2. *Assume that $\mathcal{C} \in D^{\leq d}(\mathcal{O}_T)$. We have a natural map in $D(\mathcal{O}_T)$*
$$b \colon \mathcal{C} \otimes \mathcal{I}^{\otimes d} \to L\eta_{\mathcal{I}}(\mathcal{C})$$
3. *Assume that $\mathcal{C} \in D^{[0,d]}(\mathcal{O}_T)$ and that $H^0(\mathcal{C})$ is $\mathcal{I}$-torsion free. The cofibers of $b \circ (a \otimes \mathcal{I}^{\otimes d})$ and $a \circ b$ are $O_T/\mathcal{I}^{\otimes d}$-modules.*

Proof. These claims are all part of [BMS18, Lemma 6.9.]. □

Applying Lemma 5.5.1 to the ringed topos $(\mathfrak{X}_{\mathcal{O}_C,\mathrm{\acute{e}t}}, \mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C}})$, the invertible ideal $\mathcal{I} = (\zeta_p - 1)$, and the objects $R\nu_* \hat{\mathcal{O}}^+$ and $\tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+$, we obtain morphisms:

$$\begin{aligned} a \colon L\eta_{\mathcal{I}} R\nu_* \hat{\mathcal{O}}^+ &\to R\nu_* \hat{\mathcal{O}}^+ \\ \bar{a} \colon L\eta_{\mathcal{I}} \tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+ &\to \tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+ \\ b \colon \tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+ \otimes \mathcal{I}^{\otimes d} &\to L\eta_{\mathcal{I}}(\mathcal{C}) \end{aligned}$$

Lemma 5.5.2. *The object $\mathrm{cof}(\bar{a})$ is $\mathcal{I}^{2d}$-torsion.*

Proof. Consider the diagram of cofiber sequences

$$\begin{array}{ccccc} \mathrm{cof}(b \circ (\bar{a} \otimes \mathcal{I}^{\otimes d})) & \xrightarrow{=} & \mathrm{cof}(b \circ (\bar{a} \otimes \mathcal{I}^{\otimes d})) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{cof}(b) & \longrightarrow & \mathrm{cof}(\bar{a}b) & \longrightarrow & \mathrm{cof}(\bar{a}) \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle =} \\ \mathcal{I}^{\otimes d} \otimes \mathrm{cof}(\bar{a})[1] & \longrightarrow & N & \longrightarrow & \mathrm{cof}(\bar{a}). \end{array}$$

Here N is the cofiber of the map $\mathrm{cof}(b \circ (\bar{a} \otimes I^{\otimes d})) \to \mathrm{cof}(\bar{a}b)$. Since $\mathrm{cof}(b \circ (\bar{a} \otimes \mathcal{I}^{\otimes d}))$ and $\mathrm{cof}(\bar{a}b)$ are both $\mathcal{I}^d$-torsion, N is $\mathcal{I}^{2d}$-torsion.

By Lemma 5.5.3 below, the map $\mathrm{cof}(\bar{a}) \to \mathcal{I}^{\otimes d} \otimes \mathrm{cof}(\bar{a})[2]$ is 0, so that

$$N \cong \left(\mathcal{I}^{\otimes d} \otimes \mathrm{cof}(\bar{a})[1]\right) \oplus \mathrm{cof}(\bar{a}).$$

It follows that $\mathrm{cof}(\bar{a})$ is $\mathcal{I}^{2d}$-torsion. □

Lemma 5.5.3. *Let a_1, a_2, a_3 be morphisms in a triangulated category which fit into a diagram:*

$$A \xrightarrow{a_1} B \xrightarrow{a_2} C \xrightarrow{a_3} D.$$

Then the composition of the canonical maps

$$\operatorname{cof}(a_3) \to \operatorname{cof}(a_2)[1] \to \operatorname{cof}(a_1)[2]$$

is 0.

Proof. We have a commutative diagram of fiber sequences:

$$\begin{array}{ccccccc}
\operatorname{cof}(a_2a_1) & \longrightarrow & \operatorname{cof}(a_3a_2a_1) & \longrightarrow & \operatorname{cof}(a_3) & \longrightarrow & \operatorname{cof}(a_2a_1)[1] \\
\downarrow & & \downarrow & & \downarrow{=} & & \downarrow \\
\operatorname{cof}(a_2) & \longrightarrow & \operatorname{cof}(a_3a_2) & \longrightarrow & \operatorname{cof}(a_3) & \longrightarrow & \operatorname{cof}(a_2)[1]
\end{array}$$

It follows that the canonical map $\operatorname{cof}(a_3) \to \operatorname{cof}(a_2)[1]$ factors through $\operatorname{cof}(a_2a_1)[1]$, which is exactly the fiber of the canonical map $\operatorname{cof}(a_2)[1] \to \operatorname{cof}(a_1)[2]$. Thus the composition $\operatorname{cof}(a_3) \to \operatorname{cof}(a_1)[2]$ is zero. ☐

Lemma 5.5.4. *The complexes $\operatorname{cof}(a)$ and $\operatorname{cof}(\bar{a})$ are almost isomorphic. In particular $\operatorname{cof}(a)$ is $(\zeta_p - 1)^{2d+1}$-torsion.*

Proof. Let

$$t \colon \tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+ \to R\nu_* \hat{\mathcal{O}}^+$$

be the natural map. Consider the diagram of cofiber sequences:

$$\begin{array}{ccccc}
L\eta_I \tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+ & \longrightarrow & \tau^{\leq d} R\nu_* \hat{\mathcal{O}}^+ & \longrightarrow & \operatorname{cof}(\bar{a}) \\
\downarrow{\scriptstyle L\eta_I t} & & \downarrow{\scriptstyle t} & & \downarrow \\
L\eta_I R\nu_* \mathcal{O}^+ & \longrightarrow & R\nu_* \mathcal{O}^+ & \longrightarrow & \operatorname{cof}(a) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{cof}(L\eta_I t) & \longrightarrow & \operatorname{cof}(t) & \longrightarrow & D.
\end{array}$$

We claim that t and $L\eta_{\mathcal{J}} t$ are almost isomorphisms. We begin with t. It suffices to verify that t is an almost isomorphism after evaluating the source and target complexes of sheaves on an open. Given an open $U \to \mathfrak{X}_{\text{ét}}$, $(R\nu_* \mathcal{O}^+)(U) \cong R\Gamma(\nu^{-1}(U)_{\text{proét}}, \mathcal{O}^+)$ is now subject to the constraints imposed by the discussion around [vK19, Equation 3.3.1], in which Δ has cohomological dimension d, so the cofiber of the map e is almost zero. Applied to our situation, this implies that the cofiber of t is almost zero. A similar argument applying the source and target of $L\eta_I t$ to an open allows us to apply [vK19, Theorem 3.9] to see that the cofiber of $L\eta_I t$ is almost zero.

Therefore $\operatorname{cof}(L\eta_I t)$ and $\operatorname{cof}(t)$ are both almost zero. This implies that D is almost zero, and this further implies that $\operatorname{cof}(\bar{a}) \to \operatorname{cof}(a)$ is an almost isomorphism. Since an almost zero sheaf of complexes is $\mathcal{I}$-torsion, we have that $\operatorname{cof}(a)$ is $\mathcal{I}^{2d+1}$-torsion. ☐

Consider now the map α appearing in the proof of Theorem 5.3.1:

$$\alpha \colon R\Gamma(\mathfrak{X}_{\mathcal{O}_C,\text{ét}}, L\eta_{(1-\zeta_p)} R\nu_* \hat{\mathcal{O}}^+)^{h\Gamma_K} \to R\Gamma(\mathfrak{X}_{\mathcal{O}_C,\text{ét}}, \hat{\mathcal{O}}^+)^{h\Gamma_K}$$

Its cofiber is $R\Gamma(\mathfrak{X}_{\mathcal{O}_C,\text{ét}}, \operatorname{cof}(a))^{h\Gamma_K}$. Since $\operatorname{cof}(a)$ is $(\zeta_p - 1)^{2d+1}$-torsion, we (rather crudely) conclude that it is p^{2d+1}-torsion, and thecalageore so are the cohomology groups of the cofiber of α. When combined with Proposition 5.4.2, we obtain the bounds appearing in Theorem 5.3.1.

5.6. **The integral comparison theorem for general semistable formal schemes.** We now drop the assumption that $\mathfrak{X}$ is affine or even quasi-compact.

Theorem 5.6.1. *Let $\mathfrak{X}$ be a semistable formal scheme of dimension d over $\mathcal{O}_K$ with generic fiber X. Let $A_{\mathfrak{X}}$ be the cofiber of the natural map*

$$R\Gamma(\mathfrak{X}, \mathcal{O}_{\mathrm{cond}}) \otimes R\Gamma_{\mathrm{cond}}(K_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) \to R\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+).$$

Then:

$$H^i(A_{\mathfrak{X}}) = \begin{cases} 0, & i = 0, \\ p^{dM_K+(4+a_p)(d+1)^2}\text{-torsion}, & 1 \le i \le 2d+1, \\ p^{(4+a_p)(d+1)^2}\text{-torsion}, & i \ge 2d+2. \end{cases}$$

In particular, all cohomology groups of $A_{\mathfrak{X}}$ are killed by $p^{5(d+1)^2+dM_K}$.

Proof. Let $\{U_i\}_{i\in I}$ be an affine cover of $\mathfrak{X}$. For a finite non-empty set of indices $J \subset I$, we will write $U_J = \bigcap_{i\in J} U_i$; assume these are all affine. We will write A_{U_J} for the corresponding complex.

For every $i \ge 0$, $H^i(A_{U_J})$ is $p^{M_{K,d,i}}$-torsion, where $M_{K,d,i}$ is the bound appearing in Theorem 5.3.1. The cover $\{U_i\}_{i\in I}$ induces a cosimplicial diagram whose totalization computes $A_{\mathfrak{X}}$:

$$A_{\mathfrak{X}} \cong \lim_{[n]\in\Delta} \prod_{\substack{J\subset I \\ \#J=n+1}} A_{U_J}.$$

Taking the associated spectral sequence (and noting that the nerve of the cover has dimension $\le d$) shows that $H^i(A_X)$ is $p^{m(i)}$-torsion for $m(i) = \sum_{0\le j\le d} M_{K,d,j-i}$. When $i \ge 2d+2$, the terms of the sum are all $(4+a_p)d+2 \le (4+a_p)(d+1)$, so the sum is bounded by $(4+a_p)(d+1)^2$. On the other hand if $1 \le i \le 2d+1$, the sum is bounded by

$$dM_K + \mathrm{ord}_p(d!) + (2+a_p)\sum_{i=1}^{d} i + 2(d+1)^2 \le dM_K + (4+a_p)(d+1)^2.$$

Here we used the bound $\mathrm{ord}_p(d!) \le d$. □

5.7. **Tame descent.** A smooth rigid-analytic space X/K doesn't necessarily have a semistable model; it may be necessary to extend scalars. The best result we know along these lines is the main theorem of [Har03], which states that $X_{\acute{e}t}$ admits an open cover by rigid-analytic generic fibers of semistable formal schemes defined over finite extensions L/K. If X is not quasi-compact then there may not be a single L/K that suffices for this purpose.

In the special case that all the L/K are *tamely ramified* extensions (or, at least, if the wild ramification of the L/K is bounded), then it becomes possible to get uniform control over the pro-étale cohomology of X.

Lemma 5.7.1. *Let K be a local field of characteristic $(0,p)$ and $d \in \mathbb{Z}_{\ge 0}$. Let X be a smooth affinoid rigid-analytic space of dimension d over K. Assume there exists a tamely ramified finite extension L/K and a semistable affine formal scheme $\mathfrak{X}/\mathcal{O}_L$ whose rigid-analytic generic fiber is X_L. Then the cofiber of the natural map*

$$H^0(X, \mathcal{O}^+_{\mathrm{cond}}) \otimes R\Gamma_{\mathrm{cond}}(K_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) \to R\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)$$

has cohomology groups killed by p^{M_K+6d}, where M_K is the constant from Theorem 4.0.4.

Proof. (For the purposes of this proof we drop the "cond" subscripts.) By Theorem 5.6.1 applied to $\mathfrak{X}/\mathcal{O}_L$, the cofiber of

$$R\Gamma(\mathfrak{X}, \mathcal{O}) \otimes R\Gamma(L_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) \to R\Gamma(X_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)$$

has cohomology groups killed by p^M for a constant $M = M_{K,d}$. Since $\mathfrak{X}$ is affine and semistable we have

$$R\Gamma(\mathfrak{X}, \mathcal{O}) = H^0(\mathfrak{X}, \mathcal{O}) \cong H^0(X_L, \mathcal{O}^+).$$

The idea now is to descend the result through L/K. Without loss of generality let us assume that L/K is Galois. Let K'/K be the maximal unramified subextension of L/K. Now we have

$$R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}^+) \cong R\Gamma(X_{L,\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{Gal}(L/K)}.$$

Since $R\Gamma(L_{\text{proét}}, \hat{\mathcal{O}}^+)$ is isomorphic to $\mathcal{O}_L^{h\mathbb{Z}_p}$ up to torsion bounded in terms of K (Theorem 4.0.3), it suffices to show that the natural map

$$H^0(X_K, \mathcal{O}^+) \to H^0(X_L, \mathcal{O}^+)^{h\,\mathrm{Gal}(L/K)}$$

is an isomorphism. Since the order of $\mathrm{Gal}(L/K')$ is invertible in $\mathcal{O}_K$, we have

$$H^0(X_L, \mathcal{O}^+)^{h\,\mathrm{Gal}(L/K')} = H^0(X_L, \mathcal{O}^+)^{\mathrm{Gal}(L/K')} = H^0(X_{K'}, \mathcal{O}^+)$$

Now since K'/K is unramified, we have

$$H^0(X, \mathcal{O}^+) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \cong H^0(X_{K'}, \mathcal{O}^+)$$

(since the right side is integrally closed in $H^0(X_{K'}, \mathcal{O}^+) \otimes K'$), and so

$$H^0(X_{K'}, \mathcal{O}^+)^{h\,\mathrm{Gal}(K'/K)} \cong H^0(X_K, \mathcal{O}^+) \otimes \mathcal{O}_{K'}^{h\,\mathrm{Gal}(K'/K)} = H^0(X_K, \mathcal{O}^+).$$

□

6. Proof of Theorem B

Let us recall the main players in Theorem B: a prime number p, an integer $n \geq 1$, the ring of p-typical Witt vectors $W = W(\overline{\mathbb{F}}_p)$, and the Lubin-Tate ring $A \cong W[\![u_1, \dots, u_{n-1}]\!]$, which admits a continuous action of the Morava stabilizer group $\mathbb{G}_n$. Proposition 2.5.1 states that the inclusion $W \hookrightarrow A$ admits a continuous $\mathbb{G}_n$-equivariant additive splitting, say with complement A^c. Theorem B is the statement that the continuous cohomology $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c)$ is p-power torsion.

In Section 3.5 we explained how to reduce Theorem B to a statement (Theorem 3.5.3) controlling the pro-étale cohomology of the open ball and Drinfeld's symmetric space. After a detour on the integral cohomology of p-adic Lie groups, We examine these cases in turn.

6.1. **Continuous cohomology of p-adic Lie groups with integral coefficients.** Let G be a p-adic Lie group. In Section 3.4 we reviewed Lazard's isomorphism $H^*_{\mathrm{cts}}(G, \mathbb{Q}_p) \cong H^*(\mathrm{Lie}\, G, \mathbb{Q}_p)$, where $\mathrm{Lie}\, G$ is Lazard's (rational) Lie algebra. For our purposes we will need the integral refinement of this isomorphism described in [HKN11]. First we recall a definition from [DdSMS99]. Let U be a pro-p group. We say U is *uniform* if it satisfies the conditions:

(1) U is topologically finitely generated.
(2) For p odd (resp., $p = 2$), U/U^p is abelian (resp., U/U^4 is abelian). Here U^n is the closure of the subgroup of U generated by nth powers.
(3) Let $U = U_1 \supset U_2 \supset \cdots$ be the lower p-series. Then $[U_i : U_{i+1}]$ is independent of i.

For a uniform pro-p group U, [DdSMS99, §8.2] defines the integral Lie algebra $\mathcal{L}(U)$ over $\mathbb{Z}_p$; this is a lattice in $\mathrm{Lie}\, U$. The following is [HKN11, Theorem 3.3.3].

Theorem 6.1.1. *Let U be a uniform pro-p group acting continuously on a finitely generated free $\mathbb{Z}_p$-module M. Assume that the action map $U \to \mathrm{Aut}\, M$ factors through $1 + p\, \mathrm{End}\, M$ if p is odd (resp., through $1 + 4\, \mathrm{End}\, M$ if $p = 2$). Then $\mathcal{L}(U)$ acts on M, and there is an isomorphism of graded $\mathbb{Z}_p$-modules:*

$$H^*_{\mathrm{cts}}(U, M) \cong H^*(\mathcal{L}(U), M)$$

If G is a $\mathbb{Q}_p$-analytic group acting continuously on M, then G contains a uniform pro-p subgroup U whose action on M satisfies the hypothesis of Theorem 6.1.1. Therefore the question of computing continuous cohomology of G with integral coefficients can be reduced to a question about the integral Lie algebra $\mathcal{L}(G)$.

Lemma 6.1.2. *Let G be either of the groups $\mathrm{GL}_n(\mathbb{Z}_p)$ or $\mathcal{O}_D^\times$.*

(1) Let G act trivially on $\mathbb{Z}_p$. Then

$$H^i_{\mathrm{cts}}(G, \mathbb{Z}_p) \cong \mathbb{Z}_p^{\oplus r_i} \oplus S_i,$$

where r_i is the dimension of the degree i part of $\Lambda_{\mathbb{Q}}(x_1, x_3, \dots, x_{2n-1})$, and where S_i is annihilated by a uniform power of p (that is, the power does not depend on i).

(2) Let C be a complex of solid $\mathbb{Z}_p$-modules admitting an action of G. Assume that $H^i(C) = 0$ for all $i \leq 1$ and that $H^i(C)$ is annihilated by a uniform power of p. Then $H^i(C^{hG})$ is also annihilated by a uniform power of p.

Proof. In each case, G is the group of units of a $\mathbb{Z}_p$-algebra A such that $U = 1 + p^2 A$ is a uniform pro-p subgroup of G.

For part (1): Theorem 6.1.1 gives an isomorphism $H^*_{\mathrm{cts}}(U, \mathbb{Z}_p) \cong H^*(\mathcal{L}(U), \mathbb{Z}_p)$. After tensoring with $\mathbb{Q}_p$, we have

$$H^*(\mathcal{L}(U), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^*(\mathrm{Lie}\, U, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1}).$$

Since each $H^*(\mathcal{L}(U), \mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$, we may write

$$H^i_{\mathrm{cts}}(U, \mathbb{Z}_p) \cong H^i(\mathcal{L}(U), \mathbb{Z}_p) \cong \mathbb{Z}_p^{r_i} \oplus S_i,$$

where T_i is finite. Since the cohomology is 0 for $i > n^2$, there is a uniform power of p which annihilates S_i for all i.

To extend this statement from U to G, we use the existence of the restriction and corestriction maps between $H^*_{\text{cts}}(G, W)$ and $H^*_{\text{cts}}(U, W)$; their composition is multiplication by $\#G/U$.

For part (2): Consider the spectral sequence

$$H^i_{\text{cts}}(U, H^j(C)) \implies H^{i+j}(C^{hU}).$$

Since U has cohomological dimension n^2, the left side is nonzero only when $0 \le i \le n^2$ and $j > 0$. Therefore if p^r annihilates $H^i(C)$ for all i, then p^{rn^2} annihilates $H^i(C^{hU})$ for all i. Once again, the restriction and corestriction maps can be used to extend the result from U to G. □

6.2. **Pro-étale cohomology of the open ball.** Let K be a local field of characteristic $(0, p)$. For an integer $d \ge 1$, recall that we had defined the d-dimensional rigid-analytic open ball over K:

$$B^{\circ,d} = (\operatorname{Spa} \mathcal{O}_K[\![T_1, \dots, T_d]\!]) \setminus \{|p| = 0\}$$

The ball $B^{\circ,d}$ is not quasi-compact. It is exhausted by affinoid (closed) balls of increasing radius:

$$B^{\circ,d} = \varinjlim_{r<1} B^d_r, \tag{6.2.1}$$

where r runs over real numbers in $|p|^{\mathbb{Q}_+}$, and for each $r = |p|^{m/n}$ with m, n relatively prime positive integers, B^d_r is the rational subset defined by the inequality $|T|^n \le |p|^m$.

The goal of this section is to control $R\Gamma(B^{\circ,d}_{\text{proét}}, \hat{\mathcal{O}}^+)$. The idea is to apply Lemma 5.7.1 to a collection of affinoid balls B^d_r which cover $B^{\circ,d}$. A convenient choice of radii is $r_\ell = |p|^{1/\ell}$, where ℓ runs over prime numbers $\neq p$. The closed ball $B^d_{r_\ell}$ admits a smooth formal model after passage from K to the tamely ramified extension

$$L_\ell = K(p^{1/\ell}).$$

Indeed $B^d_{r_\ell}$ is the rigid-analytic generic fiber of the smooth formal scheme $\mathfrak{B}^d_{r_\ell}$, where

$$\mathfrak{B}^d_{r_\ell} = \operatorname{Spf} \mathcal{O}_{L_\ell} \left\langle \frac{T_1}{p^{1/\ell}}, \dots, \frac{T_d}{p^{1/\ell}} \right\rangle.$$

Let A_ℓ be the cofiber of

$$H^0(B^d_{r_\ell}, \mathcal{O}^+_{\text{cond}}) \otimes \mathcal{O}_C^{h\Gamma_K} \to R\Gamma_{\text{cond}}(B^r_{d,\text{proét}}, \hat{\mathcal{O}}^+).$$

By Lemma 5.7.1, the solid abelian groups $H^i(A_\ell)$ are p^{6d+M_K}-torsion, where M_K is the constant appearing in Theorem 4.0.4. Importantly, these bounds do not depend on ℓ. We obtain an exact triangle:

$$R\varprojlim H^0(B^d_{r_\ell}, \mathcal{O}^+_{\text{cond}}) \otimes \mathcal{O}_C^{h\Gamma_K} \to R\Gamma_{\text{cond}}(B^{\circ,d}_{\text{proét}}, \hat{\mathcal{O}}^+) \to R\varprojlim A_\ell \tag{6.2.2}$$

Regarding the $R\varprojlim A_\ell$ term, its cohomology groups are computed by a spectral sequence

$$H^i(R^j \varprojlim A_\ell) \implies H^{i+j}(R\varprojlim A_\ell).$$

Since the indexing set is totally ordered, the left side is nonzero only for $j = 0, 1$. Therefore $H^i(R\varprojlim A_\ell)$ is p^{12d+2M_K}-torsion.

Turning to the term $R\varprojlim H^0(B^d_{r_\ell}, \mathcal{O}^+)$, we claim there is no $R^1 \lim$. Indeed this a general property, relying on the fact that K is spherically complete.

Lemma 6.2.3. *Let F be a nonarchimedean field. Assume that F is spherically complete, meaning that every sequence of nested closed intervals has a nonempty intersection. Let B^d_r be the rigid-analytic closed ball of radius r over F. Then*

$$R^1 \varprojlim_{r<1} H^0(B^d_r, \mathcal{O}^+) = 0.$$

Proof. The condition of being spherically complete implies that

$$R^1 \varprojlim_n I_n = 0,$$

where $I_1 \supset I_2 \supset \cdots$ is a decreasing sequence of fractional ideals of F. Indeed, the complex which computes $R\varprojlim_n I_n$ is

$$\begin{array}{rcl} d\colon \prod_n I_n & \to & \prod_I I_n \\ (a_n) & \mapsto & (a_n - a_{n+1}) \end{array}$$

We claim the map d is surjective. Given a sequence $(b_n) \in \prod_I I_n$, let $A_n = b_1 + \cdots + b_n + I_{n+1}$. Then the A_n are a sequence of nested closed intervals. Let $a_1 \in \bigcap_n A_n$, and let $a_n = a_1 - (b_1 + \cdots + b_{n-1})$. Then $d(a_n) = (b_n)$.

For $r \in \mathbb{R}_{>0}$, let $I(r)$ denote the fractional ideal of elements $f \in F$ with $|f| \leq r$. Then

$$H^0(B_r, \mathcal{O}^+) = \left\{ \sum_i b_i T^i \;\middle|\; b_i \in I(r^{-|i|}),\ |b_i|\, r^{|i|} \to 0 \right\}.$$

Here the indices i are tuples $(i_1, \dots, i_d)$ of nonnegative integers, and $|i| = \sum_j i_i$. Let $r_1, r_2, \dots$ be an increasing sequence in $(0,1)$ with limit 1. Consider the complex

$$d\colon \prod_n H^0(B_{r_n}, \mathcal{O}^+) \to \prod_n H^0(B_{r_n}, \mathcal{O}^+)$$

which computes $R\varprojlim_{r<1} H^0(B_r, \mathcal{O}^+)$. Let (b_n) be a sequence in $\prod_n H^0(B_{r_n}, \mathcal{O}^+)$, so that $b_n = \sum_i b_{ni} T^i$, with $b_{ni} \in I(r_n^{-|i|})$ and $|b_{ni}| \to 0$ as $i \to \infty$. For each i we have $R^1 \lim_n I(r_n^{-|i|}) = 0$, so there exists $a_{ni} \in I(r_n^{-|i|})$ such that $a_{ni} - a_{(n+1)i} = b_{ni}$.

Since $a_{1i} \in b_{1i} + I(r_2^{-|i|})$ we have $|a_{1i}| \leq \max\left\{ |b_{1i}|, r_2^{-|i|} \right\}$. Therefore $\lim_{|i|\to\infty} |a_{1i}|\, r_1^{|i|} = 0$, and so $a_1 = \sum_i a_{1i} T^i$ defines an element of $H^0(B_{r_1}, \mathcal{O}^+)$. Defining $a_n = a_1 - (b_1 + \cdots + b_{n-1})$, we find a sequence $(a_n) \in \prod_n H^0(B_{r_n}, \mathcal{O}^+)$ with $d(a_n) = b_n$. □

By Lemma 6.2.3, the exact triangle in $D(\mathrm{Solid})$ from (6.2.2) reduces to:

$$\mathcal{O}_K[\![T_1, \dots, T_d]\!] \otimes \mathcal{O}_C^{h\Gamma_K} \to R\Gamma_{\mathrm{cond}}(B^{d,\circ}_{\text{proét}}, \hat{\mathcal{O}}^+) \to A, \tag{6.2.4}$$

where each $H^i(A)$ is p^{12d+2M_K}-torsion.

Theorem 6.2.5. *There is a fiber sequence*

$$\mathcal{O}_K[\![T_1, \dots, T_d]\!][\varepsilon] \to R\Gamma_{\text{proét}}(B^{d,\circ}_{\text{proét}}, \hat{\mathcal{O}}^+) \to E_{\mathrm{ball}},$$

where the cohomology groups of the "error term" E_{ball} are killed by p^{12d+2M_K+5}. In particular if $p \nmid e_K$, they are killed by p^{12d+10}.

Proof. By Theorem 4.2.5, we have an exact triangle

$$\mathcal{O}_K[\varepsilon] \to \mathcal{O}_C^{h\Gamma_K} \to Z,$$

where the cohomology groups of Z are p^5-torsion, or p^4-torsion if $p \nmid e_K$. Tensoring with the flat module $\mathcal{O}_K[\![T_1, \dots, T_d]\!]$, we obtain an exact triangle

$$\mathcal{O}_K[\![T_1, \dots, T_d]\!][\varepsilon] \to \mathcal{O}_K[\![T_1, \dots, T_d]\!] \otimes \mathcal{O}_C^{h\Gamma_K} \to \mathcal{O}_K[\![T_1, \dots, T_d]\!] \otimes Z.$$

Using the octahedral axiom to combine with (6.2.4), we find an exact triangle

$$\mathcal{O}_K[\![T_1, \dots, T_d]\!] \otimes Z \to E_{\mathrm{ball}} \to A$$

from which we can extract the bounds appearing in the theorem. (Note that if $p \nmid e_K$ then we may take $M_K = 3$.) $\square$

Corollary 6.2.6. *Let $W = W(\overline{\mathbb{F}}_p)$, let $K = W[1/p]$, and let LT_K be Lubin-Tate space in height n, with $A = H^0(\mathrm{LT}_K, \mathcal{O}^+) \cong \mathcal{O}_K[\![u_1, \dots, u_{n-1}]\!]$ the Lubin-Tate ring.*

(1) There is an exact triangle in $D(\mathrm{Solid})$:

$$A^{h\mathcal{O}_D^\times}[\varepsilon] \to R\Gamma_{\mathrm{cond}}(\mathrm{LT}_{K,\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)^{h\mathcal{O}_D^\times} \to E_1,$$

where the cohomology groups of E_1 are killed by a uniform power of p.

(2) For each $i \geq 0$, let r_i be the dimension of the ith graded piece of $\Lambda_{\mathbb{Q}}(x_1, x_3, \dots, x_{2n-1})$. We have an exact sequence:

$$0 \to W^{(r_i \oplus r_{i-1})} \oplus S_i \to H^i(R\Gamma(\mathrm{LT}_{K,\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+)^{h\mathcal{O}_D^\times}) \to S_i' \to 0,$$

where S_i' is killed by a uniform power of p, and S_i is a quotient:

$$S_i = \frac{R_i \oplus H^i_{\mathrm{cts}}(\mathcal{O}_D^\times, A^c)}{R_i'}$$

Here R_i and R_i' are killed by a uniform power of p.

Proof. Identifying LT_K with the $(n-1)$-dimensional open ball over K, Theorem 6.2.5 states that the cohomology groups of E_{ball} are killed by a uniform power of p. In the statement of the corollary we have $E_1 = E_{\mathrm{ball}}^{h\mathcal{O}_D^\times}$. Now apply Lemma 6.1.2(2) to conclude that the cohomology groups of E_1 are killed by a uniform power of p. This is (1).

Part (2) follows from applying the long exact sequence in cohomology to the exact triangle in (1). Note that $H^i(A^{h\mathcal{O}_D^\times}[\varepsilon]) = H^i(A^{h\mathcal{O}_D^\times}) \oplus H^{i-1}(A^{h\mathcal{O}_D^\times})$. We have also applied the decomposition $A = W \oplus A^c$, together with the results on $H^i_{\mathrm{cts}}(\mathcal{O}_D^\times, \mathbb{Z}_p)$ from Lemma 6.1.2(1). $\square$

6.3. **The pro-étale cohomology of Drinfeld's symmetric space.** Let $\mathcal{H}$ be Drinfeld's symmetric space of dimension $n-1$. Then $\mathcal{H}$ admits a semistable formal model $\mathfrak{H}/\mathbb{Z}_p$. For an efficient construction of $\mathfrak{H}$, see [GK05a, §6]. The formal scheme $\mathfrak{H}/\mathbb{Z}_p$ admits an action of $\mathrm{GL}_n(\mathbb{Q}_p)$ which is compatible with the isomorphism $\mathfrak{H}_{\mathbb{Q}_p} \cong \mathcal{H}$.

Our goal is to control the pro-étale cohomology of $\mathcal{H}$. Key to that calculation is the following acyclicity result of Grosse-Klonne:

Theorem 6.3.1. *We have*

$$H^i(\mathfrak{H}, \mathcal{O}) = \begin{cases} \mathbb{Z}_p, & i = 0 \\ 0, & i \geq 1 \end{cases}$$

Proof. This is a special case of [GK05b, Theorem 4.5], who establishes acyclicity for coherent sheaves on $\mathfrak{H}$ appearing as integral structures in dominant representations of $\mathrm{GL}_n(\mathbb{Q}_p)$; we only use here the trivial representation. In the case $i = 0$, [GK05b, Theorem 4.5(iii)] theorem states that $H^0(\mathfrak{H}, \mathcal{O}) \otimes \mathbb{F}_p \cong H^0(\mathfrak{H}_{\mathbb{F}_p}, \mathcal{O})$. The latter is $\mathbb{F}_p$ because $\mathfrak{H}_{\mathbb{F}_p}$ is a connected scheme whose irreducible components are all copies of $\mathbb{P}^{d-1}_{\mathbb{F}_p}$. Since $H^0(\mathfrak{H}, \mathcal{O})$ is p-adically complete and torsion-free, this forces $H^0(\mathfrak{H}, \mathcal{O}) = \mathbb{Z}_p$. $\square$

Theorem 6.3.2. *There is a $\mathrm{GL}_n(\mathbb{Z}_p)$-equivariant exact triangle in $D(\mathrm{Solid})$:*

$$\mathbb{Z}_p[\varepsilon] \to R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\mathrm{pro\acute{e}t}}, \hat{\mathcal{O}}^+) \to E_{\mathcal{H}},$$

where the cohomology groups of $E_{\mathcal{H}}$ are killed by p^{5n^2+3n}.

Proof. Applying Theorem 5.6.1 to the semistable formal scheme $\mathfrak{H}_W$, we find an exact triangle

$$\mathcal{O}_C^{h\Gamma_K} \to R\Gamma_{\mathrm{cond}}(\mathcal{H}_{K,\text{proét}}, \hat{\mathcal{O}}^+) \to A,$$

where the cohomology groups of A are killed by p^{5n^2+3n-3}. As in the proof of Theorem 6.2.5, we combine with Theorem 4.2.5 to obtain the bound appearing in the theorem. □

Corollary 6.3.3. *We have the following bounds on the continuous* $\mathrm{GL}_n(\mathbb{Z}_p)$*-cohomology of the pro-étale cohomology of* $\mathcal{H}_K$*:*

(1) *There is an exact triangle*

$$\mathbb{Z}_p^{h\,\mathrm{GL}_n(\mathbb{Z}_p)}[\varepsilon] \to R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} \to E_2,$$

where the cohomology groups of E_2 *are annihilated by a uniform power of* p*.*

(2) *For each* $i \geq 0$*, we have an isomorphism*

$$H^i\left(R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)}\right) \cong W^{\oplus(r_i+r_{i-1})} \oplus Q_i,$$

where r_i *is the dimension of the degree* i *part of* $\Lambda_{\mathbb{Q}}(x_1, x_3, \ldots, x_{2n-1})$*, and where* Q_i *is annihilated by a uniform power of* p*.*

Proof. For (1): referring to Theorem 6.3.2, we have $E_2 = E_{\mathcal{H}}^{h\,\mathrm{GL}_n(\mathbb{Z}_p)}$; the bounds on its cohomology are derived from Lemma 6.1.2(2).

For (2), let H^i be the ith cohomology group of $R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)}$. Using the results of part (1) and the description of $\mathbb{Z}_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ in Lemma 6.1.2(1), we have an exact sequence

$$0 \to \mathbb{Z}_p^{\oplus(r_i+r_{i-1})} \oplus Q_i' \to H^i \to Q_i'' \to 0,$$

where Q_i' and Q_i'' are annihilated by a uniform power of p. Thus H^i contains a free $\mathbb{Z}_p$-module L^i of rank $r_i + r_{i-1}$, such that H^i/L^i is annihilated by a uniform power of p. Since H^i is p-adically separated and complete, we get an isomorphism as in (2). □

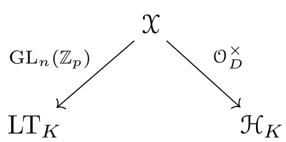
Remark 6.3.4. [GK05b] also computes the cohomology of each of the sheaves $\Omega^j_{\log}$ on $\mathfrak{H}$ in terms of certain lattices in Steinberg representations of $\mathrm{PGL}_{d+1}(K)$. This would allow us to compute the $\hat{\mathcal{O}}^+$-cohomology of $\mathcal{H}^d_{\text{proét},C}$. From this we could ultimately compute the continuous cohomology $H^i_{\mathrm{cts}}(\mathcal{O}_D^1, A)[1/p]$, where $\mathcal{O}_D^1 \subset \mathcal{O}_D^\times$ is the subgroup of elements of reduced norm 1. We do not pursue these computations here.

6.4. **Conclusion of the proof.**

We now complete the proof of Theorem B.

Theorem 6.4.1. *The cohomology groups* $H^i_{\mathrm{cts}}(\mathcal{O}_D^\times, A^c)$ *and* $H^i_{\mathrm{cts}}(\mathbb{G}_n, A^c)$ *are annihilated by a power of* p *which does not depend on* i*.*

Proof. We leverage the isomorphism between the two towers, as in Theorem 3.5.1. We have a diagram of adic spaces over $\operatorname{Spa} K$:



Here $\mathcal{H}_K$ is the base change of $\mathcal{H}$ to K. The diagram induces an isomorphism of solid complexes:

$$R\Gamma_{\mathrm{cond}}(\mathrm{LT}_{K,\text{proét}}, \hat{\mathcal{O}}^+)^{h\mathcal{O}_D^\times} \cong R\Gamma_{\mathrm{cond}}(\mathcal{H}_{K,\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} = R\Gamma_{\mathrm{cond}}(\mathcal{H}_{\text{proét}}, \hat{\mathcal{O}}^+)^{h\,\mathrm{GL}_n(\mathbb{Z}_p)} \otimes W.$$

Applying H^i and combining Corollary 6.2.6(2) and Corollary 6.3.3(2), we find an injection

$$W^{\oplus(r_i+r_{i-1})} \oplus S_i \hookrightarrow W^{\oplus(r_i+r_{i-1})} \oplus (Q_i \otimes_{\mathbb{Z}_p} W).$$

Since Q_i is annihilated by a uniform power of p, so must be S_i, and therefore so must be $H^i_{\mathrm{cts}}(\mathcal{O}_D^\times, A^c)$. Considering the description of S_i in Corollary 6.2.6(2), we find that $H^i_{\mathrm{cts}}(\mathcal{O}_D^\times, A^c)$ is annihilated by a uniform power of p.

The claim extends from $\mathcal{O}_D^\times$ to $\mathbb{G}_n$, using the Hochschild-Serre spectral sequence combined with the fact that the cohomological dimension of $\mathbb{G}_n/\mathcal{O}_D^\times \cong \hat{\mathbb{Z}}$ is 1. □

Proof of Theorem A. This is now immediate from Theorem B and Proposition 2.6.3. □

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