Journal of Topology 10 (2017) 1124–1144
doi:10.1112/topo.12032

The generalized slices of Hermitian K-theory

Tom Bachmann

Abstract

We compute the generalized slices (as defined by Spitzweck–Østvær) of the motivic spectrum KO (representing Hermitian K-theory) in terms of motivic cohomology and (a version of) generalized motivic cohomology, obtaining good agreement with the situation in classical topology and the results predicted by Markett–Schlichting. As an application, we compute the homotopy sheaves of (this version of) generalized motivic cohomology, which establishes a version of a conjecture of Morel.

1. *Introduction*

K-theory was invented by algebraic geometers and taken up by topologists. As a result of Bott periodicity, the homotopy groups of the (topological) complex K-theory spectrum KU are alternatingly $\mathbb{Z}$ and 0. Consequently, the (sped up) Postnikov tower yields a filtration of KU with layers all equal to the Eilenberg–MacLane spectrum $H\mathbb{Z}$ (which is also the zeroth Postnikov layer of the topological sphere spectrum). From this one obtains the Atiyah–Hirzebruch spectral sequence, which has the singular cohomology of a space X on the E_2 page and converges to the (higher) topological K-theory of X.

Much research has been put into replicating this picture in algebraic geometry. In its earliest form, this meant trying to find a cohomology theory for algebraic varieties called *motivic cohomology* which is related via a spectral sequence to higher algebraic K-theory. There is now a very satisfactory version of this picture. The motivic analog of the stable homotopy category $\mathbf{SH}$ is the motivic stable homotopy category $\mathbf{SH}(k)$ [**13**, Section 5]. Following Voevodsky [**27**, Section 2], this category is filtered by effectivity, yielding a kind of $\mathbb{G}_m$-Postnikov tower called the *slice filtration* and denoted

$$\cdots \to f_{n+1}E \to f_nE \to f_{n-1}E \to \cdots \to E.$$

The cofibres $f_{n+1}E \to f_nE \to s_nE$ are called the *slices* of E, and should be thought of as one kind of replacement of the (stable) homotopy groups from classical topology in motivic homotopy theory.

In $\mathbf{SH}(k)$ there are (at least) two special objects (for us): the sphere spectrum $S \in \mathbf{SH}(k)$ which is the unit of the symmetric monoidal structure, and the algebraic K-theory spectrum $KGL \in \mathbf{SH}(k)$ representing algebraic K-theory. One may show that up to twisting, all the slices of KGL are isomorphic, and in fact isomorphic to the zero-slice of S [**9**, Sections 6.4 and 9]. Putting $H_\mu\mathbb{Z} = s_0S$, this spectrum can be used to *define* motivic cohomology, and then the sought-after picture is complete.

Nonetheless there are some indications that the slice filtration is not quite right in certain situations. We give three examples. (1) We have said before that the homotopy groups of KU are alternatingly given by $\mathbb{Z}$ and 0. Thus in order to obtain a filtration in which all the layers are given by $H\mathbb{Z}$, one has to 'speed up' the Postnikov filtration by slicing 'with respect to S^2 instead of S^1'. Since the slice filtration is manifestly obtained by slicing with respect to $\mathbb{G}_m$

Received 13 January 2017; revised 28 August 2017; published online 20 November 2017.
2010 *Mathematics Subject Classification* 14F42 (primary), 19E08, 55P42, 14C25 (secondary).
The author would like to acknowledge support by the DFG through SPP 1786.

which is (at best) considered an analog of S^1, and yet the layers are already given at double speed, something seems amiss. (2) In classical topology there is another version of K-theory, namely the K-theory of real (not complex) vector bundles, denoted KO. There is also Bott periodicity, this time resulting in the computation that the homotopy groups of KO are given by $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ and then repeating periodically. There is an analog of topological KO in algebraic geometry, namely *Hermitian K-theory* [**6**] and (also) denoted $KO \in \mathbf{SH}(k)$. It satisfies an appropriate form of Bott periodicity, but this is not captured accurately by its slices, which are also very different from the topological analog [**21**]. (3) The slice filtration does not always *converge*. Thus just considering slices is not enough, for example, to determine if a morphism of spectra is an isomorphism.

Problem (3) has lead Spitzweck–Østvær [**25**] to define a refined version of the effectivity condition yielding the slice filtration which they call being 'very effective'. In this article we will argue that their filtration also solves issues (1) and (2).

To explain the ideas, recall that the category $\mathbf{SH}(k)^{\mathrm{eff}}$ is the localizing (so triangulated!) subcategory generated by objects of the form $\Sigma^\infty X_+$ for $X \in Sm(k)$ (that is, no desuspension by $\mathbb{G}_m$). Then one defines $\mathbf{SH}(k)^{\mathrm{eff}}(n) = \mathbf{SH}(k) \wedge T^{\wedge n}$ and for $E \in \mathbf{SH}(k)$ the n-effective cover $f_n E \in \mathbf{SH}(k)^{\mathrm{eff}}(n)$ is the universal object mapping to E. (Note that since $\mathbf{SH}(k)^{\mathrm{eff}}$ is triangulated, we have $\mathbf{SH}(k)^{\mathrm{eff}} \wedge T^{\wedge n} = \mathbf{SH}(k)^{\mathrm{eff}} \wedge \mathbb{G}_m^{\wedge n}$.) In contrast, Spitzweck–Østvær define the subcategory of very effective spectra $\mathbf{SH}(k)^{\mathrm{veff}}$ to be the subcategory generated under homotopy colimits and extensions by $\Sigma^\infty X_+ \wedge S^n$ where $X \in Sm(k)$ and $n \geqslant 0$. This subcategory is *not* triangulated! As before we put $\mathbf{SH}(k)^{\mathrm{veff}}(n) = \mathbf{SH}(k)^{\mathrm{veff}} \wedge T^{\wedge n}$. (Note that now, crucially, $\mathbf{SH}(k)^{\mathrm{veff}} \wedge T^{\wedge n} \neq \mathbf{SH}(k)^{\mathrm{veff}} \wedge \mathbb{G}_m^{\wedge n}$.) Then as before the very n-effective cover $\tilde{f}_n E \in \mathbf{SH}(k)^{\mathrm{veff}}(n)$ is the universal object mapping to E. The cofibres $\tilde{f}_{n+1} E \to \tilde{f}_n E \to \tilde{s}_n E$ are called the *generalized slices* of E.

As pointed out by Spitzweck–Østvær, the connectivity of $\tilde{f}_n E$ *in the homotopy t-structure* increases with n, so the generalized slice filtration automatically converges. Moreover, it is easy to see that $\tilde{f}_n KGL = f_n KGL$ (that is, the n-effective cover of KGL is 'accidentally' already very n-effective) and thus $\tilde{s}_n KGL = s_n(KGL)$. This explains how the generalized slice filtration solves problem (1): we see that the '$\mathbb{G}_m$-slices' (that is, ordinary slices) of KGL agree 'by accident' with the 'T-slices' (that is, generalized slices). But note that T is an analog of S^2, explaining the double-speed convergence.

The main point of this article is that the generalized slices of KO can be computed, and have a form which is very similar to the classical analog, thus solving problem (2). Of course this leads to Atiyah–Hirzebruch type spectral sequences for Hermitian K-theory. Heuristically, the generalized slices of KO are (supposed to be) like the S^2-Postnikov layers of the topological spectrum KO. We thus expect that they are 4-periodic (up to twist). Moreover, we expect that $\tilde{s}_i$ for $i \equiv 1, 2, 3 \pmod 4$ should just 'accidentally' be ordinary zero-slices (corresponding to the fact that $\pi_i KO = 0$ for $i = 3, 5, 7$), whereas $\tilde{s}_0 KO$ should be an extension of two objects (corresponding to $\pi_1 KO \neq 0 \neq \pi_0 KO$). This is indeed the case:

Theorem (see Theorem 16). *The generalized slices of Hermitian K-theory are given as follows:*

$$\tilde{s}_n KO \simeq T^{\wedge n} \wedge \begin{cases} \tilde{s}_0(KO) & n \equiv 0 \pmod 4 \\ H_\mu \mathbb{Z}/2 & n \equiv 1 \pmod 4 \\ H_\mu \mathbb{Z} & n \equiv 2 \pmod 4 \\ 0 & n \equiv 3 \pmod 4. \end{cases}$$

What about the 'conglomerate' $\tilde{s}_0 KO$? We offer two ways of decomposing it, either using the effectivity (slice) filtration or using the homotopy t-structure. The relevant triangles are

$$H_\mu \mathbb{Z}/2[1] \to \tilde{s}_0 KO \to \tilde{H}\mathbb{Z}$$

$$H_W \mathbb{Z} \wedge \mathbb{G}_m \to \tilde{s}_0 KO \to H_\mu \mathbb{Z}.$$

See Lemma 11 and Theorem 16 again. Here $\tilde{H}\mathbb{Z}$ is a spectrum which we call *generalized motivic cohomology*, and $H_W\mathbb{Z}$ is a spectrum which we call *Witt-motivic cohomology*. They can be characterized abstractly as the effective covers of certain objects in the heart of the homotopy t-structure on $\mathbf{SH}(k)$.

The boundary maps in the above two triangles are very interesting and will be subject of further investigation. Also the computation of generalized slices of other spectra is an interesting topic which we will take up in future work.

Relationship to other works. All our computations are done abstractly in the motivic homotopy category. This is not really satisfactory, since in general it is essentially impossible to compute cohomology with coefficients in some abstract spectrum. For the motivic spectral sequence, there is a parallel and much more computational story to the one we have outlined above: Voevodsky has defined motivic cohomology via a category $\mathbf{DM}(k)$ which is reasonably computable [**11**], and in fact motivic cohomology in this sense coincides with motivic cohomology in the sense of higher Chow groups [**11**, Theorem 19.1] which is certainly very explicit. Grayson has defined an explicit spectral sequence converging to algebraic K-theory [**5**, Section 5]. Work of Voevodsky [**30**] and Levine [**9**] shows that the explicit definitions of motivic cohomology mentioned above agree with the abstract definition $H_\mu\mathbb{Z} = s_0 S$. Work of Suslin [**26**] shows that the Grayson spectral sequence has layers given by the explicit form of motivic cohomology, which by what we just said is the same as the abstract form. Work of Garkusha–Panin [**3**] shows that the abstract and explicit motivic spectral sequences agree.

A similar picture is expected for Hermitian K-theory. Calmès–Fasel [**2**] have defined a variant $\widetilde{\mathbf{DM}}(k)$ of $\mathbf{DM}(k)$ and an associated theory $\tilde{H}'\mathbb{Z}$ which they call generalized motivic cohomology. Markett–Schlichting [in preparation] have defined a version of the Grayson filtration for Hermitian K-theory and they hope to show that the layers are of the same form as in our Theorem 16, with $\tilde{H}\mathbb{Z}$ replaced by $\tilde{H}'\mathbb{Z}$. The author contends that it will eventually be shown that $\tilde{H}\mathbb{Z} = \tilde{H}'\mathbb{Z}$ and that the Market–Schlichting spectral sequence coincides with the generalized slice spectral sequence[†].

We note that an obvious modification of the Calmés–Fasel construction yields a spectrum $H'_W\mathbb{Z}$. Again the author contends that $H'_W\mathbb{Z} = H_W\mathbb{Z}$, but this is not currently known.

More about $\tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$. In the meantime, we propose to study the spectra $\tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$ abstractly. Taking intuition from classical topology, that is, comparing the two decompositions of $\tilde{s}_0 KO$ with $(\pi_1 KO, \pi_0 KO) = (\mathbb{Z}/2, \mathbb{Z})$ we see that $\tilde{H}\mathbb{Z}$ should be a 'variant' of $\mathbb{Z}$ and $H_W\mathbb{Z}$ should be a 'variant' of $\mathbb{Z}/2$. This is a familiar game in motivic homotopy theory: the standard unoriented variant of $\mathbb{Z}$ is the homotopy module $\underline{K}_*^{MW}$ of Milnor–Witt K-theory, that is, $\underline{\pi}_0 S_*$, and the standard unoriented variant of $\mathbb{Z}/2$ is the homotopy module $\underline{K}_*^W = \underline{K}_*^{MW}/h$ of Witt K-theory [**17**, Chapter 3]. (The standard oriented variants are Milnor K-theory $\underline{K}_*^M$ and its mod-2 version.) Thus the following result confirms a very optimistic guess.

THEOREM (Morel's structure conjecture; see Theorem 17). *The homotopy sheaves of $\tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$ are given as follows:*

$$\underline{\pi}_i(\tilde{H}\mathbb{Z})_* = \begin{cases} \underline{K}_*^{MW} & i = 0 \\ \underline{\pi}_i(H_\mu\mathbb{Z})_* & i \neq 0 \end{cases} \qquad \underline{\pi}_i(H_W\mathbb{Z})_* = \begin{cases} \underline{K}_*^{W} & i = 0 \\ \underline{\pi}_i(H_\mu\mathbb{Z}/2)_* & i \neq 0. \end{cases}$$

Organization of this article. In the preliminary Section 2 we recall some basic facts about stable motivic homotopy theory, and in particular the homotopy t-structures.

[†]Added later: the isomorphism $\tilde{H}\mathbb{Z} \simeq \tilde{H}'\mathbb{Z}$ has now been established and will appear in forthcoming joint work of the author and Jean Fasel.

In Section 3 we collect some results about the category $\mathbf{SH}(k)^{\mathrm{eff}}$ of effective spectra. In particular we show that it carries a t-structure, show that the effectivization functor $r : \mathbf{SH}(k) \to \mathbf{SH}(k)^{\mathrm{eff}}$ is exact, and provide some results about the heart $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. Note that by definition $\mathbf{SH}(k)^{\mathrm{veff}} = \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$.

In Section 4 we define the generalized slice filtration and establish some basic results. In particular we show that there are two canonical ways of decomposing a generalized slice, similar to how we decomposed $\tilde{s}_0 KO$. We also give precise definitions of the spectra $H_\mu\mathbb{Z}, H_\mu\mathbb{Z}/2, \tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$ we use.

In Section 5 we prove our main theorem computing the generalized slices of the Hermitian K-theory spectrum KO. This uses crucially a lemma of Voevodsky [**28**, Proposition 4.4], the detailed study of the geometry of quaternionic Grassmannians by Panin–Walter [**20**] and the geometric representability of symplectic K-theory by Quaternionic Grassmannians, as proved by Panin–Walter [**19**] and Schlichting–Tripathi [**24**].

Finally in Section 6 we compute the homotopy sheaves of $\tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$. The computation of $\underline{\pi}_i(\tilde{H}\mathbb{Z})_j$ and $\underline{\pi}_i(H_W\mathbb{Z})_j$ for $i \leqslant 0$ or $j \leqslant 0$ is a rather formal consequence of results in Section 3. Thus the main work is in computing the higher homotopy sheaves in positive weights. The basic idea is to play off the two triangles $H_\mu\mathbb{Z}/2[1] \to \tilde{s}_0 KO \to \tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z} \wedge \mathbb{G}_m \to \tilde{s}_0 KO \to H_\mu\mathbb{Z}$ against each other. For example, an immediate consequence of the first triangle is that $\underline{\pi}_i(\tilde{s}_0 KO)_0$ is given by $\underline{GW}$ if $i = 0$, by $\mathbb{Z}/2$ if $i = 1$, and by 0 else. This implies that $\underline{\pi}_1(H_W\mathbb{Z})_1 = \mathbb{Z}/2$, which is a very special case of Theorem 17. The general case proceeds along the same lines. We should mention that this pulls in many more dependencies than the previous sections, including the resolution of the Milnor conjectures and the computation of the motivic Steenrod algebra.

Conventions. Throughout, k is perfect base field. This is because we will make heavy use of the homotopy t-structure on $\mathbf{SH}(k)$, the heart of which is the category of homotopy modules [**12**, Section 5.2]. We denote unit of the symmetric monoidal structure on $\mathbf{SH}(k)$ by S, this is also known as the motivic sphere spectrum.

We denote by $Sm(k)$ the category of smooth k-schemes. If $X \in Sm(k)$ we write $X_+ \in Sm(k)_*$ for the pointed smooth scheme obtained by adding a disjoint base point.

We use *homological* grading for our t-structures, see for example [**10**, Definition 1.2.1.1].

Whenever we say 'triangle', we actually mean 'distinguished triangle'.

2. *Recollections on motivic stable homotopy theory*

We write $\mathbf{SH}^{S^1}(k)$ for the S^1-stable motivic homotopy category [**12**, Section 4.1] and $\mathbf{SH}(k)$ for the $\mathbb{P}^1$-stable motivic homotopy category [**12**, Section 5.1]. We let $\Sigma^\infty_{S^1} : Sm(k)_* \to \mathbf{SH}^{S^1}(k)$ and $\Sigma^\infty : Sm(k)_* \to \mathbf{SH}(k)$ denote the infinite suspension spectrum functors. Note that there exists an essentially unique adjunction

$$\Sigma^\infty_s : \mathbf{SH}^{S^1}(k) \leftrightarrows \mathbf{SH}(k) : \Omega^\infty_s$$

such that $\Sigma^\infty_s \circ \Sigma^\infty_{S^1} \cong \Sigma^\infty$.

For $E \in \mathbf{SH}^{S^1}(k)$ and $i \in \mathbb{Z}$ we define $\underline{\pi}_i(E)$ to be the Nisnevich sheaf on $Sm(k)$ associated with the presheaf $X \mapsto [\Sigma^\infty_{S^1} X_+ \wedge S^i, E]$. For $E \in \mathbf{SH}(k)$ and $i, j \in \mathbb{Z}$ we put $\underline{\pi}_i(E)_j = \underline{\pi}_i(\Omega^\infty_s(E \wedge \mathbb{G}_m^{\wedge j}))$. We define

$$\mathbf{SH}^{S^1}(k)_{\geqslant 0} = \{E \in \mathbf{SH}^{S^1}(k) | \underline{\pi}_i(E) = 0 \text{ for all } i < 0\}$$

$$\mathbf{SH}^{S^1}(k)_{\leqslant 0} = \{E \in \mathbf{SH}^{S^1}(k) | \underline{\pi}_i(E) = 0 \text{ for all } i > 0\}$$

$$\mathbf{SH}(k)_{\geqslant 0} = \{E \in \mathbf{SH}(k) | \underline{\pi}_i(E)_j = 0 \text{ for all } i < 0, j \in \mathbb{Z}\}$$

$$\mathbf{SH}(k)_{\leqslant 0} = \{E \in \mathbf{SH}(k) | \underline{\pi}_i(E)_j = 0 \text{ for all } i > 0, j \in \mathbb{Z}\}.$$

As was known already to Voevodsky, this defines t-structures on $\mathbf{SH}^{S^1}(k), \mathbf{SH}(k)$ [**12**, Theorems 4.3.4 and 5.2.6], called the *homotopy t-structures*. The most important ingredient in the proof of this fact is the *stable connectivity theorem*. The unstable proof in [**12**, Lemma 3.3.9] is incorrect; this has been fixed in [**15**, Theorem 6.1.8]. It implies that if $X \in Sm(k)$ then $\Sigma^\infty X_+ \in \mathbf{SH}(k)_{\geqslant 0}$ and $\Sigma^\infty_{S^1} X_+ \in \mathbf{SH}^{S^1}(k)_{\geqslant 0}$ [**12**, Examples 4.1.16 and 5.2.1]. If $E \in \mathbf{SH}(k)$ then we denote its truncations by $E_{\geqslant 0} \in \mathbf{SH}(k)_{\geqslant 0}, E_{\leqslant 0} \in \mathbf{SH}(k)_{\leqslant 0}$ and so on. We will not explicitly use the truncation functors of $\mathbf{SH}^{S^1}(k)$, and so do not introduce a notation.

The hearts $\mathbf{SH}^{S^1}(k)^\heartsuit, \mathbf{SH}(k)^\heartsuit$ can be described explicitly. Indeed $\mathbf{SH}^{S^1}(k)^\heartsuit$ is equivalent to the category of Nisnevich sheaves of abelian groups which are strictly homotopy invariant (that is, sheaves F such that the map $H^p_{Nis}(X, F) \to H^p_{Nis}(X \times \mathbb{A}^1, F)$ obtained by pullback along the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism, for every $X \in Sm(k)$) [**12**, Lemma 4.3.7(2)]. On the other hand $\mathbf{SH}(k)^\heartsuit$ is equivalent to the category of *homotopy modules* [**12**, Theorem 5.2.6]. Let us recall that a homotopy module F_* consists of a sequence of sheaves $F_i \in Shv(Sm(k)_{Nis})$ which are strictly homotopy invariant, and isomorphisms $F_i \to (F_{i+1})_{-1}$. Here for a sheaf F the *contraction* F_{-1} is as usual defined as $F_{-1}(X) = F(X \times (\mathbb{A}^1 \setminus 0))/F(X)$. The morphisms of homotopy modules are the evident compatible systems of morphisms. One then shows that in fact for $E \in \mathbf{SH}(k)$, the homotopy sheaves $\underline{\pi}_i(E)_*$ form (for each i) a homotopy module in a natural way [**12**, Lemma 5.2.5].

We will mostly not distinguish the category $\mathbf{SH}(k)^\heartsuit$ from the (equivalent) category of homotopy modules, and so may write things like 'let $F_* \in \mathbf{SH}(k)^\heartsuit$ be a homotopy module'.

Because there can be some confusion about the meaning of epimorphism and so on when several abelian categories are being used at once, let us include the following observation. It implies in particular that not much harm will come from confusing for $E \in \mathbf{SH}(k)$ the homotopy module $\underline{\pi}_i(E)_* \in \mathbf{SH}(k)^\heartsuit$ with the family of Nisnevich sheaves $(i \mapsto \underline{\pi}_i(E)_i)$.

LEMMA 1. *Write $Ab(Shv(Sm(k)_{Nis}))$ for the category of Nisnevich sheaves of abelian groups on $Sm(k)$, and $Ab(Shv(Sm(k)_{Nis}))^{\mathbb{Z}}$ for the category of $\mathbb{Z}$-graded families of sheaves of abelian groups.*

(1) *The category $\mathbf{SH}^{S^1}(k)^\heartsuit$ has all limits and colimits and the functor $\mathbf{SH}^{S^1}(k)^\heartsuit \to Ab(Shv(Sm(k)_{Nis})), E \mapsto \underline{\pi}_0(E)$ is fully faithful and preserves limits and colimits.*

(2) *The category $\mathbf{SH}(k)^\heartsuit$ has all limits and colimits and the functor $\mathbf{SH}(k)^\heartsuit \to Ab(Shv(Sm(k)_{Nis}))^{\mathbb{Z}}, E \mapsto (\mathbb{Z} \ni i \mapsto \underline{\pi}_0(E)_i)$ is conservative and preserves limits and colimits.*

In particular, both functors are exact and detect epimorphisms. Let us also note that a conservative exact functor is faithful (two morphisms are equal if and only if their equalizer maps isomorphically to the source).

Before the proof we have two lemmas, which surely must be well known.

LEMMA 2. *Let $\mathcal{C}$ be a t-category and write $j : \mathcal{C}^\heartsuit \to \mathcal{C}$ for the inclusion of the heart. Let $\{E_i \in \mathcal{C}\}_{i \in I}$ be a family of objects. If $\bigoplus_i j(E_i) \in \mathcal{C}$ exists then $\bigoplus_i E_i \in \mathcal{C}^\heartsuit$ exists and is given by $(\bigoplus_i j(E_i))_{\leqslant 0}$. Similarly, if $\prod_i j(E_i) \in \mathcal{C}$ exists then $\prod_i E_i \in \mathcal{C}^\heartsuit$ exists and is given by $(\prod_i j(E_i))_{\geqslant 0}$.*

Proof. The second statement is dual to the first (under passing to opposite categories), so we need only prove the latter. Note first that $\bigoplus_i j(E_i) \in \mathcal{C}_{\geqslant 0}$. Indeed if $E \in \mathcal{C}_{<0}$ then $[\bigoplus_i j(E_i), E] = \prod_i [j(E_i), E] = 0$. Consequently if $E \in \mathcal{C}^\heartsuit$ then $[(\bigoplus_i j(E_i))_{\leqslant 0}, E] = [\bigoplus_i j(E_i), jE] = \prod_i [E_i, E]$, since $jE \in \mathcal{C}_{\leqslant 0}$ and j is fully faithful. This concludes the proof. $\square$

Let $\mathcal{C}, \mathcal{D}$ be provided with subcategories $\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0}, \mathcal{D}_{\geqslant 0}, \mathcal{D}_{\leqslant 0}$; for example, $\mathcal{C}, \mathcal{D}$ could be t-categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *right* (respectively, *left*) *t-exact* if $F(\mathcal{C}_{\geqslant 0}) \subset$

$\mathcal{D}_{\geqslant 0}$ (respectively, $F(\mathcal{C}_{\leqslant 0}) \subset \mathcal{D}_{\leqslant 0}$). It is called t-exact if it is both left and right t-exact.

LEMMA 3. *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ be an adjunction of t-categories, and assume that U is t-exact. Then the induced functor $U^\heartsuit : \mathcal{D}^\heartsuit \to \mathcal{C}^\heartsuit$ preserves limits and finite colimits. If $\mathcal{C}$ is compactly generated, F preserves compact objects, and $\mathcal{D}$ has arbitrary coproducts, then $U^\heartsuit$ preserves all colimits.*

Proof. There is an induced adjunction $F^\heartsuit : \mathcal{C}^\heartsuit \leftrightarrows \mathcal{D}^\heartsuit : U^\heartsuit$, cf. [**1**, Proposition 1.3.17(iii)]. It follows that $U^\heartsuit$ preserves all limits. Since U is right t-exact, $U^\heartsuit$ is right exact, that is, preserves finite colimits [**1**, Proposition 1.3.17(i)].

Under the additional assumptions, U preserves arbitrary coproducts, and so Lemma 2 implies that $\mathcal{D}^\heartsuit$ has arbitrary coproducts and that $U^\heartsuit$ preserves them. The result follows since all colimits can be built from coproducts and finite colimits. □

Proof of Lemma 1. The category $\mathbf{SH}(k)$ is compactly generated [**7**, Proposition 6.4(3)], and hence has all products and coproducts. It follows from Lemma 2 that $\mathbf{SH}(k)^\heartsuit$ has all products and coproducts, and hence all limits and colimits. The same argument applies to $\mathbf{SH}^{S^1}(k)^\heartsuit$.

Let $\mathbf{SH}^{S^1}_s(k)$ denote the stable Nisnevich-local homotopy category, built in the same way as $\mathbf{SH}^{S^1}(k)$, but without performing $\mathbb{A}^1$-localization. It is also compactly generated. Then for $E \in \mathbf{SH}^{S^1}_s(k)$ we may define $\underline{\pi}_i(E)$ in just the same way as before, and we may also define $\mathbf{SH}^{S^1}_s(k)_{\geqslant 0}, \mathbf{SH}^{S^1}_s(k)_{\leqslant 0}$ in the same way as before. Then $\mathbf{SH}^{S^1}_s(k)$ is a t-category with heart $Shv(Sm(k)_{Nis})$ [**15**, Proposition 3.3.2]. We have the localization adjunction $L : \mathbf{SH}^{S^1}_s(k) \leftrightarrows \mathbf{SH}^{S^1}(k) : i$. By construction i is fully faithful and t-exact. The functor from (1) is given by $i^\heartsuit$, so in particular is fully faithful. It preserves all limits and colimits by Lemma 3.

To prove (2), denote the functor by u. Consider for $d \in \mathbb{Z}$ the adjunction $\Sigma^{\infty+d}_s : \mathbf{SH}^{S^1}(k) \leftrightarrows \mathbf{SH}(k) : \Omega^{\infty+d}_s$ given by $\Sigma^{\infty+d}_s(E) = \Sigma^\infty_s(E) \wedge \mathbb{G}_m^{\wedge d}$, $\Omega^{\infty+d}_s(F) = \Omega^\infty_s(F \wedge \mathbb{G}_m^{\wedge -d})$. Then $\Omega^{\infty+d}$ is t-exact by construction, and so Lemma 3 applies. Note that for $E \in \mathbf{SH}(k)^\heartsuit$ we have $u(E)_d = i^\heartsuit(\Omega^{\infty-d}_s)^\heartsuit(E)$, where $i^\heartsuit$ is the functor from (1). It follows that $E \mapsto u(E)_d$ preserves all limits and colimits, and hence so does u. Note also that u detects zero objects [**12**, Proposition 5.1.14], and hence is conservative (since it detects vanishing of kernel and cokernel of a morphism).

3. *The category of effective spectra*

We write $\mathbf{SH}(k)^{\mathrm{eff}}$ for the localizing subcategory of $\mathbf{SH}(k)$ generated by the objects $\Sigma^\infty X_+$, with $X \in Sm(k)$. By Neeman's version of Brown representability, the inclusion $i : \mathbf{SH}(k)^{\mathrm{eff}} \to \mathbf{SH}(k)$ has a right adjoint which we denote by r.

For $E \in \mathbf{SH}(k)^{\mathrm{eff}}$ we let $\underline{\pi}_i(E)_0 \in Shv(Sm(k)_{Nis})$ denote $\underline{\pi}_i(E)_0 := \underline{\pi}_i(iE)_0$. In general we may drop application of the functor i when no confusion seems likely. We define

$$\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0} = \{E \in \mathbf{SH}(k)^{\mathrm{eff}} | \underline{\pi}_i(E)_0 = 0 \text{ for all } i < 0\}$$

$$\mathbf{SH}(k)^{\mathrm{eff}}_{\leqslant 0} = \{E \in \mathbf{SH}(k)^{\mathrm{eff}} | \underline{\pi}_i(E)_0 = 0 \text{ for all } i > 0\}.$$

Some or all of the following was already known to Spitzweck–Østvær [**25**, paragraph before Lemma 5.6].

PROPOSITION 4. (1) *The functors $\underline{\pi}_i(\bullet)_0 : \mathbf{SH}(k)^{\mathrm{eff}} \to Shv(Sm(k)_{Nis})$ form a conservative collection.*

(2) *The category $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$ is generated under homotopy colimits and extensions by $\Sigma^\infty X_+ \wedge S^n$, where $n \geqslant 0, X \in Sm(k)$.*

(3) *The functor r is t-exact and i is right-t-exact.*

(4) *The subcategories $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}, \mathbf{SH}(k)^{\mathrm{eff}}_{\leqslant 0}$ constitute a non-degenerate t-structure on $\mathbf{SH}(k)^{\mathrm{eff}}$.*

Proof. For $X \in Sm(k)$ and $E \in \mathbf{SH}(k)$ we have the strongly convergent Nisnevich descent spectral sequence $H^p_{Nis}(X, \underline{\pi}_{-q}(E)_0) \Rightarrow [\Sigma^\infty X_+, E[p+q]]$. Consequently if $E \in \mathbf{SH}(k)^{\mathrm{eff}}$ and $\underline{\pi}_i(E)_0 = 0$ for all i then $[\Sigma^\infty X_+, E[n]] = 0$ for all $X \in Sm(k)$ and all n. It follows that $E = 0$, since the $\Sigma^\infty X_+$ generate $\mathbf{SH}(k)^{\mathrm{eff}}$ as a localizing subcategory (by definition). Thus the $\underline{\pi}_i(\bullet)_0$ form a conservative collection, that is, we have proved (1).

As recalled in the previous section, we have $\Sigma^\infty X_+ \wedge S^n \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant n}$ for $n \geqslant 0$. Thus if $E \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$, then the homotopy sheaves $\underline{\pi}_i(E)_0$ can be killed off by attaching cells of the form $\Sigma^\infty X_+ \wedge S^n$ for $n \geqslant 0$, $X \in Sm(k)$. Consequently $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$ is generated under homotopy colimits and extensions by objects of the form claimed in (2). We give more details on this standard argument at the end of the proof.

It follows from adjunction that for $E \in \mathbf{SH}(k)$ we have $\underline{\pi}_i(r(E))_0 = \underline{\pi}_i(E)_0$. Consequently r is t-exact. Since i is a left adjoint it commutes with homotopy colimits and so $i(\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}) \subset \mathbf{SH}(k)_{\geqslant 0}$ by (2), that is, i is right-t-exact. Thus we have shown (3).

It remains to show (4), that is, that we have a non-degenerate t-structure. If $E \in \mathbf{SH}(k)^{\mathrm{eff}}$ then $riE \simeq E$. Since r is t-exact, the triangle $r[(iE)_{\geqslant 0}] \to riE \simeq E \to r[(iE)_{<0}]$ coming from the decomposition of iE in the homotopy t-structure is a decomposition of E into non-negative and negative part as required for a t-structure.

Next we need to show that if $E \in \mathbf{SH}(k)^{\mathrm{eff}}_{>0}$ and $F \in \mathbf{SH}(k)^{\mathrm{eff}}_{\leqslant 0}$ then $[E, F] = 0$. The natural map $F \to r[(iF)_{\leqslant 0}]$ induces an isomorphism on all $\underline{\pi}_i(\bullet)_0$, so is a weak equivalence (by the conservativity result (1)). Thus $[E, F] = [E, r[(iF)_{\leqslant 0}]] = [iE, (iF)_{\leqslant 0}] = 0$ since i is right-t-exact and so $iE \in \mathbf{SH}(k)_{>0}$.

We have thus shown that $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}, \mathbf{SH}(k)^{\mathrm{eff}}_{<0}$ form a t-structure. It is non-degenerate by (1). This concludes the proof. $\square$

Details on killing cells. We explain in more detail how to prove (2). Let $\mathcal{C}$ be the subcategory of $\mathbf{SH}(k)^{\mathrm{eff}}$ generated under homotopy colimits and extensions by $\Sigma^\infty X_+ \wedge S^n$, where $n \geqslant 0, X \in Sm(k)$. We wish to show that $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0} \subset \mathcal{C}$. As a first step, we claim that if $E \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant n}$ (with $n \geqslant 0$) there exists $R(E) \in \mathcal{C} \cap \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant n}$ together with $R(E) \to E$ inducing a surjection on $\underline{\pi}_i(\bullet)_0$ for all $i \geqslant 0$. Indeed, just let $R(E)$ be the sum $\bigoplus_{\Sigma^\infty X_+ \wedge S^k \to E} \Sigma^\infty X_+ \wedge S^k$, where the sum is over $k \geqslant n$, a suitably large set of varieties X, and all maps in $\mathbf{SH}(k)$ as indicated.

Now let $E \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$. We will construct a diagram $E_0 \to E_1 \to \cdots \to E$ with $E_i \in \mathcal{C}$ and $E_i \to E$ inducing an isomorphism on $\underline{\pi}_j(\bullet)_0$ for all $j < i$. Clearly then $\mathrm{hocolim}_i\, E_i \to E$ is an equivalence, showing that $E \in \mathcal{C}$, and concluding the proof.

We will also arrange that $\underline{\pi}_j(E_i)_0 \to \underline{\pi}_j(E)_0$ is surjective for all j and i. Take $E_0 = R(E)$. Suppose that E_i has been constructed and let us construct E_{i+1}. Consider the homotopy fibre $F \to E_i \to E$. Then $F \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant i}$ and $\underline{\pi}_i(F)_0 \to \underline{\pi}_i(E_i)_0 \to \underline{\pi}_i(E)_0 \to 0$ is an exact sequence $(*)$. Let E_{i+1} be a cone on the composite $R(F) \to F \to E_i$. Since the composite $R(F) \to F \to E_i \to E$ is zero, the map $E_i \to E$ factors through $E_i \to E_{i+1}$. It is now easy to see, using $(*)$, that E_{i+1} has the desired properties.

Remark. The paragraph on killing cells in fact shows that $\mathbf{SH}(k)^{\mathrm{veff}}$ is generated by $\Sigma^\infty_+ Sm(k)$ under homotopy colimits; no extensions are needed. We will not use this observation.

Terminology. In order to distinguish the t-structure on $\mathbf{SH}(k)^{\mathrm{eff}}$ from the t-structure of $\mathbf{SH}(k)$, we will sometimes call the former the *effective* (*homotopy*) *t-structure*. We denote the truncations of $E \in \mathbf{SH}(k)^{\mathrm{eff}}$ by $E_{\geqslant_e 0} \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}, E_{\leqslant_e 0} \in \mathbf{SH}(k)^{\mathrm{eff}}_{\leqslant 0}$ and so on. For $E \in \mathbf{SH}(k)^{\mathrm{eff}}$, we denote by $\underline{\pi}_i^{\mathrm{eff}}(E) \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ the homotopy objects. We prove below that the functor $\underline{\pi}_0(\bullet)_0 : \mathbf{SH}(k)^{\mathrm{eff},\heartsuit} \to Ab(Shv(Sm(k)_{Nis}))$ is conservative and preserves all limits and colimits. It is thus usually no problem to confuse $\underline{\pi}_i^{\mathrm{eff}}(E)$ and $\underline{\pi}_i(E)_0$.

Remark. By Proposition 4(3), we have $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0} \subset \mathbf{SH}(k)_{\geqslant 0} \cap \mathbf{SH}(k)^{\mathrm{eff}}$. Since the reverse inclusion is clear by definition, we conclude that $\mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0} = \mathbf{SH}(k)_{\geqslant 0} \cap \mathbf{SH}(k)^{\mathrm{eff}}$.

Remark. We call the heart $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ the category of *effective homotopy modules*. We show below that $i^\heartsuit : \mathbf{SH}(k)^{\mathrm{eff},\heartsuit} \to \mathbf{SH}(k)^\heartsuit$ is fully faithful, justifying this terminology. Note, however, that this is *not* the same category as $\mathbf{SH}(k)^\heartsuit \cap \mathbf{SH}(k)^{\mathrm{eff}}$. It follows from work of Garkusha–Panin [**4**] that $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ is equivalent to the category of homotopy invariant, quasi-stable, Nisnevich sheaves of abelian groups with linear framed transfers. We contend that this category is equivalent to the category of homotopy invariant Nisnevich sheaves with generalized transfers in the sense of Calmès–Fasel [**2**] and also in the sense of Morel [**16**, Definition 5.7].

Except for Proposition 5(3), the remainder of this section is not used in the computation of the generalized slices of Hermitian K-theory, only in the last section.

PROPOSITION 5. (1) *For* $E \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$ *we have* $\underline{\pi}_0(iE)_* = i^\heartsuit \underline{\pi}_0^{\mathrm{eff}}(E)$, *where* $i^\heartsuit : \mathbf{SH}(k)^{\mathit{eff},\heartsuit} \to \mathbf{SH}(k)^\heartsuit$ *is the induced functor* $i^\heartsuit(M) = (iM)_{\leqslant 0}$.

(2) *The functor* $i^\heartsuit : \mathbf{SH}(k)^{\mathit{eff},\heartsuit} \to \mathbf{SH}(k)^\heartsuit$ *is fully faithful.*

(3) *The category* $\mathbf{SH}(k)^{\mathit{eff},\heartsuit}$ *has all limits and colimits, and functor* $\mathbf{SH}(k)^{\mathit{eff},\heartsuit} \to Ab(Shv(Sm(k)_{Nis})), E \mapsto \underline{\pi}_i(E)_0$ *is conservative and preserves limits and colimits.*

Proof. If $E \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$ then we have the triangle $E_{\geqslant_e 1} \to E \to \underline{\pi}_0^{\mathrm{eff}} E$ and consequently get the triangle $i(E_{\geqslant_e 1}) \to iE \to i\underline{\pi}_0^{\mathrm{eff}} E$. But i is right-t-exact, so the end of the associated long exact sequence of homotopy shaves is $0 = \underline{\pi}_0(iE_{\geqslant_e 1})_* \to \underline{\pi}_0(iE)_* \to \underline{\pi}_0(i\underline{\pi}_0^{\mathrm{eff}}(E))_* = i^\heartsuit \underline{\pi}_0^{\mathrm{eff}} E \to \underline{\pi}_{-1}(i(E_{\geqslant_e 1}))_* = 0$, whence the claimed isomorphism of (1).

Let us now prove (2). If $E \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ then $E = riE \cong (riE)_{\leqslant_e 0} \cong r[(iE)_{\leqslant 0}] = ri^\heartsuit(E)$, where the last equality holds by definition, and the second to last one by t-exactness of r. Thus $ri^\heartsuit \cong \mathrm{id}$. Consequently if $E, F \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ then $[i^\heartsuit E, i^\heartsuit F] = [E, ri^\heartsuit F] \cong [E, F]$, so $i^\heartsuit$ is fully faithful as claimed. Here we have used the well-known fact that a t-exact adjunction between triangulated categories induces an adjunction of the hearts [**1**, Proposition 1.3.17(iii)].

Now we prove (3). Since $\mathbf{SH}(k)^{\mathrm{eff}}$ is compactly generated, existence of limits and colimits in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ follows from Lemma 2. Consider the adjunction $\Sigma_s^\infty : \mathbf{SH}^{S^1}(k) \leftrightarrows \mathbf{SH}(k)^{\mathrm{eff}} : \Omega_s^\infty$. Then Lemma 3 applies and we find that $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit} \to \mathbf{SH}^{S^1}(k)^\heartsuit$ preserves limits and colimits. Since $\mathbf{SH}^{S^1}(k)^\heartsuit \to Ab(Shv(Sm(k)_{Nis}))$ preserves limits and colimits by Lemma 1, we conclude that $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit} \to Ab(Shv(Sm(k)_{Nis}))$ also preserves limits and colimits. Since the functor also detects zero objects by Proposition 4(1), we conclude that it is conservative. $\square$

Remark. Parts (1) and (2) of the above proof do not use any special properties of $\mathbf{SH}(k)$ and in fact show more generally the following: If $\mathcal{C}, \mathcal{D}$ are presentable stable ∞-categories provided with t-structures and $\mathcal{C} \to \mathcal{D}$ is a right-t-exact, fully faithful functor, then the induced functor $\mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is fully faithful (in fact a colocalization). This was pointed out to the author by Benjamin Antieau.

Thus the functor $i^\heartsuit$ embeds $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ into $\mathbf{SH}(k)^\heartsuit$, explaining our choice of the name 'effective homotopy module'. We will call a homotopy module $F_* \in \mathbf{SH}(k)^\heartsuit$ *effective* if it is in the essential image of $i^\heartsuit$, that is, if there exists $E \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ such that $i^\heartsuit E \cong F$.

If F_* is a homotopy module then we denote by $F_*\langle i \rangle$ the homotopy module $\underline{\pi}_0(F \wedge \mathbb{G}_m^{\wedge i})_*$, which satisfies $F\langle i \rangle_* = F_{*+i}$ and has the same structure maps (just shifted by i places).

Lemma 6. *The homotopy module $\underline{K}_*^{MW}$ of Milnor–Witt K-theory is effective. Moreover, if F_* is an effective homotopy module then so are $F_*\langle i\rangle$ for all $i \geqslant 0$. Also cokernels of morphisms of effective homotopy modules are effective, as are (more generally) colimits of effective homotopy modules.*

In particular, the following homotopy modules are effective: $\underline{K}_*^M = coker(\eta : \underline{K}_*^{MW}\langle 1\rangle \to \underline{K}_*^{MW})$, $\underline{K}_*^W = coker(h : \underline{K}_*^{MW} \to \underline{K}_*^{MW})$ as are, for example, $K_*^M/p, K_*^{MW}[1/p]$, etc.

Proof. The sphere spectrum S is effective, non-negative and satisfies $\underline{\pi}_0(S)_* = \underline{K}_*^{MW}$. Hence Milnor–Witt K-theory is effective by Proposition 5 part (1).

Now let E in $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. Then for $i \geqslant 0$ we have $i(E \wedge \mathbb{G}_m^{\wedge i}) \in \mathbf{SH}(k)_{\geqslant 0}$ and so $E \wedge \mathbb{G}_m^{\wedge i} \in \mathbf{SH}(k)_{\geqslant 0}^{\mathrm{eff}}$. Consequently $(i^\heartsuit E)\langle i\rangle = E_{\leqslant 0} \wedge \mathbb{G}_m^{\wedge i} = (E \wedge \mathbb{G}_m^{\wedge i})_{\leqslant 0} = i^\heartsuit[(E \wedge \mathbb{G}_m^{\wedge i})_{\leqslant_e 0}]$ is effective. Here the last equality is by Proposition 5(1).

Let $E \to F \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ be a morphism and form the right exact sequence $E \to F \to C \to 0$. Since $i^\heartsuit$ has a right adjoint it is right exact, whence $i^\heartsuit E \to i^\heartsuit F \to i^\heartsuit C \to 0$ is right exact. It follows that the cokernel of $i^\heartsuit(E) \to i^\heartsuit(F)$ is $i^\heartsuit(C)$, which is effective.

Finally $i^\heartsuit$ preserves colimits, again since it has a right adjoint, so colimits of effective homotopy modules are effective by a similar argument. □

4. *The generalized slice filtration*

We put $\mathbf{SH}(k)^{\mathrm{veff}} = \mathbf{SH}(k)_{\geqslant 0}^{\mathrm{eff}}$ and for $n \in \mathbb{Z}$, $\mathbf{SH}(k)^{\mathrm{eff}}(n) = \mathbf{SH}(k)^{\mathrm{eff}} \wedge T^{\wedge n}$, $\mathbf{SH}(k)^{\mathrm{veff}}(n) = \mathbf{SH}(k)^{\mathrm{veff}} \wedge T^{\wedge n}$. Here $T = \mathbb{A}^1/\mathbb{G}_m \simeq (\mathbb{P}^1, \infty) \simeq S^1 \wedge \mathbb{G}_m$ denotes the Tate object. These are the categories of (very) n-effective spectra (we just say '(very) effective' if $n = 0$).

Write $i_n : \mathbf{SH}(k)^{\mathrm{eff}}(n) \to \mathbf{SH}(k)$ for the inclusion, $r_n : \mathbf{SH}(k) \to \mathbf{SH}(k)^{\mathrm{eff}}(n)$ for the right adjoint, put $f_n = i_n r_n$ and define s_n as the cofibre $f_{n+1}E \to f_n E \to s_n E$. This is of course the slice filtration [**27**, Section 2].

Similarly we write $\tilde{i}_n : \mathbf{SH}(k)^{\mathrm{veff}}(n) \to \mathbf{SH}(k)$ for the inclusion. There is a right adjoint $\tilde{r}_n$ (see for example the proof of Lemma 10), and we put $\tilde{f}_n = \tilde{i}_n \tilde{r}_n$. This is the generalized slice filtration [**25**, Definition 5.5]. We denote by $\tilde{s}_n(E)$ a cone on $\tilde{f}_{n+1}E \to \tilde{f}_n E$. This depends functorially on E.

Lemma 7. *There exist a functor $\tilde{s}_0 : \mathbf{SH}(k) \to \mathbf{SH}(k)$ and natural transformations $p : \mathrm{id} \Rightarrow \tilde{s}_0$ and $\partial : \tilde{s}_0 \Rightarrow \tilde{f}_1[1]$, all determined up to unique isomorphism, such that for each $E \in \mathbf{SH}(k)$ the following triangle is distinguished: $\tilde{f}_1(E) \to \tilde{f}_0(E) \xrightarrow{p_E} \tilde{s}_0(E) \xrightarrow{\partial_E} \tilde{f}_1 E[1]$.*

Moreover, for $E, F \in \mathbf{SH}(k)$ we have $[\tilde{f}_1(E)[1], \tilde{s}_0 F] = 0$.

Proof. By [**1**, Proposition 1.1.9] it suffices to show the 'moreover' part. We may as well show that if $E \in \mathbf{SH}(k)^{\mathrm{veff}}(1), F \in \mathbf{SH}(k)$ then $[E[1], \tilde{s}_0 F] = 0$. Considering the long exact sequence

$$[E[1], \tilde{f}_1 F] \xrightarrow{\alpha} [E[1], \tilde{f}_0 F] \to [E[1], \tilde{s}_0 F] \to [E, \tilde{f}_1 F] \xrightarrow{\beta} [E, \tilde{f}_0 F]$$

it is enough to show that α and β are isomorphisms. This is clear since $E, E[1] \in \mathbf{SH}(k)^{\mathrm{veff}}(1)$ and $\tilde{f}_1 \tilde{f}_0 F \simeq \tilde{f}_1 F$. □

The following lemmas will feature ubiquitously in the sequel. Recall that the f_i are triangulated functors.

Lemma 8. *For $E \in \mathbf{SH}(k)$ we have*

$$T \wedge f_n(E) \simeq f_{n+1}(T \wedge E)$$

and

$$T \wedge \tilde{f}_n(E) \simeq \tilde{f}_{n+1}(T \wedge E).$$

Proof. We have $\mathbf{SH}(k)^{\mathrm{eff}}(n+1) = \mathbf{SH}(k)^{\mathrm{eff}}(n) \wedge T$. Now for $X \in \mathbf{SH}(k)^{\mathrm{eff}}(n)$ we compute

$$[X \wedge T, T \wedge f_n E] \cong [X, f_n E] \cong [X, E] \cong [X \wedge T, E \wedge T] \cong [X \wedge T, f_{n+1}(E \wedge T)].$$

Thus $T \wedge f_n E \simeq f_{n+1}(T \wedge E)$ by the Yoneda lemma.

The proof for $\tilde{f}_n$ is exactly the same, with $\mathbf{SH}(k)^{\mathrm{eff}}$ replaced by $\mathbf{SH}(k)^{\mathrm{veff}}$ and $f_\bullet$ replaced by $\tilde{f}_\bullet$. □

We obtain a t-structure on $\mathbf{SH}(k)^{\mathrm{eff}}(n)$ by 'shifting' the t-structure on $\mathbf{SH}(k)^{\mathrm{eff}}$ by $\mathbb{G}_m^{\wedge n}$. In other words, if $E \in \mathbf{SH}(k)^{\mathrm{eff}}(n)$ then $E \in \mathbf{SH}(k)^{\mathrm{eff}}(n)_{\geqslant 0}$ if and only if $\underline{\pi}_i(E)_{-n} = 0$ for all $i < 0$. Since $\wedge \mathbb{G}_m^{\wedge n} : \mathbf{SH}(k)^{\mathrm{eff}} \to \mathbf{SH}(k)^{\mathrm{eff}}(n)$ is an equivalence of categories, all the properties established in the previous section apply to $\mathbf{SH}(k)^{\mathrm{eff}}(n)$ as well, suitably reformulated. In particular, $\mathbf{SH}(k)^{\mathrm{eff}}(n)_{\geqslant 0}$ *is* the non-negative part of a t-structure. We denote the associated truncation by $E \mapsto E_{\geqslant_{e,n} 0} \in \mathbf{SH}(k)^{\mathrm{eff}}(n)_{\geqslant 0}$, and so on.

Lemma 9. *Denote by $j_n : \mathbf{SH}(k)^{\mathrm{eff}}(n+1) \to \mathbf{SH}(k)^{\mathrm{eff}}(n)$ the canonical inclusion. The restricted functor $f_{n+1} : \mathbf{SH}(k)^{\mathrm{eff}}(n) \to \mathbf{SH}(k)^{\mathrm{eff}}(n+1)$ right adjoint to j_n and is t-exact, and j_n is right t-exact.*

Proof. Adjointness is clear. Let $E \in \mathbf{SH}(k)^{\mathrm{eff}}(n)$. We have $\underline{\pi}_i(f_{n+1}E)_{-n-1} = \underline{\pi}_i(E)_{-n-1} = (\underline{\pi}_i(E)_{-n})_{-1}$. In particular if $\underline{\pi}_i(E)_{-n} = 0$ then $\underline{\pi}_i(f_{n+1}E)_{-n-1} = 0$, which proves that f_{n+1} is t-exact. Then j_n is right t-exact, being left adjoint to a left t-exact functor. □

Lemma 10. *Let $E \in \mathbf{SH}(k)$. Then*

$$\tilde{f}_n E \simeq i_n(r_n(E)_{\geqslant_{e,n} n}) \simeq f_n(E_{\geqslant n}).$$

Proof. Note that $\wedge T : \mathbf{SH}(k)^{\mathrm{eff}}(n) \to \mathbf{SH}(k)^{\mathrm{eff}}(n+1)$ induces equivalences $\mathbf{SH}(k)^{\mathrm{eff}}(n)_{\geqslant n} \to \mathbf{SH}(k)^{\mathrm{eff}}(n+1)_{\geqslant n+1}$ and similarly for the non-negative parts. It follows that for $E \in \mathbf{SH}(k)^{\mathrm{eff}}(n)$ we have $E_{\geqslant_{e,n} n} \wedge T \simeq (E \wedge T)_{\geqslant_{e,n+1} n+1}$. Similarly we find that for $E \in \mathbf{SH}(k)$ we have $(E \wedge T)_{\geqslant n+1} \simeq E_{\geqslant n} \wedge T$. Together with Lemma 8 this implies that the current lemma holds for some n if and only if it holds for $n+1$ (and all E). We may thus assume that $n = 0$.

We have a factorization of inclusions $\mathbf{SH}(k)^{\mathrm{veff}}(0) \to \mathbf{SH}(k)^{\mathrm{eff}}(0) \to \mathbf{SH}(k)$ and hence the right adjoint factors similarly. But the right adjoint to $\mathbf{SH}(k)^{\mathrm{veff}}(0) \to \mathbf{SH}(k)^{\mathrm{eff}}(0)$ is truncation in the effective t-structure by definition, whence the first equivalence. The second equivalence follows from t-exactness of r, that is, Proposition 4 part (3). □

From now on, we will write $f_n E_{\geqslant m}$ when convenient. This will always mean $f_n(E_{\geqslant m})$ and never $f_n(E)_{\geqslant m}$. This is the same as $i_n(r_n(E)_{\geqslant_{e,n} m})$. In particular in calculations, we will similarly write $s_n E_{\geqslant m}$ to mean $s_n(E_{\geqslant m})$, never $s_n(E)_{\geqslant m}$. We will also from now on mostly write f_0 in place of r_0; whenever we make statements like 'f_0 is t-exact' we really mean that $f_0 : \mathbf{SH}(k) \to \mathbf{SH}(k)^{\mathrm{eff}}$ is t-exact.

Lemma 11. *For $E \in \mathbf{SH}(k)$ there exist natural triangles*

$$s_0(E_{\geqslant 1}) \to \tilde{s}_0(E) \to f_0(\underline{\pi}_0(E)_*) \tag{1}$$

and

$$f_1(\underline{\pi}_0(E)_*) \to \tilde{s}_0(E) \to s_0(E_{\geqslant 0}). \tag{2}$$

Of course, there are variants of this lemma for $\tilde{s}_n$, obtained, for example, using $\tilde{s}_n(E) \simeq \tilde{s}_0(E \wedge T^{-\wedge n}) \wedge T^{\wedge n}$.

Proof. We claim that there are canonical isomorphisms (1) $s_0(E_{\geqslant 1}) \simeq \tilde{s}_0(E)_{\geqslant_e 1}$, (2) $f_0(\underline{\pi}_0(E)_*) \simeq \underline{\pi}_0^{\mathrm{eff}}(\tilde{s}_0 E)$, (3) $f_1(\underline{\pi}_0(E)_*) \simeq f_1\tilde{s}_0(E)$ and (4) $s_0(E_{\geqslant 0}) \simeq s_0\tilde{s}_0(E)$. Since $\tilde{s}_0(E) \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$, the two purported triangles are thus the functorial triangles $\tilde{s}_0(E)_{\geqslant_e 1} \to \tilde{s}_0(E) \simeq \tilde{s}_0(E)_{\geqslant_e 0} \to \underline{\pi}_0^{\mathrm{eff}}\tilde{s}_0(E)$ and $f_1\tilde{s}_0(E) \to \tilde{s}_0(E) \simeq f_0\tilde{s}_0(E) \to s_0\tilde{s}_0(E)$.

(1) We have $s_0(E_{\geqslant 1}) \simeq s_0 f_0(E_{\geqslant 1}) \simeq s_0\tilde{f}_0(E)_{\geqslant_e 1}$, since f_0 is t-exact. In particular, we may assume that $E \in \mathbf{SH}(k)^{\mathrm{veff}}$. Consider the following commutative diagram

$$\begin{tikzcd}
 & & \tilde{s}_0(E)_{\geq_e 1} \arrow[d, "p'"'] & & \\
\tilde{f}_1 E \arrow[r] & E \arrow[r] & \tilde{s}_0 E \arrow[r] & \tilde{f}_1(E)[1] \arrow[r] & E[1] \\
f_1(E_{\geq_e 1}) \arrow[r] \arrow[u, equal] & E_{\geq_e 1} \arrow[r] \arrow[u, "p"] & s_0(E_{\geq_e 1}) \arrow[r] \arrow[u, "h"] & f_1(E_{\geq_e 1})[1] \arrow[r] \arrow[u, equal] & E_{\geq_e 1}[1]. \arrow[u, "p[1]"']
\end{tikzcd}$$

Here the rows are triangles and the maps p and p' are the canonical ones. The map h is induced. Using that $[f_1(E_{\geqslant_e 1})[1], \tilde{s}_0 E]$ by Lemma 7 and [**1**, Proposition 1.1.9], we see that h is in fact the *unique* morphism rendering the diagram commutative. We have $f_1(E_{\geqslant_e 1}) \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 1}$, by Lemma 9. It follows that $s_0(E_{\geqslant_e 1}) \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 1}$. Of course also $\tilde{s}_0(E)_{\geqslant_e 1} \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 1}$. Let $T \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 1}$. Then p' induces $[T, \tilde{s}_0(E)_{\geqslant_e 1}] \xrightarrow{\cong} [T, s_0(E_{\geqslant_e 1})]$. Hence by Yoneda, it suffices to show that h induces $[T, s_0(E_{\geqslant_e 1})] \xrightarrow{\cong} [T, \tilde{s}_0 E]$. By the 5-lemma it suffices to show that $[T, E_{\geqslant_e 1}] \xrightarrow{\cong} [T, E]$ which is clear by definition, and that $[T, E_{\geqslant_e 1}[1]] \to [T, E[1]]$ is injective. This follows from the exact sequence $0 = [T, E_{\leqslant_e 0}] \to [T, E_{\geqslant_e 1}[1]] \to [T, E[1]]$.

(2) We have $\underline{\pi}_0^{\mathrm{eff}}(\tilde{s}_0 E) \simeq \underline{\pi}_0^{\mathrm{eff}}(\tilde{f}_0 E) \simeq \underline{\pi}_0^{\mathrm{eff}}(f_0 E)$, since $\underline{\pi}_0^{\mathrm{eff}}(\tilde{f}_1 E) = 0 = \underline{\pi}_{-1}^{\mathrm{eff}}(\tilde{f}_1(E))$, again by Lemma 9. We conclude since $f_0 : \mathbf{SH}(k) \to \mathbf{SH}(k)^{\mathrm{eff}}$ is t-exact by Proposition 4(3).

(3) We have the two canonical triangles $\tilde{f}_0(E)_{\geqslant_e 1} \to \tilde{f}_0 E \to \underline{\pi}_0^{\mathrm{eff}}(f_0 E) \simeq f_0\underline{\pi}_0(E)_*$ (using t-exactness of f_0) and $\tilde{f}_1 E \to \tilde{f}_0 E \to \tilde{s}_0 E$. The middle terms are canonically isomorphic, and the left terms become canonically isomorphic after applying f_1. There is thus an induced isomorphism $f_1\underline{\pi}_0(E)_* \to f_1\tilde{s}_0(E)$, which is in fact unique by [**1**, Proposition 1.1.9], provided we show that $[\tilde{f}_1(E)[1], f_1\tilde{s}_0(E)] = 0$. This follows from the exact sequence $[\tilde{f}_1(E)[1], s_0(\tilde{s}_0(E))[1]] \to [\tilde{f}_1(E)[1], f_1\tilde{s}_0(E)] \to [\tilde{f}_1(E)[1], \tilde{s}_0(E)]$, Lemma 7, and the analog of Lemma 7 for ordinary slices. We can quickly prove this analog: if $E \in \mathbf{SH}(k)^{\mathrm{eff}}(1)$ and $F \in \mathbf{SH}(k)$, then $[E, s_0(F)] = 0$, since $[E, f_1 F[i]] \cong [E, f_0 F[i]]$ for all i.

(4) We have $s_0(E_{\geqslant 0}) \simeq s_0 f_0(E_{\geqslant 0}) \simeq s_0(f_0(E)_{\geqslant_e 0}) \simeq s_0\tilde{f}_0(E)$. Since s_0 is a triangulated functor, we have a canonical triangle $s_0\tilde{f}_1 E \to s_0\tilde{f}_0 E \to s_0\tilde{s}_0 E$. Since $\tilde{f}_1 E \in \mathbf{SH}(k)^{\mathrm{eff}}(1)$ we have $s_0\tilde{f}_1 E \simeq 0$, and hence we obtain the required isomorphism. □

Notation. We define spectra for (generalized) motivic cohomology theory as effective covers:

$$H_\mu\mathbb{Z} := f_0\underline{K}^M_*$$

$$H_\mu\mathbb{Z}/2 := f_0\underline{K}^M_*/2$$

$$\tilde{H}\mathbb{Z} := f_0\underline{K}^{MW}_*$$

$$H_W\mathbb{Z} = f_0\underline{K}^W_*.$$

Here $\underline{K}_*^{MW} \in \mathbf{SH}(k)^\heartsuit$ denotes the homotopy module of Milnor–Witt K-theory [**17**, Chapter 3], that is, $\underline{\pi}_0(S)_*$, where S is the sphere spectrum [**17**, Theorem 6.40]. Also $\underline{K}_*^W := \underline{K}_*^{MW}/h$ is the homotopy module of Witt K-theory [**17**, Example 3.33]. Similarly for the other right-hand sides. This terminology is justified by the following result.

LEMMA 12. *Let $S \to \underline{K}_*^M$ denote the composite $S \to \underline{\pi}_0(S)_* \simeq \underline{K}_*^{MW} \to \underline{K}_*^M$. Then the canonical maps $s_0(S) \to s_0(\underline{K}_*^M) \leftarrow f_0\underline{K}_*^M$ are equivalences. Moreover, there is a canonical equivalence $f_0(\underline{K}_*^M)/2 \simeq f_0(\underline{K}_*^M/2)$, where on the left-hand side we mean a cone on a morphism in $\mathbf{SH}(k)$ and on the right-hand side $\underline{K}_*^M/2 \in \mathbf{SH}(k)^\heartsuit$ denotes the cokernel.*

In particular, the spectra $H_\mu\mathbb{Z}$ and $H_\mu\mathbb{Z}/2$ represent motivic cohomology in the sense of Bloch's higher Chow groups [**9**, Theorems 6.5.1 and 9.0.3]. This agrees with Voevodsky's definition of motivic cohomology [**11**, Theorem 19.1] (recall that our base field is perfect). Note that if C is a cone on $\underline{K}_*^M \xrightarrow{2} \underline{K}_*^M$, then $f_0(C)$ is a cone on $H_\mu\mathbb{Z} \xrightarrow{2} H_\mu\mathbb{Z}$, that is, also canonically isomorphic to $H_\mu\mathbb{Z}/2$. In other words in the notation $f_0\underline{K}_*^M/2$ it does not matter if we view $\underline{K}_*^M/2$ as a cone or cokernel.

Proof. Since $\underline{K}_{-1}^M = 0$ we have $f_1\underline{K}_*^M = 0$ and so $f_0\underline{K}_*^M \to s_0\underline{K}_*^M$ is an equivalence. The spectrum s_0S represents motivic cohomology [**9**, Theorems 6.5.1 and 9.0.3; **11**, Theorem 19.1] and hence $\underline{\pi}_0(s_0S)_0 = \mathbb{Z}$, whereas $\underline{\pi}_i(s_0S)_0 = 0$ for $i \neq 0$. Similarly we have just from the definitions that $\underline{\pi}_0(f_0\underline{K}_*^M)_0 = \mathbb{Z}$ and $\underline{\pi}_i(f_0\underline{K}_*^M)_0 = 0$ for $i \neq 0$. Thus the map $s_0S \to s_0\underline{K}_*^M \simeq f_0\underline{K}_*^M$ induces an isomorphism on all $\underline{\pi}_i(\bullet)_0$, provided that $[S, s_0S] \to [S, s_0\underline{K}_*^M]$ is an epimorphism. But $[S,S] \to [S, s_0S]$ and $[S,S] \to [S, \underline{K}_*^M] = [S, f_0\underline{K}_*^M] = [S, s_0\underline{K}_*^M]$ are both epimorphisms, so this is true. The first claim now follows from Proposition 4(1).

For the second claim, note that it follows from Propositions 5(3) and Lemma 1 that $f_0 : \mathbf{SH}(k)^\heartsuit \to \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ is exact. Thus $f_0(\underline{K}_*^M/2)$ is the cokernel of $\alpha : f_0(\underline{K}_*^M) \xrightarrow{2} f_0(\underline{K}_*^M) \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ and it remains to show that this cokernel is isomorphic to the cone of α. This happens if and only if α is injective, which is clear since under the conservative exact functor from Proposition 5(3), α just corresponds to $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$. □

Philosophy. The complex realization of $\mathbb{G}_m$ is S^1 and the complex realization of T is S^2. We propose to think of the generalized slices as some kind of 'motivic (stable) 1-types'. Note that ordinary slices, as well as objects of $\mathbf{SH}(k)^\heartsuit$ and $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ would all be reasonable candidates for 'motivic 0-types'. Triangle (1) shows that every motivic 1-type can be canonically decomposed into an element of $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ and a zero-slice (that is, a birational motive) — both of which we think of as different kinds motivic 0-types. Thus in triangle (1) we think of $f_0\underline{\pi}_0(E)_*$ as the π_0 part of the motivic 1-type $\tilde{s}_0E$ and of $s_0(E_{\geqslant 1})$ as the π_1 part of the motivic 1-type. Of course, triangle (2) shows that every motivic 1-type can be canonically decomposed into an element of $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit} \wedge \mathbb{G}_m$ and a zero-slice, which are again motivic 0-types. Thus in triangle (2) we think of $s_0E_{\geqslant 0}$ as the π_0-part of the motivic 1-type and of $f_1\underline{\pi}_0(E)_*$ as the π_1-part.

5. *The generalized slices of Hermitian K-theory*

We will now compute $\tilde{s}_n(KO)$. Recall that there exist motivic spaces $GW^{[n]} \in Spc_*(k)$ which represent Hermitian K-theory and come with a canonical weak equivalence $\Omega_T GW^{[n]} \simeq GW^{[n-1]}$. Thus they can be assembled into a motivic T-spectrum $KO = (GW^{[0]}, GW^{[1]}, \dots) \in \mathbf{SH}(k)$ also representing Hermitian K-theory [**6**]. We remind that this spectrum is *not* connective. We will write $KO^{[n]} := KO \wedge T^{\wedge n}$. Observe that under the adjunction $\Sigma^\infty : Ho(Spc_*(k)) \leftrightarrows \mathbf{SH}(k) : \Omega^\infty$ we have $\Omega^\infty(KO^{[n]}) \simeq GW^{[n]}$.

Table 1. *Low degree Hermitian K-groups as strictly homotopy invariant Nisnevich sheaves.*

n	$\underline{\pi}_0(GW^{[n]})$
0	$\underline{GW}$
1	0
2	$\mathbb{Z}$
3	$\mathbb{Z}/2$

By Bott periodicity, we have $GW^{[n+4]} \simeq GW^{[n]}$ and so $KO^{[n+4]} \simeq KO^{[n]}$ [**22**, Proposition 7]. Also recall the low-degree Hermitian K-groups from Table 1. This table can be deduced for example from the identification of $GW_0^{[n]}$ with the Balmer–Walter Grothendieck–Witt groups [**23**, Lemma 8.2; **31**, Theorem 10.1].

Lemma 13. *We have $f_0\underline{\pi}_0(KO^{[2]})_* \simeq H_\mu\mathbb{Z}$, and the natural induced map $s_0 KO^{[2]}_{\geqslant 0} = s_0 f_0 KO^{[2]}_{\geqslant 0} \to s_0 f_0 \underline{\pi}_0(KO^{[2]})_* \simeq s_0 H_\mu \mathbb{Z} \simeq H_\mu\mathbb{Z}$ is an isomorphism.*

Proof. Consider the map $S \to KO^{[2]}$ corresponding to $1 \in \mathbb{Z} = [S, KO^{[2]}]$. It induces $\alpha : \tilde{H}\mathbb{Z} = f_0\underline{\pi}_0(S)_* \to f_0\underline{\pi}_0(KO^{[2]})_* \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. We also have the canonical map $\beta : \tilde{H}\mathbb{Z} \to H_\mu\mathbb{Z} \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. We claim that α and β are surjections with equal kernels. Indeed this may be checked after applying the conservative exact functor from Proposition 5(3), where both maps correspond to the canonical map $\underline{GW} \to \mathbb{Z}$. It follows that there is a canonical isomorphism $f_0\underline{\pi}_0 KO^{[2]} \simeq H_\mu\mathbb{Z}$. (The point of this elaboration is that even though we know that $f_0\underline{\pi}_0 KO^{[2]}$ and $H_\mu\mathbb{Z}$ are objects of $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ which have the same underlying homotopy sheaf, a priori the transfers could be different.)

The rest of the proof essentially uses an argument of Voevodsky [**28**, Section 4]. Write

$$\Sigma_s^\infty \mathbf{SH}^{S^1}(k) \leftrightarrows \mathbf{SH}(k) : \Omega_s^\infty$$

for the canonical adjunction. Note that one may define a slice filtration for $\mathbf{SH}^{S^1}(k)$ in just the same way as for $\mathbf{SH}(k)$: Let $\mathbf{SH}^{S^1}(k)(n) \subset \mathbf{SH}^{S^1}(k)$ denote the localizing subcategory generated by $T^n \wedge E$ for $E \in \mathbf{SH}^{S^1}(k)$. Then the inclusion $i'_n : \mathbf{SH}^{S^1}(k)(n) \to \mathbf{SH}^{S^1}(k)$ has a right adjoint r'_n, one puts $f'_n = i'_n r'_n$, and $s'_n E$ is defined to be the cofiber of $f'_{n+1}E \to f'_n E$ [**9**, Section 7.1].

It is enough to show that $f_0(KO^{[2]}_{\geqslant 1}) \in \mathbf{SH}(k)^{\mathrm{eff}}(1)$, that is, that $s_0(KO^{[2]}_{\geqslant 1}) \simeq 0$. The functor $\Omega_s^\infty : \mathbf{SH}(k)^{\mathrm{eff}} \to \mathbf{SH}^{S^1}(k)$ is conservative [**28**, Lemma 3.3] and commutes with taking slices [**9**, Theorems 9.0.3 and 7.1.1]. Since also $\Omega_s^\infty H_\mu\mathbb{Z} = \mathbb{Z}$ we find that it is enough to show that $s_0^{S^1}(\Omega_s^\infty KO^{[2]}_{\geqslant 0}) \simeq \mathbb{Z}$. (This is precisely how Voevodsky computes $s_0 KGL$, but since his conjectures have been proved by Levine our result is unconditional.)

We claim that $\Omega^\infty KO^{[2]}_{\geqslant 0} \simeq \Omega^\infty KO^{[2]} \simeq GW^{[2]} \in Spc_*(k)$. To see this, note that for any $E \in \mathbf{SH}(k)$ and $U \in Spc_*(k)$ we have $[U, \Omega^\infty E] = [\Sigma^\infty U, E] = [\Sigma^\infty U, E_{\geqslant 0}] = [U, \Omega^\infty E_{\geqslant 0}]$, where for the middle equality we have used that $\Sigma^\infty U \in \mathbf{SH}(k)_{\geqslant 0}$. Indeed as explained in Section 2 this holds for $U = X_+$ with $X \in Sm(k)$, the category $Spc_*(k)$ is generated under homotopy colimits by spaces of the form X_+, the functor Σ^∞ preserves homotopy colimits, and $\mathbf{SH}(k)_{\geqslant 0}$ is closed under homotopy colimits.

The geometric representability theorem of Panin–Walter [**19**] (for the case of symplectic K-theory, which is all we need here) and Schlichting–Tripathi [**24**] (for the general case) implies that $GW^{[2]} \simeq \mathbb{Z} \times HGr$. Thus by [**28**, Proposition 4.4 and proof of Lemma 4.6] the required computation $s_0^{S^1}(\Omega^\infty KO^{[2]}_{\geqslant 0}) \simeq \mathbb{Z}$ follows from the next result (which is completely analogous to [28, Lemma 4.7]). □

LEMMA 14. *Let $HGr(m,n)$ denote the quaternionic Grassmannian of* [20], *with its canonical base point. Then $\Sigma_s^\infty HGr(m,n) \in \mathbf{SH}^{S^1}(k)(1)$; in other words there exists $E \in \mathbf{SH}^{S^1}(k)$ such that $\Sigma_s^\infty HGr(m,n) \simeq E \wedge T$.*

Proof. If X is a smooth scheme, U an open subscheme and Z the closed complement which also happens to be smooth, then by homotopy purity [18, Theorem 3.2.23] there is a triangle

$$\Sigma_s^\infty U_+ \to \Sigma_s^\infty X_+ \to \Sigma_s^\infty Th(N_{Z/X}),$$

where $N_{Z/X}$ denotes the normal bundle and Th the Thom space (which is canonically pointed). It follows from the octahedral axiom (for example) that we may also use a base point inside $U \subset X$, that is, that there is a triangle

$$\Sigma_s^\infty U \to \Sigma_s^\infty X \to \Sigma_s^\infty Th(N_{Z/X})$$

(provided that U is pointed, of course).

As a next step, if E is a trivial vector bundle (of positive rank r) on Z then $\Sigma_s^\infty Th(E) \simeq T^{\wedge r} \wedge \Sigma^\infty Z_+ \in \mathbf{SH}^{S^1}(k) \wedge T$ is 1-effective. Since all vector bundles are Zariski-locally trivial and $\mathbf{SH}^{S^1}(k) \wedge T$ is closed under homotopy colimits, the same holds for an arbitrary vector bundle (of everywhere positive rank). Consequently $Th(N_{Z/X}) \in \mathbf{SH}^{S^1}(k) \wedge T$ and so $\Sigma_s^\infty X \in \mathbf{SH}^{S^1}(k) \wedge T$ if and only if $\Sigma_s^\infty U \in \mathbf{SH}^{S^1}(k) \wedge T$.

We finally come to quaternionic Grassmannians. The space $HGr(m,n)$ has a closed subscheme $N^+(m,n)$, with open complement $Y(m,n)$ everywhere of positive codimension [20, Introduction]. The space $N^+(m,n)$ is a vector bundle over $HGr(m,n-1)$ [20, Theorem 4.1(a)] and so smooth. The open complement $Y(m,n)$ is $\mathbb{A}^1$-weakly equivalent to $HGr(m-1,n-1)$ [20, Theorem 5.1].

Consequently $\Sigma_s^\infty HGr(m,n) \in \mathbf{SH}^{S^1}(k) \wedge T$ if and only if $\Sigma_s^\infty HGr(m-1,n-1) \in \mathbf{SH}^{S^1}(k) \wedge T$. The claim is clear if $m=0$, so the general case follows. □

We will use the following result, which is surely well known.

LEMMA 15. *Let $\mathcal{C}$ be a non-degenerate t-category and*

$$Z[-1] \xrightarrow{\partial} X \to Y \to Z$$

a triangle. If $\pi_0^{\mathcal{C}}(Y) \to \pi_0^{\mathcal{C}}(Z)$ is an epimorphism in $\mathcal{C}^\heartsuit$, then there is a unique map $\partial' : Z_{\geqslant 0}[-1] \to X_{\geqslant 0}$ such that the following diagram commutes, where all the unlabelled maps are the canonical ones

$$\begin{tikzcd} Z_{\geq 0}[-1] \arrow[r, "\partial'"] \arrow[d] & X_{\geq 0} \arrow[r] \arrow[d] & Y_{\geq 0} \arrow[r] \arrow[d] & Z_{\geq 0} \arrow[d] \\ Z[-1] \arrow[r, "\partial"] & X \arrow[r] & Y \arrow[r] & Z. \end{tikzcd}$$

Moreover, the top row is also a triangle.

Proof. Let F be a homotopy fibre of $Y_{\geqslant 0} \to Z_{\geqslant 0}$. Since $\pi_0^{\mathcal{C}} Y \to \pi_0^{\mathcal{C}} Z$ is epi we have $\pi_{-1}^{\mathcal{C}} F = 0$ and $F \in \mathcal{C}_{\geqslant 0}$. We will show that $F \simeq X_{\geqslant 0}$.

By TR3, there is a commutative diagram

$$\begin{array}{ccccccc} Z_{\geq 0}[-1] & \longrightarrow & F & \xrightarrow{\beta} & Y_{\geq 0} & \longrightarrow & Z_{\geq 0} \\ \downarrow & & \alpha\downarrow & & \downarrow & & \downarrow \\ Z[-1] & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z. \end{array}$$

The composite $F \xrightarrow{\alpha} X \to X_{<0}$ is zero because $F \in \mathcal{C}_{\geqslant 0}$ and consequently α factors as $F \xrightarrow{\alpha'} X_{\geqslant 0} \to X$. By the five lemma, α' induces isomorphisms on all homotopy objects, so is an isomorphism by non-degeneracy. This shows that in the above diagram, we may replace F by $X_{\geqslant 0}$ in such a way that α becomes the canonical map. We need to show that then β is also the canonical map. But since $X_{\geqslant 0} \in \mathcal{C}_{\geqslant 0}$ we have $[X_{\geqslant 0}, Y_{\geqslant 0}] \cong [X_{\geqslant 0}, Y]$, and the image of β in this latter group is the canonical map, since the diagram commutes. This proves existence.

For uniqueness, note that the triangle

$$X_{<0}[-1] \to X_{\geqslant 0} \to X \to X_{<0}$$

induces an exact sequence

$$0 = \mathrm{Hom}(Z_{\geqslant 0}[-1], X_{<0}[-1]) \to \mathrm{Hom}(Z_{\geqslant 0}[-1], X_{\geqslant 0}) \to \mathrm{Hom}(Z_{\geqslant 0}[-1], X),$$

whence there is indeed at most one map ∂' making the square commute. □

We now come to the main result.

THEOREM 16. *The generalized slices of Hermitian K-theory are given as follows:*

$$\tilde{s}_n KO \simeq T^{\wedge n} \wedge \begin{cases} \tilde{s}_0(KO) & n \equiv 0 \pmod 4 \\ H_\mu \mathbb{Z}/2 & n \equiv 1 \pmod 4 \\ H_\mu \mathbb{Z} & n \equiv 2 \pmod 4 \\ 0 & n \equiv 3 \pmod 4). \end{cases}$$

Moreover, the canonical decomposition (1) from Lemma 11 of $\tilde{s}_0(KO)$ is given by

$$H_\mu \mathbb{Z}/2[1] \to \tilde{s}_0(KO) \to \tilde{H}\mathbb{Z},$$

and decomposition (2) is given by

$$\mathbb{G}_m \wedge H_W \mathbb{Z} \to \tilde{s}_0 KO \to H_\mu \mathbb{Z}.$$

Proof. Since $KO \wedge T^{\wedge 4} \simeq KO$, the periodicity is clear, and we need only deal with $n \in \{0,1,2,3\}$. It follows from Lemma 8 that $\tilde{s}_n KO = T^{\wedge n} \wedge \tilde{s}_0(T^{\wedge -n} \wedge KO) = T^{\wedge n} \wedge \tilde{s}_0 KO^{[-n]}$, and similarly for s_n.

We first deal with $n \in \{1,2,3\}$. Since then $\underline{\pi}_0(KO^{[-n]})_{-1} = 0$ (see again Table 1) we have $f_1 \underline{\pi}_0(KO^{[-n]})_* = 0$ and hence by decomposition (2) from Lemma 11 it is enough to show $s_0(KO^{[-1]}_{\geqslant 0}) = H_\mu \mathbb{Z}/2$, $s_0(KO^{[-2]}_{\geqslant 0}) = H_\mu \mathbb{Z}$ and $s_0(KO^{[-3]}_{\geqslant 0}) = 0$. The case $n = 2$ is Lemma 13.

We will now use the triangle [**21**, Theorem 4.4]

$$\mathbb{G}_m \wedge KO \xrightarrow{\eta} KO \xrightarrow{f} KGL \xrightarrow{h} KO \wedge T = KO^{[1]}. \tag{3}$$

Smashing with $T^{\wedge 2}$ and applying f_0 we get (using that $T \wedge KGL \simeq KGL$)

$$f_0 KO^{[2]} \xrightarrow{f} f_0 KGL \xrightarrow{h} f_0 KO^{[3]}.$$

We claim that $h : \underline{\pi}_0^{\mathrm{eff}} f_0 KGL \to \underline{\pi}_0^{\mathrm{eff}} f_0 KO^{[3]} \in \mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$ is epi. By Proposition 5(3), it suffices to show that the induced map of Nisnevich sheaves $h : \mathbb{Z} = \underline{\pi}_0(KGL)_0 \to \underline{\pi}_0(KO^{[3]})_0 = \mathbb{Z}/2$

is an epimorphism. All symplectic forms have even rank, so the map $f : \underline{\pi}_0(KO^{[2]})_0 \to \underline{\pi}_0(KGL)_0 = \mathbb{Z}$ has image $2\mathbb{Z}$ and thus non-zero cokernel. This implies the claim. We may thus apply Lemma 15 to $\mathcal{C} = \mathbf{SH}(k)^{\mathrm{eff}}$ and this triangle. Consequently there is a triangle

$$s_0 KO^{[2]}_{\geqslant 0} \to s_0 KGL_{\geqslant 0} \to s_0 KO^{[3]}_{\geqslant 0}.$$

Note that $f_0 KGL \in \mathbf{SH}(k)^{\mathrm{eff}}_{\geqslant 0}$ since K-theory of smooth schemes is connective. Thus the triangle is isomorphic to

$$H_\mu \mathbb{Z} \xrightarrow{2} H_\mu \mathbb{Z} \to s_0 KO^{[3]}_{\geqslant 0}$$

yielding the required computation $s_0 KO^{[3]}_{\geqslant 0} \simeq H_\mu \mathbb{Z}/2$.

Throughout the proof we will keep using Proposition 5(3) all the time. To simplify notation we will no longer talk about $\underline{\pi}_i^{\mathrm{eff}}$ but only about $\underline{\pi}_i(\bullet)_0$; any statement about the latter should be understood to correspond to a statement about the former.

By a similar argument, smashing with T instead of $T^{\wedge 2}$, and using that $\mathbb{Z} = \underline{\pi}_0(KGL)_0 \xrightarrow{h} \underline{\pi}_0(KO^{[2]})_0 = \mathbb{Z}$ is an isomorphism, we conclude from

$$f_0 KO^{[1]} \to f_0 KGL \to f_0 KO^{[2]}$$

that $s_0 KO^{[1]}_{\geqslant 0} = 0$.

We have thus handled the cases $n \in \{1, 2, 3\}$. Consider the triangle

$$f_0 KO^{[3]} \to f_0 KGL \to f_0 KO^{[4]},$$

obtained by smashing triangle (3) with $T^{\wedge 3}$ and applying f_0. We have $\underline{\pi}_0(KO^{[3]})_0 = \mathbb{Z}/2$ and $\underline{\pi}_0(KGL)_0 = \mathbb{Z}$, whence $\underline{\pi}_0(KO^{[3]})_0 \to \underline{\pi}_0(KGL)_0$ must be the zero map and consequently $\underline{\pi}_1(KO^{[4]})_0 \to \underline{\pi}_0(KO^{[3]})_0$ must be epi. It follows that we may apply Lemma 15 to the rotated triangle

$$f_0 KGL[-1] \to f_0 KO^{[4]}[-1] \to f_0 KO^{[3]}.$$

Now $(E[-1])_{\geqslant 0} \simeq E_{\geqslant 1}[-1]$ and so, rotating back, we get a triangle

$$s_0 KO^{[3]}_{\geqslant 0} \to s_0 KGL_{\geqslant 1} \to s_0 KO^{[4]}_{\geqslant 1}.$$

We claim that $s_0 KGL_{\geqslant 1} = 0$. Indeed we have a triangle

$$s_0 KGL_{\geqslant 1} \to s_0 KGL_{\geqslant 0} \to s_0 \underline{\pi}_0(KGL)_*,$$

and the two terms on the right are isomorphic by what we have already said.

We thus conclude that $s_0 KO^{[0]}_{\geqslant 1} \simeq s_0 KO^{[4]}_{\geqslant 1} \simeq s_0 KO^{[3]}_{\geqslant 0}[1] \simeq H_\mu \mathbb{Z}/2[1]$. The unit map $S \to KO$ induces an isomorphism $\tilde{H}\mathbb{Z} = f_0 \underline{\pi}_0(S)_* \to f_0 \underline{\pi}_0(KO)_*$, and hence the decomposition (1) of $\tilde{s}_0 KO$ follows.

It remains to establish the second decomposition. We first show that $s_0(KO_{\geqslant 0}) = H_\mu \mathbb{Z}$. For this we consider the triangle

$$f_0 KO^{[0]} \to f_0 KGL \to f_0 KO^{[1]}$$

obtained by applying f_0 to triangle (3). Since $\underline{\pi}_0(KO^{[1]})_0 = 0$, by Lemma 15 we get a triangle

$$s_0 KO^{[0]}_{\geqslant 0} \to s_0 KGL_{\geqslant 0} \to s_0 KO^{[1]}_{\geqslant 0}$$

and we have already seen that $s_0 KO^{[1]}_{\geqslant 0} = 0$ and $s_0 KGL_{\geqslant 0} = H_\mu \mathbb{Z}$. Thus $s_0(KO_{\geqslant 0}) = H_\mu \mathbb{Z}$ as claimed.

Finally the map $\underline{K}^W_* \wedge \mathbb{G}_m \xrightarrow{\eta} \underline{K}^{MW}_* \to \underline{\pi}_0(KO)_*$ (which makes sense since $\eta h = 0$ and so $\eta : \underline{K}^{MW}_{*+1} \to \underline{K}^{MW}_*$ factors through $\underline{K}^{MW}_{*+1}/h = \underline{K}^W_{*+1}$; see also [**14**]) induces an isomorphism on $\underline{\pi}_*(\bullet)_{-1}$ and consequently

$$\mathbb{G}_m \wedge H_W\mathbb{Z} \simeq f_1(\underline{K}^W_* \wedge \mathbb{G}_m) \simeq f_1\underline{K}^{MW}_* \simeq f_1\underline{\pi}_0(KO)_*,$$

where the first equivalence is by Lemma 8. This concludes the proof. □

6. *The homotopy sheaves of $\tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$*

In this section we prove the following result.

THEOREM 17 (Morel's structure conjecture[†]). *Let k be a perfect field of characteristic different from two.*

The natural maps $\tilde{H}\mathbb{Z} = f_0\underline{K}^{MW}_* \to \underline{K}^{MW}_*$ and $H_W\mathbb{Z} = f_0\underline{K}^W_* \to \underline{K}^W_*$ induce isomorphisms on $\underline{\pi}_0(\bullet)_*$. Moreover, the natural maps $\tilde{H}\mathbb{Z} \to H_\mu\mathbb{Z}$ and $H_W\mathbb{Z} \to H_\mu\mathbb{Z}/2$ (obtained by applying f_0 to $\underline{K}^{MW}_* \to \underline{K}^{MW}_*/\eta \simeq \underline{K}^M_*$ and $\underline{K}^W_* \to \underline{K}^W_*/\eta \simeq \underline{K}^M_*/2$, respectively) induce isomorphisms on $\underline{\pi}_i(\bullet)_*$ for $i \neq 0$.

The proof will proceed through a series of lemmas. Note that $H_W\mathbb{Z}, \tilde{H}\mathbb{Z} \in \mathbf{SH}(k)_{\geqslant 0}$ by Proposition 4(3), so in the theorem only $i \geqslant 0$ is interesting.

Throughout this subsection, we fix the perfect field k of characteristic not two. Actually the only place where we explicitly use the assumption on the characteristic is in Lemma 20.

LEMMA 18. *We have $\underline{\pi}_0(\tilde{H}\mathbb{Z})_* = \underline{K}^{MW}_*$ and $\underline{\pi}_0(H_W\mathbb{Z})_* = \underline{K}^W_*$.*

Proof. This follows from the results of the second half of Section 3. Namely the homotopy modules $\underline{K}^{MW}_*$ and $\underline{K}^W_*$ are effective (Lemma 6), so

$$\underline{\pi}_0(f_0\underline{K}^{MW}_*)_* = \underline{\pi}_0(ir\underline{K}^{MW}_*)_* \cong i^\heartsuit r\underline{K}^{MW}_* \cong \underline{K}^{MW}_*,$$

(where the first equality is by definition, the second is by Proposition 5 part (1), and the third is by part (2) of that proposition and effectivity of $\underline{K}^{MW}_*$) and similarly for $\underline{K}^W_*$. □

LEMMA 19. *We have*

$$\underline{\pi}_i(\mathbb{G}_m \wedge H_W\mathbb{Z})_0 = \begin{cases} \underline{K}^W_1 & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & \text{else.} \end{cases}$$

The canonical map $H_W\mathbb{Z} \to H_\mu\mathbb{Z}/2$ induces an isomorphism on $\underline{\pi}_1(\bullet)_1$.

Proof. By Theorem 16, triangle (2) from Lemma 11 for $\tilde{s}_0KO$ reads

$$\mathbb{G}_m \wedge H_W\mathbb{Z} \to \tilde{s}_0KO \to H_\mu\mathbb{Z}.$$

We know the $\underline{\pi}_i(\tilde{s}_0KO)_0$ from Theorem 16 (use triangle (1) from Lemma 11 for $\tilde{s}_0KO$), and we know $\underline{\pi}_i(H_\mu\mathbb{Z})_0$ from the definition. We also know $\underline{\pi}_0(\mathbb{G}_m \wedge H_W\mathbb{Z})_0$ from Lemma 18, and $\underline{\pi}_i(\mathbb{G}_m \wedge H_W\mathbb{Z})_0 = 0$ for $i < 0$ by Proposition 4(3). The claim about the remaining $\underline{\pi}_i(H_W\mathbb{Z})_0$ follows from the long exact sequence of the triangle.

[†]Morel conjectured a form of this result in personal communication.

To prove the last claim, consider the diagram of homotopy modules

$$\begin{tikzcd} \underline{K}_*^M \arrow[r, "h"] \arrow[d, equal] & \underline{K}_*^{MW} \arrow[r] \arrow[d] & \underline{K}_*^W \arrow[d] \\ \underline{K}_*^M \arrow[r, "2"] & \underline{K}_*^M \arrow[r] & \underline{K}_*^M/2. \end{tikzcd}$$

Applying f_0, both sequences become exact as sequences in the abelian category $\mathbf{SH}(k)^{\mathrm{eff},\heartsuit}$. Equivalently, by Proposition 5(3), the sequences of sheaves $\mathbb{Z} \to \underline{GW} \to \underline{W}$ and $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$ are exact. Thus we get a morphism of triangles

$$\begin{tikzcd} H_\mu\mathbb{Z} \arrow[r, "h"] \arrow[d, equal] & \tilde{H}\mathbb{Z} \arrow[r] \arrow[d] & H_W\mathbb{Z} \arrow[d, "c"'] \\ H_\mu\mathbb{Z} \arrow[r, "2"] & H_\mu\mathbb{Z} \arrow[r] & H\mu\mathbb{Z}/2, \end{tikzcd}$$

where c is the canonical map we are interested in. A diagram chase (using Lemma 18) concludes that $\underline{\pi}_1(c)_1$ is an isomorphism as claimed. □

Recall that the motivic Hopf map $\eta : \mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$ induces a stable map of the same name $\eta : \mathbb{G}_m \to S$ [**12**, Section 6.2]. For $E \in \mathbf{SH}(k)$ we write E/η for a cone on the map $\mathrm{id} \wedge \eta : E \wedge \mathbb{G}_m \to E \wedge S \simeq E$. We denote by $\tau : H_\mu\mathbb{Z}/2[1] \to H_\mu\mathbb{Z}/2 \wedge \mathbb{G}_m$ the unique non-zero map [**8**, top of p. 33].

LEMMA 20. *There is an isomorphism $H_W\mathbb{Z}/\eta \simeq H_\mu\mathbb{Z}/2 \oplus H_\mu\mathbb{Z}/2[2]$, such that $H_W\mathbb{Z} \to H_\mu\mathbb{Z}/2$ is the canonical map, and the boundary map $H_\mu\mathbb{Z}/2[2] \to H_W\mathbb{Z} \wedge \mathbb{G}_m[1]$ has the property that the composite*

$$H_\mu\mathbb{Z}/2[2] \to H_W\mathbb{Z} \wedge \mathbb{G}_m[1] \to H_\mu\mathbb{Z}/2 \wedge \mathbb{G}_m[1]$$

is $\tau[1]$.

Proof. It follows from the long exact sequence of homotopy sheaves and Lemma 19 that $\underline{\pi}_i(H_W\mathbb{Z}/\eta)_0$ is given by $\mathbb{Z}/2$ if $i \in \{0, 2\}$ and is zero otherwise. Consider the triangle

$$(H_W\mathbb{Z}/\eta)_{\geqslant 1} \to H_W\mathbb{Z}/\eta \to (H_W\mathbb{Z}/\eta)_{\leqslant 0}.$$

We have $H_W\mathbb{Z}/\eta = s_0(H_W\mathbb{Z})$ (see also Proposition 23(2)) and so $H_W\mathbb{Z}/\eta$ is an *effective motive*. We may consider the above triangle as coming from the homotopy t-structure on $\mathbf{DM}^{\mathrm{eff}}$. Since the heart of the category of effective motives can be modeled as homotopy invariant sheaves with transfers (here we need the assumption on the characteristic), we find that $(H_W\mathbb{Z}/\eta)_{\geqslant 1} \simeq H_\mu\mathbb{Z}/2[2]$ and $(H_W\mathbb{Z}/\eta)_{\leqslant 0} \simeq H_\mu\mathbb{Z}/2$. Since $\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}}(H_\mu\mathbb{Z}/2, H_\mu\mathbb{Z}/2[3]) = 0$ the triangle splits.

The composite $\alpha : H_\mu\mathbb{Z}/2[2] \to H_W\mathbb{Z} \wedge \mathbb{G}_m[1] \to H_\mu\mathbb{Z}/2 \wedge \mathbb{G}_m[1]$ defines a cohomology operation of weight $(0, 1)$. By the computation of the motivic Steenrod algebra [**8**, Theorem 1.1(1)], this is either $\tau[1]$ or 0. Here again we use the assumption that $char(k) \neq 2$. Consider again the triangle $H_W\mathbb{Z} \wedge \mathbb{G}_m \to H_W\mathbb{Z} \to H_W\mathbb{Z}/\eta \simeq H_\mu\mathbb{Z}/2 \oplus H_\mu\mathbb{Z}/2[2]$, and its morphism to the triangle for $H_\mu\mathbb{Z}/2/\eta$. Since $\underline{\pi}_1(H_W\mathbb{Z})_0 = 0$ and $\underline{\pi}_1(H_W\mathbb{Z} \wedge \mathbb{G}_m) = \mathbb{Z}/2$, the boundary map $\mathbb{Z}/2 = \underline{\pi}_2(\mathbb{Z}/2[2])_0 \to \underline{\pi}_1(H_W\mathbb{Z} \wedge \mathbb{G}_m)_0$ must be an isomorphism. Since also $\underline{\pi}_1(H_W\mathbb{Z} \wedge \mathbb{G}_m)_0 \to \underline{\pi}_1(H_\mu\mathbb{Z}/2 \wedge \mathbb{G}_m)_0$ is an isomorphism by the last sentence of Lemma 19, we conclude that α is not the zero map. This was to be shown. □

We will use the following easy fact about 'split triangles'.

Lemma 21. *Let $\mathcal{C}$ be a triangulated category and*

$$A \to B \oplus C \to D$$

a triangle. Then $A \to B$ is an isomorphism if and only if $C \to D$ is an isomorphism.

In particular if $0 \to A \to B \oplus C \to D \to 0$ is an exact sequence in an abelian category, then $A \to B$ is an isomorphism if and only if $C \to D$ is an isomorphism.

Proof. Since the axioms of triangulated categories are self-dual, it suffices to show one implication. Thus suppose that $C \to D$ is an isomorphism.

Let $E \in \mathcal{C}$. Consider the long exact sequence

$$\cdots \to [E, D[-1]] \xrightarrow{\partial} [E, A] \to [E, B] \oplus [E, C] \to [E, D] \xrightarrow{\partial} [E, A[1]] \to \cdots .$$

Since $C \to D$ is an isomorphism $[E, C[i]] \to [E, D[i]]$ is surjective, and thus $\partial \equiv 0$. We hence get a short exact sequence

$$0 \to [E, A] \xrightarrow{(i,j)} [E, B] \oplus [E, C] \xrightarrow{(p\, q)} [E, D] \to 0.$$

But in such a situation surjectivity of q implies surjectivity of i, and injectivity of q implies injectivity of i. Hence $[E, A] \to [E, B]$ is an isomorphism. Since E was arbitrary, we conclude by the Yoneda lemma. $\square$

Lemma 22. *The canonical map $H_W\mathbb{Z} \to H_\mu\mathbb{Z}/2$ induces an isomorphism on $\underline{\pi}_k(\bullet)_n$ for $k \geqslant 1$ and $n \in \mathbb{Z}$.*

Proof. Consider the long exact sequence of homotopy sheaves associated with the triangle $H_W\mathbb{Z} \wedge \mathbb{G}_m \to H_W\mathbb{Z} \to H_\mu\mathbb{Z}/2 \oplus H_\mu\mathbb{Z}/2[2]$:

$$\cdots \to \underline{\pi}_{k+1}(H_W\mathbb{Z})_n \to \underline{\pi}_{k+1}(H_\mu\mathbb{Z}/2)_n \oplus \underline{\pi}_{k-1}(H_\mu\mathbb{Z}/2)_n \to \underline{\pi}_k(H_W\mathbb{Z})_{n+1}$$
$$\to \underline{\pi}_k(H_W\mathbb{Z})_n \to \underline{\pi}_k(H_\mu\mathbb{Z}/2)_n \oplus \underline{\pi}_{k-2}(H_\mu\mathbb{Z}/2)_n \to \cdots$$

We will use induction on n. If $n = 0$ (or $n \leqslant 0$) the claim is clear. We may assume by induction that $\underline{\pi}_k(H_W\mathbb{Z})_n \to \underline{\pi}_k(H_\mu\mathbb{Z}/2)_n$ is an isomorphism for $k \geqslant 1$. It follows that (still for $k \geqslant 1$) we get short exact sequences

$$0 \to \underline{\pi}_{k+1}(H_W\mathbb{Z})_n \to \underline{\pi}_{k+1}(H_\mu\mathbb{Z}/2)_n \oplus \underline{\pi}_{k-1}(H_\mu\mathbb{Z}/2)_n \to \underline{\pi}_k(H_W\mathbb{Z})_{n+1} \to 0.$$

Lemma 21 now implies that $\underline{\pi}_{k-1}(H_\mu\mathbb{Z}/2)_n \to \underline{\pi}_k(H_W\mathbb{Z})_{n+1}$ is an isomorphism. By the last part of Lemma 20 the composite

$$\underline{\pi}_{k-1}(H_\mu\mathbb{Z}/2)_n \to \underline{\pi}_k(H_W\mathbb{Z})_{n+1} \to \underline{\pi}_k(H_\mu\mathbb{Z}/2)_{n+1}$$

(where the last map is the canonical one) is τ and hence is an isomorphism for $k \geqslant 1$ by Voevodsky's resolution of the Milnor conjectures [**29**] (which implies that $H^{*,*}(k, \mathbb{Z}/2) = K_*^M/2[\tau]$).

It follows that $\underline{\pi}_k(H_W\mathbb{Z})_{n+1} \to \underline{\pi}_k(H_\mu\mathbb{Z}/2)_{n+1}$ must also be an isomorphism. This concludes the induction and hence the proof. $\square$

Proof of Theorem 17. It remains to prove the claim about $\underline{\pi}_i(\tilde{H}\mathbb{Z})_*$ for $i > 0$. The exact sequence $0 \to \underline{K}_*^{MW} \to \underline{K}_*^M \oplus \underline{K}_*^W \to \underline{K}_*^M/2 \to 0$ [**14**, Théorème 5.3] induces a triangle

$$\tilde{H}\mathbb{Z} \to H_W\mathbb{Z} \oplus H_\mu\mathbb{Z} \to H_\mu\mathbb{Z}/2.$$

By Lemma 18, the map $\underline{\pi}_0(\tilde{H}\mathbb{Z})_* \to \underline{\pi}_0(H_W\mathbb{Z} \oplus H_\mu\mathbb{Z})_*$ is just the canonical map $\underline{K}_*^{MW} \to \underline{K}_*^M \oplus \underline{K}_*^W$ and thus injective. Hence by Lemma 15 we get a triangle

$$(\tilde{H}\mathbb{Z})_{\geqslant 1} \to (H_W\mathbb{Z})_{\geqslant 1} \oplus (H_\mu\mathbb{Z})_{\geqslant 1} \to (H_\mu\mathbb{Z}/2)_{\geqslant 1}.$$

Since $(H_W\mathbb{Z})_{\geqslant 1} \to (H_\mu\mathbb{Z}/2)_{\geqslant 1}$ is an equivalence by Lemma 22, we find that $(\tilde{H}\mathbb{Z})_{\geqslant 1} \to (H_\mu\mathbb{Z})_{\geqslant 1}$ is an equivalence by Lemma 21. This concludes the proof.

The slices of $\tilde{H}\mathbb{Z}$ and $H_W\mathbb{Z}$. A minor extension of part the argument of the above proof also yields the following.

Proposition 23. (1) *The canonical map $\tilde{H}\mathbb{Z} \to H_W\mathbb{Z}$ induces an isomorphism on f_n and s_n for $n \geqslant 1$.*

(2) *For $n \geqslant 0$ we have $f_n H_W\mathbb{Z} \simeq \mathbb{G}_m^{\wedge n} H_W\mathbb{Z}$.*

(3) *We have $s_0\tilde{H}\mathbb{Z} \simeq H_\mu\mathbb{Z} \oplus H_\mu\mathbb{Z}/2[2]$, and for $n \geqslant 0$ we have $s_n H_W\mathbb{Z} \simeq \mathbb{G}_m^{\wedge n}(H_\mu\mathbb{Z}/2 \oplus H_\mu\mathbb{Z}/2[2])$.*

Proof. Since $\underline{\pi}_i(\tilde{H}\mathbb{Z})_{-n} \to \underline{\pi}_i(H_W\mathbb{Z})_{-n}$ is an isomorphism for $n \geqslant 1$, claim (1) is clear. For claim (2), it follows from Lemma 8 that $f_n H_W\mathbb{Z} \simeq \mathbb{G}_m^{\wedge n} f_0(H_W\mathbb{Z} \wedge \mathbb{G}_m^{\wedge -n})$. But $\eta^n : H_W\mathbb{Z} \to H_W\mathbb{Z} \wedge \mathbb{G}_m{-}n$ induces an isomorphism on $\underline{\pi}_i(\bullet)_0$ for $n \geqslant 0$, so $f_0(H_W\mathbb{Z} \wedge \mathbb{G}_m^{\wedge -n}) \simeq f_0(H_W\mathbb{Z}) = H_W\mathbb{Z}$.

To prove (3), using the triangle $\tilde{H}\mathbb{Z} \to H_W\mathbb{Z} \oplus H_\mu\mathbb{Z} \to H_\mu\mathbb{Z}/2$ one finds using Lemma 21 that the claim for $s_0\tilde{H}\mathbb{Z}$ reduces to the one for $H_W\mathbb{Z}$. By (2), $s_n H_W\mathbb{Z} \simeq (\mathbb{G}_m^{\wedge n+1} H_W\mathbb{Z})/\eta \simeq \mathbb{G}_m^{\wedge n+1}(H_W\mathbb{Z}/\eta)$, and $H_W\mathbb{Z}/\eta \simeq H_\mu\mathbb{Z}/2 \oplus H_\mu\mathbb{Z}/2[2]$ by Lemma 20. □

Acknowledgements. The main impetus for writing this article was a talk by Marco Schlichting where he presented his results about a Grayson-type spectral sequence for Hermitian K-theory, and Markus Spitzweck's question if this spectral sequence can be recovered using the generalized slice filtration.

I would also like to thank Fabien Morel for teaching me most of the things I know about motivic homotopy theory, and for encouragement and support during my attempts at proving these results. Finally I thank Benjamin Antieau for comments on a draft of this article.

References

1. A. A. Beilinson, J. Bernstein and P. Deligne, 'Faisceaux pervers', *Analysis and topology on singular spaces, I (Luminy, 1981)*, Astérisque 100 (Société Mathématique de France, Paris, 1982) 5–171.
2. B. Calmès and J. Fasel, 'Finite Chow-Witt correspondences', Preprint, 2017, arXiv:1412.2989.
3. G. Garkusha and I. Panin, 'On the motivic spectral sequence', *J. Inst. Math. Jussieu* (2015), https://doi.org/10.1017/S1474748015000419.
4. G. Garkusha and I. Panin, 'Homotopy invariant presheaves with framed transfers', Preprint, 2015, arXiv:1504.00884.
5. D. R. Grayson, 'The motivic spectral sequence', *Handbook of K-theory* (Springer, Berlin–New York, 2005) 39–69.
6. J. Hornbostel, 'A1-representability of Hermitian K-theory and Witt groups', *Topology* 44 (2005) 661–687.
7. M. Hoyois, 'The six operations in equivariant motivic homotopy theory', *Adv. Math.* 305 (2017) 197–279.
8. M. Hoyois, S. Kelly and P. Arne Østvær, 'The motivic Steenrod algebra in positive characteristic', *J. Eur. Math. Soc.*, to appear.
9. M. Levine, 'The homotopy coniveau tower', *J. Topol.* 1 (2008) 217–267.
10. J. Lurie, *Higher algebra*, 2017, http://www.math.harvard.edu/~lurie/papers/HA.pdf.
11. C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology* (American Mathematical Society, Providence, RI, 2006).
12. F. Morel, 'An introduction to $\mathbb{A}^1$-homotopy theory', *ICTP Trieste Lecture Note Ser.* 15 (2003) 357–441.
13. F. Morel, 'On the motivic π_0 of the sphere spectrum', *Axiomatic, enriched and motivic homotopy theory* (Springer, Berlin, 2004) 219–260.
14. F. Morel, 'Sur les puissances de l'idéal fondamental de l'anneau de Witt', *Comment. Math. Helv.* 79 (2004) 689–703.
15. F. Morel, 'The stable $\mathbb{A}^1$-connectivity theorems', *K-Theory* 35 (2005) 1–68.
16. F. Morel, 'On the Friedlander–Milnor conjecture for groups of small rank', *Current developments in mathematics* (International Press, Somerville, MA, 2011) 45–93.
17. F. Morel, *$\mathbb{A}^1$-Algebraic Topology over a Field*, Lecture Notes in Mathematics (Springer, Berlin–Heidelberg, 2012).

18. F. Morel and V. Voevodsky, '$\mathbb{A}^1$-homotopy theory of schemes', *Publ. Math. Inst. Hautes Études Sci.* 90 (1999) 45–143.

19. I. Panin and C. Walter, 'On the motivic commutative ring spectrum BO', Preprint, 2010, arXiv:1011.0650.

20. I. Panin and C. Walter, 'Quaternionic Grassmannians and Pontryagin classes in algebraic geometry', Preprint, 2010, arXiv:1011.0649.

21. O. Röndigs and P. A. Østvær, 'Slices of hermitian K-theory and Milnor's conjecture on quadratic forms', *Geom. Topol.* 20 (2016) 1157–1212.

22. M. Schlichting, 'The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes', *Invent. Math.* 179 (2010) 349–433.

23. M. Schlichting, 'Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem', *J. Pure Appl. Algebra* 221 (2017) 1729–1844.

24. M. Schlichting and G. S. Tripathi, 'Geometric models for higher Grothendieck–Witt groups in $\mathbb{A}^1$-homotopy theory', *Math. Ann.* 362 (2015) 1143–1167.

25. M. Spitzweck and P. A. Østvær, 'Motivic twisted K–theory', *Algebr. Geom. Topol.* 12 (2012) 565–599.

26. A. Suslin, 'On the Grayson spectral sequence', *Proceedings of the Steklov Institute of Mathematics-Interperiodica Translation* 241 (2003) 202–237.

27. V. Voevodsky, 'Open problems in the motivic stable homotopy theory, I', *International Press Conference on Motives, Polylogarithms and Hodge Theory* (International Press, Somerville, MA, 2002).

28. V. Voevodsky, 'A possible new approach to the motivic spectral sequence for algebraic K-theory', *Recent progress in homotopy theory* (*Baltimore, MD, 2000*), Contemporary Mathematics 293 (American Mathematical Society, Providence, RI, 2002) 371–379.

29. V. Voevodsky, 'Motivic cohomology with $\mathbf{Z}/2$-coefficients', *Publ. Math. Inst. Hautes Études Sci.* 98 (2003) 59–104.

30. V. Voevodsky, 'On the zero slice of the sphere spectrum', *Proc. Steklov Inst. Math.* 246 (2004) 93–102.

31. C. Walter, 'Grothendieck-Witt groups of triangulated categories', Preprint, 2003, https://faculty.math.illinois.edu/K-theory/0643/TriGW.pdf.

Tom Bachmann
Fakultät Mathematik
Universität Duisburg-Essen
Thea-Leymann-Straße 9
45127 Essen
Germany

tom.bachmann@zoho.com