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Algebraic chromatic homotopy theory for BP_*BP-comodules

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Abstract

In this paper, we study the global structure of an algebraic avatar of the derived category of ind-coherent sheaves on the moduli stack of formal groups. In analogy with the stable homotopy category, we prove a version of the nilpotence theorem as well as the chromatic convergence theorem, and construct a generalized chromatic spectral sequence. Furthermore, we discuss analogs of the telescope conjecture and chromatic splitting conjecture in this setting, using the local duality techniques established earlier in joint work with Valenzuela.

Contents

1. *Introduction*

The chromatic approach to stable homotopy theory is a powerful tool both for understanding the local and global structure of the stable homotopy as well as for making explicit computations. The goal of this paper is to study an algebraic version of this theory, based on the category of BP_*BP-comodules. As such, it is deeply intertwined with recent efforts to implement the chromatic perspective in motivic homotopy theory as well as in more algebraic contexts.

More specifically, we work with a suitable version Stable_{BP_*BP} of the derived category of BP_*BP-comodules, which is an algebraic avatar of the category of ind-coherent sheaves on the moduli stack of formal groups. This category was introduced by Hovey [**26**, **27**] and in related work of Palmieri [**55**], and then further studied by the authors and Valenzuela [**8**, Section 8]. From an axiomatic point of view, Stable_{BP_*BP} is a prominent example of a non-Noetherian stable homotopy theory in the sense of [**28**], so that many of the standard techniques do not apply. The importance of this category is due to the fact that it sits at the intersection of three different areas, so that its local and global structure provides new insights in each of them:

(1) *Stable homotopy theory: As an approximation to the category of spectra.* Many structural patterns of the stable homotopy category are visible through the lens of the Adams–Novikov spectral sequence (ANSS). The E_2-term of this spectral sequence for the sphere

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spectrum is isomorphic to $\pi_* BP_*$ in Stable_{BP_*BP}, so that this category is a very close algebraic approximation to the category of spectra. In particular, the chromatic filtration in stable homotopy theory provides a filtration of Stable_{BP_*BP} by full subcategories $\mathrm{Stable}_{E(n)_*E(n)}$, whose suitably defined limit over p is essentially equivalent to the limit of the category of $E(n)$-local spectra [**9**].

(2) *Algebraic geometry: The relationship to ind-coherent sheaves on the moduli stack of formal groups $\mathcal{M}_{fg}$.* By work of Quillen [**58**] there is a close connection between stable homotopy theory and the theory of formal groups. More specifically, our results can be translated into properties of the category of ind-coherent sheaves over a certain moduli stack $\mathcal{M}_{fg}$ of formal groups. The stack $\mathcal{M}_{fg}$ is stratified by height, and this height filtration corresponds to the chromatic filtration in stable homotopy. Thus, studying the category $\mathrm{Stable}_{E(n)_*E(n)}$ corresponds to geometrically studying ind-coherent sheaves on open substacks of $\mathcal{M}_{fg}$. One may therefore consider Stable_{BP_*BP} as a toy example of a category of ind-coherent sheaves on stratified stacks, which are for instance relevant in the geometric Langlands program [**19**].

(3) *Motivic homotopy theory: Motivic module spectra over the cofiber of τ.* Via work of Isaksen [**34**], $\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*)$ also appears naturally in motivic homotopy theory as the homotopy groups of $C\tau$, the motivic cofiber of τ over $\mathrm{Spec}(\mathbb{C})$.† Joint work of Gheorghe–Xu–Wang (a forthcoming paper) shows that this isomorphism extends to an equivalence between Stable_{BP_*BP} and a category closely related to the category $\mathrm{Mod}^{\mathrm{cell}}_{C\tau}$ of cellular motivic $C\tau$-modules. Thus, our results about Stable_{BP_*BP} can be translated to results in the stable motivic homotopy category.

This exhibits Stable_{BP_*BP} as an important test case for the more in-depth study of related categories in these areas.

Main results

In [**52**] Miller, Ravenel and Wilson introduced the chromatic spectral sequence, which converges to the E_2-term of the ANSS. Based on systematic algebraic patterns seen in this work, Ravenel was lead to his famous nilpotence and periodicity conjectures [**59**], later proved by Devinatz, Hopkins and Smith [**14**, **23**], giving rise to the field of chromatic homotopy theory. In this paper we develop and prove algebraic analogs of Ravenel's conjectures in the category of BP_*BP-comodules.

As noted previously we work with the category Stable_{BP_*BP} instead of the usual derived category $\mathcal{D}_{BP_*BP}$. As is already clear from work of Hovey [**26**], the usual derived category is homotopically poorly behaved; for example, the tensor unit BP_* is not compact, and this necessitates working with the more complicated category Stable_{BP_*BP}.

In order to construct Stable_{BP_*BP} we must first study the abelian category of BP_*BP-comodules. We do this in more generality in Section 2, by recalling some basic properties of the abelian category of comodules over a flat Hopf algebroid. We quickly specialize to the case of BP_*BP and $E(n)_*E(n)$, giving a classification of hereditary torsion theories for $\mathrm{Comod}_{E(n)_*E(n)}$. In Section 3 we recall the construction of the stable category Stable_Ψ associated to a flat Hopf algebroid, and give a change of rings theorem for Hopf algebroids associated to faithfully flat extensions.

With the stable category associated to a flat Hopf algebroid defined, we move on to the study of the global structure of Stable_{BP_*BP} and $\mathrm{Stable}_{E(n)_*E(n)}$. On the abelian level, the structure of the category of $E(n)_*E(n)$-comodules is known to be much simpler when the prime is large compared to n. For example, $\pi_*E(n)_* \cong \mathrm{Ext}^s_{E(n)_*E(n)}(E(n)_*, E(n)_*)$ vanishes for $s > n^2 + n$,

†Recall that, working in the p-complete setting, the motivic cohomology of a point over $\mathrm{Spec}(\mathbb{C})$ is isomorphic to $\mathbb{F}_p[\tau]$, where τ has bidegree (0,1), and that this gives rise to an essential map $\tau\colon S^{0,-1} \to S^{0,0}$.

whenever $p > n+1$. This is reflected in Theorem 4.11, where we prove the following; here we denote by $(K(n)_*, \Sigma(n))$ the Hopf algebroid studied extensively by Miller, Ravenel and Wilson.

THEOREM. *If $p > n+1$, then there is an equivalence $\mathrm{Stable}_{E(n)_*E(n)} \simeq \mathcal{D}_{E(n)_*E(n)}$, between the stable category of $E(n)_*E(n)$-comodules and the usual derived category of $E(n)_*E(n)$-comodules. Similarly, if n does not divide $p-1$, then there is an equivalence $\mathrm{Stable}_{\Sigma(n)} \simeq \mathcal{D}_{\Sigma(n)}$.*

In stable homotopy theory, the Morava K-theories $K(n)$ detect nilpotence. In our algebraic setting we use the BP_*BP-comodule $\mathrm{Tel}(n)_* = v_n^{-1}BP_*/I_n$ as our detecting family for nilpotence, proving the following version of the nilpotence theorem in Section 5. This result crucially relies on the use of the category Stable_{BP_*BP} instead of the derived category, as here BP_* is compact.

THEOREM (Algebraic nilpotence theorem — weak version).

(1) *Suppose $F, X \in \mathrm{Stable}_{BP_*BP}$ with F compact, then a map $f\colon F \to X$ is smash nilpotent, that is, $f^{(m)} = 0$ for some $m \gg 0$, if $\mathrm{Tel}(n)_* \otimes_{BP_*} f = 0$ for all $0 \leqslant n \leqslant \infty$.*

(2) *A self-map $f\colon \Sigma^i F \to F$ for F compact in Stable_{BP_*BP} is nilpotent, in the sense that $f^j\colon \Sigma^{ij}F \to F$ is null for some $j \gg 0$, if and only if $\mathrm{Tel}(n)_* \otimes_{BP_*} f$ is nilpotent for all $0 \leqslant n \leqslant \infty$.*

(3) *Suppose $X \in \mathrm{Stable}_{BP_*BP}$, then a map $f\colon BP_* \to X$ is smash nilpotent if $\pi_*(\mathrm{Tel}(n)_* \otimes_{BP_*} f) = 0$ for all $0 \leqslant n \leqslant \infty$.*

(4) *Let R be a ring object in Stable_{BP_*BP}. Then an element $\alpha \in \pi_*R \cong \mathrm{Ext}_{BP_*BP}(BP_*, R)$ is nilpotent if and only if $\pi_*(\mathrm{Tel}(n)_* \otimes_{BP_*} \alpha)$ is nilpotent for all $0 \leqslant n \leqslant \infty$.*

We call this a weak version of the algebraic nilpotence theorem, because the results are not as strong as those in [**23**]. Indeed, they do not account for all periodic elements in π_*BP_*, but only those appearing in Adams–Novikov filtration 0, which is a reflection of the fact that $\mathrm{Tel}(n)_*$ is not a field object. Indeed, the nilpotence theorem implies that there is a vanishing curve on the E_∞-page of the ANSS for the sphere that has slope tending to zero as $t-s$ approaches ∞. However, this vanishing curve is not present on the E_2-page and in fact there are non-nilpotent elements of positive Adams–Novikov filtration. It follows that there are many more non-nilpotent elements in Stable_{BP_*BP} than in stable homotopy theory. This complicates the structure of Stable_{BP_*BP}; there appear to be many more thick subcategories than in stable homotopy theory. We will return to the systematic study of self-maps and thick subcategories of Stable_{BP_*BP} in forthcoming work with Achim Krause.

One formulation of the telescope conjecture in stable homotopy is that $\langle \mathrm{Tel}(m)\rangle = \langle K(m)\rangle$ [**24**], where $\langle \mathrm{Tel}(m)\rangle$ denotes the Bousfield class of the telescope of a finite spectrum of type m. Since $K(n)_*$ is not a BP_*BP-comodule, strictly speaking this question does not make sense in Stable_{BP_*BP}. Nonetheless, it is a BP_*-module, and so one can formulate a variant of the telescope conjecture, which we show in Theorem 5.13 does hold. This gives some explanation for the use of $\mathrm{Tel}(n)_*$ in the nilpotence theorem above.

THEOREM. *For all $n \geqslant 0$, there is an identity of Bousfield classes $\langle K(n)_*\rangle = \langle \mathrm{Tel}(n)_*\rangle$.*

In Section 6 we move on to the study of the local structure of Stable_{BP_*BP}. We begin by constructing localization functors L_n for $0 \leqslant n \leqslant \infty$ which in particular define an exhaustive filtration of the full subcategory of compact objects. Their essential images $L_n\,\mathrm{Stable}_{BP*BP}$ are algebraic counterparts of the categories of $E(n)$-local spectra, which in turn form the building blocks of chromatic homotopy theory. In geometric terms, L_n corresponds to the restriction to an open substack of $\mathcal{M}_{fg}$. Such functors have previously been studied by Hovey and Strickland [27, 31, 32], who proved that there is an equivalence of categories between $L_n \operatorname{Stable}_{BP_*BP}$ and $\operatorname{Stable}_{E(n)_*E(n)}$. As Hovey points out in [27, p. 171] an alternative formulation of the telescope conjecture, namely that L_n is the same as the Bousfield localization with the homology theory associated with $E(n)_*$, is false in general in $\operatorname{Stable}_{BP_*BP}$, however we note that this holds when $n < p-1$, see Remark 6.9.

The algebraic localization functors L_n assemble into the algebraic chromatic tower

$$\ldots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0,$$

precisely as in stable homotopy theory. Hopkins and Ravenel have shown [61] that a compact spectrum is the limit of its chromatic tower. We prove the following variant of this in Section 7.

Theorem (Chromatic convergence). *If $M \in \operatorname{Stable}_{BP_*BP}$ has finite projective dimension, then there is a natural equivalence $M \simeq \lim_n L_n M$.*

In particular, this implies that all compact objects of $\operatorname{Stable}_{BP_*BP}$ satisfy chromatic convergence. The strength of this algebraic chromatic convergence theorem is akin to that of the first author's generalization of the chromatic convergence theorem in stable homotopy theory [5]. We also show that $\lim_n L_n \simeq L_\infty$ where the latter is the localization functor associated to $BP_*/I_\infty \cong \mathbb{Z}/p$.

The Bousfield–Kan spectral sequence associated to the chromatic tower in stable homotopy leads to a spectral sequence of the form $E_1 = \pi_k M_n S^0 \Rightarrow \pi_k S^0$, where M_n is the fiber of $L_n \to L_{n-1}$. Associated to the algebraic chromatic tower, we can similarly construct a spectral sequence. This recovers, and indeed generalizes, the classical chromatic spectral sequence, which is obtained by setting $X = Y = S^0$.

Theorem (The chromatic spectral sequence). *For any spectra X, Y, there is a natural convergent spectral sequence*

$$E_1^{n,s,t} = \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*X, M_n BP_*Y) \Rightarrow \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*X, L_\infty BP_*Y).$$

Furthermore, if BP_*Y satisfies the conditions of Section 1, then the spectral sequence converges to $\operatorname{Ext}_{BP_*BP}(BP_*X, BP_*Y)$.

By truncating the chromatic tower, we can also build a height n analog of the chromatic spectral sequence which, as a special case, recovers the truncated chromatic spectral sequence constructed by Hovey and Sadofsky [29, Theorem 5.1].

As a concrete application of our results, we obtain the following transchromatic comparison between the E_2-terms of the BP-Adams spectral sequence and the E-Adams spectral sequence at height n, see Corollary 7.19.

Corollary. *If X is a p-local bounded below spectrum such that BP_*X has projective BP_*-dimension $\operatorname{pdim}(BP_*X) \leqslant r$, then the natural map*

$$\operatorname{Ext}^s_{BP_*BP}(BP_*, BP_*X) \longrightarrow \operatorname{Ext}^s_{E_*E}(E_*, E_*X)$$

is an isomorphism if $s < n-r-1$ and injective for $s = n-r-1$.

A related result can be found in work of Goerss [20, Theorem 8.24], however the authors are unaware of a result of this generality in the literature.

Relation to other work

The present paper is a natural continuation of work of Hovey and Strickland [**27**, **31**, **32**] as well as unpublished work of Goerss [**20**]. In contrast to our algebraic approach, Goerss works more geometrically, studying the derived category of quasi-coherent sheaves on the moduli stack of p-typical formal group laws. However, both approaches are equivalent and consequently some of our results are equivalent to those obtained by Goerss. For example, Goerss' chromatic convergence theorem [**20**, Theorem 8.22] translates into a special case of Theorem 7.12. Similar geometric approaches have been studied by Hollander [**22**], Naumann [**53**], Pribble [**57**], Sitte [**64**] and Smithling [**65**].

Conventions

In this paper we work with stable ∞-categories in the quasi-categorical setting developed by Joyal [**38**] and Lurie [**44**, **45**]. For simplicity, we will refer to a quasi-category as an ∞-category throughout this paper.

Unless otherwise noted, all categorical constructions are implicitly considered derived. For example, the tensor product $\otimes$ usually refers to the derived tensor product, and limits and colimits mean homotopy limits and homotopy colimits, respectively. The symbol $\boxtimes$ is reserved for the underived tensor product.

If $\mathcal{C}$ is a closed symmetric monoidal stable ∞-category, the internal function object will be denoted by $\underline{\mathrm{Hom}}_{\mathcal{C}}$ to distinguish it from the merely spectrally enriched categorical mapping object $\mathrm{Hom}_{\mathcal{C}}$. This is related to the usual mapping space via a natural weak equivalence $\Omega^\infty \mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq \mathrm{Map}_{\mathcal{C}}(X,Y)$. If no confusion is likely to arise, the subscript $\mathcal{C}$ will be omitted from the notation.

When dealing with chain complexes, we will always employ homological grading, that is, complexes are written as

$$\dots \xrightarrow{d} X_1 \xrightarrow{d} X_0 \xrightarrow{d} X_{-1} \xrightarrow{d} \dots$$

with the differential d lowering degree by 1. As usual, taking cohomology of a chain complex X reverses the sign of the homology, that is $H^*(X) = H_{-*}(X)$.

We work with Hopf algebroids (A,Ψ) over a commutative ring K throughout; that is, A and Ψ are both commutative K-algebras. We will always assume that Ψ is a flat A-module, and we call such Hopf algebroids flat.

2. *Hopf algebroids and the structure of* Comod_{BP_*BP}

In this section we will prove some basic results about the abelian category Comod_Ψ of comodules over a flat Hopf algebroid (A,Ψ). We assume the reader is familiar with the notion of comodules over a Hopf algebroid, for which good references include [**26**; **60**, Appendix A]. We finish with a classification of the hereditary torsion theories for Landweber exact BP_*-algebras of height n, extending work of Hovey and Strickland [**31**].

We note that since we work with abelian categories in this section, all functors are assumed to be underived.

2.1. *Recollections on Hopf algebroids and comodules*

Given a flat Hopf algebroid (A,Ψ) over a commutative ring K (that is, A and Ψ are commutative K-algebras), we will write Comod_Ψ for the abelian category of Ψ-comodules. The following proposition, which is essentially a compendium of results in [**26**, Section 1], establishes the basic properties of the category Comod_Ψ.

PROPOSITION 2.1. *The abelian category* Comod_Ψ *of comodules over a flat Hopf algebroid* (A, Ψ) *is a complete and cocomplete locally presentable Grothendieck abelian category with a closed symmetric monoidal structure. A* Ψ*-comodule is compact or dualizable if and only if the underlying* A*-module is finitely presented or finitely presented and projective, respectively.*[†] *Moreover, the forgetful functor*

$$\epsilon_*\colon \mathrm{Comod}_\Psi \longrightarrow \mathrm{Mod}_A$$

is exact, faithful, symmetric monoidal and preserves all colimits. The corresponding right adjoint ϵ^**, which sends an* A*-module* M *to the cofree* Ψ*-comodule* $\Psi \otimes M$*, is exact as well.*

Given Ψ-comodules M and N, we write $M \otimes_A N$ for the monoidal product, and $\underline{\mathrm{Hom}}_\Psi(M, N)$ for the internal Hom object. We will often omit the subscript if it is clear from context. No confusion should arise with the use of $M \otimes_A N$; for example, given a Ψ-comodule M, $\Psi \otimes_A M$ could be interpreted as the extended comodule on M or as the symmetric monoidal product of the comodules Ψ and M, but these turn out to be the naturally isomorphic, see [**26**, Lemma 1.1.5].

We say that a Ψ-comodule I is relatively injective if $\mathrm{Hom}_\Psi(-, I)$ takes A-split short exact sequences to short exact sequences.

LEMMA 2.2. *A* Ψ*-comodule* M *is injective if and only if it is a retract of an extended comodule* $\Psi \otimes I$ *on an injective* A*-module* I*. A comodule* M *is relatively injective if and only if it is a retract of an extended comodule.*

Proof. See, for example, [**32**, Lemma 2.1] for the first statement, and [**26**, Lemma 3.1.2] for the latter. □

The link between Hopf algebroids and topology arises from the observation that if F is a ring spectrum with F_*F flat over F_*, then $(F_, F_*F)$ is a flat Hopf algebroid, and F_*X is an F_*F-comodule for any spectrum X. We will be particularly interested in Hopf algebroids that are Landweber exact over BP_* in the following sense, see [**31**, Definitions 2.1 and 4.1]

DEFINITION 2.3. Suppose $f\colon BP_* \to E_*$ is a ring homomorphism, then E_* is said to be a Landweber exact BP_*-algebra of height $n \in \mathbb{N} \cup \{\infty\}$ if the following conditions are satisfied.

(1) The BP_*-module E_*/I_n is non-zero, and $E_*/I_k \cong 0$ for all $k > n$. If E_*/I_n is non-zero for all n, then the height is set to be ∞.
(2) The functor from BP_*BP-comodules to E_*-modules induced by $M \mapsto E_* \otimes_{BP_*} M$ is exact.

Typical examples include Johnson–Wilson theories $E(n)_*$, Morava E-theory $(E_n)_*$ and $v_n^{-1}BP_*$, all of which have height n, see also Section 2.4. Given such an E_*, we can associate a Hopf algebroid $(E_*, E_*E) = (E_*, E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_*)$. By [**31**, Theorem C] any two Landweber exact BP_*-algebras of the same height have equivalent categories of comodules.

Recall that a flat Hopf algebroid (A, Ψ) is an Adams Hopf algebroid if Ψ is a filtered colimit of comodules Ψ_i that are finitely generated and projective as A-modules. By a theorem of Schäppi [**62**, Theorem 1.3.1], the Adams condition is equivalent to the statement that Comod_Ψ is generated by dualizable comodules. The following result is due to Hovey [**26**, Section 1.4].

[†] Recall that a projective A-module is finitely generated if and only if it is finitely presented.

Proposition 2.4 (Hovey). *The Hopf algebroids* (BP_*, BP_*BP) *and* (E_*, E_*E), *where* E_* *is any height* n *Landweber exact* BP_**-algebra, are Adams Hopf algebroids, so the corresponding category of comodules is generated by the dualizable comodules.*

2.2. *The cotensor product and Cotor*

In this section we recall some basic facts about the cotensor product and its derived functor Cotor. Our approach is slightly non-standard, in that we prefer to work with a relative version of Cotor, which agrees with the one defined using standard homological algebra only when the first variable is flat.

To begin, recall that given a right Ψ-comodule M and a left Ψ-comodule N, the cotensor product $M\square_\Psi N$ is defined as the equalizer

$$M\square_\Psi N \longrightarrow M \otimes_A N \underset{1\otimes\psi_N}{\overset{\psi_M\otimes 1}{\rightrightarrows}} M \otimes_A \Psi \otimes_A N.$$

Note that this only inherits the structure of a K-module.

Lemma 2.5. *If* $N = \Psi \otimes_A N'$ *is an extended comodule, then* $M\square_\Psi N \cong M \otimes_A N'$.

Proof. It is easy to check that the map $M \otimes_A N' \xrightarrow{\Psi_M\otimes 1} M \otimes_A \Psi \otimes_A N'$ is the kernel of $\psi_M \otimes 1 - 1 \otimes \psi_N$. $\square$

The following definition naturally arises when using the methods of relative homological algebra, see [**7**, Section 2; **15**, p. 15].

Definition 2.6. A proper injective resolution of a comodule M is a resolution of M by relative injectives such that each map in the resolution is split as map of A-modules.

Such resolutions always exist; indeed, for a comodule M, the standard cobar resolution

$$C^*_\Psi(M) = \left(M \xrightarrow{\psi_M} \Psi \otimes_A M \underset{1\otimes\Psi_M}{\overset{\Delta\otimes 1}{\rightrightarrows}} \Psi \otimes_A \Psi \otimes_A M \Rrightarrow \cdots \right)$$

is a resolution by relative injectives, which is split using the map $\epsilon\colon \Psi \to A$. Moreover, such resolutions are unique up to chain homotopy [**33**, Theorem 2.2].

Definition 2.7. Given Ψ-comodules M and N, let $0 \to M \xrightarrow{i} J^\bullet$ and $0 \to N \xrightarrow{i'} L^\bullet$ be proper injective resolutions. We define

$$\mathrm{Cotor}^n_\Psi(M, N) = H^n(\mathrm{Tot}^\oplus(J^\bullet\square_\Psi L^\bullet)),$$

where $\mathrm{Tot}^\oplus$ is the totalization of the bicomplex with respect to the direct sum.

The next result says that it is enough to take a resolution of either of the variables.

Lemma 2.8. *The maps*

$$J^\bullet\square_\Psi N \xrightarrow{1\square i'} J^\bullet\square_\Psi L^\bullet \xleftarrow{i\square 1} M\square_\Psi J^\bullet$$

induce isomorphisms on homology.

Proof. This is proved in the context of comodules over a coalgebra in [**16**]. By Lemma 2.2 each J^i is a retract of an extended comodule $\tilde{J}^i \otimes_A \Psi$, so we have $J^i \square_\Psi N \cong \tilde{J}^i \otimes_A N$ by Lemma 2.5. It follows that $J^i \square_\Psi (-)$ preserves A-split exact sequences.

Filter $J^\bullet$ by $F_n(J^\bullet) = J^{\leqslant n}$, which induces filtrations on $J^\bullet \square_\Psi N$ and $J^\bullet \square_\Psi L^\bullet$. One checks, using the fact that $J^i \square_\Psi (-)$ preserves A-split exact sequences, that in the associated spectral sequence the map $J^\bullet \square_\Psi N \xrightarrow{1\square i} J^\bullet \square_\Psi L^\bullet$ induces an isomorphism on E_1-terms, and so is an isomorphism on homology. The argument for $i\square 1$ is similar. □

REMARK 2.9. Note that this result implies that $\mathrm{Cotor}^*_\Psi(M,N)$ is independent of the choice of resolution of M or N.

Since Comod_Ψ has enough injectives, it is more customary to define $\mathrm{Cotor}^*_\Psi(M,N)$ by taking an injective resolution I^* of N to construct the derived functors of $M\square_\Psi -$. Let us temporarily write $\widetilde{\mathrm{Cotor}}_\Psi(M,N)$ for this functor.

LEMMA 2.10. *For $M, N \in \mathrm{Comod}_\Psi$ with M flat, then there is an isomorphism*

$$\mathrm{Comod}^*_\Psi(M,N) \cong \widetilde{\mathrm{Comod}}^*_\Psi(M,N).$$

Proof. This is proved in [**60**, Lemma A1.2.8]: Ravenel assumes M projective, but it is clear from the proof that M flat is sufficient. □

We prefer to use the relative version of Cotor since it allows us to dispense with flatness hypothesis in certain results, such as Lemma 2.12.

Given a left (respectively, right) Ψ-comodule, we can always turn it into a right (respectively, left) Ψ-comodule, by conjugating the action by the antipode χ of Ψ. We use that implicitly in the next result.

LEMMA 2.11. *Suppose given two comodules $M, N \in \mathrm{Comod}_\Psi$, then there is a natural isomorphism*

$$M\square_\Psi N \xrightarrow{\sim} A\square_\Psi (M \otimes_A N)$$

of K-modules.

Proof. This follows by comparing the coequalizers defining the two cotensor products, and a careful diagram chase. We note that if we write $\psi_M(m) = \Sigma_i m_i \otimes x_i$ and $\psi_N(n) = \Sigma_j y_j \otimes n_j$, then the comodule structure map on $M \otimes_A N$ is given by $\psi_{M\otimes_A N}(m \otimes n) = \Sigma_{i,j}(\chi(x_i)y_j \otimes m_i \otimes n_j)$. □

This leads to the following.

LEMMA 2.12. *Suppose given two comodules $M, N \in \mathrm{Comod}_\Psi$, then there is a natural isomorphism*

$$\mathrm{Cotor}^*_\Psi(M,N) \xrightarrow{\sim} \mathrm{Cotor}^*_\Psi(A, M \otimes_A N)$$

of K-modules.

Proof. As noted above, given a comodule X, the Ψ-cobar complex $C^*_\Psi(X)$ is a proper injective resolution, and so can be used to compute Cotor. The isomorphism

$M \otimes_A N \cong M\square_\Psi(\Psi \otimes_A N)$ of Lemma 2.5, along with Lemma 2.11 shows that

$$\begin{aligned} M\square_\Psi C^k_\Psi(N) &= M\square_\Psi(\Psi \otimes_A \Psi^{\otimes k} \otimes_A N) \\ &\cong M \otimes_A \Psi^{\otimes k} \otimes_A N \\ &\cong \Psi\square_\Psi(M \otimes_A \Psi^{\otimes k} \otimes_A N) \\ &\cong A\square_\Psi(M \otimes_A \Psi^{\otimes (k+1)} \otimes_A N). \end{aligned}$$

for all k. This leads to a quasi-isomorphism

$$M\square_\Psi C^*_\Psi(N) \simeq A\square_\Psi C^*_\Psi(M \otimes_A N),$$

hence the desired isomorphism of Cotor groups. $\square$

REMARK 2.13. If we were to use $\widetilde{\mathrm{Cotor}}^*_\Psi(M,N)$ instead of $\mathrm{Cotor}^*_\Psi(M,N)$, then we only know how to prove this when M is a flat A-module.

2.3. *Hereditary torsion theories*

In this subsection we give a brief introduction to hereditary torsion theories, and prove a result relating hereditary torsion theories under certain localizations of categories. We use the terminology of hereditary torsion theories in order to distinguish it from the notion of localizing subcategory used in the context of stable ∞-categories in later sections.

DEFINITION 2.14. Let $\mathcal{A}$ be a cocomplete abelian category. A full subcategory $\mathcal{T}$ of $\mathcal{A}$ is said to be a hereditary torsion theory if it is closed under subobjects, quotient objects, extensions and arbitrary coproducts in $\mathcal{A}$.

We recall that given a class of maps $\mathcal{E}$ in a category $\mathcal{A}$, we say that an object $M \in \mathcal{A}$ is $\mathcal{E}$-local if $\mathrm{Hom}_\mathcal{A}(f, M)$ is an isomorphism for all $f \in \mathcal{E}$, and we denote the full subcategory of $\mathcal{E}$-local objects by $L_\mathcal{E}\,\mathcal{A}$. Given such a class of maps it is known (for example, by [**66**]) that there exists a localization functor $L\colon \mathcal{A} \to \mathcal{A}$ such that for each $M \in \mathcal{A}$ we have $LM \in L_\mathcal{E}\,\mathcal{A}$.

Given a hereditary torsion theory $\mathcal{T}$, we let $\mathcal{E}_\mathcal{T}$ denote the class of $\mathcal{T}$-equivalences, that is, those maps whose kernel and cokernel are in $\mathcal{T}$.

DEFINITION 2.15. Let $\mathcal{A}$ be an abelian category and $\mathcal{T}$ a hereditary torsion theory in $\mathcal{A}$, then the Gabriel localization $L_\mathcal{T}\colon \mathcal{A} \to \mathcal{A}$ is the localization functor associated to the class $\mathcal{E}_\mathcal{T}$ of $\mathcal{T}$-equivalences.

THEOREM 2.16. *Suppose $\mathcal{A}$ is a Grothendieck abelian category. There is a natural bijection between hereditary torsion theories of $\mathcal{A}$ and Gabriel localizations of $\mathcal{A}$:*

$$\begin{gathered} \mathrm{Her}(\mathcal{A}) \longleftrightarrow \mathrm{Loc}^G(\mathcal{A}) \\ \mathcal{T} \longmapsto L_\mathcal{T} \\ \ker(L) \longleftarrow L. \end{gathered}$$

This bijection is realized by sending a hereditary torsion theory $\mathcal{T}$ to the localization $L_\mathcal{T}$ defined above; conversely, a Gabriel localization functor L determines a hereditary torsion theory $\mathcal{T}_L = \ker(L)$.

Proof. See [**11**, Theorem 1.13.5]. $\square$

Proposition 2.17. *Suppose $\mathcal{T} \subseteq \mathcal{A}$ is a hereditary torsion theory, and let $\mathcal{A}/\mathcal{T}$ be the associated local category with localization functor $\Phi_*\colon \mathcal{A} \to \mathcal{A}/\mathcal{T}$. If $\mathcal{S} \subseteq \mathcal{A}/\mathcal{T}$ is a hereditary torsion theory in $\mathcal{A}/\mathcal{T}$, then there exists a hereditary torsion theory $\overline{\mathcal{S}} \subseteq \mathcal{A}$ with $\mathcal{T} \subseteq \overline{\mathcal{S}}$ and such that $\mathcal{S} = \Phi_*(\overline{\mathcal{S}})$.*

Proof. Write for (Φ_*, Φ^*) and $(F_{\mathcal{S}}, G_{\mathcal{S}})$ for the localization adjunction corresponding to $\mathcal{T}$ and $\mathcal{S}$, respectively, so that we have a diagram

$$\mathcal{A} \underset{\Phi^*}{\overset{\Phi_*}{\rightleftarrows}} \mathcal{A}/\mathcal{T} \underset{G_{\mathcal{S}}}{\overset{F_{\mathcal{S}}}{\rightleftarrows}} (\mathcal{A}/\mathcal{T})/\mathcal{S}.$$

We first claim that $(\Phi_*^{\mathcal{S}}, \Phi_{\mathcal{S}}^*) = (F_{\mathcal{S}}\Phi_*, \Phi^* G_{\mathcal{S}})$ is a localization adjunction. Indeed, $\Phi_*^{\mathcal{S}}$ is exact and there are natural isomorphisms

$$\Phi_*^{\mathcal{S}}\Phi_{\mathcal{S}}^* = F_{\mathcal{S}}\Phi_*\Phi^* G_{\mathcal{S}} \xrightarrow{\sim} F_{\mathcal{S}} G_{\mathcal{S}} \xrightarrow{\sim} \mathrm{Id}\,.$$

Therefore, there exists a hereditary torsion theory $\overline{\mathcal{S}} \subseteq \mathcal{A}$ corresponding to $(\Phi_*^{\mathcal{S}}, \Phi_{\mathcal{S}}^*)$; in particular, $\mathcal{T} = \ker(\Phi_*) \subseteq \ker(\Phi_*^{\mathcal{S}}) = \overline{\mathcal{S}}$.

It thus remains to show that $\mathcal{S} = \Phi_*(\overline{\mathcal{S}})$. Clearly, $F_{\mathcal{S}}\Phi_*(\overline{\mathcal{S}}) = \Phi_*^{\mathcal{S}}(\overline{\mathcal{S}}) = 0$, so $\Phi_*(\overline{\mathcal{S}}) \subseteq \ker(F_{\mathcal{S}}) = \mathcal{S}$. Conversely, for $X \in \mathcal{S}$ we have

$$0 = F_{\mathcal{S}}X = F_{\mathcal{S}}\Phi_*\Phi^* X = \Phi_*^{\mathcal{S}}\Phi^* X,$$

which implies $\Phi^* X \in \ker(\Phi_*^{\mathcal{S}}) = \overline{\mathcal{S}}$. Consequently, $X = \Phi_*\Phi^* X \in \Phi_*(\overline{\mathcal{S}})$, hence $\mathcal{S} \subseteq \Phi_*(\overline{\mathcal{S}})$. □

Remark 2.18. With notation as above, $\mathcal{A}/\mathcal{T}$ inherits the structure of a Grothendieck abelian category, see [**56**, Corollary 4.6.2].

2.4. *Height n cohomology theories and classification of hereditary torsion theories*

In this section we introduce the Hopf algebroids $(E(n)_*, E(n)_*E(n))$ and $(K(n)_*, \Sigma(n))$ closely related to (BP_*, BP_*BP), and give a classification of the hereditary torsion theories of the former.

In stable homotopy theory the geometric counterpart of the Brown–Peterson spectrum is the moduli stack of p-typical formal group laws. If we restrict to open substacks of formal group laws of height at most n, then the corresponding spectrum is Johnson–Wilson E-theory $E(n)$, with coefficient ring

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}],$$

with $|v_i| = 2(p^i - 1)$. This gives rise to a flat Hopf algebroid $(E(n)_*, E(n)_*E(n))$ where, by Landweber exactness of $E(n)$, we have $E(n)_*E(n) \cong E(n)_* \otimes BP_*BP \otimes E(n)_*$.

The geometric point associated to the open substack corresponds to Morava K-theory $K(n)$, whose coefficient ring is the graded field

$$K(n)_* \cong \mathbb{F}_{p^n}[v_n^{\pm 1}].$$

Importantly $K(n)_*$ satisfies a Künneth isomorphism: for spectra X and Y there is an isomorphism

$$K(n)_*(X \otimes Y) \cong K(n)_*X \otimes_{K(n)_*} K(n)_*Y. \tag{2.19}$$

Correspondingly one would expect to study the Hopf algebroid $(K(n)_*, K(n)_*K(n))$. However, $K(n)_*$ is not Landweber exact, and it turns out to be slightly more convenient to work with the Hopf algebroid $(K(n)_*, \Sigma(n))$ with $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$; note that if $K(n)_*$ were Landweber exact, then this would be precisely $(K(n)_*, K(n)_*K(n))$. For example, it is the latter Hopf algebroid that appears in the important change of rings theorem of Miller and Ravenel [**51**, Theorem 2.10].

As noted previously, $E(n)_*$ is an example of a Landweber exact BP_*-algebra of height n in the sense of Definition 2.3. Other examples include $v_n^{-1}BP_*$ or E_*, where $E = E_n$ denotes the n-th Morava E-theory, with coefficient ring

$$E_* \cong \mathbb{W}(\mathbb{F}_{p^n})[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}],$$

where $|u_i| = 0$ and $|u| = -2$. Geometrically this corresponds to the universal deformation of the geometric point associated to Morava K-theory. Since the associated comodule categories are equivalent, the hereditary torsion theories are equivalent for any Landweber exact BP_*-algebra of height n.

The geometry of the moduli stack of formal groups is reflected in the global structure of the associated Hopf algebroids, more precisely in the poset of their hereditary torsion theories. A (partial) classification of hereditary torsion theories for Comod_{BP_*BP} was proved by Hovey and Strickland [**31**], following the classification of thick subcategories (or Serre classes) of finitely presented BP_*BP-comodules by Jeanneret, Landweber and Ravenel [**36**]. We use this classification of hereditary torsion theories for BP_*BP-comodules and the results of the previous subsection to classify the hereditary torsion theories for E_*E-comodules.

In what follows, let $\mathcal{T}_n$ denote the full subcategory of all graded BP_*BP-comodules that are v_n-torsion, with the convention that $\mathcal{T}_{-1} = \mathrm{Comod}_{BP_*BP}$. The next result is [**31**, Theorems B and C].

Theorem 2.20 (Hovey–Strickland). *Let $\mathcal{T} \subseteq \mathrm{Comod}_{BP_*BP}$ be a hereditary torsion theory containing a non-trivial compact comodule, then $\mathcal{T} = \mathcal{T}_n$ for some $-1 \leqslant n$. Moreover, if $n \geqslant 0$, then the local category corresponding to $\mathcal{T}_n$ is naturally equivalent to Comod_{E_*E} with E_* a Landweber exact BP_*-algebra of height n.*

Fix $-1 \leqslant n$ and let

$$\mathrm{Comod}_{BP_*BP} \underset{\Phi_*}{\overset{\Phi^*}{\rightleftarrows}} \mathrm{Comod}_{E_*E}$$

be the localization adjunction corresponding to $\mathcal{T}_n$.

Corollary 2.21. *Let $\mathcal{S} \subseteq \mathrm{Comod}_{E_*E}$ be a hereditary torsion theory, then $\mathcal{S} = \mathcal{T}_m$ for some $-1 \leqslant m \leqslant n$.*

Proof. By Theorem 2.20 and Proposition 2.17, there exists a hereditary torsion theory $\overline{\mathcal{S}} \subseteq \mathrm{Comod}_{BP_*BP}$ such that $\mathcal{T}_n^{BP} \subseteq \overline{\mathcal{S}}$ and $\mathcal{S} = \Phi_*(\overline{\mathcal{S}})$. The first property implies that $\overline{\mathcal{S}}$ contains a non-zero compact BP_*BP-comodule, hence Theorem 2.20 shows that there exists an m with $\overline{\mathcal{S}} = \mathcal{T}_m^{BP}$. By definition, $\Phi_*\mathcal{T}_m^{BP} = \mathcal{T}_m^E$, so the claim follows. $\square$

We note that, in contrast to Theorem 2.20, we do not require that $\mathcal{S}$ contains a non-trivial compact comodule; instead, this condition is automatically satisfied in this case.

3. *The Stable category of comodules*

In this section we study the stable category of Ψ-comodules, previously introduced in [**8**, **26**]. In particular, we use a derived version of the cotensor product considered in the last section to derive Ravenel's base-change spectral sequence for Cotor [**60**, Appendix A.1.3.11], as well as a variant of a change of rings theorem of Hovey and Sadofsky [**29**, Theorem 3.3].

3.1. *The definition of* Stable_Ψ

As noted by Hovey [26], the category of chain complexes of comodules should be thought of as like topological spaces, in the sense that there is both a notion of homology and homotopy, and to form the 'correct' version of the derived category we should invert the homotopy, not homology, isomorphisms. In [26] Hovey constructed such a category Stable_Ψ associated to a Hopf algebroid† (A, Ψ). In [8, Section 4] we gave an alternative construction, which agrees with Hovey's model under some very mild conditions on the Hopf algebroid. We give a brief review of our construction here, referring the reader to [8] for the details.

For some motivation, we start with an observation of Hovey [27, Section 3]: in the derived category of BP_*BP-comodules the tensor unit BP_* is not compact (essentially due to the existence of non-nilpotent elements in $\mathrm{Ext}_{BP_*}(BP_*, BP_*)$). The idea of the following definition is to force the tensor unit (and indeed, all dualizable comodules) to be compact. Thus, let (A, Ψ) be a flat Hopf algebroid, and write $\mathcal{G} = \mathcal{G}_\Psi$ for the set of dualizable Ψ-comodules and $\mathcal{D}_\Psi$ for the usual derived category of comodules.

Definition 3.1. We define the stable ∞-category of Ψ-comodules as the ind-category of the thick subcategory of $\mathcal{D}_\Psi$ generated by $\mathcal{G}$ viewed as complexes concentrated in degree 0, that is,

$$\mathrm{Stable}_\Psi = \mathrm{Ind}(\mathrm{Thick}_\Psi(\mathcal{G})).$$

The next proposition summarizes some basic properties of Stable_Ψ. Proofs are given in [8, Section 4].

Proposition 3.2. *Let (A, Ψ) be a flat amenable Hopf algebroid.*

(1) Stable_Ψ *is a presentable stable ∞-category compactly generated by $\mathcal{G}$, equipped with a closed symmetric monoidal product preserving colimits in both variables.*

(2) *There is a cocontinuous functor $\omega\colon \mathrm{Stable}_\Psi \to \mathcal{D}_\Psi$ to the ordinary derived category, which is a (symmetric monoidal) equivalence when (A, A) is a discrete Hopf algebroid.*

(3) *The functor ω is given by Bousfield localization at the homology isomorphisms, that is, those morphisms which induce an isomorphism on H_*.*

(4) *For a Hopf algebroid (A, Ψ) there is a canonical equivalence between* Stable_Ψ *and the underlying ∞-category of the model category constructed by Hovey in* [26].

(5) *There is an adjunction of stable categories*

$$\epsilon_*\colon \mathrm{Stable}_\Psi \rightleftarrows \mathcal{D}_A : \epsilon^*$$

extending the adjunction from Proposition 2.1.

Since Point (3) is perhaps not clearly outlined in [8], we note that it follows from Hovey's construction of Stable_Ψ and Point (4) above.

As noted in Proposition 3.2, Stable_Ψ is compactly generated by the set of (isomorphism classes of) dualizable Ψ-comodules. We will say that it is monogenic if it is compactly generated by A itself. The following implies that many of the categories we study in this paper are monogenic; in particular Stable_{BP_*BP} itself is.

Proposition 3.3 [26, Corollary 6.7]. *If E is a ring spectrum that is Landweber exact over MU or BP and E_*E is commutative, then* Stable_{E_*E} *is monogenic.*

---

†Actually, Hovey constructed Stable_Ψ for amenable Hopf algebroids, see [26, Definition 2.3.2], but all the Hopf algebroids we consider in this paper are amenable.

Given $M, N \in \mathrm{Stable}_\Psi$ we will again write $M \otimes_A N$ for the monoidal product, and $\underline{\mathrm{Hom}}_\Psi(M, N)$ for the internal Hom object (sometimes we will omit the subscripts if the context is clear). Because of Proposition 3.2, we always assume our Hopf algebroids are amenable.

For technical reasons, it is sometimes useful to restrict to a certain subclass of Hopf algebroids.

Definition 3.4 [8, Definition 4.14]. Let (A, Ψ) be a flat Hopf algebroid, and write $\mathrm{Comod}_\Psi^\omega[0]$ for the image of the nerve of the abelian category of compact comodules in $\mathcal{D}_\Psi$. We call (A, Ψ) a Landweber Hopf algebroid if $\mathrm{Comod}_\Psi^\omega[0]$ is contained in $\mathrm{Thick}_\Psi(A)$.

This definition includes all the commonly used Hopf algebroids in stable homotopy theory, see [8, Section 4.3]. The next result was mentioned without proof in [8, Remark 4.30].

Lemma 3.5. *If (A, Ψ) is a Landweber Hopf algebroid, then Stable_Ψ is monogenic.*

Proof. It suffices to show that $\mathrm{Thick}_\Psi(A) = \mathrm{Thick}_\Psi(\mathcal{G})$; we always have $\mathrm{Thick}_\Psi(A) \subseteq \mathrm{Thick}_\Psi(\mathcal{G})$, and so we must show the other inclusion. Let $\mathcal{D}_0 \subset \mathcal{D}_\Psi$ be the full subcategory of complexes Q with homology concentrated in finitely many degrees such that $H_d(Q) \in \mathrm{Comod}_\Psi$ is compact. In [8, Lemma 4.16] we showed that $\mathcal{D}_0 = \mathrm{Thick}_\Psi(A)$. But since $G \in \mathcal{D}_0$ for each $G \in \mathcal{G}$ there is an inclusion $\mathrm{Thick}_\Psi(\mathcal{G}) \to \mathcal{D}_0 = \mathrm{Thick}_\Psi(A)$, completing the lemma. □

Landweber Hopf algebroids have another important property, which rests on a theorem due to Krause [39].

Proposition 3.6. *Assume (A, Ψ) is a Landweber Hopf algebroid with A coherent. There is a natural t-structure on Stable_Ψ such that the inclusion functor $\iota\colon \mathcal{D}_\Psi \to \mathrm{Stable}_\Psi$ is t-exact and induces natural equivalences*

$$\mathcal{D}_\Psi^{\leqslant k} \xrightarrow{\sim} \mathrm{Stable}_\Psi^{\leqslant k}$$

on the full subcategories of k-coconnective objects for all $k \in \mathbb{Z}$. The inverse equivalence is given by inverting the homology isomorphisms.

Proof. This is the content of [8, Proposition 4.17], where we proved this under the hypothesis that A is Noetherian. This can be generalized to the case that A is coherent using the work of Krause [39], as extended to the ∞-categorical setting by Lurie [46, Appendix C.5.8]. □

Definition 3.7. Let $\mathrm{Stable}_\Psi^{<\infty}$ be the full subcategory of those $M \in \mathrm{Stable}_\Psi$ for which there exists some k such that $M \in \mathrm{Stable}_\Psi^{\leqslant k}$.

Since Stable_Ψ is a stable ∞-category, $\mathrm{Hom}_\Psi(A, M)$ canonically has the structure of a spectrum. To avoid confusion in the following definition, we write π_*^{st} for the homotopy groups of a spectrum.

Definition 3.8. For $M \in \mathrm{Stable}_\Psi$, we define the homotopy groups of M as $\pi_* M = \pi_*^{\mathrm{st}} \mathrm{Hom}_\Psi(A, M)$.

Remark 3.9. By [26, Proposition 6.10], $\pi_* A \cong \mathrm{Ext}_\Psi^*(A, A)$, so that $\pi_* M$ is always a graded module over the graded-commutative ring $\mathrm{Ext}_\Psi^*(A, A)$. More generally given any discrete Ψ-comodule M, thought of as an object of Stable_Ψ, Hovey's result shows that $\pi_* M \cong \mathrm{Ext}_\Psi^*(A, M)$. By [60, A1.1.6] this is in turn isomorphic to $\mathrm{Cotor}_\Psi^*(A, M)$.

The relation between homology and homotopy in Stable_Ψ is given by the following:

Lemma 3.10. *For any $M \in \mathrm{Stable}_\Psi$ we have $\pi_*(\Psi \otimes M) \cong H_* M$.*

Proof. This follows easily by the adjunction between Stable_Ψ and $\mathcal{D}_A$ stated in Proposition 3.2(5); indeed, we have

$$\mathrm{Hom}_\Psi(N, \Psi \otimes M) \simeq \mathrm{Hom}_{\mathcal{D}_A}(\epsilon_* N, M)$$

for $N \in \mathrm{Stable}_\Psi$ and $M \in \mathcal{D}_A$, so that in particular $\pi_*(\Psi \otimes M) \cong H_* M$. $\square$

3.2. *Some derived functors*

Given a morphism $\Phi\colon (A, \Psi) \to (B, \Sigma)$ of Hopf algebroids, there exists a functor $\Phi_*\colon \mathrm{Comod}_\Psi \to \mathrm{Comod}_\Sigma$ induced by $M \mapsto B \otimes_A M$, with a right adjoint Φ^*. We shall see in the next lemma that both of these exist in the associated stable categories and that, interestingly, there is a third adjoint.

Lemma 3.11. *If $\Phi\colon (A, \Psi) \to (B, \Sigma)$ is a map of Hopf algebroids, then there exist adjoint functors*

$$\begin{tikzcd} \mathrm{Stable}_\Psi \arrow[r, shift left=2, "\Phi_!"] \arrow[r, shift right=2, "\Phi_*"'] & \mathrm{Stable}_\Sigma \arrow[l, "\Phi^*" description] \end{tikzcd}$$

where Φ_* is left adjoint to Φ^*, which in turn is left adjoint to $\Phi_!$.

Proof. If M is finitely generated and projective over A (and hence dualizable in Comod_Ψ, see Proposition 2.1), then $B \otimes_A M$ is finitely generated and projective over B, so that

$$B \otimes_A -\colon\ \mathrm{Comod}_\Psi \longrightarrow \mathrm{Comod}_\Sigma$$

preserves dualizable comodules. Using Hovey's model structure as in [**26**], it follows that there is an induced exact functor $\Phi_*\colon \mathrm{Thick}_\Psi(\mathcal{G}_\Psi) \to \mathrm{Thick}_\Psi(\mathcal{G}_\Sigma)$. Applying Ind, we get a functor $\Phi_*\colon \mathrm{Stable}_\Psi \to \mathrm{Stable}_\Sigma$ that preserves all colimits and compact objects. Thus, by [**4**, Theorem 1.7; **44**, Proposition 5.3.5.13], Φ_* has a right adjoint Φ^*, which has a further right adjoint $\Phi_!$. $\square$

The canonical map from the initial Hopf algebroid (K, K) to any Hopf algebroid (A, Ψ) will always be denoted by $\gamma_\Psi\colon (K, K) \to (A, \Psi)$; if the Hopf algebroid is clear from context, the subscript Ψ will be omitted. We now give a simple proof of the fact that γ_Ψ^* is the functor of derived primitives.

Lemma 3.12. *For $M \in \mathrm{Stable}_\Psi$ there is a natural equivalence*

$$\gamma_\Psi^* M \simeq \mathrm{Hom}_\Psi(A, M).$$

Proof. By Proposition 3.2(2) there is a symmetric monoidal equivalence of ∞-categories $\mathrm{Stable}_K \simeq \mathcal{D}_K$. Let $M \in \mathrm{Stable}_\Psi$, then

$$\gamma_\Psi^* M \simeq \mathrm{Hom}_{\mathcal{D}_K}(K, \gamma_\Psi^* M) \simeq \mathrm{Hom}_\Psi((\gamma_\Psi)_* K, M) \simeq \mathrm{Hom}_\Psi(A, M).$$

$\square$

Note that by definition we have $\pi_*M = \pi_*^{\mathrm{st}}(\gamma_\Psi^* M)$. Moreover, given a map $\Phi\colon (A,\Psi) \to (B,\Sigma)$ there is a commutative diagram of Hopf algebroids

$$\begin{tikzcd} (K,K) \arrow[r, "\gamma_\Psi"] \arrow[dr, "\gamma_\Sigma"'] & (A,\Psi) \arrow[d, "\Phi"] \\ & (B,\Sigma), \end{tikzcd}$$

so that $\gamma_\Sigma^* \simeq \gamma_\Psi^* \Phi^*$.

The next result is known as the projection formula.

LEMMA 3.13 (Projection formula). *For $M \in \mathrm{Stable}_\Sigma$ and $N \in \mathrm{Stable}_\Psi$, there is a natural equivalence*

$$(\Phi^* M) \otimes_A N \xrightarrow{\sim} \Phi^*(M \otimes_B \Phi_*(N)).$$

Proof. The map is constructed as the adjoint of the natural transformation

$$\Phi_*(\Phi^*(M) \otimes_A N) \xleftarrow{\sim} \Phi_*\Phi^*(M) \otimes_B \Phi_* N \xrightarrow{\epsilon \otimes \mathrm{Id}} M \otimes_B \Phi_* N,$$

where ϵ is the counit of the adjunction (Φ_*, Φ^*). Since all functors involved preserve colimits, it suffices to verify the claim for $M = B$ and $N = A$, for which it is clear. □

We can give an explicit formula for the right adjoint Φ^*.

LEMMA 3.14. *For $\Phi\colon (A,\Psi) \to (B,\Sigma)$ a map of Hopf algebroids, the right adjoint Φ^* of Φ_* can be identified as the derived primitives of the extended Ψ-comodule functor, that is,*

$$\Phi^* M \simeq \mathrm{Hom}_\Sigma(B, M \otimes_A \Psi),$$

for any $M \in \mathrm{Stable}_\Sigma$.

Proof. We first note that the statement makes sense: $M \otimes_A \Psi$ obtains the structure of a Σ-comodule via the comodule structure on M. It is also clearly a Ψ-comodule, and $\mathrm{Hom}_\Sigma(B, M \otimes_A \Psi)$ obtains a Σ-comodule structure by an argument similar to [**60**, 1.3.11(a)].

There are natural equivalences

$$\begin{aligned} \gamma_\Sigma^*(M \otimes_A \Psi) &\simeq \gamma_\Sigma^*(M \otimes_B \Phi_*(\Psi)) && \text{(since } \Phi_*(\Psi) \simeq B \otimes_A \Psi) \\ &\simeq \gamma_\Psi^* \Phi^*(M \otimes_B \Phi_*(\Psi)) && \text{(since } \gamma_\Sigma^* \simeq \gamma_\Psi^*\Phi^*) \\ &\simeq \gamma_\Psi^*(\Phi^*(M) \otimes_A \Psi) && \text{(by Lemma 3.13)} \\ &\simeq \gamma_\Psi^* \epsilon^*(\Phi^*(M)) && \text{(since } \epsilon^*(-) = \Psi \otimes_A -) \\ &\simeq \Phi^*(M) \end{aligned}$$

where ϵ^* is as in Proposition 3.2(5). The same argument as in [**60**, 1.3.11(a)] shows that these equivalences are compatible with the comodule structures. □

In virtue of Lemma 2.12, the following definition is a natural generalization of the classical construction of the Cotor groups of discrete comodules.

DEFINITION 3.15. We define the derived cotensor product of any two objects $M, N \in \mathrm{Stable}_\Psi$ as the derived primitives of their tensor product,

$$\mathrm{Cotor}_\Psi(M, N) = \gamma_\Psi^*(M \otimes_A N),$$

viewed as an object of Stable_K.

We then define $\mathrm{Cotor}_\Psi^i(M, N) = \pi_i^{\mathrm{st}} \mathrm{Cotor}_\Psi(M, N) = \pi_i(M \otimes_A N)$. If M and N are discrete comodules then, by Remark 3.9 and Lemma 2.12, this agrees with the definition of Cotor given in Definition 2.7. Furthermore,

LEMMA 3.16. *If (A, Ψ) is a Landweber Hopf algebroid with A coherent, then for $M, N \in \mathrm{Stable}_\Psi^{<\infty}$ we have $\pi_* \underline{\mathrm{Hom}}_\Psi(M, N) \cong \mathrm{Ext}_\Psi^*(\omega M, \omega N)$, where ω is the functor from Proposition 3.2(2).*

Proof. By adjunction $\pi_* \underline{\mathrm{Hom}}_\Psi(M, N) \cong \pi_*^{\mathrm{st}} \mathrm{Hom}_\Psi(M, N)$. Now apply [8, Corollary 4.19], using Proposition 3.6. $\square$

As an easy application of the results of this section, we can reinterpret the base-change spectral sequence for Cotor constructed by Ravenel in [60, Appendix A.1.3.11]. Note that we can dispense of the hypothesis that M is flat by our use of the relative Cotor functor.

COROLLARY 3.17. *Let $f: (A, \Psi) \to (B, \Sigma)$ be a map of Hopf algebroids. If M is a discrete (right) Ψ-comodule and N is a discrete (left) Σ-comodule, then there is a natural convergent spectral sequence*

$$\mathrm{Cotor}_\Psi^s(M, \mathrm{Cotor}_\Sigma^t(B \otimes_A \Psi, N)) \Rightarrow \mathrm{Cotor}_\Sigma^{s+t}(M \otimes_A B, N)$$

with differentials $d^r: E_r^{s,t} \to E_r^{s+r, t-r+1}$.

Proof. First, using Lemma 3.12 and Lemma 3.14, we obtain equivalences

$$\gamma_\Sigma^*((B \otimes_A \Psi) \otimes_B N) \simeq \mathrm{Hom}_\Sigma(B, (B \otimes_A \Psi) \otimes_B N) \simeq \mathrm{Hom}_\Sigma(B, N \otimes_A \Psi) \simeq f^* N.$$

The projection formula Lemma 3.13 then gives natural equivalences

$$\begin{aligned}
\gamma_\Sigma^*(f_*(M) \otimes_B N) &\simeq \gamma_\Psi^* f^*(f_*(M) \otimes_B N) \\
&\simeq \gamma_\Psi^*(M \otimes_A f^*(N)) \\
&\simeq \gamma_\Psi^*(M \otimes_A (\gamma_\Sigma^*((B \otimes_A \Psi) \otimes_B N))).
\end{aligned}$$

By testing on extended Σ-comodules as in [7, Section 6], the Grothendieck spectral sequence associated to the two functors

$$\gamma_\Psi^*(M \otimes_A -) \qquad \text{and} \qquad \gamma_\Sigma^*((B \otimes_A \Psi) \otimes_B -)$$

exists and converges [67, Theorem 5.8.3]. By construction and Lemma 2.12, the resulting spectral sequence recovers the Cotor spectral sequence. $\square$

REMARK 3.18 (Geometric interpretation). We recall from [53] that to a flat Hopf algebroid (A, Ψ) we can associate an algebraic stack $\mathfrak{X}$ with a fixed presentation $\mathrm{Spec}(A) \to \mathfrak{X}$, and that this gives rise to an equivalence of 2-categories between flat Hopf algebroids and rigidified algebraic stacks [53, Theorem 8]. Moreover, there is an equivalence of abelian categories between $\mathrm{QCoh}(\mathfrak{X})$, the category of quasi-coherent sheaves on $\mathfrak{X}$ and Comod_Ψ. Using this we can define the category $\mathrm{Ind}\,\mathrm{Coh}_{\mathfrak{X}}$ of ind-coherent sheaves on $\mathfrak{X}$, and show that it is equivalent to Stable_Ψ, see [8, Proposition 5.40]. This equivalence is symmetric monoidal. Geometrically, this means that Cotor as defined in Definition 3.15 corresponds to the derived global sections of the tensor product of ind-coherent sheaves.

3.3. *Change of rings*

As another application, we will prove a change of rings theorem for Hopf algebroids associated to faithfully flat extensions. For precursors of this result, see [**2**, Proposition 3.2; **25**, Theorem D; **29**, Theorem 3.3; **31**, Theorem 6.2].

Given a Hopf algebroid (A, Ψ) and morphism $\Phi\colon A \to B$ of K-algebras, let $\Sigma_\Phi = B \boxtimes_A \Psi \boxtimes_A B$, where we use the symbol $\boxtimes$ to denote the underived tensor product. Note that (B, Σ_Φ) forms a Hopf algebroid, and there is a natural morphism of Hopf algebroids $\Phi\colon (A, \Psi) \to (B, \Sigma_\Phi)$. In general (B, Σ_Φ) need not be a flat Hopf algebroid, even when (A, Ψ) is. It is, however, when $B \boxtimes_A \Psi$ is a flat A-module.

Lemma 3.19. *Suppose (A, Ψ) is a Landweber Hopf algebroid with A coherent. If T is a faithfully flat A-module, then the composite*

$$\mathrm{Stable}_\Psi^{<\infty} \xrightarrow{\epsilon_*} \mathrm{Stable}_A^{<\infty} \xrightarrow{T\otimes_A -} \mathcal{D}_A$$

is conservative.

Proof. By Proposition 3.6 there is an equivalence of categories $\mathrm{Stable}_\Psi^{<\infty} \simeq \mathcal{D}_\Psi^{<\infty}$, so ϵ_* restricted to $\mathrm{Stable}_\Psi^{<\infty}$ is conservative. Now let $f\colon M \to N$ be a morphism in $\mathrm{Stable}_A^{<\infty} \simeq \mathcal{D}_A^{<\infty}$ such that $T \otimes_A f$ is an equivalence. The morphism f gives rise to a cofiber sequence $M \xrightarrow{f} N \to \mathrm{cofib}(f)$ where, by assumption, $T \otimes_A \mathrm{cofib}(f) \simeq 0$. Since T is faithfully flat over A, this implies that $\mathrm{cofib}(f) \simeq 0$, so that f was an equivalence to begin with. □

For the following compare [**31**, Theorem 6.2].

Proposition 3.20. *Let $\Phi\colon A \to B$ be as above, and suppose that (A, Ψ) is a Landweber Hopf algebroid with A coherent. Suppose the composite*

$$A \xrightarrow{\eta_R} \Psi \xrightarrow{1\otimes\Phi} \Psi \otimes_A B$$

is a faithfully flat extension of A, then Φ^* induces an equivalence

$$\mathrm{Stable}_{\Sigma_\Phi} \xrightarrow{\sim} \mathrm{Stable}_\Psi .$$

Proof. For this proof, we use the notation $M \boxtimes N$ to denote the underived tensor product between two modules M and N.

We will first show that the unit $u\colon \mathrm{id} \to \Phi^*\Phi_*$ is an equivalence. Since Φ_* and Φ^* preserve all colimits and Stable_Ψ is monogenic by Lemma 3.5, the unit u is a natural equivalence if and only if it is so when evaluated on A. Moreover, $u_A\colon A \to \Phi^*\Phi_*A$ is a map between objects in $\mathrm{Stable}_\Psi^{\leqslant 0}$ and so by Lemma 3.19 it suffices to show that $(\Psi \otimes_A B) \otimes_A u_A$ is an equivalence. To see this, first observe that the projection formula Lemma 3.13 together with Lemma 3.14 give

$$\begin{aligned}(\Psi \otimes_A B) \otimes_A \Phi^*\Phi_*A &\simeq \Phi^*(\Phi_*(\Psi \otimes_A B) \otimes_B \Phi_*A)\\ &\simeq \Phi^*\Phi_*(\Psi \otimes_A B)\\ &\simeq \mathrm{Hom}_{\Sigma_\Phi}(B, \Psi \otimes_A (B \otimes_A \Psi \otimes_A B)).\end{aligned}$$

Note that $B \otimes_A \Psi \simeq B \boxtimes_A \Psi$ since Ψ is flat over A. Then, since $B \boxtimes_A \Psi$ is assumed to be flat over A, we deduce an equivalence $B \otimes_A \Psi \otimes_A B \simeq B \boxtimes_A \Psi \boxtimes_A B = \Sigma_\Phi$. Thus, we have

$$\begin{aligned}(\Psi \otimes_A B) \otimes_A \Phi^* \Phi_* A &\simeq \mathrm{Hom}_{\Sigma_\Phi}(B, \Psi \otimes_A \Sigma_\Phi)\\ &\simeq \mathrm{Hom}_{\Sigma_\Phi}(B, (\Psi \otimes_A B) \otimes_B \Sigma_\Phi)\\ &\simeq \mathrm{Hom}_B(B, \Psi \otimes_A B)\\ &\simeq \Psi \otimes_A B.\end{aligned}$$

It is standard to verify that this equivalence is induced by u_A, that is, $(\Psi \otimes_A B) \otimes_A u_A$ is an equivalence, as required.

Let c denote the counit of the adjunction (Φ_*, Φ^*) and suppose $Y \in \mathrm{Stable}_{\Sigma_\Phi}$. In order to show that $c\colon \Phi_*\Phi^*Y \to Y$ is an equivalence, it suffices to prove that the top morphism in the following commutative diagram is an equivalence

$$\begin{tikzcd} \mathrm{Hom}_{\Sigma_\Phi}(B, \Phi_*\Phi^*Y) \arrow[r, "\mathrm{Hom}(B, c_Y)"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\Sigma_\Phi}(B, Y) \arrow[d, "\sim"] \\ \mathrm{Hom}_\Psi(A, \Phi^*\Phi_*\Phi^*Y) \arrow[r, "\mathrm{Hom}(A, \Phi^* c_Y)"'] & \mathrm{Hom}_\Psi(A, \Phi^*Y), \end{tikzcd}$$

since $B = \Phi_*A$ is a compact generator of $\mathrm{Stable}_{\Sigma_\Phi}$. Because the unit of the adjunction is an equivalence, the triangle identity implies that the bottom horizontal map is an equivalence as well, and the claim follows. □

Remark 3.21. This demonstrates how working systematically on the derived level can help to considerably simplify arguments, cf. the proof of [**25**, Theorem D].

4. *Morava theories and generic primes*

In this section we study the stable categories $\mathrm{Stable}_{\Sigma(n)}$ and Stable_{E_*E} associated to the Hopf algebroids $(K(n)_*, \Sigma(n))$ and (E_*, E_*E) introduced in Section 2.4, proving that for certain primes they are equivalent to their respective derived categories. In particular, we show that this is true whenever p is large with respect to n. This implies that in these cases the stable category of comodules is much simpler, an algebraic manifestation of the well-known fact that chromatic homotopy at height n simplifies when the prime p is much larger than n.

Recall that the homology theory E_* is complex-oriented and the associated formal group law over E_* is the universal deformation of the Honda formal group, the formal group law associated to Morava K-theory. We define the Morava stabilizer group $\mathbb{S}_n$ to be the group of automorphisms of the Honda formal group law of height n. If n is not divisible by $p-1$, then $\mathbb{S}_n$ is of finite cohomological dimension n^2, which implies that $\mathrm{Ext}^{s,t}_{\Sigma(n)}(K(n)_*, K(n)_*)$ is zero for $s > n^2$. This leads to the following definition, where as usual E_* denotes any height n Landweber exact BP_*-algebra.

Definition 4.1. For any n, the set of $K(n)$-generic primes is the set of primes p for which n is not divisible by $p-1$, and the set of E-generic primes is the intersection of the sets of $K(i)$-generic primes for $0 \leqslant i \leqslant n$.

In the case of Morava E-theory, the E-based chromatic spectral sequence can be used to show that if p is an E-generic prime, then $\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) = 0$ for $s > n^2 + n$ [**29**, Theorem 5.1]. The main result of this section is that the natural functors

$$\mathrm{Stable}_{\Sigma(n)} \to \mathcal{D}_{\Sigma(n)} \quad \text{and} \quad \mathrm{Stable}_{E_*E} \to \mathcal{D}_{E_*E}$$

are equivalences for the set of $K(n)$-generic and E-generic primes, respectively. Note that such a statement is not true for Stable_{BP_*BP} since, for example, BP_* is compact in Stable_{BP_*BP}

but not in $\mathcal{D}_{BP_*BP}$. This is shown by Hovey [**27**, Section 3] using the existence of non-nilpotent elements in $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ of positive cohomological degree.

4.1. *Field theories*

Let (K, Υ) be a Hopf algebroid over a field K, so that Υ is in fact a Hopf algebra over K. There are two important types of examples. First, for any finite group G, the group ring of G over the field k has the structure of a Hopf algebra, so that (k, kG) is a Hopf algebroid. Second, for any field object K in the category of spectra, (K_*, K_*K) is a Hopf algebroid over K_*. In particular, we can consider the Steenrod algebra $(\mathbb{F}_p, \mathcal{A}_*)$ and $(K(n)_*, \Sigma(n))$ corresponding to $H\mathbb{F}_p$ and Morava K-theory $K(n)$ for a given prime p and height $n \geqslant 0$, respectively. As a consequence of the nilpotence theorem, these are essentially all fields of the stable homotopy category [**23**, Proposition 1.9].

The following two lemmata generalize [**55**, Corollary 1.2.10 and Lemma 1.3.9]. As is standard, we define the homology theory associated to $E \in \mathrm{Stable}_\Upsilon$ via the assignment $X \mapsto \pi_*(E \otimes X)$ for any $X \in \mathrm{Stable}_\Upsilon$.

Lemma 4.2. *Let (K, Υ) be a Hopf algebroid over a field K.*

(1) *The homology theory represented by Υ is ordinary (chain) homology H_*, and this satisfies the Künneth formula.*

(2) *For any $M \in \mathrm{Stable}_\Upsilon$, $\Upsilon \otimes M$ decomposes as a direct sum of suspensions of Υ.*

Proof. That Υ represents homology is a special case of Lemma 3.10. Since $\pi_*(\Upsilon \otimes M) \cong H_*M$ is a free graded K-module, it satisfies the Künneth formula, and so (1) holds. Moreover, we can construct a map

$$\bigoplus_{b \in H_*M} \Sigma^{|b|}\Upsilon \longrightarrow \Upsilon \otimes M$$

in Stable_Υ, where the direct sum is indexed by a K-basis of H_*M. By construction, this map is an equivalence, and (2) follows. □

Lemma 4.3. *Assume that Stable_Υ is monogenic and suppose that $\mathcal{D}$ is a localizing subcategory of Stable_Υ containing a non-acyclic object M_0, then $\mathrm{Loc}(\Upsilon) \subseteq \mathcal{D}$.*

Proof. Because Stable_Υ is monogenic, the localizing ideals coincide with the localizing subcategories. Since $0 \not\simeq \Upsilon \otimes M_0 \in \mathcal{D}$, we get $\Upsilon \in \mathcal{D}$ by Lemma 4.2. □

In order to apply this to the examples of interest, we need the following.

Proposition 4.4. *The category $\mathrm{Stable}_{\Sigma(n)}$ is monogenic.*

Proof. In this proof we will again use the symbol $\boxtimes$ to denote the underived tensor product.

Let $N \in \mathrm{Stable}_{\Sigma(n)}$ be compact. By construction of $\mathrm{Stable}_{\Sigma(n)}$, N is in the thick subcategory generated by the dualizable $\Sigma(n)$-comodules. We will show that it is in the thick subcategory generated by $K(n)_*$. We note that by Proposition 2.1 each dualizable discrete $\Sigma(n)$-comodule is finitely generated and projective as a $K(n)_*$-module, that is, as a $K(n)_*$-module it is isomorphic to a finite direct sum of copies of $K(n)_*$, up to suspension.

Let E be the Landweber exact cohomology theory with $E_* \cong \mathbb{W}(\mathbb{F}_{p^n})[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$, where $\mathbb{W}(\mathbb{F}_{p^n})$ denotes the ring of Witt vectors on $\mathbb{F}_{p^n}$. It follows that $E_*/I_n \cong K(n)_*$ and $E_*E/I_n \cong K(n)_*E \cong \Sigma(n)$ [**30**, p. 15]. Let $f\colon (E_*, E_*E) \to (K(n)_*, \Sigma(n))$ denote the quotient morphism of Hopf algebroids. Then, for a $\Sigma(n)$-comodule M there are equivalences

$$\begin{aligned}\Sigma(n) \boxtimes_{K(n)_*} M &\cong E_*E \boxtimes_{E_*} K(n)_* \boxtimes_{K(n)_*} M \\ &\cong E_*E \boxtimes_{E_*} M.\end{aligned}$$

In particular, M is also an E_*E-comodule, with comodule structure map given by the composite

$$M \xrightarrow{\psi_M} \Sigma(n) \boxtimes_{K(n)_*} M \cong E_*E \boxtimes_{E_*} M.$$

We will write $M^\sharp$ when we think of M as an E_*E-comodule.

For arbitrary $M \in \mathrm{Stable}_{\Sigma(n)}^{<\infty}$, Lemma 3.14 gives equivalences

$$\begin{aligned}f^*M \simeq \mathrm{Hom}_{\Sigma(n)}(K(n)_*, M \otimes_{E_*} E_*E) &\simeq \mathrm{Hom}_{\Sigma(n)}(K(n)_*, M \boxtimes_{E_*} E_*E) \\ &\simeq \mathrm{Hom}_{\Sigma(n)}(K(n)_*, M \boxtimes_{K(n)_*} \Sigma(n)) \\ &\simeq \mathrm{Hom}_{\Sigma(n)}(K(n)_*, M \otimes_{K(n)_*} \Sigma(n)) \\ &\simeq M,\end{aligned}$$

with E_*E-comodule structure given as above, where we have used that our Hopf algebroids are flat.

It follows that $f^*M \simeq M^\sharp$, and in particular that $f^*(K(n)_*) \simeq K(n)_*^\sharp \simeq (E_*/I_n)^\sharp$. Since I_n is a finitely generated invariant ideal of E_*, it follows that E_*/I_n is a finitely presentable E_*-module, and hence so is $f^*(P) \simeq P^\sharp$ for any dualizable $\Sigma(n)$-comodule P. We have shown in [**8**, Section 4.3] that this implies that f^*P is compact and hence dualizable in Stable_{E_*E}.

Since f^* is exact, this implies that if $N \in \mathrm{Stable}_{\Sigma(n)}^{<\infty}$ is in the thick subcategory generated by the dualizable $\Sigma(n)$-comodules, then $f^*N \simeq N^\sharp$ is in the thick subcategory generated by the dualizable E_*E-comodules, that is, $N^\sharp$ is compact in Stable_{E_*E}. Again, using the fact that Stable_{E_*E} is monogenic, we see that $N^\sharp$ is in the thick subcategory generated by E_*. It follows that f_*f^*N is in the thick subcategory generated by $K(n)_*$. Now we have cofiber sequences

$$E_*/I_k \otimes K(n)_* \xrightarrow{\cdot v_k} E_*/I_k \otimes K(n)_* \longrightarrow E_*/I_{k+1} \otimes K(n)_*,$$

and since $K(n)_*$ is killed by I_n, these give rise to equivalences

$$E_*/I_k \otimes K(n)_* \simeq (E_*/I_{k-1} \otimes K(n)_*) \oplus (E_*/I_{k-1} \otimes \Sigma K(n)_*)$$

for all $0 \leqslant k \leqslant n$, hence

$$K(n)_* \otimes_{E_*} K(n)_* \cong E_*/I_n \otimes_{E_*} K(n)_* \cong \bigoplus_{0 \leqslant j \leqslant n} \Sigma^j K(n)_*^{\lambda_n(j)},$$

where $\lambda_n(j) = \binom{n}{j}$. Therefore,

$$\begin{aligned}f_*f^*N \simeq K(n)_* \otimes_{E_*} N &\simeq K(n)_* \otimes_{E_*} K(n)_* \otimes_{K(n)_*} N \\ &\simeq N \oplus \bigoplus_{1 \leqslant j \leqslant n} \Sigma^j N^{\oplus \lambda_n(j)}\end{aligned}$$

is in the thick subcategory generated by $K(n)_*$. It follows that $N \in \mathrm{Thick}(K(n)_*)$ as required. □

4.2. *Generic primes*

We now focus on the behavior of $\mathrm{Stable}_{\Sigma(n)}$ and Stable_{E_*E} at the set of $K(n)$-generic and E-generic primes, respectively. We start with $\mathrm{Stable}_{\Sigma(n)}$.

LEMMA 4.5. *If $p-1\nmid n$, then $K(n)_* \in \mathrm{Thick}_{\Sigma(n)}(\Sigma(n))$.*

Proof. If $p-1\nmid n$, the cohomological p-dimension of the Morava stabilizer group $\mathbb{S}_n$ is n^2, so there exists a length n^2 projective resolution

$$0 \longrightarrow P_\bullet \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

of the trivial $\mathbb{Z}_p[\![\mathbb{S}_n]\!]$-module $\mathbb{Z}_p$, see [21, Theorem 4]. As shown in [21], this resolution can be lifted to a finite resolution of $K(n)_*$ as a $\Sigma(n)$-comodule, such that each term is a direct summand of a finite wedge of copies of $\Sigma(n)$. In the usual way, we can split the long exact sequence into short exact sequences. Starting from the final term and working our way back to $K(n)_*$, the claim follows inductively. □

REMARK 4.6. For $p-1\mid n$, while $\mathbb{S}_n$ has infinite cohomological p-dimension, it is still of virtual cohomological dimension n^2. In stable homotopy theory, this fact manifests itself in the existence of a finite spectrum $X_{p,n}$ of type 0 such that $K(n)^*X_{p,n}$ has projective dimension n^2 over $\Sigma(n)^*$, see [30, proof of Theorem 8.9]. Such complexes were constructed by Hopkins, Ravenel and Smith as explained in [61, Section 8.3]; note, however, that $X_{p,n}$ cannot be taken to be S^0 if $p-1\mid n$.

PROPOSITION 4.7. *Suppose p is a $K(n)$-generic prime, that is $p-1\nmid n$, then the natural functor*

$$\omega\colon \mathrm{Stable}_{\Sigma(n)} \xrightarrow{\sim} \mathcal{D}_{\Sigma(n)}$$

is an equivalence of symmetric monoidal stable ∞-categories.

Proof. The functor ω exhibits $\mathcal{D}_{\Sigma(n)}$ as the localization of $\mathrm{Stable}_{\Sigma(n)}$ at the quasi-isomorphisms, that is, it is localization at the localizing subcategory of all $M \in \mathrm{Stable}_{\Sigma(n)}$ such that $\pi_*(\Sigma(n)\otimes M) \cong H_*M = 0$, see Proposition 3.2. It follows from Lemma 4.5 that $\pi_*(M) \cong \pi_*(K(n)_*\otimes M) = 0$, hence $M \simeq 0$ since $\mathrm{Stable}_{\Sigma(n)}$ is monogenic by Proposition 4.4. Therefore, ω is localization at (0). □

REMARK 4.8. Combining the proof of Proposition 4.7 with Remark 4.6, we see that, for any prime p, $M \in \mathrm{Stable}_{\Sigma(n)}$ being acyclic implies $\pi_*(K(n)_*X_{p,n}\otimes M) = 0$.

For the following we let $\mathrm{Spc}(\mathrm{Stable}_{\Sigma(n)})$ denote the Balmer spectrum associated to $\mathrm{Stable}_{\Sigma(n)}$, see [3].

COROLLARY 4.9. *Suppose $p-1\nmid n$. If $\mathcal{D} \subseteq \mathrm{Stable}_{\Sigma(n)}$ is a localizing subcategory containing a non-zero object M_0, then $\mathcal{D} = \mathrm{Stable}_{\Sigma(n)}$. In particular, there are no non-trivial proper thick subcategories in $\mathrm{Stable}^{\omega}_{\Sigma(n)}$, that is, $\mathrm{Spc}(\mathrm{Stable}_{\Sigma(n)}) = \{*\}$.*

Proof. By Proposition 4.7 such an M_0 corresponds to a non-acyclic object of $\mathrm{Stable}_{\Sigma(n)}$. Hence, combining Lemma 4.3, Proposition 4.4 and Lemma 4.5, we get

$$\mathrm{Stable}_{\Sigma(n)} = \mathrm{Loc}(K(n)_*) \subseteq \mathrm{Loc}(\Sigma(n)) \subseteq \mathcal{D},$$

so $\mathrm{Stable}_{\Sigma(n)} = \mathcal{D}$.

In order to prove the second claim, consider a non-trivial thick subcategory $\mathcal{T} \subseteq \mathrm{Stable}^{\omega}_{\Sigma(n)}$ and write $\mathcal{L} = \mathrm{Loc}(\mathcal{T})$ for the corresponding localizing subcategory of $\mathrm{Stable}_{\Sigma(n)}$. It follows from the first part that $\mathcal{L} = \mathrm{Stable}_{\Sigma(n)}$ and therefore, by [54, Theorem 2.1(3)], that $\mathcal{T} = \mathrm{Loc}(\mathcal{T})^{\omega} = \mathrm{Stable}^{\omega}_{\Sigma(n)}$. □

QUESTION 4.10. Is it possible to classify the thick subcategories of $\mathrm{Stable}_{\Sigma(n)}$ for $p-1 \mid n$?

Let E_* be any height n Landweber exact BP_*-algebra. As noted previously these give rise to a category of comodules (E_*, E_*E), and the comodule categories of any two such BP_*-algebras are equivalent. Thinking of E_* as the coefficient ring of Morava E-theory, the following result gives a lift of Proposition 4.7 from Morava K-theory to Morava E-theory.

THEOREM 4.11. *If p is an E-generic prime, that is $p > n+1$, then the localization functor $\omega\colon \mathrm{Stable}_{E_*E} \xrightarrow{\sim} \mathcal{D}_{E_*E}$ is an equivalence of symmetric monoidal stable ∞-categories.*

Proof. We will first show that $E_* \in \mathrm{Thick}^{\otimes}(E_*E)$, the thick tensor ideal in $\mathcal{D}_{E_*E}$ generated by E_*E. To this end, let $E_* \to I^\bullet$ be a resolution of E_* by injective E_*E-comodules. The assumption that $n < p-1$ implies that there exists some $N \geqslant 0$ such that $\mathrm{Ext}^s_{E_*E}(E_*, E_*) = 0$ for all $s > N$, see the [**59**, proof of Theorem 10.9]. Induction on k then shows that N can be chosen large enough so that $\mathrm{Ext}^s_{E_*E}(E_*/I_k, E_*) = 0$ for all $s > N$ and all $0 \leqslant k \leqslant n$ as well. Since every dualizable discrete comodule P is finitely presented and projective by Proposition 2.1, it thus follows from the Landweber filtration theorem [**31**, Theorem D] and the long exact sequence in Ext that

$$\mathrm{Ext}^s_{E_*E}(P, E_*) = 0 \tag{4.12}$$

for all $s > N$. Now consider the exact sequence

$$0 \longrightarrow E_* \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \xrightarrow{f^2} \dots \xrightarrow{f^N} I^N \xrightarrow{g} \mathrm{coker}(f^N) \longrightarrow 0. \tag{4.13}$$

Recall from Proposition 2.4 that the dualizable discrete comodules generate Comod_{E_*E}, so (4.12) forces the map g to be split, as is carefully proven, for example, in [**17**, Section 3.4, Lemma 2]. Therefore, $\mathrm{coker}(f^N)$ is a retract of an injective comodule and hence itself injective. But every injective comodule is a retract of an extended comodule by Lemma 2.2, so the resolution (4.13) is spliced together from short exact sequences involving only extended comodules. Because short exact sequences induce fiber sequences in $\mathcal{D}_{E_*E}$, this yields $E_* \in \mathrm{Thick}^{\otimes}(E_*E)$.

To finish the argument, recall from Proposition 3.2 that $\omega\colon \mathrm{Stable}_{E_*E} \to \mathcal{D}_{E_*E}$ is the localization with respect to the homology isomorphisms. Since Stable_{E_*E} is stable, it suffices to show that any $M \in \mathrm{Stable}_{E_*E}$ with $H_*M = 0$ must be trivial. Suppose $M \in \mathrm{Stable}_{E_*E}$ with $H_*M = 0$. Define a full subcategory $\mathcal{C}(M) \subseteq \mathrm{Stable}_{E_*E}$ consisting of those $X \in \mathrm{Stable}_{E_*E}$ with $\pi_*(X \otimes M) = 0$; note that $\mathcal{C}(M)$ is a thick tensor ideal. Since E_*E represents homology, that is, there is a natural equivalence $H_*(-) \cong \pi_*(E_*E \otimes -)$, we get $E_*E \in \mathcal{C}(M)$, hence

$$E_* \in \mathrm{Thick}^{\otimes}(E_*E) \subseteq \mathcal{C}(M).$$

This means that $\pi_*M = 0$, thus $M \simeq 0$, and the claim follows. □

REMARK 4.14. More conceptually, the fact that E_* is contained in the thick tensor ideal generated by E_*E is equivalent to the morphism $E_* \to E_*E$ being descendable in the language of [**48**]. The latter statement, in turn, can be shown to be equivalent to the existence of a horizontal vanishing line in the (collapsing) Adams spectral sequence, see [**48**, Section 4], that is, to the finite cohomological dimension of $E_* \in \mathrm{Comod}_{E_*E}$.

5. *The nilpotence theorem*

In this section we present an algebraic version of the nilpotence theorem in Stable_{BP_*BP}. Our results are not as strong as the nilpotence theorem in stable homotopy theory given by

Devinatz, Hopkins and Smith [14, 23], principally due to the fact that the detecting family we use does not consist of field objects in Stable_{BP_*BP}.

5.1. *Equivalent statements of the algebraic nilpotence theorem*

In [23] Hopkins and Smith prove that the Morava K-theories $K(n)$ can be used to detect nilpotence: a map $f\colon F \to X$ from a finite spectrum to a p-local spectrum X is smash nilpotent, that is, $f^{(m)} = 0$ for some $m \gg 0$, if and only if $K(n)_*f = 0$ for all $0 \leqslant n \leqslant \infty$.

In this section we prove a Stable_{BP_*BP} variant of this. Our results are more like the nilpotence theorems given in [28, Section 5], although we note that they do not follow automatically from their work, since (5.1.2) of [28] is not satisfied in our case.

Recall that, for $0 \leqslant n \leqslant \infty$, I_n denotes the ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$ (with the convention that $I_0 = (0)$); by [40] these are the only invariant prime ideals in BP_*. In analogy with the notation for a type n complex in stable homotopy theory, we let $F(n)_* = BP_*/I_n$.† We then define $\mathrm{Tel}(n)_*$ as the localization $v_n^{-1}F(n)_*$ (by convention we set $\mathrm{Tel}(0)_* = \mathbb{Q}$ and $\mathrm{Tel}(\infty)_* = \mathbb{F}_p$). These play the role of the detecting theories in this context (see also Theorem 5.13).

Our version of the nilpotence theorem takes the following form.

THEOREM 5.1 (Algebraic nilpotence Theorem I — weak version).

(1) *Suppose $F, X \in \mathrm{Stable}_{BP_*BP}$ with F compact, then a map $f\colon F \to X$ is smash nilpotent, that is, $f^{(m)} = 0$ for some $m \gg 0$, if $\mathrm{Tel}(n)_* \otimes_{BP_*} f = 0$ for all $0 \leqslant n \leqslant \infty$.*

(2) *A self-map $f\colon \Sigma^i F \to F$ for $F \in \mathrm{Stable}^{\omega}_{BP_*BP}$ is nilpotent, in the sense that $f^j : \Sigma^{ij} F \to F$ is null for some $j \gg 0$, if and only if $\mathrm{Tel}(n)_* \otimes_{BP_*} f$ is nilpotent for all $0 \leqslant n \leqslant \infty$.*

THEOREM 5.2 (Algebraic nilpotence Theorem II — weak version).

(1) *Suppose $X \in \mathrm{Stable}_{BP_*BP}$, then a map $f\colon BP_* \to X$ is smash nilpotent, that is, $f^{(m)} = 0$ for some $m \gg 0$, if $\pi_*(\mathrm{Tel}(n)_* \otimes_{BP_*} f) = 0$ for all $0 \leqslant n \leqslant \infty$.*

(2) *Let R be a ring object in Stable_{BP_*BP}. Then an element $\alpha \in \pi_*R \cong \mathrm{Ext}_{BP_*BP}(BP_*, R)$ is nilpotent if and only if $\pi_*(\mathrm{Tel}(n)_* \otimes_{BP_*} \alpha)$ is nilpotent for all $0 \leqslant n \leqslant \infty$.*

We will prove these in Section 5.2. Our proof follows closely the ideas of the original proof of Hopkins and Smith, and we start by building a Bousfield decomposition similar to that seen in the ordinary stable homotopy category.

REMARK 5.3. We refer to these theorems as weak versions of the nilpotence theorem because they do not account for all periodic elements in π_*BP_*, but only those of Adams–Novikov filtration 0, which in turn correspond to the classical periodic elements v_n in ordinary stable homotopy theory. This manifests itself in the fact that the telescopes $\mathrm{Tel}(n)_*$ are not field objects, and we thus cannot deduce a description of the Balmer spectrum of Stable_{BP_*BP}.

However, in forthcoming work with A. Krause, we will study the global structure of Stable_{BP_*BP} in more detail. In particular, we will establish a much more refined description of the thick subcategories of compact objects by constructing a more sophisticated detecting family.

5.2. *The proof of the algebraic nilpotence theorem*

We start by recalling the basic definition of a Bousfield class, specialized to the category Stable_{BP_*BP}.

†This often appears in the literature as $P(n)_*$.

Definition 5.4. Let $M, N \in \mathrm{Stable}_{BP_*BP}$. We say that M and N are Bousfield equivalent if, given any $X \in \mathrm{Stable}_{BP_*BP}$, we have $M \otimes_{BP_*} X \simeq 0$ if and only if $N \otimes_{BP_*} X \simeq 0$. We write $\langle M \rangle$ for the Bousfield class of M.

Lemma 5.5. *There is an equivalence* $\mathrm{colim}_m F(m)_* \simeq \mathbb{F}_p$ *in* Stable_{BP_*BP}.

Proof. There is a natural map $\mathrm{colim}_m F(m)_* \to \mathbb{F}_p$. Since this map is in $\mathrm{Stable}^{\leqslant 0}_{BP_*BP}$, it suffices to check that it is a quasi-isomorphism, which is clear: Indeed, this map is even an isomorphism in Comod_{BP_*BP}. □

Recall that we denote $\mathrm{Tel}(n)_* = v_n^{-1}F(n)_*$.

Lemma 5.6. *For any $m \geqslant 0$, we have an identity of Bousfield classes*

$$\langle BP_* \rangle = \langle F(m+1)_* \rangle \oplus \bigoplus_{i=0}^{m} \langle \mathrm{Tel}(i)_* \rangle.$$

Proof. By virtue of the general formula $\langle M \rangle = \langle M/v \rangle \oplus \langle v^{-1}M \rangle$ for any self-map $v\colon \Sigma^d M \to M$, see [**28**, Proposition 3.6.9(d); **59**, Lemma 1.34], we see that $\langle F(m)_* \rangle = \langle F(m+1)_* \rangle \oplus \langle \mathrm{Tel}(m)_* \rangle$. Since $F(0)_* = BP_*$, the result then follows inductively. □

Remark 5.7. This result also appears in the proof of [**27**, Lemma 4.10].

For the remainder of this subsection, we will omit all suspensions from the notation. Let $f\colon BP_* \to X$ be a map in Stable_{BP_*BP}. We write

$$T_f = \mathrm{colim}(BP_* \xrightarrow{f} X \simeq BP_* \otimes X \xrightarrow{f \otimes 1} X \otimes X \xrightarrow{f \otimes 1 \otimes 1} \ldots)$$

for the corresponding telescope and $f^{(\infty)}\colon BP_* \to T_f$ for the canonical map.

Lemma 5.8. *Let $R \in \mathrm{Stable}_{BP_*BP}$ be a ring object with unit $\iota\colon BP_* \to R$ and $f\colon BP_* \to X$ some map in Stable_{BP_*BP}. The following statements are equivalent:*

(1) $R \otimes T_f \simeq 0$;
(2) $\iota \otimes f^{(\infty)}\colon BP_* \to R \otimes T_f$ *is zero;*
(3) $\iota \otimes f^{(n)}\colon BP_* \to R \otimes X^{(n)}$ *is zero for $n \gg 0$;*
(4) $1_R \otimes f^{(n)}\colon R \to R \otimes X^{(n)}$ *is zero for $n \gg 0$.*

Proof. This is proven as in [**23**, Lemma 2.4]. □

Remark 5.9. The proof of this result uses the compactness of BP_*, and hence it is crucial that we work in Stable_{BP_*BP}, and not just $\mathcal{D}_{BP_*BP}$.

We now prove the first algebraic nilpotence theorem.

Proof of Theorem 5.1. The 'only if' direction of Part (2) is clear, and we note that the other direction follows from Part (1). Indeed, this is the same argument as in [**23**], namely we replace $f\colon \Sigma^i F \to F$ with its adjoint $f^{\#}\colon \Sigma^i BP_* \to DF \otimes F$. To prove part (1) we can similarly replace $f\colon F \to X$ with its adjoint $f^{\#}\colon BP_* \to DF \otimes X$, and so reduce to the case[†] where $F = BP_*$.

[†]In particular, the claim follows from Theorem 5.2(2). However, the proof of the latter relies on Theorem 5.1, so we cannot apply it here.

Let T_f be the telescope associated to f. We first start by assuming that $1_{\mathrm{Tel}(n)_*} \otimes f = 0$ for all $0 \leqslant n \leqslant \infty$, so that $\mathrm{Tel}(n)_* \otimes T_f \simeq 0$ for all n. By Lemma 5.8 we have to show $BP_* \otimes T_f \simeq T_f \simeq 0$. By the Bousfield decomposition of Lemma 5.6, it then suffices to prove that $F(n)_* \otimes T_f \simeq 0$ for $n \gg 0$. Using Theorem 5.8 again, this will follow from $BP_* \to F(n)_* \otimes T_f$ being null for sufficiently large n. To this end, compactness of BP_* gives a factorization

$$\begin{tikzcd} BP_* \arrow[rr] \arrow[d, dashed] & & \mathbb{F}_p \otimes T_f \arrow[d, "\sim"] \\ F(n) \otimes T_f \arrow[r] & \operatorname{colim}_n F(n) \otimes T_f, & \end{tikzcd}$$

where the right vertical equivalence was established in Lemma 5.5. By assumption, the top horizontal map is zero, so the claim follows. □

The proof of the second nilpotence theorem follows closely the one given in [**28**, Theorem 5.1.3].

Proof of Theorem 5.2. Once again it suffices to prove part (1). To see this, consider the commutative diagram

$$\begin{tikzcd} \Sigma^{ktm} BP_* \arrow[r, "(\alpha^t)^{\otimes m}"] \arrow[dr, "\alpha^{tm}"'] & R^{\otimes m} \arrow[d, "\mu"] \\ & R. \end{tikzcd}$$

If (1) holds, then α^{tm} is null for $m \gg 0$, so that α is nilpotent. The other direction of (2) is clear.

The proof of (1) is identical to [**28**, Theorem 5.1.3], which we repeat for the convenience of the reader. Namely, $\mathrm{Tel}(n)_*$ is a ring object in Stable_{BP_*BP}, so there exist maps $\eta\colon BP_* \to \mathrm{Tel}(n)_*$ and $\mu\colon \mathrm{Tel}(n)_* \otimes_{BP_*} \mathrm{Tel}(n)_* \to \mathrm{Tel}(n)_*$ satisfying the usual relations. Suppose $f\colon BP_* \to X$ is such that $\pi_*(\mathrm{Tel}(n)_* \otimes_{BP_*} f)$ is zero, so that the composite $BP_* \xrightarrow{\eta} \mathrm{Tel}(n)_* \xrightarrow{1\otimes f} \mathrm{Tel}(n)_* \otimes_{BP_*} X$ is null. But $1 \otimes f$ factors as $\mathrm{Tel}(n)_* \xrightarrow{(1\otimes f)\circ\eta} \mathrm{Tel}(n)_* \otimes_{BP_*} \mathrm{Tel}(n) \otimes_{BP_*} X \xrightarrow{\mu\otimes 1} \mathrm{Tel}(n)_* \otimes_{BP_*} X$ so that $1 \otimes f$ is null. We now apply Theorem 5.1(1). □

5.3. *Base-change and the algebraic telescope conjecture*

The goal of this subsection is to generalize the main structural results of [**8**, Section 8] to the Hopf algebroid $(F(m)_*, F(m)_*F(m))$. In particular, we deduce an algebraic version of the telescope conjecture for Stable_{BP_*BP}, which is analogous to Ravenel's theorem that $L_n^f BP \simeq L_n BP$ for all $n \geqslant 0$.

There are Landweber exact $F(m)_*$-algebras $E(m,n)_* = \mathrm{Tel}(m)_*/(v_{n+1}, v_{n+2}, \ldots)$ for all $m \leqslant n$, giving rise to Hopf algebroids $(E(m,n)_*, E(m,n)_*E(m,n))$. These theories come with a natural base-change functor

$$\Phi(m,n)_*\colon \mathrm{Stable}_{F(m)_*F(m)} \longrightarrow \mathrm{Stable}_{E(m,n)_*E(m,n)},$$

defined by $\Phi(m,n)_*(M) = E(m,n)_* \otimes_{F(m)_*} M$ for any $M \in \mathrm{Stable}_{F(m)_*F(m)}$. This functor clearly preserves arbitrary colimits and thus admits a right adjoint $\Phi(m,n)^*$.

For a fixed integer $n \geqslant m$, we will write (E_*, E_*E) for $(E(m,n)_*, E(m,n)_*E(m,n))$ and similarly (Φ_*, Φ^*) for the base-change adjunction just constructed.

PROPOSITION 5.10. *The functor* $\Phi^*\colon \mathrm{Stable}_{E_*E} \to \mathrm{Stable}_{F(m)_*F(m)}$ *is bimonadic, in the sense that it satisfies the following properties:*

(1) Φ^* *has a left adjoint* Φ_**;*
(2) Φ^* *has a right adjoint* $\Phi_!$*;*
(3) *the counit map* $\Phi_*\Phi^* \to \mathrm{Id}$ *is an equivalence, so* Φ^* *is conservative.*

In particular, the pairs $(\Phi_* \dashv \Phi^*)$ *and* $(\Phi^* \dashv \Phi_!)$ *are monadic and comonadic, respectively.*

Proof. The proof is the same as [8, Proposition 8.13]. □

Let $m \leqslant k$ and consider the compact object $F(m)_*/I_k \in \mathrm{Stable}_{F(m)_*F(m)}$. In the terminology of [8], the pair $(\mathrm{Stable}_{F(m)_*F(m)}, F(m)_*/I_k)$ forms a local duality context.

THEOREM 5.11. *Let* E_* *be a Landweber exact* $F(m)_*$*-algebra of height* $n \geqslant m$*, then the ring map* $F(m)_* \to E_*$ *induces a natural equivalence*

$$\mathrm{Stable}^{I_k-\mathrm{loc}}_{F(m)_*F(m)} \xrightarrow{\sim} \mathrm{Stable}^{I_k-\mathrm{loc}}_{E_*E}$$

for any $m \leqslant k \leqslant n+1$.

Proof. The same as [8, Theorem 8.19]. □

COROLLARY 5.12. *For any* n *and* $m \leqslant k \leqslant n+1$ *and* E_* *as above, there is a natural equivalence of stable categories*

$$\mathrm{Stable}^{I_k-\mathrm{loc}}_{E_*E} \xrightarrow{\sim} \mathrm{Stable}_{F_*F}$$

for any Landweber exact $F(m)_*$*-algebra* F_* *of height* $k-1$*. Furthermore, there is a natural equivalence* $\Phi_*L^{F(m)}_{I_k} \simeq L^E_{I_k}\Phi_*$*, that is, the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Stable}_{F(m)_*F(m)} \arrow[r, "L^{F(m)}_{I_k}"] \arrow[d, "\Phi_*"'] & \mathrm{Stable}^{I_k-\mathrm{loc}}_{F(m)_*F(m)} \arrow[d, "\Phi_*"] \\ \mathrm{Stable}_{E_*E} \arrow[r, "L^E_{I_k}"'] & \mathrm{Stable}^{I_k-\mathrm{loc}}_{E_*E}. \end{tikzcd}$$

Proof. The first claim follows from the theorem since $\mathrm{Stable}^{I_k-\mathrm{loc}}_{F_*F} \simeq \mathrm{Stable}_{F_*F}$, while for the second claim argues as in [8, Corollary 8.23]. □

Recall that the telescope conjecture in stable homotopy theory is equivalent to the statement that there is an equivalence of Bousfield classes of spectra $\langle \mathrm{Tel}(m)\rangle = \langle K(m)\rangle$ [24], where $\langle \mathrm{Tel}(m)\rangle$ denotes the Bousfield class of the telescope of a finite spectrum of type m.

One can ask the same question here: Is $\langle \mathrm{Tel}(m)_*\rangle = \langle K(m)_*\rangle$?[†] The main problem in asking this question is that $K(m)_*$ is not an object of Stable_{BP_*BP}. If one modifies the definition

[†]However, note that $\mathrm{Tel}(m)_*$ is not isomorphic to $\pi_*\,\mathrm{Tel}(m)$, but rather $BP_*\,\mathrm{Tel}(m)$ in case the corresponding Smith–Toda complex exists.

of Bousfield class so that $- \otimes_{BP_*} -$ refers to the tensor product in the derived category of BP_*-modules only, then one can show that the algebraic telescope conjecture holds.

THEOREM 5.13 (The algebraic telescope conjecture). *With the above definition, there is an identity of Bousfield classes* $\langle K(m)_* \rangle = \langle \mathrm{Tel}(m)_* \rangle$ *for all* $m \geqslant 0$.

Proof. Both $\mathrm{Tel}(m)_*$ and $K(m)_*$ are Landweber exact $F(m)_*$-algebras, so Corollary 5.12 for $m = k = n+1$ provides a commutative diagram

$$\begin{tikzcd} \mathrm{Stable}_{F(m)_*F(m)} \arrow[d, "\Phi_*"'] \arrow[dr, "\Phi_*"] & \\ \mathrm{Stable}_{v_m^{-1}F(m)_*F(m)} \arrow[r, "\sim"] & \mathrm{Stable}_{K(m)_*K(m)} . \end{tikzcd}$$

It follows that $\mathrm{Tel}(m)_* \otimes_{F(m)_*} N = 0$ if and only if $K(m)_* \otimes_{F(m)_*} N = 0$ for all $N \in \mathrm{Stable}_{F(m)_*F(m)}$. Because $K(m)_* \otimes_{BP_*} M \simeq K(m)_* \otimes_{F(m)_*} F(m)_* \otimes_{BP_*} M$ for any $M \in \mathrm{Stable}_{BP_*BP}$, we do indeed have the claimed equivalence $\langle \mathrm{Tel}(m)_* \rangle = \langle K(m)_* \rangle$. □

REMARK 5.14. For an alternative formulation of the algebraic telescope conjecture, see Remark 6.9.

REMARK 5.15. There is an algebraic analog of Freyd's generating hypothesis [**18**] for Stable_{BP_*BP}; to wit, the algebraic generating hypothesis asks whether the functor

$$\pi_* \colon \mathrm{Stable}^{\omega}_{BP_*BP} \to \mathrm{Mod}_{\pi_*BP_*}$$

is faithful. As a special case of Lockridge's result [**43**, Proposition 2.2.1], the algebraic generating hypothesis holds if and only if the E_2-page of the ANSS for the sphere, π_*BP_*, is totally incoherent as a ring. However, we are not aware of any results about the ring structure of π_*BP_*.

Similarly, one can consider the local algebraic generating hypothesis as in [**6**]. Using an algebraic version of Brown–Comenetz duality, we suspect that this local version fails for all positive heights, but we will leave the details to the interested reader.

6. *Local duality and chromatic splitting for* Stable_{BP_*BP}

In this section, we introduce the algebraic analogs of Bousfield localization at Morava K-theories and Morava E-theories, which play a fundamental role in chromatic homotopy theory. Combined with the local duality theory developed in [**8**], this provides a convenient framework in which we can study the local structure of Stable_{BP_*BP}. As one instance of this, we discuss an algebraic version of the chromatic splitting conjecture.

6.1. *Local cohomology and local homology at height* n

We begin with some recollections from [**8**, Section 8]. Recall from the previous section that, for $0 \leqslant n < \infty$, we let I_n denote the ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$; in particular, $I_0 = (0)$. If $n = \infty$, we define $I_\infty = (p, v_1, \ldots) = \bigcup_n I_n$. We refer to [**8**, Section 2] for background material on localization and colocalization functors.

DEFINITION 6.1. Let $\mathrm{Stable}^{I_{n+1}-\mathrm{tors}}_{BP_*BP}$ be the localizing subcategory of Stable_{BP_*BP} generated by BP_*/I_{n+1}. The associated colocalization and localization functors will be denoted by Γ_n and L_n, respectively.

We can represent the categories and functors constructed via the following diagram:

$$\begin{tikzcd}
& \mathrm{Stable}^{I_{n+1}-\mathrm{loc}}_{BP_*BP} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
& \mathrm{Stable}_{BP_*BP} \arrow[u, shift left, "L_n"] \arrow[dl, shift left, "\Gamma_n"] \arrow[dr, shift left] & \\
\mathrm{Stable}^{I_{n+1}-\mathrm{tors}}_{BP_*BP} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\sim"'] & & \mathrm{Stable}^{I_{n+1}-\mathrm{comp}}_{BP_*BP} \arrow[ul, shift left, "\Lambda^n"']
\end{tikzcd} \tag{6.2}$$

Remark 6.3. In [**8**, Theorem 2.21] we denoted Γ_n and L_n by $\Gamma_{I_{n+1}}$ and $L_{I_{n+1}}$. In order to emphasize the structural similarity with the stable homotopy category and as no confusion is likely to arise, we have changed the notation to Γ_n and L_n.

For all $n \geqslant 0$ we have morphisms $\Phi\colon BP_* \to E(n)_*$, which give rise to adjoint pairs

$$\Phi_* = \Phi(n)_*\colon \mathrm{Stable}_{BP_*BP} \rightleftarrows \mathrm{Stable}_{E(n)_*E(n)} \colon \Phi(n)^* = \Phi^*.$$

The next result summarizes some of the main results of [**8**, Section 8; cf. Theorem 5.11].

Theorem 6.4. *Let n be a non-negative integer.*

(1) *There is a natural equivalence of functors $L_n \xrightarrow{\sim} \Phi^*\Phi_*$ and $\Phi_*\Phi^* \xrightarrow{\sim} \mathrm{Id}$.*
(2) *For any $k \leqslant n+1$, the maps $BP_* \to E(n)_* \to v_{k-1}^{-1}E(n)_*$ induce symmetric monoidal equivalences*

$$\mathrm{Stable}^{I_k-loc}_{BP_*BP} \xrightarrow{\sim} \mathrm{Stable}^{I_k-loc}_{E(n)_*E(n)} \xrightarrow{\sim} \mathrm{Stable}_{v_{k-1}^{-1}E(n)_*E(n)}$$

and there is an equivalence $\mathrm{Stable}_{v_{k-1}^{-1}E(n)_*E(n)} \simeq \mathrm{Stable}_{E(k-1)_*E(k-1)}$.

In geometric terms, the localization functor L_n corresponds to the restriction of a sheaf to the open substack of $\mathcal{M}_{fg}$ of formal groups of height at most n. The second part of Theorem 6.4 can thus be interpreted as giving a presentation of this open substack in terms of the Johnson–Wilson theories $E(n)$.

The inclusions $\mathrm{Loc}(BP_*/I_{n+1}) \subset \mathrm{Loc}(BP_*/I_n)$ give rise to an algebraic chromatic tower

$$L_\infty \longrightarrow \ldots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0, \tag{6.5}$$

which is the algebraic analog of the chromatic tower in stable homotopy theory. Our next goal is to study the layers of this tower in more detail. Recall that we can inductively construct objects $BP_*/I_n^\infty \in \mathrm{Stable}_{BP_*BP}$ for $n \geqslant 0$ via cofiber sequences

$$BP_*/I_n^\infty \to v_n^{-1}BP_*/I_n^\infty \to BP_*/I_{n+1}^\infty, \tag{6.6}$$

under the usual convention $v_0 = p$.

Proposition 6.7. *For any $M \in \mathrm{Stable}_{BP_*BP}$, we have $L_nM \simeq M \otimes L_nBP_*$, and L_nBP_* can be computed inductively by $L_0BP_* \simeq p^{-1}BP_*$ and cofiber sequences*

$$\Sigma^{-(n+1)}v_{n+1}^{-1}BP_*/I_{n+1}^\infty \to L_{n+1}BP_* \to L_nBP_*$$

for all $n \geqslant 0$.

Proof. The first statement is just that L_n is smashing, which follows from the fact that L_n is a finite localization, see [**28**, Lemma 3.3.1]. By [**8**, Corollary 8.9] we have $\Gamma_0BP_* \simeq$

$\Sigma^{-1}BP_*/p^\infty$. By definition, L_0BP_* fits in a cofiber sequence $\Gamma_0 BP_* \to BP_* \to L_0BP_*$. Comparison with (6.6) shows that $L_0BP_* \simeq p^{-1}BP_*$ as claimed. In order to prove the final claim, consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(g_n) \arrow[r] \arrow[d] & \Gamma_{n+1}BP_* \arrow[r, "g_n"] \arrow[d] & \Gamma_n BP_* \arrow[d] \\
0 \arrow[r] \arrow[d] & BP_* \arrow[r, "\simeq"] \arrow[d] & BP_* \arrow[d] \\
\mathrm{fib}(l_n) \arrow[r] & L_{n+1}BP_* \arrow[r, "l_n"'] & L_n BP_*
\end{tikzcd}$$

in which all rows and columns are cofiber sequences. The fiber of g_n can be identified with $\Sigma^{-(n+2)}v_{n+1}^{-1}BP_*/I_{n+1}^\infty$ by [8, Corollary 8.9] and (6.6). Therefore, $\mathrm{fib}(l_n) \simeq \Sigma^{-(n+1)}v_{n+1}^{-1}BP_*/I_{n+1}^\infty$ and the claim follows. ☐

As is standard, we denote the fiber of $L_nM \to L_{n-1}M$ by M_nX, and call this the nth (algebraic) monochromatic layer.

Corollary 6.8. *The nth monochromatic layer satisfies the formula $M_nBP_* \simeq \Sigma^{-n}v_n^{-1}BP_*/I_n^\infty$ and is smashing, that is, for any $X \in \mathrm{Stable}_{BP_*BP}$ there is an equivalence*

$$M_nX \simeq \Sigma^{-n}v_n^{-1}BP_*/I_n^\infty \otimes X.$$

In particular, $M_nE(n)_* \simeq \Sigma^{-n}E(n)_*/I_n^\infty$.

Proof. Since M_n is a fiber of smashing functors, it is smashing as well. The stated formula follows directly from Proposition 6.7, and the rest is clear. ☐

Remark 6.9 (The algebraic telescope conjecture revisited). With the introduction of the functor L_n we can give another version of the algebraic telescope conjecture Theorem 5.13. Recall that in stable homotopy an equivalent formulation of the telescope conjecture is that finite localization with respect to a finite type n-spectrum, denoted L_n^f, is equivalent to Bousfield localization with respect to $E(n)$, denoted L_n, see [24, 47]. Here we formulate an algebraic version of this conjecture.

We say that $X \in \mathrm{Stable}_{BP_*BP}$ is $E(n)_*$-local if, for any $T \in \mathrm{Stable}_{BP_*BP}$ with $E(n)_* \otimes_{BP_*} T \simeq 0$, the space of maps $\mathrm{Hom}_{BP_*BP}(T, X)$ is contractible. These form a colocalizing subcategory of Stable_{BP_*BP} and by the ∞-categorical version of Bousfield localization [44, Section 5.5.4] the inclusion of this full subcategory has a left adjoint, which we denote by $L_{E(n)_*}$. An alternative formulation of the algebraic telescope conjecture is that $L_n \simeq L_{E(n)_*}$.

It follows from Theorem 6.4 that $E(n)_* \otimes_{BP_*} X \simeq 0$ if and only if $L_nX \simeq L_nBP_* \otimes_{BP_*} X \simeq 0$, or equivalently that $\langle L_nBP_* \rangle = \langle E(n)_* \rangle$. But by [50, Corollary 11] L_n is Bousfield localization with respect to L_nBP_*, and hence $L_n \simeq L_{E(n)_*}$. It follows that this version of the algebraic telescope conjecture holds in Stable_{BP_*BP}.

In [27] Hovey considers yet another version of the algebraic telescope conjecture, comparing L_n with the functor given by Bousfield localization at the homology theory corresponding to $E(n)_*$. By [27, Proposition 3.11] this functor is given by $H_*(E(n)_* \otimes_{BP_*} X)$ for $X \in \mathrm{Stable}_{BP_*BP}$. Hovey proves that this cannot agree in general with L_n, as the former has essential image $\mathcal{D}_{BP_*BP}$, while the latter has essential image Stable_{BP_*BP}. Nonetheless, the proof of Theorem 4.11 shows that when $n < p-1$, so that $\mathrm{Stable}_{E_*E} \simeq \mathcal{D}_{E_*E}$, these two localizations do agree.

Similar to the case of BP_*BP above, we can consider the localizing subcategory of $\mathrm{Stable}_{E(n)_*E(n)}$ generated by $E(n)_*/I_n$. There is an associated localization functor L_{n-1}^E which by [8, Corollary 8.23] has the property that $\Phi_* L_{n-1} \simeq L_{n-1}^E \Phi_*$. We let Δ_n denote the functor that is right adjoint to L_{n-1}^E, viewed as endofunctors of $\mathrm{Stable}_{E(n)_*E(n)}$, which exists by [8, Theorem 2.21]; in particular, there is a local duality equivalence

$$\underline{\mathrm{Hom}}_{E(n)_*E(n)}(L_{n-1}^E E(n)_*, M) \simeq \Delta_n M \tag{6.10}$$

for $M \in \mathrm{Stable}_{E(n)_*E(n)}$.

6.2. *Local cohomology at height* ∞

We now turn to the height ∞ analog of the theory presented above.

Definition 6.11. Let $\mathrm{Stable}_{BP_*BP}^{I_\infty-\mathrm{tors}}$ be the localizing subcategory of Stable_{BP_*BP} generated by $BP_*/I_\infty \cong \mathbb{Z}/p$. The associated colocalization and localization functors will be denoted by Γ_∞ and L_∞, respectively.

Note that $BP_*/I_\infty \in \mathrm{Stable}_{BP_*BP}$ is not compact, but we still have a diagram of adjunctions

$$\mathrm{Stable}_{BP_*BP}^{I_\infty-\mathrm{tors}} \underset{\Gamma_\infty}{\overset{\iota_\infty}{\rightleftarrows}} \mathrm{Stable}_{BP_*BP} \overset{L_\infty}{\rightleftarrows} \mathrm{Stable}_{BP_*BP}^{I_\infty-\mathrm{loc}},$$

where the left adjoints are displayed on top. Recall from (6.5) the algebraic chromatic tower

$$\ldots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0.$$

The next result identifies the limit of this tower.

Proposition 6.12. *There is a natural equivalence of functors* $L_\infty \xrightarrow{\sim} \lim_n L_n$.

Proof. First we note that $BP_*/I_\infty \simeq \mathrm{colim}_n\, BP_*/I_n$, where the colimit is taking along the canonical quotient maps. The inclusion $\mathrm{Loc}(BP_*/I_\infty) \subseteq \mathrm{Loc}(BP_*/I_n)$ induces natural transformations $L_\infty \to L_n$ for all n. Therefore, we have a natural morphism of cofiber sequences of functors

$$\begin{tikzcd} \Gamma_\infty \arrow[r] \arrow[d, "\phi"'] & \mathrm{Id} \arrow[r] \arrow[d, "\sim"'] & L_\infty \arrow[d] \\ \lim_n \Gamma_n \arrow[r] & \mathrm{Id} \arrow[r] & \lim_n L_n, \end{tikzcd}$$

so it suffices to show that ϕ is an equivalence. We will show that $\lim_n \Gamma_n$ is right adjoint to the inclusion functor ι_∞. To this end, let $M \in \mathrm{Stable}_{BP_*BP}^{I_\infty-\mathrm{tors}}$ and $N \in \mathrm{Stable}_{BP_*BP}$; we get

$$\begin{aligned} \mathrm{Hom}(M, \lim_n \Gamma_n N) &\simeq \lim_n \mathrm{Hom}(M, \Gamma_n N) \\ &\simeq \lim_n \mathrm{Hom}(\iota_n M, N) \\ &\simeq \mathrm{Hom}(\mathrm{colim}_n\, \iota_n M, N) \\ &\simeq \mathrm{Hom}(\iota_\infty M, N), \end{aligned}$$

where the last equivalence from the construction, using the commutative triangle

$$\begin{tikzcd}
 & \mathrm{Stable}_{BP_*BP} & \\
\mathrm{Loc}(BP_*/I_\infty) = \mathrm{Stable}_{BP_*BP}^{I_\infty-\mathrm{tors}} \arrow[ur, "\iota_\infty"] \arrow[rr, hook] & & \mathrm{Stable}_{BP_*BP}^{I_n-\mathrm{tors}} = \mathrm{Loc}(BP_*/I_n). \arrow[ul, "\iota_n"']
\end{tikzcd}$$

The claim follows. ☐

We will see in Section 7 that L_∞ is equivalent to the identity functor on a large subcategory of Stable_{BP_*BP}, that is, we will prove an algebraic version of the chromatic convergence theorem of Hopkins and Ravenel [**61**].

6.3. *The algebraic chromatic splitting conjecture*

The goal of this section is to explore an algebraic version of Hopkins' chromatic splitting conjecture for Stable_{BP_*BP}. To this end, we recall that we let $F(n)_*$ denote the quotient BP_*/I_n. Note that since L_n is smashing, $L_nF(n)_*$ is a compact object of $\mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$.

Definition 6.13. We define the functor $L_{K(n)}$ to be the composite $\Lambda^{L_nF(n)_*}L_n$, where $\Lambda^{L_nF(n)_*}$ is the completion functor associated to the local duality context $(\mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}, L_nF(n)_*)$. This definition makes sense because L_n takes essential image in $\mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$.

Of course, $L_nF(n)_*$ is also an object of Stable_{BP_*BP} via the canonical inclusion, and we have the following.

Lemma 6.14. *$L_{K(n)}$ is Bousfield localization on Stable_{BP_*BP} with respect to the theory $L_nF(n)_*$.*

Proof. The argument is similar to the proof of [**8**, Proposition 2.31]. It is easy to verify that $L_{K(n)} = \Lambda^{L_nF(n)_*}L_n$ is a localization functor, so it suffices to identify the corresponding category of acyclics. For $X \in \mathrm{Stable}_{BP_*BP}$, we have $L_{K(n)}X \simeq 0$ if and only if $L_nF(n)_* \otimes_{BP_*} L_nX \simeq 0$, because $L_nF(n) \in \mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$ is compact. This in turn is equivalent to $L_nF(n)_* \otimes_{BP_*} X \simeq 0$, and the claim follows. ☐

Proposition 6.15. *For any $X \in \mathrm{Stable}_{BP_*BP}$ there is a pullback square*

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\
L_{n-1}X \arrow[r, "\iota_X"'] & L_{n-1}L_{K(n)}X,
\end{tikzcd}$$

with horizontal fibers equivalent to $\underline{\mathrm{Hom}}_{BP_*BP}(L_{n-1}BP_*, L_nX) \simeq \Delta_n(\Phi_*X)$.

Proof. Applying the fracture square [**8**, Corollary 2.26] associated to the local duality context $(\mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}, L_nF(n)_*)$ to L_nX we get a pullback square

$$\begin{tikzcd} L_nX \arrow[r] \arrow[d] & \Lambda^{L_nF(n)_*}L_nX \arrow[d] \\ L_{L_nF(n)_*}L_nX \arrow[r, "\iota_X"'] & L_{L_nF(n)_*}\Lambda^{L_nF(n)_*}L_nX \end{tikzcd} \tag{6.16}$$

for any $X \in \mathrm{Stable}_{BP_*BP}$. Since $L_{K(n)}X = \Lambda^{L_nF(n)_*}L_nX$ by definition, we must show that $L_{L_nF(n)_*}L_nX \simeq L_{n-1}X$. To see this, let us denote by $\mathcal{M}_n$ the essential image of the functor $\Gamma_{L_nF(n)_*}L_n$ on $\mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$. The same argument as in the first part of [8, Lemma 7.14] shows that there is a commutative diagram of adjunctions

$$\begin{tikzcd} \mathrm{Stable}_{BP_*BP}^{I_n-\mathrm{tors}} \arrow[r, shift right] \arrow[d, "L_n"'] & \mathrm{Stable}_{BP_*BP} \arrow[l, shift right, "\Gamma_{n-1}"'] \arrow[d, "L_n"] \\ \mathcal{M}_n \arrow[r, shift right] & \mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}} . \arrow[l, shift right, "\Gamma_{L_nF(n)_*}"'] \end{tikzcd}$$

We have a fiber sequence

$$\Gamma_{L_nF(n)_*}L_nX \longrightarrow L_nX \longrightarrow L_{L_nF(n)_*}L_nX$$

which, using the diagram above, is equivalent to the fiber sequence

$$L_n\Gamma_{n-1}X \longrightarrow L_nX \longrightarrow L_{L_nF(n)_*}L_nX.$$

By comparing with the defining cofiber sequence $\Gamma_{n-1}X \to X \to L_{n-1}X$, we deduce that there are equivalences $L_{L_nF(n)_*}L_nX \simeq L_nL_{n-1}X \simeq L_{n-1}X$.

To compute the fiber, we work with the fiber sequence associated to the top map in (6.16). By [8, Theorem 2.21] there is a right adjoint $\Delta^{BP_*}_{L_nF(n)_*}$ to $L_{L_nF(n)_*}$ on Stable_{BP_*BP}, fitting into a fiber sequence

$$\Delta^{BP_*}_{L_nF(n)_*} \longrightarrow \mathrm{id} \longrightarrow \Lambda^{L_nF(n)_*}_{BP_*}.$$

Moreover, Δ satisfies the local duality formula $\Delta^{BP_*}_{L_nF(n)_*}(-) \simeq \underline{\mathrm{Hom}}_{BP_*BP}(L_{L_nF(n)_*}BP_*, -)$. Therefore, the fiber is equivalent to

$$\begin{aligned} \Delta^{BP_*}_{L_nF(n)_*}(L_nX) &\simeq \underline{\mathrm{Hom}}_{BP_*BP}(L_{L_nF(n)_*}BP_*, L_nX) \\ &\simeq \underline{\mathrm{Hom}}_{BP_*BP}(L_{L_nF(n)_*}L_nBP_*, L_nX) \\ &\simeq \underline{\mathrm{Hom}}_{BP_*BP}(L_{n-1}BP_*, L_nX) \end{aligned}$$

by the previous paragraph.

For the final equivalence of the statement, note that $\underline{\mathrm{Hom}}_{BP_*BP}(-,-)$ is equivalent to the internal Hom in $\mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$. Indeed, if $M, N \in \mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$, then

$$\mathrm{Hom}_{BP_*BP}(X, \underline{\mathrm{Hom}}_{BP_*BP}(M,N)) \simeq \mathrm{Hom}_{BP_*BP}(X \otimes M, N) \simeq 0,$$

for all $X \in \mathrm{Loc}(BP_*/I_n)$, since Stable_{BP_*BP} is monogenic and $N \in \mathrm{Stable}_{BP_*BP}^{I_{n+1}-\mathrm{loc}}$. This implies that $\underline{\mathrm{Hom}}_{BP_*BP}(M,N)$ is I_{n+1}-local, from which the claim easily follows.

By the equivalence of categories of Theorem 6.4 and using (6.10), we thus see that, via the natural inclusion, the fiber in question is equivalent to

$$\underline{\mathrm{Hom}}_{E_*E}(\Phi_*L_{n-1}BP_*, \Phi_*L_nX) \simeq \underline{\mathrm{Hom}}_{E_*E}(L^E_{n-1}E_*, \Phi_*X) \simeq \Delta_n(\Phi_*X),$$

where we have used the fact that $\Phi_*L_nX \simeq \Phi_*X$, see Theorem 6.4(1). $\square$

The algebraic chromatic fracture square of Proposition 6.15 describes how objects in Stable_{BP_*BP} are assembled from their local pieces $L_{K(n)}X$. In analogy to Hopkins' chromatic splitting conjecture [24, Conjecture 4.2], one can ask if the map ι_X is split for compact X and, if so, how to further decompose its cofiber.

In fact, there are various versions of the algebraic chromatic splitting conjecture, corresponding to the analogous statements in chromatic homotopy theory. The most conceptual form asks whether ι_X is a split monomorphism for any $X \in \mathrm{Stable}^{\omega}_{BP_*BP}$.[†] However, we are interested in the more refined statement that also describes the other summand in the splitting. Furthermore, we will focus on the so-called edge case of the algebraic chromatic splitting conjecture corresponding to Hopkins' chromatic splitting conjecture at height n for a type $n-1$ complex.

To this end, fix $n \geqslant 0$ and note that the algebraic chromatic fracture square of Proposition 6.15 remains unchanged when X is localized at $E_* = (E_n)_*$. Therefore, by base-change we may assume without loss of generality that we are working in Stable_{E_*E} with, and we write L_n for what was previously denoted L_n^E. In [13, Theorem 6], Devinatz, Hopkins and Miller construct a class $\zeta \in \pi_{-1}L_{K(n)}S^0$ by lifting the determinant class $\det \in \mathrm{Hom}_{\mathrm{cts}}(\mathbb{G}_n, E_*)$, the set of continuous functions from the (extended) Morava stabilizer group $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ to E_*. If X is a finite spectrum of type $n-1$, then Hopkins' chromatic splitting conjecture stipulates that there is an equivalence

$$L_{n-1}X \oplus \Sigma^{-1}L_{n-1}X \xrightarrow{\sim} L_{n-1}L_{K(n)}X$$

induced by the natural inclusion and ζ. This conjecture is known to hold for $n=1$ as well as $n=2$ and $p \geqslant 3$, but needs to be modified for $n=2$ and $p=2$ by work of Beaudry [10]. Therefore, we will assume that p is large with respect to n for the remainder of this section.

In work in progress of the first author with Beaudry and Peterson, we explain how to construct an algebraic class $\zeta \in \mathrm{Hom}_{E_*E}(E_*, \lim^1_i E_*/I_n^i)$ associated to $\det \in E_*^{\vee}E = \pi_*L_{K(n)}(E \otimes E) \cong \mathrm{Hom}_{\mathrm{cts}}(\mathbb{G}_n, E_*)$.[‡] In order to lift this class to an analog in Stable_{E_*E} of the topological class ζ, we need the following lemma, which was proven in [8, Theorem 8.31].

Lemma 6.17. *For a finitely presented E_*E-comodule M and any $s \geqslant 0$, there is a canonical isomorphism $H^sL_{K(n)}M \cong \lim^s_i M/I_n^i$ of E_*E-comodules.*

Therefore, the convergent hyperext spectral sequence yields a (potentially trivial) class $\zeta \in \mathrm{Ext}^1_{E_*E}(E_*, L_{K(n)}E_*)$, that is, a map

$$\zeta_{E_*}\colon \Sigma^{-1}E_* \longrightarrow L_{K(n)}E_*$$

in Stable_{E_*E}. Note that, since p is assumed to be large with respect to n, Theorem 4.11 implies that $\mathrm{Stable}_{E_*E} \simeq \mathcal{D}_{E_*E}$. It follows that there are corresponding maps $\zeta_M\colon \Sigma^{-1}L_{n-1}M \to L_{n-1}L_{K(n)}M$ for any $M \in \mathrm{Stable}^{\omega}_{E_*E}$. We may thus state an algebraic version of the chromatic splitting conjecture.

Conjecture 6.18 (Algebraic chromatic splitting conjecture). For any $M \in \mathrm{Thick}(E_*/I_{n-1})$ there is an equivalence

$$L_{n-1}M \oplus \Sigma^{-1}L_{n-1}M \xrightarrow{\sim} L_{n-1}L_{K(n)}M$$

induced by the maps ι_M and ζ_M.

[†] Similar but inequivalent questions have been investigated by Hovey [24] and Devinatz [12].

[‡] The skeptical reader may consider the existence of this class as being part of the conjecture throughout this section.

Note that a thick subcategory argument reduces this conjecture to the case $M = E_*/I_{n-1}$. We will therefore restrict attention to the case that M is a finitely presented I_{n-1}-torsion E_*E-comodule viewed as an object of Stable_{E_*E} concentrated in degree 0. There are then two other equivalent formulations of this conjecture, in particular relating it to the version of the algebraic chromatic splitting conjecture proposed in unpublished work by Hopkins and Sadofsky. Combining the following result with a forthcoming paper by Barthel, Beaudry and Peterson, this would show that Conjecture 6.18 is equivalent to the topological chromatic splitting conjecture.

PROPOSITION 6.19. *For a finitely presented I_{n-1}-torsion E_*E-comodule M the following three statements are equivalent.*

(1) *The algebraic chromatic splitting conjecture holds for M.*
(2) *The maps ι_M and ζ_M induce isomorphisms*

$$\lim_i^s M/v_{n-1}^i \cong \begin{cases} M & \text{if } s=0 \\ v_{n-1}^{-1}M & \text{if } s=1 \\ 0 & \text{otherwise.} \end{cases}$$

(3) *The class ζ_M induces an equivalence $D(L_{n-1}E_*) \otimes M \simeq \Sigma^{-2}L_{n-1}M$, where D denotes internal duality in the stable category* Stable_{E_*E}.

Proof. Let $M \in \mathrm{Thick}(E_*/I_{n-1})$ and consider the fiber sequence

$$\Delta_n M \longrightarrow L_{n-1}M \xrightarrow{\iota_M} L_{n-1}L_{K(n)}M,$$

which follows from Proposition 6.15 (recall that we assume that $M \in \mathrm{Stable}_{E_*E}$). On the one hand, if the algebraic chromatic splitting conjecture holds for M, then we obtain an equivalence

$$\Sigma\Delta_n M \simeq \Sigma^{-1}L_{n-1}M.$$

On the other hand, (6.10) provides a natural equivalence $\Delta_n M \simeq \underline{\mathrm{Hom}}(L_{n-1}E_*, M)$, hence $\Delta_n M \simeq D(L_{n-1}E_*) \otimes M$ by compactness of M. This shows that (1) implies (3).

Now assume Statement (3), which is equivalent to $\Delta_n M \simeq \Sigma^{-2}L_{n-1}M$ as just shown. From the long exact sequence in cohomology associated to the fiber sequence $\Delta_n M \to M \to L_{K(n)}M$ we thus obtain

$$H^s L_{K(n)}M \cong \begin{cases} M & \text{if } s=0 \\ L_{n-1}M & \text{if } s=1 \\ 0 & \text{otherwise.} \end{cases}$$

The isomorphisms in (2) follow from this by virtue of Lemma 6.17 and Proposition 6.7, because M is I_{n-1}-torsion.

Finally, Condition (2) implies that the map

$$(\iota_M, \zeta_M)\colon L_{n-1}M \oplus \Sigma^{-1}L_{n-1}M \xrightarrow{\sim} L_{n-1}L_{K(n)}M$$

is a quasi-isomorphism. Since $\mathrm{Stable}_{E_*E} \simeq \mathcal{D}_{E_*E}$ for large p by Theorem 4.11, this gives the algebraic chromatic splitting conjecture for M. ☐

REMARK 6.20. Statement (3) of the previous proposition says in particular that $L_{n-1}(E_*/I_{n-1})$ is reflexive (or weakly dualizable) as an object in the derived category of $(E_*/I_{n-1}, E_*E/I_{n-1})$-comodules, that is, that $L_{n-1}E_*/I_{n-1} \simeq D^2_{I_{n-1}}(L_{n-1}E_*/I_{n-1})$ via the canonical map, where $D_{I_{n-1}} = \underline{\mathrm{Hom}}_{E_*E/I_{n-1}}(-, E_*/I_{n-1})$. This is remarkable, since $L_{n-1}E_*/I_{n-1} \in \mathrm{Stable}_{E_*E/I_{n-1}}$ is not compact and hence not dualizable.

REMARK 6.21. There is also a version of Proposition 6.19 that is independent of the existence of the algebraic analog of ζ. In this case, the proof still gives the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

7. *The algebraic chromatic convergence theorem*

The chromatic convergence theorem shows that a finite spectrum F can be recovered from its chromatic localizations L_nF. The goal of this section to establish an algebraic analog of this result for Stable_{BP_*BP}.

7.1. *The theory of algebraic n-buds and comodules*

In this section, we present an analog of the parts of the theory of n-buds of formal groups as developed by Goerss [**20**, Section 3.3] to the setting of BP_*BP-comodules, and then generalize it to Stable_{BP_*BP}. This will provide an appropriate setting for the first version of our algebraic chromatic convergence theorem, see Theorem 7.8.

DEFINITION 7.1. For any $0 \leqslant n \leqslant \infty$ let (B_n, W_n) be the Hopf algebroid representing $(n+1)$-buds of formal groups. Explicitly, $B_n = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ and $W_n = B_n[a_1, \ldots, a_n]$; viewing (B_n, W_n) as a sub-Hopf algebroid of (BP_*, BP_*BP) via the natural inclusion map

$$q_n\colon (B_n, W_n) \longrightarrow (BP_*, BP_*BP)$$

determines the structure maps. These functors induce a natural isomorphism

$$\mathrm{colim}_n (B_n, W_n) \cong (BP_*, BP_*BP) \tag{7.2}$$

of Hopf algebroids, which motivates to write $(B_\infty, W_\infty) = (B, W) = (BP_*, BP_*BP)$.

The map q_n gives rise to functors of abelian categories

$$(q_n)_*\colon \mathrm{Comod}_{W_n} \rightleftarrows \mathrm{Comod}_{BP_*BP}\colon (q_n)^*,$$

where the left adjoint is given by $(q_n)_*M = BP_* \otimes_{B_n} M$ with its natural comodule structure. As BP_* is flat as a B_n-module, q_n is exact. Note that in Goerss' algebro-geometric language [**20**], the left adjoint is denoted by $(q_n)^*$, whereas our choice of notation is consistent with the one in Section 6. The next result relates two important properties of a comodule to the categories Comod_{W_n}. Recall that a BP_*BP-comodule M is said to have projective BP_*-dimension $n \geqslant 0$ if the underlying BP_*-module $\epsilon_*(M)$ has projective dimension n.

LEMMA 7.3. *For a comodule $M \in \mathrm{Comod}_{BP_*BP}$, consider the following conditions.*

(1) *M is in the essential image of $(q_r)_*$.*
(2) *The projective BP_*-dimension of M is at most $r+1$.*
(3) *M is v_{r+2}-torsion free. Equivalently, M is v_i-torsion free for all $i \geqslant r+2$.*

Then Condition (1) implies the Condition (2). If M is additionally bounded below, then Condition (2) implies Condition (3).

Proof. Suppose first that M is in the essential image of $(q_r)_*$, say $M \cong (q_r)_*N$. Since the homological dimension of B_r is $r+1$, N admits a projective resolution by B_r-modules of length at most $r+1$. Since $(q_r)_*$ preserves projective objects, it follows that (1) implies (2).

As shown in [**37**, Proposition 2.5], a BP_*BP-comodule M is v_{r+1}-torsion free if and only if it is v_m-torsion free for all $m > r$, which gives the last claim in Condition (3). Moreover, Johnson and Yosimura prove that for bounded below M, this condition follows from M having homological BP_*-dimension $\leqslant r+1$, see [**37**, Proposition 3.7], hence (2) implies (3). □

In an earlier version of Goerss' manuscript [20], he uses the functors q_n to compare Comod_{BP_*BP} to an appropriately defined colimit of the categories Comod_{W_n}, see also [65]. In order to prove the algebraic chromatic convergence theorem, we will use a derived version of this theory. To this end, let Stable_{W_n} for $0 \leqslant n < \infty$ denote the stable category associated to Comod_{W_n}.

Lemma 7.4. *The stable category* Stable_{W_n} *is monogenic for all* n.

Proof. By Lemma 3.5 it suffices to show that (B_n, W_n) is a Landweber Hopf algebroid, and by the argument given in [26, Theorem 6.6] this will follow if we can show that every finitely presented W_n-comodule has a Landweber filtration. The proof for this is similar to that for BP_*BP-comodules; in fact, it is simpler because B_n is Noetherian. First, the invariant radical ideals in W_n are given by $I_k \cap B_n$ for $k \leqslant n$ [22, Ex. 5.10]. We then apply [41, Theorem 3.3] with $R = B_n \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n], S \cong \mathbb{Z}_{(p)}[a_1, \ldots, a_n]$ (so that $R \otimes S \cong B_n[a_1, \ldots, a_n] = W_n$), and $\Psi \colon B_n \to W_n$ given by the right unit of the Hopf algebroid (B_n, W_n). □

Proposition 7.5. *The maps* q_n *introduced above induce an exact functor* $q_* \colon \mathrm{colim}_n \mathrm{Stable}_{W_n} \to \mathrm{Stable}_{BP_*BP}$*, which restricts to an equivalence*

$$q_*^\omega \colon \mathrm{colim}_n \mathrm{Stable}_{W_n}^\omega \xrightarrow{\sim} \mathrm{Stable}_{BP_*BP}^\omega$$

of stable ∞*-categories.*

Proof. For any pair (m, n) with $0 \leqslant m \leqslant n \leqslant \infty$, the map $q_{m,n} \colon (B_m, W_m) \to (B_n, W_n)$ of Hopf algebroids induces a functor

$$(q_{m,n})_* \colon \mathrm{Stable}_{W_m} \longrightarrow \mathrm{Stable}_{W_n}$$

which preserves colimits and compact objects. These functors are compatible with each other, hence we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_n \mathrm{Stable}_{W_n} \arrow[r, "q_*"] \arrow[d, "\sim"'] & \mathrm{Stable}_{BP_*BP} \arrow[d, "\sim"] \\
\mathrm{colim}_n \mathrm{Mod}_{\mathrm{End}_{W_n}(B_n)} \arrow[r, "q_*"'] & \mathrm{Mod}_{\mathrm{End}_{BP_*BP}(BP_*)},
\end{tikzcd}$$

where the vertical equivalences follow from derived Morita theory and the previous lemma, see [45, Theorem 7.1.2.1; 63, Theorem 3.1.1]. Passing to compact objects and using that the functor $\mathrm{Mod}_{-}^{\omega} \colon \mathrm{Alg}_{\mathbb{E}_\infty} \to \mathrm{Cat}_\infty$ preserves filtered colimits, as is shown, for example, in the proof of [49, Proposition 2.4.1], this gives a functor

$$q_*^\omega \colon \mathrm{Mod}^\omega_{\mathrm{colim}_n \mathrm{End}_{W_n}(B_n)} \simeq \mathrm{colim}_n \mathrm{Mod}^\omega_{\mathrm{End}_{W_n}(B_n)} \longrightarrow \mathrm{Mod}^\omega_{\mathrm{End}_{BP_*BP}(BP_*)}.$$

Unraveling the construction, note that q_*^ω is induced by the natural map

$$\phi \colon \mathrm{colim}_n \mathrm{End}_{W_n}(B_n) \to \mathrm{End}_{BP_*BP}(BP_*),$$

so it suffices to prove that ϕ is an equivalence. To this end, let $C^*(B_n)$ be the cobar construction on B_n in Comod_{W_n}. Using (7.2) and exactness of q_*, we compute

$$\begin{aligned}
\mathrm{colim}_n \mathrm{Ext}^*_{W_n}(B_n, B_n) &\cong \mathrm{colim}_n H^*(\mathrm{Hom}_{W_n}(B_n, C^*(B_n))) \\
&\cong H^*(\mathrm{Hom}_{\mathrm{colim}_n W_n}(\mathrm{colim}_n B_n, C^*(\mathrm{colim}_n B_n))) \\
&\cong H^*(\mathrm{Hom}_{BP_*BP}(BP_*, C^*(BP_*))) \\
&\cong \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*),
\end{aligned}$$

hence ϕ is an equivalence. □

7.2. *Chromatic convergence*

Before we can come to the proof of the algebraic chromatic convergence theorem, we need a technical lemma regarding the vanishing of derived functors of inverse limits of comodules. We remind the reader about our grading conventions, see Section 1.

LEMMA 7.6. *Suppose $d \in \mathbb{Z}$ and $M = (M_n, \phi_n)_n \in (\mathrm{Stable}_{BP_*BP}^{\leqslant d})^{\mathbb{N}^{\mathrm{op}}}$ is an inverse system with structure maps $\phi_n \colon M_{n+1} \to M_n$. If for any $q \in \mathbb{Z}$ there exists $m(q)$ such that the induced map $H^q(\phi_n)$ is zero for all $n > m(q)$, then $\lim_n M_n \simeq 0$.*

Proof. Since $\mathrm{Stable}_{BP_*BP}^{\leqslant d} \simeq \mathcal{D}_{BP_*BP}^{\leqslant d}$, it suffices to show that $H^k \lim(M_n) = 0$ for all k. To this end, note that the convergent hypercohomology spectral sequence takes the form

$$E_2^{p,q} \cong \lim{}^p H^q(M_n) \Rightarrow \lim{}^{p+q} M_n,$$

where the derived limits on the E_2-page are computed with respect to the structure maps $H^q(\phi_n)$. By assumption, these morphisms are zero for all $n > m(q)$, so it follows from [**35**, Lemma 1.11] that $E_2^{p,q} = 0$ for all p and q. Therefore, $H^k \lim(M_n) \cong \lim^k M_n = 0$ for all $k \in \mathbb{Z}$. ☐

LEMMA 7.7. *If $X \in \mathcal{D}_{B_r}$ for some $r \geqslant 0$, then the natural map*

$$H_*\Gamma_n BP_* \otimes_{B_r} X \longrightarrow H_*\Gamma_{n-1} BP_* \otimes_{B_r} X$$

of BP_*-modules is zero for all $n > r$.

Proof. Consider the following segment of the long exact sequence in homology corresponding to the cofiber sequence $BP_*/I_n^\infty \to BP_*/I_n^\infty[v_n^{-1}] \to BP_*/I_{n+1}^\infty$:

$$H_*(BP_*/I_{n+1}^\infty \otimes_{B_r} X) \xrightarrow{\delta_n} H_{*-1}(BP_*/I_n^\infty \otimes_{B_r} X) \longrightarrow H_{*-1}(BP_*/I_n^\infty \otimes_{B_r} X)[v_n^{-1}].$$

By [**8**, Corollary 8.10], $\Gamma_{n-1} Y \simeq \Sigma^{-n} BP_*/I_n^\infty \otimes Y$ for all n and $Y \in \mathcal{D}_{BP_*}$. Applying this to $Y = BP_* \otimes_{B_r} X$, we need to show that δ_n is zero. But $H_*(BP_*/I_n^\infty \otimes_{B_r} X)$ is v_n-torsion free as $X \in \mathcal{D}_{B_r}$ and $n > r$, hence the second map in the above diagram is injective. ☐

THEOREM 7.8. *If $M \simeq q_* N$ for some $N \in \mathrm{Stable}_{W_r}^{<\infty}$, then there is a natural equivalence*

$$M \xrightarrow{\sim} \lim L_n M.$$

Proof. The natural cofiber sequences $\Gamma_n \to \mathrm{Id} \to L_n$ of functors induce a cofiber sequence

$$\lim_n \Gamma_n M \longrightarrow M \longrightarrow \lim_n L_n M$$

for any $M \in \mathrm{Stable}_{BP_*BP}$. Therefore, the claim is equivalent to the statement that $\lim_n \Gamma_n M \simeq 0$ whenever M satisfies the assumptions of the theorem. Because $\Gamma_n M \in \mathrm{Stable}_{BP_*BP}^{<\infty}$ for all $n \geqslant 0$, this will follow from Lemma 7.6 once we have shown that the morphism

$$H_*\Gamma_n(q_*N) \longrightarrow H_*\Gamma_{n-1}(q_*N)$$

is zero for all $n > r$. Since ϵ_* is faithful, it suffices to show that the left vertical map in the following commutative diagram

$$\begin{tikzcd}
\epsilon_* H_*\Gamma_n(q_*N) \arrow[r, "\sim"] \arrow[d] & H_*\epsilon_*\Gamma_n(q_*N) \arrow[r, "\sim"] \arrow[d] & H_*\Gamma_n^{BP_*}\epsilon_*(q_*N) \arrow[r, "\sim"] \arrow[d] & H_*\Gamma_n^{BP_*}BP_* \otimes_{B_r} \epsilon_*N \arrow[d] \\
\epsilon_* H_*\Gamma_{n-1}(q_*N) \arrow[r, "\sim"'] & H_*\epsilon_*\Gamma_{n-1}(q_*N) \arrow[r, "\sim"'] & H_*\Gamma_{n-1}^{BP_*}\epsilon_*(q_*N) \arrow[r, "\sim"'] & H_*\Gamma_{n-1}^{BP_*}BP_* \otimes_{B_r} \epsilon_*N
\end{tikzcd} \tag{7.9}$$

is zero for $n > r$. The commutativity of the first square is clear, while the second square commutes by [8, Lemma 5.20]; here, the superscript BP_* in Γ^{BP_*} indicates that those local cohomology functors are taken in $\mathcal{D}_{BP_*}$. Finally, the rightmost square commutes because Γ_n is smashing together with the commutative diagram

$$\begin{tikzcd}
\mathrm{Stable}_{W_r} \arrow[r, "q_*"] \arrow[d, "\epsilon_*"'] & \mathrm{Stable}_{BP_*BP} \arrow[d, "\epsilon_*"] \\
\mathcal{D}_{B_r} \arrow[r, "BP_*\otimes_{B_r}-"'] & \mathcal{D}_{BP_*} .
\end{tikzcd}$$

It therefore remains to show that the right vertical map in (7.9) is zero, which is the content of Lemma 7.7. $\square$

Corollary 7.10. *If $M \in \mathrm{Stable}_{BP_*BP}$ is compact, then there is a natural equivalence*

$$M \xrightarrow{\sim} \lim L_n M.$$

Proof. By Proposition 7.5, every compact object in Stable_{BP_*BP} is given by $(q_r)_*N$ for some r and $N \in \mathrm{Stable}_{W_r}^{<\infty}$. The result thus follows from Theorem 7.8. $\square$

In fact, the algebraic chromatic convergence theorem, Theorem 7.8, can be generalized to comodules with finite projective BP_*-dimension, by reducing the statement to its analog for BP_*-modules. This argument is essentially due to Hollander; since it has not appeared in print yet, we sketch the argument.

As in the proof of the previous theorem, let $\Gamma_n^{BP_*}$ and $L_n^{BP_*}$ denote the local cohomology functors on $\mathcal{D}_{BP_*}$ and write $\lim^{BP_*}$ for the total derived functor of inverse limit in this category.

Lemma 7.11. *Suppose $M \in \mathcal{D}_{BP_*}$ has finite projective dimension, then $M \simeq \lim_n^{BP_*} L_n^{BP_*} M$.*

Proof. To simplify the notation, in this proof only we write $\lim$ for $\lim^{BP_*}$. Without loss of generality, assume that M is represented by a complex of projective BP_*-modules concentrated in degrees between 0 and $-k$ for some $k \geqslant 0$. By [8, Lemma 5.33],

$$\Gamma_n^{BP_*} M \simeq \Sigma^{-n} M \otimes BP_*/I_{n+1}^{\infty}$$

is then concentrated in degrees between n and $n-k$. Consequently, $H^s(\Gamma_n^{BP_*} M) = 0$ for all $s < n-k$, that is, whenever $n > s+k$. The Milnor sequence

$$0 \longrightarrow \lim\nolimits_n^1 H^{s-1}(\Gamma_n^{BP_*} M) \longrightarrow H^s(\lim\nolimits_n \Gamma_n^{BP_*} M) \longrightarrow \lim\nolimits_n^0 H^s(\Gamma_n^{BP_*} M) \longrightarrow 0$$

thus implies $\lim_n \Gamma_n^{BP_*} M \simeq 0$ and the claim follows from the usual fiber sequence relating $\Gamma_n^{BP_*}$ and $L_n^{BP_*}$. $\square$

Theorem 7.12. *If $M \in \mathrm{Stable}_{BP_*BP}$ has finite projective BP_*-dimension, then there is a natural equivalence $M \simeq \lim_n L_n M$.*

Proof. Consider the cosimplicial Amitsur complex

$$C^\bullet(M) = (BP_*BP \otimes M \rightrightarrows BP_*BP^{\otimes 2} \otimes M \cdots)$$

of M. By [8, Theorem 4.29; 26, Corollary 5.2.4], the canonical map $M \to \mathrm{Tot}(C^\bullet(M))$ is an equivalence in Stable_{BP_*BP}. Note that the Amitsur complex is functorial in M and that $C^s(M) = BP_*BP^{\otimes s+1} \otimes M \simeq (\epsilon^*\epsilon_*)^{s+1}M$, where (ϵ_*, ϵ^*) is the forgetful-cofree adjunction between Stable_{BP_*BP} and $\mathcal{D}_{BP_*}$. Moreover, if M is of finite projective BP_*-dimension, then so is $C^s(M)$ for all $s \geqslant 0$ as BP_*BP is free over BP_*.

Recall that we denote the total derived limit in Stable_{BP_*BP} and $\mathcal{D}_{BP_*}$ by lim and $\lim^{BP_*}$, respectively. Using the fact that ϵ^* is a right adjoint as well as [8, Proposition 5.22], we obtain a sequence of natural equivalences

$$\begin{aligned}
\lim_n L_n M &\simeq \lim_n \mathrm{Tot}(C^\bullet(L_n M)) \\
&\simeq \mathrm{Tot}\lim_n((\epsilon^*\epsilon_*)^{\bullet+1}L_n M) \\
&\simeq \mathrm{Tot}\,\epsilon^*\lim_n^{BP_*}(\epsilon_*(\epsilon^*\epsilon_*)^\bullet L_n M) \\
&\simeq \mathrm{Tot}\,\epsilon^*\lim_n^{BP_*} L_n(\epsilon_*(\epsilon^*\epsilon_*)^\bullet M) \\
&\simeq \mathrm{Tot}\,\epsilon^*\epsilon_*(\epsilon^*\epsilon_*)^\bullet M \\
&\simeq \mathrm{Tot}\,C^\bullet(M) \\
&\simeq M,
\end{aligned}$$

where the fifth equivalence comes from Lemma 7.11. It is straightforward to verify that the composite of these natural maps are compatible with the canonical map $M \to \lim_n M$. □

Remark 7.13. Theorem 7.12 generalizes the algebraic chromatic convergence theorems of Goerss [20] and Sitte [64]. The generality of the theorem is analogous to the generalized (topological) chromatic convergence theorem of [5]. However, the topological chromatic convergence theorem does not follow formally from the algebraic version, due to the potential non-convergence of the corresponding inverse limit spectral sequence.

Corollary 7.14. *Suppose that either*

(1) *M is a bounded below BP_*BP-comodule which is flat as a BP_*-module; or*
(2) *X is a finite complex.*

Then, with $M = BP_*X$ in (2), there is a natural equivalence $M \simeq \lim_n L_n M$.

Proof. The previous theorem reduces the claim to showing that M has finite projective dimension. This follows from [68, Theorem 4.5] in the case of (1), and [42, Corollary 7] in the case of (2). □

7.3. *Further results*

In this subsection, we prove a vanishing result for local cohomology in Stable_{BP_*BP} and then deduce a comparison theorem for the E_2-terms of the Adams–Novikov and E-based Adams spectral sequence. Similar, but inequivalent results were originally proven by Goerss in the setting of quasi-coherent sheaves on the moduli of formal groups $\mathcal{M}_{fg}$.

Proposition 7.15. *Suppose $N \in \mathrm{Stable}_{W_r}^{\leqslant d}$ for some $d \in \mathbb{Z}$, then for all $s > r-n+d$ we have*

$$H_s(\Gamma_{n-1}BP_* \otimes_{B_r} N) = 0.$$

Proof. As in the proof of Theorem 7.8, this is readily reduced to the analogous statement in $\mathcal{D}_{BP_*}$, namely

$$H_s(\Gamma_{n-1}BP_* \otimes_{B_r} X) = 0$$

for $X \in \mathcal{D}_{B_r}$ and $s > r-n+d$. In order to prove this, we distinguish two cases. First, assume that $n \leqslant r$. The hypertor spectral sequence [**67**, 5.7.9 and Theorem 10.6.3] takes the form

$$E_2^{p,q} \cong \bigoplus_{i+j=q} \mathrm{Tor}_p^{BP_*}(H_i(\Gamma_{n-1}BP_*), H_j(BP_* \otimes_{B_r} X)) \Rightarrow H_{p+q}(\Gamma_{n-1}BP_* \otimes_{B_r} X).$$

Since $H_*(\Gamma_{n-1}BP_*) \cong \Sigma^{-n}BP_*/I_n^\infty$ has flat dimension n, then $E_2^{p,q} \neq 0$ only if $p \leqslant n$ and $q \leqslant -n+d$. Therefore, $H_{p+q}(\Gamma_{n-1}BP_* \otimes_{B_r} X) = 0$ if $p+q > d$, so certainly $H_s(\Gamma_{n-1}BP_* \otimes_{B_r} X) = 0$ for $s > r-n+d \geqslant d$.

For the second case, let $n > r$, and consider the exact sequence

$$H_s(\Gamma_{n-1}BP_* \otimes X) \longrightarrow H_s(\Gamma_{n-1}BP_* \otimes X)[v_n^{-1}] \longrightarrow H_s(\Sigma\Gamma_n BP_* \otimes X) \xrightarrow{\delta_n} \dots.$$

By Lemma 7.7, $\delta_n = 0$. Inductively, we know that $H_s(\Gamma_{n-1}BP_* \otimes X) = 0$ if $s > r-n+d$, so it follows that $H_s(\Sigma\Gamma_n BP_* \otimes X) = 0$ in the same range. In other words,

$$H_s(\Gamma_n BP_* \otimes X) = 0$$

for $s > r-(n+1)+d$. □

Proposition 7.16. *Let $E = E_n$ be Morava E-theory of height n. If $M \in \mathrm{Stable}_{BP_*BP}$ satisfies one of the following two conditions:*

1. *there exists $N \in \mathrm{Stable}_{W_r}^{\leqslant d}$ such that $M \simeq q_*N$, or*
2. *$M \in \mathrm{Comod}_{BP_*BP}$ (so $d = 0$) is of finite projective dimension at most $r-1$,*

then the natural localization morphism

$$\mathrm{Ext}^s_{BP_*BP}(BP_*, M) \xrightarrow{l^s} \mathrm{Ext}^s_{E_*E}(E_*, E_* \otimes_{BP_*} M)$$

is an isomorphism for $s < n-r-d$ and injective for $s = n-r-d$.

Proof. Throughout this proof, we will write $\mathrm{Ext}_\Psi(-)$ for the derived primitives $\mathrm{Ext}_\Psi(A,-)$ of a Hopf algebroid (A,Ψ). We also use the notation of Section 6.1; in particular, (Φ_*, Φ^*) denotes the base-change adjunction corresponding to $BP_* \to E_*$.

The morphism l^s is part of an exact sequence

$$\mathrm{Ext}^s_{BP_*BP}(\Gamma_n M) \longrightarrow \mathrm{Ext}^s_{BP_*BP}(M) \longrightarrow \mathrm{Ext}^s_{BP_*BP}(L_n M), \tag{7.17}$$

which is induced by the cofiber sequence $\Gamma_n M \to M \to L_n M$. Indeed, since $L_n M = \Phi^*\Phi_* M$ by Theorem 6.4, the last term can be rewritten as

$$\mathrm{Ext}^s_{BP_*BP}(L_n M) \cong \mathrm{Ext}^s_{E_*E}(E_* \otimes_{BP_*} M),$$

so that the second map in (7.17) can be identified with l^s. By Proposition 7.15, Condition (1) on M implies that $H^q\Gamma_n M = H_{-q}\Gamma_n M = 0$ if $-q > r-(n+1)+d$, that is, for $q \leqslant n-r-d$. If M satisfies the second condition instead, then the same argument as in the proof of Lemma 7.11

shows that $H^q\Gamma_n M = 0$ if $q < n-(r-1)$, that is, $q \leqslant n-r$. Plugging these computations into the hyperext spectral sequence

$$\mathrm{Ext}^p_{BP_*BP}(H^q\Gamma_n M) \Rightarrow \mathrm{Ext}^{p+q}_{BP_*BP}(\Gamma_n M),$$

we see that $\mathrm{Ext}^s_{BP_*BP}(\Gamma_n M) = 0$ for $s \leqslant n-r-d$, so the claim follows. □

REMARK 7.18. For discrete comodules, the first condition in Proposition 7.16 is weaker than the second condition, in the following sense: Suppose $M = q_*N$ for some $N \in \mathrm{Comod}_{W_r}$, so $d = 0$. By Lemma 7.3, M has projective dimension at most $r+1$, so Condition (2) gives an isomorphism l_s for all $s < n-r-2$, while appealing to Condition (1) gives it for $s < n-r$.

As an immediate consequence, we obtain

COROLLARY 7.19. *If X is a p-local bounded below spectrum such that BP_*X has projective BP_*-dimension $\mathrm{pdim}(BP_*X) \leqslant r$, then the natural map*

$$\mathrm{Ext}^s_{BP_*BP}(BP_*, BP_*(X)) \longrightarrow \mathrm{Ext}^s_{E_*E}(E_*, E_*(X))$$

is an isomorphism if $s < n-r-1$ and injective for $s = n-r-1$.

8. *The chromatic spectral sequence*

The chromatic spectral sequence was introduced by Miller, Ravenel and Wilson [**52**] as a tool for computing and organizing the E_2-term of the Adams–Novikov spectral for the sphere. Splicing together short exact sequences gives the chromatic resolution

$$\begin{tikzcd}[column sep=small] BP_* \arrow[r] & p^{-1}BP_* \arrow[rr] \arrow[dr] & & v_1^{-1}BP_*/p^\infty \arrow[rr] \arrow[dr] & & \cdots \\ & & BP_*/p^\infty \arrow[ur] & & BP_*/(p^\infty, v_1^\infty) \arrow[ur] & & \cdots \end{tikzcd}$$

and the resulting spectral sequence is the chromatic spectral sequence. As remarked, for example, in [**1**, **60**], one can proceed similarly for any bounded below spectrum X with BP_*X flat.

In this section, we will provide a different construction of the chromatic spectral sequence which works for an arbitrary object $M \in \mathrm{Stable}^{<\infty}_{BP_*BP}$, hence in particular for the BP-homology of *any* spectrum $X \in \mathrm{Sp}$. In the case that M is a bounded below flat comodule concentrated in a single degree, our spectral sequence recovers the classical one. However, our approach has several advantages over the classical one, as we will see shortly.

8.1. *The construction*

We will construct our generalization of the chromatic spectral sequence as the Bousfield–Kan spectral sequence associated to the algebraic chromatic tower (6.5).

THEOREM 8.1. *For any $M, N \in \mathrm{Stable}^{<\infty}_{BP_*BP}$, there is a natural convergent spectral sequence*

$$E_1^{n,s,t} \cong \mathrm{Ext}^{s,t}_{BP_*BP}(M, M_nN) \Rightarrow \mathrm{Ext}^{s,t}_{BP_*BP}(M, L_\infty N).$$

Furthermore, if N satisfies the conditions of Theorem 7.8 or Theorem 7.12, then the spectral sequence converges to $\mathrm{Ext}_{BP_*BP}(M, N)$.

Proof. Applying the functor $\underline{\mathrm{Hom}}(M,-)$ to the chromatic tower (6.5) of N yields a tower

$$\begin{tikzcd} \cdots \arrow[r] & \underline{\mathrm{Hom}}(M, L_2N) \arrow[r] & \underline{\mathrm{Hom}}(M, L_1N) \arrow[r] & \underline{\mathrm{Hom}}(M, L_0N) \\ \cdots & \underline{\mathrm{Hom}}(M, M_2N) \arrow[u] & \underline{\mathrm{Hom}}(M, M_1N) \arrow[u] & \underline{\mathrm{Hom}}(M, M_0N). \arrow[u] \end{tikzcd}$$

The Bousfield–Kan spectral sequence associated to this diagram, for example, in the form constructed by Lurie in [**45**, Proposition 1.2.2.14], thus takes the form

$$E_1 = \pi_*\underline{\mathrm{Hom}}(M, M_nN) \Rightarrow \pi_*\underline{\mathrm{Hom}}(M, \lim_n L_nN).$$

We claim that both M_nN and $L_\infty N$ are in $\mathrm{Stable}_{BP_*BP}^{<\infty}$. The first claim follows from the fact $M_nN \simeq M_nBP_* \otimes_{BP_*} N$ and Corollary 6.8. For the second claim it suffices by [**45**, Corollary 1.2.1.6] and Proposition 6.12 to show that $L_nN \in \mathrm{Stable}_{BP_*BP}^{<\infty}$. Using the fiber sequence $\Gamma_nN \to N \to L_nN$ we can in turn reduce to showing that $\Gamma_nN \in \mathrm{Stable}_{BP_*BP}^{<\infty}$, which follows from the formula $\Gamma_nN \simeq \Sigma^{-n}BP_*/I_n^\infty \otimes_{BP_*} N$, see [**8**, Proposition 8.9].

Then, using Lemma 3.16 and Proposition 6.12, we can rewrite this spectral sequence as

$$E_1 \cong \mathrm{Ext}_{BP_*BP}(M, M_nN) \Rightarrow \mathrm{Ext}_{BP_*BP}(M, L_\infty N).$$

To see the last part of the claim, it remains to note that $L_\infty N \simeq \lim_n L_nN \simeq N$ by Proposition 6.12 and Theorem 7.8. □

Remark 8.2. Presented in this form, it becomes transparent that the chromatic spectral sequence is completely analogous to the Bousfield–Kan spectral sequence associated to the topological chromatic tower in Sp:

$$\begin{tikzcd} \cdots \arrow[r] & L_2 \arrow[r] & L_1 \arrow[r] & L_0 \\ \cdots & M_2 \arrow[u] & M_1 \arrow[u] & M_0. \arrow[u] \end{tikzcd}$$

Evaluated on $X \in \mathrm{Sp}$, this spectral sequence takes the form

$$\pi_*M_nX \Rightarrow \pi_* \lim_n L_nX, \tag{8.3}$$

where M_n denotes the nth monochromatic functor and L_n is Bousfield localization at Johnson–Wilson theory $E(n)$. If X is chromatically complete [**5**], then the abutment is equivalent to π_*X. We will refer to this spectral sequence as the topological chromatic spectral sequence (TCSS).

When specialized to the BP-homology of the sphere, we recover the classical chromatic spectral sequence.

Corollary 8.4. *Suppose $M = N = BP_*$, then the spectral sequence of Theorem 8.1 takes the form*

$$E_1 = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, v_n^{-1}BP_*/I_n^\infty) \Rightarrow \mathrm{Ext}_{BP_*BP}^{s+n,t}(BP_*, BP_*)$$

Proof. By Corollary 6.8 there is an equivalence $M_nBP_* \simeq \Sigma^{-n}v_n^{-1}BP_*/I_n^\infty$. The result then follows from Theorem 8.1 and Corollary 7.10. □

Remark 8.5. More generally, since M_n is smashing, for any BP_*BP-comodule N there is a spectral sequence of the form

$$E_1^{s,t} = \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n^\infty \otimes N) \Rightarrow \operatorname{Ext}^{s+n,t}_{BP_*BP}(BP_*, L_\infty N).$$

Here the tensor product must be considered in the derived sense. Suppose that X is a spectrum such that $N = BP_*X$ is a bounded below flat BP_*-module, then the tensor product is automatically derived, and by Corollary 7.14 the spectral sequence abuts to $\operatorname{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*X)$.

Suppose now that X is a spectrum such that $BP_*(M_nX) \cong \Sigma^{-n}v_n^{-1}BP_*/I_n^\infty \otimes BP_*X$; for example, by [**61**, Chapter 8] this is true for the sphere, and hence also whenever BP_*X is a flat BP_*-module. If additionally $L_\infty BP_*X \simeq BP_*X$ (for example, if X is a finite complex, or if BP_*X is bounded below and flat), then it follows that there is a commutative diagram of spectral sequences

$$\begin{tikzcd}
\operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n^\infty \otimes BP_*X) \arrow[r, Rightarrow, "\mathrm{CSS}"] \arrow[d, "\cong"'] & \operatorname{Ext}^{s+n,t}_{BP_*BP}(BP_*, BP_*X) \arrow[dd, Rightarrow, "\mathrm{ANSS}"] \\
\operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, \Sigma^n BP_*M_nX) \arrow[d, Rightarrow, "\mathrm{ANSS}"'] & \\
\pi_{t-s-n}M_nX \arrow[r, Rightarrow, "\mathrm{TCSS}"'] & \pi_{t-n-s}X,
\end{tikzcd}$$

relating the chromatic spectral sequence (CSS) with the ANSS and the TCSS.

8.2. *The finite height chromatic spectral sequence*

It is easy to derive a finite height analog of the CSS from Theorem 8.1. First, we need a base-change lemma. We use the notation of Section 6.1; in particular, (Φ_*, Φ^*) denotes the base-change adjunction corresponding to $BP_* \to E(n)_*$.

Lemma 8.6. *For any $X, Y \in \mathrm{Stable}_{BP_*BP}$, there is a natural equivalence*

$$\operatorname{Hom}_{BP_*BP}(X, M_nY) \simeq \operatorname{Hom}_{E(n)_*E(n)}(E(n) \otimes X, E(n)_*/I_n^\infty \otimes Y),$$

with $E(n)_*/I_n^\infty \otimes Y \simeq \Gamma_{n-1}^{E(n)_*}\Phi_*Y$.

Proof. There are natural equivalences

$$M_n = L_n^{BP_*}\Gamma_{n-1}^{BP_*} \simeq \Phi^*\Phi_*\Gamma_{n-1}^{BP_*} \simeq \Phi^*\Gamma_{n-1}^{E(n)_*}\Phi_*,$$

the last one resulting from the equivalence $E(n)_* \otimes BP_*BP_*/I_n^\infty \simeq E(n)_*/I_n^\infty$. Consequently, by adjunction we obtain

$$\operatorname{Hom}_{BP_*BP}(X, M_nY) \simeq \operatorname{Hom}_{E(n)_*E(n)}(\Phi_*X, \Gamma_{n-1}^{E(n)_*}\Phi_*Y),$$

and the claim follows. $\square$

Proposition 8.7. *Fix an integer $n \geqslant 0$. For any $X, Y \in \mathrm{Stable}^{<\infty}_{BP_*BP}$, there is a natural strongly convergent spectral sequence of the form*

$$E_1^{k,s,t} = \begin{cases} \operatorname{Ext}^{s,t}_{E(k)_*E(k)}(E(k)_* \otimes X, E(k)_*/I_k^\infty \otimes Y) & k \leqslant n \\ 0 & k > n \end{cases}$$

converging to $\operatorname{Ext}^{s,t}_{BP_*BP}(X, L_nY) \cong \operatorname{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_* \otimes_{BP_*} X, E(n)_* \otimes_{BP_*} Y)$.

Proof. Truncating the chromatic tower at height n and using the same argument as in the proof of Theorem 8.1, we obtain a strongly convergent spectral sequence

$$E_1^{k\leqslant n} = \mathrm{Ext}_{BP_*BP}(X, M_kY) \Rightarrow \mathrm{Ext}_{BP_*BP}(X, L_nY).$$

Applying Lemma 8.6 to this, we obtain the desired E_1-term. The identification of the abutment follows a similar argument. □

We thus recover [**29**, Theorem 5.1]:

Corollary 8.8. *The CSS converging to* $\mathrm{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$ *has* E_1*-term*

$$E_1^{k,s,t} = \begin{cases} \mathrm{Ext}^{s,t}_{E(k)_*E(k)}(E(k)_*, E(k)_*/I_k^\infty) & k \leqslant n \\ 0 & k > n. \end{cases}$$

This spectral sequence was used by Hovey and Sadofsky in their calculations of the $E(n)$-local Picard group, see [**29**].

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