A NOTE ON STABLE RECOLLEMENTS

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Abstract. In this short étude, we observe that the full structure of a recollement on a stable ∞-category can be reconstructed from minimal data: that of a reflective and coreflective full subcategory. The situation has more symmetry than one would expect at a glance. We end with a practical lemma on gluing equivalences along a recollement.

Let $\mathbf{X}$ be a stable ∞-category and let $\mathbf{U}$ be a full subcategory of $\mathbf{X}$ that is stable under equivalences and is both reflective and coreflective – that is, its inclusion admits both a left and a right adjoint. We'll denote the inclusion functor $\mathbf{U} \subseteq \mathbf{X}$ by j_* and its two adjoints by j^* and $j^\times$, so that we have a chain of adjunctions

$$j^* \dashv j_* \dashv j^\times.$$

Let $\mathbf{Z}^\wedge \subseteq \mathbf{X}$ denote the right orthogonal complement of $\mathbf{U}$ – that is, the full subcategory of $\mathbf{X}$ spanned by those objects M such that $\mathrm{Map}_{\mathbf{X}}(N, M) = *$ for every $N \in \mathbf{U}$. Dually, let $\mathbf{Z}^\vee \subseteq \mathbf{X}$ denote the left orthogonal complement of $\mathbf{U}$ – that is, the full subcategory of $\mathbf{X}$ spanned by those objects M such that $\mathrm{Map}_{\mathbf{X}}(M, N) = *$ for every $N \in \mathbf{U}$. The inclusions of $\mathbf{Z}^\wedge \subseteq \mathbf{X}$ and $\mathbf{Z}^\vee \subseteq \mathbf{X}$ will be denoted $i_\wedge$ and $i_\vee$ respectively.

Warning 1. Our notation is chosen to evoke a geometric idea, but the role of open and closed is reversed from recollements that arise in the theory of constructible sheaves.

In our thinking, we imagine $\mathbf{X}$ as the ∞-category $\mathbf{D}_{qcoh}(X)$ of quasicoherent complexes over a suitably nice scheme X, which is decomposed as an open subscheme U together with a closed complement Z. In this analogy, we think of $\mathbf{U}$ as the ∞-category of quasicoherent modules on U, embedded via the (derived) pushforward. The subcategory $\mathbf{Z}^\vee$ is then the ∞-category of quasicoherent complexes on X that are set-theoretically supported on Z, and the subcategory $\mathbf{Z}^\wedge$ is the ∞-category of quasicoherent complexes on X that are complete along Z.

Lemma 2. *In this situation, $\mathbf{Z}^\wedge$ is reflective and $\mathbf{Z}^\vee$ is coreflective.*

Proof. Denote by κ the cofiber of the counit $j_* j^\times \to \mathrm{id}_{\mathbf{X}}$. Then $\kappa(\mathbf{X}) \subseteq \mathbf{Z}^\wedge$, so we factor

$$\kappa = i_\wedge i^\wedge$$

with $i^\wedge \in \mathrm{Fun}(\mathbf{X}, \mathbf{Z}^\wedge)$. We claim that $i^\wedge$ is left adjoint to $i_\wedge$. Indeed, for any $M \in \mathbf{X}$ and $N \in \mathbf{Z}^\wedge$, we have a cofiber sequence of spectra

$$F_{\mathbf{Z}^\wedge}(i^\wedge M, N) \simeq F_{\mathbf{X}}(i_\wedge i^\wedge M, i_\wedge N) \to F_{\mathbf{X}}(M, i_\wedge N) \to F_{\mathbf{X}}(j_* j^\times M, i_\wedge N) \simeq 0.$$

The proof that $\mathbf{Z}^\vee$ is coreflective is dual, and we'll denote the right adjoint of $i_\vee$ by $i^\vee$. ☐

Lemma 3. *In the sense of [2, Df. 3.4],*

$$\mathfrak{S}(\{0\}) = \mathbf{Z}^\wedge,\ \mathfrak{S}(\{1\}) = \mathbf{U},\ \mathfrak{S}(\Delta^1) = \mathbf{X},\ \mathfrak{S}(\emptyset) = 0$$

is a stratification of $\mathbf{X}$ *along* Δ^1.

Proof. After unravelling the notation, one sees that this amounts to the following two claims.

- First, $i^\wedge j_* j^* = 0$. This point is obvious.
- The usual fracture square

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & i_\wedge i^\wedge \\ \downarrow & & \downarrow \\ j_* j^* & \longrightarrow & j_* j^* i_\wedge i^\wedge \end{array}$$

is cartesian. To see this, take fibers of the horizontal maps to get the map

$$j_* j^\times \to j_* j^* j_* j^\times,$$

which is an equivalence since $j^* j_*$ is homotopic to the identity. □

Remark 4. Conversely, if $\mathfrak{S}$ is a stratification of $\mathbf{X}$ along Δ^1, then $\mathfrak{S}(\{0\})$ is coreflective as well as reflective. Indeed, the fracture square together with the argument of Lm. 2 shows that the fiber of $\mathrm{id} \to \mathcal{L}_1$ defines a right adjoint to the inclusion of $\mathfrak{S}(\{0\})$.

Lemma 5. *In the sense of [3, Df. A.8.1],* $\mathbf{X}$ *is a recollement of* $\mathbf{U}$ *and* $\mathbf{Z}^\wedge$.

Proof. The only claim that isn't obvious is point e): that j^* and $i^\wedge$ are jointly conservative. But since they are exact functors of stable ∞-categories, this is equivalent to the claim that if j^*M and $i^\wedge M$ are both zero, then M is zero, and this is clear from the fracture square. □

Remark 6. Again there's a converse; indeed, if a stable ∞-category $\mathbf{X}$ is a recollement of $\mathbf{U}$ and $\mathbf{Z}$, then $\mathbf{U}$ is coreflective [3, Rk. A.8.5]. We thus conclude that the following three pieces of data are essentially equivalent:

- reflective and coreflective subcategories of $\mathbf{X}$,
- stratifications $\mathfrak{S}$ along Δ^1 in the sense of [2, Df. 3.4] with $\mathfrak{S}(\Delta^1) = \mathbf{X}$, and
- recollements of $\mathbf{X}$ in the sense of [3, Df. A.8.1].

As we have described this structure, there's a surprising intrinsic symmetry that traditional depictions of recollements don't really bring out:

Proposition 7. *The functors* $i^\wedge i_\vee$ *and* $i^\vee i_\wedge$ *define inverse equivalences of categories between* $\mathbf{Z}^\wedge$ *and* $\mathbf{Z}^\vee$.

This proposition is an extreme abstraction of prior results, such as those of [1], giving equivalences between categories of complete objects and categories of torsion objects.

Proof. Let's show that the counit map

$$\eta\colon i^\wedge i_\vee i^\vee i_\wedge \to \mathrm{id}$$

is an equivalence; the other side will of course be dual. The counit factors as

$$i^\wedge i_\vee i^\vee i_\wedge \xrightarrow{\eta_0} i^\wedge i_\wedge \xrightarrow{\eta_1} \mathrm{id},$$

but of course η_1 is an equivalence since $i_\wedge$ is fully faithful. But η_0 fits into a cofiber sequence

$$i^\wedge i_\vee i^\vee i_\wedge \xrightarrow{\eta_0} i^\wedge i_\wedge \to i^\wedge j_* j^* i_\wedge,$$

and the final term is zero since $i^\wedge j_* = 0$. $\square$

Finally, we give a useful criterion for when a morphism of recollements gives rise to an equivalence, the proof of which is unfortunately a little more technical than the foregoing.

Proposition 8. *Let $\mathbf{X}$ and $\mathbf{X}'$ be stable ∞-categories with reflective, coreflective subcategories $\mathbf{U} \subseteq \mathbf{X}$ and $\mathbf{U}' \subseteq \mathbf{X}'$ and ancillary subcategories*

$$\mathbf{Z}^\vee \subseteq \mathbf{X},\ \mathbf{Z}^\wedge \subseteq \mathbf{X},\ (\mathbf{Z}')^\vee \subseteq \mathbf{X}',\ (\mathbf{Z}')^\wedge \subseteq \mathbf{X}'.$$

Suppose $F\colon \mathbf{X} \to \mathbf{Y}$ is a functor with

$$F(\mathbf{U}) \subseteq \mathbf{U}',\ F(\mathbf{Z}^\wedge) \subseteq (\mathbf{Z}')^\wedge,\ F(\mathbf{Z}^\vee) \subseteq (\mathbf{Z}')^\vee.$$

Suppose moreover that $F|_{\mathbf{U}}$ and at least one of $F|_{\mathbf{Z}^\wedge}$ and $F|_{\mathbf{Z}^\vee}$ is an equivalence. Then F is an equivalence.

Proof. Let's suppose that $F|_{\mathbf{Z}^\wedge}$ is an equivalence; once again, the other case is dual.

Lemma 9. *Set*

$$\mathbf{Z}^\wedge \downarrow_{\mathbf{X}} \mathbf{U} = \mathbf{Z}^\wedge \times_{\mathbf{X}} \mathrm{Fun}(\Delta^1, \mathbf{X}) \times_{\mathbf{X}} \mathbf{U}$$

be the ∞-category of morphisms in $\mathbf{X}$ whose source is in $\mathbf{Z}^\wedge$ and whose target is in $\mathbf{U}$; we claim that the functor

$$k\colon \mathbf{Z}^\wedge \downarrow_{\mathbf{X}} \mathbf{U} \to \mathbf{X}$$

that maps a morphism to its cofiber is an equivalence.

Proof. The functor k is really constructed as a zigzag

$$\mathbf{Z}^\wedge \downarrow_{\mathbf{X}} \mathbf{U} \xleftarrow{\sim} \mathbf{E} \xrightarrow{t} \mathbf{X},$$

where $\mathbf{E}$ is the ∞-category of cofiber sequences $M \to N \to P$ in $\mathbf{X}$ for which $(M \to N) \in \mathbf{Z}^\wedge \downarrow_{\mathbf{X}} \mathbf{U}$. The leftward arrow is a trivial Kan fibration. We'd like to prove that the right hand arrow, t, is also a trivial Kan fibration. It's clearly a cartesian fibration, and so it suffices to show that each fiber of t is a contractible Kan complex.

The fiber of t over P is the ∞-category of cofiber sequences

$$M \to N \to P$$

with $M \in \mathbf{Z}^\wedge$ and $N \in \mathbf{U}$. Since fibers are unique, this is equivalent to the ∞-category of morphisms $\phi\colon N \to P$ with $N \in \mathbf{U}$ and $\mathrm{fib}(\phi) \in \mathbf{Z}^\wedge$. But $\mathrm{fib}(\phi) \in \mathbf{Z}^\wedge$ if and only if ϕ exhibits N as the $\mathbf{U}$-colocalization of P, and such a ϕ exists uniquely. $\square$

Corollary 10. *The ∞-category $\mathbf{X}$ is equivalent to the ∞-category of sections of the map*

$$p\colon \mathbf{C} \to \Delta^1$$

where $\mathbf{C} \subseteq \mathbf{X} \times \Delta^1$ is the full subcategory spanned by objects of $\mathbf{Z}^\wedge \times \{0\}$ or $\mathbf{U} \times \{1\}$. $\square$

Observe here that p is a cocartesian fibration, and the cocartesian edges correspond to morphisms $f\colon M \to N$ in $\mathbf{X}$ which exhibit N as the $\mathbf{U}$ localization of M.

Now we finish the proof of Pr. 8. In fact, $F\colon \mathbf{X} \to \mathbf{X}'$ induces a functor over Δ^1

$$\overline{F}\colon \mathbf{C} \to \mathbf{C}',$$

where $\mathbf{C}' \subseteq \mathbf{X}' \times \Delta^1$ is the full subcategory spanned by objects of $(\mathbf{Z}')^{\wedge} \times \{0\}$ or $\mathbf{U}' \times \{1\}$. By hypothesis, $\overline{F}$ induces equivalences on the fibers over $\{0\}$ and $\{1\}$. If $\overline{F}$ moreover preserves cocartesian edges, we'll be able to conclude that $\overline{F}$ is an equivalence of ∞-categories, inducing an equivalence on ∞-categories of sections, whence the result.

The claim that $\overline{F}$ preserves cocartesian edges is equivalent to the claim that the naturally lax-commutative square

$$\begin{array}{ccc} \mathbf{Z}^{\wedge} & \xrightarrow{j^* i_{\wedge}} & \mathbf{U} \\ {\scriptstyle F|_{\mathbf{Z}^{\wedge}}}\downarrow & & \downarrow{\scriptstyle F|_{\mathbf{U}}} \\ (\mathbf{Z}')^{\wedge} & \xrightarrow[(j')^*(i')^{\wedge}]{} & \mathbf{U}' \end{array}$$

is in fact commutative up to equivalence. In fact, the stronger claim that the lax-commutative square

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{j^*} & \mathbf{U} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F|_{\mathbf{U}}} \\ \mathbf{X}' & \xrightarrow[(j')^*]{} & \mathbf{U}' \end{array}$$

commutes up to equivalence is equivalent to the claim that F takes j^*-equivalences to $(j')^*$-equivalences. But this is the case if and only if F takes left orthogonal objects to $\mathbf{U}$ – that is, objects of $\mathbf{Z}^{\vee}$ – to left orthogonal objects to $\mathbf{U}'$ – that is, objects of $(\mathbf{Z}')^{\vee}$. Since this was one of our hypotheses, the proof is complete.

References

[1] William G Dwyer and John Patrick Campbell Greenlees. Complete modules and torsion modules. *American Journal of Mathematics*, 124(1):199–220, 2002.

[2] Saul Glasman. Stratified categories, geometric fixed points and a generalized Arone-Ching theorem. *arXiv preprint arXiv:1507.01976*, 2015.

[3] Jacob Lurie. *Higher Algebra*. 2012.