

Power operations for $H\underline{\mathbb{F}}_2$ and a cellular construction of BPR

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Abstract

We study some power operations for ordinary C_2 -equivariant homology with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$. In addition to a few foundational results, we calculate the action of these power operations on a C_2 -equivariant dual Steenrod algebra. As an application, we give a cellular construction of the C_2 -spectrum BPR and deduce its slice tower.

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Introduction

From a user’s perspective, the primary goal of this paper is to construct equivariant power operations for $\mathbb{H}\mathbb{F}_2$ and compute their action on an equivariant dual Steenrod algebra. This comprises the bulk of the work below, and the main results are contained in Theorem 4.4 and Theorem 5.4 below. They can be summarized as follows:

Theorem. *If X is a C_2 -spectrum admitting an equivariant symmetric multiplication, $\mathbb{P}_2(X) \rightarrow X$, then $(\mathbb{H}\mathbb{F}_2)_*X$ admits natural operations $Q^{n\rho}$ of degree $n\rho$ for $n \in \mathbb{Z}$ and satisfying analogues of the standard properties of the non-equivariant Dyer-Lashof operations as in, e.g., [Bru+86, Ch. III].*

Theorem. *As an algebra over the ring of equivariant power operations, the equivariant dual Steenrod algebra $\pi_*(\mathbb{H}\mathbb{F}_2 \wedge \mathbb{H}\mathbb{F}_2)$ is generated by $\tau_0 \in \pi_1^{C_2}(\mathbb{H}\mathbb{F}_2 \wedge \mathbb{H}\mathbb{F}_2)$.*

There are several departures from the classical story in the proofs. Any non-equivariant result that depends on the fact that $\pi_*\mathbb{H}\mathbb{F}_2$ vanishes away from π_0 is now much more delicate, since the equivariant homology of a point is far from trivial. For example, in general there is no Thom isomorphism which computes the homology of extended powers of representation spheres.

Nevertheless, it is possible to give a complete analysis of the extended powers of *regular* representation spheres, $S^{n\rho}$, and single desuspensions of these— in other words, slice cells. This aligns with the emerging philosophy on equivariant homotopy theory:

The Slice Filtration: It’s ordinary homotopy theory, only twisted.¹

That said, we would still like to have operations for classes in degree V for an arbitrary representation sphere. After all, the purported generator τ_0 in the dual Steenrod algebra is in degree 1, which is not of the form $n\rho - 1$. To do this, we use a trick inspired by Goodwillie calculus. The idea is to replace the extended powers $\mathbb{P}_2(S^{n\rho})$ by the co-linearization:

$$\text{Ops}_{\mathbb{H}\mathbb{F}_2} := \text{holim}_n(\mathbb{H}\mathbb{F}_2 \wedge \Sigma^{n\rho}\mathbb{P}_2(S^{-n\rho})).$$

Since regular representation spheres are cofinal amongst all representation spheres, this allows us to define power operations in general degrees. This trick, has a long history, and can be found in various degrees of overttness in the references [JW83; Lur11; Lur07; KM13].

The use of this spectrum $\text{Ops}_{\mathbb{H}\mathbb{F}_2}$ has more advantages. In §4 we identify this spectrum with the desuspension of a Tate construction, and this allows us to easily deduce the Steenrod coaction on the homotopy of this spectrum. In §5 we apply this calculation to proving the Nishida relations and computing the action of the power operations on the dual Steenrod algebra. This approach is worthwhile even in the non-equivariant case, where it gives a cleaner derivation of Steinberger’s formulas in [Bru+86, p. III.2].

Before describing the second main goal of this paper, it is perhaps helpful to give some motivation.

In the course of their resolution of the Kervaire invariant problem, Hill, Hopkins, and Ravenel used a machine-crafted C_8 -equivariant spectrum. This machine had organic input, however: the C_2 -equivariant spectrum **MUR** of **R**eal cobordism, which is found in nature.

In the odd-primary setting, we are not so lucky. Doug Ravenel, in unpublished work, has indicated that, if we had a C_p -spectrum that displayed some of the same excellent behavior as **MUR** does in the C_2 -equivariant case, then we could likely resolve the 3-primary Kervaire invariant problem. Moreover, **MUR** and spectra built from it have had applications in homotopy theory at the prime 2 aside from the Kervaire invariant problem, and we could hope to achieve similar success at odd primes if we had an analog of **MUR**.

Before tackling the odd primary story, it seems prudent to revisit the prime 2 with an eye towards generalization. The original motivation for this paper was to answer the following:

¹This slogan is inspired by the 1997 ad campaign for the orange-flavored soft drink ‘Slice’.

Question. Is it possible to construct \mathbf{MUR} , or the 2-local summand \mathbf{BPR} , without using anything about manifolds or formal group laws?

The answer to this question non-equivariantly is yes: the original construction of \mathbf{BP} by Brown and Peterson used only homological properties of the Steenrod algebra. Their technique was to build \mathbf{BP} by writing down its Adams resolution. Later, a dual construction was found by Priddy [Pri80] that was even simpler:

Theorem (Priddy, [Pri80]). *The spectrum obtained from $S_{(p)}^0$ by non-trivially attaching even, p -local cells in order to kill all odd homotopy groups is equivalent to \mathbf{BP} .*

In §6 below, we prove the following as Theorem 6.24.

Theorem. *The C_2 -spectrum obtained from $S_{(2)}^0$ by non-trivially attaching even dimensional, 2-local slice cells in order to kill all homotopy Mackey functors in degrees of the form $n\rho - 1$ is equivalent to \mathbf{BPR} .*

It is important to stress that if this theorem were all we were after, the paper would be very short. Since we know \mathbf{BPR} exists, we can compare it to this cellular construction and see immediately that it gives the right answer, just as in Priddy [Pri80, p. 31]. But this paper is a thought experiment: can we deduce the basic properties of \mathbf{BPR} directly from its cellular construction? Let $\mathbf{Even}(S^0)$ denote the 2-local, C_2 -equivariant spectrum obtained as in the theorem. We prove the next result as Theorem 6.23 in the text.

Theorem. *Without invoking the existence of \mathbf{BPR} , it is possible to establish the following properties of $\mathbf{Even}(S^0)$ from its construction:*

- (i) *The spectrum underlying $\mathbf{Even}(S^0)$ is \mathbf{BP} ,*
- (ii) *the geometric fixed points of $\mathbf{Even}(S^0)$ are equivalent to \mathbf{HF}_2 ,*
- (iii) *the homotopy Mackey functors $\pi_{*,\rho}\mathbf{Even}(S^0)$ are constant and given by $\mathbb{Z}_{(2)}[v_1, v_2, \dots]$ where $|v_i| = (2^i - 1)\rho$, and ρ is the regular representation of C_2 .*
- (iv) *the homotopy Mackey functors $\pi_{*,\rho-1}\mathbf{Even}(S^0)$ vanish.*

These last two properties determine the slice tower of $\mathbf{Even}(S^0)$.

The main hurdle is in identifying the homology of $\mathbf{Even}(S^0)$. To do so, we use obstruction theory on the slice tower to inductively construct power operations on $\mathbf{Even}(S^0)$ which constrain the homology, and in turn the homotopy, and use this new information to show the next obstruction vanishes. This argument is far more involved than the classical case, and requires some delicate arguments in $RO(C_2)$ -graded homological algebra.

Acknowledgements. This paper benefitted greatly from conversations with Mike Hill. I thank Doug Ravenel for sharing his notes on the odd primary Kervaire invariant problem; this served as a guide for what sorts of results we want in the C_p -equivariant case, and thus influenced the choice of methods and presentation used here. I am very grateful to Mike Hopkins for telling me about Priddy's result. I would also like to thank Jim McClure for suggesting that, in order to compute the homology of more general extended powers, I reduce to the case of a sphere instead of arguing on the chain level. I thank Sean Tilson and Cary Malkewich for helpful comments on an earlier version of this paper. I am also grateful to Nick Kuhn for telling me about some of the history of the ideas I used in §3. Finally, none of this would be possible without the encouragement and wisdom of my advisor, Paul Goerss.

Notation and conventions. We display Mackey functors M for C_2 by diagrams:

$$\begin{array}{c} M(C_2/C_2) \\ \text{res} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \text{tr} \\ M(C_2) \end{array}$$

where $M(C_2)$ is a C_2 -module. As usual \underline{A} will denote the Burnside Mackey functor.

If M is an abelian group, \underline{M} will denote the associated constant Mackey functor. When we say “ $x \in \underline{M}$ ” we mean x is an element of either $\underline{M}(C_2)$ or $\underline{M}(C_2/C_2)$. Throughout we write \mathbb{F} to mean \mathbb{F}_2 .

Unless otherwise stated, the group in the background is C_2 . For the author’s sanity, we distinguish between C_2 and Σ_2 ; hopefully this does not have the opposite effect on the reader.

1 Preliminaries

1.1 C_2 -equivariant homotopy and homology

Fixed points

Let EC_2 denote a contractible space with a free C_2 -action, and define \tilde{EC}_2 by the cofiber sequence of pointed C_2 -spaces:

$$EC_{2+} \rightarrow S^0 \rightarrow \tilde{EC}_2.$$

If E is a C_2 -spectrum, then we will use the following shorthand:

$$E^h := F(EC_{2+}, E), \quad E_h := EC_{2+} \wedge E, \quad E^t := \tilde{EC}_2 \wedge F(EC_{2+}, E), \quad \Phi E := \tilde{EC}_2 \wedge E.$$

We will sometimes refer to E^h as the *Borel completion* of E . The genuine fixed points of each spectrum above are then:

- the homotopy fixed points E^{hC_2} ,
- the homotopy orbits E_{hC_2} ,
- the Tate spectrum E^{tC_2} , and
- the geometric fixed points $\Phi^{C_2} E$.

This is true by definition in each case except for the equivalence:

$$(E_h)^{C_2} \cong E_{hC_2}$$

which is a consequence of the Adams isomorphism.

Homotopy Mackey functors

Given a representation V and a C_2 -spectrum E we define a Mackey functor by the following formula for a C_2 -set T :

$$\pi_V(E) : T \mapsto [T_+ \wedge S^V, E].$$

Given two representations V, W , we define

$$\pi_{V-W}(E) : T \mapsto [T_+ \wedge S^V, S^W \wedge E].$$

We will denote the value of this Mackey functor on $*$ and C_2 by:

$$\pi_{V-W}(E)(*) =: \pi_{V-W}^{C_2}(E),$$

$$\pi_{V-W}(E)(C_2) =: \pi_{V-W}^u(E).$$

For every choice of isomorphism $V \oplus W' \cong V' \oplus W$ we get an isomorphism of Mackey functors

$$\pi_{V-W}(E) \cong \pi_{V'-W'}(E)$$

but this isomorphism depends on the first choice. We choose once and for all a preferred isomorphism of a virtual representation to one of the form $a + b\sigma$, where σ is the sign representation of C_2 . In this way the homotopy Mackey functors of E are $RO(C_2)$ -graded and we denote the whole collection by

$$\pi_* E$$

When E admits a pairing $E \wedge E \rightarrow E$, then $\pi_* E$ admits a ring structure which is graded commutative in the following sense: If the degrees of x and y are written as $|x| = a + b\sigma$ and $|y| = a' + b'\sigma$ then

$$xy = (-1)^{aa'} \epsilon^{bb'} yx$$

where $\epsilon = 1 - [C_2]$ in the Burnside Mackey functor $\underline{A}(C_2)$. For a proof, see, for example, [HK01, Lem. 2.12]. In particular, whenever $[C_2] - 2$ acts by zero, ϵ acts by -1 . This holds for any module spectrum over $H\underline{\mathbb{Z}}$, for example, which covers every example of interest in this paper.

Homology of a point

We now wish to describe the homology of a point. Before we do so we will need to name some elements. For any representation V , let $a_V \in \pi_{-V}^{C_2} S^0$ denote the element

$$a_V : S^0 \rightarrow S^V$$

given by the inclusion of $\{0, \infty\}$. This is sometimes called an *Euler class*.

For any representation V of dimension d , the restriction map

$$\mathbb{H}_d^{C_2}(S^V) \rightarrow \mathbb{H}_d^u(S^V)$$

is an isomorphism ([HHR16, Ex. 3.10]), and we denote the unique nonzero class by u_V . In particular, we have a class

$$u_\sigma \in \mathbb{H}_1^{C_2}(S^\sigma) = \pi_{1-\sigma}^{C_2} \mathbb{H}\mathbb{F}.$$

We'll start by stating the calculation in the Borel complete, Tate, and geometric cases.

Proposition 1.1. *As $RO(C_2)$ -graded Green functors, we have*

$$\begin{aligned} \pi_* \mathbb{H}\mathbb{F}^h &= \begin{array}{c} \mathbb{F}_2[u_\sigma^{\pm 1}, a_\sigma] \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\ \mathbb{F}_2[\text{res}(u_\sigma)^{\pm 1}] \end{array} & \quad \pi_* \mathbb{H}\mathbb{F}^t &= \begin{array}{c} \mathbb{F}_2[u_\sigma^{\pm 1}, a_\sigma^{\pm 1}] \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\ 0 \end{array} \\ \\ \pi_* \Phi \mathbb{H}\mathbb{F} &= \begin{array}{c} \mathbb{F}_2[u_\sigma, a_\sigma^{\pm 1}] \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\ 0 \end{array} \end{aligned}$$

where $\text{res}(a_\sigma) = 0$.

The case we'll be most interested in is recorded in the next proposition.

Proposition 1.2. *As an $RO(C_2)$ -graded ring we have the square-zero extension:*

$$\pi_*^{C_2} \mathbb{H}\mathbb{F} = \mathbb{F}[u_\sigma, a_\sigma] \oplus \mathbb{F} \left\{ \frac{1}{a_\sigma^k u_\sigma^{n+2}} \mid k, n \geq 0 \right\}.$$

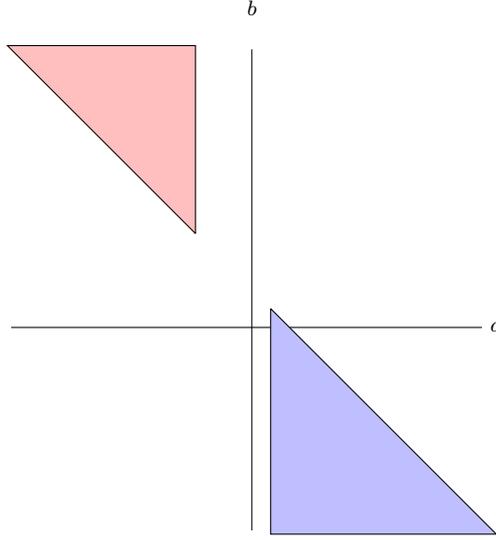
The underlying homotopy groups are given by

$$\pi_*^u \mathbb{H}\mathbb{F} = \mathbb{F}[\text{res}(u_\sigma)^{\pm 1}]$$

and the Mackey functor structure is determined by $\text{res}(a_\sigma) = 0$ and $\text{tr}(\text{res}(u_\sigma)^{-n-2}) = u_\sigma^{-n-2}$ for $n \geq 0$.

Warning 1.3. The element $\text{res}(u_\sigma)^{-n}$ is not in the image of the restriction map when $n \geq 1$.

Here is a picture of the groups $\pi_{a+b\sigma}^{C_2} \underline{\mathbb{H}\mathbb{F}}$:



The blue region is the polynomial part $\mathbb{F}[a_\sigma, u_\sigma]$, and the red region is the Pontryagin dual piece. Notice the important gap $\pi_{-1+*\sigma}^{C_2} \underline{\mathbb{H}\mathbb{F}} = \pi_{*\rho-1}^{C_2} \underline{\mathbb{H}\mathbb{F}} = 0$. Also note that the only nonzero group $\pi_{*\rho}^{C_2} \underline{\mathbb{H}\mathbb{F}}$ is $\pi_0^{C_2} \underline{\mathbb{H}\mathbb{F}} = \mathbb{F}$.

The elements $\frac{1}{a_\sigma^k u_\sigma^{n+2}}$ come from the boundary map:

$$\partial : \underline{\mathbb{H}\mathbb{F}}_*^t \rightarrow \Sigma \underline{\mathbb{H}\mathbb{F}}_{h*} \rightarrow \Sigma \underline{\mathbb{H}\mathbb{F}}_*$$

Specifically:

$$\partial \left(\frac{1}{a_\sigma^{k+1} u_\sigma^{n+1}} \right) = \begin{cases} \frac{1}{a_\sigma^k u_\sigma^{n+2}} & k, n \geq 0 \\ 0 & \text{else} \end{cases}$$

1.2 C_2 -Steenrod algebra

There are lots of versions of the dual Steenrod algebra equivariantly. We will describe the geometric, Borel-complete, Tate, and genuine versions below. The calculations can be found in Hu-Kriz, and the Mackey structure was computed in [Ric15].

Convention 1.4. We let $\zeta_i \in \mathcal{A}_*$ denote Milnor's generators of the dual Steenrod algebra (*not*, as the convention has become, their conjugates!) To be absolutely clear, the right, completed coaction on $\mathbb{H}^*(\mathbb{R}P^\infty) = \mathbb{F}_2[w]$ is given by

$$w \mapsto \sum w^{2^i} \otimes \zeta_i.$$

Proposition 1.5. *As a left $\Phi \underline{\mathbb{H}\mathbb{F}}_*$ -algebra,*

$$\pi_* \Phi(\underline{\mathbb{H}\mathbb{F}} \wedge \underline{\mathbb{H}\mathbb{F}}) = \Phi \underline{\mathbb{H}\mathbb{F}}_*[\zeta_i, \bar{u}_\sigma]$$

where the inclusion of the usual dual Steenrod algebra is a map of Hopf algebroids and $\eta_R(a_\sigma) = \eta_L(a_\sigma)$.

Proposition 1.6. *As a left $\underline{\mathbb{H}\mathbb{F}}_*^h$ -algebra, we have*

$$\pi_*(\underline{\mathbb{H}\mathbb{F}} \wedge \underline{\mathbb{H}\mathbb{F}})^h = \underline{\mathbb{H}\mathbb{F}}_*^h[\zeta_i]_{a_\sigma}.$$

The inclusion of the dual Steenrod algebra is a map of (completed) Hopf algebroids, and

$$\eta_R(u_\sigma^{-1}) = \sum_{i \geq 0} u_\sigma^{-2^i} \zeta_i a_\sigma^{2^i - 1}, \quad \eta_r(a_\sigma) = a_\sigma.$$

The Mackey functor structure is determined by declaring that ζ_i restricts to the usual ζ_i .

Corollary 1.7. *As a left $\mathbb{H}\mathbb{F}_\star^t$ -algebra, we have*

$$\pi_\star(\mathbb{H}\mathbb{F} \wedge \mathbb{H}\mathbb{F})^t = \mathbb{H}\mathbb{F}_\star^h[\zeta_i]_{a_\sigma}[a_\sigma^{-1}].$$

The inclusion of the dual Steenrod algebra is a map of (completed) Hopf algebroids, and

$$\eta_R(u_\sigma^{-1}) = \sum_{i \geq 0} u_\sigma^{-2^i} \zeta_i a_\sigma^{2^i - 1}, \quad \eta_r(a_\sigma) = a_\sigma.$$

In this paper we will mostly be concerned with the Hopf algebroid $\pi_\star(\mathbb{H}\mathbb{F} \wedge \mathbb{H}\mathbb{F})$, which we now name.

Definition 1.8. The C_2 -equivariant dual Steenrod algebra is the Hopf algebroid in Green functors $(\mathbb{H}\mathbb{F}_\star, \pi_\star(\mathbb{H}\mathbb{F} \wedge \mathbb{H}\mathbb{F}))$. We will denote it by \mathcal{A}_\star .

We will need to define a few elements in \mathcal{A}_\star before stating the computation, and for that we need the equivariant analog of BC_2 .

Definition 1.9. Let $E\mu_2$ be the $C_2 \times \Sigma_2$ -homotopy type uniquely determined by the property that

$$(E\mu_2)^H = \begin{cases} \ast & H = 1, C_2, \Delta \\ \emptyset & H = \Sigma_2, C_2 \times \Sigma_2 \end{cases}$$

where $\Delta \subset C_2 \times \Sigma_2$ is the graph of the unique nontrivial homomorphism from C_2 to Σ_2 . Let $B\mu_2$ be the C_2 -homotopy type $E\mu_2/\Sigma_2$.

Remark 1.10. The space underlying $B\mu_2$ is $\mathbb{R}P^\infty$, but the C_2 action is nontrivial. Indeed, $(B\mu_2)^{C_2} \cong \mathbb{R}P^\infty \amalg \mathbb{R}P^\infty$. This space has many other names. In [HK01] it is called $B'\mathbb{Z}/2$, and it is also the C_2 -equivariant classifying space for C_2 -equivariant Σ_2 -bundles, which is often written $B_{C_2}\Sigma_2$.

Before describing the dual Steenrod algebra, we'll need a few preliminaries on the cohomology of $B\mu_2$. Consider $\mathbb{C}P^\infty$ as a C_2 -space under complex conjugation. Then the Euler class of the canonical bundle lifts canonically to an element in $\mathbb{H}\mathbb{Z}^{\rho}\mathbb{C}P^\infty$. The natural inclusion $B\mu_2 \hookrightarrow \mathbb{C}P^\infty$ is equivariant and so defines an element $\tilde{b} \in \mathbb{H}\mathbb{Z}^{\rho}B\mu_2$ by pulling back this Euler class. Changing coefficients yields an element $b \in \mathbb{H}\mathbb{F}^{\rho}B\mu_2$.

Let β denote the Bockstein associated to the exact sequence of Mackey functors

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{F}.$$

Then there is a unique element $c \in \mathbb{H}\mathbb{F}^{\sigma}B\mu_2$ such that $\beta c = b$ and c vanishes when restricting to a point. The existence of some such c follows from the fact that $2\tilde{b} = 0$, which one proves by trivializing the square of the line bundle classified by \tilde{b} . There are exactly two choices of c , differing by addition of a_σ , and this is detected by restricting to a point.

Hu-Kriz show that $\mathbb{H}\mathbb{F}_\star^{\rho}B\mu_2$ is free over $\mathbb{H}\mathbb{F}_\star$ with basis elements $c^\epsilon b^i$ for $i \geq 0$ and $\epsilon = 0, 1$.² We give a different proof of this fact below, as well (see Proposition 3.2, below).

The key observation is that the right, completed Steenrod coaction on c takes the form

$$\psi(c) = c \otimes 1 + \sum_{i \geq 0} b^{2^i} \otimes \tau_i,$$

for some elements $\tau_i \in \mathcal{A}_\star$ of degree $(2^i - 1)\rho + 1$. The next theorem is the content of [HK01, Thm. 6.41].

²In [HK01], they denote by b' what we have denoted by b , and they never quite pin down which choice of c they use.

Theorem 1.11 (Hu-Kriz). *Let $\xi_i := \beta\tau_i$. Then, as a left $\mathbb{H}\mathbb{F}_*$ -algebra,*

$$\mathcal{A}_* = \mathbb{H}\mathbb{F}_*[\tau_i, \xi_i, \overline{u}_\sigma] / \tau_0 a_\sigma = \overline{u}_\sigma + u_\sigma, \tau_i^2 = \tau_{i+1} a_\sigma + \xi_{i+1} \overline{u}_\sigma.$$

Here $|\tau_i| = (2^i - 1)\rho + 1$, and $|\xi_i| = (2^i - 1)\rho$. The behavior of the right unit can be written in terms of the boundary map $\partial : \mathbb{H}\mathbb{F}_*^t \rightarrow \Sigma\mathbb{H}\mathbb{F}_{h*} \rightarrow \Sigma\mathbb{H}\mathbb{F}_*$:

$$\eta_R \left(\frac{1}{a_\sigma^k u_\sigma^{n+2}} \right) = \partial \left(\frac{a_\sigma^{k+1}}{(u_\sigma + \tau_0 a_\sigma)^{n+1}} \right)$$

when $k, n \geq 0$.

Remark 1.12. For us, the most important element is τ_0 , and that has a more elementary description. The left and right units on u_σ restrict to the same element on underlying homotopy, so $u_\sigma + \overline{u}_\sigma$ is in the kernel of the restriction. Thus there is some element $\tau_0 = \frac{1}{a_\sigma}(u_\sigma + \overline{u}_\sigma)$. Since the transfer is zero in this degree, the element τ_0 is uniquely determined by this property.

Actually, this basic idea lies behind the whole computation of the Dyer-Lashof and dual Steenrod algebras: write down some classical elements and some equivariant elements that restrict to the same thing and keep dividing by a_σ until you can't any more. This business of dividing by a_σ corresponds, on the cellular side, to extending a map over a CW -complex with a free cell in each dimension. Such things end up looking like representation spheres in the case of C_2 and for C_p they're stranger objects. These cell complexes appear in the Hill, Hopkins, and Ravenel computation [HHR] of $\pi_* EO_{2(p-1)}$ and in unpublished work by the same authors.

2 Extended Powers

In this section we refine the skeletal filtration of $E\Sigma_2$ to an equivariant filtration of $E\mu_2$. It's important that we do not use the standard equivariant skeletal filtration on $E\mu_2$. To see why, notice that the 0-skeleton of $E\mu_2$ must contain at least two 0-cells:

$$\Sigma_{2+}, \quad \frac{C_2 \times \Sigma_2}{\Delta}.$$

In terms of power operations, the first of these cells is eventually responsible for the *squaring* map, while the second is responsible for the *norm* map. We choose a filtration which privileges the squaring map in order to get formulas which look like the classical case.

Another departure from the classical case is that the resulting filtration on $\mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(X)$ does not obviously split in general. However, the filtration does split when $X = S^{n\rho}$ or $S^{n\rho-1}$, and this ends up being all we need in the sequel.

2.1 Definitions and first properties

Let \mathcal{O} denote an operad in C_2 -spaces such that $\mathcal{O}(n)$ is a universal space for the family of subgroups $\Gamma \subset C_2 \times \Sigma_n$ which are graphs of homomorphisms $C_2 \rightarrow \Sigma_n$. This is a complete N_∞ -operad for the group C_2 in the terminology of [BH13], or, a G - E_∞ -operad in more classical terminology. There is a unique such operad up to equivalence so we are not concerned with the choice. Notice that, in particular, we have an equivalence of $C_2 \times \Sigma_2$ -spaces

$$\mathcal{O}(2) \cong E\mu_2.$$

Let R be an equivariant commutative ring spectrum, and let Mod_R denote the category of modules over (a cofibrant replacement of) R . We have a functor

$$\mathbb{P} : \text{Mod}_R \rightarrow \text{Alg}_{\mathcal{O}}(\text{Mod}_R)$$

homotopically left adjoint to the forgetful functor.

The underlying object in $\text{Mod}_R, \mathbb{P}(X)$, admits a canonical filtration by arity with subquotients $\mathbb{P}_m(X)$ for $m \geq 0$ and we have

$$\mathbb{P}_2(X) = \mathcal{O}(2)_+ \otimes_{\Sigma_2} X^{\otimes 2} \cong E\mu_2 \otimes_{\Sigma_2} X^{\otimes 2}.$$

We will refer to this as the **genuine** or **complete extended power** of X . Notice that it differs from the standard extended power $X_{h\Sigma_2}^{\otimes 2}$. If we wish to emphasize the ring we'll use the notation \mathbb{P}_2^R .

Warning 2.1. While Φ^H does take N_∞ -algebras to N_∞ -algebras (for a different operad), it *does not preserve extended powers* unless $H = e$ is trivial. That is:

$$\Phi^H(\mathbb{P}_m^\mathcal{O}(X)) \not\cong \mathbb{P}_m^{\mathcal{O}^H}(\Phi^H X),$$

where we've momentarily decorated the extended powers to indicate which operad we're using. For example, when $X = S^0$, $G = H = C_2$, and $m = 2$, the left hand side is (the suspension spectrum of) $\mathbb{R}P^\infty \amalg \mathbb{R}P^\infty$ and the right hand side is just (the suspension spectrum of) $\mathbb{R}P^\infty$.

Since change of base ring is monoidal, we get the following.

Lemma 2.2. *If $R \rightarrow R'$ is a map of commutative ring spectra, then there is an equivalence in the homotopy category:*

$$R' \wedge_R \mathbb{P}_2^R(X) \cong \mathbb{P}_2^{R'}((R' \wedge_R X)).$$

We also record the behavior of complete extended powers with respect to indexed wedges, for later use. To prove it one may, for example, model the extended power by honest symmetric powers after cofibrant replacement [HHR16, B.117] and then argue using the distributive law [HHR16, A.37].

Lemma 2.3. *For any equivariant commutative ring R , and R -modules X and Y we have*

$$\mathbb{P}_2(X \vee Y) \cong \mathbb{P}_2(X) \vee \mathbb{P}_2(Y) \vee (X \vee Y),$$

$$\mathbb{P}_2(C_{2+} \wedge X) \cong C_{2+} \wedge \mathbb{P}_2(X) \vee (C_{2+} \wedge X).$$

2.2 A filtration on $E\mu_2$

We will be interested in filtering $\mathbb{P}_2(X)$ when X is a representation sphere in order to get a handle on its homology. These filtrations will be induced by a filtration on $E\mu_2$ itself which refines the non-equivariant skeletal filtration. We give two descriptions of the filtration- one geometric, and the other combinatorial.

Construction 2.4. Let τ denote the sign representation of Σ_2 , and define

$$F_{2k}E\mu_2 := S((k\rho + 1) \otimes \tau), \quad F_{2k+1}E\mu_2 := S((k+1)\rho \otimes \tau).$$

There are evident equivariant inclusions $F_i E\mu_2 \subset F_{i+1} E\mu_2$ which yield a filtered $C_2 \times \Sigma_2$ -space $F_\infty E\mu_2$. We write $\text{gr}_i E\mu_2$ for the i th-layer of the associated graded. Recall that $\Delta \subset C_2 \times \Sigma_2$ denotes the diagonal subgroup.

We'll now build equivariant maps

$$g_{2k} : S^{k\rho} \wedge \Sigma_{2+} \longrightarrow \text{gr}_{2k} E\mu_2,$$

$$g_{2k+1} : S^{(k\rho+1)\tau} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)_+ \longrightarrow \text{gr}_{2k+1} E\mu_2.$$

For g_{2k} , let $D(k\rho) \longrightarrow S(k\rho + 1)$ be the C_2 -equivariant inclusion of the graph of the function $D(k\rho) \longrightarrow \mathbb{R}$ given by $v \mapsto \sqrt{1 - |v|^2}$. By adjunction, this extends to a map of pairs

$$(D(k\rho) \times \Sigma_2, S(k\rho) \times \Sigma_2) \longrightarrow (S((k\rho + 1) \otimes \tau), S(k\rho \otimes \tau)) = (F_{2k}E\mu_2, F_{2k-1}E\mu_2)$$

and so descends to define g_{2k} .

We define g_{2k+1} similarly. That is, we use the functions $v \mapsto \pm\sqrt{1-|v|^2}$ to define a map

$$D((k\rho + 1) \otimes \tau) \times \left(\frac{C_2 \times \Sigma_2}{\Delta} \right) \longrightarrow S((k+1)\rho \otimes \tau),$$

which includes into the top hemisphere on the trivial coset and the bottom hemisphere on the nontrivial coset. This map descends to the quotient and yields g_{2k+1} .

Proposition 2.5. *The $C_2 \times \Sigma_2$ space $F_\infty E\mu_2 = S(\infty(\rho \otimes \tau))$ is a model for $E\mu_2$, and the maps*

$$\begin{aligned} g_{2k} : S^{k\rho} \wedge \Sigma_{2+} &\longrightarrow \text{gr}_{2k} E\mu_2, \\ g_{2k+1} : S^{(k\rho+1)\tau} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)_+ &\longrightarrow \text{gr}_{2k+1} E\mu_2. \end{aligned}$$

are equivalences of pointed $(C_2 \times \Sigma_2)$ -spaces.

Proof. For the first claim note that the underlying space is the sphere in \mathbb{R}^∞ , the Δ -fixed points are given by $S(\infty(\sigma \otimes \tau))$, and the C_2 -fixed points are given by $S(\infty(1 \otimes \tau))$, all of which are contractible. It is also evident that Σ_2 acts freely.

For the latter claim, it's clear that g_{2k} and g_{2k+1} are homeomorphisms, so we'll just double-check that they are equivariant. Let γ be the generator of C_2 and χ the generator of Σ_2 . We will abuse notation and evaluate g_{2k} (resp. g_{2k+1}) on points of $D(k\rho) \times \Sigma_2$ (resp. $D((k\rho + 1) \otimes \tau) \times \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)$).

$$\begin{aligned} g_{2k}(\gamma \cdot (v_1, v_2, \dots, v_{2k}, 1)) &= g_{2k}(v_1, -v_2, \dots, v_{2k-1}, -v_{2k}, 1) \\ &= (v_1, -v_2, \dots, v_{2k-1}, -v_{2k}, \sqrt{1-|v|^2}) \\ &= \gamma \cdot (v_1, v_2, \dots, v_{2k}, \sqrt{1-|v|^2}) \\ &= \gamma \cdot g_{2k}(v_1, \dots, v_{2k}, 1) \end{aligned}$$

$$\begin{aligned} g_{2k}(\chi \cdot (v_1, \dots, v_{2k}, 1)) &= g_{2k}(v_1, v_2, \dots, v_{2k}, \chi) \\ &= (-v_1, \dots, -v_{2k}, -\sqrt{1-|v|^2}) \\ &= \chi \cdot (v_1, \dots, v_{2k}, \sqrt{1-|v|^2}) \\ &= \chi \cdot g_{2k}(v_1, \dots, v_{2k}, 1) \end{aligned}$$

$$\begin{aligned} g_{2k+1}(\gamma \cdot (v_1, \dots, v_{2k}, v_{2k+1}, [1])) &= g_{2k+1}(v_1, -v_2, \dots, -v_{2k}, v_{2k+1}, [\gamma]) \\ &= (v_1, -v_2, \dots, v_{2k+1}, -\sqrt{1-|v|^2}) \\ &= \gamma \cdot (v_1, \dots, v_{2k+1}, \sqrt{1-|v|^2}) \\ &= \gamma \cdot g_{2k+1}(v_1, \dots, v_{2k+1}, [1]) \end{aligned}$$

$$\begin{aligned} g_{2k+1}(\chi \cdot (v_1, \dots, v_{2k+1}, [1])) &= g_{2k+1}(-v_1, \dots, -v_{2k+1}, [\chi]) \\ &= (-v_1, \dots, -v_{2k+1}, -\sqrt{1-|v|^2}) \\ &= \chi \cdot (v_1, \dots, v_{2k+1}, \sqrt{1-|v|^2}) \\ &= \chi \cdot g_{2k+1}(v_1, \dots, v_{2k+1}, [1]) \end{aligned}$$

□

Remark 2.6. It's worth pointing out that if X is a pointed G -space and $H \subset G$ is a subgroup then $X \wedge (G/H)_+$ is equivalent to $i_H^* X \wedge_H G_+$ as G -spaces. Indeed, the adjunction between restriction and induction gives a G -equivariant map from the right-hand side to the left which is evidently a homeomorphism.

Now, σ and τ both restrict to the sign representation of Δ , and $\sigma\tau$ restricts to the trivial representation of Δ . Combining these two observations yields an equivalence:

$$S^{\sigma V} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)_+ \cong S^{\sigma V} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)_+.$$

Now we sketch a combinatorial approach to the above filtration. This has the benefit of generalizing more readily to the odd primary case, but other than that it is much the same.

Construction 2.7. Let $\mathbf{E} \frac{C_2 \times \Sigma_2}{\Delta}$ denote the $(C_2 \times \Sigma_2)$ -simplicial set with n -simplices given by

$$\left(\mathbf{E} \frac{C_2 \times \Sigma_2}{\Delta} \right)_n = \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)^{\times(n+1)}$$

and boundaries and degeneracies given by projection and diagonal maps, respectively.

Similarly let $\mathbf{E}\Sigma_2$ denote the nerve of the transport category for Σ_2 , i.e. the $(C_2 \times \Sigma_2)$ -simplicial set with n -simplices given by

$$(\mathbf{E}\Sigma_2)_n = \Sigma_2^{\times(n+1)}$$

and boundaries and degeneracies given by projections and diagonals.

Remark 2.8. This is slightly non-standard: we are thinking of the tuple (g_0, \dots, g_n) as specifying the string of morphisms $g_0 \rightarrow g_1 \rightarrow \dots \rightarrow g_n$ and the Σ_2 -action is diagonal. We could equally well use the tuples (g_0, h_1, \dots, h_n) where g_0 is the source of the first arrow, and we have $g_0 h_1 = g_1$, $g_1 h_2 = g_2$, etc. Then Σ_2 acts only on the first coordinate.

Finally, let $\mathbf{E}\mu_2$ denote the join of these two simplicial sets. So the n -simplices are:

$$(\mathbf{E}\mu_2)_n = \Sigma_2^{\times(n+1)} \cup \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)^{\times(n+1)} \cup \bigcup_{i+j=n-1} \Sigma_2^{\times(i+1)} \times \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)^{\times(j+1)}.$$

Lemma 2.9. *The geometric realization of $\mathbf{E}\mu_2$ is a model for $\mathbf{E}\mu_2$.*

Proof. This is immediate from the fact that taking fixed points commutes with the join and that joining with a contractible space yields a contractible space. \square

Lemma 2.10. *The non-degenerate simplices of $\mathbf{E}\mu_2$ are given by*

$$\begin{aligned} N(\mathbf{E}\mu_2)_n &= \Sigma_2 \cup \left(\frac{C_2 \times \Sigma_2}{\Delta} \right) \cup \bigcup_{i+j=n-1} \Sigma_2 \times \left(\frac{C_2 \times \Sigma_2}{\Delta} \right) \\ &= \Sigma_2 \cup \left(\frac{C_2 \times \Sigma_2}{\Delta} \right) \cup \bigcup_{i+j=n-1} C_2 \times \Sigma_2. \end{aligned}$$

Proof. This follows from the computation of the non-degenerate simplices in each factor of the join, which is classical:

$$\begin{aligned} N(\mathbf{E} \frac{C_2 \times \Sigma_2}{\Delta})_n &= \left(\frac{C_2 \times \Sigma_2}{\Delta} \right), \\ N(\mathbf{E}\Sigma_2)_n &= \Sigma_2. \end{aligned}$$

\square

To give the filtration in this combinatorial setting we need some notation for simplices in $\mathbf{E}\mu_2$. Let x_n denote a generator of the non-degenerate orbit Σ_2 in dimension n , and let y_n denote a generator of the non-degenerate orbit $\frac{C_2 \times \Sigma_2}{\Delta}$ in dimension n . Denote by $x_i * y_j$ the corresponding generators of $C_2 \times \Sigma_2$ in dimension $i + j + 1$.

Now define $F_{2k}\mathbf{E}\mu_2$ as the subcomplex spanned by the simplices $x_0, \dots, x_k, y_0, \dots, y_{k-1}$ and their pairwise joins, and define $F_{2k+1}\mathbf{E}\mu_2$ as the subcomplex spanned by the simplices $x_0, \dots, x_k, y_0, \dots, y_k$ and their pairwise joins.

Construction 2.11. Begin with the pointed simplicial C_2 -set $(\text{sk}_n \mathbf{E}C_2)_+$, take the Kan suspension, induce up to a pointed simplicial $(C_2 \times \Sigma_2)$ -set by smashing with Σ_{2+} , and then smash with $\Delta^k / \partial\Delta^k$. The result is a simplicial $(C_2 \times \Sigma_2)$ -set \mathbf{G}_{2k} with the following properties:

- (i) the geometric realization of \mathbf{G}_{2k} is equivalent to $S^k \wedge S^{k\sigma} \wedge \Sigma_{2+}$,
- (ii) the non-degenerate simplices are: the basepoint, a copy of Σ_2 in dimension k generated by a simplex a_k , a copy of $C_2 \times \Sigma_2$ in each dimension $k + 1, \dots, k + 2$ generated by simplices z_0, \dots, z_{k-1} , respectively.

One can construct an equivariant map

$$\mathbf{G}_{2k} \longrightarrow \text{gr}_{2k}\mathbf{E}\mu_2$$

which sends $a_k \mapsto x_k$ and $z_i \mapsto x_k * y_i$ and induces an equivariant map

$$h_{2k} : S^k \wedge S^{k\sigma} \wedge \Sigma_{2+} \longrightarrow \text{gr}_{2k}\mathbf{E}\mu_2$$

upon taking geometric realization.

The same construction with $\mathbf{E}\Sigma_2$ replacing $\mathbf{E}C_2$ and $\left(\frac{C_2 \times \Sigma_2}{\Delta}\right)_+$ replacing Σ_{2+} gives a map

$$h_{2k+1} : S^k \wedge S^{(k+1)\tau} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta}\right)_+ \longrightarrow \text{gr}_{2k+1}\mathbf{E}\mu_2.$$

Proposition 2.12. *The maps h_i are equivalences of pointed $(C_2 \times \Sigma_2)$ -spaces.*

Proof. By construction the maps induce isomorphisms on non-degenerate simplices, which implies the result. \square

Remark 2.13. To connect the combinatorial and geometric constructions notice that the subcomplex spanned by x_0, \dots, x_k corresponds to a cell structure on $S(k(1 \otimes \tau))$ and the subcomplex spanned by y_0, \dots, y_k corresponds to a cell structure on $S(k(\sigma \otimes \tau))$. The join of spheres has the following property: if V and W are vector spaces then there is a natural homeomorphism

$$S(V) * S(W) \cong S(V \oplus W).$$

From this it should be clear that the two filtrations and the identifications of their associated graded spaces are compatible up to re-ordering coordinates and applying Remark 2.6.

2.3 Extended powers of spheres

The filtration on $\mathbf{E}\mu_2$ introduced in the previous section gives rise to a filtration $F_k \mathbb{P}_2 X$ on complete extended powers which is natural in X . We'll be particularly interested in the case when X is a (virtual) representation sphere. The main result is:

Theorem 2.14. *The associated graded space of $\mathbb{P}_2(S^V)$ with respect to the above filtration is given by*

$$\begin{aligned}\mathrm{gr}_{2k}\mathbb{P}_2(S^V) &\cong S^{k\rho} \wedge S^{2V}, \\ \mathrm{gr}_{2k+1}\mathbb{P}_2(S^V) &\cong S^{k\rho+\sigma} \wedge S^{\rho V}.\end{aligned}$$

If, moreover, $V = m\rho$ or $V = m\rho - 1$, this filtration splits upon smashing with $\mathbb{H}\mathbb{F}$ and we get an equivalence of C_2 -spectra:

$$\begin{aligned}\mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{m\rho}) &\cong S^{2m\rho} \wedge \left(\bigvee_{k \geq 0} \mathbb{H}\mathbb{F} \wedge S^{k\rho} \vee \bigvee_{k \geq 0} \mathbb{H}\mathbb{F} \wedge S^{k\rho+\sigma} \right), \\ \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{m\rho-1}) &\cong S^{2m\rho-2} \wedge \left(\bigvee_{k \geq 0} \mathbb{H}\mathbb{F} \wedge S^{k\rho} \vee \bigvee_{k \geq 0} \mathbb{H}\mathbb{F} \wedge S^{k\rho+1} \right).\end{aligned}$$

Notice that this situation differs from its classical counterpart. We only have a ‘Thom isomorphism’ for regular representation spheres. This is analogous to the classical odd primary situation where only the extended powers of even spheres have canonical orientations.

Remark 2.15. It is nevertheless true that the homology of $\mathbb{P}_2(S^V)$ is free over the homology of a point when V is a representation. This follows from the surprisingly general freeness theorem of Kronholm [Kro10]. However, it is not clear a priori in which degrees we may choose our generators and, in any case, we do not need this fact for any of our arguments. We are grateful to Sean Tilson for telling us about this reference, which we did not know of while writing the first version of this paper.

Before giving the proof, let’s record the following consequence of the computation of $\mathbb{H}\mathbb{F}_*$.

Lemma 2.16. *We have vanishing ranges:*

$$\begin{aligned}[\underline{S}^{t\rho+\sigma}, \mathbb{H}\mathbb{F}] &= 0, \quad t \in \mathbb{Z}, \\ [\underline{S}^{t\rho-1}, \mathbb{H}\mathbb{F}] &= 0, \quad t \in \mathbb{Z}, \\ [\underline{S}^{t\rho} \wedge S^{(1-\sigma)(a+b\sigma)}, \mathbb{H}\mathbb{F}] &= 0, \quad t \geq 1, \quad a - b \geq -2\end{aligned}$$

Proof. The first two are equivalent and straightforward. For the last one, notice that, since $\sigma^2 = 1$, we get $(1 - \sigma)(a + b\sigma) = (a - b)(1 - \sigma)$. The coefficients $\mathbb{H}\mathbb{F}_*$ don’t necessarily vanish on this line, but they do as soon as you add a positive multiple of the regular representation when $a - b \geq -2$. \square

Proof of Theorem 2.14. We begin by identifying the even gradations. Recall that τ denotes the sign representation of Σ_2 . By Proposition 2.5, we have an equivalence of $(C_2 \times \Sigma_2)$ -spaces

$$\mathrm{gr}_{2k}\mathbb{E}\mu_2 \wedge S^{(1+\tau)V} \cong S^{k\rho} \wedge \Sigma_{2+} \wedge S^{(1+\tau)V}.$$

Since Σ_2 acts freely on the middle factor we may untwist the action on the right to get an equivalence of $(C_2 \times \Sigma_2)$ -spaces

$$\mathrm{gr}_{2k}\mathbb{E}\mu_2 \wedge S^{(1+\tau)V} \cong S^{k\rho} \wedge \Sigma_{2+} \wedge S^{2V}.$$

So the first result follows upon taking the quotient by the Σ_2 action.

For the odd gradations we compute:

$$\begin{aligned}\mathrm{gr}_{2k+1}\mathbb{E}\mu_2 \wedge S^{(1+\tau)V} &\cong S^{(k\rho+1)\tau} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)_+ \wedge S^{(1+\tau)V} \\ &\cong S^{(k\rho+1)\sigma} \wedge S^{(1+\sigma)V} \wedge \left(\frac{C_2 \times \Sigma_2}{\Delta} \right)_+ \quad (\text{Remark 2.6}) \\ &\cong S^{k\rho+\sigma} \wedge S^{\rho V} \quad \text{Since } \rho\sigma = \rho.\end{aligned}$$

For the remainder of the proof, let $V = a + b\sigma$. It remains to show that, when $a - b = 0, -1$, this filtration splits after smashing with $\underline{\mathbb{H}\mathbb{F}}$. This is equivalent to showing that $\underline{\mathbb{H}\mathbb{F}} \wedge \text{gr}_n \mathbb{P}_2 S^V \rightarrow \underline{\mathbb{H}\mathbb{F}} \wedge \Sigma F_{n-1} \mathbb{P}_2 S^V$ is null for all $n \geq 0$. We'll prove this by inductively lifting up the tower

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\underline{\mathbb{H}\mathbb{F}} \wedge \Sigma F_1 \mathbb{P}_2 S^V & \longrightarrow & \underline{\mathbb{H}\mathbb{F}} \wedge \Sigma \text{gr}_1 \mathbb{P}_2 S^V \\
\downarrow & & \\
\underline{\mathbb{H}\mathbb{F}} \wedge \Sigma F_2 \mathbb{P}_2 S^V & \longrightarrow & \underline{\mathbb{H}\mathbb{F}} \wedge \Sigma \text{gr}_2 \mathbb{P}_2 S^V \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
\text{gr}_n \mathbb{P}_2 S^V & \longrightarrow & \underline{\mathbb{H}\mathbb{F}} \wedge \Sigma F_{n-1} \mathbb{P}_2 S^V \longrightarrow \underline{\mathbb{H}\mathbb{F}} \wedge \Sigma \text{gr}_{n-1} \mathbb{P}_2 S^V
\end{array}$$

First consider the case $n = 2k$ is even. Then the obstructions to lifting live in the groups

$$[S^{k\rho} \wedge S^{2V}, \Sigma(S^{\ell\rho} \wedge S^{2V} \wedge \underline{\mathbb{H}\mathbb{F}})] = [S^{(k-\ell)\rho-1}, \underline{\mathbb{H}\mathbb{F}}], \quad 1 \leq \ell \leq k-1,$$

$$[S^{k\rho} \wedge S^{2V}, \Sigma(S^{\ell\rho+\sigma} \wedge S^{\rho V} \wedge \underline{\mathbb{H}\mathbb{F}})] = [S^{(k-\ell-1)\rho} \wedge S^{(1-\sigma)V}, \underline{\mathbb{H}\mathbb{F}}], \quad 0 \leq \ell \leq k-1.$$

The first of these groups always vanishes, and the latter vanishes under our constraints when $\ell < k-1$. So we are left with a single obstruction, which is a map

$$S^{(1-\sigma)V} \rightarrow \underline{\mathbb{H}\mathbb{F}}.$$

When $a - b = -1$ this homotopy group vanishes, otherwise, when $a - b \geq 0$ we are in the situation where the restriction map is an isomorphism so we need only check that the obstruction vanishes on underlying homotopy groups. This, in turn, follows from the classical computation of the mod 2 homology of extended powers.

Finally, consider the case when $n = 2k + 1$ is odd. Then the obstructions live in the groups

$$[S^{k\rho+\sigma} \wedge S^{\rho V}, \Sigma(S^{\ell\rho} \wedge S^{2V} \wedge \underline{\mathbb{H}\mathbb{F}})] = [S^{(k-\ell)\rho} \wedge S^{(1-\sigma)(-V-1)}, \underline{\mathbb{H}\mathbb{F}}], \quad 0 \leq \ell \leq k,$$

$$[S^{k\rho+\sigma} \wedge S^{\rho V}, \Sigma(S^{\ell\rho+\sigma} \wedge S^{\rho V} \wedge \underline{\mathbb{H}\mathbb{F}})] = [S^{(k-\ell)\rho-1}, \underline{\mathbb{H}\mathbb{F}}], \quad 0 \leq \ell \leq k-1.$$

The second groups always vanish. The first groups vanish when $\ell < k$ and $a - b \leq 0$ (which certainly holds for $V = m\rho$ or $m\rho - 1$). When $k = \ell$ and $a - b \leq 0$, the restriction map is injective and we argue as before. \square

Remark 2.17. We have actually shown something stronger. When $V = a + b\sigma$, the even gradations split off after smashing with $\underline{\mathbb{H}\mathbb{F}}$ as long as $a - b \geq -1$, and the odd gradations split off after smashing with $\underline{\mathbb{H}\mathbb{F}}$ as long as $a - b \leq 0$.

Remark 2.18. The splitting of $\underline{\mathbb{H}\mathbb{F}} \wedge \mathbb{P}_2(S^{m\rho-\varepsilon})$ which is constructed inductively like this is actually canonical, for $\varepsilon = 0, 1$. That is, at each step after the first, there is a unique $\underline{\mathbb{H}\mathbb{F}}$ -linear homotopy section of $\underline{\mathbb{H}\mathbb{F}} \wedge F_n \mathbb{P}_2(S^{m\rho-\varepsilon}) \rightarrow \underline{\mathbb{H}\mathbb{F}} \wedge \text{gr}_n \mathbb{P}_2(S^{m\rho-\varepsilon})$.

In the next section we'll study the homology of specific extended powers, but before we do there is one useful observation we can make in general that will be used later.

Corollary 2.19. *When $V = a + b\sigma$, the Mackey functors $\mathbb{H}\mathbb{F}_{s\rho+V}\mathbb{P}_2(S^V)$ vanish for $s < \min\{a, \frac{a+b}{2}\}$.*

Proof. It suffices to check the vanishing for the associated graded $\Sigma^{-V}\text{gr}_*\mathbb{P}_2(S^V)$ where it follows, after some analytic geometry, from the vanishing ranges for the homology of a point. \square

3 Homology of Extended Powers

It is well known that power operations for homology are parameterized by elements in $\mathbb{H}\mathbb{F}_V\mathbb{P}_2(S^W)$. Unfortunately, we don't know the homology of general extended powers of representation spheres. On the other hand, if $W = a + b\sigma$ there is always a map

$$\Sigma^{a+b\sigma}\mathbb{P}_2(S^0) \rightarrow \mathbb{P}_2(S^{a+b\sigma})$$

which we might try to use to study the homology of the target. Asking that our power operations behave well with respect to this suspension map is essentially equivalent to asking that they factor through the inverse limit

$$\text{holim}_V \Sigma^V \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{-V}).$$

The representations $n\rho$ are cofinal amongst all representations, so we are free to restrict attention to these. In §3.1 we study the homotopy of the inverse limit above.

This inverse limit has another useful feature: we can identify it with a suspension of a localization of the ring $F(\mathbb{B}\mu_2, \mathbb{H}\mathbb{F})$. We carry this out in §3.2 and then use this identification to compute the Steenrod coaction on the homotopy of the inverse limit above. We will use this later to study the relationship between power operations and Steenrod operations, as well as to compute the action of the power operations on the dual Steenrod algebra.

Historical remarks

On the algebraic side, the relationship between Tate-like constructions and operations goes back to Singer [Sin81]. Jones and Wegmann realized this algebraic construction topologically in [JW83] (see especially their results 5.2(a) and 5.2(b) and compare with our 3.7 and 3.20). More recent references include course notes of Lurie [Lur07] and a paper of Kuhn and McCarty [KM13, Cor. 2.13].

3.1 Homology, suspension, and the spectrum of operations

We can re-write the splitting of the previous section as a calculation of homology groups.

Theorem 3.1. *For $m \in \mathbb{Z}$, we have*

$$\begin{aligned} \mathbb{H}\mathbb{F}_*\mathbb{P}_2(S^{m\rho}) &= \bigoplus_{s \geq m} \mathbb{H}\mathbb{F}_*\{e_{s\rho}^{m\rho}\} \oplus \bigoplus_{s \geq m} \mathbb{H}\mathbb{F}_*\{e_{s\rho+\sigma}^{m\rho}\} \\ \mathbb{H}\mathbb{F}_*\mathbb{P}_2(S^{m\rho-1}) &= \bigoplus_{s \geq m-1} \mathbb{H}\mathbb{F}_*\{e_{s\rho+\sigma}^{m\rho-1}\} \oplus \bigoplus_{s \geq m} \mathbb{H}\mathbb{F}_*\{e_{s\rho}^{m\rho-1}\} \end{aligned}$$

where the generator e_W^V has degree $V + W$.

Notice that when $m = 0$ we recover the Hu-Kriz calculation of $\mathbb{H}\mathbb{F}^*B\mu_2$. We can even describe the whole ring, whose structure is just like the motivic case.

Proposition 3.2. ³ As an algebra over $\mathbb{H}\mathbb{F}_*$, we have:

$$\mathbb{H}\mathbb{F}^* B\mu_2 = \mathbb{H}\mathbb{F}^* \llbracket c, b \rrbracket / (c^2 = a_\sigma c + u_\sigma b).$$

Proof. We just need to check the relation, which can be done in Borel cohomology. The Borel cohomology spectral sequence yields:

$$(\mathbb{H}\mathbb{F}^h)^* B\mu_2 = (\mathbb{H}\mathbb{F}^h)^* \llbracket w \rrbracket, \quad |w| = 1.$$

And, in particular:

$$\left(\mathbb{H}\mathbb{F}^h \right)^\sigma B\mu_2 = \mathbb{F}\{u_\sigma w, a_\sigma\}.$$

We know that the restriction of c to the underlying cohomology must be $u_\sigma w$ since there is only one generator in that degree. So, modulo a_σ , $c \mapsto u_\sigma w$. Since we have required that c maps to zero after restriction to a point, we can actually conclude that $c \mapsto u_\sigma w$.

Now recall that $(\mathbb{H}\mathbb{F}^{hC_2})^* = \mathbb{H}^*(\mathbb{R}P^\infty, \mathbb{F}_2) = \mathbb{F}_2 \llbracket x \rrbracket$ where $x = \frac{a_\sigma}{u_\sigma}$. From the computation $\beta x = \text{Sq}^1 x = x^2$ we deduce that

$$b \mapsto \beta(u_\sigma w) = u_\sigma w^2 + a_\sigma w.$$

The relation now follows from inspection. \square

As we mentioned in the introduction to this section, we're interested in power operations in an inverse limit involving maps like

$$\mathbb{H}\mathbb{F} \wedge \Sigma^V \mathbb{P}_2(S^{-V}) \longrightarrow \mathbb{H}\mathbb{F} \wedge \Sigma^W \mathbb{P}_2(S^{-W}),$$

where V contains at least as many copies of each irreducible representation as W . This map comes from a diagonal

$$S^V \wedge (S^{-V})^{\wedge 2} \longrightarrow S^W \wedge (S^{V-W} \wedge S^{-V})^{\wedge 2},$$

which is $(C_2 \times \Sigma_2)$ -equivariant. We are free to restrict to a cofinal system of representations, so we'll focus on the inverse system:

$$\dots \longrightarrow \mathbb{H}\mathbb{F} \wedge \Sigma^{m\rho} \mathbb{P}_2(S^{-m\rho}) \longrightarrow \mathbb{H}\mathbb{F} \wedge \Sigma^{(m-1)\rho+1} \mathbb{P}_2(S^{-(m-1)\rho-1}) \longrightarrow \mathbb{H}\mathbb{F} \wedge \Sigma^{(m-1)\rho} \mathbb{P}_2(S^{-(m-1)\rho}) \longrightarrow \dots$$

Definition 3.3. We define the **spectrum of quadratic operations** by the homotopy limit

$$\text{Ops}_{\mathbb{H}\mathbb{F}} = \text{holim}_V \mathbb{H}\mathbb{F} \wedge \Sigma^V \mathbb{P}_2(S^{-V}).$$

Warning 3.4. This spectrum is not the same as $\mathbb{H}\mathbb{F} \wedge \text{holim}_V \Sigma^V \mathbb{P}_2(S^{-V})$. Indeed, it is a theorem of Lin that the spectrum $\text{holim}_n \Sigma^n \mathbb{P}_2(S^{-n})$ is S^{-1} .

We want to compute the homotopy groups of $\text{Ops}_{\mathbb{H}\mathbb{F}}$. That means we need to study the effect on homology of the maps:

$$\begin{aligned} \theta : \Sigma \mathbb{P}_2(S^{m\rho-1}) &\longrightarrow \mathbb{P}_2(S^{m\rho}), \\ \theta^\sigma : \Sigma^\sigma \mathbb{P}_2(S^{m\rho}) &\longrightarrow \mathbb{P}_2(S^{m\rho+\sigma}) = \mathbb{P}_2(S^{(m+1)\rho-1}). \end{aligned}$$

Proposition 3.5. With notation as above, the effect of θ and θ^σ on homology is determined by:

$$\begin{aligned} \theta(e_{s\rho}^{m\rho-1}) = e_{s\rho}^{m\rho}, \quad \theta(e_{s\rho+\sigma}^{m\rho-1}) &= \begin{cases} e_{s\rho+\sigma}^{m\rho} & s \geq m \\ 0 & s = m-1. \end{cases} \\ \theta^\sigma(e_{s\rho+\sigma}^{m\rho}) = e_{s\rho+\sigma}^{(m+1)\rho-1}, \quad \theta^\sigma(e_{s\rho}^{m\rho}) &= \begin{cases} e_{s\rho}^{(m+1)\rho-1} & s \geq m+1 \\ 0 & s = m \end{cases}. \end{aligned}$$

³This is essentially contained in the proof of Theorem 6.22 of [HK01].

Proof. It is clear from construction that the maps θ and θ^σ are filtration preserving and so preserve this filtration degree on homology. In other words, θ and θ^σ induce maps $\underline{\mathbb{H}\mathbb{F}} \wedge S^V \rightarrow \underline{\mathbb{H}\mathbb{F}} \wedge S^V$ and the question is whether generators are sent to generators. Such maps are determined by an element in $\underline{\mathbb{H}\mathbb{F}}_0$, and all such elements are determined by their restriction. So the effects of θ and θ^σ are determined by the maps on *underlying spectra*, where the answer is as indicated and known classically. \square

Remark 3.6. We don't know whether the map $\underline{\mathbb{H}\mathbb{F}}_* \Sigma^V \mathbb{P}_2(S^W) \rightarrow \underline{\mathbb{H}\mathbb{F}}_* \mathbb{P}_2(S^{V+W})$ is surjective more generally.

Theorem 3.7. *The homotopy Mackey functors $\pi_* \text{Ops}_{\underline{\mathbb{H}\mathbb{F}}}$ are given, as a module over $\underline{\mathbb{H}\mathbb{F}}_*$, by*

$$\pi_* \text{Ops}_{\underline{\mathbb{H}\mathbb{F}}} = \bigoplus_{s \in \mathbb{Z}} \underline{\mathbb{H}\mathbb{F}}_* \{e_{s\rho}\} \oplus \bigoplus_{s \in \mathbb{Z}} \underline{\mathbb{H}\mathbb{F}}_* \{e_{s\rho+\sigma}\}$$

where the generator e_V is in dimension V .

Proof. The proposition implies that the inverse system of homotopy Mackey functors is Mittag-Leffler (since each map is surjective), and this almost gives the result. The caveat is that the actual limit is a completion of the stated module which is determined by the requirement

$$\lim_{s \rightarrow -\infty} e_{s\rho} = \lim_{s \rightarrow -\infty} e_{s\rho+\sigma} = 0.$$

In other words, we need to rule out the possibility of an infinite sum converging to a weird element in degree $a + b\sigma$. This can't happen because, if we fix a and b , then $\underline{\mathbb{H}\mathbb{F}}_* \{e_{s\rho}\}$ vanishes in degree $a + b\sigma$ for sufficiently negative s , e.g. $s < \min(a, b)$. \square

Remark 3.8. The completion does arise in the geometric case, where a_σ is inverted. For example, there is the element

$$\sum_{s \geq 0} \frac{u_\sigma^s}{a_\sigma^{2s}} e_{-s\rho}$$

in degree 0.

3.2 Relationship to a Tate construction

In the non-equivariant case, $\text{Ops}_{\underline{\mathbb{H}\mathbb{F}}_2}$ has the curious feature that $\Sigma(\text{Ops}_{\underline{\mathbb{H}\mathbb{F}}_2})$ is a ring spectrum and receives a ring map from $F(\mathbb{R}P_+^\infty, \underline{\mathbb{H}\mathbb{F}}_2)$. The best way to prove this is to identify the former spectrum with the Tate spectrum $\underline{\mathbb{H}\mathbb{F}}^{t\Sigma^2}$. Here is a low-tech construction of this map and the ring structure.

First observe that the generator $x \in \underline{\mathbb{H}\mathbb{F}}_2^1(\mathbb{R}P^1)$ defines a filtered system of towers:

$$\{\underline{\mathbb{H}\mathbb{F}}_2 \wedge D\mathbb{R}P_+^m\} \xrightarrow{-x} \{\Sigma(\underline{\mathbb{H}\mathbb{F}}_2 \wedge D\mathbb{R}P_+^{m+1})\} \xrightarrow{-x} \{\Sigma^2(\underline{\mathbb{H}\mathbb{F}}_2 \wedge D\mathbb{R}P_+^{m+2})\} \longrightarrow \dots$$

Atiyah duality and a Thom isomorphism allows one to compare with the filtered system of towers:

$$\{\underline{\mathbb{H}\mathbb{F}}_2 \wedge \Sigma \mathbb{R}P_{-m-1}^{-1}\} \longrightarrow \{\underline{\mathbb{H}\mathbb{F}}_2 \wedge \Sigma \mathbb{R}P_{-m-1}^0\} \longrightarrow \{\underline{\mathbb{H}\mathbb{F}}_2 \wedge \Sigma \mathbb{R}P_{-m-1}^1\} \longrightarrow \dots$$

After checking some finiteness and vanishing conditions, it turns out you can take the homotopy limit and homotopy colimit in either order. Doing it one way yields $F(\mathbb{R}P_+^\infty, \underline{\mathbb{H}\mathbb{F}}_2)[x^{-1}]$, while doing it the other gives $\Sigma(\text{Ops}_{\underline{\mathbb{H}\mathbb{F}}_2})$, and you're done. The equivalence preserves the completed Steenrod coactions on both sides, so we can use this to compute the coaction on $\pi_* \text{Ops}_{\underline{\mathbb{H}\mathbb{F}}_2}$.

We'd like to carry over this argument to the equivariant case.

Theorem 3.9. *The C_2 -spectrum $\Sigma(\text{Ops}_{\mathbb{H}\mathbb{F}})$ receives a map*

$$\phi : F((B\mu_2)_+, \mathbb{H}\mathbb{F}) \longrightarrow \Sigma(\text{Ops}_{\mathbb{H}\mathbb{F}})$$

with the following properties:

- (a) *The induced map on π_* is a homomorphism of (complete) $(A_*)_{\star}$ -comodules,*
- (b) *ϕ factors through an equivalence*

$$F((B\mu_2)_+, \mathbb{H}\mathbb{F})[b^{-1}] \xrightarrow{\cong} \Sigma(\text{Ops}_{\mathbb{H}\mathbb{F}}),$$

- (c) *this equivalence takes b^{-s} to the element $\Sigma e_{(s-1)\rho+\sigma}$ and cb^{-s} to the element $\Sigma e_{(s-1)\rho}$.*

Remark 3.10. One might wonder what role the *odd* steps in the filtration of $B\mu_2$ have in the process of inverting b . The fact that multiplication by b can be expressed as this composite is essentially equivalent to the observation that b is the Bockstein on the class c .

The proof will require a few preliminaries. Up to now, we have avoided interpreting our filtration on extended powers in terms of Thom spectra, as in [Til13]. We can no longer get away with this, since we'll need to identify Spanier-Whitehead duals via Atiyah duality.

Definition 3.11. Let $\mathbb{R}P^{k\rho}$ and $\mathbb{R}P^{k\rho-1}$ denote $S((k\rho+1)\otimes\tau)/\Sigma_2$ and $S(k\rho\otimes\tau)/\Sigma_2$, respectively. Note that $\text{hocolim}_k \mathbb{R}P^{k\rho} = B\mu_{2+}$.

Given a $C_2 \times \Sigma_2$ -representation, W , we get a C_2 -equivariant vector bundle on $\mathbb{R}P^{k\rho-\varepsilon}$ defined by

$$S((k\rho+1-\varepsilon)\otimes\tau) \times_{\Sigma_2} W \longrightarrow \mathbb{R}P^{k\rho-\varepsilon},$$

for $\varepsilon = 0, 1$. This extends to an assignment of a virtual bundle to a virtual representation. We'll denote the Thom spectrum associated to some virtual representation W by

$$(\mathbb{R}P^{k\rho-\varepsilon})^W.$$

Immediate from the geometric definition of our filtration on extended powers, we get the next lemma.

Lemma 3.12. *With notation as in (2.4) and $\varepsilon = 0, 1$, there are canonical equivalences*

$$F_{2j-\varepsilon}\mathbb{P}_2(S^{k\rho}) \cong (\mathbb{R}P^{2j-\varepsilon})^{k\rho(1+\tau)},$$

$$F_{2j-\varepsilon}\mathbb{P}_2(S^{k\rho-1}) \cong (\mathbb{R}P^{2j-\varepsilon})^{(k\rho-1)(1+\tau)}.$$

We'll also need to know the duals of these Thom spectra. The tangent bundle of $\mathbb{R}P^{k\rho-\varepsilon}$ is given by $(k\rho-\varepsilon+1)\tau-1$, so equivariant Atiyah duality [Wir75] implies the following proposition.

Proposition 3.13. *The Spanier-Whitehead dual of $(\mathbb{R}P^{k\rho-\varepsilon})^W$ is given by*

$$D(\mathbb{R}P^{k\rho-\varepsilon})^W \cong (\mathbb{R}P^{k\rho-\varepsilon})^{1-(k\rho-\varepsilon+1)\tau-W}.$$

for $\varepsilon = 0, 1$. In particular,

$$D(\mathbb{R}P_+^{k\rho-\varepsilon}) \cong (\mathbb{R}P^{k\rho-\varepsilon})^{1-(k\rho-\varepsilon+1)\tau}.$$

Combining the previous two results, we get a canonical equivalence

$$D\mathbb{R}P_+^{k\rho} \cong (\mathbb{R}P^{k\rho})^{1-(k\rho+1)\tau} = \Sigma^{k\rho+2}(\mathbb{R}P^{k\rho})^{(-k\rho-1)(1+\tau)} \cong \Sigma^{k\rho+2}F_{2k}\mathbb{P}_2(S^{-k\rho-1}) \quad (3.13.1)$$

Now we are interested in a certain filtered system of towers. The structure maps of the tower are refinements of the maps θ and θ^σ we used earlier when studying the suspension map. We'll denote by θ^ρ the composite $\theta^\sigma \circ \theta$.

Lemma 3.14. *The following diagram commutes:*

$$\begin{array}{ccc}
D\mathbb{R}P_+^{(k+1)\rho} & \xrightarrow{\text{incl}^\vee} & D\mathbb{R}P_+^{k\rho} \\
\cong \downarrow & & \downarrow \cong \\
\Sigma^{(k+1)\rho+2} F_{2k+2} \mathbb{P}_2(S^{-(k+1)\rho-1}) & & \Sigma^{k\rho+2} F_{2k} \mathbb{P}_2(S^{-k\rho-1}) \\
& \searrow \theta^\rho & \downarrow \text{incl} \\
& & \Sigma^{k\rho+2} F_{2k+2} \mathbb{P}_2(S^{-k\rho-1})
\end{array}$$

Proof. Recall that, given a closed, equivariant inclusion $N \hookrightarrow M$ with normal bundle ν and a virtual equivariant vector bundle W on M , there is a twisted Pontryagin-Thom collapse

$$M^W \longrightarrow N^{W|_N \oplus \nu}.$$

When $W = -TM$ is the stable normal bundle, then the collapse has the form

$$\text{coll} : M^{-TM} \longrightarrow N^{-TM|_N \oplus \nu} \cong N^{-TN}.$$

This models the dual $DM_+ \rightarrow DN_+$ of the original inclusion. In our case this translates into the commutativity of the diagram

$$\begin{array}{ccc}
D\mathbb{R}P_+^{(k+1)\rho} & \xrightarrow{\text{incl}^\vee} & D\mathbb{R}P_+^{k\rho} \\
\cong \downarrow & & \downarrow \cong \\
(\mathbb{R}P^{(k+1)\rho})^{1-((k+1)\rho+1)\tau} & \xrightarrow{\text{coll}} & (\mathbb{R}P^{k\rho})^{1-(k\rho+1)\tau}
\end{array}$$

Now, under the identification of Lemma 3.12, the map θ^ρ corresponds to the map of Thom spectra induced by adding $\rho\tau$ to the stable normal bundle. So we are left with checking the commutativity of:

$$\begin{array}{ccc}
(\mathbb{R}P^{(k+1)\rho})^{1-((k+1)\rho+1)\tau} & \xrightarrow{\text{coll}} & (\mathbb{R}P^{k\rho})^{1-(k\rho+1)\tau} \\
& \searrow \oplus \rho\tau & \downarrow \text{incl} \\
& & (\mathbb{R}P^{(k+1)\rho})^{1-(k\rho+1)\tau}
\end{array}$$

By naturality of collapse, this is equivalent to the commutativity of:

$$\begin{array}{ccc}
(\mathbb{R}P^{(k+1)\rho})^{1-((k+1)\rho+1)\tau} & & \\
\text{incl} \downarrow & \searrow \oplus \rho\tau & \\
(\mathbb{R}P^{(k+2)\rho})^{1-((k+1)\rho+1)\tau} & \xrightarrow{\text{coll}} & (\mathbb{R}P^{(k+1)\rho})^{1-(k\rho+1)\tau}
\end{array}$$

Finally, this diagram commutes by the definition of the collapse together with the fact that the normal bundle of the inclusion $\mathbb{R}P^{(k+1)\rho} \hookrightarrow \mathbb{R}P^{(k+2)\rho}$ is $\rho\tau$. \square

The above lemma ensures that the dual of the inclusions of projective spaces gives a lift of θ^ρ which we denote by $\tilde{\theta}^\rho$. More generally, we'll denote by $\tilde{\theta}^{j\rho}$ the j -fold composite of $\tilde{\theta}^\rho$, or any of its suspensions.

Construction 3.15. Define a functor $\text{Ops}_*^* : \mathbb{Z}_{\geq 0}^{op} \times \mathbb{Z}_{\geq 0} \longrightarrow \text{Sp}^{C_2}$ on objects by

$$\text{Ops}_j^i := \Sigma^{i\rho+1} F_{2(i+j)} \mathbb{P}_2 (S^{-i\rho-1}),$$

and on morphisms $(i, j) \rightarrow (k, \ell)$ by:

$$\text{incl} \circ \tilde{\theta}^{(k-i)\rho} : \Sigma^{i\rho+1} F_{2(i+j)} \mathbb{P}_2 (S^{-i\rho-1}) \longrightarrow \Sigma^{k\rho+1} F_{2(k+\ell)} \mathbb{P}_2 (S^{-k\rho-1}).$$

By design:

$$\text{Ops}_{\mathbb{H}\mathbb{F}} \cong \text{holim}_i \text{hocolim}_j (\mathbb{H}\mathbb{F} \wedge \text{Ops}_j^i).$$

The next thing to do is build a map of towers

$$\{D\mathbb{R}P_+^{k\rho}\} \longrightarrow \left\{ \Sigma \left(\text{Ops}_0^k \right) \right\}.$$

By (2.26.1), and our definition of Ops_0^k , we have a canonical equivalence $D\mathbb{R}P_+^{k\rho} \cong \Sigma \left(\text{Ops}_0^k \right)$. This gives:

$$\mathbb{H}\mathbb{F} \wedge D\mathbb{R}P_+^{k\rho} \longrightarrow \text{hocolim}_j \Sigma \mathbb{H}\mathbb{F} \wedge \text{Ops}_j^k.$$

By our definition of $\tilde{\theta}^\rho$, these maps fit into a map of towers and yield:

$$\phi : F(B\mu_{2+}, \mathbb{H}\mathbb{F}) \longrightarrow \Sigma \left(\text{Ops}_{\mathbb{H}\mathbb{F}} \right).$$

The induced homomorphism on π_* respects the comodule structure since we started with maps of spectra.

Lemma 3.16. *The map $\phi : F(B\mu_{2+}, \mathbb{H}\mathbb{F}) \longrightarrow \Sigma \left(\text{Ops}_{\mathbb{H}\mathbb{F}} \right)$ has the following effect on homotopy, for $s \leq 0$ and $\varepsilon = 0, 1$:*

$$\phi_* c^\varepsilon b^{-s} = \Sigma e_{(s-1)\rho+\varepsilon\sigma}.$$

Proof. This is a restatement of the fact that $c^\varepsilon b^{-s}$ is dual to $e_{s\rho+\varepsilon\sigma}^0$, using the notation of Theorem 3.1. \square

The remainder of this section will be devoted to showing that this map identifies the target with $F(B\mu_{2+}, \mathbb{H}\mathbb{F})[b^{-1}]$.

First we need to interchange the homotopy limit and the homotopy colimit.

Lemma 3.17. *The canonical map*

$$\text{hocolim}_j \text{holim}_i \mathbb{H}\mathbb{F} \wedge \text{Ops}_j^i \longrightarrow \text{holim}_i \text{hocolim}_j \mathbb{H}\mathbb{F} \wedge \text{Ops}_j^i$$

is an equivalence.

Proof. We showed earlier that the inverse systems of homology groups are Mittag-Leffler, so we are reduced to showing that the map

$$\text{colim}_j \lim_i \mathbb{H}\mathbb{F}_* \text{Ops}_j^i \longrightarrow \lim_i \text{colim}_j \mathbb{H}\mathbb{F}_* \text{Ops}_j^i$$

is an equivalence. This is immediate from the calculation of $\pi_* \text{Ops}_{\mathbb{H}\mathbb{F}}$ (3.7) and the fact that the elements $e_{s\rho}$ and $e_{s\rho+\sigma}$ are detected in $\mathbb{H}\mathbb{F}_* \text{Ops}_j^i$ for i and j sufficiently large. \square

Finally, we identify the homotopy limit of each tower with a suspension of a dual projective space via a Thom isomorphism and determine the resulting map. The main theorem of the section follows.

Proposition 3.18. *There is an equivalence*

$$\operatorname{holim}_i \mathbb{H}\mathbb{F} \wedge \Sigma (\operatorname{Ops}_j^i) \cong \Sigma^{j\rho} F(B\mu_{2+}, \mathbb{H}\mathbb{F})$$

such that the composite:

$$F(B\mu_{2+}, \mathbb{H}\mathbb{F}) \longrightarrow \operatorname{holim}_i \mathbb{H}\mathbb{F} \wedge \Sigma (\operatorname{Ops}_j^i) \cong \Sigma^{j\rho} F(B\mu_{2+}, \mathbb{H}\mathbb{F})$$

induces multiplication by b^j on homotopy.

Proof. We need to check that the following diagram commutes:

$$\begin{array}{ccc} \pi_\star \mathbb{H}\mathbb{F} \wedge D\mathbb{R}P_+^{k\rho} & \xrightarrow{\quad \cdot b \quad} & \pi_\star \Sigma \mathbb{H}\mathbb{F} \wedge D\mathbb{R}P_+^{k\rho+1} \\ \cong \downarrow & & \downarrow \cong \\ \pi_\star \mathbb{H}\mathbb{F} \wedge \Sigma^{k\rho+2} F_{2k} \mathbb{P}_2(S^{-k\rho-1}) & \xrightarrow{\text{incl}} \pi_\star \mathbb{H}\mathbb{F} \wedge \Sigma^{k\rho+2} F_{2k+2} \mathbb{P}_2(S^{-k\rho-1}) \xrightarrow{\cong} \pi_\star \mathbb{H}\mathbb{F} \wedge \Sigma^{k\rho+4} F_{2k+2} \mathbb{P}_2(S^{-(k+1)\rho-1}) & \end{array}$$

where the bottom horizontal isomorphism comes from the splitting of Theorem 2.14. Specifically, this isomorphism takes $e_{s\rho+\varepsilon\sigma}^{-k\rho-1}$ to $e_{(s+1)\rho+\varepsilon\sigma}^{-(k+1)\rho-1}$ where $\varepsilon = 0, 1$. Commutativity now follows from Lemma 3.16, since this implies that the two ways around the diagram agree on generators. \square

3.3 Comodule structure

We can now determine the completed comodule structure on $\pi_\star \operatorname{Ops}_{\mathbb{H}\mathbb{F}}$. It will be helpful to introduce some notation.

Notation 3.19. Let t be a formal variable of degree $-\rho$, and dt a variable of degree -1 . Then set

$$\xi(t) = \sum_{i \geq 0} \xi_i t^{2^i}, \quad \tau(t) = \sum_{i \geq 0} \tau_i t^{2^i}$$

so that $|\xi(t)| = |\tau(t)dt| = -\rho$. Similarly define

$$\bar{\xi}(t) = \sum_{i \geq 0} \bar{\xi}_i t^{2^i}, \quad \bar{\tau}(t) = \sum_{i \geq 0} \bar{\tau}_i t^{2^i}.$$

Given a formal power series $f(t)$ we use $[f(t)]_{t^r}$ to denote the coefficient of t^r .

Theorem 3.20. *The completed, left \mathcal{A}_\star -comodule structure on $\pi_\star \operatorname{Ops}_{\mathbb{H}\mathbb{F}}$ is given by the following formulas:*

$$\begin{aligned} \psi_L(e_{s\rho+\sigma}) &= \sum_{r \in \mathbb{Z}} [\xi(t)^r]_{t^s} \otimes e_{r\rho+\sigma}, \\ \psi_L(e_{s\rho}) &= \sum_{r \in \mathbb{Z}} [\xi(t)^r]_{t^s} \otimes e_{r\rho} + \sum_{r \in \mathbb{Z}} [\xi(t)^r \tau(t)]_{t^s} \otimes e_{r\rho+\sigma}. \end{aligned}$$

Before proving this let's collect a few preliminaries.

Lemma 3.21. *The right, completed \mathcal{A}_\star -comodule structure on $\pi_\star F((B\mu_2)_+, \mathbb{H}\mathbb{F})[b^{-1}]$ is given, for $s \in \mathbb{Z}$, by:*

$$\begin{aligned} \psi_R(b^s) &= \sum_{i \in \mathbb{Z}} b^i \otimes [\xi(t)^s]_{t^i}, \\ \psi_R(cb^s) &= \sum_{i \in \mathbb{Z}} cb^i \otimes [\xi(t)^s]_{t^i} + \sum_{i \in \mathbb{Z}} b^i \otimes [\tau(t)\xi(t)^s]_{t^i}. \end{aligned}$$

Proof. We already know these formulas hold for $\psi_R(c)$ and $\psi_R(b)$, by the definition of ξ_i and τ_i , and the general case follows since ψ_R is a map of algebras. \square

Changing from left to right comodule structures introduces a conjugate, so we will need to identify certain coefficients of, say, $\bar{\xi}(t)^s$. This is the content of the following two lemmas.

Lemma 3.22. *We have the following identities of formal power series:*

$$\begin{aligned}\xi(\bar{\xi}(t)) &= \bar{\xi}(\xi(t)) = t \\ \tau(\bar{\xi}(t)) &= \bar{\tau}(t) \\ \bar{\tau}(\xi(t)) &= \tau(t).\end{aligned}$$

Proof. These are restatements of the identities:

$$\tau_i + \sum \xi_{i-j}^{2^j} \bar{\tau}_j = \sum \xi_{i-j}^{2^j} \bar{\xi}_j = 0,$$

which follow from the formula for comultiplication and the definition of conjugation. \square

Lemma 3.23. *We have*

$$\begin{aligned}[\bar{\xi}(t)^{-s-1}]_{t^{-r-1}} &= [\xi(t)^r]_{t^s} \\ [\bar{\tau}(t)\bar{\xi}(t)^{-s-1}]_{t^{-r-1}} &= [\xi(t)^r \tau(t)]_{t^s}.\end{aligned}$$

Proof. We will prove the second identity- the first one is easier. We'll use $\oint(-)dz$ to denote the operation of taking the coefficient of z^{-1} in a formal Laurent series. Then we compute

$$\begin{aligned}[\bar{\tau}(t)\bar{\xi}(t)^{-s-1}]_{t^{-r-1}} &= \oint t^r \bar{\tau}(t)\bar{\xi}(t)^{-s-1} dt \\ &= \oint \xi(u)^r \tau(u) u^{-s-1} \xi'(u) du && \text{(change of variables and (3.22))} \\ &= \oint \xi(u)^r \tau(u) u^{-s-1} du && (\xi'(u) = 1 \text{ because } 2 = 0) \\ &= [\xi(u)^r \tau(u)]_{u^s}.\end{aligned}$$

\square

Proof of Theorem 3.20. Identifying $\Sigma e_{s\rho}$ with cb^{-s-1} and $\Sigma e_{s\rho+\sigma}$ with b^{-s-1} using Theorem 3.9, we are reduced by Lemma 3.21 to Lemma 3.23. \square

Remark 3.24. (i) This section is inspired by the description of the comodule structure on truncated projective spaces given in [Bak15]. (Though the actual derivation of the formulas was not given there, so it's possible the proof above is not the most efficient.)

(ii) The notation is meant to suggest a relationship between the dual Steenrod algebra and the Hopf algebroid of automorphisms of some kind of 'equivariant, formal, additive supergroup' corresponding to the addition map $B\mu_2 \times B\mu_2 \rightarrow B\mu_2$. Compare [Ino06], for example. The situation here is complicated by the fact that $\mathbb{H}\mathbb{F}^*B\mu_2$ isn't the trivial square zero extension of $\mathbb{F}[b]$, amongst other things.

4 Power Operations

The definition of power operations we give below is slightly nonstandard. It appears explicitly in the non-equivariant context, for example, in [Lur11, (2.2.6)].

4.1 Definition and statement of main theorem

Recall that we can define extended powers over any C_2 -equivariant commutative ring spectrum. In particular, if A is an $\mathbb{H}\mathbb{F}$ -module, we can make sense of the definition

$$\mathbb{P}_2^{\mathbb{F}}(A) := (E\mu_{2+} \wedge_{\mathbb{H}\mathbb{F}} A^{\wedge_{\mathbb{H}\mathbb{F}} 2}) / \Sigma_2.$$

In the case $A = \mathbb{H}\mathbb{F} \wedge X$ for some C_2 -spectrum X , this simplifies (2.2):

$$\mathbb{P}_2^{\mathbb{F}}(\mathbb{H}\mathbb{F} \wedge X) \cong \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(X).$$

Just as before, there's a map $\Sigma^V \mathbb{P}_2^{\mathbb{F}}(A) \rightarrow \mathbb{P}_2^{\mathbb{F}}(\Sigma^V A)$ and we can define a homotopical functor

$$\text{Ops}_{\mathbb{H}\mathbb{F}}(-) : \text{Mod}_{\mathbb{H}\mathbb{F}} \longrightarrow \text{Mod}_{\mathbb{H}\mathbb{F}}$$

such that $\text{Ops}_{\mathbb{H}\mathbb{F}}(\mathbb{H}\mathbb{F}) = \text{Ops}_{\mathbb{H}\mathbb{F}}$.

Construction 4.1. The functors Σ_+^{∞} and $\text{Ops}_{\mathbb{H}\mathbb{F}}(-)$ induce morphisms of derived mapping spaces, natural in a space X :

$$X \rightarrow \text{Map}_{\text{Spaces}}(*, X) \rightarrow \text{Map}_{\text{Spectra}}(S^0, \Sigma_+^{\infty} X) \rightarrow \text{Map}_{\mathbb{F}}(\mathbb{H}\mathbb{F}, \mathbb{H}\mathbb{F} \wedge \Sigma_+^{\infty} X) \rightarrow \text{Map}_{\mathbb{F}}(\text{Ops}_{\mathbb{H}\mathbb{F}}, \text{Ops}_{\mathbb{H}\mathbb{F}}(\mathbb{H}\mathbb{F} \wedge \Sigma_+^{\infty} X)).$$

By adjunction we get a natural map

$$\alpha : \text{Ops}_{\mathbb{H}\mathbb{F}} \wedge_{\mathbb{H}\mathbb{F}} (\mathbb{H}\mathbb{F} \wedge \Sigma_+^{\infty} X) \rightarrow \text{Ops}_{\mathbb{H}\mathbb{F}}(\mathbb{H}\mathbb{F} \wedge \Sigma_+^{\infty} X).$$

There is also a natural map $\Sigma^{-V} \text{Ops}_{\mathbb{H}\mathbb{F}}(A) \rightarrow \text{Ops}_{\mathbb{H}\mathbb{F}}(\Sigma^V A)$, coming from the map on extended powers used to construct the functor $\text{Ops}_{\mathbb{H}\mathbb{F}}(-)$. Since every $\mathbb{H}\mathbb{F}$ -module A is a filtered colimit of modules of the form $\Sigma^{-V} \mathbb{H}\mathbb{F} \wedge \Sigma_+^{\infty} X$, we can extend our definition of α to an **assembly map** for A :

$$\alpha : \text{Ops}_{\mathbb{H}\mathbb{F}} \wedge_{\mathbb{H}\mathbb{F}} A \longrightarrow \text{Ops}_{\mathbb{H}\mathbb{F}}(A).$$

We can define power operations in a fairly weak setting.

Definition 4.2. Let R be an equivariant commutative ring. We say that an R -module A has an **equivariant symmetric multiplication** if it is equipped with R -linear maps $\mathbb{P}_2^R(A) \rightarrow A$ and $R \rightarrow A$ such that

$$\begin{array}{ccc} \mathbb{P}_2^R(R) & \longrightarrow & \mathbb{P}_2^R(A) \\ \downarrow & & \downarrow \\ R & \longrightarrow & A \end{array}$$

commutes up to homotopy.

Definition 4.3. Let A be an $\mathbb{H}\mathbb{F}$ -module with an equivariant symmetric multiplication, and let $e \in \pi_V \text{Ops}_{\mathbb{H}\mathbb{F}}$ be represented by some map $\mathbb{H}\mathbb{F} \rightarrow \Sigma^{-V} \text{Ops}_{\mathbb{H}\mathbb{F}}$ of right $\mathbb{H}\mathbb{F}$ -modules. Then define the power operation $Q^e : A \rightarrow \Sigma^{-V} A$ by the composite:

$$Q^e : A \cong \mathbb{H}\mathbb{F} \wedge_{\mathbb{H}\mathbb{F}} A \longrightarrow \Sigma^{-V} \text{Ops}_{\mathbb{H}\mathbb{F}} \wedge_{\mathbb{H}\mathbb{F}} A \xrightarrow{\alpha} \Sigma^{-V} \text{Ops}_{\mathbb{H}\mathbb{F}}(A) \longrightarrow \Sigma^{-V} \mathbb{P}_2^{\mathbb{F}}(A) \longrightarrow \Sigma^{-V} A.$$

In the special case when $e = e_{s\rho}$ or $e = e_{s\rho-1}$ we denote the resulting operations by $Q^{s\rho}$ and $Q^{s\rho-1}$, respectively. We will also use the same notation for the induced map on homotopy groups.

The remainder of this section is devoted to proving the first properties of these operations $Q^{k\rho-\varepsilon}$. For convenience we collect them all in the next theorem. Throughout, $\varepsilon = 0, 1$.

Theorem 4.4. *The operations $Q^{k\rho-\varepsilon}$ on the homotopy groups of $\mathbb{H}\mathbb{F}$ -modules with an equivariant symmetric multiplication satisfy the following properties.*

- (a) (Vanishing) *If $x \in \pi_{a+b\sigma}A$, then $Q^{s\rho}x = 0$ for all $s < \min\left\{a, \frac{a+b}{2}\right\}$. In particular, if $x \in \pi_{n\rho}A$ then $Q^{s\rho}x = 0$ for $s < n$.*
- (b) (Squaring) *If $x \in \pi_{n\rho-\varepsilon}A$, then $Q^{n\rho-\varepsilon}x = x^2$.*
- (c) (Bockstein) *Let β denote the Bockstein associated to $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{F}$. Then, for all s , $\beta Q^{s\rho} = Q^{s\rho-1}$.*
- (d) (Product formula) *For any $x \in \pi_*A$ and $y \in \pi_*B$ we have*

$$Q^{k\rho}(x \wedge y) = \sum_{i+j=k} Q^{i\rho}x \wedge Q^{j\rho}y \in \pi_*(A \wedge_{\mathbb{H}\mathbb{F}} B).$$

- (e) (Additivity) *The operations $Q^{s\rho}$ commute with addition, restriction, and transfers.*
- (f) (Action on $\mathbb{H}\mathbb{F}_*$) *Q^0 acts by the identity on $\mathbb{H}\mathbb{F}_*$, and $Q^{-1}u_\sigma = a_\sigma$.*
- (g) (Unstable condition) *If X is a space, then Q^0 acts by the identity on $\pi_*F(X, \mathbb{H}\mathbb{F})$ and $Q^{k\rho}$ acts by zero for $k > 0$.*
- (h) (Co-Nishida relations) *Let E be a C_2 -spectrum with an equivariant symmetric multiplication and let $x \in \mathbb{H}\mathbb{F}_*E$, then, for any $s \in \mathbb{Z}$, we have an identity of formal power series:*

$$\sum_{r \in \mathbb{Z}} \psi_R(Q^r x) t^r = \sum_{r \in \mathbb{Z}} Q^{r\rho}(\psi_R(x)) \bar{\xi}(t)^r + \sum_{r \in \mathbb{Z}} Q^{r\rho+\sigma}(\psi_R(x)) \bar{\xi}(t)^r \bar{\tau}(t).$$

Remark 4.5. It is not true that analogues of the Adem relations hold in this generality. For such relations to hold, we need the extra structure of a map $\mathbb{P}_4^R(A) \rightarrow A$ compatible with the equivariant symmetric multiplication. Since we have not examined higher extended powers in this paper, the Adem relations will have to wait for another time.

We'll prove each of these below, except additivity which is trivial from the definition: the operation $Q^{s\rho}$ on homotopy is induced by a map of spectra.

4.2 Proofs of first properties

Lemma 4.6 (Vanishing). *If $x \in \pi_{a+b\sigma}A$, then $Q^{s\rho}x = 0$ for all $s < \min\left\{a, \frac{a+b}{2}\right\}$. In particular, if $x \in \pi_{n\rho}A$ then $Q^{s\rho}x = 0$ for $s < n$.*

Proof. Choose a map $S^{a+b\sigma} \rightarrow A$ representing x . Then we get a diagram

$$\begin{array}{ccccccc} S^0 \wedge S^{a+b\sigma} & \xrightarrow{e_{s\rho} \wedge \text{id}} & \Sigma^{-s\rho} \text{Ops}_{\mathbb{H}\mathbb{F}} \wedge S^{a+b\sigma} & \longrightarrow & \Sigma^{-s\rho} \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{a+b\sigma}) \\ \downarrow x & & \downarrow & & \downarrow \\ A & \longrightarrow & \Sigma^{-s\rho} \text{Ops}_{\mathbb{H}\mathbb{F}} \wedge_{\mathbb{H}\mathbb{F}} A & \longrightarrow & \Sigma^{-s\rho} \mathbb{P}_2^{\mathbb{F}}(A) & \longrightarrow & \Sigma^{-s\rho} A \end{array}$$

The longest composite is $Q^{s\rho}x$, by definition, so a sufficient condition for its vanishing is that the top horizontal composite vanishes. But the top horizontal composite is an element in $\mathbb{H}\mathbb{F}_{-s\rho+a+b\sigma} \mathbb{P}_2(S^{a+b\sigma})$. By Lemma 2.19, this group is zero in the range indicated. \square

Remark 4.7. Perhaps a better mnemonic is that the operations vanish on x for “ $s < \frac{|x|}{\rho}$ ”, in analogy with the odd primary Dyer-Lashof operations.

Lemma 4.8 (Squaring). *Let $\varepsilon = 0, 1$. For $x \in \pi_{n, \rho - \varepsilon} A$, $Q^{n\rho - \varepsilon} x = x^2 \in \pi_{2n\rho - 2\varepsilon} A$.*

Proof. The inclusion $\Sigma_{2+} = F_0 E\mu_{2+} \hookrightarrow E\mu_{2+}$ induces a diagram

$$\begin{array}{ccc} \mathbb{H}\mathbb{F} \wedge (S^{n\rho - \varepsilon} \wedge S^{n\rho - \varepsilon}) & \longrightarrow & \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{n\rho - \varepsilon}) \\ \downarrow & & \downarrow \\ A \wedge_{\mathbb{F}} A & \longrightarrow & \mathbb{P}_2^{\mathbb{F}}(A) \longrightarrow A \end{array}$$

The bottom horizontal composite is by definition the multiplication on the N_{∞} -algebra A , and the top horizontal arrow is the element $e_{n\rho - \varepsilon}^{n\rho - \varepsilon}$, again by definition. The result follows. \square

Remark 4.9. The reason why we only get a squaring result in these degrees is because our distinguished power operations are related to a homology basis for the extended powers of spheres like $S^{n\rho - \varepsilon}$. We do not know the homology of extended powers of arbitrary representation spheres S^V , but this bottom class is certainly there so one could define an operation Q^V . It’s unclear whether this class lifts to an element in $\pi_{\star} \text{Ops}_{\mathbb{H}\mathbb{F}}$, but if it did then the operations Q^V would be a linear combination of the operations we’ve already listed.

Remark 4.10. Had we started our filtration of $E\mu_2$ with $\frac{C_2 \times \Sigma_2}{\Delta}$ instead, then the bottom operation would be the *norm* instead of the square.

Lemma 4.11 (Bockstein). *For all $s \in \mathbb{Z}$, $\beta Q^{s\rho} = Q^{s\rho - 1} = Q^{(s-1)\rho + \sigma}$.*

Proof. Recall that $\beta c = b \in \mathbb{H}\mathbb{F}^{\rho} B\mu_2$. Since $\beta^2 = 0$ and β is a derivation on cohomology, it follows that $\beta(cb^{-s+1}) = b^{-s}$ in $\mathbb{H}\mathbb{F}^{\star}(B\mu_2)[b^{-1}]$. By Theorem 3.9 we conclude that $\beta e_{s\rho} = e_{s\rho - 1}$ and the result follows from naturality of the Bockstein and the definition of the power operations. \square

The universal example of power operations applied to a product of classes in degree V and W is captured by a diagonal map

$$\delta : \mathbb{P}_2(S^V \wedge S^W) \longrightarrow \mathbb{P}_2(S^V) \wedge \mathbb{P}_2(S^W).$$

Concretely, this arises from the diagonal of the space $E\mu_2$, which fits into a composite:

$$E\mu_{2+} \wedge (S^V \wedge S^W)^{\wedge 2} \longrightarrow E\mu_{2+} \wedge E\mu_{2+} \wedge (S^V \wedge S^W)^{\wedge 2} \cong (E\mu_{2+} \wedge (S^V)^{\wedge 2}) \wedge (E\mu_{2+} \wedge (S^W)^{\wedge 2}).$$

Interpreting the source and target as Thom spectra, this diagonal arises from the diagram:

$$\begin{array}{ccc} (V \oplus W)(1 \oplus \tau) & \longrightarrow & V(1 + \tau) \boxplus W(1 + \tau) \\ \downarrow & & \downarrow \\ B\mu_2 & \xrightarrow{\Delta} & B\mu_2 \times B\mu_2 \end{array}$$

where we’ve used \boxplus to denote an exterior sum. The next proposition follows.

Proposition 4.12. *The map*

$$\delta_{\star} : \mathbb{H}\mathbb{F}_{\star} \mathbb{P}_2(S^{k\rho} \wedge S^{\ell\rho}) \longrightarrow \mathbb{H}\mathbb{F}_{\star} \mathbb{P}_2(S^{k\rho}) \otimes_{\mathbb{H}\mathbb{F}_{\star}} \mathbb{H}\mathbb{F}_{\star} \mathbb{P}_2(S^{\ell\rho})$$

is given by the following formula together with the known action of the Bockstein:

$$\delta_{\star}(e_{r\rho}^{(k+\ell)\rho}) = \sum_{s+t=r} e_{s\rho}^{k\rho} \otimes e_{t\rho}^{\ell\rho}.$$

The diagonal maps taken together yield a map

$$\delta : \text{Ops}_{\mathbb{H}\mathbb{F}} \longrightarrow \text{holim}_{k,\ell} \Sigma^{(k+\ell)\rho} \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{-k\rho}) \wedge \mathbb{P}_2(S^{-\ell\rho})$$

and our computations of the homotopy of extended powers in this tower show that the canonical map

$$\text{Ops}_{\mathbb{H}\mathbb{F}} \wedge_{\mathbb{H}\mathbb{F}} \text{Ops}_{\mathbb{H}\mathbb{F}} \longrightarrow \text{holim}_{k,\ell} \Sigma^{(k+\ell)\rho} \mathbb{H}\mathbb{F} \wedge \mathbb{P}_2(S^{-k\rho}) \wedge \mathbb{P}_2(S^{-\ell\rho})$$

is an equivalence. So we get the following corollary.

Corollary 4.13. *The map*

$$\delta_* : \pi_* \text{Ops}_{\mathbb{H}\mathbb{F}} \longrightarrow \pi_* \text{Ops}_{\mathbb{H}\mathbb{F}} \otimes_{\mathbb{H}\mathbb{F}_*} \pi_* \text{Ops}_{\mathbb{H}\mathbb{F}}$$

is given by

$$\delta_*(e_{k\rho}) = \sum_{i+j=k} e_{i\rho} \otimes e_{j\rho}.$$

The next proposition then follows from a diagram chase from the definitions of our power operations.

Proposition 4.14 (Product formula). *For any $x \in \pi_* A$ and $y \in \pi_* B$ we have*

$$Q^{k\rho}(x \wedge y) = \sum_{i+j=k} Q^{i\rho}x \wedge Q^{j\rho}y \in \pi_*(A \wedge_{\mathbb{H}\mathbb{F}} B).$$

Remark 4.15. The fact that x and y are permitted to be in an arbitrary degree is a nice consequence of the use of $\text{Ops}_{\mathbb{H}\mathbb{F}}$ in our definition of power operations. This spectrum keeps track of all the relations implied by the interactions between power operations and $RO(C_2)$ -graded suspensions, which allows us to forego knowledge of the homology of $\mathbb{P}_2(S^V)$ for arbitrary V .

In order to compute the action of our power operations on $\mathbb{H}\mathbb{F}_* S^0$, we'll need to understand the induced map on homology of

$$\mathbb{P}_2(S^0) \longrightarrow S^0.$$

Algebraically, this is some additive map

$$\bigoplus \mathbb{H}\mathbb{F}_* \{e_{s\rho}^0\} \oplus \bigoplus \mathbb{H}\mathbb{F}_* \{e_{s\rho+\sigma}^0\} \rightarrow \mathbb{H}\mathbb{F}_*.$$

But, for degree reasons, all the generators must map to zero except possibly e_0^0 . On the span of this generator, we get the identity, which can be seen from the fact that squaring $S^0 \rightarrow S^0 \wedge S^0 \rightarrow S^0$ factors through the extended power.

Remark 4.16. If we had used $\mathbb{H}R$ for some \mathbb{F} -algebra, R , then e_0^0 would map to 1, but re_0^0 would map to r^2 for $r \in R$.

Together with vanishing, and the Bockstein lemma, we get

Lemma 4.17 (Action on $\mathbb{H}\mathbb{F}_*$). *Q^0 acts by the identity on $\mathbb{H}\mathbb{F}_*$, and $Q^{-1}u_\sigma = a_\sigma$.*

Finally, we turn to the action of power operations on the cohomology of a C_2 -space. First we record a fact which is immediate from the definition of the power operations.

Lemma 4.18. *If X is a C_2 -space, then the action of power operations on $\pi_* F(X_+, \mathbb{H}\mathbb{F})$ commutes with the suspension isomorphism.*

Proposition 4.19 (Unstable condition). *If X is a C_2 -space, then Q^0 acts by the identity on $\pi_* F(X_+, \mathbb{H}\mathbb{F})$ and $Q^{k\rho}$ acts by zero for $k > 0$.*

Proof. Using the suspension isomorphism, it suffices to check this result for elements in degree $\mathbb{H}\mathbb{F}^V(X)$ where V is a representation containing at least two copies of the trivial representation. By naturality, we may also assume that $X = K(\mathbb{F}, V)$ and the cohomology class in question is the universal one. The vanishing result now follows from the equivariant Hurewicz theorem (which is the source of our assumption on V) applied to the fundamental map

$$S^V \longrightarrow K(\mathbb{F}, V).$$

Also by the Hurewicz theorem we are reduced to checking the claim about Q^0 on the fundamental class in S^V . By the suspension isomorphism, this reduces to the action of Q^0 on $1 \in \mathbb{H}\mathbb{F}^0 S^0 = \mathbb{H}\mathbb{F}_0 S^0$. We have already seen that $Q^0 1 = 1$, so we're done. \square

5 The action of power operations on the dual Steenrod algebra

Consider the following diagram of adjoints between homotopy theories:

$$\begin{array}{ccc} N_\infty\text{-Alg}_{\mathbb{H}\mathbb{F}} & \text{---} & N_\infty\text{-Alg} \\ \downarrow & & \downarrow \\ \text{Mod}_{\mathbb{H}\mathbb{F}} & \text{---} & \text{Sp}^{C_2} \end{array}$$

Objects in the top right corner have two equivalent descriptions in terms of $\text{Mod}_{\mathbb{H}\mathbb{F}}$ with extra structure. Specifically, let C denote the comonad on $\text{Mod}_{\mathbb{H}\mathbb{F}}$ whose comodules are ordinary spectra, and let $\mathbb{P}^{\mathbb{F}}$ denote the monad on $\text{Mod}_{\mathbb{H}\mathbb{F}}$ whose algebras are N_∞ -algebras over $\mathbb{H}\mathbb{F}$. Then an object $X \in N_\infty\text{-Alg}$ is:

- a C -comodule with a compatible $\mathbb{P}^{\mathbb{F}}$ -module structure,
- a $\mathbb{P}^{\mathbb{F}}$ -module with a compatible C -comodule structure.

To formulate the notion of *compatible* one must write down a certain distributor, in the sense of Beck, which gives a relationship between the monads and comonads derived from the square above.

In our case, this distributor is essentially given by either of the following pieces of data:

- the \mathcal{A}_\star -comodule structure on $\pi_\star \text{Ops}_{\mathbb{H}\mathbb{F}}$,
- the action of power operations on \mathcal{A}_\star .

In particular, the abstract considerations above lead us to suspect that these two pieces of information determine one another. We have already computed the \mathcal{A}_\star -coaction on $\pi_\star \text{Ops}_{\mathbb{H}\mathbb{F}}$, and in this section we carry out the deduction of the action of power operations on \mathcal{A}_\star . We first establish an abstract characterization of the action in the form of Nishida relations, along the lines of [BJ97] and [Bak15], and then we concretize this characterization as a computation. The main result is Theorem 5.4.

5.1 Nishida relations

Lemma 5.1. *Let E be a C_2 -spectrum with an equivariant symmetric multiplication, and let $e \in \pi_\star \text{Ops}_{\mathbb{H}\mathbb{F}}$ be arbitrary. Write $\psi_R(e) = \sum e_i \otimes a_i \in \pi_\star \text{Ops}_{\mathbb{H}\mathbb{F}} \widehat{\otimes} \mathcal{A}_\star$. Then, for any $x \in \pi_\star E$,*

$$\psi_R(Q^e x) = \sum (1 \otimes a_i) Q^{e_i}(\psi_R(x)) \in \mathbb{H}\mathbb{F}_\star E \otimes \mathcal{A}_\star.$$

Proof. Represent x by a map $S^V \longrightarrow E$. This yields compatible maps of spectra

$$\Sigma^{V+W} \mathbb{P}_2(S^{-W}) \longrightarrow E,$$

and hence a map

$$\pi_* \Sigma^V \text{Ops}_{\mathbb{H}\mathbb{F}} \longrightarrow \mathbb{H}\mathbb{F}_* E$$

which respects the completed, right comodule structure over the equivariant dual Steenrod algebra. By definition of the power operations, an element $\Sigma^V e \in \pi_* \Sigma^V \text{Ops}_{\mathbb{H}\mathbb{F}}$ maps to $Q^e x$ in the target. The formula is now just a restatement of the preceding two observations. \square

The following reformulation of the Nishida relations in the classical case is due to Bisson-Joyal [BJ97] (for $p = 2$) and Baker [Bak15] (for $p > 2$). Recall that we had defined:

$$\bar{\xi}(t) = \sum_{i \geq 0} \bar{\xi}_i t^{2^i}, \quad \bar{\tau}(t) = \sum_{i \geq 0} \bar{\tau}_i t^{2^i}.$$

Theorem 5.2 (Co-Nishida relations). *Let E be a C_2 -spectrum with an equivariant symmetric multiplication and let $x \in \mathbb{H}\mathbb{F}_* E$, then, for any $s \in \mathbb{Z}$, we have an identity of formal power series:*

$$\sum_{r \in \mathbb{Z}} \psi_R(Q^{r\rho} x) t^r = \sum_{r \in \mathbb{Z}} Q^{r\rho}(\psi_R(x)) \bar{\xi}(t)^r + \sum_{r \in \mathbb{Z}} Q^{r\rho+\sigma}(\psi_R(x)) \bar{\xi}(t)^r \bar{\tau}(t).$$

Proof. Combine Theorem 3.20 with Lemma 5.1, and recall that switching from left to right comodule structures introduces a conjugation. \square

5.2 Computing the action

Recall that $\pi_* F(B\mu_{2+}, \mathbb{H}\mathbb{F})$ is generated by $c \in \mathbb{H}\mathbb{F}^\sigma B\mu_2$ and $b \in \mathbb{H}\mathbb{F}^\rho B\mu_2$.

Lemma 5.3. *The action of power operations on $\pi_* F(B\mu_{2+}, \mathbb{H}\mathbb{F})$ is determined by the unstable condition, the Cartan formula, vanishing, and the following:*

$$\sum (Q^{r\rho} b) t^r = b + b^2 t^{-1},$$

$$Q^{-1} c = b.$$

In particular:

$$Q^{i\rho}(b^{2^k}) = \begin{cases} b^{2^k} & i = 0 \\ b^{2^{k+1}} & i = -2^k \\ 0 & i \neq 0, -2^k \end{cases}$$

Theorem 5.4. *We have the following formulas describing the action of power operations on the equivariant dual Steenrod algebra.*

$$\begin{aligned} Q^{2^k \rho} \tau_k &= \tau_{k+1} + \tau_0 \xi_{k+1}, & Q^{2^k \rho} \bar{\tau}_k &= \bar{\tau}_{k+1}, \\ \beta Q^{2^k \rho} \tau_k &= \xi_{k+1}, & \beta Q^{2^k \rho} \bar{\tau}_k &= \bar{\xi}_{k+1} \\ Q^{2^k \rho} \xi_k &= \xi_{k+1} + \xi_1 \xi_k^2, & Q^{2^k \rho} \bar{\xi}_k &= \bar{\xi}_{k+1} \end{aligned}$$

Before turning to the proof, we record a corollary.

Corollary 5.5. *The equivariant dual Steenrod algebra is generated by τ_0 as an algebra over the ring of power operations. The polynomial subalgebra $P_* = \mathbb{H}\mathbb{F}_*[\xi_1, \xi_2, \dots]$ is generated by ξ_1 over the ring of power operations.*

One can use this corollary to deduce splitting results analogous to those of Steinberger [Bru+86, p. III.2]. We will not explore this here, though our main application of the theorem is a homological splitting result in the course of our cellular construction of BPR below.

Proof of the theorem. First we show how that the right-hand formulas follow from the left-hand formulas. Indeed, $\bar{\tau}_k$ is defined inductively by

$$\bar{\tau}_k = \tau_k + \sum_{0 \leq j < k} \xi_{k-j}^{2^j} \bar{\tau}_j$$

We want to apply $Q^{2^k \rho}$ to both sides, and assume by induction that $Q^{2^k \rho} \bar{\tau}_j = \bar{\tau}_{j+1}$ for $j < k$. Observe that, by the Cartan formula and degree considerations,

$$\begin{aligned} Q^{2^k \rho}(\xi_{k-j}^{2^j} \bar{\tau}_j) &= Q^{(2^k - 2^j) \rho}(\xi_{k-j}^{2^j}) Q^{2^j \rho} \bar{\tau}_j \\ &= \xi_{k-j}^{2^{j+1}} Q^{2^j \rho} \bar{\tau}_j && \text{(Squaring)} \\ &= \xi_{k-j}^{2^{j+1}} \bar{\tau}_{j+1} && \text{(Induction)}. \end{aligned}$$

Thus,

$$\begin{aligned} Q^{2^k \rho} \bar{\tau}_k &= Q^{2^k \rho} \tau_k + \sum_{j < k} \xi_{k-j}^{2^{j+1}} \bar{\tau}_{j+1} \\ &= Q^{2^k \rho} \tau_k + \xi_k^2 \bar{\tau}_1 + \cdots \\ &= \tau_{k+1} + \xi_{k+1} \tau_0 + \xi_k^2 \bar{\tau}_1 + \cdots \\ &= \tau_{k+1} + \sum_{0 \leq j < k+1} \xi_{k+1-j}^{2^j} \bar{\tau}_j \\ &= \bar{\tau}_{k+1}. \end{aligned}$$

Here we've used that $\bar{\tau}_0 = \tau_0$. Applying the Bockstein to the definition of $\bar{\tau}_k$ and using induction and the identity $\beta \tau_j = \xi_j$ yields the next formula on the right-hand column. The last formula is proved just like the first one, except we need to know that

$$Q^{(2^k - 2^j + 1) \rho} \xi_{k-j}^{2^j} = 0,$$

which follows from the Cartan formula and vanishing of some binomial coefficients mod 2.

Now we turn to the proof of the identities:

$$\begin{aligned} Q^{2^k \rho} \tau_k &= \tau_{k+1} + \tau_0 \xi_{k+1}, \\ \beta Q^{2^k \rho} \tau_k &= \xi_{k+1}, \\ Q^{2^k \rho} \xi_k &= \xi_{k+1} + \xi_1 \xi_k^2. \end{aligned}$$

The second follows immediately from the first, so we will only prove the first and last identities. We will apply the co-Nishida relations to the N_∞ -algebra $F(B\mu_{2^+}, \mathbf{H}\mathbb{F})$. For the element $c \in \underline{\pi}_{-\sigma} F(\mu_{2^+}, \mathbf{H}\mathbb{F})$ the co-Nishida relations read

$$\psi_R(c) = \psi_R(Q^0 c) = \left[\sum_{r \in \mathbb{Z}} Q^{r \rho}(\psi_R(c)) \bar{\xi}(t)^r + \sum_{r \in \mathbb{Z}} Q^{r \rho + \sigma}(\psi_R(c)) \bar{\xi}(t)^r \bar{\tau}(t) \right]_{t^0}$$

We are free to ignore terms with $r \geq 0$ since these do not contribute to the constant term. Also, each term in $\psi_R(c)$ has degree $-\sigma$, so, by vanishing, we need only consider the operations Q^0 and $\beta = Q^{-1} = Q^{-\rho + \sigma}$. So we can rewrite the expression as:

$$\psi_R(c) = [Q^0(\psi_R(c)) + \beta(\psi_R(c)) \bar{\xi}(t)^{-1} \bar{\tau}(t)]_{t^0}.$$

Now recall that $\psi_R(c) = c \otimes 1 + \sum b^{2^k} \otimes \tau_k$, so we can expand out the right hand side:

$$\begin{aligned} Q^0(\psi_R(c)) + \beta(\psi_R(c)) \bar{\xi}(t)^{-1} \bar{\tau}(t) &= c \otimes 1 + \sum_{k \geq 0} \sum_{i+j=0} Q^{i \rho}(b^{2^k}) \otimes Q^{j \rho} \tau_k \\ &\quad + \sum_{k \geq 0} \beta(b^{2^k}) \otimes \tau_k \bar{\xi}(t)^{-1} \bar{\tau}(t) + \sum_{k \geq 0} b^{2^k} \otimes \beta \tau_k \bar{\xi}(t)^{-1} \bar{\tau}(t) \end{aligned}$$

Since $\beta c = b$, $\beta b^{2^k} = 0$. Also, the constant term in $\bar{\xi}(t)^{-1}\bar{\tau}(t)$ is $\bar{\tau}_0 = \tau_0$. Indeed:

$$\begin{aligned} \frac{\bar{\tau}(t)}{\bar{\xi}(t)} &= \frac{\bar{\tau}(t)}{t + \bar{\xi}_1 t^2 + \bar{\xi}_2 t^4 + \dots} \\ &= \frac{1}{t} \frac{\bar{\tau}(t)}{1 + \bar{\xi}_1 t + \bar{\xi}_2 t^3 + \dots} \\ &= \frac{1}{t} [(\bar{\tau}(t))] \cdot [1 + (\bar{\xi}_1 t + \dots) + (\bar{\xi}_1 t + \dots)^2 + \dots] \\ &= \frac{1}{t} (\bar{\tau}_0 t + \bar{\tau}_1 t^2 + \dots)(1 + \bar{\xi}_1 t + \dots) \\ &= \bar{\tau}_0 + (\bar{\tau}_0 \bar{\xi}_1 + \bar{\tau}_1) t + \dots \end{aligned}$$

So, at this point we get the identity

$$\psi_R(c) = c \otimes 1 + \sum_{k \geq 0} \sum_{i+j=0} Q^{i\rho}(b^{2^k}) \otimes Q^{j\rho} \tau_k + \sum_{k \geq 0} b^{2^k} \otimes \xi_k \tau_0.$$

Now we expand the left hand side and simplify the right hand side using Lemma 5.3 to get

$$c \otimes 1 + \sum b^{2^k} \otimes \tau_k = c \otimes 1 + \sum_{k \geq 0} b^{2^{k+1}} \otimes Q^{2^k \rho} \tau_k + \sum_{k \geq 0} b^{2^k} \otimes \xi_k \tau_0$$

Comparing coefficients of $b^{2^{k+1}}$ yields the desired result:

$$Q^{2^k \rho} \tau_k = \tau_{k+1} + \tau_0 \xi_{k+1}.$$

The second identity follows from the fact that $\beta \tau_i = \xi_i$ and $\beta \xi_j = 0$, and third identity is proved just like the first. \square

6 A cellular construction of BPR

To motivate the maneuvers in the next section, we recall Priddy's approach for constructing BP. First, he builds a spectrum X by attaching even cells to $S_{(2)}^0$ to kill odd homotopy. There is a canonical map $X \rightarrow \mathbb{H}\mathbb{F}_2$ classifying the unit in cohomology, and Priddy wants to show that

$$\mathbb{H}\mathbb{F}_{2*} X \rightarrow \mathcal{A}_*$$

is surjective. In order to do so, he notes that ζ_1 is automatically hit because the unit map $S^0 \rightarrow X$ extends over the cofiber on η , which is detected by ζ_1 . In order to hit the remaining ζ_i , he purports to construct a map

$$\mathbb{P}_2(X) \rightarrow X$$

by obstruction theory. The obstructions live in $H^{n+1}(\mathbb{P}_2(X); \pi_n X)$. Half of these obstructions vanish by construction because $\pi_{\text{odd}} X = 0$. The other obstructions live in *odd* cohomology of $\mathbb{P}_2(X)$ with coefficients in $\pi_*(X)$. If we knew that $\pi_*(X)$ was torsion-free, we would be done. The author does not know how to prove that $\pi_*(X)$ is torsion-free without assuming the result we are trying to prove, or assuming the existence of BP, so it seems we have encountered a problem in our obstruction theory.

In the equivariant case, things are even worse. Let $\text{Even}(S^0)$ denote the C_2 -spectrum obtained from $S_{(2)}^0$ by inductively killing the Mackey functors $\pi_{*\rho-1}$. We might hope to construct a map

$$\mathbb{P}_2(\text{Even}(S^0)) \rightarrow \text{Even}(S^0)$$

by inducting up the slice tower of $\text{Even}(S^0)$. We find that the obstructions live in

$$[\mathbb{P}_2(\text{Even}(S^0)), \Sigma P_{2n}^{2n} \text{Even}(S^0)]^{C_2}$$

where $P_{2n}^{2n} \text{Even}(S^0)$ denotes the $2n$ -slice of $\text{Even}(S^0)$. Just as in the non-equivariant case, we need to know something about the homotopy Mackey functor $\underline{\pi}_{n,\rho} \text{Even}(S^0)$ in order to conclude that these groups vanish. This time it is not enough to know that these Mackey functors are torsion free, we must check that their underlying groups are torsion-free *and* that they have a surjective restriction map.

The observation that saves the argument is as follows: we do not need the map $\mathbb{P}_2(\text{Even}(S^0)) \rightarrow \text{Even}(S^0)$ all at once. Instead, we can inductively define maps from a sub-complex of $\mathbb{P}_2(\text{Even}(S^0))$ into $\text{Even}(S^0)$, and use these maps to define *some* of the power operations we need. Something similar works, and more easily, in the non-equivariant case.

The induction requires some rather technical homological algebra, for which we give proofs that work, but are perhaps not so elegant. We imagine that developing the theory of the slice filtration in the context of chain complexes of $RO(C_2)$ -graded Mackey functors would result in more conceptual proofs of these results, but we do not see immediately just how this should go.

Finally, we use this characterization of the homology of $\text{Even}(S^0)$ to determine its homotopy, its geometric fixed points, its slice tower, and to identify it with **BPR**. Though, once again, we emphasize that the fact that $\text{Even}(S^0) \cong \text{BPR}$ admits a much simpler proof but it is not in the spirit of this paper.

6.1 Review of the slice filtration

We will need to build maps using obstruction theory. In the classical case, obstruction theory comes from Postnikov towers. We will need an equivariant version of these originating in work of Dugger and developed much further by Hill-Hopkins-Ravenel and Hill.

Definition 6.1. The set of **slice cells** is

$$\{C_{2+} \wedge S^m, S^{m\rho}, S^{m\rho-1} | m \in \mathbb{Z}\}.$$

The dimension of a slice cell is the dimension of the underlying spheres.

Define a full sub-homotopy theory $\text{Sp}_{>n}^{C_2} \subset \text{Sp}^{C_2}$ containing slice cells of dimension $> n$ and closed under equivalence, homotopy colimits, and extension.

The inclusion $\text{Sp}_{>n}^{C_2} \hookrightarrow \text{Sp}^{C_2}$ admits a homotopical right adjoint $P_{n+1} : \text{Sp}^{C_2} \rightarrow \text{Sp}_{>n}^{C_2}$ and we define P^n by the natural cofiber sequence

$$P_{n+1} \rightarrow \text{id} \rightarrow P^n.$$

The inclusions $\text{Sp}_{>n}^{C_2} \subset \text{Sp}_{>n-1}^{C_2}$ yield natural maps $P^n \rightarrow P^{n-1}$ so every $E \in \text{Sp}^{C_2}$ has an associated tower:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & P^n E & \longleftarrow & P_n E \\
 & \nearrow & \downarrow & & \\
 & & P^{n-1} E & \longleftarrow & P_{n-1} E \\
 & \nearrow & \downarrow & & \\
 & & \vdots & & \\
 & \nearrow & \downarrow & & \\
 E & \longrightarrow & P^0 E & \longleftarrow & P_0 E
 \end{array}$$

The fiber $P_n^n E$ is called the n -**slice** of E , and the tower is called the **slice tower** of E . The natural map $E \rightarrow \text{holim } P^n E$ is a weak equivalence and, for each k , the tower of homotopy Mackey functors $\{\pi_k P^n E\}$ is pro-isomorphic to the constant tower $\{\pi_k E\}$ [HHR16, Thm 4.42].

We will need a formula for the n -slices in terms of the homotopy Mackey functors of E , but to describe this formula we need a definition.

Definition 6.2. If \underline{M} is a C_2 Mackey functor let $P^0 \underline{M}$ denote the Mackey functor:

$$P^0 \underline{M} = \begin{array}{c} \text{im}(\text{res} : \underline{M}(C_2/C_2) \rightarrow \underline{M}(C_2)) \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\ \underline{M}(C_2) \end{array}$$

Theorem 6.3 (HHR, Hill). *For any C_2 -spectrum E , the slices are given by*

$$\begin{aligned} P_{2n}^{2n} E &= \Sigma^{n\rho} \text{HP}^0 \pi_{n\rho} E, \\ P_{2n-1}^{2n-1} E &= \Sigma^{n\rho-1} \text{H}\pi_{n\rho-1} E. \end{aligned}$$

6.2 Construction of $\text{Even}(S^0)$

The remainder of §6 is implicitly localized at 2.

Suppose given a C_2 -spectrum Y . We will say that Z is obtained from Y by **non-trivially attaching $n\rho$ -cells** if there is a cofiber sequence

$$\bigvee W_\alpha \xrightarrow{f} Y \longrightarrow Z$$

where each W_α is a slice cell of dimension $2n - 1$ and $\ker(f_*) \subset ([C_2], 2) \cdot \bigoplus_\alpha \pi_{n\rho-1}(W_\alpha)$.

Construction 6.4. Let $\text{Even}(S^0)_0 = S^0$. Given $\text{Even}(S^0)_k$, non-trivially attach cells of dimension $2k + 2$ to obtain $\text{Even}(S^0)_{k+1}$ with $\pi_{(k+1)\rho-1} \text{Even}(S^0)_{k+1} = 0$. Let $\text{Even}(S^0) = \text{colim } \text{Even}(S^0)_k$. Note that $\text{Even}(S^0)$ comes equipped with a map $S^0 \rightarrow \text{Even}(S^0)$ which we'll call the unit.

An immediate consequence of the construction is a splitting result in homology.

Lemma 6.5. $\text{H}\underline{\mathbb{Z}} \wedge \text{Even}(S^0) \cong \text{H}\underline{\mathbb{Z}} \wedge W$ as $\text{H}\underline{\mathbb{Z}}$ -modules, where W is a wedge of even-dimensional slice cells.

Proof. Consider the cofiber sequence:

$$\begin{array}{ccccccc} \bigvee W_\alpha & \longrightarrow & \text{Even}(S^0)_k & \longrightarrow & \text{Even}(S^0)_{k+1} & \longrightarrow & \bigvee \Sigma W_\alpha \\ & \searrow & \downarrow & & & & \\ & & \text{H}\underline{\mathbb{Z}} \wedge \text{Even}(S^0)_k & & & & \end{array}$$

By induction we may assume that $\text{H}\underline{\mathbb{Z}} \wedge \text{Even}(S^0)_k \cong \text{H}\underline{\mathbb{Z}} \wedge Y$ where Y is a wedge of even-dimensional slice cells. Thus, the diagonal map is a sum of elements in $\pi_{m\rho-1} \text{H}\underline{\mathbb{Z}}$ for various m . But these Mackey functors are all zero, so the map is null and we get the desired splitting. \square

A slightly less immediate consequence of the construction is the vanishing of the mod 2 Hurewicz map.

Lemma 6.6. *The Hurewicz map $\pi_{k\rho} \text{Even}(S^0) \rightarrow \text{H}\underline{\mathbb{F}}_{k\rho} \text{Even}(S^0)$ is zero for $k > 0$.*

Proof. We argue as in Priddy. We have a commutative diagram:

$$\begin{array}{ccccc}
\pi_{n\rho}\text{Even}(S^0)_n & \longrightarrow & \bigoplus \pi_{n\rho}\Sigma W_\alpha & \xrightarrow{\partial} & \pi_{n\rho}\Sigma\text{Even}(S^0)_{n-1} \\
\downarrow & & \downarrow & & \\
\mathbb{H}\mathbb{F}_{n\rho}\text{Even}(S^0)_n & \longrightarrow & \bigoplus \mathbb{H}\mathbb{F}_{n\rho}\Sigma W_\alpha & &
\end{array}$$

The bottom horizontal arrow is an isomorphism for $n > 0$ because $\mathbb{H}\mathbb{F}_{n\rho}\text{Even}(S^0)_{n-1} = 0$ by induction on n . Thus $j(g)$ is in the kernel of ∂ but does not lie in $([C_2], 2) \cdot \bigoplus \pi_{n\rho}\Sigma W_\alpha$, otherwise it would vanish under the Hurewicz map.

For obstruction theory purposes, we will need to know a bit about the homotopy groups of $\text{Even}(S^0)$. \square

Lemma 6.7. $\pi_{k\rho-1}\text{Even}(S^0) = 0$ for all $k \geq 0$.

Proof. Consider the cofiber sequence

$$\bigvee W_\alpha \longrightarrow \text{Even}(S^0)_n \longrightarrow \text{Even}(S^0)_{n+1}$$

where each W_α is a slice cell of dimension $2n - 1$. This yields an exact sequence:

$$\pi_{k\rho-1}\text{Even}(S^0)_n \longrightarrow \pi_{k\rho-1}\text{Even}(S^0)_{n+1} \longrightarrow \bigoplus \pi_{k\rho-2}W_\alpha.$$

Recall that

$$\pi_{k\rho-2}S^{n\rho-1} = \pi_0 S^{(n-k)\rho+1} = 0, \quad n \geq k.$$

It follows that

$$\pi_{k\rho-1}\text{Even}(S^0) = \pi_{k\rho-1}\text{Even}(S^0)_k = 0.$$

\square

We can also compute the first nonvanishing homotopy group.

Lemma 6.8. $\pi_0\text{Even}(S^0) = \mathbb{Z}_{(2)}$.

Proof. The same connectivity argument as above shows that $\pi_0\text{Even}(S^0) = \pi_0\text{Even}(S^0)_1$. The spectrum $\text{Even}(S^0)_1$ is built from S^0 by killing the elements in $\pi_{\rho-1}S^0$. This group is generated by the Hopf map $\eta : S^\sigma \rightarrow S^0$ so $\text{Even}(S^0)_1 = \text{cofib}(\eta)$. The attaching map then gives an exact sequence:

$$\begin{array}{ccccccc}
\pi_0(S^\sigma) & \longrightarrow & \pi_0(S^0) & \longrightarrow & \pi_0(\text{Even}(S^0)_1) & \longrightarrow & \pi_{-1}(S^\sigma) \\
& & \uparrow \cong & & \parallel & & \parallel \\
\underline{A} & \longrightarrow & \pi_0(\text{Even}(S^0)_1) & \longrightarrow & 0 & &
\end{array}$$

where \underline{A} denotes the (2-local) Burnside Mackey functor. The image of η_* is precisely the ideal $([C_2] - 2)$, so $\pi_0(\text{Even}(S^0)_1) = \underline{A}/([C_2] - 2) \cong \mathbb{Z}_{(2)}$, which was to be shown. \square

The vanishing in (6.7) is enough to build a ring structure on $\text{Even}(S^0)$.

Lemma 6.9. *The spectrum $\text{Even}(S^0)$ admits a multiplication $\text{Even}(S^0) \wedge \text{Even}(S^0) \rightarrow \text{Even}(S^0)$ compatible with the unit $S^0 \rightarrow \text{Even}(S^0)$.*

Proof. We build the map by induction up the slice tower of $\text{Even}(S^0)$. Since the odd slices of $\text{Even}(S^0)$ are zero, we are only concerned with obstructions in

$$[\text{Even}(S^0) \wedge \text{Even}(S^0), \Sigma^{n\rho+1}\mathbf{HP}^0 \underline{\pi}_{n\rho} \text{Even}(S^0)]^{C_2}.$$

Every Mackey functor with injective restriction maps is a module over $\underline{\mathbb{Z}}$. So we need only show that

$$\pi_{k\rho-1}^{C_2} \text{Map}_{\mathbf{H}\underline{\mathbb{Z}}}(\mathbf{H}\underline{\mathbb{Z}} \wedge (\text{Even}(S^0) \wedge \text{Even}(S^0)), \mathbf{H}\underline{M}) = 0.$$

where \underline{M} has injective restriction maps.

By (6.5), we know that $\mathbf{H}\underline{\mathbb{Z}} \wedge \text{Even}(S^0) \wedge \text{Even}(S^0)$ splits as a wedge of even-dimensional slice cells. So it's enough to check that $\pi_{k\rho-1}^{C_2} \mathbf{H}\underline{M} = 0$. That's essentially true by the definition of the slice filtration. We can verify it just to be sure, though. It amounts to showing that $\pi_{k\rho-1}^{C_2} \mathbf{H}\underline{M} = 0$. By connectivity, we need only check that case when $k = 1$, so we want to know $\pi_{\sigma}^{C_2} \mathbf{H}\underline{M}$. From the cofiber sequence

$$C_{2+} \rightarrow S^0 \rightarrow S^\sigma$$

we see that $\pi_{\sigma}^{C_2} \mathbf{H}\underline{M} = \ker(\text{res})$. This vanishes by assumption on \underline{M} . \square

6.3 Some algebraic preliminaries on P_\star

Let $P_\star \subset \mathcal{A}_\star$ denote the left $\mathbf{H}\underline{\mathbb{F}}_\star$ -submodule $\mathbf{H}\underline{\mathbb{F}}_\star[\xi_1, \xi_2, \dots]$. This is a left \mathcal{A}_\star -comodule algebra, but it is *not* a Hopf algebroid. The trouble is that

$$\eta_R(u_\sigma) = a_\sigma \tau_0 + u_\sigma,$$

so P_\star is not even a right $\mathbf{H}\underline{\mathbb{F}}_\star$ -module. If we want a Hopf algebroid, we can take $(\underline{\mathbb{F}}, P_\star^\circ)$ defined by $P_\star^\circ = \underline{\mathbb{F}}[\xi_1, \xi_2, \dots]$.

There is an extension of Hopf algebroids

$$(\underline{\mathbb{F}}[a_\sigma], P_\star^\circ[a_\sigma]) \rightarrow (\mathbf{H}\underline{\mathbb{F}}_\star, \mathcal{A}_\star) \rightarrow (\mathbf{H}\underline{\mathbb{F}}_\star, \Lambda)$$

where

$$\Lambda = \mathbf{H}\underline{\mathbb{F}}_\star[\tau_0, \tau_1, \dots] / (\tau_i^2 = a_\sigma \tau_{i+1}).$$

Lemma 6.10. *As a left \mathcal{A}_\star -comodule algebra,*

$$P_\star \cong \mathcal{A}_\star \square_\Lambda \mathbf{H}\underline{\mathbb{F}}_\star.$$

Proof. Since $P_\star = \mathbf{H}\underline{\mathbb{F}}_\star \otimes_{\underline{\mathbb{F}}[a_\sigma]} P_\star^\circ[a_\sigma]$ the result follows from [Rav86, A1.1.16]. \square

Corollary 6.11. $\underline{\text{Ext}}_{\mathcal{A}_\star}(P_\star) \cong \underline{\text{Ext}}_\Lambda(\mathbf{H}\underline{\mathbb{F}}_\star)$.

We'll need to compute something about these Ext groups for the next section. First we record the answer:

Proposition 6.12. (i) *The Mackey functors $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s, s+n\rho-1}(P_\star)$ vanish,*

(ii) *the Mackey functors $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s, s+n\rho+1}(P_\star)$ vanish when evaluated at C_2 ,*

(iii) *if we let $v_i := [\tau_i]$ in the cobar complex, then*

$$\bigoplus \underline{\text{Ext}}_{\mathcal{A}_\star}^{s, s+n\rho}(P_\star) \cong \underline{\mathbb{F}}[v_0, v_1, \dots].$$

Proof. Consider the τ_0 -adic filtration on Λ , and the trivial filtration on $\mathbb{H}\mathbb{F}_*$. Then $(\mathbb{H}\mathbb{F}_*, \text{gr}(\Lambda))$ is now a Hopf algebra since η_L and η_R agree modulo τ_0 (cf. [HK01, (6.44)]). We have:

$$\text{gr}(\Lambda) = \mathbb{H}\mathbb{F}_*[x_0, x_1, \dots]/(x_i^2 = a_\sigma x_{i+1}, a_\sigma^{2^i} x_i = 0).$$

Notice that this Hopf algebra is not flat over $\mathbb{H}\mathbb{F}_*$, but that does not affect the remaining argument. It is still possible to define Ext-groups using relative homological algebra with respect to the class of extended comodules, and the resolutions we use are resolutions by relative injectives.

All the x_i are primitive, and the additive structure of Ext only depends on the *coalgebra* structure of $\text{gr}(\Lambda)$. So, using the standard Koszul resolution for a primitively generated exterior algebra, we conclude that *additively*,

$$\underline{\text{Ext}}_{\text{gr}(\Lambda)}(\mathbb{H}\mathbb{F}_*) = \mathbb{H}\mathbb{F}_*[y_0, y_1, \dots]/(a_\sigma^{2^i} y_i)$$

where the y_i have homological degree 1. Using the spectral sequence

$$\underline{\text{Ext}}_{\text{gr}(\Lambda)}(\mathbb{H}\mathbb{F}_*) \Rightarrow \underline{\text{Ext}}_\Lambda(\mathbb{H}\mathbb{F}_*)$$

Each claim (i)-(iii) is already true on the E_2 -page. So (i) and (ii) follow immediately. The claim (iii) follows because the classes $[\tau_{i_1} | \dots | \tau_{i_k}]$ are clearly cocycles in the cobar complex and are detected by the corresponding monomials in the y_j . Thus, every monomial in the y_j is a permanent cycle in the spectral sequence above, and the claim follows because there are no multiplicative extensions for a polynomial algebra. \square

Remark 6.13. The first instinct is to use the a_σ -Bockstein spectral sequence, but it's not at all obvious that this converges to the right thing. After all, $\mathbb{H}\mathbb{F}_*$ has elements which are both a_σ -torsion and a_σ -divisible. Nonetheless, it turns out that these Ext-groups are a_σ -adically complete. In fact, $\underline{\text{Ext}}_{\mathcal{A}_*}(P_*)$ is an associated graded of $\underline{\pi}_* \text{BPR}$, and the a_σ -Bockstein spectral sequence agrees with the associated graded of the slice spectral sequence. The author learned this surprising fact from Zhouli Xu and Guchuan Li.

In the next section, we will be concerned with the \mathcal{A}_* -comodule structure of $\mathbb{H}\mathbb{F}_* \text{Even}(S^0)$. For the sake of inductive arguments, it will be helpful to record two lemmas which allow one to conclude information based off of the behavior of a comodule in a range of dimensions. The first is trivial, and the second is less so.

Lemma 6.14 (Approximate Milnor-Moore). *Let (B, Σ) be a graded connected Hopf algebroid, M a graded connected right Σ -comodule algebra, and $C = M \square_\Sigma B$. Suppose*

- (i) *there is a comodule algebra map $f : M \rightarrow \Sigma$ surjective in degrees $\leq N$,*
- (ii) *C is a B -module and as such is a direct summand of M in degrees $\leq N$.*

Then $M \cong C \otimes_B \Sigma$ as a left C -module and right Σ -comodule in degrees $\leq N$.

Proof. The proof is the same as in A.1.1.17 of [Rav86]. \square

Definition 6.15. We will say that an $RO(C_2)$ -graded $\mathbb{H}\mathbb{F}_*$ -module M is **pure** if it is of the form

$$M \cong \bigoplus \mathbb{H}\mathbb{F}_*\{e_\alpha\} \oplus \bigoplus [\mathbb{H}\mathbb{F}_*]_{C_2}\{f_\beta\}$$

where $|e_\alpha| = n_\alpha \rho$ and $|f_\beta| = 2m_\beta$ for some integers n_α and m_β and some indexing set. If moreover, we can choose the e_α and f_β such that $n_\alpha, m_\beta \geq 0$ for all α, β then we will say that M is **pure and connected**.

Lemma 6.16 (Approximate Ext). *Let M and M' be \mathcal{A}_* -comodules which are pure and connected as $\mathbb{H}\mathbb{F}_*$ -modules, and fix a positive integer N . Suppose that there is an isomorphism on the submodules*

$$\langle e_\alpha, f_\beta | n_\alpha, m_\beta \leq N \rangle \cong \langle e'_\alpha, f'_\beta | n'_\alpha, m'_\beta \leq N \rangle$$

for some decomposition as in the definition above. Suppose, moreover, that this is an isomorphism of co-modules modulo the generators not listed. Then the Mackey functors

$$\underline{\text{Ext}}_{\mathcal{A}_*}^{s,V}(M) \quad \underline{\text{Ext}}_{\mathcal{A}_*}^{s,V}(M')$$

are

$$\begin{cases} \text{isomorphic} & V - s = k\rho - 1, k \leq N \\ \text{isomorphic mod } a_\sigma & V - s = k\rho, k \leq N \\ \text{isomorphic on underlying groups} & V - s = k\rho, k\rho + 1 \text{ and } k \leq N \end{cases}$$

Proof. We will show that the terms of the reduced cobar complex agree in the appropriate dimensions, and then make sure that the cycles and boundaries remain the same. Recall that

$$\text{Cobar}^s(M) := \overline{\mathcal{A}}_*^{\otimes s} \otimes_{\text{HF}_*} M.$$

As an \mathbb{F}_2 -vector space, this is generated by elements of the form

$$c[a_1|a_2|\cdots|a_s]e \quad \text{and} \quad c[a_1|a_2|\cdots|a_s]\frac{f}{u_\sigma^r}$$

where $c \in \text{HF}_*$, the a_i are non-constant monomials in τ_i and ξ_i , $e = e_\alpha$ for some α and $f = f_\beta$ for some β .

Step 1. The degree $s + k\rho$ part of $\text{Cobar}^s(M)$ is contained in the span of e_α and f_β with $n_\alpha, m_\beta \leq k$.

If x is an element in an $RO(C_2)$ -graded module, we will write $\deg(x) = x^0 + x^1\sigma$. Then

$$\begin{aligned} \deg(c[a_1|a_2|\cdots|a_s]e) &= (c^0 + \sum a_i^0 + e^0) + (c^1 + \sum a_i^1 + e^1)\sigma, \\ \deg\left(c[a_1|a_2|\cdots|a_s]\frac{f}{u_\sigma^r}\right) &= (c^0 + \sum a_i^0 + f^0 - r) + (c^1 + \sum a_i^1 + f^1 + r)\sigma. \end{aligned}$$

Observe that $\sum a_i^0 \geq s$ since the minimal a_i^0 that can occur for a monomial a_i in $\overline{\mathcal{A}}_*$ is 1. Also observe that $\sum a_i^1 \geq 0$ since every non-constant monomial has this property. If we restrict to those basis elements in degree $s + k\rho$, we see

$$\begin{cases} e^0 \leq k - c^0 \\ e^1 \leq k - c^1 \\ f^0 \leq k - c^0 + r \\ f^1 \leq k - c^1 - r \end{cases}$$

Now we use the fact that $e = e_\alpha$ and $f = f_\beta$ so that $e^0 = e^1 = n$ for some $n \geq 0$ and $f^1 = 0$ and $f^0 = 2m$ for some $m \geq 0$. Then these inequalities become:

$$\begin{cases} n \leq k - c^0 \\ n \leq k - c^1 \\ 2m \leq k - c^0 + r \\ 0 \leq k - c^1 - r \end{cases}$$

But $c \in \text{HF}_*$ is only nonzero when c^0 and c^1 have opposite sign, so at least one of these is nonnegative and we conclude that $n \leq k$. If $c^0 < 0$ then c becomes a_σ -divisible, while $[\text{HF}_*]_{C_2}$ is a_σ -torsion and we conclude that our generator vanishes in the second case. If $c^0 \geq 0$ and divisible by a_σ we're also okay. If $c^0 \geq 0$ and not divisible by a_σ , then it must be of the form u_σ^ℓ , in which case $c^0 = -c^1$. Adding the last two inequalities then yields $2m \leq 2k$ and hence $m \leq k$.

Step 2. The degree $s + k\rho - 1$ part of $\text{Cobar}^s(M)$ is contained in the span of

The same argument for degree $s + k\rho - 1$ yields the inequalities

$$\begin{cases} n \leq k - c^0 - 1 \\ n \leq k - c^1 \end{cases}$$

$$\begin{cases} 2m \leq k - c^0 + r - 1 \\ 0 \leq k - c^1 - r \end{cases}$$

If $c^0 \geq 0$ then $n \leq k - 1$. But if $c^1 \geq 0$ then in fact $c^1 \geq 2$ by the gap in the equivariant homology of a point, so then $n \leq k - 2$. In either case, $n \leq k - 1$. The same argument as before shows that $m \leq k - \frac{1}{2}$. But m is an integer, so $m \leq k - 1$.

Combining the results of these two arguments, and the identical result for M' in place of M , we have shown that

$$(\text{Cobar}^s(M))_V \cong (\text{Cobar}^s(M'))_V$$

when

$$V - s = \begin{cases} k\rho & k \leq N, \\ k\rho - 1 & k \leq N + 1 \end{cases}$$

Now we need to check that this induces an isomorphism in homology in the degrees claimed in the statement of the lemma. The only difficult thing to check is that the witnesses for boundaries do not have too high of a degree.

Step 3. Both M and M' have identical $\underline{\text{Ext}}^{s,V}$ when $V - s = k\rho - 1$ for $k \leq N$.

Suppose $y \in \text{Cobar}^{s+1}(M)$ has degree $s + 1 + (N\rho - 1) = s + N\rho$, and there is some $x \in \text{Cobar}^s(M)$ with $dx = y$. Since x has degree $s + N\rho$, what we've done so far guarantees that y remains a boundary in $\text{Cobar}^{s+1}(M')$.

Step 4. Both M and M' have the same $\underline{\text{Ext}}^{s,V}$ modulo a_σ , when $V - s = k\rho$ for $k \leq N$.

It follows from what we've done so far that the cycles in homological degree s are the same for both M and M' . The only thing we have to worry about are the boundaries.

Suppose $y \in \text{Cobar}^{s+1}(M)$ has degree $s + 1 + N\rho$ and $x \in \text{Cobar}^s(M)$ has $dx = y$. Then we claim there is some x' in the span of $\overline{\mathcal{A}}_\star^{\otimes s} \otimes \{e_\alpha, f_\beta\}$ for $n_\alpha, m_\beta \leq N$ such that $dx' = y$ modulo a_σ .

To that end, let's consider elements in $\text{Cobar}^s(M)$ of degree $s + 1 + N\rho$. Using notation as before, we must have

$$\begin{cases} n \leq N - c^0 + 1 \\ n \leq N - c^1 \end{cases}$$

$$\left\{ m \leq N + \frac{1}{2} \right.$$

Since m is an integer, $m \leq N$. Thus, writing x in terms of generators, the only possible components which are in high degrees are those with e_α having $n_\alpha = N + 1$. By the inequalities above, this implies $c^0 = 0$ and $\sum a_i^0 = s$. So we are concerned about generators of the form

$$a_\sigma^r [a_1 | \cdots | a_s] e, \quad \text{where } |e| = (N + 1)\rho, \quad a_i \in \{\tau_0, \xi_1\}.$$

In order for this element to have degree $s + 1 + N\rho$, we must have $r \geq 1$, so this element is divisible by a_σ . The differentials are linear over $\mathbb{F}[a_\sigma]$ (since a_σ is primitive), so the claim follows.

Step 5. Both M and M' have the same *underlying* group for $\underline{\text{Ext}}^{s,V}$ when $V - s = k\rho + 1$ and $k \leq N$.

This follows from the same vanishing argument as in the previous step using the free generators. \square

Remark 6.17. One should think that things shouldn't be so hard. The author welcomes any suggestions.

6.4 Properties of $\text{Even}(S^0)$ and equivalence with BPR

Lemma 6.18. *There is a factorization:*

$$\begin{array}{ccc} \underline{\mathbb{F}}\{e_\alpha^i, f_\beta^j | i, j \leq k\} & \dashrightarrow & P_\star^\circ \otimes_{\underline{\mathbb{F}}} \underline{\mathbb{F}}\{e_\alpha^i, f_\beta^j | i, j \leq k\} \\ \downarrow & & \downarrow \\ \underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0)_k & \xrightarrow{\psi} & \mathcal{A}_\star \otimes_{\underline{\mathbb{H}\mathbb{F}}_\star} \underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0) \end{array}$$

This implies that $\underline{\mathbb{F}}\{e_\alpha^i, f_\beta^j | i, j \leq k\}$ is a P_\star° -comodule and that the full coaction factors as:

$$\underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0)_k \rightarrow P_\star \otimes_{\underline{\mathbb{H}\mathbb{F}}_\star} \underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0)_k \rightarrow \mathcal{A}_\star \otimes_{\underline{\mathbb{H}\mathbb{F}}_\star} \underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0).$$

Proof. This follows formally for degree reasons. □

We would like to show that

$$\underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0) = P_\star.$$

We will prove this by a simultaneous induction on the following properties:

- (i) $\pi_{\star\rho} \text{Even}(S^0)$ is a constant Mackey functor in a range,
- (ii) $\text{Even}(S^0)$ admits a partially defined map $\mathbb{P}_2(\text{Even}(S^0)) \rightarrow \text{Even}(S^0)$,
- (iii) the canonical map $\underline{\mathbb{H}\mathbb{F}}_\star \text{Even}(S^0) \rightarrow P_\star$ hits all the $\bar{\xi}_k$ in a range.

Property (i) will allow us, by obstruction theory, to extend our symmetric multiplication in (ii) a bit further. This will allow us to define a certain power operation on the top available $\bar{\xi}_k$ to get the next one, which extends the range in (iii). This gives new input for the Adams spectral sequence, and allows us to extend the range in (i) and repeat the process.

The next two lemmas will be used to show the vanishing of certain obstructions.

Lemma 6.19. *Suppose X has the property that*

$$\underline{\mathbb{H}\mathbb{Z}} \wedge X \cong \underline{\mathbb{H}\mathbb{Z}} \wedge W$$

where W is a wedge of even-dimensional slice cells. Let \underline{M} be a torsion-free, constant Mackey functor. Then, for all $n \in \mathbb{Z}$,

$$\underline{\mathbb{H}\mathbb{M}}^{k\rho+1}(\mathbb{P}_2(X)) = 0.$$

Proof. We are immediately reduced to the case where $\underline{M} = \underline{\mathbb{Z}}$. Notice that

$$\underline{\mathbb{H}\mathbb{Z}}^\star(\mathbb{P}_2(X)) = \pi_{-\star} \text{Map}_{\underline{\mathbb{H}\mathbb{Z}}}(\underline{\mathbb{H}\mathbb{Z}} \wedge \mathbb{P}_2(X), \underline{\mathbb{H}\mathbb{Z}})$$

where $\text{Map}_{\underline{\mathbb{H}\mathbb{Z}}}(-, -)$ is the derived mapping spectrum in the homotopy theory of $\underline{\mathbb{H}\mathbb{Z}}$ -modules. We will split the source and verify the vanishing on each piece.

Recall (2.2) that

$$\underline{\mathbb{H}\mathbb{Z}} \wedge \mathbb{P}_2(X) \cong \mathbb{P}_2^{\underline{\mathbb{Z}}}(\underline{\mathbb{H}\mathbb{Z}} \wedge X).$$

By our assumption on X , repeated application of (2.3) yields a splitting

$$\underline{\mathbb{H}\mathbb{Z}} \wedge \mathbb{P}_2(\text{Even}(S^0)) \cong (\underline{\mathbb{H}\mathbb{Z}} \wedge W_1) \vee (\underline{\mathbb{H}\mathbb{Z}} \wedge W_2) \vee (\underline{\mathbb{H}\mathbb{Z}} \wedge W_3) \vee (\underline{\mathbb{H}\mathbb{Z}} \wedge W_4)$$

where:

- W_1 is a wedge of extended powers $\mathbb{P}_2(S^{m\rho})$,

- W_2 is a wedge of spheres of the form $S^{m\rho}$,
- W_3 is a wedge of extended powers $C_{2+} \wedge \mathbb{P}_2(S^{2m})$,
- W_4 is a wedge of spheres of the form $C_{2+} \wedge S^{2m}$.

We know that $\mathbb{H}\mathbb{Z}^{k\rho+1}(S^{m\rho})$ and $\mathbb{H}\mathbb{Z}^{k\rho+1}(C_{2+} \wedge S^{2m})$ vanish. The groups $\mathbb{H}\mathbb{Z}^{k\rho+1}(C_{2+} \wedge \mathbb{P}_2(S^{2m}))$ are just the underlying cohomology groups $\mathbb{H}\mathbb{Z}^{2k+1}(\mathbb{P}_2(S^{2m}))$ which vanish classically (group cohomology of Σ_2 with torsion-free coefficients vanishes in even degrees). So we are left with showing that $\mathbb{H}\mathbb{Z}^{k\rho+1}(\mathbb{P}_2(S^{m\rho})) = 0$. This follows from our computation of the action of the Bockstein on $\mathbb{H}\mathbb{F}_*(\mathbb{P}_2(S^{m\rho}))$ (recall that we have implicitly localized at 2). \square

Lemma 6.20. *Suppose X admits a filtration with associated graded $\text{gr}(X) = W$, a wedge of even-dimensional slice cells of dimension $\leq 2N$. Then, for any Mackey functor \underline{M} ,*

$$[F_{2j}\mathbb{P}_2(X), \Sigma^{k\rho+1}\underline{HM}] = 0,$$

whenever $k \geq j + N$.

Proof. Combining the filtration of X with the filtration of \mathbb{P}_2 , we are reduced to checking that each of the following vanish for $m \leq j + N \leq k$:

$$[S^{m\rho}, \Sigma^{k\rho+1}\underline{HM}], \quad [S^{m\rho-1}, \Sigma^{k\rho+1}\underline{HM}], \quad [C_{2+} \wedge S^{2m}, \Sigma^{k\rho+1}\underline{HM}], \quad [C_{2+} \wedge S^{2m-1}, \Sigma^{k\rho+1}\underline{HM}].$$

But we have $m - k \leq 0$ so this follows by connectivity. \square

Finally, we are ready for the main technical theorem pinning down the homology of $\text{Even}(S^0)$.

Theorem 6.21. *The map $\text{Even}(S^0) \rightarrow \mathbb{H}\mathbb{F}$ induces an isomorphism of \mathcal{A}_* -comodules:*

$$\mathbb{H}\mathbb{F}_*\text{Even}(S^0) \xrightarrow{\cong} P_*.$$

Proof. We will reduce to inductively showing that the image of $\mathbb{H}\mathbb{F}_*\text{Even}(S^0)$ contains all the $\bar{\xi}_i$. Then we'll reduce this to an Ext calculation which we perform by comparing $\underline{\text{Ext}}_{\mathcal{A}_*}(\mathbb{H}\mathbb{F}_*\text{Even}(S^0))$ to $\underline{\text{Ext}}_{\mathcal{A}_*}(P_*)$ in a range. Throughout, we will use the equivariant Adams spectral sequence for the cohomology theory $\mathbb{H}\mathbb{F}$ found in [HK01, Cor. 6.47].

Step 1. It's enough to show that the image of $\mathbb{H}\mathbb{F}_*\text{Even}(S^0)$ contains $\bar{\xi}_i$ for all $i \geq 1$.

Indeed, in this case we conclude that, in the notation of Lemma 6.18, the map $\mathbb{F}\{e_\alpha^i, f_\beta^j\} \rightarrow P_*^\circ$ is surjective. By the comodule splitting lemma, we conclude that $\mathbb{F}\{e_\alpha^i, f_\beta^j\}$ is cofree over P_*° . But then $\mathbb{H}\mathbb{F}_* \otimes_{\mathbb{F}} \mathbb{F}\{e_\alpha^i, f_\beta^j\} = \mathbb{H}\mathbb{F}_*\text{Even}(S^0)$ is of the form $P_* \otimes C$ as a left \mathcal{A}_* -comodule, where C is primitive and generated in even degrees. The Adams spectral sequence then takes the form

$$\underline{\text{Ext}}_{\mathcal{A}_*}(P_*) \otimes C \Rightarrow \underline{\pi}_*\text{Even}(S^0).$$

Since these Ext-Mackey functors vanish in degrees $V - s = n\rho - 1$, every element in homotopy dimension $n\rho$ is a permanent cycle. In particular, all of the positive degree elements of C would survive to homotopy classes which are detected by the mod 2 Hurewicz map. There are no such elements, by Lemma 6.6, so C is generated in degree 0. But there is only room for one such generator and we conclude $\mathbb{H}\mathbb{F}_*\text{Even}(S^0) = P_*$, as desired. This completes the first step.

Note that the unit map $1 : S^0 \rightarrow \text{Even}(S^0)_1$ extends over $\text{cofib}(\eta)$, by construction. The left comodule action on the top class e_ρ is:

$$e_\rho \mapsto 1 \otimes e_\rho + \bar{\xi}_1 \otimes 1.$$

So we may assume, by induction on $k \geq 1$, that

- The elements $\bar{\xi}_1, \dots, \bar{\xi}_k$ are in the image of the map $\mathbf{HF}_\star \text{Even}(S^0)_{2^k-1} \rightarrow P_\star$.

Step 2. If $\bar{\xi}_1, \dots, \bar{\xi}_k$ are in the image, then $\bar{\xi}_{k+1}$ is in the image if we can construct a dotted arrow making the following diagram commute:

$$\begin{array}{ccc} F_{2^{k+1}}(\mathbb{P}_2(\text{Even}(S^0)_{2^k-1})) & \longrightarrow & \mathbb{P}_2(\mathbf{HF}) \\ \downarrow & & \downarrow \\ \text{Even}(S^0) & \longrightarrow & \mathbf{HF} \end{array}$$

Indeed, this subcomplex of the extended power is just large enough to contain the operation $Q^{2^k\rho}$. Moreover, $\mathbf{HF}_\star \text{Even}(S^0)_{2^k-1} \rightarrow \mathbf{HF}_\star \text{Even}(S^0)$ induces an isomorphism on degree $(2^k - 1)\rho$ so there is an element $x \in \mathbf{HF}_\star \text{Even}(S^0)_{2^k-1}$ which hits $\bar{\xi}_k$. So, by commutativity of the diagram, there is an element $Q^{2^k\rho}x \in \mathbf{HF}_\star \text{Even}(S^0)$ which hits $Q^{2^k\rho}\bar{\xi}_k = \bar{\xi}_{k+1} \in P_\star$, by Theorem 5.4, which was to be shown.

Step 3. It suffices to show that $P^0\pi_{n\rho}\text{Even}(S^0)$ is constant and torsion-free for $n \leq 2^{k+1} - 2$.

We'll show that, in this case, we can build the diagram from Step 2. We construct the map

$$F_{2^{k+1}}(\mathbb{P}_2(\text{Even}(S^0)_{2^k-1})) \rightarrow \text{Even}(S^0)$$

by inducting up the slice tower of $\text{Even}(S^0)$. Since the odd slices vanish, the obstructions live in

$$[F_{2^{k+1}}(\mathbb{P}_2(\text{Even}(S^0)_{2^k-1})), \Sigma^{n\rho+1}\mathbf{HP}^0\pi_{n\rho}\text{Even}(S^0)]^{C_2}.$$

When $n \leq 2^{k+1} - 2$, we are, by hypothesis, in the realm of Lemma 6.19 and these groups vanish. When $n \geq 2^k + 2^k - 1 = 2^{k+1} - 1$ these vanish by Lemma 6.20. The homotopy commutativity of the diagram in Step 2 is a statement about classes in \mathbf{HF}^0 . These are detected on the bottom cell, and the result follows since $S^0 \rightarrow \text{Even}(S^0) \rightarrow \mathbf{HF}$ is the unit.

Step 4. It suffices to show that

- (i) $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0)) = 0$ for $V - s = (2^{k+1} - 2)\rho - 1$,
- (ii) $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0))$ has surjective restriction map for $V - s = (2^{k+1} - 2)\rho$,
- (iii) the *underlying* group of $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0))$ is v_0 -torsion-free when $V - s = (2^{k+1} - 2)\rho$,
- (iv) the *underlying* group of $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0))$ is zero for $V - s = (2^{k+1} - 2)\rho + 1$.

By Step 3, we need to know that $P^0\pi_{n\rho}\text{Even}(S^0)$ is constant and torsion-free in some range. This is equivalent to checking that the underlying groups of $\pi_{n\rho}$ are torsion-free and the restriction map is surjective. Using the fact that v_0 corresponds to multiplication by 2, we can check this property on the E_∞ -page of the Adams spectral sequence. The conditions listed above ensure that the E_2 -page is equal to the E_∞ -page in homotopy dimension $n\rho$, modulo a_σ , for n in the requisite range. This is all we need, since a_σ vanishes under the restriction map.

Step 5

- (a) $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0)) = \underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(P_\star)$ for $V - s = (2^{k+1} - 2)\rho - 1$,
- (b) modulo a_σ , we have $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0)) = \underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(P_\star)$ when $V - s = (2^{k+1} - 2)\rho$,
- (c) the *underlying* groups $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(\mathbf{HF}_\star \text{Even}(S^0))$ agree with the *underlying* groups of $\underline{\text{Ext}}_{\mathcal{A}_\star}^{s,V}(P_\star)$ for $V - s = (2^{k+1} - 2)\rho$ and $V - s = (2^{k+1} - 2)\rho + 1$.

First, by our induction hypothesis, Lemma 6.14 implies that $\mathbb{H}\mathbb{F}_\star \text{Even}(S^0)$ is of the form $P_\star \otimes C$ in a range, where C is primitive on even generators. Then Lemma 6.16 implies that, in our range, the Ext groups in homotopy dimension $n\rho - 1$ vanish. It follows that elements of C in positive degree and in our range must survive to homotopy classes, and the same argument as in Step 1 shows that this can't happen. So $\mathbb{H}\mathbb{F}_\star \text{Even}(S^0)$ and P_\star are isomorphic as comodules modulo terms of high degree. So, again by Lemma 6.16 and Proposition 6.12 the result follows. \square

Remark 6.22. I learned from Paul Goerss that it is possible to modify this induction to give a construction of **BPR** without using power operations. The idea is to put a bit more control over the attaching maps in Priddy's construction, and keep track of the comodule structure and homotopy as you go along using Lemma 6.16 and Proposition 6.12. The present approach has the advantage of efficiency, and we simultaneously construct and compute the action of power operations on **BPR**, which is useful in proving splitting theorems.

Theorem 6.23. *Without invoking the existence of **BPR**, it is possible to establish the following properties of $\text{Even}(S^0)$ from its construction:*

- (i) *The spectrum underlying $\text{Even}(S^0)$ is **BP**,*
- (ii) *the geometric fixed points of $\text{Even}(S^0)$ are equivalent to $\mathbb{H}\mathbb{F}_2$,*
- (iii) *the homotopy Mackey functors $\pi_{*\rho} \text{Even}(S^0)$ are constant and given by $\mathbb{Z}_{(2)}[v_1, v_2, \dots]$ where $|v_i| = (2^i - 1)\rho$, and ρ is the regular representation of C_2 .*
- (iv) *the homotopy Mackey functors $\pi_{*\rho-1} \text{Even}(S^0)$ vanish.*

These last two properties determine the slice tower of $\text{Even}(S^0)$.

Proof. Part (iii) follows from Hu-Kriz's equivariant version of the Adams spectral sequence together with Proposition 6.12.

For parts (i) and (ii), standard splitting theorems apply to show that $\Phi^{C_2} \text{Even}(S^0)$ splits as a wedge of suspensions of $\mathbb{H}\mathbb{F}$ and $\Phi^e \text{Even}(S^0)$ splits as a wedge of suspensions of **BP**. But there is only room for one copy in each case.

Part (iv) holds by construction, and the last statement follows from Theorem 6.3. \square

At long last, we invoke the existence of **BPR** and its known properties to deduce a final sanity check: we have produced the correct spectrum, in the end.

Theorem 6.24. $\text{Even}(S^0) \cong \text{BPR}$.

Proof. A map extending the unit exists by obstruction theory applied to the known description of the slice tower of **BPR**. To see this is an equivalence, one can check on underlying spectra and geometric fixed points. Both **BP** and $\mathbb{H}\mathbb{F}$ have the property that any endomorphism inducing an isomorphism on π_0 is an equivalence. Indeed, both spectra are connective and have cohomology which is generated by the unit over the Steenrod algebra. The result follows.

Alternatively, one can check that the map induces an isomorphism of homology, hence Adams E_2 -terms, and hence an equivalence on $\pi_{*\rho}$. Since $\pi_{*\rho-1} = 0$ for both spectra, we can apply an $RO(C_2)$ -graded version of the Whitehead theorem due to Hill-Meier [HM15, Lem 2.4]. \square

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