A DESCENT VIEW ON MITCHELL'S THEOREM

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ABSTRACT. In this short note, we give a new proof of Mitchell's theorem that $L_{T(n)}K(\mathbf{Z}) \simeq 0$ for $n \ge 2$. Instead of reducing the problem to delicate representation theory, we use recently established hyperdescent technology for chromatically-localized algebraic K-theory.

1. INTRODUCTION AND BACKGROUND

In this note, we give an alternate proof of the following result:

Theorem 1.0.1 (Mitchell). For all primes p and $n \ge 2$, $K(n)_*K(\mathbf{Z}) = 0$.

The proof Theorem 1.0.1 given in [Mit87] is relatively self-contained and depends on showing that the unit map $\mathbf{1} \to K(\mathbf{Z})$ factors through the "image of J." This factoring relies on delicate representation theory by way of the Barratt-Priddy-Quillen theorem. Since the latter spectrum is known to be acyclic for the Morava's K(n) for $n \ge 2$, the result follows. The value of the present note is that it locates the proof in its natural environment — Rognes' redshift philosophy in algebraic K-theory.

The starting point of Theorem 1.0.1 is Thomason's seminal result that K(1)-localized algebraic K-theory satisfies étale descent [Tho85]. Combined with the rigidity theorems of Suslin [Sus83] and Gabber [Gab92], one concludes that K(1)-local algebraic K-theory is, more or less, topological K-theory; we also refer the reader to [DFST82] for further elaboration on this point of view.

One can view Thomason's theorem as a "Bott-asymptotic" version of a more refined statement — the Quillen-Lichtenbaum conjecture (now the Voevodsky-Rost theorem [Voe03, Voe11]) which asserts that algebraic and étale K-theory agrees in high enough degrees. By analogy with the Quillen-Lichtenbaum conjectures, Rognes has formulated the idea that taking algebraic Ktheory increases "chromatic complexity" — demonstrating a "redshift"; we refer the reader to [Rog14] for a discussion.

In line with this ideology, Thomason's theorem is then viewed as saying that taking algebraic K-theory of a discrete commutative ring (which is K(1)-acyclic) yields an interesting answer K(1)-locally. At the next height, results of Ausoni-Rognes [Aus10, AR02] who confirmed that v_2 acts invertibly on K(K(\mathbf{C})^{\wedge}_p) \otimes V(1) where V(1) is a type 2 complex, hence is interesting K(2)-locally.

We can view Mitchell's result anachronistically as a demonstration of the strictness of redshift: while the 2-fold algebraic K-theory of a discrete ring is interesting K(2)-locally, the algebraic K-theory thereof itself is not.

The value of our proof, if there is one, is that it is born in the same spirit as Thomason's results: we confirm Mitchell's vanishing by way of étale hyperdescent. We believe that proving the result this way places it within its proper context, at the cost of using more technology.

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2. On Mitchell's Theorem

2.1. A *p*-adic version of Mitchell's theorem. As a warm-up, we first give a very simple proof of the *p*-adic version of Mitchell's theorem. So fix a prime p > 0; here our T(n)-localizations will be at this implicit prime.

Theorem 2.1.1. For $n \ge 2$, we have that $L_{T(n)}K(\mathbf{Z}_p) \simeq 0$.

To begin, let C be the completion of the algebraic closure of \mathbf{Q}_p and \mathcal{O}_C be its ring of integers which is an integral perfectoid ring.

Lemma 2.1.2. For $n \ge 2$, we have that $L_{T(n)}K(\mathcal{O}_C) \simeq 0$.

Proof. Consider the zig-zag of maps

$$ku \to K(C) \xleftarrow{j^*} K(\mathcal{O}_C)$$

The maps above are all *p*-adic equivalences:

- (1) for j^* this is [HN19, Lemma 1.3.7],
- (2) for the unlabeled arrow, we have Suslin rigidity [Sus84].

Hence we conclude that $L_{T(n)}K(\mathcal{O}_C) \simeq 0$.

Remark 2.1.3. Note that [HN19, Lemma 1.3.7] only uses very basic facts about perfectoid rings (that we can choose a pseudouniformizer π such that $\mathcal{O}_{\mathrm{C}}/\pi \simeq \mathcal{O}_{\mathrm{C}^{\flat}}/\pi^{\flat}$) and the fact that the positive homotopy groups of the K-theory of local perfect \mathbf{F}_p -algebras are all *p*-divisible by [Hil81, Kra80].

Proof of Theorem 2.1.1. Since K-theory is a finitary invariant¹, we can write

$$\mathrm{K}\left(\mathcal{O}_{\mathrm{C}};\mathbf{Z}_{p}\right)\simeq\operatorname{colim}_{\mathbf{Q}_{p}\subset\mathrm{E}\subset\mathrm{C}}\mathrm{K}\left(\mathcal{O}_{\mathrm{E}};\mathbf{Z}_{p}\right),$$

a colimit of \mathbf{E}_{∞} -rings, and the colimit ranges along finite extensions of \mathbf{Q}_p contained in C. Now, the source vanishes after applying $\mathcal{L}_{\mathcal{T}(n)}$, whence so is the target. Since $\mathcal{L}_{\mathcal{T}(n)}$ commutes with filtered colimits, we may find a finite extension E of \mathbf{Q}_p such that $\mathcal{L}_{\mathcal{T}(n)}\mathcal{K}(\mathcal{O}_{\mathrm{E}}) \simeq 0$ since the colimit in sight is computed in \mathbf{E}_{∞} -ring; indeed, vanishing is equivalent to the unit being nullhomotopic. Since the morphism of rings $\mathbf{Z}_p \to \mathcal{O}_{\mathrm{E}}$ is finite flat, by the descent results of [CMNN20] we get that

$$L_{T(n)}K(\mathbf{Z}_p) \simeq Tot\left(L_{T(n)}K\left(\mathcal{O}_E^{\otimes \mathbf{Z}_p \bullet + 1}\right)\right).$$

We are now done, since all terms of the limit on the right hand side are modules over $L_{T(n)}K(\mathcal{O}_E) = 0$.

To conclude note that if A is T(n)-acyclic, then it is also K(n)-acyclic. Though we will not need it, we can also reverse the implication if A is, furthermore, an \mathbf{E}_{∞} -ring spectrum by [LMT20, Lemma 2.3].

Corollary 2.1.4. If $n \ge 2$, we have that $L_{K(n)}K(\mathbf{Z}_p) \simeq 0$. In particular $K(n)_*K(\mathbf{Z}_p) = 0$.

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¹In more details: \mathcal{O}_{C} is *p*-adically isomorphic to the colimit of the \mathcal{O}_{E} 's and K-theory preserves *p*-adic equivalences in this setting. This follows from, for example, the fact that given a morphism of rings $A \rightarrow B$, fib(GL(A) \rightarrow GL(B)) \simeq fib(M(A) \rightarrow M(B)) and the formation of matrix rings evidently preserves local equivalences.

2.2. Mitchell's theorem. We now give a proof of Mitchell's theorem. We phrase this as.

Theorem 2.2.1. For all primes p and $n \ge 2$, $L_{T(n)}K(\mathbf{Z}) \simeq 0$. Equivalently, $L_{K(n)}K(\mathbf{Z}) \simeq 0$ and, in particular, $K(n)_*K(\mathbf{Z}) = 0$.

Proof. The "equivalently" part follows from [LMT20, Lemma 2.3]. We first claim that that we can work with rings which are (p)-local. To see this, we claim that the map

$$L_{T(n)}K(\mathbf{Z}) \to L_{T(n)}K\left(\mathbf{Z}\left[\frac{1}{p}\right]\right),$$

is an equivalence. Indeed, by localization and dévissage [Qui10] we have a cofiber sequence

$$\mathbf{K}(\mathbf{F}_p) \to \mathbf{K}(\mathbf{Z}) \to \mathbf{K}\left(\mathbf{Z}\left[\frac{1}{p}\right]\right),$$

and $L_{T(n)}K(\mathbf{F}_p) \simeq L_{T(n)}H\mathbf{Z}_p \simeq 0$ where the first equivalence is [Qui72] and the second equivalence follows since $n \ge 2$. Therefore, it suffices to show that $L_{T(n)}K\left(\mathbf{Z}\left[\frac{1}{p}\right]\right) \simeq 0$ which follows from the next two claims.

Claim 2.2.2. For any $n \ge 1$, the presheaf,

$$L_{T(n)}K(-): Et^{op}_{\mathbf{Z}\left[\frac{1}{p}\right]} \to Spt$$

is a hypercomplete sheaf of spectra.

Proof. While this is an immediate application of [CM19, Theorem 7.14], we will give a more detailed proof here to highlight the ingredients. Since telescopic localization is invariant under taking connective covers ([LMT20, Lemma 2.3(iii)]) we obtain a map of presheaves of \mathbf{E}_{∞} -rings:

$$\mathbf{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_{p}\right) \rightarrow \mathbf{L}_{\mathrm{T}(n)}\mathbf{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_{p}\right) \cong \mathbf{L}_{\mathrm{T}(n)}\mathbf{K}\left(-\right)$$

Moreover $L_{T(n)}K(-)$ is an étale sheaf, thus this map factors through a canonical \mathbf{E}_{∞} -map

$$\mathrm{K}^{\mathrm{cn}}\left(-;\mathbf{Z}_{p}\right)^{\mathrm{\acute{e}t}} \to \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}\left(-\right)$$

Since hypercompletion is smashing by [CM19, Corollary 4.39], it suffices to then prove that $K^{cn}(-; \mathbf{Z}_p)^{\text{ét}}$ is a hypersheaf on $\text{Et}_{\mathbf{Z}[1]}$.

As is proved by Thomason in [Tho85, Theorem 4.1] for odd primes (which relies on the Suslin-Merkerjuev theorem [MS82]) and Rosenschon and Østvær [ROsr05] for the prime 2 (which does rely on the Milnor conjecture [Voe03]), $L_{T(1)}K$ does satisfy étale hyperdescent. We consider the canonical map $K^{cn}(-; \mathbf{Z}_p)^{\text{ét}} \rightarrow L_{T(1)}K$ and \mathcal{F} be the fiber. The claim then follows if one can show that the fiber \mathcal{F} has étale hyperdescent.

Let \mathcal{G} denote the fiber of the map $\mathrm{K}^{\mathrm{cn}}(-; \mathbf{Z}_p) \to \mathrm{L}_{\mathrm{T}(1)}\mathrm{K}$. By the full strength of the Bloch-Kato conjectures [Voe11, Voe03] (see [CM19, Theorem 6.18], noting that Spec \mathbf{Z} admits the desired global bound by [Ser02, I.3.2]) we see that the fiber is truncated. Therefore the sheaffication, $\mathcal{F} \simeq \mathcal{G}^{\mathrm{\acute{e}t}}$ is Postnikov complete, whence is indeed hypercomplete as desired.

Claim 2.2.3. For $n \ge 2$ and for all strictly hensel local ring R with residue field κ of characteristic $\ell > 0$ and $(p, \ell) = 1$

$$L_{T(n)}K(R) \simeq 0$$

Indeed, since strictly henselian local rings are the points in the étale topology and vanishing of étale hypersheaves are detected on points [Lur09, Remark 6.5.4.7], the claim implies our result.

Proof of Claim 2.2.3. By Gabber rigidity [Gab92], the map $K^{cn}(R) \to K^{cn}(\kappa)$ is a *p*-adic equivalence, while by Suslin rigidity [Sus84] we have a further *p*-adic equivalence $K^{cn}(\kappa) \leftarrow ku$. Since telescopic localization for $n \ge 1$ is invariant under taking connective covers by [LMT20, Lemma 2.3(iii)] we conclude:

$$\mathcal{L}_{\mathcal{T}(n)}\mathcal{K}\left(\mathcal{R}\right)\simeq\mathcal{L}_{\mathcal{T}(n)}\mathcal{K}^{\mathrm{cn}}\left(\mathcal{R}\right)\simeq\mathcal{L}_{\mathcal{T}(n)}\mathbf{k}\mathbf{u}\simeq0.$$

Remark 2.2.4. Because of the appeal to [CM19, Theorem 7.14], our proof is not "simple", in contrast to the *p*-adic situation. This is because the Clausen-Mathew theorem depends on the resolution of the Quillen-Lichtenbaum conjectures. Specifically, the version of [CM19, Theorem 7.14] that we need, requires [CM19, Theorem 6.13] which ultimately proves that the restriction of K^{ét} to $\text{Et}_{\mathbf{Z}[1/p]}$ is a étale hypersheaf. This latter result, in turn, relies on Rost-Voevodsky's resolution of the Bloch-Kato conjectures. Note that, in contrast to the *p*-adic situation, Theorem 2.2.1 asserts something global which prompts us to argue via stalks, whence an appeal to hyperdescent.

Remark 2.2.5 (Personal communication by J. Rognes). Morally speaking, our proof is a cleaner packaging of the following method to prove Mitchell's theorem using the Rost-Voevodsky results. For simplicity let p be an odd prime and let $V(1) := 1/(p, v_1)$ be a finite complex which admits a v_2 -self map. Our goal is to prove that $K(\mathbf{Z}) \otimes V(1)$ is a bounded complex which suffices to prove Mitchell's theorem since inverting a positive degree self-map on a bounded complex annihilates it. Using the localization sequence, we reduce to proving the following assertions:

- (1) if $\ell \neq p$, then v_1 acts invertibly on $K_*(\mathbf{F}_\ell)/p$ for * large enough,
- (2) v_1 acts invertibly on $K_*(\mathbf{Q})/p$ for * large enough, and
- (3) if $\ell = p$, then $K_*(\mathbf{F}_p)/p$ is bounded above.

The last statement follows from Quillen's computation [Qui72] and the first statement follows by a further application of Suslin rigidity [Sus83]. The second statements is where Rost-Voevodsky's resolution of the Bloch-Kato conjectures is needed [Voe03, Voe11] (this is where the "overkill happens"), the motivic spectral sequence [FS02] and the fact that v_1 is detected by a periodicity operator on étale cohomology.

As a consequence of the main theorem we can easily obtain two corollaries:

Corollary 2.2.6. The functor $L_{T(n)}K|_{Cat_{\mathbf{z}}^{Perf}}$ is zero for $n \ge 2$.

Corollary 2.2.7. For $n \ge 2$, $L_{T(n)}TC(\mathbf{Z}) \simeq 0$. Consequently, $L_{T(n)}TC|_{Cat_{\mathbf{Z}}^{perf}}$ is zero for $n \ge 2$.

Proof. Via the trace map, $L_{T(n)}TC(\mathbf{Z})$ is an $L_{T(n)}K(\mathbf{Z})$ - \mathbf{E}_{∞} -algebra, whence the result follows from Theorem 2.2.1.

References

- [AR02] C. Ausoni and J. Rognes, Algebraic K-theory of topological K-theory, Acta Math. 188 (2002), no. 1, pp. 1–39, https://doi.org/10.1007/BF02392794
- [Aus10] C. Ausoni, On the algebraic K-theory of the complex K-theory spectrum, Invent. Math. 180 (2010), no. 3, pp. 611–668, https://doi.org/10.1007/s00222-010-0239-x
- [CM19] D. Clausen and A. Mathew, Hyperdescent and étale K-theory, 2019, arXiv:1905.06611
- [CMNN20] D. Clausen, A. Mathew, N. Naumann, and J. Noel, Descent in algebraic K-theory and a conjecture of Ausoni-Rognes, J. Eur. Math. Soc. (JEMS) 22 (2020), no. 4, pp. 1149–1200, https://doi-org.ezp-prod1.hul.harvard.edu/10.4171/JEMS/942
- [DFST82] W. G. Dwyer, E. M. Friedlander, V. P. Snaith, and R. W. Thomason, Algebraic K-theory eventually surjects onto topological K-theory, Invent. Math. 66 (1982), no. 3, pp. 481-491, http://dx.doi.org/10.1007/BF01389225
- [FS02] E. M. Friedlander and A. Suslin, The spectral sequence relating algebraic K-theory to motivic cohomology, Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 6, pp. 773-875, https://doi.org/10.1016/S0012-9593(02)01109-6

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[Gab92]	O. Gabber, K-theory of Henselian local rings and Henselian pairs, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989),
	Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 59–70,
[11:10:1]	https://doi-org.ezp-prod1.hul.harvard.edu/10.1090/conm/126/00509
[Hil81] [HN19]	H. L. Hiller, λ -rings and algebraic K-theory, J. Pure Appl. Algebra 20 (1981), no. 3, pp. 241–266,
	https://doi.org/10.1016/0022-4049(81)90062-1
	L. Hesselholt and T. Nikolaus, <i>Topological cyclic homology</i> , Handbook of homotopy theory (Haynes Miller, ed.), CRC Press/Chapman and Hall, 2019
[Kra80] [LMT20]	Kiner, ed.), CAC Press/Chapman and Han, 2019 C. Kratzer, λ -structure en K-théorie algébrique, Comment. Math. Helv. 55 (1980), no. 2, pp. 233–
	254, https://doi.org/10.1007/BF02566684
	M. Land, L. Meier, and G. Tamme, Vanishing results for chromatic localizations of algebraic K-
[LIM120]	theory, 2020, arXiv:2001.10425
[Lur09]	J. Lurie, <i>Higher Topos Theory</i> , Annals of Mathematics Studies, vol. 170, Princeton University Press,
	Princeton, NJ, 2009
[Mit87]	S. A. Mitchell, The Morava K-theory of algebraic K-theory spectra, K-Theory 1 (1987), no. 2,
	pp. 197-205, https://doi.org/10.1007/BF00533419
[MS82]	A. S. Merkurjev and A. A. Suslin, K-cohomology of Severi-Brauer varieties and the norm residue
	homomorphism, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 5, pp. 1011–1046, 1135–1136
[Qui72]	D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Ann. of
	Math. (2) 96 (1972), pp. 552–586, https://doi.org/10.2307/1970825
[Qui10]	, Higher algebraic K-theory: I, pp. 413–478
[Rog14]	J. Rognes, Chromatic redshift, 2014, arXiv:1403.4838
[ROsr05]	A. Rosenschon and P. A. Ø stvær, The homotopy limit problem for two-primary algebraic K-theory,
	Topology 44 (2005), no. 6, pp. 1159–1179, https://doi.org/10.1016/j.top.2005.04.004
[Ser02]	JP. Serre, Galois cohomology, Springer-Verlag, 2002
[Sus83]	A. Suslin, On the K-theory of algebraically closed fields, Invent. Math. 73 (1983), no. 2, pp. 241-245,
	https://doi-org.ezp-prod1.hul.harvard.edu/10.1007/BF01394024
[Sus84]	A. A. Suslin, On the K-theory of local fields, Proceedings of the Luminy con-
	ference on algebraic K-theory (Luminy, 1983), vol. 34, 1984, pp. 301-318,
	https://doi-org.ezp-prod1.hul.harvard.edu/10.1016/0022-4049(84)90043-4
[Tho 85]	R. W. Thomason, Algebraic K-theory and étale cohomology, Ann. Sci. École Norm. Sup. (4) 18
	(1985), no. 3, pp. 437-552, http://www.numdam.org/item?id=ASENS_1985_4_18_3_437_0
[Voe03]	V. Voevodsky, Motivic cohomology with Z/2-coefficients, Publ. Math. I.H.É.S. 98 (2003), no. 1,
	pp. 59–104
[Voe11]	, On motivic cohomology with \mathbf{Z}/l -coefficients, Ann. Math. 174 (2011), no. 1, pp. 401–438,
	preprint arXiv:0805.4430
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