Decomposition spaces, incidence algebras and Möbius inversion III: The decomposition space of Möbius intervals ☆

Imma Gálvez-Carrillo [a], Joachim Kock [b], Andrew Tonks [c],*

[a] *Departament de Matemàtiques, Universitat Politècnica de Catalunya*
[b] *Departament de Matemàtiques, Universitat Autònoma de Barcelona*
[c] *Department of Mathematics, University of Leicester*

A B S T R A C T

Decomposition spaces are simplicial ∞-groupoids subject to a certain exactness condition, needed to induce a coalgebra structure on the space of arrows. Conservative ULF functors (CULF) between decomposition spaces induce coalgebra homomorphisms. Suitable added finiteness conditions define the notion of Möbius decomposition space, a far-reaching generalisation of the notion of Möbius category of Leroux. In this paper, we show that the Lawvere–Menni Hopf algebra of Möbius intervals, which contains the universal Möbius function (but is not induced by a Möbius category), can be realised as the homotopy cardinality of a Möbius decomposition space U of all Möbius intervals, and that in a certain sense U is universal for Möbius decomposition spaces and CULF functors.

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Contents

Introduction

This paper is the third of a trilogy dedicated to the study of decomposition spaces and their incidence algebras.

In [8] we introduced the notion of decomposition space as a general framework for incidence algebras and Möbius inversion. (Independently, Dyckerhoff and Kapranov [5], motivated by geometry, representation theory and homological algebra, had discovered the same notion, but formulated quite differently.) A decomposition space is a simplicial ∞-groupoid X satisfying a certain exactness condition, weaker than the Segal condition. Just as the Segal condition expresses composition, the new condition expresses decomposition, and there is an abundance of examples from combinatorics. It is just the condition needed for a canonical coalgebra structure to be induced on the slice ∞-category over X_1. The comultiplication is given by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1,$$

which can be interpreted as saying that comultiplying an edge $f \in X_1$ returns the sum of all pairs of edges (a, b) that are the short edges of a triangle with long edge f. If X is the nerve of a category, so f is an arrow, then the (a, b) are all pairs of arrows such that $b \circ a = f$.

In [9] we arrived at the notion of Möbius decomposition space, a far-reaching generalisation of the notion of Möbius category of Leroux [23], by imposing suitable finiteness conditions on decomposition spaces. These notions will be recalled below.

The present paper introduces the Möbius decomposition space of Möbius intervals, subsuming discoveries made by Lawvere in the 1980s, and establish that it is in a precise sense a universal Möbius decomposition space.

After Rota [26] and his collaborators [15] had demonstrated the great utility of incidence algebras and Möbius inversion in locally finite posets, and Cartier and Foata [3] had developed a similar theory for monoids with the finite-decomposition property, it was Leroux who found the common generalisation, that of Möbius categories [23]. These are categories with two finiteness conditions imposed: the first ensures that an incidence

coalgebra exists; the second ensures a general Möbius inversion formula. Conservative ULF functors (CULF) induce coalgebra homomorphisms [4], [8].

Lawvere (in 1988, unpublished until Lawvere–Menni [22]) observed that there is a universal coalgebra $\mathcal{H}$ (in fact a Hopf algebra) spanned by isomorphism classes of Möbius intervals. From any incidence coalgebra of a Möbius category there is a canonical coalgebra homomorphism to $\mathcal{H}$, and the Möbius inversion formula in the former is induced from a master inversion formula in $\mathcal{H}$.

Here is the idea: a *Möbius interval* is a Möbius category with an initial object 0 and terminal object 1 (not necessarily distinct). The category of factorisations of any arrow a in a Möbius category $\mathbb{C}$ determines ([21]) a Möbius interval $I(a)$ with 0 given by the factorisation id-followed-by-a, and 1 by the factorisation a-followed-by-id. There is a canonical CULF functor $I(a) \to \mathbb{C}$ sending $0 \to 1$ to a, and since the arrow $0 \to 1$ in $I(a)$ has the same decomposition structure as the arrow a in $\mathbb{C}$, the comultiplication of a can be calculated in $I(a)$.

Any collection of Möbius intervals that is closed under subintervals defines a coalgebra, and it is an interesting integrability condition for such a collection to come from a single Möbius category. The Lawvere–Menni coalgebra is simply the collection of *all* isomorphism classes of Möbius intervals.

Now, the coalgebra of Möbius intervals cannot be the coalgebra of a single Segal space, because such a Segal space U would have to have U_1 the space of all Möbius intervals, and U_2 the space of all subdivided Möbius intervals. But a Möbius interval with a subdivision (i.e. a 'midpoint') contains more information than the two parts of the subdivision: one from 0 to the midpoint, and one from the midpoint to 1:
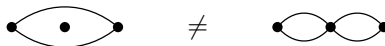
This is to say that the Segal condition is not satisfied: we have

$$U_2 \neq U_1 \times_{U_0} U_1.$$

We shall prove that the simplicial space of all intervals and their subdivisions *is* a decomposition space, as suggested by this figure:
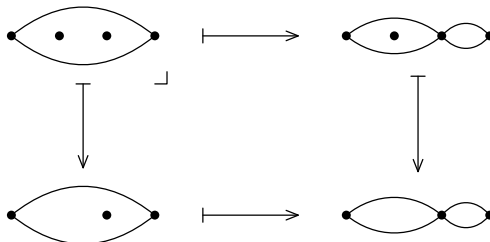
meant to indicate that this diagram is a pullback:

$$
\begin{array}{ccc}
U_3 & \xrightarrow{\;(d_3,d_0d_0)\;} & U_2 \times_{U_0} U_1 \\
\; {\scriptstyle d_1}\downarrow & {\scriptstyle \lrcorner} & \downarrow {\scriptstyle d_1\times\mathrm{id}} \\
U_2 & \xrightarrow[\;(d_2,d_0)\;]{} & U_1 \times_{U_0} U_1
\end{array}
$$

which in turn is one of the conditions involved in the decomposition-space axiom.

While the ideas outlined have a clear intuitive content, a considerable amount of machinery is needed actually to construct the universal decomposition space, and to get sufficient hold of its structural properties to prove the desired results about it. We first work out the theory without finiteness conditions, which we impose at the end.

Let us outline our results in more detail.

First of all we need to develop a theory of intervals in the framework of decomposition spaces. Lawvere's idea [21] is that to an arrow one may associate its category of factorisations, which is an interval. To set this up, we exploit factorisation systems and adjunctions derived from them, and start out in Section 1 with some general results about factorisation systems, some results of which are already available in Lurie's book [24]. Specifically we describe a situation in which a factorisation system lifts across an adjunction to produce a new factorisation system, and hence a new adjunction.

Before coming to intervals in Section 3, we need flanked decomposition spaces (Section 2): these are certain presheaves on the category Ξ of nonempty finite linear orders with a top and a bottom element. The ∞-category of flanked decomposition spaces features the important *stretched-cartesian* factorisation system, where 'stretched' is to be thought of as endpoint-preserving, and cartesian is like 'distance-preserving'. There is also the basic adjunction between decomposition spaces and flanked decomposition spaces, which in fact is the double decalage construction (this is interesting since decalage already plays an important part in the theory of decomposition spaces [8]). Intervals are first defined as certain flanked decomposition spaces which are contractible in degree -1 (this condition encodes an initial and a terminal object) (3.4), and via the basic adjunction we obtain the definitive ∞-category of intervals as a full subcategory of the ∞-category of complete decomposition spaces (4.1); it features the stretched-CULF factorisation system (4.2), which extends the active-inert (a.k.a. generic-free) factorisation system on Δ (4.3). The factorisation-interval construction can now finally be described (Theorem 5.1) as a coreflection from complete decomposition spaces to intervals (or more precisely, on certain coslice ∞-categories). We show that every interval is a Segal space (2.18). The simplicial space U of intervals (which lives in a bigger universe) can finally (4.5) be defined very formally in terms of a natural right fibration over Δ whose total space has objects stretched interval maps from an ordinal. In plain words, U consists of subdivided intervals.

With these various preliminary technical constructions having taken up two thirds of the paper, we can finally state and prove the main results:

Theorem 4.8. *U is a complete decomposition space.*

The factorisation-interval construction yields, for every decomposition space X, a canonical functor $X \to U$, called the *classifying map*.

Theorem 5.2. *The classifying map is CULF.*

We conjecture that U is universal for complete decomposition spaces and CULF maps, and prove the following partial result:

Theorem 5.5. *For each complete decomposition space X, the space $\mathrm{Map}_{\mathbf{cDcmp}^{\mathrm{culf}}}(X, U)$ is connected.*

We finish in Section 6 by imposing the Möbius condition, obtaining the corresponding finite results. A *Möbius interval* is an interval which is Möbius as a decomposition space. We show that every Möbius interval is a Rezk complete Segal space (6.6). There is a decomposition space of *all* Möbius intervals, and it is shown to be small.

Our final theorem is now:

Theorem 6.14. *The decomposition space of all Möbius intervals is Möbius.*

It follows that it admits a Möbius inversion formula with coefficients in finite ∞-groupoids or in $\mathbb{Q}$, and since every Möbius decomposition space admits a canonical CULF functor to it, we find that Möbius inversion in every incidence algebra (of a Möbius decomposition space) is induced from this master formula.

Note. This work was originally Section 7 of a large single manuscript *Decomposition spaces, incidence algebras and Möbius inversion* [6]. For publication, this manuscript has been split into six papers:

(0) Homotopy linear algebra [7]
(1) Decomposition spaces, incidence algebras and Möbius inversion I: basic theory [8]
(2) Decomposition spaces, incidence algebras and Möbius inversion II: completeness, length filtration, and finiteness [9]
(3) Decomposition spaces, incidence algebras and Möbius inversion III: the decomposition space of Möbius intervals [present paper]
(4) Decomposition spaces and restriction species [10]
(5) Decomposition spaces in combinatorics [11]

0. Decomposition spaces

We briefly recall from [8] the notions of decomposition space and CULF functors, and a few key results needed.

0.1. The setting: ∞-categories. We work in the ∞-category of ∞-categories, and refer to Lurie's *Higher Topos Theory* [24] for background. Thanks to the monumental effort of Joyal [18], [19] and Lurie [24], it is now possible to work model-independently, at least as long as the category theory involved is not too sophisticated. This is the case in the present work, where most of the constructions are combinatorial, dealing as they do with various configurations of ∞-groupoids, and it is feasible to read most of the paper substituting the word set for the word ∞-groupoid. In fact, even at that level of generality, the results are new and interesting.

Working model-independently has a slightly different flavour than many of the arguments in the works of Joyal and Lurie, who, in order to bootstrap the theory and establish all the theorems we now harness, had to work in the category of simplicial sets with the Joyal model structure. For example, throughout when we refer to a slice ∞-category $\mathcal{C}_{/X}$ (for X an object of an ∞-category $\mathcal{C}$), we only refer to an ∞-category determined up to equivalence of ∞-categories by a certain universal property (Joyal's insight of defining slice categories as adjoint to a join operation [18]). In the Joyal model structure for quasi-categories, this category can be represented by an explicit simplicial set. However, there is more than one possibility, depending on which explicit version of the join operator is employed (and of course these are canonically equivalent). In the works of Joyal and Lurie, these different versions are distinguished, and each has some technical advantages. In the present work we shall only need properties that hold for both, and we shall not distinguish between them.

0.2. Linear algebra with coefficients in ∞-groupoids [7]. Let $\mathcal{S}$ denote the ∞-category of ∞-groupoids. The slice ∞-categories $\mathcal{S}_{/I}$ form the objects of a symmetric monoidal ∞-category ***LIN***, described in detail in [7]: the morphisms are the linear functors, meaning that they preserve homotopy sums, or equivalently indeed all colimits. Such functors are given by spans: the span

$$I \xleftarrow{p} M \xrightarrow{q} J$$

defines the linear functor

$$q_! \circ p^* : \mathcal{S}_{/I} \longrightarrow \mathcal{S}_{/J}$$

given by pullback along p followed by composition with q. The ∞-category ***LIN*** can play the role of the category of vector spaces, although to be strict about that interpretation, finiteness conditions should be imposed, as we do later in this paper (Section 6). The symmetric monoidal structure on ***LIN*** is given on objects by

$$\mathcal{S}_{/I} \otimes \mathcal{S}_{/J} = \mathcal{S}_{I \times J},$$

just as the tensor product of vector spaces with bases indexed by sets I and J is the vector space with basis indexed by $I \times J$. The neutral object is $\mathcal{S}$.

0.3. Active and inert maps (generic and free maps). The category $\mathbb{\Delta}$ of nonempty finite ordinals and monotone maps has an active-inert factorisation system: an arrow $a : [m] \to [n]$ in $\mathbb{\Delta}$ is *active* (also called *generic*) when it preserves end-points, $a(0) = 0$ and $a(m) = n$; and it is *inert* (also called *free*) if it is distance preserving, $a(i+1) = a(i)+1$ for $0 \leq i \leq m - 1$. The active maps are generated by the codegeneracy maps and the inner coface maps, while the inert maps are generated by the outer coface maps. Every morphism in $\mathbb{\Delta}$ factors uniquely as an active map followed by an inert map.

(The notions of generic and free maps are general notions in category theory, introduced by Weber [28,29], who extracted the notions from earlier work of Joyal [16]; a recommended entry point to the theory is Berger–Melliès–Weber [1]. The more recent terminology 'active/inert' is due to Lurie [25], and is more suggestive for the role the two classes of maps play.)

Lemma 0.4. *Active and inert maps in $\mathbb{\Delta}$ admit pushouts along each other, and the resulting maps are again active and inert.*

0.5. Decomposition spaces [8]. A simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is called a *decomposition space* when it takes active-inert pushouts in $\mathbb{\Delta}$ to pullbacks.

Every Segal space is a decomposition space. The main construction in the present paper, the decomposition space of intervals, is an example which is not a Segal space.

The notion of decomposition space can be seen as an abstraction of coalgebra: it is precisely the condition required to obtain a counital coassociative comultiplication on $\mathcal{S}_{/X_1}$. The following is the main theorem of [8].

Theorem 0.6. *[8] For X a decomposition space, the slice ∞-category $\mathcal{S}_{/X_1}$ has the structure of a strong homotopy comonoid in the symmetric monoidal ∞-category **LIN**, with the comultiplication defined by the span*

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

If X is the nerve of a locally finite category (for example a poset), then (the cardinality of) this comultiplication is that of the classical incidence coalgebra,

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b.$$

0.7. CULF maps. For the present purposes, the relevant notion of morphism is that of conservative ULF map: A simplicial map is called *ULF* (unique lifting of factorisations) if

it is cartesian on active face maps, and it is called *conservative* if cartesian on degeneracy maps. We write *CULF* for conservative and ULF, that is, cartesian on all active maps.

The CULF maps induce coalgebra homomorphisms.

0.8. Decalage. (See Illusie [14].) Given a simplicial space X as in the top row of the following diagram, the *lower dec* $\mathrm{Dec}_\perp X$ is a new simplicial space (the bottom row of the diagram) obtained by deleting X_0 and shifting everything one place down, deleting also all d_0 face maps and all s_0 degeneracy maps. It comes equipped with a simplicial map, the *dec map*, $d_\perp : \mathrm{Dec}_\perp X \to X$ given by the original d_0:
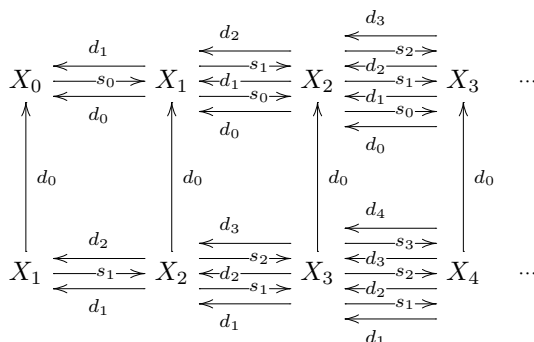
Similarly, the upper dec, denoted $\mathrm{Dec}_\top X$ is obtained by instead deleting, in each degree, the last face map $d_\top$ and the last degeneracy map $s_\top$.

The functor $\mathrm{Dec}_\perp$ can be described more conceptually as follows (see Lawvere [20]). There is an 'add-bottom' endofunctor $b : \mathbb{\Delta} \to \mathbb{\Delta}$, which sends $[k]$ to $[k+1]$ by adding a new bottom element. This is in fact a monad; the unit $\varepsilon : \mathrm{Id} \Rightarrow b$ is given by the bottom coface map $d^\perp$. The lower dec is given by precomposition with b:

$$\mathrm{Dec}_\perp X = b^* X$$

Hence $\mathrm{Dec}_\perp$ is a comonad, and its counit is the bottom face map $d_\perp$.

Similarly, the upper dec is obtained from the 'add-top' monad on $\mathbb{\Delta}$. Below we shall exploit crucially the combination of the two comonads.

The following result from [8, Theorem 4.10] will be invoked several times:

Theorem 0.9. *X is a decomposition space if and only if $\mathrm{Dec}_\top X$ and $\mathrm{Dec}_\perp X$ are Segal spaces, and the dec maps $d_\top : \mathrm{Dec}_\top X \to X$ and $d_\perp : \mathrm{Dec}_\perp X \to X$ are CULF.*

0.10. Complete decomposition spaces [9]. A decomposition space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is *complete* when $s_0 : X_0 \to X_1$ is a monomorphism (i.e. is (-1)-truncated). It follows from the decomposition space axiom that in this case *all* degeneracy maps are monomorphisms.

A Rezk complete Segal space is a complete decomposition space. The motivation for the completeness notion is to get a good notion of nondegenerate simplices, in turn

needed to obtain the Möbius inversion principle. The completeness condition is also needed to formulate the 'tightness' condition, locally finite length, which we come to in 6.5 below.

1. Factorisation systems and cartesian fibrations

In this section, which makes no reference to decomposition spaces, we prove some general results in category theory to the effect of lifting factorisation systems along an adjunction, and the like. For background to this section, see Lurie [24, § 5.2.8].

1.1. Factorisation systems. A *factorisation system* on an ∞-category $\mathcal{D}$ consists of two classes E and F of maps, that we shall depict as $\twoheadrightarrow$ and $\rightarrowtail$, such that

(1) The classes E and F are closed under equivalences.
(2) The classes E and F are orthogonal, $E \perp F$. That is, given $e \in E$ and $f \in F$, for every solid square
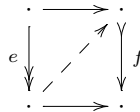
the space of fillers is contractible.
(3) Every map h admits a factorisation
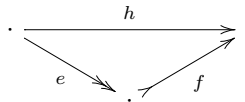
with $e \in E$ and $f \in F$.

(Note that in [24, Definition 5.2.8.8], the first condition is given as 'stability under formation of retracts'. In fact this stability follows from the three conditions above. Indeed, suppose $h \perp F$; factor $h = f \circ e$ as above. Since $h \perp f$, there is a diagonal filler in
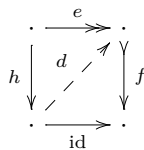
Now d belongs to $^{\perp}F$ since e and h do, and d belongs to $E^{\perp}$ since f and id do. Hence d is an equivalence, and therefore $h \in E$, by equivalence stability of E. Hence $E = {}^{\perp}F$, and is therefore closed under retracts. Similarly for F. It also follows that the two classes are closed under composition.)

1.2. Set-up. In this section, fix an ∞-category $\mathcal{D}$ with a factorisation system (E, F) as above. Let $\mathrm{Ar}(\mathcal{D}) = \mathrm{Fun}(\Delta[1], \mathcal{D})$, whose 0-simplices we depict vertically, then the domain projection $\mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$ (induced by the inclusion $\{0\} \hookrightarrow \Delta[1]$) is a cartesian fibration; the cartesian arrows are the squares of the form

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \underset{\sim}{\longrightarrow} & \cdot
\end{array}
$$
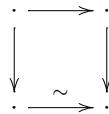
Let $\mathrm{Ar}^E(\mathcal{D}) \subset \mathrm{Ar}(\mathcal{D})$ denote the full subcategory spanned by the arrows in the left-hand class E.

Lemma 1.3. *The domain projection $\mathrm{Ar}^E(\mathcal{D}) \to \mathcal{D}$ is a cartesian fibration. The cartesian arrows in $\mathrm{Ar}^E(\mathcal{D})$ are given by squares of the form*

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \rightarrowtail & \cdot
\end{array}
$$
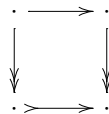
Proof. The essence of the argument is to provide uniquely the dashed arrow in
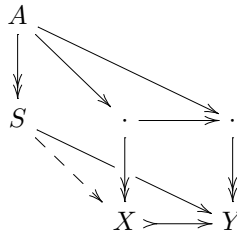
which amounts to filling

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
S & \longrightarrow & Y,
\end{array}
$$
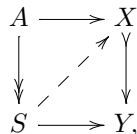
in turn uniquely fillable by orthogonality $E \perp F$. $\quad\square$

Lemma 1.4. *The inclusion $\mathrm{Ar}^E(\mathcal{D}) \to \mathrm{Ar}(\mathcal{D})$ admits a right adjoint w. This right adjoint $w : \mathrm{Ar}(\mathcal{D}) \to \mathrm{Ar}^E(\mathcal{D})$ sends an arrow a to its E-factor. In other words, if a factors as $a = f \circ e$ then $w(a) = e$.*

Proof. This is dual to [24, 5.2.8.19]. $\square$

Lemma 1.5. *The right adjoint w sends cartesian arrows in $\mathrm{Ar}(\mathcal{D})$ to cartesian arrows in $\mathrm{Ar}^E(\mathcal{D})$.*

Proof. This can be seen from the factorisation:
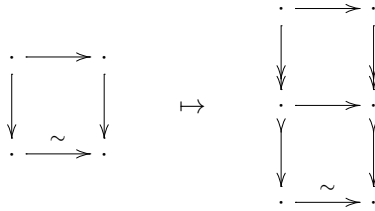
The middle horizontal arrow is forced into F by the closure properties of right classes. $\square$

Let $\mathrm{Fun}'(\Lambda_2^1, \mathcal{D}) = \mathrm{Ar}^E(\mathcal{D}) \times_{\mathcal{D}} \mathrm{Ar}^F(\mathcal{D})$ denote the ∞-category whose objects are pairs of composable arrows where the first arrow is in E and the second in F. Let $\mathrm{Fun}'(\Delta[2], \mathcal{D})$ denote the ∞-category of 2-simplices in $\mathcal{D}$ for which the two 'short' edges are in E and F respectively. The projection map $\mathrm{Fun}'(\Delta[2], \mathcal{D}) \to \mathrm{Fun}'(\Lambda_2^1, \mathcal{D})$ is always a trivial Kan fibration, just because $\mathcal{D}$ is an ∞-category.

Proposition 1.6. ([24, 5.2.8.17].) *The projection $\mathrm{Fun}'(\Delta[2], \mathcal{D}) \to \mathrm{Fun}(\Delta[1], \mathcal{D})$ induced by the long edge $d^1 : [1] \to [2]$ is a trivial Kan fibration.*

Corollary 1.7. *There is an equivalence of ∞-categories*

$$\mathrm{Ar}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Ar}^E(\mathcal{D}) \times_{\mathcal{D}} \mathrm{Ar}^F(\mathcal{D})$$

given by (E, F)-factoring an arrow.

Proof. Pick a section to the map in 1.6 and compose with the projection discussed just prior. $\square$

Let x be an object in $\mathcal{D}$, and denote by $\mathcal{D}^E_{x/}$ the ∞-category of E-arrows out of x. More formally it is given by the pullback

Corollary 1.8. *We have a pullback*

$$
\begin{array}{ccc}
\mathcal{D}_{x/} & \longrightarrow & \mathrm{Ar}^F(\mathcal{D}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{dom}} \\
\mathcal{D}_{x/}^E & \longrightarrow & \mathcal{D}
\end{array}
$$

Proof. In the diagram

$$
\begin{array}{ccccc}
\mathcal{D}_{x/} & \longrightarrow & \mathrm{Ar}(\mathcal{D}) & \longrightarrow & \mathrm{Ar}^F(\mathcal{D}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle w} & \lrcorner & \downarrow {\scriptstyle \mathrm{dom}} \\
\mathcal{D}_{x/}^E & \longrightarrow & \mathrm{Ar}^E(\mathcal{D}) & \xrightarrow{\ \mathrm{codom}\ } & \mathcal{D} \\
\downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{dom}} & & \\
* & \xrightarrow[\ \ulcorner x \urcorner\ ]{} & \mathcal{D} & &
\end{array}
$$

the right-hand square is a pullback by 1.7; the bottom square and the left-hand rectangle are clearly pullbacks, hence the top-left square is a pullback, and hence the top rectangle is too. □

Lemma 1.9. *Let $e : x \to x'$ be an arrow in the class E. Then we have a pullback square*

$$
\begin{array}{ccc}
\mathcal{D}_{x'/} & \xrightarrow{\ e^!\ } & \mathcal{D}_{x/} \\
{\scriptstyle w} \downarrow & \lrcorner & \downarrow {\scriptstyle w} \\
\mathcal{D}_{x'/}^E & \xrightarrow[\ e^!\ ]{} & \mathcal{D}_{x/}^E
\end{array}
$$

Here $e^!$ means 'precompose with e'.

Proof. In the diagram

$$
\begin{array}{ccccc}
\mathcal{D}_{x'/} & \xrightarrow{\ e^!\ } & \mathcal{D}_{x/} & \longrightarrow & \mathrm{Ar}^F(\mathcal{D}) \\
{\scriptstyle w} \downarrow & & {\scriptstyle w} \downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{dom}} \\
\mathcal{D}_{x'/}^E & \xrightarrow[\ e^!\ ]{} & \mathcal{D}_{x/}^E & \xrightarrow[\ \mathrm{codom}\ ]{} & \mathcal{D}
\end{array}
$$

the functor $\mathcal{D}_{x/} \to \mathrm{Ar}^F(\mathcal{D})$ is 'taking F-factor'. Note that the horizontal composites are again 'taking F-factor' and codomain, respectively, since precomposing with an E-map

does not change the F-factor. Since both the right-hand square and the rectangle are pullbacks by 1.8, the left-hand square is a pullback too.   □

1.10. Restriction. We shall need a slight variation of these results. We continue the assumption that $\mathcal{D}$ is an ∞-category with a factorisation system (E, F). Given a full subcategory $\mathbb{A} \subset \mathcal{D}$, we denote by $\mathbb{A}{\downarrow}\mathcal{D}$ the 'comma ∞-category of arrows in $\mathcal{D}$ with domain in $\mathbb{A}$'. More precisely it is defined as the pullback

$$
\begin{array}{ccc}
\mathbb{A}{\downarrow}\mathcal{D} & \xrightarrow{\text{f.f}} & \mathrm{Ar}(\mathcal{D}) \\
{\scriptstyle\text{dom}}\downarrow & \llcorner & \downarrow{\scriptstyle\text{dom}} \\
\mathbb{A} & \xrightarrow{\text{f.f}} & \mathcal{D}
\end{array}
$$

The map $\mathbb{A}{\downarrow}\mathcal{D} \to \mathbb{A}$ is a cartesian fibration. Similarly, let $\mathrm{Ar}^E(\mathcal{D})_{|\mathbb{A}}$ denote the comma ∞-category of E-arrows with domain in $\mathbb{A}$, defined as the pullback

$$
\begin{array}{ccc}
\mathrm{Ar}^E(\mathcal{D})_{|\mathbb{A}} & \xrightarrow{\text{f.f}} & \mathrm{Ar}^E(\mathcal{D}) \\
{\scriptstyle\text{dom}}\downarrow & \llcorner & \downarrow{\scriptstyle\text{dom}} \\
\mathbb{A} & \xrightarrow{\text{f.f}} & \mathcal{D}
\end{array}
$$

Again $\mathrm{Ar}^E(\mathcal{D})_{|\mathbb{A}} \to \mathbb{A}$ is a cartesian fibration (where the cartesian arrows are squares whose top part is in $\mathbb{A}$ and whose bottom horizontal arrow belongs to the class E). These two fibrations are just the restriction to $\mathbb{A}$ of the fibrations $\mathrm{Ar}(\mathcal{D}) \to \mathcal{D}$ and $\mathrm{Ar}^E(\mathcal{D}) \to \mathcal{D}$. Since the coreflection $\mathrm{Ar}(\mathcal{D}) \to \mathrm{Ar}^E(\mathcal{D})$ is vertical for the domain fibrations, it restricts to a coreflection $w : \mathbb{A}{\downarrow}\mathcal{D} \to \mathrm{Ar}^E(\mathcal{D})_{|\mathbb{A}}$.

Just as in the unrestricted situation (Corollary 1.7), we have a pullback square

$$
\begin{array}{ccc}
\mathbb{A}{\downarrow}\mathcal{D} & \longrightarrow & \mathrm{Ar}^F(\mathcal{D}) \\
{\scriptstyle w}\downarrow & \llcorner & \downarrow \\
\mathrm{Ar}^E(\mathcal{D})_{|\mathbb{A}} & \longrightarrow & \mathcal{D}
\end{array}
$$

saying that an arrow in $\mathcal{D}$ factors like before, also if it starts in an object in $\mathbb{A}$. Corollary 1.8 is the same in the restricted situation — just assume that x is an object in $\mathbb{A}$. Lemma 1.9 is also the same, just assume that $e : x' \to x$ is an E-arrow between $\mathbb{A}$-objects.

The following easy lemma expresses the general idea of extending a factorisation system.

Lemma 1.11. *Given an adjunction $L : \mathcal{D} \rightleftarrows \mathcal{C} : R$ and given a factorisation system (E, F) on $\mathcal{D}$ with the properties*

- *RL preserves the class F;*
- *$R\varepsilon$ belongs to F;*

consider the full subcategory $\widetilde{\mathcal{D}} \subset \mathcal{C}$ spanned by the image of L. Then there is an induced factorisation system $(\widetilde{E}, \widetilde{F})$ on $\widetilde{\mathcal{D}} \subset \mathcal{C}$ with $\widetilde{E} := L(E)$ (saturated by equivalences), and $\widetilde{F} := R^{-1}F \cap \widetilde{\mathcal{D}}$.

Proof. It is clear that the classes $\widetilde{E}$ and $\widetilde{F}$ are closed under equivalences. The two classes are orthogonal: given $Le \in \widetilde{E}$ and $\tilde{f} \in \widetilde{F}$ we have $Le \perp \tilde{f}$ in the full subcategory $\widetilde{\mathcal{D}} \subset \mathcal{C}$ if and only if $e \perp R\tilde{f}$ in $\mathcal{D}$, and the latter is true since $R\tilde{f} \in F$ by definition of $\widetilde{F}$. Finally, every map $g : LA \to X$ in $\widetilde{\mathcal{D}}$ admits an $(\widetilde{E}, \widetilde{F})$-factorisation: indeed, it is transpose to a map $A \to RX$, which we simply (E, F)-factor in $\mathcal{D}$,

$$A \xrightarrow{\hspace{4cm}} RX,$$
$$\searrow_e \qquad \nearrow_f$$
$$D$$

and transpose back the factorisation (i.e. apply L and postcompose with the counit): g is now the composite

$$LA \xrightarrow{Le} LD \xrightarrow{Lf} LRX \xrightarrow{\varepsilon} X,$$

where clearly $Le \in \widetilde{E}$, and we also have $\varepsilon \circ Lf \in \widetilde{F}$ because of the two conditions imposed. $\square$

1.12. Remarks. By general theory (1.4), having the factorisation system $(\widetilde{E}, \widetilde{F})$ implies the existence of a right adjoint to the inclusion

$$\mathrm{Ar}^{\widetilde{E}}(\widetilde{\mathcal{D}}) \longrightarrow \mathrm{Ar}(\widetilde{\mathcal{D}}).$$

This right adjoint returns the $\widetilde{E}$-factor of an arrow.

Inspection of the proof of 1.11 shows that we have the same factorisation property for other maps in $\mathcal{C}$ than those between objects in $\mathrm{Im}\, L$, namely giving up the requirement that the codomain should belong to $\mathrm{Im}\, L$: it is enough that the domain belongs to $\mathrm{Im}\, L$: *every map in $\mathcal{C}$ whose domain belongs to $\mathrm{Im}\, L$ factors as a map in $\widetilde{E}$ followed by a map in $\widetilde{F} := R^{-1}F$, and we still have $\widetilde{E} \perp \widetilde{F}$, without restriction on the codomain in the right-hand class.* This result amounts to a coreflection:

Theorem 1.13. *In the situation of Lemma 1.11, let $\widetilde{\mathcal{D}} \downarrow \mathcal{C} \subset \mathrm{Ar}(\mathcal{C})$ denote the full subcategory spanned by the maps with domain in* $\mathrm{Im}\, L$. *The inclusion functor*

$$\mathrm{Ar}^{\widetilde{E}}(\widetilde{\mathcal{D}}) \hookrightarrow \widetilde{\mathcal{D}} \downarrow \mathcal{C}$$

has a right adjoint, given by factoring any map with domain in $\mathrm{Im}\, L$ *and returning the $\widetilde{E}$-factor. Furthermore, the right adjoint preserves cartesian arrows (for the domain projections).*

Proof. Given that the factorisations exist as explained above, the proof now follows the proof of Lemma 5.2.8.18 in Lurie [24], using the dual of his Proposition 5.2.7.8.   □

The following restricted version of these results will be useful.

Lemma 1.14. *In the situation of Lemma 1.11, assume there is a full subcategory $J : \mathbb{A} \hookrightarrow \mathcal{D}$ such that*

– *All arrows in $\mathbb{A}$ belong to E.*
– *If an arrow in $\mathcal{D}$ has its domain in $\mathbb{A}$, then its E-factor also belongs to $\mathbb{A}$.*

Consider the full subcategory $\widetilde{\mathbb{A}} \subset \mathcal{C}$ spanned by the image of LJ. Then there is induced a factorisation system $(\widetilde{E}, \widetilde{F})$ on $\widetilde{\mathbb{A}} \subset \mathcal{C}$ with $\widetilde{E} := LJ(E)$ (saturated by equivalences) and $\widetilde{F} := R^{-1}F \cap \widetilde{\mathbb{A}}$.

Proof. The proof is the same as before.   □

1.15. A basic factorisation system. Suppose $\mathcal{C}$ is any ∞-category with pullbacks, and $\mathcal{D}$ is an ∞-category with a terminal object 1. Then evaluation on 1 defines a cartesian fibration

$$\mathrm{ev}_1 : \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathcal{C}$$

for which the cartesian arrows are precisely the cartesian natural transformations. The vertical arrows are the natural transformations whose component at 1 is an equivalence. Hence the functor ∞-category has a factorisation system in which the left-hand class is the class of vertical natural transformations, and the right-hand class is the class of cartesian natural transformations:

Finally we shall need the following general result (not related to factorisation systems):

Lemma 1.16. *Let $\mathcal{D}$ be any ∞-category. Then the functor*

$$F : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

$$D \longmapsto (\mathcal{D}_{D/})^{\mathrm{eq}},$$

corresponding to the right fibration $\mathrm{Ar}(\mathcal{D})^{\mathrm{cart}} \to \mathcal{D}$, preserves pullbacks.

Proof. Observe first that $F \simeq \mathrm{colim}_{X \in \mathcal{D}^{\mathrm{eq}}} \mathrm{Map}(-, X)$, a homotopy sum of representables. Given now a pushout in $\mathcal{D}$,

$$
\begin{array}{ccc}
D & \longleftarrow & B \\
\uparrow & \lrcorner & \uparrow \\
A & \longleftarrow & C
\end{array}
$$

we compute, using the distributive law:

$$
\begin{aligned}
F\left(A \coprod_C B\right) &\simeq \operatorname*{colim}_{X \in \mathcal{D}^{\mathrm{eq}}} \mathrm{Map}\left(A \coprod_C B, X\right) \\
&\simeq \operatorname*{colim}_{X \in \mathcal{D}^{\mathrm{eq}}} \left(\mathrm{Map}(A, X) \times_{\mathrm{Map}(C, X)} \mathrm{Map}(B, X)\right) \\
&\simeq \operatorname*{colim}_{X \in \mathcal{D}^{\mathrm{eq}}} \mathrm{Map}(A, X) \times_{\mathrm{colim}\, \mathrm{Map}(C, X)} \operatorname*{colim}_{X \in \mathcal{D}^{\mathrm{eq}}} \mathrm{Map}(B, X) \\
&\simeq F(A) \times_{F(C)} F(B). \quad \square
\end{aligned}
$$

2. Flanked decomposition spaces

2.1. Idea. The idea is that 'interval' should mean complete decomposition space (equipped) with both an initial and a terminal object. An object $x \in X_0$ is *initial* if the projection map $X_{x/} \to X$ is a levelwise equivalence. Here the *coslice* $X_{x/}$ is defined as the pullback of the lower dec $\mathrm{Dec}_\perp X$ along $1 \overset{\ulcorner x \urcorner}{\to} X_0$. Terminal objects are defined similarly with slices, i.e. pullbacks of the upper dec. It is not difficult to see (compare Proposition 2.18 below) that the existence of an initial or a terminal object forces X to be a Segal space.

While this intuition may be helpful, it turns out to be practical to approach the notion of interval from a more abstract viewpoint, which will allow us to get hold of various adjunctions and factorisation systems that are useful to prove things about intervals. We come to intervals in the next section. First we have to deal with flanked decomposition spaces.

2.2. The category Ξ of finite strict intervals. We denote by Ξ the category of finite strict intervals (cf. [17]), that is, a skeleton of the category whose objects are nonempty finite linear orders with a bottom and a top element, required to be distinct, and whose arrows

are the maps that preserve both the order and the bottom and top elements. We imagine the objects as columns of dots, with the bottom and top dot white, then the maps are the order-preserving maps that send white dots to white dots, but are allowed to send black dots to white dots.

There is a forgetful functor $u : \Xi \to \mathbb{\Delta}$ which forgets that there is anything special about the white dots, and just makes them black. This functor has a left adjoint $i : \mathbb{\Delta} \to \Xi$ which to a linear order (column of black dots) adjoins a bottom and a top element (white dots).

Our indexing convention for Ξ follows the free functor i: the object in Ξ with k black dots (and two outer white dots) is denoted $[k - 1]$. Hence the objects in Ξ are $[-1]$, $[0]$, $[1]$, etc. Note that $[-1]$ is an initial object in Ξ. The two functors can therefore be described on objects as $u([k]) = [k + 2]$ and $i([k]) = [k]$, and the adjunction is given by the following isomorphism:

$$\Xi([n], [k]) = \mathbb{\Delta}([n], [k+2]) \qquad n \geq 0, k \geq -1. \tag{1}$$

2.3. New outer degeneracy maps. Compared to $\mathbb{\Delta}$ via the inclusion $i : \mathbb{\Delta} \to \Xi$, the category Ξ has one extra coface map in Ξ, namely $[-1] \to [0]$. It also has, in each degree, two extra *outer codegeneracy maps*: $s^{\perp-1} : [n] \to [n-1]$ sends the bottom black dot to the bottom white dot, and $s^{\top+1} : [n] \to [n-1]$ sends the top black dot to the top white dot. (Both maps are otherwise bijective.)

2.4. Basic adjunction. The adjunction $i \dashv u$ induces an adjunction $i^* \dashv u^*$

$$\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S}) \underset{u^*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$$

which will play a central role in all the constructions in this section.

The functor i^* takes *underlying simplicial space*: concretely, applied to a Ξ^{op}-space A, the functor i^* deletes A_{-1} and removes all the extra outer degeneracy maps.

On the other hand, the functor u^*, applied to a simplicial space X, deletes X_0 and removes all outer face maps (and then reindexes).

The comonad

$$i^* u^* : \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$$

is precisely the double-dec construction $\mathrm{Dec}_\perp \mathrm{Dec}_\top$, and the counit of the adjunction is precisely the double-dec map

$$\varepsilon_X = d_\top d_\perp : i^* u^* X = \mathrm{Dec}_\perp \mathrm{Dec}_\top X \longrightarrow X.$$

On the other hand, the monad

$$u^* i^* : \mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$$

is also a kind of double-dec, removing first the extra outer degeneracy maps, and then the outer face maps. The unit

$$\eta_A = s_{\perp-1} s_{\top+1} : A \to u^* i^* A$$

will also play an important role.

Lemma 2.5. *If $f : Y \to X$ is a CULF map of simplicial spaces, then $u^* f : u^* Y \to u^* X$ is cartesian.*

Proof. The CULF condition on f says it is cartesian on 'everything' except outer face maps, which are thrown away when taking $u^* f$. □

Note that the converse is not always true: if $u^* f$ is cartesian then f is ULF, but there is no information about $s_0 : Y_0 \to Y_1$, so we cannot conclude that f is conservative.

Dually:

Lemma 2.6. *If a map of Ξ^{op}-spaces $g : B \to A$ is cartesian (or just cartesian on inner face and degeneracy maps), then $i^* g : i^* B \to i^* A$ is cartesian.*

2.7. Representables. The representables on Ξ we denote by $\Xi[-1]$, $\Xi[0]$, etc. By convention we will also denote the terminal presheaf on Ξ by $\Xi[-2]$, although it is not representable since we have chosen not to include $[-2]$ (a single white dot) in our definition of Ξ. Note that (1) says that i^* preserves representables:

$$i^*(\Xi[k]) \simeq \Delta[k+2], \qquad k \geq -1. \tag{2}$$

2.8. Stretched-cartesian factorisation system. Call an arrow in $\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ *stretched* if its $[-1]$-component is an equivalence. Call an arrow *cartesian* if it is a cartesian natural transformation of Ξ^{op}-spaces. By general theory (1.15) we have a factorisation system on $\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ where the left-hand class is formed by the stretched maps and the right-hand class consists of the cartesian maps. In concrete terms, given any map $B \to A$, since $[-1]$ is terminal in Ξ^{op}, one can pull back the whole diagram A along the map $B_{-1} \to A_{-1}$. The resulting Ξ^{op}-space A' is cartesian over A by construction, and by the universal property of the pullback it receives a map from B which is manifestly the identity in degree -1, hence stretched.
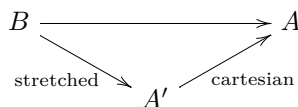
2.9. Flanked Ξ^{op}-spaces. A Ξ^{op}-space A is called *flanked* if the extra outer degeneracy maps form cartesian squares with opposite outer face maps. Precisely, for $n \geq 0$

$$
\begin{array}{ccc}
A_{n-1} & \xleftarrow{\;d_\top\;} & A_n \\
{\scriptstyle s_{\bot-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{\bot-1}} \\
A_n & \xleftarrow{\;d_\top\;} & A_{n+1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
A_{n-1} & \xleftarrow{\;d_\bot\;} & A_n \\
{\scriptstyle s_{\top+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{\top+1}} \\
A_n & \xleftarrow{\;d_\bot\;} & A_{n+1}
\end{array}
$$

Here we have included the special extra face map $A_{-1} \leftarrow A_0$ both as a top face map and a bottom face map.

2.10. Example. If $\mathcal{C}$ is a small category with an initial object and a terminal object, then its nerve is naturally a Ξ^{op}-space (contractible in degree -1): the extra bottom degeneracy maps add the initial object to the beginning of a sequence of composable arrows, and the extra top degeneracy maps add the terminal object to the end of a sequence of composable arrow. The flanking condition then states precisely that these two objects are initial and terminal.

Lemma 2.11 (*'Bonus pullbacks' for flanked spaces*). *In a flanked Ξ^{op}-space A, all the following squares are pullbacks:*

$$
\begin{array}{ccc}
A_{n-1} & \xleftarrow{\;d_i\;} & A_n \\
{\scriptstyle s_{\bot-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{\bot-1}} \\
A_n & \xleftarrow{\;d_{i+1}\;} & A_{n+1}
\end{array}
\;\;
\begin{array}{ccc}
A_{n-1} & \xrightarrow{\;s_j\;} & A_n \\
{\scriptstyle s_{\bot-1}}\downarrow & \llcorner & \downarrow{\scriptstyle s_{\bot-1}} \\
A_n & \xrightarrow{\;s_{j+1}\;} & A_{n+1}
\end{array}
\;\;
\begin{array}{ccc}
A_{n-1} & \xleftarrow{\;d_i\;} & A_n \\
{\scriptstyle s_{\top+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{\top+1}} \\
A_n & \xleftarrow{\;d_i\;} & A_{n+1}
\end{array}
\;\;
\begin{array}{ccc}
A_{n-1} & \xrightarrow{\;s_j\;} & A_n \\
{\scriptstyle s_{\top+1}}\downarrow & \llcorner & \downarrow{\scriptstyle s_{\top+1}} \\
A_n & \xrightarrow{\;s_j\;} & A_{n+1}
\end{array}
$$

This is for all $n \geq 0$, and the running indices are $0 \leq i \leq n$ and $-1 \leq j \leq n$.

Proof. Easy argument with pullbacks, similar to [8, 3.10]. □

Note that in the upper rows, all face or degeneracy maps are present, whereas in the lower rows, there is one map missing in each case. In particular, all the 'new' outer degeneracy maps appear as pullbacks of 'old' degeneracy maps.

2.12. Flanked decomposition spaces. By definition, a *flanked decomposition space* is a Ξ^{op}-space $A : \Xi^{\mathrm{op}} \to \mathcal{S}$ that is flanked and whose underlying Δ^{op}-space i^*A is a decomposition space. Let **FD** denote the full subcategory of $\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ spanned by the flanked decomposition spaces.

Lemma 2.13. *If X is a decomposition space, then u^*X is a flanked decomposition space.*

Proof. The underlying simplicial space is clearly a decomposition space (in fact a Segal space), since all we have done is to throw away some outer face maps and reindex. The flanking condition comes from the 'bonus pullbacks' of X, cf. [8, 3.10]. □

It follows that the basic adjunction $i^* \dashv u^*$ restricts to an adjunction

$$i^* : \mathbf{FD} \rightleftarrows \mathbf{Dcmp} : u^*$$

between flanked decomposition spaces (certain Ξ^{op}-spaces) and decomposition spaces.

Lemma 2.14. *The counit*

$$\varepsilon_X : i^* u^* X \to X$$

is CULF, when X is a decomposition space.

Proof. This follows from Theorem 0.9. □

Lemma 2.15. *The unit*

$$\eta_A : A \to u^* i^* A$$

is cartesian, when A is flanked.

Proof. The map η_A is given by $s_{\perp-1}$ followed by $s_{\top+1}$. The asserted pullbacks are precisely the 'bonus pullbacks' of Lemma 2.11. □

From Lemma 2.15 and Lemma 2.14 we get:

Corollary 2.16. *The monad $u^* i^* : \mathbf{FD} \to \mathbf{FD}$ preserves cartesian maps.*

Lemma 2.17. *$i^* A \to X$ is CULF in $\mathbf{Dcmp}$ if and only if the transpose $A \to u^* X$ is cartesian in $\mathbf{FD}$.*

Proof. This follows since the unit is cartesian (2.15), the counit is CULF (2.14), and u^* and i^* send those two classes to each other (2.5 and 2.6). □

Proposition 2.18. *If A is a flanked decomposition space, then $i^* A$ is a Segal space.*

Proof. Put $X = i^* A$. We have the maps

$$i^* A \xrightarrow{i^* \eta_A} i^* u^* i^* A = u^* i^* X \xrightarrow{\varepsilon_X} X = i^* A$$

Now X is a decomposition space by assumption, so $i^* u^* X = \mathrm{Dec}_\perp \mathrm{Dec}_\top X$ is a Segal space and the counit is CULF (both statements by Theorem 0.9). On the other hand, since A is flanked, the unit η is cartesian by Lemma 2.15, hence $i^* \eta$ is cartesian by Lemma 2.6. Since $i^* A$ is thus cartesian over a Segal space, it is itself a Segal space ([8, 2.11]). □

Lemma 2.19. *If $B \to A$ is a cartesian map of Ξ^{op}-spaces and A is a flanked decomposition space then so is B.*

Corollary 2.20. *The stretched-cartesian factorisation system restricts to a factorisation system on* ***FD***.

Lemma 2.21. *The representable functors $\Xi[k]$ are flanked.*

Proof. Since the pullback squares required for a presheaf to be flanked are images of pushouts in Ξ, this follows since representable functors send colimits to limits.   □

3. Intervals and the factorisation-interval construction

3.1. Complete Ξ^{op}-spaces. A Ξ^{op}-space is called *complete* if all degeneracy maps are monomorphisms. We are mostly interested in this notion for flanked decomposition spaces. In this case, if just $s_0 : A_0 \to A_1$ is a monomorphism, then all the degeneracy maps are monomorphisms. This follows because on the underlying decomposition space, we know [9, 2.5] that $s_0 : A_0 \to A_1$ being a monomorphism implies that all the simplicial degeneracy maps are monomorphisms, and by flanking we then deduce that also the new outer degeneracy maps are monomorphisms. Denote by ***cFD*** $\subset$ ***FD*** the full subcategory spanned by the complete flanked decomposition spaces.

It is clear that if X is a complete decomposition space, then u^*X is a complete flanked decomposition space, and if A is a complete flanked decomposition space then i^*A is a complete decomposition space. Hence the fundamental adjunction $i^* : $ ***FD*** $\rightleftarrows$ ***Dcmp*** $: u^*$ between flanked decomposition spaces and decomposition spaces restricts to an adjunction

$$i^* : \textit{\textbf{cFD}} \rightleftarrows \textit{\textbf{cDcmp}} : u^*$$

between complete flanked decomposition spaces and complete decomposition spaces.

Note that anything cartesian over a complete Ξ^{op}-space is again complete.

3.2. Reduced Ξ^{op}-spaces. A Ξ^{op}-space $A : \Xi^{\mathrm{op}} \to \mathcal{S}$ is called *reduced* when $A[-1] \simeq 1$.

Lemma 3.3. *If $A \to B$ is a stretched map of Ξ^{op}-spaces and A is reduced then B is reduced.*

3.4. Algebraic intervals. An *algebraic interval* is by definition a reduced complete flanked decomposition space. We denote by ***aInt*** the full subcategory of $\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ spanned by the algebraic intervals. In other words, a *morphism of algebraic intervals* is just a natural transformation of functors $\Xi^{\mathrm{op}} \to \mathcal{S}$. Note that the underlying decomposition space of an interval is always a Segal space.

Lemma 3.5. *All representables* $\Xi[k]$ *are algebraic intervals (for* $k \geq -1$*), and also the terminal presheaf* $\Xi[-2]$ *is an algebraic interval.*

Proof. It is clear that all these presheaves are contractible in degree -1, and they are flanked by Lemma 2.21. It is also clear from (2) that their underlying simplicial spaces are complete decomposition spaces (they are even Rezk complete Segal spaces). □

Lemma 3.6. $\Xi[-1]$ *is an initial object in* **aInt***.*

Lemma 3.7. *Every morphism in* **aInt** *is stretched.*

Corollary 3.8. *If a morphism of algebraic intervals is cartesian, then it is an equivalence.*

3.9. The factorisation-interval construction. We now come to the important notion of factorisation interval $I(a)$ of a given arrow a in a decomposition space X. In the case where X is a 1-category the construction is due to Lawvere [21]: the objects of $I(a)$ are the two-step factorisations of a, with initial object id-followed-by-a and terminal object a-followed-by-id. The 1-cells are arrows between such factorisations, or equivalently 3-step factorisations, and so on.

For a general (complete) decomposition space X, the idea is this: taking the double-dec of X gives a simplicial object starting at X_2, but equipped with an augmentation $X_1 \leftarrow X_2$. Pulling back this simplicial object along $\ulcorner a \urcorner : 1 \to X_1$ yields a new simplicial object which is $I(a)$. This idea can be formalised in terms of the basic adjunction as follows.

By Yoneda, to give an arrow $a \in X_1$ is to give $\Delta[1] \to X$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$, or in the full subcategory **cDcmp**. By adjunction, this is equivalent to giving $\Xi[-1] \to u^*X$ in **cFD**. Now factor this map as a stretched map followed by a cartesian map:

$$\Xi[-1] \xrightarrow{\hspace{3cm}} u^*X.$$

with $\Xi[-1] \xrightarrow{\text{stretched}} A \xrightarrow{\text{cart}} u^*X$

The object appearing in the middle is an algebraic interval since it is stretched under $\Xi[-1]$ (3.3). By definition, the factorisation interval of a is $I(a) := i^*A$, equipped with a CULF map to X, as seen in the diagram

$$\Delta[1] \xrightarrow{\hspace{3cm}} i^*u^*X \xrightarrow[\text{culf}]{\varepsilon} X.$$

with $\Delta[1] \to I(a) \xrightarrow{\text{culf}} i^*u^*X$

The map $\Delta[1] \to I(a)$ equips $I(a)$ with two endpoints, and a longest arrow between them. The CULF map $I(a) \to X$ sends the longest arrow of $I(a)$ to a.

More generally, by the same adjunction argument, given an k-simplex $\sigma : \Delta[k] \to X$ with long edge a, we get a k-subdivision of $I(a)$, i.e. a stretched map $\Delta[k] \to I(a)$.

The construction shows, remarkably, that as far as comultiplication is concerned, any decomposition space is locally a Segal space, in the sense that the comultiplication of an arrow a may as well be performed inside $I(a)$, which is a Segal space by 2.18. So while there may be no global way to compose arrows even if their source and targets match, the *decompositions* that exist do compose again.

We proceed to formalise the factorisation-interval construction.

3.10. Coreflections. Inside the ∞-category of arrows $\mathrm{Ar}(\boldsymbol{cFD})$, denote by $\mathrm{Ar}^{\mathrm{s}}(\boldsymbol{cFD})$ the full subcategory spanned by the stretched maps. The stretched-cartesian factorisation system amounts to a coreflection

$$w : \mathrm{Ar}(\boldsymbol{cFD}) \longrightarrow \mathrm{Ar}^{\mathrm{s}}(\boldsymbol{cFD});$$

it sends an arrow $A \to B$ to its stretched factor $A \to B'$, and in particular can be chosen to have A as domain again (1.4). In particular, for each algebraic interval $A \in \boldsymbol{aInt} \subset \boldsymbol{cFD}$, the adjunction restricts to an adjunction between coslices, with coreflection

$$w_A : \boldsymbol{cFD}_{A/} \longrightarrow \boldsymbol{cFD}^{\mathrm{s}}_{A/}.$$

The first ∞-category is that of flanked decomposition spaces under A, and the second ∞-category is that of flanked decomposition spaces with a stretched map from A. Now, if a flanked decomposition space receives a stretched map from an algebraic interval then it is itself an algebraic interval (3.3), and all maps of algebraic intervals are stretched (3.7). So in the end the cosliced adjunction takes the form of the natural full inclusion functor

$$v_A : \boldsymbol{aInt}_{A/} \to \boldsymbol{cFD}_{A/}$$

and a right adjoint

$$w_A : \boldsymbol{cFD}_{A/} \to \boldsymbol{aInt}_{A/}.$$

3.11. Remark. These observations amount to saying that the functor $v : \boldsymbol{aInt} \to \boldsymbol{cFD}$ is a *colocal left adjoint*. This notion is dual to the important concept of local right adjoint [29, 12].

We record the following obvious lemmas:

Lemma 3.12. *The coreflection w sends cartesian maps to equivalences.*

Lemma 3.13. *The counit is cartesian.*

3.14. Factorisation-interval as a comonad. We also have the basic adjunction $i^* \dashv u^*$ between complete decomposition spaces and complete flanked decomposition spaces. Applied to coslices over an algebraic interval A and its underlying decomposition space $\underline{A} = i^*A$, we get the adjunction

$$L : \textbf{\textit{cFD}}_{A/} \; \underset{\longleftarrow}{\overset{\longrightarrow}{}} \; \textbf{\textit{cDcmp}}_{\underline{A}/} : R.$$

Here L is simply the functor i^*, while the right adjoint R is given by applying u^* and precomposing with the unit η_A. Note that the unit of this adjunction $L \dashv R$ at an object $f : A \to X$ is given by

$$
\begin{array}{ccc}
 & A & \\
 {}^{f}\swarrow & & \searrow^{u^*i^*f \circ \eta_A} \\
X & \xrightarrow[\eta_X]{} & u^*i^*X
\end{array}
$$

We now combine the two adjunctions:

$$\textbf{\textit{aInt}}_{A/} \; \underset{w}{\overset{v}{\rightleftarrows}} \; \textbf{\textit{cFD}}_{A/} \; \underset{R}{\overset{L}{\rightleftarrows}} \; \textbf{\textit{cDcmp}}_{\underline{A}/} \, .$$

The factorisation-interval functor is the $\underline{A} = \Delta[k]$ instantiation:

$$I := L \circ v \circ w \circ R.$$

Indeed, this is precisely what we said in the construction, just phrased more functorially. It follows that the factorisation-interval construction is a comonad on $\textbf{\textit{cDcmp}}_{\underline{A}/}$.

Lemma 3.15. *The composed counit is CULF.*

Proof. This follows readily from 2.14. $\square$

Proposition 3.16. *The composed unit $\eta : \mathrm{Id} \Rightarrow w \circ R \circ L \circ v$ is an equivalence.*

Proof. The result of applying the four functors to an algebraic interval map $f : A \to B$ is the stretched factor in

$$
\begin{array}{ccc}
A & \xrightarrow{} & u^*i^*B \\
{}_{\text{stretched}}\searrow & & \nearrow_{\text{cart}} \\
 & D &
\end{array}
$$

The unit on f sits in this diagram

$$
\begin{array}{ccc}
 & A & \\
 f \swarrow & & \searrow \\
 B \dashrightarrow{\eta_f} & & D \\
 \eta_B \searrow & & \swarrow \\
 & u^*i^*B, &
\end{array}
$$
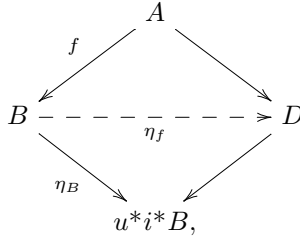
where η_B is cartesian by 2.15. It follows now from orthogonality of the stretched-cartesian factorisation system that η_f is an equivalence. □

Corollary 3.17. *The functor $i^* \circ v : \mathbf{aInt} \to \mathbf{cDcmp}_{\Delta[1]/}$ is fully faithful.*

Proposition 3.18. *I sends CULF maps to equivalences. In detail, for a CULF map $F : Y \to X$ and any arrow $a \in Y_1$ we have a natural equivalence of intervals (and hence of underlying Segal spaces)*

$$
I(a) \overset{\sim}{\to} I(Fa).
$$

Proof. R sends CULF maps to cartesian maps, and w send cartesian maps to equivalences. □

Corollary 3.19. *If X is an interval, with longest arrow $a \in X_1$, then $X \simeq I(a)$.*

Proposition 3.20. *The composed functor*

$$
\mathbf{aInt} \to \mathbf{cDcmp}_{\Delta[1]/} \to \mathbf{cDcmp}
$$

is faithful (i.e. induces a monomorphism on mapping spaces).

Proof. Given two algebraic intervals A and B, denote by $f : \Delta[1] \to i^*A$ and $g : \Delta[1] \to i^*B$ the images in $\mathbf{cDcmp}_{\Delta[1]/}$. The claim is that the map

$$
\mathrm{Map}_{\mathbf{aInt}}(A, B) \longrightarrow \mathrm{Map}_{\mathbf{cDcmp}_{\Delta[1]/}}(f, g) \longrightarrow \mathrm{Map}_{\mathbf{cDcmp}}(i^*A, i^*B)
$$

is a monomorphism. We already know that the first part is an equivalence (by Corollary 3.17). The second map will be a monomorphism because of the special nature of f and g. We have a pullback diagram (mapping space fibre sequence for coslices):

$$\begin{array}{ccc}
\mathrm{Map}_{\textbf{\textit{cDcmp}}_{\Delta[1]/}}(f,g) & \longrightarrow & \mathrm{Map}_{\textbf{\textit{cDcmp}}}(i^*A, i^*B) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{precomp}.f} \\
1 & \xrightarrow{\ulcorner g \urcorner} & \mathrm{Map}_{\textbf{\textit{cDcmp}}}(\Delta[1], i^*B).
\end{array}$$

Since $g : \Delta[1] \to i^*B$ is the image of the canonical map $\Xi[-1] \to B$, the map

$$1 \xrightarrow{\ulcorner g \urcorner} \mathrm{Map}_{\textbf{\textit{cDcmp}}}(\Delta[1], i^*B)$$

can be identified with

$$B_{-1} \xrightarrow{s_{\perp -1} s_{\top +1}} B_1,$$

which is a monomorphism since B is complete. It follows that the top map in the above pullback square is a monomorphism, as asserted. (Note the importance of completeness.) $\square$

4. The decomposition space of intervals

4.1. Interval category as a full subcategory in cDcmp. We now invoke the general results about extension of factorisation systems (Lemmas 1.11 and 1.14). Let

$$\textbf{\textit{Int}} := \widetilde{\textbf{\textit{aInt}}}$$

denote the image of $\textbf{\textit{aInt}} \subset \textbf{\textit{cFD}}$ under the left adjoint i^* in the adjunction

$$i^* : \textbf{\textit{cFD}} \rightleftarrows \textbf{\textit{cDcmp}} : u^*$$

as in 1.14. Hence $\textbf{\textit{Int}} \subset \textbf{\textit{cDcmp}}$ is the full subcategory of decomposition spaces underlying algebraic intervals. Say a map in $\textbf{\textit{Int}}$ is *stretched* if it is the i^* image of a map in $\textbf{\textit{aInt}}$ (i.e. a stretched map in $\textbf{\textit{cFD}}$).

Proposition 4.2. *The stretched maps as left-hand class and the CULF maps as right-hand class form a factorisation system on* $\textbf{\textit{Int}}$.

Proof. The stretched-cartesian factorisation system on $\textbf{\textit{cFD}}$ is compatible with the adjunction $i^* \dashv u^*$ and the subcategory $\textbf{\textit{Int}}$ precisely as required to apply the general Lemma 1.14. Namely, we have:

- u^*i^* preserves cartesian maps by Corollary 2.16.
- $u^*\varepsilon$ is cartesian by 2.5, since ε is CULF by 2.14.
- If $A \to B$ is stretched, A an algebraic interval, then so is B, by 3.3.

Lemma 1.14 now tells us that there is a factorisation system on **Int** where the left-hand class are the maps of the form i^* of a stretched map. The right-hand class of **Int**, described by Lemma 1.14 as those maps f for which u^*f is cartesian, is seen by Lemma 2.17 to be precisely the CULF maps. □

As in 1.14, we can further restrict to the image of the category $\Xi_+ \subset$ **aInt** consisting of the representables together with the terminal object $\Xi[-2]$:

Lemma 4.3. *The restriction (as in 1.14) to $\Xi_+ \subset$ **aInt** gives $\triangle \subset$ **Int**:*

$$
\begin{array}{ccccc}
\triangle & \xrightarrow{\text{f.f.}} & \textbf{Int} & \xrightarrow{\text{f.f.}} & \textbf{cDcmp} \\
\uparrow & & \uparrow & & {\scriptstyle i^*}\big\uparrow\big\downarrow{\scriptstyle u^*} \\
\Xi_+ & \longrightarrow & \textbf{aInt} & \longrightarrow & \textbf{cFD,}
\end{array}
$$

*and the stretched-CULF factorisation systems on **Int** restricts to the active-inert factorisation system on $\triangle$.*

Proof. By construction the objects are $[-2], [-1], [0], [1], \dots$ and the mapping spaces are

$$
\mathrm{Map}_{\textbf{Int}}(\Xi[k], \Xi[n]) \simeq \mathrm{Map}_{\textbf{Dcmp}}(i^*\Xi[k], i^*\Xi[n])
$$
$$
\simeq \mathrm{Map}_{\widehat{\triangle}}(\Delta[k+2], \Delta[n+2])
$$
$$
\simeq \triangle([k+2], [n+2]).
$$

It is clear by the explicit description of i^* that it takes the maps in Ξ_+ to the active maps in $\triangle$. On the other hand, it is clear that the CULF maps in $\triangle$ are the inert maps. □

4.4. Arrow ∞-category and restriction to $\triangle$. Let $\mathrm{Ar}^{\mathrm{s}}(\textbf{Int}) \subset \mathrm{Ar}(\textbf{Int})$ denote the full subcategory of the arrow ∞-category spanned by the stretched maps. Recall (from 1.3) that $\mathrm{Ar}^{\mathrm{s}}(\textbf{Int})$ is a cartesian fibration over **Int** via the domain projection. We now restrict this cartesian fibration to $\triangle \subset$ **Int** as in 1.10:

$$
\begin{array}{ccc}
\mathrm{Ar}^{\mathrm{s}}(\textbf{Int})_{|\triangle} & \xrightarrow{\text{f.f.}} & \mathrm{Ar}^{\mathrm{s}}(\textbf{Int}) \\
{\scriptstyle \text{dom}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \text{dom}} \\
\triangle & \xrightarrow[\text{f.f.}]{} & \textbf{Int}
\end{array}
$$

We put

$$
\mathcal{U} := \mathrm{Ar}^{\mathrm{s}}(\textbf{Int})_{|\triangle}.
$$

$\mathcal{U} \to \mathbb{A}$ is the *Cartesian fibration of subdivided intervals*: the objects of $\mathcal{U}$ are the stretched interval maps $\Delta[k] \to A$, which we think of as subdivided intervals. The arrows are commutative squares

$$\begin{array}{ccc} \Delta[k] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where the downwards maps are stretched, and the rightwards maps are in $\mathbb{A}$ and in ***cDcmp***, respectively. (These cannot be realised in the world of Ξ^{op}-spaces, and the necessity of having them was the whole motivation for constructing ***Int***.) By 1.3, the cartesian maps are squares

$$\begin{array}{ccc} \Delta[k] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ \text{culf}\ } & B. \end{array}$$

Hence, cartesian lifts are performed by precomposing and then coreflecting (i.e. stretched-CULF factorising and keeping only the stretched part). For a fixed domain $\Delta[k]$, we have (in virtue of Proposition 3.20)

$$\mathbf{\textit{Int}}^{\mathrm{s}}_{\Delta[k]/} \simeq \mathbf{\textit{aInt}}_{\Xi[k-2]/}.$$

4.5. The (large) decomposition space of intervals. The cartesian fibration $\mathcal{U} = \mathrm{Ar}^{\mathrm{s}}(\mathbf{\textit{Int}})_{|\mathbb{A}} \to \mathbb{A}$ determines a right fibration, $\mathcal{U}^{\mathrm{cart}} = \mathrm{Ar}^{\mathrm{s}}(\mathbf{\textit{Int}})_{|\mathbb{A}}^{\mathrm{cart}} \to \mathbb{A}$, and hence by straightening ([24], Ch. 2) a simplicial ∞-groupoid

$$U : \mathbb{A}^{\mathrm{op}} \to \widehat{\mathcal{S}},$$

where $\widehat{\mathcal{S}}$ is the very large ∞-category of not necessarily small ∞-groupoids. We shall see that it is a complete decomposition space.

We shall postpone the straightening as long as possible, as it is more convenient to work directly with the right fibration $\mathcal{U}^{\mathrm{cart}} \to \mathbb{A}$. Its fibre over $[k] \in \mathbb{A}$ is the ∞-groupoid U_k of k-subdivided intervals. That is, an interval $\underline{A}$ equipped with a stretched map $\Delta[k] \to A$. Note that U_1 is equivalent to the ∞-groupoid $\mathbf{\textit{Int}}^{\mathrm{eq}}$. Similarly, U_2 is equivalent to the ∞-groupoid of subdivided intervals, more precisely intervals with a stretched map from $\Delta[2]$. Somewhat more exotic is U_0, the ∞-groupoid of intervals with a stretched map from $\Delta[0]$. This means that the endpoints must coincide. This does not imply that the

interval is trivial. For example, any ∞-category with a zero object provides an example of an object in U_0.

4.6. A remark on size. The fibres of the right fibration $\mathcal{U}^{\mathrm{cart}} \to \mathbb{\Delta}$ are large ∞-groupoids. Indeed, they are all variations of U_1, the ∞-groupoid of intervals, which is of the same size as the ∞-category of simplicial spaces, which is of the same size as $\mathcal{S}$. Accordingly, the corresponding presheaf takes values in large ∞-groupoids, and U is therefore a large decomposition space. These technicalities do not affect the following results, but will play a role from 5.4 and onwards.

Among the active maps in U, in each degree the unique map $g : U_r \to U_1$ consists in forgetting the subdivision. The space U also has the codomain projection $U \to \textbf{Int}$. In particular we can describe the g-fibre over a given interval A:

Lemma 4.7. *We have a pullback square*

$$
\begin{CD}
(A_r)_a @>>> U_r \\
@VVV @VV{g}V \\
* @>>{\ulcorner A \urcorner}> U_1
\end{CD}
$$

where $a \in A_1$ denotes the longest edge.

Proof. Indeed, the fibre over a coslice is the mapping space, so the pullback is at first

$$\mathrm{Map}_{\mathrm{stretched}}(\Delta[r], A)$$

But that's the full subgroupoid inside $\mathrm{Map}(\Delta[r], A) \simeq A_r$ consisting of the stretched maps, but that means those whose restriction to the long edge is a. $\quad\square$

Theorem 4.8. *The simplicial space $U : \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ is a (large) complete decomposition space.*

Proof. We first show it is a decomposition space. We need to show that for an active-inert pullback square in $\mathbb{\Delta}^{\mathrm{op}}$, the image under U is a pullback:

$$
\begin{CD}
U_k @>{f'}>> U_m \\
@V{g'}VV @VV{g}V \\
U_n @>>{f}> U_s
\end{CD}
$$

This square is the outer rectangle in

$$
\begin{array}{ccccccc}
\boldsymbol{Int}^{\mathrm{s}}_{\Delta[k]/} & \xrightarrow{\ j\ } & \boldsymbol{Int}_{\Delta[k]/} & \xrightarrow{\ f'\ } & \boldsymbol{Int}_{\Delta[m]/} & \xrightarrow{\ w\ } & \boldsymbol{Int}^{\mathrm{s}}_{\Delta[m]/} \\
{\scriptstyle g'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\
\boldsymbol{Int}^{\mathrm{s}}_{\Delta[n]/} & \xrightarrow[\ j\ ]{} & \boldsymbol{Int}_{\Delta[n]/} & \xrightarrow[\ f\ ]{} & \boldsymbol{Int}_{\Delta[s]/} & \xrightarrow[\ w\ ]{} & \boldsymbol{Int}^{\mathrm{s}}_{\Delta[s]/}
\end{array}
$$

(Here we have omitted taking maximal ∞-groupoids, but it doesn't affect the argument.) The first two squares consist in precomposing with the inert maps f, f'. The result will no longer be a stretched map, so in the middle columns we allow arbitrary maps. But the final step just applies the coreflection to take the stretched part. Indeed this is how cartesian lifting goes in $\mathrm{Ar}^{\mathrm{s}}(\boldsymbol{Int})$. The first square is a pullback since j is fully faithful. The last square is a pullback since it is a special case of Lemma 1.9. The main point is the second square which is a pullback by Lemma 1.16 — this is where we use that the active-inert square in $\triangle^{\mathrm{op}}$ is a pullback.

To establish that U is complete, we need to check that the map $U_0 \to U_1$ is a monomorphism. This map is just the forgetful functor

$$
(\boldsymbol{Int}^{\mathrm{s}}_{*/})^{\mathrm{eq}} \to \boldsymbol{Int}^{\mathrm{eq}}.
$$

The claim is that its fibres are empty or contractible. The fibre over an interval $\underline{A} = i^*A$ is

$$
\mathrm{Map}_{\mathrm{stretched}}(1, \underline{A}) \simeq \mathrm{Map}_{\boldsymbol{aInt}}(\Xi[-2], A) \simeq \mathrm{Map}_{\widehat{\Xi}}(\Xi[-2], A).
$$

Note that in spite of the notation, $\Xi[-2]$ is not a representable: it is the terminal object, and it is hence the colimit of all the representables. It follows that $\mathrm{Map}_{\widehat{\Xi}}(\Xi[-2], A) = \lim A$. This is the limit of a cosimplicial diagram

$$
\lim A \xrightarrow{\ e\ } * \rightrightarrows A_0 \cdots
$$

In general the limiting map of a cosimplicial diagram does not have to be a monomorphism, but in this case it is, as all the coface maps (these are the degeneracy maps of A) are monomorphisms by completeness of A, and since A_{-1} is contractible. Since finally e is a monomorphism into the contractible space A_{-1}, the limit must be empty or contractible. Hence $U_0 \to U_1$ is a monomorphism, and therefore U is complete. $\quad\square$

5. Universal property of U

The refinements discussed in 1.12 now pay off to give us the following main result. Let $\boldsymbol{Int}{\downarrow}\boldsymbol{cDcmp}$ denote the comma ∞-category (as in 1.13). It is the full subcategory

in Ar(***cDcmp***) spanned by the maps whose domain is in ***Int***. Let Ar$^{\mathrm{s}}$(***Int***) denote the full subcategory of Ar(***Int***) spanned by the stretched maps. Recall (from 1.3) that both ***Int↓cDcmp*** and Ar$^{\mathrm{s}}$(***Int***) are cartesian fibrations over ***Int*** via the domain projections, and that the inclusion Ar$^{\mathrm{s}}$(***Int***) → ***Int↓cDcmp*** commutes with the projections (but does not preserve cartesian arrows).

Theorem 5.1. *The inclusion functor* Ar$^{\mathrm{s}}$(***Int***) ↪ ***Int↓cDcmp*** *has a right adjoint*

$$I : \mathbf{Int{\downarrow}cDcmp} \to \mathrm{Ar^s}(\mathbf{Int}),$$

which takes cartesian arrows to cartesian arrows.

Proof. We have already checked, in the proof of 4.2, that the conditions of the general Theorem 1.13 are satisfied by the adjunction $i^* \dashv u^*$ and the stretched-cartesian factorisation system on ***cFD***. It remains to restrict this adjunction to the full subcategory ***aInt*** ⊂ ***cFD***.   □

Note that over an interval $\underline{A}$, the adjunction restricts to the adjunction of 3.14 as follows:

$$
\begin{array}{ccc}
\mathbf{Int}^{\mathrm{s}}_{\underline{A}/} & \underset{I}{\overset{}{\rightleftarrows}} & \mathbf{cDcmp}_{\underline{A}/} \\
{\scriptstyle \simeq}\Big\downarrow & & L\Big\upharpoonleft\Big\downharpoonright R \\
\mathbf{aInt}_{A/} & \underset{w}{\overset{v}{\rightleftarrows}} & \mathbf{cFD}_{A/}
\end{array}
$$

We now restrict these cartesian fibrations further to $\triangle \subset$ ***Int***. We call the coreflection I, as it is the factorisation-interval construction:

$$
\begin{array}{ccc}
\mathcal{U} = \mathrm{Ar^s}(\mathbf{Int})_{|\triangle} & \underset{I}{\overset{}{\rightleftarrows}} & \triangle{\downarrow}\mathbf{cDcmp} \\
 {\scriptstyle \mathrm{dom}}\searrow & & \swarrow {\scriptstyle \mathrm{dom}} \\
 & \triangle &
\end{array}
$$

The coreflection

$$I : \triangle{\downarrow}\mathbf{cDcmp} \to \mathcal{U}$$

is a morphism of cartesian fibrations over $\triangle$ (i.e. preserves cartesian arrows). Hence it induces a morphism of right fibrations $I : (\triangle{\downarrow}\mathbf{cDcmp})^{\mathrm{cart}} \to \mathcal{U}^{\mathrm{cart}}$.

Theorem 5.2. *The morphism of right fibrations*

$$I : (\triangle{\downarrow}\mathbf{cDcmp})^{\mathrm{cart}} \to \mathcal{U}^{\mathrm{cart}}$$

is CULF.

Proof. We need to establish that, for $g : \Delta[k] \to \Delta[1]$ the unique active map in degree k, the following square is a pullback:

$$
\begin{array}{ccc}
\boldsymbol{cDcmp}_{\Delta[k]/}{}^{\mathrm{eq}} & \xrightarrow{\;\mathrm{pre}.g\;} & \boldsymbol{cDcmp}_{\Delta[1]/}{}^{\mathrm{eq}} \\
{\scriptstyle I_k} \downarrow & & \downarrow {\scriptstyle I_1} \\
\boldsymbol{Int}^{\mathrm{s}}_{\Delta[k]/}{}^{\mathrm{eq}} & \xrightarrow[\mathrm{pre}.g]{} & \boldsymbol{Int}^{\mathrm{s}}_{\Delta[1]/}{}^{\mathrm{eq}}.
\end{array}
$$

Here the functors I_1 and I_k are the coreflections of Theorem 5.1. We compute the fibres of the horizontal maps over a point $a : \Delta[1] \to X$. For the first row, the fibre is

$$\mathrm{Map}_{\boldsymbol{cDcmp}_{\Delta[1]/}}(g, a).$$

For the second row, the fibre is

$$\mathrm{Map}_{\boldsymbol{Int}^{\mathrm{s}}_{\Delta[1]/}}(g, I_1(a)).$$

But these two spaces are equivalent by the adjunction of Theorem 5.1. $\quad\square$

Inside $\mathbb{\Delta}{\downarrow}\boldsymbol{cDcmp}$, we have the fibre over X, for the codomain fibration (which is a cocartesian fibration). This fibre is just $\mathbb{\Delta}_{/X}$, the Grothendieck construction of the presheaf X. This fibre clearly includes into the cartesian part of $\mathbb{\Delta}{\downarrow}\boldsymbol{cDcmp}$.

Lemma 5.3. *The associated morphism of right fibrations*

$$\mathbb{\Delta}_{/X} \to (\mathbb{\Delta}{\downarrow}\boldsymbol{cDcmp})^{\mathrm{cart}}$$

is CULF.

Proof. For $g : \Delta[k] \to \Delta[1]$ the unique active map in degree k, consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Map}(\Delta[k], X) & \xrightarrow{\;\mathrm{pre}.g\;} & \mathrm{Map}(\Delta[1], X) & \longrightarrow & 1 \\
\downarrow & {\scriptstyle \lrcorner} & \downarrow & {\scriptstyle \lrcorner} & \downarrow {\scriptstyle \ulcorner X \urcorner} \\
\boldsymbol{cDcmp}_{\Delta[k]/}{}^{\mathrm{eq}} & \xrightarrow[\mathrm{pre}.g]{} & \boldsymbol{cDcmp}_{\Delta[1]/}{}^{\mathrm{eq}} & \xrightarrow[\mathrm{codom}]{} & \boldsymbol{cDcmp}^{\mathrm{eq}}.
\end{array}
$$

The right-hand square and the outer rectangle are obviously pullbacks, as the fibres of coslices are the mapping spaces. Hence the left-hand square is a pullback, which is precisely to say that the vertical map is CULF. $\quad\square$

So altogether we have CULF map

$$\mathbb{A}_{/X} \to (\mathbb{A}{\downarrow}\boldsymbol{cDcmp})^{\mathrm{cart}} \to \mathcal{U}^{\mathrm{cart}},$$

or, by straightening, a CULF map of complete decomposition spaces

$$I : X \to U,$$

the *classifying map*. It takes a k-simplex in X to a k-subdivided interval, as already detailed in Section 3.

The following conjecture expresses the idea that U should be terminal in the ∞-category of complete decomposition spaces and CULF maps, but since U is large this cannot literally be true, and we have to formulate it slightly differently.

5.4. Conjecture. *U is the universal complete decomposition space for CULF maps. That is, for each (legitimate) complete decomposition space X, the space $\mathrm{Map}_{\boldsymbol{cDcmp}^{\mathrm{culf}}}(X, U)$ is contractible.*

At the moment we are only able to prove the following weaker statement.

Theorem 5.5. *For each (legitimate) complete decomposition space X, the space $\mathrm{Map}_{\boldsymbol{cDcmp}^{\mathrm{culf}}}(X, U)$ is connected.*

Proof. Suppose $J : X \to U$ and $J' : X \to U$ are two CULF functors. As in the proof of Theorem 5.2, CULFness is equivalent to saying that we have a pullback

$$
\begin{array}{ccc}
\mathrm{Map}_{\boldsymbol{cDcmp}}(\Delta[k], X) & \xrightarrow{\mathrm{pre}.g} & \mathrm{Map}_{\boldsymbol{cDcmp}}(\Delta[1], X) \\
{\scriptstyle J_k}\downarrow\ \ \ \ \ \ \lrcorner & & \ \ \ \ \downarrow{\scriptstyle J_1} \\
\boldsymbol{Int}^{\mathrm{s}}_{\Delta[k]/}{}^{\mathrm{eq}} & \xrightarrow[\mathrm{pre}.g]{} & \boldsymbol{Int}^{\mathrm{s}}_{\Delta[1]/}{}^{\mathrm{eq}}.
\end{array}
$$

We therefore have equivalences between the fibres over a point $a : \Delta[1] \to X$:

$$\mathrm{Map}_{\boldsymbol{cDcmp}_{\Delta[1]/}}(g, a) \simeq \mathrm{Map}_{\boldsymbol{Int}^{\mathrm{s}}_{\Delta[1]/}}(g, J_1(a)).$$

But the second space is equivalent to $\mathrm{Map}_{\boldsymbol{Int}^{\mathrm{s}}}(\Delta[k], J_1(a))$. Since these equivalences hold also for J', we get

$$\mathrm{Map}_{\boldsymbol{Int}^{\mathrm{s}}}(\Delta[k], J_1(a)) \simeq \mathrm{Map}_{\boldsymbol{Int}^{\mathrm{s}}}(\Delta[k], J'_1(a)),$$

naturally in k. This is to say that $J_1(a)$ and $J'_1(a)$ are levelwise equivalent simplicial spaces. But a CULF map is determined by its 1-component, so J and J' are equivalent

in the functor ∞-category. In particular, every object in $\mathrm{Map}^{\mathrm{culf}}(X, U)$ is equivalent to the canonical I constructed in the previous theorems. $\quad\square$

5.6. Size issues and cardinal bounds. We have observed that the decomposition space of intervals is large, in the sense that it takes values in the very large ∞-category of large ∞-groupoids. This size issue prevents U from being a terminal object in the ∞-category of decomposition spaces and CULF maps.

A more refined analysis of the situation is possible by standard techniques, by imposing cardinal bounds, as we briefly explain. For κ a regular uncountable cardinal, say that a simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is κ-*bounded*, when for each $n \in \mathbb{\Delta}$ the space X_n is κ-compact. In other words, X takes values in the (essentially small) ∞-category $\mathcal{S}^\kappa$ of κ-compact ∞-groupoids. Hence the ∞-category of κ-bounded simplicial spaces is essentially small. The attribute κ-bounded now also applies to decomposition spaces and intervals. Hence the ∞-categories of κ-bounded decomposition spaces and κ-bounded intervals are essentially small. Carrying the κ-bound through all the constructions, we see that there is an essentially small ∞-category U_1 of κ-bounded intervals, and a legitimate presheaf $U^\kappa : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ of κ-bounded intervals.

It is clear that if X is a κ-bounded decomposition space, then all its intervals are κ-bounded too. It follows that if Conjecture 5.4 is true then it is also true that U^κ, the (legitimate) decomposition space of all κ-bounded intervals, is universal for κ-bounded decomposition spaces, in the sense that for any κ-bounded decomposition space X, the space $\mathrm{Map}_{\boldsymbol{cDcmp}^{\mathrm{culf}}}(X, U^\kappa)$ is contractible.

6. Möbius intervals and the universal Möbius function

We finally impose the Möbius condition.

6.1. Nondegeneracy. Recall from [9, 2.12] that for a complete decomposition space X we have

$$\vec{X}_r \subset X_r$$

the full subgroupoid of r-simplices none of whose principal edges are degenerate. These can also be described as the full subgroupoid

$$\vec{X}_r \simeq \mathrm{Map}_{\mathrm{nondegen}}(\Delta[r], X) \subset \mathrm{Map}(\Delta[r], X) \simeq X_r$$

consisting of the nondegenerate maps.

Now assume that A is an interval. Inside

$$\mathrm{Map}_{\mathrm{nondegen}}(\Delta[r], A) \simeq \vec{A}_r$$

we can further require the maps to be stretched. It is clear that this corresponds to considering only nondegenerate simplices whose longest edge is the longest edge $a \in A_1$:

Lemma 6.2.

$$\mathrm{Map}_{\mathrm{stretched+nondegen}}(\Delta[r], A) \simeq (\vec{A}_r)_a.$$

6.3. Nondegeneracy in U. In the case of $U : \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathcal{S}}$, it is easy to describe the spaces $\vec{U}_r$. They consist of stretched maps $\Delta[r] \to A$ for which none of the restrictions to principal edges $\Delta[1] \to A$ are degenerate. In particular we can describe the fibre over a given interval A (in analogy with 4.7):

Lemma 6.4. *We have a pullback square*

$$
\begin{array}{ccc}
(\vec{A}_r)_a & \longrightarrow & \vec{U}_r \\
\downarrow & & \downarrow \\
* & \xrightarrow[\ulcorner A \urcorner]{} & U_1.
\end{array}
$$

6.5. Finiteness conditions and Möbius intervals. Recall from [9] that a decomposition space X is called *locally finite* when X_1 is locally finite, and both $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps. Recall also that a complete decomposition space X is called of locally finite length, or just *tight*, when for each $a \in X_1$, there is an upper bound on the dimension of simplices with long edge a. Recall finally that a complete decomposition space is called Möbius when it is locally finite and of locally finite length (i.e. tight). The Möbius condition can also be formulated by saying that X_1 is locally finite and the 'long-edge' map

$$\sum \vec{X}_r \to X_1$$

is finite.

A *Möbius interval* is an interval which is Möbius as a decomposition space.

Proposition 6.6. *Any Möbius interval is a Rezk complete Segal space.*

Proof. Just by being an interval it is a Segal space (by 2.18). Now the filtration condition implies the Rezk condition by [9, Corollary 8.7]. □

Lemma 6.7. *If X is a tight decomposition space, then for each $a \in X_1$, the interval $I(a)$ is a tight decomposition space.*

Proof. We have a CULF map $I(a) \to X$, and anything CULF over tight is again tight (see [9, Proposition 6.5]). □

Lemma 6.8. *If X is a locally finite decomposition space then for each $a \in X_1$, the interval $I(a)$ is a locally finite decomposition space.*

Proof. The morphism of decomposition spaces $I(a) \to X$ was constructed by pullback of the map $1 \overset{\ulcorner a \urcorner}{\to} X_1$ which is finite since X_1 is locally finite (see [7, Lemma 3.16]). Hence $I(a) \to X$ is a finite morphism of decomposition spaces, and therefore $I(a)$ is locally finite since X is. $\square$

From these two lemmas we get

Corollary 6.9. *If X is a Möbius decomposition space, then for each $a \in X_1$, the interval $I(a)$ is a Möbius interval.*

Proposition 6.10. *If A is a Möbius interval then for every r, the space A_r is finite.*

Proof. The squares

$$
\begin{array}{ccccc}
A_0 & \overset{s_{\top+1}}{\longrightarrow} & A_1 & \overset{s_{\perp-1}}{\longrightarrow} & A_2 \\
\downarrow & \lrcorner & \downarrow {\scriptstyle d_0} & \lrcorner & \downarrow {\scriptstyle d_1} \\
1 & \overset{s_{\top+1}}{\longrightarrow} & A_0 & \overset{s_{\perp-1}}{\longrightarrow} & A_1 \\
& & \underset{\ulcorner a \urcorner}{\longrightarrow} & &
\end{array}
$$

are pullbacks by the flanking condition 2.9 (the second is a bonus pullback, cf. 2.11). The bottom composite arrow picks out the long edge $a \in A_1$. (That the outer square is a pullback can be interpreted as saying that the 2-step factorisations of a are parametrised by their midpoint, which can be any point in A_0.) Since the active maps of A are finite (simply by the assumption that A is locally finite) in particular the map $d_1 : A_2 \to A_1$ is finite, hence the fibre A_0 is finite. The same argument works for arbitrary r, by replacing the top row by $A_r \to A_{r+1} \to A_{r+2}$, and letting the columns be d_0^r, d_0^r and d_1^r. $\square$

(This can be seen as a homotopy version of [22, Lemma 2.3].)

Corollary 6.11. *For a Möbius interval, the total space of all nondegenerate simplices $\sum_r \vec{A}_r$ is finite.*

Proof. This follows from Proposition 6.10 and the fact that a complete decomposition space is Möbius if and only if the map

$$
\sum_r d_1{}^{r-1} : \sum_r \vec{X}_r \to X_1
$$

is finite (see 6.5). $\square$

Corollary 6.12. *A Möbius interval is κ-bounded for any uncountable cardinal κ.*

6.13. The decomposition space of Möbius intervals. There is a decomposition space $M \subset U$ consisting of all Möbius intervals. In each degree, M_k is the full subgroupoid of U_k consisting of the stretched maps $\Delta[k] \to A$ for which A is Möbius. While U is large, M is a legitimate decomposition space by 6.12 and 5.6.

Theorem 6.14. *The decomposition space M is Möbius.*

Proof. We first prove that the map $\sum_r \vec{M}_r \to M_1$ is a finite map. Just check the fibre: fix a Möbius interval $A \in M_1$, with longest edge $a \in A_1$. From Lemma 6.4 we see that the fibre over A is $(\sum_r \vec{A}_r)_a = \sum_r (\vec{A}_r)_a$. But this is the fibre over $a \in A_1$ of the map $\sum_r \vec{A}_r \to A_1$, which is finite by the assumption that A is Möbius.

Next we show that the ∞-groupoid M_1 is locally finite. But M_1 is the space of Möbius intervals, a full subcategory of the space of all decomposition spaces, so we need to show, for any Möbius interval A, that $\mathrm{Eq}_{\textbf{\textit{Dcmp}}}(A)$ is finite. Now we exploit an important property of Möbius decomposition spaces, namely that they are *split* [9]: this means that face maps preserve nondegenerate simplices. The key feature of split decomposition spaces is that they are essentially semi-decomposition spaces (i.e. $\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$-spaces satisfying the decomposition-space axioms for face maps) with degeneracies freely added. More formally, restriction along $\mathbb{\Delta}_{\mathrm{inj}} \to \mathbb{\Delta}$ yields an equivalence of ∞-categories between split decomposition spaces and CULF maps, and semi-decomposition spaces and ULF maps [9, 5.20].

Since A is split, we can compute $\mathrm{Eq}_{\textbf{\textit{Dcmp}}}(A)$ inside the ∞-groupoid of split decomposition spaces, which is equivalent to the ∞-groupoid of semi-decomposition spaces. So we have reduced to computing

$$\mathrm{Map}_{\mathrm{Fun}(\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathcal{S})}(\vec{A}, \vec{A}).$$

Now we know that all $\vec{A}_k$ are finite, so the mapping space can be computed in the functor ∞-category with values in $\mathcal{F}$, the ∞-category of finite ∞-groupoids. On the other hand we also know that these ∞-groupoids are empty for k big enough, say $\vec{A}_k = \varnothing$ for $k > r$. Hence we can compute this mapping space as a functor ∞-category on the truncation $\mathbb{\Delta}_{\mathrm{inj}}^{\leq r}$. So we are finally talking about a functor ∞-category over a finite simplicial set (finite in the sense: only finitely many nondegenerate simplices), and with values in finite ∞-groupoids. So we are done by the following lemma. $\square$

Recall that $\mathcal{F}$ denotes the ∞-category of finite ∞-groupoids.

Lemma 6.15. *Let K be a finite simplicial set, and let X and Y be presheaves on K valued in finite ∞-groupoids. Then*

$$\mathrm{Map}_{\mathrm{Fun}(K^{\mathrm{op}}, \mathcal{F})}(X, Y)$$

is finite.

Proof. This mapping space may be calculated as the limit of the diagram

$$\widetilde{K} \xrightarrow{f} K \times K^{\mathrm{op}} \xrightarrow{X^{\mathrm{op}} \times Y} \mathcal{F}^{\mathrm{op}} \times \mathcal{F} \xrightarrow{\mathrm{Map}} \mathcal{S}.$$

See for example [13, Proposition 2.3] for a proof. Here $\widetilde{K}$ is the edgewise subdivision of K, introduced in [27, Appendix 1] as follows:

$$\widetilde{K}_n = K_{2n+1}, \qquad \widetilde{d}_i = d_i d_{2n+1-i}, \quad \widetilde{s}_i = s_i s_{2n+1-i},$$

and $f : \widetilde{K} \to K \times K^{\mathrm{op}}$ is defined by $(d_{n+1}, d_0)^{n+1} : K_{2n+1} \to K_n \times K_n$. Now $\widetilde{K}$ is also finite: for each nondegenerate simplex k of K, only a finite number of the degeneracies $s_{i_j} \ldots s_{i_1} k$ will be nondegenerate in $\widetilde{K}$. Furthermore, mapping spaces between finite ∞-groupoids are again finite, since $\mathcal{F}$ is cartesian closed (see [7, Proposition 3.17]). Thus the mapping space in question can be computed as a finite limit of finite ∞-groupoids, so it is again finite (see [7, Proposition 3.9]). $\square$

Proposition 6.16. *Let X be a decomposition space with locally finite X_1. Then the following are equivalent.*

(1) *X is Möbius.*
(2) *All the intervals in X are Möbius.*
(3) *Its classifying map factors through $M \subset U$.*

Proof. If the classifying map factors through $X \to M$, then X is CULF over a Möbius space, hence is itself tight (by [9, Proposition 6.5]), and has finite active maps. Since we have assumed X_1 locally finite, altogether X is Möbius. We already showed (6.9) that if X is Möbius then so are all its intervals. Finally if all the intervals are Möbius, then clearly the classifying map factors through M. $\square$

Remark 6.17. For 1-categories, Lawvere and Menni [22] show that a category is Möbius if and only if all its intervals are Möbius. This is not quite true in our setting: even if all the intervals of X are Möbius, and in particular finite, there is no guarantee that X_1 is locally finite.

6.18. Conjecture. The decomposition space M is terminal in the ∞-category of Möbius decomposition spaces and CULF maps.

This would follow from Conjecture 5.4, but could be strictly weaker.

6.19. Möbius functions. Recall from [9] that for a complete decomposition space X, for each $k \geq 0$, we have the linear functor Φ_k defined by the span $X_1 \leftarrow \vec{X}_k \to 1$, and that these assemble into the Möbius function, namely the formal difference

$$\mu = \Phi_{\text{even}} - \Phi_{\text{odd}},$$

which is convolution inverse to the zeta functor ζ given by the span $X_1 \leftarrow X_1 \rightarrow 1$. Since we cannot directly make sense of the minus sign, the actual Möbius inversion formula is expressed as a canonical equivalence of ∞-groupoids

$$\zeta * \Phi_{\text{even}} \simeq \varepsilon + \zeta * \Phi_{\text{odd}}.$$

When furthermore X is a Möbius decomposition space, then this equivalence admits a cardinality (see [9, Theorem 8.9]), which is the Möbius inversion formula in $\mathbb{Q}$-vector spaces (where the minus sign *can* be interpreted).

6.20. The universal Möbius function. The decomposition space U of all intervals is complete, hence it has Möbius inversion at the objective level as just described. Note that the map $m : \vec{U}_k \rightarrow U_1$ in $\widehat{\mathcal{S}}$, that defines Φ_k, has fibres in $\mathcal{S}$ by Lemma 4.7. Now it is a general fact that for a CULF map $f : X \rightarrow Y$ between complete decomposition spaces, we have $f^*\Phi_k = \Phi_k$ (see [9, 3.9]). Since every complete decomposition space X has a canonical CULF map to U, it follows that the Möbius function of X is induced from that of U. The latter can therefore be called the *universal Möbius function.*

The same reasoning works in the Möbius situation, and implies the existence of a universal Möbius function numerically. Namely, since M is Möbius, its Möbius inversion formula admits a cardinality.

Theorem 6.21. *In the incidence algebra* $\mathbb{Q}^{\pi_0 M}$*, the zeta function* $|\zeta| : \pi_0 M \rightarrow \mathbb{Q}$ *is invertible under convolution, and its inverse is the universal Möbius function*

$$|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}| \,.$$

The Möbius function in the (numerical) incidence algebra of any Möbius decomposition space is induced from this universal Möbius function via the classifying map.

6.22. Comparison with Lawvere–Menni. The idea of a universal Hopf algebra of Möbius intervals is due to Lawvere, and an objective construction of it was first given by Lawvere–Menni [22]. We comment on the differences between their setting and approach and ours.

Our setting is the symmetric monoidal ∞-category $(\boldsymbol{LIN}, \otimes, \mathcal{S})$ whose objects are slices of $\mathcal{S}$. The slice $\mathcal{S}_{/X}$ is the homotopy-sum completion of the ∞-groupoid X. Our coalgebras live in $\boldsymbol{LIN}$, and our convolution algebras live in the linear dual, whose objects are presheaf categories $\mathcal{S}^X$ (also the homotopy-sum completion of the ∞-groupoid X). This means our coefficients are ∞-groupoids. (To be precise, finiteness conditions should be imposed, as we do. For all details, see [7].) The incidence coalgebra of Möbius intervals is thus the slice $\mathcal{S}_{/U_1}$, where U_1 is the ∞-groupoid of Möbius intervals. To arrive at ordinary algebra, we take just take homotopy cardinality. Our 'objectification' is thus the most simple-minded, replacing numbers by sets, groupoids, ∞-groupoids.

Lawvere and Menni do not go in the homotopy direction of ∞-groupoids, but consider instead a somewhat more subtle objectification which involves a level of non-invertible arrows. Their setting is the symmetric monoidal 2-category (***EXT***, $\otimes$, ***Set***) of extensive categories, i.e. categories $\mathcal{E}$ for which the canonical functor $\mathcal{E}/(A + B) \to \mathcal{E}/A \times \mathcal{E}/B$ is an equivalence [2]. Their objective coalgebra is the extensive category Fam(***sMöI***), the finite-sum completion of the category of Möbius intervals and their stretched maps (in our terminology), and the convolution algebra is the extensive dual, ***Cat***(***sMöI***, ***Set***). To arrive at ordinary algebra, they apply the Burnside-rig construction for extensive categories, which amounts to taking isomorphism classes. To bypass the 2-categorical hassle of (***EXT***, $\otimes$, ***Set***), they actually work with *extensive procomonoids* rather than comonoids. In the case at hand, this means a functor $\Delta : $ ***sMöI*** $\to$ Fam(***sMöI*** $\times$ ***sMöI***). Here the Fam on the right is what allows to write formal sums. This induces the monoidal structure on ***Cat***(***sMöI***, ***Set***) by Day convolution.

An important thing to note in the Lawvere–Menni set-up is that ***sMöI*** is not just a groupoid, and that their construction is therefore functorial also in some maps that are not invertible, namely the stretched interval maps. The full significance of this extra functoriality is not clear to us. It is invisible at the algebraic level. Regarding the universal property, note that the Möbius categories for which it is supposed to be universal do *not* have non-invertible interval maps: the varying incidence coalgebras are of the form Fam(X_1), where X_1 is the set of arrows of a Möbius category, and in particular is discrete.

The non-invertible aspect is only implicit in our construction. Namely, the universal decomposition space U was constructed by taking the right fibration associated to $\mathcal{U} \to \triangle$, which in turn involves the stretched maps. The reason for discarding these maps was just to get a decomposition *space*, not a simplicial ∞-category, so as to be able to take cardinality and relate to classical theory. We leave the development of a theory of *decomposition categories* for another occasion.

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