Lax Colimits and Free Fibrations in ∞-Categories

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Abstract. We define and discuss lax and weighted colimits of diagrams in ∞-categories and show that the coCartesian fibration corresponding to a functor is given by its lax colimit. A key ingredient, of independent interest, is a simple characterization of the free Cartesian fibration on a functor of ∞-categories. As an application of these results, we prove that 2-representable functors are preserved under exponentiation, and also that the total space of a presentable Cartesian fibration between is presentable, generalizing a theorem of Makkai and Paré to the ∞-categories setting. Lastly, in an appendix, we observe that pseudofunctors between (2,1)-categories give rise to functors between ∞-categories via the Duskin nerve. setting and the Duskin nerve.

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Contents

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1. Introduction

In the context of ordinary category theory, Grothendieck's theory of fibrations [Gro63] can be used to give an alternative description of functors to the category Cat of categories. This has been useful, for example, in the theory of stacks in algebraic geometry, as the fibration setup is usually more flexible. When working with ∞-categories, however, the analogous notion of *Cartesian* fibrations is far more important: since defining a functor to the ∞-category Cat_∞ of ∞-categories requires specifying an infinite amount of coherence data, it is in general not feasible to "write down" definitions of functors, so that manipulating Cartesian fibrations is often the only reasonable way to define key functors.

For ordinary categories, the *Grothendieck construction* gives a simple description of the fibration classified by a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathrm{Cat}$; this can also be described formally as a certain weighted colimit, namely the lax colimit of the functor F. For ∞-categories, on the other hand, the equivalence between Cartesian fibrations and functors has been proved by Lurie using the *straightening functor*, a certain left Quillen functor between model categories. This leaves the corresponding right adjoint, the *unstraightening* functor, quite inexplicit.

One of our main goals in this paper is to show that Lurie's unstraightening functor is a model for the ∞-categorical analogue of the Grothendieck construction. More precisely, we introduce ∞-categorical versions of lax and oplax limits and colimits and prove the following:

Theorem 1.1.

(i) *Suppose $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ is a functor of ∞-categories, and $\mathcal{E} \to \mathcal{C}$ is a coCartesian fibration classified by F. Then $\mathcal{E}$ is the oplax colimit of the functor F.*

(ii) *Suppose $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a functor of ∞-categories, and $\mathcal{E} \to \mathcal{C}$ is a Cartesian fibration classified by F. Then $\mathcal{E}$ is the lax colimit of the functor F.*

To prove this we make use of an explicit description of the free Cartesian fibration on an arbitary functor of ∞-categories. More precisely, the ∞-category $\mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}$ of Cartesian fibrations over $\mathcal{C}$ is a subcategory of the slice ∞-category $\mathrm{Cat}_{\infty/\mathcal{C}}$, and we show that the inclusion admits a left adjoint given by a simple formula:

Theorem 1.2. *Let $\mathcal{C}$ be an ∞-category. For $p\colon \mathcal{E} \to \mathcal{C}$ any functor of ∞-categories, let $F(p)$ denote the map $\mathcal{E} \times_{\mathcal{C}^{\{1\}}} \mathcal{C}^{\Delta^1} \to \mathcal{C}^{\{0\}}$ (i.e. the pullback is along the map $\mathcal{C}^{\Delta^1} \to \mathcal{C}$ given by evaluation at $1 \in \Delta^1$ and the projection is induced by evaluation at 0). Then F defines a functor $\mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}$, which is left adjoint to the forgetful functor $\mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$.*

In the special case where $p\colon \mathcal{E} \to \mathcal{C}$ is a Cartesian fibration and $\mathcal{C}$ is an ∞-category equipped with a "mapping ∞-category" functor $\mathrm{MAP}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$

Cat_∞, such as is the case when $\mathcal{C}$ is the underlying ∞-category of an $(\infty, 2)$-category, then it is natural to ask when p is classified by a functor of the form $\mathrm{MAP}_{\mathcal{C}}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ for some object X of $\mathcal{C}$. We say that p is *2-representable* when this is the case. As an application of our theorems, we show that, if $p\colon \mathcal{E} \to \mathcal{C}$ is a 2-representable Cartesian fibration such that the mapping ∞-category functor $\mathrm{MAP}_{\mathcal{C}}$ is tensored and cotensored over Cat_∞, and $\mathcal{D}$ is any ∞-category, then the exponential $q\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is itself 2-representable. This is relevant in the description of the functoriality of twisted cohomology theories as discussed in joint work of the third author with U. Bunke [BN14]. More precisely it describes a converse to the construction in Section 3 and Appendix A of this paper.

The third main result of this paper provides a useful extension of the theory of presentable ∞-categories in the context of Cartesian fibrations, generalizing a theorem of Makkai and Paré [MP89] to the ∞-categorical context. More precisely, we show:

Theorem 1.3. *Suppose $p\colon \mathcal{E} \to \mathcal{C}$ is a Cartesian and coCartesian fibration such that $\mathcal{C}$ is presentable, the fibres $\mathcal{E}_x$ are presentable for all $x \in \mathcal{C}$, and the classifying functor $F\colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ preserves κ-filtered limits for some regular cardinal κ. Then the ∞-category $\mathcal{E}$ is presentable, and the projection p is an accessible functor (i.e. it preserves λ-filtered colimits for some sufficiently large cardinal λ).*

While the theory of *accessible* and *presentable* categories is already an important part of ordinary category theory, when working with ∞-categories the analogous notions turn out to be indispensable. Whereas, for example, it is often possible to give an explicit construction of colimits in an ordinary category, when working with ∞-categories we often have to conclude that colimits exist by applying general results on presentable ∞-categories. Similarly, while for ordinary categories one can frequently just write down an adjoint to a given functor, for ∞-categories an appeal to the adjoint functor theorem, which is most naturally considered in the presentable context, is often unavoidable. It is thus very useful to know that various ways of constructing ∞-categories give accessible or presentable ones; many such results are proved in [Lur09a, §5], and our result adds to these by giving a criterion for the source of a Cartesian fibration to be presentable.

1.1. Overview. In §2 we briefly review the definitions of twisted arrow ∞-categories and ∞-categorical ends and coends, and use these to define weighted (co)limits. Then in §3 we prove our main result for coCartesian fibrations over a simplex, using the *mapping simplex* defined in [Lur09a, §3.2.2]. Before we extend this result to general coCartesian fibrations we devote three sections to preliminary results: in §4 we give a description of the *free Cartesian fibration*, i.e. the left adjoint to the forgetful functor from Cartesian fibrations over $\mathcal{C}$ to the slice ∞-category $\mathrm{Cat}_{\infty/\mathcal{C}}$; in §5 we prove that the space of natural transformations between two functors is given by an end (a result first proved by

Glasman [Gla16, Proposition 2.3.]), and in §6 we prove that the straightening equivalence extends to an equivalence of the natural enrichments in Cat_∞ of the two ∞-categories involved. §7 then contains the proof of our main result: Cartesian and coCartesian fibrations are given by weighted colimits of the classifying functors. In §8 we give a simple application of our results to functors that are representable via an enrichment in Cat_∞, and in §9 we apply them to identify the functor classifying a certain simple Cartesian fibration; this is a key step in our proof in §10 that the source of a presentable fibration is presentable. Finally, in appendix A we use Duskin's nerve for strict (2,1)-categories to check that the pseudonaturality of the unstraightening functors on the level of model categories implies that they are natural on the level of ∞-categories.

1.2. Notation. Much of this paper is based on work of Lurie in [Lur09a, Lur14]; we have generally kept his notation and terminology. In particular, by an ∞-*category* we mean an $(\infty, 1)$-category or more specifically a quasicategory. We also use the following conventions, some of which differ from those of Lurie:

- Generic categories are generally denoted by single capital bold-face letters ($\mathbf{A}, \mathbf{B}, \mathbf{C}$) and generic ∞-categories by single caligraphic letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}$). Specific categories and ∞-categories both get names in the normal text font.
- If $\mathcal{C}$ is an ∞-category, we write $\iota\mathcal{C}$ for the *interior* or *underlying space* of $\mathcal{C}$, i.e. the largest subspace of $\mathcal{C}$ that is a Kan complex.
- If $f\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $g\colon \mathcal{D} \to \mathcal{C}$, we will refer to the adjunction as $f \dashv g$.
- We write Pr^{L} for the ∞-category of presentable ∞-categories and functors that are left adjoints, i.e. colimit-preserving functors, and Pr^{R} for the ∞-category of presentable ∞-categories and functors that are right adjoints, i.e. accessible functors that preserve all small limits.
- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, we will denote the ∞-category of functors $\mathcal{C} \to \mathcal{D}$ by both $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathcal{D}^{\mathcal{C}}$.
- If S is a simplicial set, we write
$$\mathrm{St}_S^+ : (\mathrm{Set}_\Delta^+)_{/S^\sharp} \rightleftarrows \mathrm{Fun}(\mathfrak{C}(S)^{\mathrm{op}}, \mathrm{Set}_\Delta^+) : \mathrm{Un}_S^+$$
for the marked (un)straightening Quillen equivalence, as defined in [Lur09a, §3.2].
- We write $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ for the subcategory of $\mathrm{Cat}_{\infty/\mathcal{C}}$ consisting of Cartesian fibrations over $\mathcal{C}$, with morphisms the functors that preserve Cartesian edges, $\mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\text{–},\text{–})$ for the mapping spaces in $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$, and $\mathrm{Fun}_{\mathcal{C}}^{\mathrm{cart}}(\text{–},\text{–})$ for the ∞-category of functors that preserve Cartesian edges, defined as a full subcategory of the ∞-category $\mathrm{Fun}_{\mathcal{C}}(\text{–},\text{–})$ of functors over $\mathcal{C}$. Similarly, we write $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cocart}}$ for the ∞-category of coCartesian fibrations over $\mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}^{\mathrm{cocart}}(\text{–},\text{–})$ for the mapping spaces in $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cocart}}$, and $\mathrm{Fun}_{\mathcal{C}}^{\mathrm{cocart}}(\text{–},\text{–})$ for the full subcategory of $\mathrm{Fun}_{\mathcal{C}}(\text{–},\text{–})$ spanned by the functors that preserve coCartesian edges.

- If $\mathcal{C}$ is an ∞-category, we write
$$\mathrm{St}_{\mathcal{C}}\colon \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \rightleftarrows \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)\colon \mathrm{Un}_{\mathcal{C}}$$
for the adjoint equivalence of ∞-categories induced by the (un)straightening Quillen equivalence via [Lur09a, Proposition 5.2.4.6].
- If S is a simplicial set, we write
$$\mathrm{St}_S^{+,\mathrm{co}} : (\mathrm{Set}_\Delta^+)_{/S^\sharp} \rightleftarrows \mathrm{Fun}(\mathfrak{C}(S), \mathrm{Set}_\Delta^+) : \mathrm{Un}_S^{+,\mathrm{co}}$$
for the coCartesian marked (un)straightening Quillen equivalence, given by $\mathrm{St}_S^{+,\mathrm{co}}(X) := (\mathrm{St}^+_{S^{\mathrm{op}}}(X^{\mathrm{op}}))^{\mathrm{op}}$.
- If $\mathcal{C}$ is an ∞-category, we write
$$\mathrm{St}_{\mathcal{C}}^{\mathrm{co}}\colon \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cocart}} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)\colon \mathrm{Un}_{\mathcal{C}}^{\mathrm{co}}$$
for the adjoint equivalence of ∞-categories induced by the coCartesian (un)straightening Quillen equivalence.
- If $\mathcal{C}$ is an ∞-category, we denote the Yoneda embedding for $\mathcal{C}$ by
$$y_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{P}(\mathcal{C}),$$
where $\mathcal{P}(\mathcal{C})$ is the presheaf ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ with $\mathcal{S}$ the ∞-category of spaces.

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2. Twisted Arrow ∞-Categories, (Co)ends, and Weighted (Co)limits

In this section we briefly recall the definitions of twisted arrow ∞-categories and (co)ends, and then use these to give a natural definition of weighted (co)limits in the ∞-categorical setting.

Definition 2.1. Let $\epsilon\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be the functor $[n] \mapsto [n] \star [n]^{\mathrm{op}}$. The *edgewise subdivision* of a simplicial set S is the composite $\epsilon^* S = S \circ \epsilon$.

Definition 2.2. Let $\mathcal{C}$ be an ∞-category. The *twisted arrow ∞-category* $\mathrm{Tw}(\mathcal{C})$ of $\mathcal{C}$ is the simplicial set $\epsilon^*\mathcal{C}$. Thus in particular
$$\mathrm{Hom}(\Delta^n, \mathrm{Tw}(\mathcal{C})) \cong \mathrm{Hom}(\Delta^n \star (\Delta^n)^{\mathrm{op}}, \mathcal{C}).$$
The natural transformations $\Delta^\bullet, (\Delta^\bullet)^{\mathrm{op}} \to \Delta^\bullet \star (\Delta^\bullet)^{\mathrm{op}}$ induce a projection $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Remark 2.3. The twisted arrow ∞-category, which was originally introduced by Joyal, has previously been extensively used by Barwick [Bar13, Bar17] and collaborators [BGN14], and by Lurie [Lur14, §5.2.1]. By [Lur14, Proposition 5.2.1.3] the projection $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a right fibration; in particular, the simplicial set $\mathrm{Tw}(\mathcal{C})$ is an ∞-category if $\mathcal{C}$ is. The functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ classified by this right fibration is the mapping space functor $\mathrm{Map}_{\mathcal{C}}(-,-)$ by [Lur14, Proposition 5.2.1.11].

Warning 2.4. There are two possible conventions for defining the edgewise subdivision (and therefore also the twisted arrow ∞-category); we follow that of Lurie in [Lur14, §5.2.1]. Alternatively, one can define the edgewise subdivision using the functor $[n] \mapsto [n]^{\mathrm{op}} \star [n]$, in which case $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is a *left* fibration — this is the convention used in the papers of Barwick cited above.

Example 2.5. The twisted arrow category $\mathrm{Tw}([n])$ of the category $[n]$ is the partially ordered set with objects (i,j) where $0 \leq i \leq j \leq n$ and with $(i,j) \leq (i',j')$ if $i \leq i' \leq j' \leq j$.

A natural definition of (co)ends in the ∞-categorical setting is then the following.

Definition 2.6. If $F\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a functor of ∞-categories, the *coend* of F is the colimit of the composite functor

$$\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}.$$

Similarly, if $G\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then the *end* of G is the limit of the composite functor

$$\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}.$$

Remark 2.7. These ∞-categorical notions of ends and coends are also discussed in [Gla16, §2]. In the context of simplicial categories, a homotopically correct notion of coends was extensively used by Cordier and Porter [CP97]; see their paper for a discussion of the history of such definitions.

Now we can consider weighted (co)limits:

Definition 2.8. Let $\mathcal{R}$ be a presentably symmetric monoidal ∞-category, i.e. a presentable ∞-category equipped with a symmetric monoidal structure such that the tensor product preserves colimits in each variable, and let $\mathcal{M}$ be a right $\mathcal{R}$-module in Pr^{L}. Then $\mathcal{M}$ is in particular tensored and cotensored over $\mathcal{R}$, i.e. there are functors

$$(-\otimes-)\colon \mathcal{M} \times \mathcal{R} \to \mathcal{M},$$
$$(-)^{(-)}\colon \mathcal{R}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M},$$

such that for every $x \in \mathcal{R}$ the functor $-\otimes x\colon \mathcal{M} \to \mathcal{M}$ is left adjoint to $(-)^x$. Given functors $F\colon \mathcal{C} \to \mathcal{M}$ and $W\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{R}$, the *$W$-weighted colimit* $\mathrm{colim}_{\mathcal{C}}^{W} F$ of F is defined to be the coend $\mathrm{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \otimes W(-)$. Similarly, given $F\colon \mathcal{C} \to \mathcal{M}$ and $W\colon \mathcal{C} \to \mathcal{R}$, the *$W$-weighted limit* $\lim_{\mathcal{C}}^{W} F$ of F is the end $\lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} F(-)^{W(-)}$.

We are interested in the case where both $\mathcal{R}$ and $\mathcal{M}$ are the ∞-category Cat_∞ of ∞-categories, with the tensoring given by Cartesian product and the cotensoring by $\mathrm{Fun}(-,-)$. In this case there are two special weights for every ∞-category $\mathcal{C}$: we have functors $\mathcal{C}_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ and $\mathcal{C}_{-/}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending $x \in \mathcal{C}$ to $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$, respectively. Precisely, these functors are obtained by straightening the source and target projections $\mathcal{C}^{\Delta^1} \to \mathcal{C}$, which are respectively Cartesian and coCartesian. Using these functors, we can define lax and oplax (co)limits:

Definition 2.9. Suppose $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ is a functor. Then:

- The *oplax colimit* of F is the colimit of F weighted by $\mathcal{C}_{-/}$, i.e.
$$\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/}.$$
- The *lax colimit* of F is the colimit of F weighted by $(\mathcal{C}^{\mathrm{op}})_{/-}$, i.e.
$$\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times (\mathcal{C}^{\mathrm{op}})_{/-}.$$
- The *lax limit* of F is the limit of F weighted by $\mathcal{C}_{/-}$, i.e.
$$\lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}_{/-}, F(-)).$$
- The *oplax limit* of F is the limit of F weighted by $(\mathcal{C}^{\mathrm{op}})_{-/}$, i.e.
$$\lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}((\mathcal{C}^{\mathrm{op}})_{-/}, F(-)).$$

3. CoCartesian Fibrations over a Simplex

In this preliminary section we study coCartesian fibrations over the simplices Δ^n, and observe that in this case the description of a coCartesian fibration as an oplax colimit follows easily from results of Lurie in [Lur09a, §3.2]. More precisely, we will prove:

Proposition 3.1. *There is an equivalence*
$$\operatorname*{colim}_{\mathrm{Tw}([n])} \phi(-) \times [n]_{-/} \xrightarrow{\sim} \mathrm{Un}^{\mathrm{co}}_{[n]}(\phi)$$
of functors $\mathrm{Fun}([n], \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$*, natural in* $\mathbf{\Delta}^{\mathrm{op}}$.

To see this we first recall fron [Lur09a, §3.2] the definition and some features of the *mapping simplex* of a functor $\phi\colon [n] \to \mathrm{Set}_\Delta^+$ and show that its fibrant replacement is a coCartesian fibration classified the corresponding functor $\Delta^n \to \mathrm{Cat}_\infty$.

Definition 3.2. Let $\phi\colon [n] \to \mathrm{Set}_\Delta^+$ be a functor. The *mapping simplex* $M_{[n]}(\phi) \to \Delta^n$ has k-simplices given by a map $\sigma\colon [k] \to [n]$ together with a k-simplex $\Delta^k \to \phi(\sigma(0))$. In particular, an edge of $M_{[n]}(\phi)$ is given by a pair of integers $0 \leq i \leq j \leq n$ and an edge f in $\phi(i)$; let S be the set of edges of $M_{[n]}(\phi)$ where the edge f is marked. Then $M^\natural_{[n]}(\phi)$ is the marked simplicial set $(M_{[n]}(\phi), S)$. This gives a functor $M^\natural_{[n]}\colon \mathrm{Fun}([n], \mathrm{Set}_\Delta^+) \to (\mathrm{Set}_\Delta^+)_{/\Delta^n}$, pseudonatural in $\mathbf{\Delta}^{\mathrm{op}}$ (with respect to composition and pullback) — see Appendix A for a discussion of pseudonatural transformations.

Definition 3.3. Let $\phi\colon [n] \to \mathrm{Set}_\Delta$ be a functor. The *relative nerve* $N_{[n]}(\phi) \to \Delta^n$ has k-simplices given by a map $\sigma\colon [k] \to [n]$ and for every ordered subset $J \subseteq [k]$ with greatest element j, a map $\Delta^J \to \phi(\sigma(j))$ such that for $J' \subseteq J$ the diagram

$$\begin{tikzcd}
\Delta^{J'} \arrow[r] \arrow[d] & \phi(\sigma(j')) \arrow[d] \\
\Delta^{J} \arrow[r] & \phi(\sigma(j))
\end{tikzcd}$$

commutes. Given a functor $\overline{\phi}\colon [n] \to \mathrm{Set}_\Delta^+$ we define $\mathrm{N}^+_{[n]}(\overline{\phi})$ to be the marked simplicial set $(\mathrm{N}_{[n]}(\phi), M)$ where ϕ is the underlying functor $[n] \to \mathrm{Set}_\Delta$ of $\overline{\phi}$, and M is the set of edges $\Delta^1 \to \mathrm{N}_{[n]}\phi$ determined by

- a pair of integers $0 \leq i \leq j \leq n$,
- a vertex $x \in \phi(i)$,
- a vertex $y \in \phi(j)$ and an edge $\phi(i \to j)(x) \to y$ that is marked in $\overline{\phi}(j)$.

This determines a functor $\mathrm{N}^+_{[n]}\colon \mathrm{Fun}([n], \mathrm{Set}_\Delta^+) \to (\mathrm{Set}_\Delta^+)_{/\Delta^n}$, pseudonatural in $\mathbf{\Delta}^{\mathrm{op}}$.

Remark 3.4. By [Lur09a, Proposition 3.2.5.18], the functor $N^+_{[n]}$ is a right Quillen equivalence from the projective model structure on $\mathrm{Fun}([n], \mathrm{Set}_\Delta^+)$ to the coCartesian model structure on $(\mathrm{Set}_\Delta^+)_{/\Delta^n}$. In particular, if $\phi\colon [n] \to \mathrm{Set}_\Delta^+$ is a functor such that $\phi(i)$ is fibrant (i.e. is a quasicategory marked by its set of equivalences) for every i, then $N^+_{[n]}(\phi)$ is a coCartesian fibration.

Definition 3.5. There is a natural transformation $\nu_{[n]}\colon M^\natural_{[n]}(-) \to N^+_{[n]}(-)$ that sends a k-simplex $(\sigma\colon [k] \to [n], \Delta^k \to \phi(\sigma(0)))$ in $M^\natural_{[n]}(\sigma)$ to the k-simplex of $N^+_{[n]}(\phi)$ determined by the composites $\Delta^J \to \Delta^k \to \phi(\sigma(0)) \to \phi(\sigma(j))$. This is clearly pseudonatural in maps in $\mathbf{\Delta}^{\mathrm{op}}$, i.e. we have a pseudofunctor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fun}([1], \mathrm{Cat})$ that to $[n]$ assigns

$$\nu_{[n]}\colon [1] \times \mathrm{Fun}([n], \mathrm{Set}_\Delta^+) \to (\mathrm{Set}_\Delta^+)_{/\Delta^n}.$$

Proposition 3.6. *Suppose $\phi\colon [n] \to \mathrm{Set}_\Delta^+$ is fibrant. Then the natural map $\nu_{[n],\phi}\colon M^\natural_{[n]}(\phi) \to N^+_{[n]}(\phi)$ is a coCartesian equivalence.*

Proof. Since $N^+_{[n]}(\phi) \to \Delta^n$ is a coCartesian fibration by [Lur09a, Proposition 3.2.5.18], it follows from [Lur09a, Proposition 3.2.2.14] that it suffices to check that $\nu_{[n],\phi}$ is a "quasi-equivalence" in the sense of [Lur09a, Definition 3.2.2.6]. Thus we need only show that the induced map on fibres $M^\natural_{[n]}(\phi)_i \to N^+_{[n]}(\phi)_i$ is a categorical equivalence for all $i = 0, \ldots, n$. But unwinding the definitions we see that this can be identified with the identity map $\phi(i) \to \phi(i)$ (marked by the equivalences). $\square$

Let $\mathrm{Un}^{+,\mathrm{co}}_{[n]}\colon \mathrm{Fun}([n], \mathrm{Set}^{+}_{\Delta}) \to (\mathrm{Set}^{+}_{\Delta})_{/\Delta^n}$ be the coCartesian version of the marked *unstraightening functor* defined in [Lur09a, §3.2.1]. By [Lur09a, Remark 3.2.5.16] there is a natural transformation $\lambda_{[n]}\colon N^{+}_{[n]} \to \mathrm{Un}^{+,\mathrm{co}}_{[n]}$, which is a weak equivalence on fibrant objects by [Lur09a, Corollary 3.2.5.20]. Since this is also pseudonatural in $\mathbf{\Delta}^{\mathrm{op}}$, combining this with Proposition 3.6 we immediately get:

Corollary 3.7. *For every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ there is a natural transformation $\lambda_{[n]}\nu_{[n]}\colon M^{\natural}_{[n]}(-) \to \mathrm{Un}^{+,\mathrm{co}}_{[n]}(-)$, and this is pseudonatural in $[n] \in \mathbf{\Delta}^{\mathrm{op}}$. If $\phi\colon [n] \to \mathrm{Set}^{+}_{\Delta}$ is fibrant, then the map $M^{\natural}_{[n]}(\phi) \to \mathrm{Un}^{+,\mathrm{co}}_{[n]}(\phi)$ is a coCartesian equivalence.*

It is immediate from the definition that $M^{\natural}_{[n]}(\phi)$ is the pushout

$$\begin{tikzcd}
\phi(0)^{\natural} \times (\Delta^{\{1,\ldots,n\}})^{\sharp} \arrow[r, hook] \arrow[d] & \phi(0)^{\natural} \times (\Delta^{n})^{\sharp} \arrow[d] \\
M^{\natural}_{[n-1]}(\phi|_{\{1,\ldots,n\}}) \arrow[r, hook] & M^{\natural}_{[n]}(\phi).
\end{tikzcd}$$

Moreover, since all objects are cofibrant in the model structure on marked simplicial sets and the top horizontal map is a cofibration, this is a homotopy pushout. Combining this with Corollary 3.7, we get the following:

Lemma 3.8. *Suppose $F\colon [n] \to \mathrm{Cat}_{\infty}$ is a functor, and that $\mathcal{E} \to \Delta^n$ is the corresponding coCartesian fibration. Let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ along the inclusion $\Delta^{\{1,\ldots,n\}} \hookrightarrow \Delta^n$. Then there is a pushout square*

$$\begin{tikzcd}
F(0) \times \Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[d] & F(0) \times \Delta^{n} \arrow[d] \\
\mathcal{E}' \arrow[r] & \mathcal{E}
\end{tikzcd}$$

in Cat_{∞}.

Unwinding the definition, we see that $M^{\natural}_{[n]}(\phi)$ is the colimit of the diagram

$$\begin{tikzcd}
 & \phi(0) \times (\Delta^{n})^{\sharp} \\
\phi(0) \times (\Delta^{\{1,\ldots,n\}})^{\sharp} \arrow[ur] \arrow[dr] & \\
 & \phi(1) \times (\Delta^{\{1,\ldots,n\}})^{\sharp} \\
\phi(1) \times (\Delta^{\{2,\ldots,n\}})^{\sharp} \arrow[ur] \arrow[dr] & \\
 & \phi(2) \times (\Delta^{\{2,\ldots,n\}})^{\sharp} \\
 & \vdots \\
 & \phi(n-1) \times (\Delta^{\{n-1,n\}})^{\sharp} \\
\phi(n-1) \times (\Delta^{\{n\}})^{\sharp} \arrow[ur] \arrow[dr] & \\
 & \phi(n) \times (\Delta^{\{n\}})^{\sharp}.
\end{tikzcd}$$

By Example 2.5 the category indexing this colimit is a cofinal subcategory of the twisted arrow category $\mathrm{Tw}([n])$ of $[n]$ — this is easy to check using [Lur09a, Corollary 4.1.3.3] since both categories are partially ordered sets. Hence we may identify $M^{\natural}_{[n]}(\phi)$ with the coend

$$\operatorname*{colim}_{\mathrm{Tw}([n])} (\phi(-) \times \mathrm{N}[n]_{-/}).$$

Moreover, since we can write this colimit as an iterated pushout along cofibrations, this is a homotopy colimit. From this we can prove Proposition 3.1 using the results of appendix A together with the following observation:

Lemma 3.9. *Let $G\colon \mathbf{C} \to \mathbf{D}$ be a right Quillen functor between model categories. Suppose $f\colon X \to \bar{X}$ and $g\colon Y \to \bar{Y}$ are weak equivalences such that $\bar{X}$ and $\bar{Y}$ are fibrant, and $G(f)$ and $G(g)$ are weak equivalences in $\mathbf{D}$. Then if $h\colon X \to Y$ is a weak equivalence, the morphism $G(h)$ is also a weak equivalence in $\mathbf{D}$.*

Proof. Choose a factorization of the composite $g \circ h\colon X \to \bar{Y}$ as a trivial cofibration $i\colon X \hookrightarrow X'$ followed by a fibration $p\colon X' \to \bar{Y}$. We then have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"] \arrow[dd, "h"'] & & \bar{X} \\ & X' \arrow[ur, dotted, "q"] \arrow[dr, "p"] & \\ Y \arrow[rr, "g"] & & \bar{Y} \end{tikzcd}$$

where the dotted arrow q exists since $\bar{X}$ is fibrant and i is a trivial cofibration, and all morphisms are weak equivalences by the 2-of-3 property. By assumption G takes f and g to weak equivalences, and as G is a right Quillen functor by Brown's Lemma it also takes q and p to weak equivalences as these are weak equivalences between fibrant objects. By the 2-of-3 property we can then conclude first that $G(i)$ is a weak equivalence and then that $G(h)$ is a weak equivalence. □

Proof of Proposition 3.1. We will prove this by applying Proposition A.30 to a relative Grothendieck fibration constructed in the same way as in Proposition A.31. The only difference is that the mapping simplex of a functor $\phi\colon [n] \to \mathrm{Set}_{\Delta}^{+}$ is not in general fibrant. We must therefore consider a larger relative subcategory of $(\mathrm{Set}_{\Delta}^{+})_{/\Delta^n}$ containing the mapping simplices of fibrant functors whose associated ∞-category is still $\mathrm{Cat}_{\infty/\Delta^n}^{\mathrm{cocart}}$.
By [Lur09a, Proposition 3.2.2.7] every mapping simplex admits a weak equivalence to a fibrant object that is preserved under pullbacks along all morphisms in $\mathbf{\Delta}$. We therefore think of $M_{[n]}^{\natural}$ and $\mathrm{Un}_{[n]}^{+,\mathrm{co}}$ as functors from fibrant objects in $\mathrm{Fun}([n], \mathrm{Set}_{\Delta}^{+})$ to objects in $(\mathrm{Set}_{\Delta}^{+})_{/\Delta^n}$ that admit a weak equivalence to a fibrant object that is preserved by pullbacks — by Lemma 3.9 all weak equivalences between such objects are preserved by pullbacks, so we still get functors of relative categories.
It remains to show that inverting the weak equivalences in this subcategory gives the same ∞-category as inverting the weak equivalences in the subcategory of fibrant objects. This follows from [BK12, 7.5], since any fibrant replacement functor gives a homotopy equivalence of relative categories. □

4. Free Fibrations

Our goal in this section is to prove that for any ∞-category $\mathcal{C}$, the forgetful functor

$$\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$$

has a left adjoint, given by the following explicit formula:

Definition 4.1. Let $\mathcal{C}$ be an ∞-category. For $p\colon \mathcal{E} \to \mathcal{C}$ any functor of ∞-categories, let $F(p)$ denote the map $\mathcal{E} \times_{\mathcal{C}^{\{1\}}} \mathcal{C}^{\Delta^1} \to \mathcal{C}^{\{0\}}$, where the pullback is along the target fibration $\mathcal{C}^{\Delta^1} \to \mathcal{C}$ given by evaluation at $1 \in \Delta^1$, and the projection $F(p)$ is induced by evaluation at 0. Then F defines a functor $\mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$.

We will call the projection $F(p)\colon \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \to \mathcal{C}$ the *free Cartesian fibration* on $p\colon \mathcal{E} \to \mathcal{C}$ — the results of this section will justify this terminology.

Example 4.2. The free Cartesian fibration on the identity $\mathcal{C} \to \mathcal{C}$ is the source fibration $F\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$, given by evaluation at $0 \in \Delta^1$.

Lemma 4.3. *The functor F factors through the subcategory* $\mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$.

Proof. By [Lur09a, Corollary 2.4.7.12] the projection $F(p) \to \mathcal{C}$ is a Cartesian fibration for any $p\colon \mathcal{E} \to \mathcal{C}$, and a morphism in $F(p)$ is Cartesian if and only if its image in $\mathcal{E}$ is an equivalence. It is thus clear that for any map $\phi\colon \mathcal{E} \to \mathcal{F}$ in $\mathrm{Cat}_{\infty/\mathcal{C}}$, the induced map $F(\phi)$ preserves Cartesian morphisms, since the diagram

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \arrow[r] \arrow[d] & \mathcal{F} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

commutes. □

Remark 4.4. If $p\colon \mathcal{E} \to \mathcal{C}$ is a functor, the objects of $F(p)$ can be identified with pairs $(e, \phi\colon c \to p(e))$ where $e \in \mathcal{E}$ and ϕ is a morphism in $\mathcal{C}$. Similarly, a morphism in $F(p)$ can be identified with the data of a morphism $\alpha\colon e' \to e$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd}
c' \arrow[r] \arrow[d] & p(e') \arrow[d] \\
c \arrow[r] & p(e).
\end{tikzcd}$$

If (e, ϕ) is an object in $F(p)$ and $\psi\colon c' \to c$ is a morphism in $\mathcal{C}$, the Cartesian morphism over ψ with target (e, ϕ) is the obvious morphism from $(e, \phi\psi)$.

Theorem 4.5. *Let $\mathcal{C}$ be an ∞-category. The functor $F\colon \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}$ is left adjoint to the forgetful functor $U\colon \mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$.*

Remark 4.6. Analogues of this result in the setting of ordinary categories (as well as enriched and internal variants) can be found in [Str80] and [Web07].

Composition with the degeneracy $s_0\colon \Delta^1 \to \Delta^0$ induces a functor $\mathcal{C} \to \mathcal{C}^{\Delta^1}$ (which sends an object of $\mathcal{C}$ to the constant functor $\Delta^1 \to \mathcal{C}$ with that value). Since the composition of this with both of the evaluation maps $\mathcal{C}^{\Delta^1} \to \mathcal{C}$ is the identity, this induces a natural map $\mathcal{E} \to \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1}$ over $\mathcal{C}$, i.e. a natural transformation

$$\eta\colon \mathrm{id} \to UF$$

of functors $\mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$. We will show that this is a *unit transformation* in the sense of [Lur09a, Definition 5.2.2.7], i.e. that it induces an equivalence

$$\mathrm{Map}^{\mathrm{Cart}}_{\mathcal{C}}(F(\mathcal{E}), \mathcal{F}) \to \mathrm{Map}_{\mathcal{C}}(UF(\mathcal{E}), U(\mathcal{F})) \to \mathrm{Map}_{\mathcal{C}}(\mathcal{E}, U(\mathcal{F}))$$

for all $\mathcal{E} \to \mathcal{C}$ in $\mathrm{Cat}_{\infty/\mathcal{C}}$ and $\mathcal{F} \to \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}$.

We first check this for the objects of $\mathrm{Cat}_{\infty/\mathcal{C}}$ with source Δ^0 and Δ^1, which (in a weak sense) generate $\mathrm{Cat}_{\infty/\mathcal{C}}$ under colimits. If a map $\Delta^0 \to \mathcal{C}$ corresponds to the object $x \in \mathcal{C}$, then its image under F is the projection $\mathcal{C}_{/x} \to \mathcal{C}$. (Strictly speaking, the image is the "alternative overcategory" $\mathcal{C}^{/x}$ in the notation of [Lur09a, §4.2.1], but this is naturally weakly equivalent to $\mathcal{C}_{/x}$ by [Lur09a, Proposition 4.2.1.5].) Thus in this case we need to show the following:

LEMMA 4.7.

(i) *For every $x \in \mathcal{C}$, the map $\mathrm{Map}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}) \to \mathrm{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}) \simeq \iota\mathcal{E}_x$ is an equivalence.*

(ii) *More generally, for any $\mathcal{X} \in \mathrm{Cat}_\infty$, the map*

$$\mathrm{Map}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{C}_{/x} \times \mathcal{X}, \mathcal{E}) \to \mathrm{Map}_{\mathcal{C}}(\{x\} \times \mathcal{X}, \mathcal{E}) \simeq \mathrm{Map}(\mathcal{X}, \mathcal{E}_x)$$

is an equivalence.

Proof. The inclusion of the ∞-category of right fibrations over $\mathcal{C}$ into $\mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}$ has a right adjoint, which sends a Cartesian fibration $p \colon \mathcal{E} \to \mathcal{C}$ to its restriction to the subcategory $\mathcal{E}_{\mathrm{cart}}$ of $\mathcal{E}$ where the morphisms are the p-Cartesian morphisms. The map $\mathrm{Map}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}) \to \iota\mathcal{E}_x$ thus factors as

$$\mathrm{Map}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}_{\mathrm{cart}}) \to \iota\mathcal{E}_x,$$

where $\mathrm{Map}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}_{\mathrm{cart}})$ is the mapping space in the ∞-category of right fibrations over $\mathcal{C}$, which is modelled by the contravariant model structure on $(\mathrm{Set}_\Delta)_{/\mathcal{C}}$ constructed in [Lur09a, §2.1.4].

By [Lur09a, Proposition 4.4.4.5], the inclusion $\{x\} \to \mathcal{C}_{/x}$ is a trivial cofibration in this model category. Since this is a simplicial model category by [Lur09a, Proposition 2.1.4.8], it follows immediately that we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}_{\mathrm{cart}}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}_{\mathrm{cart}}).$$

This proves (i). To prove (ii) we simply observe that since the model category is simplicial, the product $\{x\} \times K \to \mathcal{C}_{/x} \times K$ is also a trivial cofibration for any simplicial set K. $\square$

For the case of maps $\Delta^1 \to \mathcal{C}$, the key observation is:

PROPOSITION 4.8. *If $\Delta^1 \to \mathcal{C}$ corresponds to a map $f \colon x \to y$ in $\mathcal{C}$, then the diagram*

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathcal{C}_{/x} \times \Delta^1 \arrow[d] \\ \mathcal{C}_{/y} \arrow[r] & \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \Delta^1 \end{tikzcd}$$

is a pushout square in $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$*, where the top map is induced by the inclusion* $\{0\} \hookrightarrow \Delta^1$.

Proof. Since colimits in $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ are detected fibrewise, it suffices to show that for every $c \in \mathcal{C}$, the diagram on fibres is a pushout in Cat_∞. This diagram can be identified with

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(c,x) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(c,x) \times \Delta^1 \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(c,y) \arrow[r] & \mathcal{C}_{c/} \times_{\mathcal{C}} \Delta^1. \end{tikzcd}$$

This is a pushout by Lemma 3.8, since $\mathcal{C}_{c/} \times_{\mathcal{C}} \Delta^1 \to \Delta^1$ is the left fibration corresponding to the map of spaces $\mathrm{Map}_{\mathcal{C}}(c,x) \to \mathrm{Map}_{\mathcal{C}}(c,y)$ induced by composition with f. □

Corollary 4.9. *For every map* $\sigma \colon \Delta^1 \to \mathcal{C}$ *and every Cartesian fibration* $\mathcal{E} \to \mathcal{C}$*, the map*

$$\eta_\sigma^* \colon \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \Delta^1, \mathcal{E}) \to \mathrm{Map}_{\mathcal{C}}(\Delta^1, \mathcal{E})$$

is an equivalence.

Proof. By Proposition 4.8, if the map σ corresponds to a morphism $f \colon x \to y$ in $\mathcal{C}$, we have a pullback square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \Delta^1, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}_{/y}, \mathcal{E}) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}_{/x} \times \Delta^1, \mathcal{E}) \arrow[r] & \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}_{/x}, \mathcal{E}). \end{tikzcd}$$

The map η_σ^* fits in an obvious map of commutative squares from this to the square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(\Delta^1, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(\{y\}, \mathcal{E}) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(\{x\} \times \Delta^1, \mathcal{E}) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}), \end{tikzcd}$$

where the right vertical map is given by composition with Cartesian morphisms over f. Since $\mathcal{E} \to \mathcal{C}$ is a Cartesian fibration, this is also a pullback square (this amounts to saying morphisms in $\mathcal{E}$ over f are equivalent to composites of a morphism in $\mathcal{E}_x$ with a Cartesian morphism over f). But now, by Lemma 4.7, we have a natural transformation of pullback squares that's an equivalence everywhere except the top left corner, so the map in that corner is an equivalence too. □

To complete the proof, we now only need to observe that F preserves colimits:

LEMMA 4.10. *F preserves colimits.*

Proof. Colimits in $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ are detected fibrewise, so we need to show that for every $x \in \mathcal{C}$, the functor $\mathcal{C}_{x/} \times_{\mathcal{C}} (-): \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_\infty$ preserves colimits. But $\mathcal{C}_{x/} \to \mathcal{C}$ is a flat fibration by [Lur14, Example B.3.11], so pullback along it preserves colimits as a functor $\mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}_{x/}}$ (since on the level of model categories the pullback functor is a left Quillen functor by [Lur14, Corollary B.3.15]), and the forgetful functor $\mathrm{Cat}_{\infty/\mathcal{C}_{x/}} \to \mathrm{Cat}_\infty$ also preserves colimits. □

Proof of Theorem 4.5. By Lemma 4.10 the source and target of the natural map

$$\mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(F(\mathcal{E}), \mathcal{F}) \to \mathrm{Map}_{\mathcal{C}}(\mathcal{E}, U(\mathcal{F}))$$

both take colimits in $\mathcal{E}$ to limits of spaces. Since Cat_∞ is a localization of $\mathcal{P}(\boldsymbol{\Delta})$, every object of $\mathrm{Cat}_{\infty/\mathcal{C}}$ is canonically the colimit of a diagram consisting of objects of the form $\Delta^n \to \mathcal{C}$, so it suffices to show that the map is an equivalence for such objects. But Δ^n can in turn be identified with the colimit $\Delta^1 \amalg_{\Delta^0} \cdots \amalg_{\Delta^0} \Delta^1$, so it suffices to check that the map is an equivalence when $\mathcal{E} = \Delta^0$ and Δ^1. Thus the result follows from Lemma 4.7 and Corollary 4.9. □

PROPOSITION 4.11.

(i) Suppose $X \to S$ is a map of ∞-categories and K is an ∞-category. Then there is a natural equivalence $F(K \times X) \simeq K \times F(X)$.

(ii) The unit map $X \to F(X)$ induces an equivalence of ∞-categories

$$\mathrm{Fun}_S^{\mathrm{cart}}(F(X), Y) \xrightarrow{\sim} \mathrm{Fun}_S(X, Y).$$

Proof. (i) is immediate from the definition of F. Then (ii) follows from the natural equivalence

$$\begin{aligned}\mathrm{Map}(K, \mathrm{Fun}_S(A, B)) &\simeq \mathrm{Map}_S(K \times A, B) \simeq \mathrm{Map}_S^{\mathrm{cart}}(F(K \times A), B) \\ &\simeq \mathrm{Map}_S^{\mathrm{cart}}(K \times F(A), B) \simeq \mathrm{Map}(K, \mathrm{Fun}_S^{\mathrm{cart}}(F(A), B)).\end{aligned}$$

□

5. NATURAL TRANSFORMATIONS AS AN END

It is a familiar result from ordinary category theory that for two functors $F, G: \mathbf{C} \to \mathbf{D}$ the set of natural transformations from F to G can be identified with the *end* of the functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathrm{Set}$ that sends (C, C') to $\mathrm{Hom}_{\mathbf{D}}(F(C), G(C'))$. Our goal in this section is to prove the analogous result for ∞-categories:

Proposition 5.1. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors of ∞-categories. Then the space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G)$ of natural transformations from F to G is naturally equivalent to the end of the functor*

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{(F^{\mathrm{op}},G)} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Map}_{\mathcal{D}}} \mathcal{S}.$$

A proof of this is also given in [Gla16, Proposition 2.3]; we include a slightly different proof for completeness.

Lemma 5.2. *Suppose $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a fully faithful functor of ∞-categories. Then for any ∞-category $\mathcal{X}$ the functor $\mathrm{Fun}(\mathcal{X}, \mathcal{C}_0) \to \mathrm{Fun}(\mathcal{X}, \mathcal{C})$ is also fully faithful.*

Proof. A functor $G\colon \mathcal{A} \to \mathcal{B}$ is fully faithful if and only if the commutative square of spaces

$$\begin{tikzcd}
\mathrm{Map}(\Delta^1, \mathcal{A}) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1, \mathcal{B}) \arrow[d] \\
\iota\mathcal{A}^{\times 2} \arrow[r] & \iota\mathcal{B}^{\times 2}
\end{tikzcd}$$

is Cartesian. Thus, we must show that for any $\mathcal{X}$, the square

$$\begin{tikzcd}
\mathrm{Map}(\Delta^1 \times \mathcal{X}, \mathcal{C}_0) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1 \times \mathcal{X}, \mathcal{C}) \arrow[d] \\
\mathrm{Map}(\mathcal{X}, \mathcal{C}_0)^{\times 2} \arrow[r] & \mathrm{Map}(\mathcal{X}, \mathcal{C})^{\times 2}
\end{tikzcd}$$

is Cartesian. But this is equivalent to the commutative square of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_0^{\Delta^1} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d] \\
\mathcal{C}_0^{\times 2} \arrow[r] & \mathcal{C}^{\times 2}
\end{tikzcd}$$

being Cartesian. By [Lur09a, Corollary 2.4.7.11] the vertical maps in this diagram are bifibrations in the sense of [Lur09a, Definition 2.4.7.2], so by [Lur09a, Propositions 2.4.7.6 and 2.4.7.7] to prove that this square is Cartesian it suffices to show that for all $x, y \in \mathcal{C}_0$ the induced map on fibres $(\mathcal{C}_0^{\Delta^1})_{(x,y)} \to (\mathcal{C}^{\Delta^1})_{(ix,iy)}$ is an equivalence. But this can be identified with the map $\mathrm{Map}_{\mathcal{C}_0}(x, y) \to \mathrm{Map}_{\mathcal{C}}(ix, iy)$, which is an equivalence as i is by assumption fully faithful. $\square$

Proof of Proposition 5.1. By [Lur09a, Corollary 3.3.3.4], we can identify the limit of the functor

$$\phi\colon \mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{(F^{\mathrm{op}},G)} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Map}_{\mathcal{D}}} \mathcal{S}$$

with the space of sections of the corresponding left fibration. By [Lur14, Proposition 5.2.1.11], the left fibration classified by $\mathrm{Map}_{\mathcal{D}}$ is the projection $\mathrm{Tw}(\mathcal{D})^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$, so the left fibration classified by ϕ is the pullback of this along $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$. Thus the space of sections is equivalent to the space of commutative diagrams

$$\begin{tikzcd} \mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{D})^{\mathrm{op}} \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}, \end{tikzcd}$$

i.e. the space of maps from $\mathrm{Tw}(\mathcal{C})$ to the pullback of $\mathrm{Tw}(\mathcal{D})$ in the ∞-category of left fibrations over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Using the "straightening" equivalence between this ∞-category and that of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ we can identify our limit with the space of maps from $y_{\mathcal{C}}$ to $F^* \circ y_{\mathcal{D}} \circ G$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{C}))$. Since F^* has a left adjoint $F_!$, we have an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{C}))}(y_{\mathcal{C}}, F^* y_{\mathcal{D}} \circ G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}))}(F_! y_{\mathcal{C}}, y_{\mathcal{D}} \circ G).$$

But by [Lur09a, Proposition 5.2.6.3] the functor $F_! y_{\mathcal{C}}$ is equivalent to $y_{\mathcal{D}} \circ F$, and so the limit is equivalent to $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}))}(y_{\mathcal{D}} \circ F, y_{\mathcal{D}} \circ G)$. The Yoneda embedding $y_{\mathcal{D}}$ is fully faithful by [Lur09a, Proposition 5.1.3.1], so Lemma 5.2 implies that the functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ given by composition with $y_{\mathcal{D}}$ is fully faithful, hence we have an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}))}(y_{\mathcal{C}} \circ F, y_{\mathcal{D}} \circ G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G),$$

which completes the proof. □

6. Enhanced Mapping Functors

The Yoneda embedding for ∞-categories, constructed in [Lur09a, Proposition 5.1.3.1] or [Lur14, Proposition 5.2.1.11], gives for any ∞-category $\mathcal{C}$ a mapping space functor $\mathrm{Map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. In some cases, this is the underlying functor to spaces of an interesting functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}_\infty$ — in particular, this is the case if $\mathcal{C}$ is the underlying ∞-category of an $(\infty, 2)$-category. For a definition and a comparison of different models of $(\infty, 2)$-categories see [Lur09b].

Definition 6.1. A *mapping ∞-category functor* for an ∞-category $\mathcal{C}$ is a functor

$$\mathrm{MAP}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}_\infty$$

together with an equivalence from the composite $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}_\infty \xrightarrow{\iota} \mathcal{S}$ to the mapping space functor $\mathrm{Map}_{\mathcal{C}}$.

Lemma 6.2. *Suppose $\mathcal{C}$ is an $(\infty, 2)$-category with underlying ∞-category $\mathcal{C}'$. Then $\mathcal{C}'$ has a mapping ∞-category functor that sends (C, D) to the ∞-category of maps from C to D in $\mathcal{C}$.*

Proof. This follows from the same argument as in [Lur09a, §5.1.3], using the model of $(\infty, 2)$-categories as categories enriched in marked simplicial sets, cf. [Lur09b]. $\square$

Example 6.3. The ∞-category Cat_∞ of ∞-categories has a mapping ∞-category functor

$$\mathrm{MAP}_{\mathrm{Cat}_\infty} := \mathrm{Fun},$$

defined using the construction of Cat_∞ as the coherent nerve of the simplicial category of quasicategories.

Lemma 6.4. *Suppose $\mathcal{C}$ is an ∞-category with a mapping ∞-category functor $\mathrm{MAP}_{\mathcal{C}}$. Then for any ∞-category $\mathcal{D}$ the functor ∞-category $\mathcal{C}^{\mathcal{D}}$ has a mapping ∞-category functor $\mathrm{MAP}_{\mathcal{C}^{\mathcal{D}}}$ given by the composite*

$$(\mathcal{C}^{\mathcal{D}})^{\mathrm{op}} \times \mathcal{C}^{\mathcal{D}} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}} \times \mathcal{D}, \mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \to \mathrm{Fun}(\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\lim} \mathrm{Cat}_\infty,$$

where the second functor is given by composition with the projection $\mathrm{Tw}(\mathcal{D})^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ and $\mathrm{MAP}_{\mathcal{C}}$.

Proof. We must show that the underlying functor to spaces $\iota \circ \mathrm{MAP}_{\mathcal{C}^{\mathcal{D}}}$ is equivalent to $\mathrm{Map}_{\mathcal{C}^{\mathcal{D}}}$. Since ι preserves limits (being a right adjoint), this follows immediately from Proposition 5.1. $\square$

Definition 6.5. Suppose $\mathcal{C}$ is an ∞-category with a mapping ∞-category functor $\mathrm{MAP}_{\mathcal{C}}$. We say that $\mathcal{C}$ is *tensored over* Cat_∞ if for every $C \in \mathcal{C}$ the functor $\mathrm{MAP}_{\mathcal{C}}(C, -)$ has a left adjoint $- \otimes C \colon \mathrm{Cat}_\infty \to \mathcal{C}$; in this case these adjoints determine an essentially unique functor $\otimes \colon \mathrm{Cat}_\infty \times \mathcal{C} \to \mathcal{C}$.

Example 6.6. The ∞-category Cat_∞ is obviously tensored over Cat_∞ via the Cartesian product $\times \colon \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$.

Lemma 6.7. *Suppose $\mathcal{C}$ is an ∞-category with a mapping ∞-category $\mathrm{MAP}_{\mathcal{C}}$ that is tensored over Cat_∞. Then for any ∞-category $\mathcal{D}$, the mapping ∞-category functor for $\mathcal{C}^{\mathcal{D}}$ defined in Lemma 6.4 is also tensored over Cat_∞, via the composite*

$$\mathrm{Cat}_\infty \times \mathcal{C}^{\mathcal{D}} \to \mathrm{Cat}_\infty^{\mathcal{D}} \times \mathcal{C}^{\mathcal{D}} \simeq (\mathrm{Cat}_\infty \times \mathcal{C})^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$$

where the first functor is given by composition with the functor $\mathcal{D} \to *$ and the last by composition with the tensor functor for $\mathcal{C}$.

Proof. We must show that for every functor $F \colon \mathcal{D} \to \mathcal{C}$ there is a natural equivalence

$$\mathrm{Map}_{\mathcal{C}^{\mathcal{D}}}(\mathcal{X} \otimes F, G) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{MAP}_{\mathcal{C}^{\mathcal{D}}}(F, G)).$$

By Proposition 5.1 and the definition of $\otimes$ for $\mathcal{C}^{\mathcal{D}}$, there is a natural equivalence

$$\mathrm{Map}_{\mathcal{C}^{\mathcal{D}}}(\mathcal{X} \otimes F, G) \simeq \lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}}(\mathcal{X} \otimes F(-), G(-)).$$

Now using that $\mathcal{C}$ is tensored over Cat_∞, this is naturally equivalent to

$$\lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{MAP}_{\mathcal{C}}(F(-), G(-))).$$

Moving the limit inside, this is

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{MAP}_{\mathcal{C}}(F(-), G(-))),$$

which is $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{MAP}_{\mathcal{C}^{\mathcal{D}}}(F, G))$ by definition. □

Example 6.8. For any ∞-category $\mathcal{D}$, the ∞-category $\mathrm{Cat}_\infty^{\mathcal{D}}$ is tensored over Cat_∞: $\mathcal{X} \otimes F$ is the functor $D \mapsto \mathcal{X} \times F(D)$.

In the case where $\mathcal{C}$ is the ∞-category Cat_∞ of ∞-categories, Lemma 6.4 gives a mapping ∞-category functor

$$\mathrm{Nat}_{\mathcal{D}^{\mathrm{op}}} := \mathrm{MAP}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat}_\infty)}$$

for $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat}_\infty)$, for any ∞-category $\mathcal{D}$. However, using the equivalence $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_{\infty/\mathcal{D}}^{\mathrm{cart}}$ we can construct another such functor: the space of maps from $\mathcal{E}$ to $\mathcal{E}'$ in $\mathrm{Cat}_{\infty/\mathcal{D}}^{\mathrm{cart}}$ is the underlying ∞-groupoid of the ∞-category $\mathrm{Fun}_{\mathcal{D}}^{\mathrm{cart}}(\mathcal{E}, \mathcal{E}')$, the full subcategory of $\mathrm{Fun}_{\mathcal{D}}(\mathcal{E}, \mathcal{E}')$ spanned by the functors that preserve Cartesian morphisms. We will now prove that these two functors are equivalent:

Proposition 6.9. *For every ∞-category $\mathcal{C}$ there is a natural equivalence*

$$\mathrm{Fun}_{\mathcal{C}}^{\mathrm{cart}}(\mathcal{E}, \mathcal{E}') \simeq \mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{St}_{\mathcal{C}}\mathcal{E}, \mathrm{St}_{\mathcal{C}}\mathcal{E}').$$

Proof. By the Yoneda Lemma it suffices to show that there are natural equivalences

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{Fun}_{\mathcal{C}}^{\mathrm{cart}}(\mathcal{E}, \mathcal{E}')) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{St}_{\mathcal{C}}\mathcal{E}, \mathrm{St}_{\mathcal{C}}\mathcal{E}')).$$

It is easy to see that $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{Fun}_{\mathcal{C}}^{\mathrm{cart}}(\mathcal{E}, \mathcal{E}'))$ is naturally equivalent to $\mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}}(\mathcal{X} \times \mathcal{E}, \mathcal{E}')$ — these correspond to the same components of $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{Fun}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}'))$. The equivalence $\mathrm{St}_{\mathcal{C}}$ preserves products, so this is equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(\mathrm{St}_{\mathcal{C}}(\mathcal{X} \times \mathcal{C}) \times \mathrm{St}_{\mathcal{C}}\mathcal{E}, \mathrm{St}_{\mathcal{C}}\mathcal{E}').$$

But the projection $\mathcal{X} \times \mathcal{C} \to \mathcal{C}$ corresponds to the constant functor $c^*\mathcal{X} \colon \mathcal{C}^{\mathrm{op}} \to * \to \mathrm{Cat}_\infty$ with value $\mathcal{C}$ (since the Cartesian fibration classified by this composite is precisely the pullback of $\mathcal{X} \to *$ along $\mathcal{C} \to *$). Thus there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{Fun}_{\mathcal{C}}^{\mathrm{cart}}(\mathcal{E}, \mathcal{E}')) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(c^*\mathcal{X} \times \mathrm{St}_{\mathcal{C}}\mathcal{E}, \mathrm{St}_{\mathcal{C}}\mathcal{E}').$$

But by Lemma 6.7, the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ is tensored over Cat_∞ and this is naturally equivalent to $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{X}, \mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{St}_{\mathcal{C}}\mathcal{E}, \mathrm{St}_{\mathcal{C}}\mathcal{E}'))$, as required. □

7. Cartesian and CoCartesian Fibrations as Weighted Colimits

In ordinary category theory it is a familiar fact that the Grothendieck fibration classified by a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathrm{Cat}$ can be identified with the lax colimit of F, and the Grothendieck opfibration classified by a functor $F\colon \mathbf{C} \to \mathrm{Cat}$ with the oplax colimit of F. In this section we will show that Cartesian and coCartesian fibrations admit analogous descriptions.

It is immediate from our formula for the free Cartesian fibration that the sections of a Cartesian fibration are given by the oplax limit of the corresponding functor:

Proposition 7.1. *The ∞-category of sections of the Cartesian fibration classified by F is given by the oplax limit of F. In other words, there is a natural equivalence*

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathrm{Un}_{\mathcal{C}}(F)) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}_{-/}, F(-))$$

of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$.

Proof. By Theorem 4.5 and Proposition 6.9 we have natural equivalences

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathrm{Un}_{\mathcal{C}}(F)) \simeq \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{C}}(F(\mathcal{C}), \mathrm{Un}_{\mathcal{C}}(F)) \simeq$$
$$\simeq \mathrm{Nat}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}_{-/}, F) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}_{-/}, F(-)). \square$$

Definition 7.2. Let $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be a functor, and let $\mathcal{F} \to \mathcal{C}$ be its associated coCartesian fibration. Given an ∞-category $\mathcal{X}$, write $\Phi^F_{\mathcal{X}}$ for the simplicial set over $\mathcal{C}$ with the universal property

$$\mathrm{Hom}(K \times_{\mathcal{C}} \mathcal{F}, \mathcal{X}) \cong \mathrm{Hom}_{\mathcal{C}}(K, \Phi^F_{\mathcal{X}}).$$

By [Lur09a, Corollary 3.2.2.13] the projection $\Phi^F_{\mathcal{X}} \to \mathcal{C}$ is a Cartesian fibration.

Proposition 7.3. *The Cartesian fibration $\Phi^F_{\mathcal{X}} \to \mathcal{C}$ corresponds to the functor*

$$\mathrm{Fun}(F(-), \mathcal{X})\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

Proof. We first consider the case where $\mathcal{C}$ is a simplex Δ^n. By Proposition 3.1 there are natural equivalences

$$\operatorname*{colim}_{\mathrm{Tw}([n])} \phi(-) \times [n]_{-/} \xrightarrow{\sim} \mathrm{Un}^{\mathrm{co}}_{[n]}(\phi)$$

for any $\phi\colon [n] \to \mathrm{Cat}_\infty$, natural in $\mathbf{\Delta}^{\mathrm{op}}$. Thus by Proposition 7.1 there are natural equivalences

$$\mathrm{Fun}_{\Delta^n}(\Delta^n, \Phi^\phi_{\mathcal{X}}) \simeq \mathrm{Fun}(\mathrm{Un}^{\mathrm{co}}_{[n]}(\phi), \mathcal{X}) \simeq \lim_{\mathrm{Tw}([n])^{\mathrm{op}}} \mathrm{Fun}([n]_{-/}, \mathrm{Fun}(\phi(-), \mathcal{X}))$$
$$\simeq \mathrm{Fun}_{\Delta^n}(\Delta^n, \mathrm{Un}_{[n]}(\mathrm{Fun}(\phi(-), \mathcal{X}))).$$

Since this equivalence is natural in $\mathbf{\Delta}^{\mathrm{op}}$ and Cat_∞ is a localization of the presheaf ∞-category $\mathcal{P}(\mathbf{\Delta})$, we get by the Yoneda lemma a natural equivalence

$$\Phi^\phi_{\mathcal{X}} \simeq \mathrm{Un}^{\mathrm{cart}}_{[n]}(\mathrm{Fun}(\phi(-), \mathcal{X})).$$

Since Cat_∞ is an accessible localization of $\mathcal{P}(\mathbf{\Delta})$, any ∞-category $\mathcal{C}$ is naturally equivalent to the colimit of the diagram $\mathbf{\Delta}^{\mathrm{op}}_{/\mathcal{C}} \to \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Cat}_\infty$. Now given $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$, we have, since pullback along a Cartesian fibration preserves colimits,

$$\mathrm{Un}^{\mathrm{cart}}_{\mathcal{C}}(\mathrm{Fun}(F(-),\mathcal{X})) \simeq \operatorname*{colim}_{\sigma\in\mathbf{\Delta}^{\mathrm{op}}_{/\mathcal{C}}} \mathrm{Un}_{[n]}(\mathrm{Fun}(F\sigma(-),\mathcal{X})) \simeq \operatorname*{colim}_{\sigma\in\mathbf{\Delta}^{\mathrm{op}}_{/\mathcal{C}}} \Phi^{F\sigma}_{\mathcal{X}} \simeq \Phi^{F}_{\mathcal{X}},$$

which completes the proof. □

Theorem 7.4. *The coCartesian fibration classified by a functor $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ is given by the oplax colimit of F. In other words, there is a natural equivalence*

$$\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F) \simeq \operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/}$$

of functors $\mathrm{Fun}(\mathcal{C},\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$.

Proof. Let $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be a functor. Then by Proposition 7.3, we have a natural equivalence

$$\mathrm{Fun}(\mathrm{Un}_{\mathcal{C}}(F),\mathcal{X}) \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\Phi^{F}_{\mathcal{X}}).$$

By Proposition 7.1 we have a natural equivalence between the right-hand side and

$$\lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}_{-/},\mathrm{Fun}(F(-),\mathcal{X})) \simeq \mathrm{Fun}\big(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-)\times\mathcal{C}_{-/},\mathcal{X}\big).$$

By the Yoneda lemma, it follows that $\mathrm{Un}_{\mathcal{C}}(F)$ is naturally equivalent to $\operatorname{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/}$. □

Corollary 7.5. *Any ∞-category $\mathcal{C}$ is the oplax colimit of the constant functor $\mathcal{C} \to \mathrm{Cat}_\infty$ with value $*$.*

Proof. The identity $\mathcal{C} \to \mathcal{C}$ is the coCartesian fibration classified by this functor. □

Corollary 7.6. *The Cartesian fibration classified by a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is given by the lax colimit of F. In other words, there is a natural equivalence*

$$\mathrm{Un}_{\mathcal{C}}(F) \simeq \operatorname*{colim}_{\mathrm{Tw}(\mathcal{C}^{\mathrm{op}})} F(-) \times \mathcal{C}_{/-}$$

of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$.

Proof. We have a natural equivalence $\mathrm{Un}_{\mathcal{C}}(F) \simeq \mathrm{Un}^{\mathrm{co}}_{\mathcal{C}^{\mathrm{op}}}(F^{\mathrm{op}})^{\mathrm{op}}$. Since $(-)^{\mathrm{op}}$ is an automorphism of Cat_∞ it preserves colimits, so by Theorem 7.4 we have

$$\mathrm{Un}_{\mathcal{C}}(F) \simeq \left(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C}^{\mathrm{op}})} F(-)^{\mathrm{op}} \times (\mathcal{C}^{\mathrm{op}})_{-/}\right)^{\mathrm{op}} \simeq \operatorname*{colim}_{\mathrm{Tw}(\mathcal{C}^{\mathrm{op}})} F(-) \times \mathcal{C}_{/-}. \qquad \square$$

Similarly, dualizing Proposition 7.1 gives:

Corollary 7.7. *The ∞-category of sections of the coCartesian fibration classified by F is given by the lax limit of F. In other words, there is a natural equivalence*

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F)) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}_{/-}, F(-))$$

of functors $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$.

8. 2-Representable Functors

Suppose $\mathcal{C}$ is an ∞-category equipped with a mapping ∞-category functor $\mathrm{MAP}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}_\infty$. We say a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is *2-representable* by $C \in \mathcal{C}$ if F is equivalent to $\mathrm{MAP}_{\mathcal{C}}(-, C)$. Similarly, we say a Cartesian fibration $p \colon \mathcal{E} \to \mathcal{C}$ is *2-representable* by C if p is classified by the functor $\mathrm{MAP}_{\mathcal{C}}(-, C)$. Our goal in this section is to prove that if a Cartesian fibration $p \colon \mathcal{E} \to \mathcal{C}$ is 2-representable then, under mild hypotheses, the same is true for the induced map $\mathcal{E}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$ for any ∞-category $\mathcal{D}$. We begin by giving a somewhat unwieldy description of the functor classifying such fibrations for an arbitrary Cartesian fibration p:

Proposition 8.1. *Suppose $p \colon \mathcal{E} \to \mathcal{C}$ is a Cartesian fibration corresponding to a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Then for any ∞-category $\mathcal{D}$ the functor $\mathcal{E}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$ given by composition with p is a Cartesian fibration classified by a functor $F_{\mathcal{D}} \colon (\mathcal{C}^{\mathcal{D}})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends a functor $\phi \colon \mathcal{D} \to \mathcal{C}$ to*

$$\lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{D}_{-/}, F \circ \phi(-)).$$

Proof. The induced functor $\mathcal{E}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$ is a Cartesian fibration by [Lur09a, Proposition 3.1.2.1]. For $f \colon K \times \mathcal{D} \to \mathcal{C}$ we have a natural equivalence

$$\mathrm{Fun}_{\mathcal{C}^{\mathcal{D}}}(K, \mathcal{E}^{\mathcal{D}}) \simeq \mathrm{Fun}_{\mathcal{C}}(K \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}_{K \times \mathcal{D}}(K \times \mathcal{D}, f^*\mathcal{E}).$$

But then by Proposition 7.1 we have a natural equivalence

$$\mathrm{Fun}_{K \times \mathcal{D}}(K \times \mathcal{D}, f^*\mathcal{E}) \simeq \lim_{\mathrm{Tw}(K \times \mathcal{D})^{\mathrm{op}}} \mathrm{Fun}((K \times \mathcal{D})_{-/}, F \circ f(-)),$$

and then as Tw preserves products (being a right adjoint) we can rewrite this as

$$\lim_{\mathrm{Tw}(K)^{\mathrm{op}}} \mathrm{Fun}(K_{-/}, \lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{D}_{-/}, F \circ f(-))) \simeq \lim_{\mathrm{Tw}(K)^{\mathrm{op}}} \mathrm{Fun}(K_{-/}, F_{\mathcal{D}} \circ f(-)),$$

which we can identify, using Proposition 7.1, with

$$\mathrm{Fun}_K(K, f^*\mathrm{Un}_{\mathcal{C}^{\mathcal{D}}}(F_{\mathcal{D}})) \simeq \mathrm{Fun}_{\mathcal{C}^{\mathcal{D}}}(K, \mathrm{Un}_{\mathcal{C}^{\mathcal{D}}}(F_{\mathcal{D}})).$$

Now applying the Yoneda Lemma completes the proof. □

Definition 8.2. Suppose $\mathcal{C}$ is an ∞-category equipped with a mapping ∞-category functor. We say that $\mathcal{C}$ is *cotensored over* Cat_∞ if for every $C \in \mathcal{C}$ the functor $\mathrm{MAP}_{\mathcal{C}}(-, C) \colon \mathcal{C} \to \mathrm{Cat}_\infty^{\mathrm{op}}$ has a right adjoint $C^{(-)} \colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathcal{C}$; in this case these adjoints determine an essentially unique functor $(-)^{(-)} \colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is also tensored over Cat_∞, being cotensored is equivalent to the functor $- \otimes \mathcal{X}$ having a right adjoint $(-)^{\mathcal{X}}$ for all ∞-categories $\mathcal{X}$.

Corollary 8.3. *Let $\mathcal{C}$ be an ∞-category equipped with a mapping ∞-category functor $\mathrm{MAP}_{\mathcal{C}}$ that is tensored and cotensored over Cat_∞, and suppose $p\colon \mathcal{E} \to \mathcal{C}$ is a Cartesian fibration that is 2-representable by $C \in \mathcal{C}$. Then for any ∞-category $\mathcal{D}$ the fibration $\mathcal{E}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$ is 2-representable by the functor $C^{\mathcal{D}_{-/}}$.*

Proof. By Proposition 8.1 this fibration corresponds to the functor sending $\phi\colon \mathcal{D} \to \mathcal{C}$ to

$$\lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{D}_{-/}, F \circ \phi(-)),$$

where F is the functor corresponding to p. If F is 2-representable by C, this is equivalent to

$$\lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{D}_{-/}, \mathrm{MAP}_{\mathcal{C}}(\phi(-), C) \simeq \lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathrm{MAP}_{\mathcal{C}}(\phi(-), C^{\mathcal{D}_{-/}}),$$

which is $\mathrm{MAP}_{\mathcal{C}^{\mathcal{D}}}(\phi, C^{\mathcal{D}_{-/}})$ by definition of the mapping ∞-category of $\mathcal{C}^{\mathcal{D}}$. □

9. Some Cartesian Fibrations Identified

In this section we will use our results so far to explicitly identify the Cartesian fibrations classified by certain classes of functors. This is the key input needed to prove our presentability result in the next section. We start with some notation:

Definition 9.1. If $p\colon \mathcal{E} \to \mathcal{B}$ is a functor of ∞-categories, we denote by $\mathcal{E}^{\triangleright}_{\mathcal{B}}$ the pushout

$$\mathcal{B} \amalg_{\mathcal{E}\times\{1\}} \mathcal{E} \times \Delta^1,$$

and by $\mathcal{E}^{\triangleleft}_{\mathcal{B}}$ the pushout

$$\mathcal{B} \amalg_{\mathcal{E}\times\{0\}} \mathcal{E} \times \Delta^1.$$

Warning 9.2. The notations $\mathcal{E}^{\triangleright}_{\mathcal{B}}$ and $\mathcal{E}^{\triangleleft}_{\mathcal{B}}$ are somewhat abusive, as these simplicial sets depend on the functor p rather than just on $\mathcal{E}$ and $\mathcal{B}$. Moreover, if $\mathcal{B} = \Delta^0$ then the definition does not reduce to $\mathcal{E}^{\triangleright}$ and $\mathcal{E}^{\triangleleft}$ but rather the "alternative joins" $\mathcal{E} \diamond \Delta^0$ and $\Delta^0 \diamond \mathcal{E}$ in the notation of [Lur09a, §4.2.1]. However, these are weakly equivalent to the usual joins by [Lur09a, Proposition 4.2.1.2].

Remark 9.3. Observe that $(\mathcal{E}^{\triangleright}_{\mathcal{B}})^{\mathrm{op}} \simeq (\mathcal{E}^{\mathrm{op}})^{\triangleleft}_{\mathcal{B}^{\mathrm{op}}}$ and $(\mathcal{E}^{\triangleleft}_{\mathcal{B}})^{\mathrm{op}} \simeq (\mathcal{E}^{\mathrm{op}})^{\triangleright}_{\mathcal{B}^{\mathrm{op}}}$.

We then have the following simple observation:

Lemma 9.4. *Given a functor $p\colon \mathcal{E} \to \mathcal{B}$, write $i\colon \mathcal{B} \hookrightarrow \mathcal{E}^{\triangleleft}_{\mathcal{B}}$ and $j\colon \mathcal{B} \hookrightarrow \mathcal{E}^{\triangleright}_{\mathcal{B}}$ for the inclusions in the pushout diagrams defining $\mathcal{E}^{\triangleleft}_{\mathcal{B}}$ and $\mathcal{E}^{\triangleright}_{\mathcal{B}}$, respectively. Then for any ∞-category $\mathcal{D}$, we have:*

(i) The functor

$$i^*\colon \mathrm{Fun}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{D})$$

given by composition with i is a Cartesian fibration classified by the functor $\mathrm{Fun}(\mathcal{B}, \mathcal{D})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends F to $\mathrm{Fun}(\mathcal{E}, \mathcal{D})_{F\circ p/}$.

(ii) The functor

$$j^*\colon \mathrm{Fun}(\mathcal{E}^{\triangleright}_{\mathcal{B}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{D})$$

given by composition with j is a coCartesian fibration classified by the functor $\mathrm{Fun}(\mathcal{B}, \mathcal{D}) \to \mathrm{Cat}_\infty$ that sends F to $\mathrm{Fun}(\mathcal{E}, \mathcal{D})_{/F\circ p}$.

Proof. We will prove (i); the proof of (ii) is similar. By the definition of $\mathcal{E}^{\triangleleft}_{\mathcal{B}}$ we have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}, \mathcal{D}) \arrow[r] \arrow[d, "i^*"'] & \mathrm{Fun}(\mathcal{E} \times \Delta^1, \mathcal{D}) \arrow[d, "\mathrm{ev}_0"] \\
\mathrm{Fun}(\mathcal{B}, \mathcal{D}) \arrow[r, "p^*,"'] & \mathrm{Fun}(\mathcal{E}, \mathcal{D})
\end{tikzcd}$$

where the right vertical map can be identified with the evaluation-at-0 functor

$$\mathrm{Fun}(\mathcal{E}, \mathcal{D})^{\Delta^1} \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$$

This is the Cartesian fibration classified by the undercategory functor $\mathrm{Fun}(\mathcal{E}, \mathcal{D})_{(-)/}$, hence the pullback i^* is the Cartesian fibration classified by the composite functor $\mathrm{Fun}(\mathcal{E}, \mathcal{D})_{p^*(-)/}$. □

Remark 9.5. If $\mathcal{D}$ has pushouts, then i^* is also a coCartesian fibration, with coCartesian morphisms given by taking pushouts. Similarly, if $\mathcal{D}$ has pullbacks then j^* is also a Cartesian fibration, with Cartesian morphisms given by taking pullbacks.

In particular, given a map $p\colon \mathcal{E} \to \mathcal{B}$ we see that $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a coCartesian and Cartesian fibration (recall that $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denotes the presheaf ∞-category). The corresponding functors are given on objects by $\mathcal{P}(\mathcal{E})_{/p^*(-)}$, with functoriality determined by composition and pullbacks, respectively. Our next goal is to give an alternative description of this functor:

Proposition 9.6. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a functor of ∞-categories, and let $j\colon \mathcal{B} \to \mathcal{E}^{\triangleleft}_{\mathcal{B}}$ be the obvious inclusion. Then the functor $j^*\colon \mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration corresponding to the functor $\mathcal{P}(\mathcal{B})^{\mathrm{op}} \simeq \mathrm{RFib}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends a right fibration $\mathcal{Y} \to \mathcal{B}$ to $\mathcal{P}(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})$.*

To prove this, we need to identify the functor $\mathcal{P}(\mathcal{E})_{/p^*(-)}$ with the functor $\mathcal{P}(- \times_{\mathcal{B}} \mathcal{E})$ under the equivalence $\mathcal{P}(\mathcal{B}) \simeq \mathrm{RFib}(\mathcal{B})$, for which we use the following observation:

Proposition 9.7. *Suppose $p\colon \mathcal{K} \to \mathcal{C}$ is a right fibration of ∞-categories. Then the functor*

$$p_!\colon \mathrm{RFib}(\mathcal{K}) \to \mathrm{RFib}(\mathcal{C})_{/p}$$

given by composition with p is an equivalence. Moreover, this equivalence is natural in $p \in \mathrm{RFib}(\mathcal{C})$ (with respect to composition with maps $f\colon \mathcal{K} \to \mathcal{L}$ over $\mathcal{C}$, and also with respect to pullbacks along such maps).

Proof. The functor $p_!$ is described by the left Quillen functor $p_!\colon \mathrm{Set}_{\Delta/\mathcal{K}} \to (\mathrm{Set}_{\Delta/\mathcal{C}})_{/p}$ given by composition with p, where $\mathrm{Set}_{\Delta/\mathcal{K}}$ is equipped with the contravariant model structure of [Lur09a, §2.1.4] and $(\mathrm{Set}_{\Delta/\mathcal{C}})_{/p}$ with the model structure induced from the contravariant model structure on $\mathrm{Set}_{\Delta/\mathcal{C}}$. It therefore suffices to show that $p_!$ is a left Quillen equivalence. The functor $p_!$ is obviously an equivalence of underlying categories, and we claim it is actually an equivalence of model categories. The cofibrations clearly correspond under $p_!$, being the monomorphisms of underlying simplicial sets in both cases, so by [Joy08, Proposition E.1.10] it suffices to show the fibrant objects are the same. In $\mathrm{Set}_{\Delta/\mathcal{K}}$ these are the right fibrations $\mathcal{X} \to \mathcal{K}$ by [Lur09a, Corollary 2.2.3.12], while in $(\mathrm{Set}_{\Delta/\mathcal{C}})_{/p}$ they are the diagrams

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr] & & \mathcal{K} \arrow[dl, "p"] \\ & \mathcal{C} & \end{tikzcd}$$

where f is a fibration in $\mathrm{Set}_{\Delta/\mathcal{C}}$. But as p is a right fibration, this is equivalent to f being a right fibration by [Lur09a, Corollary 2.2.3.14]. □

COROLLARY 9.8. *Suppose $p\colon \mathcal{K} \to \mathcal{C}$ is a right fibration, corresponding to a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. Then the functor $p_!\colon \mathcal{P}(\mathcal{K}) \to \mathcal{P}(\mathcal{C})_{/F}$ given by left Kan extension along p^{op} is an equivalence, natural in $p \in \mathrm{RFib}(\mathcal{C})$ (with respect to left Kan extensions along maps $f\colon \mathcal{K} \to \mathcal{L}$ over $\mathcal{C}$ and composition with the associated natural transformation, as well as with respect to composition with f and pullback along the natural transformation).*

Proof. This follows from combining Proposition 9.7 with the naturality of the straightening equivalence between right fibrations and functors, which can be proved by the same argument as in the proof of Corollary A.32. □

Proof of Proposition 9.6. This follows from combining Lemma 9.4 with Corollary 9.8, since under the equivalence between presheaves and right fibrations the functor $p^*\colon \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{E})$ corresponds to pullback along p.

□

COROLLARY 9.9.

(i) *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a Cartesian fibration classified by a functor $F\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, and write j for the inclusion $\mathcal{B} \hookrightarrow \mathcal{E}^{\triangleleft}_{\mathcal{B}}$. Then the functor $j^*\colon \mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration classified by the functor $\mathrm{RFib}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends $\mathcal{Y} \to \mathcal{B}$ to $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{Y}^{\mathrm{op}}, \Phi^F_{\mathcal{S}})$, where $\Phi^F_{\mathcal{S}} \to \mathcal{B}^{\mathrm{op}}$ is the Cartesian fibration classified by the functor $\mathcal{P} \circ F\colon \mathcal{B} \to \mathrm{Cat}_\infty$.*

(ii) *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a coCartesian fibration classified by a functor $F\colon \mathcal{B} \to \mathrm{Cat}_\infty$, and write j for the inclusion $\mathcal{B} \hookrightarrow \mathcal{E}^{\triangleleft}_{\mathcal{B}}$. Then the functor $j^*\colon \mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration classified by the functor $\mathrm{RFib}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends $\mathcal{Y} \to \mathcal{B}$ to $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{Y}^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}})$, where $\widetilde{\Phi}^F_{\mathcal{S}} \to \mathcal{B}^{\mathrm{op}}$ is the coCartesian fibration classified by the functor $\mathcal{P} \circ F\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.*

Proof. Combine Proposition 9.6 with Proposition 7.3 and its dual. □

Our next observation lets us identify several interesting functors with full subfunctors of the functor $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}})$, which will allow us to identify the corresponding Cartesian fibrations with full subcategories of $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$.

Lemma 9.10. *Suppose $F\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a functor of ∞-categories corresponding to the Cartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$ and the coCartesian fibration $q\colon \mathcal{F} \to \mathcal{B}^{\mathrm{op}}$. Let $\widehat{F}\colon \mathcal{P}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be the unique limit-preserving functor extending F. Then:*

(i) *The functor $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{B}}(\mathcal{B}_{/-}, \mathcal{E})\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is equivalent to F.*
(ii) *The functor $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{B}}(-, \mathcal{E})\colon \mathrm{RFib}(\mathcal{B}) \to \mathrm{Cat}_\infty$ corresponds to $\widehat{F}$ under the equivalence $\mathrm{RFib}(\mathcal{B}) \simeq \mathcal{P}(\mathcal{B})$.*
(iii) *The functor $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((\mathcal{B}^{\mathrm{op}})_{-/}, \mathcal{F})\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is equivalent to F.*
(iv) *The functor $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \mathcal{F})\colon \mathrm{RFib}(\mathcal{B}) \to \mathrm{Cat}_\infty$ corresponds to $\widehat{F}$ under the equivalence $\mathrm{RFib}(\mathcal{B}) \simeq \mathcal{P}(\mathcal{B})$.*

Proof. We will prove (i) and (ii); the proofs of (iii) and (iv) are essentially the same. To prove (i), observe that the straightening equivalence between Cartesian fibrations and functors to Cat_∞ gives us a natural equivalence

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{B}}(\mathcal{B}_{/-}, \mathcal{E})) &\simeq \mathrm{Map}^{\mathrm{cart}}_{\mathcal{B}}(\mathcal{C} \times \mathcal{B}_{/-}, \mathcal{E})\\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(\mathcal{C} \times y_{\mathcal{B}}(-), F)\\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(y_{\mathcal{B}}(-), F^{\mathcal{C}}).\end{aligned}$$

Now the Yoneda Lemma implies that this is naturally equivalent to $\iota F^{\mathcal{C}}(-) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, F(-))$, and so we must have $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{B}}(\mathcal{B}_{/-}, \mathcal{E}) \simeq F$. This proves (i). To prove (ii), we first observe that the functor $\mathrm{Fun}^{\mathrm{cart}}_{\mathcal{B}}(-, \mathcal{E})$ preserves limits, since for any ∞-category $\mathcal{C}$ we have

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{B}}(-, \mathcal{E})) \simeq \mathrm{Map}^{\mathrm{cart}}_{\mathcal{B}}((\mathcal{C} \times \mathcal{B}) \times_{\mathcal{B}} (-), \mathcal{E})$$

and the Cartesian product in $\mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{B}}$ preserves colimits in each variable. Moreover, it follows from (i) that this functor extends F, since the right fibration $\mathcal{B}_{/b} \to \mathcal{B}$ corresponds to the presheaf $y_{\mathcal{B}}(b)$ under the equivalence between $\mathrm{RFib}(\mathcal{B})$ and $\mathcal{P}(\mathcal{B})$. □

Definition 9.11. Suppose $F\colon \mathcal{B} \to \mathrm{Cat}_\infty$ is a functor. Then we write $\mathcal{P}F\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ for the composite of F^{op} with $\mathcal{P}\colon \mathrm{Cat}^{\mathrm{op}}_\infty \to \widehat{\mathrm{Cat}}_\infty$, and let $\widehat{\mathcal{P}F}\colon \mathcal{P}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ be the unique limit-preserving functor extending $\mathcal{P}F$.

Proposition 9.12. *Let $F\colon \mathcal{B} \to \mathrm{Cat}_\infty$ be a functor, with $p\colon \mathcal{E} \to \mathcal{B}$ an associated coCartesian fibration. We define $\mathcal{P}^{\mathrm{cocart}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ to be the full subcategory of $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ spanned by those presheaves $\phi\colon (\mathcal{E}^{\triangleleft}_{\mathcal{B}})^{\mathrm{op}} \to \mathcal{S}$ such that for every coCartesian morphism $\bar{\alpha}\colon e \to \alpha_! e$ in $\mathcal{E}$ over $\alpha\colon b \to b'$ in $\mathcal{B}$, the commutative* *square*

$$\begin{tikzcd} \phi(\alpha_! e, 1) \arrow[r, "\phi(\bar{\alpha}, 1)"] \arrow[d] & \phi(e, 1) \arrow[d] \\ \phi(b', 0) \arrow[r, "\phi(\alpha, 0)"'] & \phi(b, 0) \end{tikzcd}$$

is a pullback square. Then the restricted projection $\mathcal{P}^{\mathrm{cocart}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration classified by the functor $\widehat{\mathcal{P}F}$.

Proof. Combining Lemma 9.10 with (the dual of) Proposition 7.3, we may identify $\widehat{\mathcal{P}F}$ with the functor $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^{F}_{\mathcal{S}})$. This is a natural full subcategory of $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^{F}_{\mathcal{S}})$, the functor classified by the Cartesian fibration $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ by Proposition 9.6. It follows that $\widehat{\mathcal{P}F}$ is classified by the projection to $\mathcal{P}(\mathcal{B})$ of the full subcategory of $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ spanned by those presheaves that correspond to objects of $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^{F}_{\mathcal{S}})$ under the identification of the fibres with $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^{F}_{\mathcal{S}})$.
By [Lur09a, Corollary 3.2.2.13], the coCartesian edges of $\widetilde{\Phi}^{F}_{\mathcal{S}}$ are those that correspond to functors $\Delta^1 \times_{\mathcal{B}^{\mathrm{op}}} \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ that take Cartesian edges of $\mathcal{E}^{\mathrm{op}}$ to equivalences in $\mathcal{S}$. Thus if $\mathcal{F} \to \mathcal{B}^{\mathrm{op}}$ is a coCartesian fibration, a functor $\mathcal{F} \to \widetilde{\Phi}^{F}_{\mathcal{S}}$ over $\mathcal{B}^{\mathrm{op}}$ preserves coCartesian morphisms precisely if the classifying functor $\mathcal{F} \times_{\mathcal{B}^{\mathrm{op}}} \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ takes morphisms of the form (ϕ, ϵ) with ϕ coCartesian in $\mathcal{F}$ and ϵ Cartesian in $\mathcal{E}^{\mathrm{op}}$ to equivalences in $\mathcal{S}$.
If $\mathcal{Y} \to \mathcal{B}$ is a right fibration, this means that $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{Y}^{\mathrm{op}}, \widetilde{\Phi}^{F}_{\mathcal{S}})$ corresponds to the full subcategory of $\mathcal{P}(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})$ spanned by the presheaves $(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ that take morphisms of the form (η, ϵ) with ϵ coCartesian in $\mathcal{E}$ to equivalences in $\mathcal{S}$.
Suppose $\phi \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}$ is the presheaf classified by $\mathcal{Y} \to \mathcal{B}$. Then unwinding the equivalence

$$\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})_{\phi} \simeq \mathcal{P}(\mathcal{E})_{/p^*\phi} \simeq \mathrm{RFib}(\mathcal{E})_{/p^*\mathcal{Y}} \simeq \mathrm{RFib}(\mathcal{E} \times_{\mathcal{B}} \mathcal{Y}) \simeq \mathcal{P}(\mathcal{E} \times_{\mathcal{B}} \mathcal{Y}),$$

we see that the presheaf $\widetilde{\Psi}$ on $\mathcal{E} \times_{\mathcal{B}} \mathcal{Y}$ classified by $\Psi \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ over $p^*\phi$ assigns to $(e, y) \in \mathcal{E} \times_{\mathcal{B}} \mathcal{Y}$ the fibre $\Psi(e)_y$ of the map $\Psi(e) \to \phi(pe)$ at y. Thus $\widetilde{\Psi}(\bar{\alpha}, \eta)$ is an equivalence for every coCartesian morphism $\bar{\alpha} \colon e \to \alpha_! e$ in $\mathcal{E}$ if and only if for every $y \in \phi(p(\alpha_! e))$ the map on fibres $\Psi(\alpha_! e)_y \to \Psi(e)_{\phi(\alpha)(e)}$ is an equivalence. This is equivalent to the commutative square

$$\begin{tikzcd} \Psi(\alpha_! e) \arrow[r] \arrow[d] & \Psi(e) \arrow[d] \\ \phi(p\alpha_! e) \arrow[r] & \phi(pe) \end{tikzcd}$$

being Cartesian. Thus $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{Y}^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}})$ corresponds to the full subcategory $\mathcal{P}^{\mathrm{cocart}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})_\phi$ of $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})_\phi$, which completes the proof. $\square$

Definition 9.13. Let $\mathcal{K}$ be a collection of small simplicial sets. We write $\mathrm{Cat}^{\mathcal{K}}_\infty$ for the subcategory of Cat_∞ with objects the small ∞-categories that admit K-indexed colimits for all $K \in \mathcal{K}$ and morphisms the functors that preserve these. Given $\mathcal{C} \in \mathrm{Cat}^{\mathcal{K}}_\infty$ we let $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by those presheaves $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ that take K-indexed colimits in $\mathcal{C}$ to limits for all $K \in \mathcal{K}$. This defines a functor $\mathcal{P}_{\mathcal{K}} \colon (\mathrm{Cat}^{\mathcal{K}}_\infty)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$.

Example 9.14. Let $\mathcal{K}(\kappa)$ be the collection of all κ-small simplicial sets. In this case we write $\mathrm{Cat}^{\kappa}_\infty$ for $\mathrm{Cat}^{\mathcal{K}(\kappa)}_\infty$ and $\mathcal{P}_\kappa$ for $\mathcal{P}_{\mathcal{K}(\kappa)}$. For $\mathcal{C} \in \mathrm{Cat}^{\kappa}_\infty$ the ∞-category $\mathcal{P}_\kappa(\mathcal{C})$ is equivalent to $\mathrm{Ind}_\kappa \mathcal{C}$ by [Lur09a, Corollary 5.3.5.4].

Definition 9.15. Suppose $\mathcal{K}$ is a collection of small simplicial sets and $F \colon \mathcal{B} \to \mathrm{Cat}^{\mathcal{K}}_\infty$ is a functor of ∞-categories. Then we write $\mathcal{P}_{\mathcal{K}}F \colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ for the composite of F^{op} with $\mathcal{P}_{\mathcal{K}} \colon (\mathrm{Cat}^{\mathcal{K}}_\infty)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$, and let $\widehat{\mathcal{P}_{\mathcal{K}}F} \colon \mathcal{P}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ be the unique limit-preserving functor extending $\mathcal{P}_{\mathcal{K}}F$.

Proposition 9.16. *Let $\mathcal{K}$ be a collection of small simplicial sets and let $F \colon \mathcal{B} \to \mathrm{Cat}^{\mathcal{K}}_\infty$ be a functor, with $p \colon \mathcal{E} \to \mathcal{B}$ an associated coCartesian fibration. We define $\mathcal{P}^{\mathrm{cocart}}_{\mathcal{K}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ to be the full subcategory of $\mathcal{P}^{\mathrm{cocart}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ spanned by those presheaves $\phi \colon (\mathcal{E}^{\triangleleft}_{\mathcal{B}})^{\mathrm{op}} \to \mathcal{S}$ such that for every colimit diagram $\bar{q} \colon K^{\triangleright} \to \mathcal{E}_b$ (for any $b \in \mathcal{B}$), the composite*

$$(K^{\triangleleft\triangleright})^{\mathrm{op}} \to (\mathcal{E}^{\triangleleft}_b)^{\mathrm{op}} \to (\mathcal{E}^{\triangleleft}_{\mathcal{B}})^{\mathrm{op}} \to \mathcal{S}$$

is a limit diagram. Then the restricted projection $\mathcal{P}^{\mathrm{cocart}}_{\mathcal{K}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration associated to the functor $\widehat{\mathcal{P}_{\mathcal{K}}F}$.

Proof. Since $\mathcal{P}_{\mathcal{K}}F(b)$ is a natural full subcategory of $\mathcal{P}F(b)$, we may identify the coCartesian fibration classified by $\mathcal{P}_{\mathcal{K}}F$ with the projection to $\mathcal{B}^{\mathrm{op}}$ of a full subcategory $\widetilde{\Phi}^F_{\mathcal{S}}(\mathcal{K})$ of $\widetilde{\Phi}^F_{\mathcal{S}}$. By Lemma 9.10 we may then identify the functor $\widehat{\mathcal{P}_{\mathcal{K}}F}$ with $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}}(\mathcal{K}))$, which is a natural full subcategory of $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}})$. By Proposition 9.12 we may therefore identify the Cartesian fibration classified by $\widehat{\mathcal{P}_{\mathcal{K}}F}$ with the projection to $\mathcal{P}(\mathcal{B})$ of a full subcategory of $\mathcal{P}^{\mathrm{cocart}}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$.

It thus remains to identify those presheaves on $\mathcal{E}^{\triangleleft}_{\mathcal{B}}$ that correspond to objects of the ∞-category $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}}(\mathcal{K}))$ under the identification of the fibres with $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}((-)^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}})$. If $\mathcal{Y} \to \mathcal{B}$ is a right fibration, it is clear that $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{Y}^{\mathrm{op}}, \widetilde{\Phi}^F_{\mathcal{S}}(\mathcal{K}))$ corresponds to the full subcategory of $\mathcal{P}(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})$ spanned by the presheaves $(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})^{\mathrm{op}} \to \mathcal{S}$ such that for every $y \in \mathcal{Y}$ over $b \in \mathcal{B}$, the restriction $(\{y\} \times_{\mathcal{B}} \mathcal{E})^{\mathrm{op}} \simeq \mathcal{E}^{\mathrm{op}}_b \to \mathcal{S}$ preserves K-indexed limits for all $K \in \mathcal{K}$. Suppose $\phi \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}$ is the presheaf classified by $\mathcal{Y} \to \mathcal{B}$. Then unwinding the equivalence

$$\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})_\phi \simeq \mathcal{P}(\mathcal{E})_{/p^*\phi} \simeq \mathrm{RFib}(\mathcal{E})_{/p^*\mathcal{Y}} \simeq \mathrm{RFib}(\mathcal{E} \times_{\mathcal{B}} \mathcal{Y}) \simeq \mathcal{P}(\mathcal{E} \times_{\mathcal{B}} \mathcal{Y}),$$

we see that these presheaves on $\mathcal{E} \times_{\mathcal{B}} \mathcal{Y}$ correspond to presheaves $\Psi \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ over $p^*\phi$ such that for every $b \in \mathcal{B}$, the restriction

$$\mathcal{E}_b^{\mathrm{op}} \to \mathcal{S}_{/\phi(b)}$$

has the property that for every $y \in \phi(b)$, the composite with the map $\mathcal{S}_{/\phi(b)} \to \mathcal{S}$ given by pullback along $\{y\} \to \phi(b)$ takes K-indexed colimits to limits for all $K \in \mathcal{K}$.

Now recall that the ∞-category $\mathcal{S}_{/\phi(b)}$ is equivalent to $\mathrm{Fun}(\phi(b), \mathcal{S})$, with pullback to $\{y\}$ corresponding to evaluation at y, and that limits in functor categories are computed pointwise. Thus we may identify our full subcategory with that of presheaves Ψ such that for every $b \in \mathcal{B}$, the restriction

$$\mathcal{E}_b^{\mathrm{op}} \to \mathcal{S}_{/\phi(b)}$$

takes K-indexed colimits to limits in $\mathcal{S}_{/\phi(b)}$.

For any ∞-category $\mathcal{C}$ and $x \in \mathcal{C}$, a diagram $K^{\triangleleft} \to \mathcal{C}_{/x}$ is a limit diagram if and only if the associated diagram $K^{\triangleleft\triangleright} \to \mathcal{C}$ is a limit diagram. Therefore, the full subcategory of $\mathcal{P}^{\mathrm{cocart}}(\mathcal{E}_{\mathcal{B}}^{\triangleleft})$ we have identified is precisely $\mathcal{P}^{\mathrm{cocart}}_{\mathcal{K}}(\mathcal{E}_{\mathcal{B}}^{\triangleleft})$, which completes the proof. $\square$

Corollary 9.17. *Suppose $F \colon \mathcal{B} \to \mathrm{Cat}_\infty^\kappa$ is a functor, with $p \colon \mathcal{E} \to \mathcal{B}$ an associated coCartesian fibration. Let $\widehat{\mathrm{Ind}}_\kappa F \colon \mathcal{P}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ be the unique limit-preserving functor extending $\mathrm{Ind}_\kappa \circ F^{\mathrm{op}} \colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. Then the restricted projection $\mathcal{P}^{\mathrm{cocart}}_\kappa(\mathcal{E}_{\mathcal{B}}^{\triangleleft}) := \mathcal{P}^{\mathrm{cocart}}_{\mathcal{K}(\kappa)}(\mathcal{E}_{\mathcal{B}}^{\triangleleft}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration classified by the functor $\widehat{\mathrm{Ind}}_\kappa F$.*

10. Presentable Fibrations are Presentable

In ordinary category theory, an *accessible fibration* is a Grothendieck fibration $p \colon \mathbf{E} \to \mathbf{C}$ such that $\mathbf{C}$ is an accessible category, the corresponding functor $F \colon \mathbf{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ factors through the category of accessible categories and accessible functors, and F preserves κ-filtered limits for κ sufficiently large.

In [MP89], Makkai and Paré prove that if p is an accessible fibration, then its source $\mathbf{E}$ is also an accessible category, and p is an accessible functor. The goal of this section is to prove an ∞-categorical variant of this result. As it makes the proof much clearer we will, however, restrict ourselves to considering only *presentable fibrations* of ∞-categories, defined as follows:

Definition 10.1. A *presentable fibration* is a Cartesian fibration $p \colon \mathcal{E} \to \mathcal{B}$ such that $\mathcal{B}$ is a presentable ∞-category, the corresponding functor $F \colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ factors through the ∞-category Pr^{R} of presentable ∞-categories and right adjoints, and F preserves κ-filtered limits for κ sufficiently large.

Remark 10.2. Suppose $p \colon \mathcal{E} \to \mathcal{B}$ is a presentable fibration. Since the morphisms of $\mathcal{B}$ are all mapped to right adjoints under the associated functor, it follows that p is also a coCartesian fibration.

The goal of this section is then to prove the following:

Theorem 10.3. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a presentable fibration. Then $\mathcal{E}$ is a presentable ∞-category.*

As in Makkai and Paré's proof of [MP89, Theorem 5.3.4], we will prove this by explicitly describing the total space of the presentable fibration classified by a special class of functors as an accessible localization of a presheaf ∞-category. To state this result we first recall some notation from [Lur09a, §5.5.7]:

Definition 10.4. Suppose κ is a regular cardinal. As before, let $\mathrm{Cat}_\infty^\kappa$ be the category of small ∞-categories that have all κ-small colimits, and functors that preserve these. Then Ind_κ gives a functor from $\mathrm{Cat}_\infty^{\kappa,\mathrm{op}}$ to the ∞-category $\mathrm{Pr}^{\mathrm{R}}_\kappa$ of κ-presentable ∞-categories and limit-preserving functors that preserve κ-filtered colimits. Using the equivalence $\mathrm{Pr}^{\mathrm{L}} \simeq (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$ we may equivalently regard this as a functor $\mathrm{Ind}^\vee_\kappa\colon \mathrm{Cat}_\infty^\kappa \to \mathrm{Pr}^{\mathrm{L}}_\kappa$, where $\mathrm{Pr}^{\mathrm{L}}_\kappa$ is the ∞-category of κ-presentable ∞-categories and colimit-preserving functors that preserve κ-compact objects.

The key step in the proof of Theorem 10.3 can then be stated as follows:

Proposition 10.5. *Suppose $F\colon \mathcal{B} \to \mathrm{Cat}_\infty^\kappa$ is a functor of ∞-categories with associated coCartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$. Let $q\colon \widehat{\mathcal{E}} \to \mathcal{P}(\mathcal{B})$ be a Cartesian fibration classified by the unique limit-preserving functor $\widehat{\mathrm{Ind}_\kappa F}\colon \mathcal{P}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}_\kappa$ extending $\mathrm{Ind}_\kappa \circ F^{\mathrm{op}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}_\kappa$. Then the ∞-category $\widehat{\mathcal{E}}$ is an accessible ∞-category, and q is an accessible functor.*

We will prove Proposition 10.5 using Corollary 9.17 together with the following simple observation:

Lemma 10.6. *Suppose $\mathcal{C}$ is a small ∞-category, and let $S = \{\overline{p}_\alpha\colon K_\alpha^{\triangleright} \to \mathcal{C}\}$ be a small set of diagrams in $\mathcal{C}$. Then the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by presheaves that take the diagrams in S to limit diagrams in $\mathcal{S}$ is accessible, and the inclusion of this into $\mathcal{P}(\mathcal{C})$ is an accessible functor.*

Proof. Let $y_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denote the Yoneda embedding. A presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ takes $\overline{p}_\alpha^{\mathrm{op}}$ to a limit diagram if and only if it is local with respect to the map of presheaves
$$\mathrm{colim}(y_{\mathcal{C}} \circ \overline{p}|_{K_\alpha}) \to y_{\mathcal{C}}(\infty),$$
where ∞ denotes the cone point. Thus if S' is the set of these morphisms for $\overline{p}_\alpha \in S$, the subcategory in question is precisely the full subcategory of S'-local objects. (This observation can also be found e.g. in the proof of [Lur09a, Proposition 5.3.6.2].) Since S, and hence S', is by assumption a small set, it follows that this subcategory is an accessible localization of $\mathcal{P}(\mathcal{C})$. In particular, it is itself accessible and the inclusion into $\mathcal{P}(\mathcal{C})$ is an accessible functor. □

Proof of Proposition 10.5. By Proposition 9.17 the Cartesian fibration $\widehat{\mathcal{E}} \to \mathcal{P}(\mathcal{B})$ can be identified with the restriction to the full subcategory $\mathcal{P}^{\mathrm{cocart}}_\kappa(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ of the functor $j^*\colon \mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ induced by composition with the inclusion $j\colon \mathcal{B} \hookrightarrow \mathcal{E}^{\triangleleft}_{\mathcal{B}}$.

The ∞-category $\mathcal{P}^{\mathrm{cocart}}_{\kappa}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ is by definition the full subcategory of $\mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ spanned by presheaves that take two classes of diagrams to limit diagrams in $\mathcal{S}$ — one indexed by coCartesian morphisms in $\mathcal{E}$, which form a set, and one indexed by κ-small colimit diagrams in the fibres of p; these do not form a set, but we can equivalently consider only pushout squares and coproducts indexed by κ-small sets, which *do* form a set. It then follows from Lemma 10.6 that $\mathcal{P}^{\mathrm{cocart}}_{\kappa}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ is accessible and the inclusion $\mathcal{P}^{\mathrm{cocart}}_{\kappa}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \hookrightarrow \mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}})$ is an accessible functor. The functor $j^*\colon \mathcal{P}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ preserves colimits, since these are computed pointwise, and so the composite $\mathcal{P}^{\mathrm{cocart}}_{\kappa}(\mathcal{E}^{\triangleleft}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is also an accessible functor. □

To complete the proof of Theorem 10.3 we now just need an easy Lemma:

Lemma 10.7. *Suppose $\pi\colon \mathcal{E} \to \mathcal{B}$ is a coCartesian fibration such that both $\mathcal{B}$ and the fibres $\mathcal{E}_b$ for all $b \in \mathcal{B}$ admit small colimits, and the functors $f_!\colon \mathcal{E}_b \to \mathcal{E}_{b'}$ preserve colimits for all morphisms $f\colon b \to b'$ in $\mathcal{B}$. Then $\mathcal{E}$ admits small colimits.*

Proof. The coCartesian fibration π satisfies the conditions of [Lur09a, Corollary 4.3.1.11] for all small simplicial sets K, and so in every diagram

$$\begin{tikzcd} K \arrow[r, "p"] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\ K^{\triangleright} \arrow[r, "\bar{q}"'] \arrow[ur, dashed, "\bar{p}"] & \mathcal{B} \end{tikzcd}$$

there exists a lift $\bar{p}$ that is a π-colimit of p. Given a diagram $p\colon K \to \mathcal{E}$ we can apply this with $\bar{q}$ a colimit of $\pi \circ p$ to get a colimit $\bar{p}\colon K^{\triangleright} \to \mathcal{E}$ of p. □

Proof of Theorem 10.3. It follows from Lemma 10.7 that $\mathcal{E}$ has small colimits. It thus remains to prove that $\mathcal{E}$ is accessible and p is an accessible functor. Let $F\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$ be a functor corresponding to p. Choose a regular cardinal κ so that $\mathcal{B}$ is κ-presentable and F preserves κ-filtered limits. Since $\mathcal{B}$ is κ-presentable, $\mathcal{B} \simeq \mathrm{Ind}_{\kappa} \mathcal{B}^{\kappa}$ is the full subcategory of $\mathcal{P}(\mathcal{B}^{\kappa})$ spanned by the presheaves that preserve κ-small limits. Let $\widehat{F}\colon \mathcal{P}(\mathcal{B}^{\kappa})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$ be the unique limit-preserving functor extending $F|_{\mathcal{B}^{\kappa,\mathrm{op}}}$; then F is equivalent to the restriction of $\widehat{F}$ to $\mathrm{Ind}_{\kappa} \mathcal{B}^{\kappa}$. If $\widehat{p}\colon \widehat{\mathcal{E}} \to \mathcal{P}(\mathcal{B}^{\kappa})$ is a Cartesian fibration classified by $\widehat{F}$ we therefore have a pullback square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \widehat{\mathcal{E}} \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{P}(\mathcal{B}^{\kappa}), \end{tikzcd}$$

where the bottom map preserves κ-filtered colimits, so by [Lur09a, Proposition 5.4.6.6] it suffices to show that $\widehat{\mathcal{E}}$ is accessible and $\widehat{p}$ is an accessible functor. Since $\mathcal{B}^\kappa$ is a small ∞-category, we can choose a cardinal λ such that $F|_{\mathcal{B}^{\kappa,\mathrm{op}}}$ factors through the ∞-category $\mathrm{Pr}^{\mathrm{R},\lambda}$ of λ-presentable ∞-categories and right adjoints that preserve λ-filtered colimits. By [Lur09a, Proposition 5.5.7.2] we can equivalently think of this, via the equivalence $\mathrm{Pr}^{\mathrm{R}} \simeq (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$, as a functor from $\mathcal{B}^\kappa$ to the ∞-category $\mathrm{Pr}^{\mathrm{L},\lambda}$ of λ-presentable ∞-categories and functors that preserve colimits and λ-compact objects. Taking λ-compact objects defines a functor $(-)^\lambda\colon \mathrm{Pr}^{\mathrm{L},\lambda} \to \mathrm{Cat}_\infty^\lambda$. Then, defining $F_0\colon (\mathcal{B}^\kappa) \to \mathrm{Cat}_\infty$ to be $(F^{\mathrm{op}}|_{\mathcal{B}^\kappa})^\lambda$, we see that $F \simeq \mathrm{Ind}_\lambda F_0$, and so $\widehat{\mathcal{E}}$ is accessible and $\widehat{p}$ is an accessible functor by Proposition 10.5. □

Appendix A. Pseudofunctors and the Naturality of Unstraightening

At several points in this paper we will need to know that the unstraightening functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \to \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ (and a number of similar constructions) are natural as we vary the ∞-category $\mathcal{C}$. The obvious way to prove this is to consider the naturality of the unstraightening $\mathrm{Un}_S^+\colon \mathrm{Fun}(\mathfrak{C}(S), \mathrm{Set}_\Delta^+) \to (\mathrm{Set}_\Delta^+)_{/S}$ as we vary the simplicial set S. However, since pullbacks are only determined up to canonical isomorphism, these functors are not natural "on the nose", but only up to natural isomorphism — i.e. they are only *pseudo-natural*. In the body of the paper we have swept such issues under the rug, but in this appendix we indulge ourselves in a bit of 2-category theory to prove that pseudo-naturality on the level of model categories does indeed give naturality on the level of ∞-categories. We begin by reviewing Duskin's nerve of bicategories [Dus02] and its basic properties. However, we will only need to consider the case of *strict* 2- and (2,1)-categories:

Definition A.1. A *strict 2-category* is a category enriched in Cat, and a *strict (2,1)-category* is a category enriched in Gpd. We write Cat_2 for the category of strict 2-categories and $\mathrm{Cat}_{(2,1)}$ for the category of strict (2,1)-categories.

Definition A.2. Suppose $\mathbf{C}$ and $\mathbf{D}$ are strict 2-categories. A *normal oplax functor* $F\colon \mathbf{C} \to \mathbf{D}$ consists of the following data:

(a) for each object $x \in \mathbf{C}$, an object $F(x) \in \mathbf{D}$,
(b) for each 1-morphism $f\colon x \to y$ in $\mathbf{C}$, a 1-morphism $F(f)\colon F(x) \to F(y)$,
(c) for each 2-morphism $\phi\colon f \Rightarrow g$ in $\mathbf{C}(x,y)$, a 2-morphism $F(\phi)\colon F(f) \Rightarrow F(g)$ in $\mathbf{D}(F(x), F(y))$,
(d) for each pair of composable 1-morphisms $f\colon x \to y$, $g\colon y \to z$ in $\mathbf{C}$, a 2-morphism $\eta_{f,g}\colon F(g \circ f) \Rightarrow F(g) \circ F(f)$,

such that:

(i) for every object $x \in \mathbf{C}$, the 1-morphism $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$,
(ii) for every 1-morphism $f\colon x \to y$ in $\mathbf{C}$, the 2-morphism $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$,
(iii) for composable 2-morphisms $\phi\colon f \Rightarrow g$, $\psi\colon g \Rightarrow h$ in $\mathbf{C}(x,y)$, we have $F(\psi \circ \phi) = F(\psi) \circ F(\phi)$,

(iv) for every morphism $f\colon x \to y$, the morphisms $\eta_{\mathrm{id}_x,f}$ and $\eta_{f,\mathrm{id}_y}\colon F(f) \to F(f)$ are both $\mathrm{id}_{F(f)}$,

(v) if $\phi\colon f \Rightarrow f'$ is a 2-morphism in $\mathbf{C}(x,y)$ and $\psi\colon g \Rightarrow g'$ is a 2-morphism in $\mathbf{C}(y,z)$, then the diagram

$$\begin{tikzcd} F(g\circ f) \arrow[r, "\eta_{f,g}"] \arrow[d, "F(\psi\circ\phi)"'] & F(g)\circ F(f) \arrow[d, "F(\psi)\circ F(\phi)"] \\ F(g'\circ f') \arrow[r, "\eta_{f',g'}"'] & F(g')\circ F(f') \end{tikzcd}$$

commutes,

(vi) for composable triples of 1-morphisms $f\colon x \to y$, $g\colon y \to z$, $h\colon z \to w$, the diagram

$$\begin{tikzcd} F(h\circ g\circ f) \arrow[r, "\eta_{f,hg}"] \arrow[d, "\eta_{gf,h}"'] & F(h\circ g)\circ F(f) \arrow[d, "\eta_{g,h}\circ \mathrm{id}"] \\ F(h)\circ F(g\circ f) \arrow[r, "\mathrm{id}\circ\eta_{f,g}"'] & F(h)\circ F(g)\circ F(f)) \end{tikzcd}$$

commutes.

We say a normal oplax functor F from $\mathbf{C}$ to $\mathbf{D}$ is a *normal pseudofunctor* if the 2-morphisms $\eta_{f,g}$ are all isomorphisms. In particular, if the 2-category $\mathbf{C}$ is a (2,1)-category, all normal oplax functors $\mathbf{C} \to \mathbf{D}$ are normal pseudofunctors.

Remark A.3. In 2-category theory one typically considers the more general notions of (not necessarily normal) oplax functors and pseudofunctors, which do not satisfy $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ but instead include the data of natural maps $F(\mathrm{id}_x) \to \mathrm{id}_{F(x)}$ (which are isomorphisms for pseudofunctors). We only consider the normal versions because, as we will see below, these correspond to maps of simplicial sets between the *nerves* of strict 2- and (2,1)-categories.

Before we recall the definition of the nerve of a strict 2-category, we first review the definition of nerves for ordinary categories and simplicial categories:

Definition A.4. Let $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{Set}_\Delta$ be the usual nerve of categories, i.e. if $\mathbf{C}$ is a category then $\mathrm{N}\mathbf{C}_k$ is the set $\mathrm{Hom}([k],\mathbf{C})$ where $[k]$ is the category corresponding to the partially ordered set $\{0,\ldots,k\}$.

Remark A.5. Since Cat has colimits, the functor N has a left adjoint $\mathrm{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}$, which is the unique colimit-preserving functor such that $\mathrm{C}(\Delta^n) = [n]$.

Lemma A.6. *The functor* $\mathrm{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}$ *takes inner anodyne morphisms to isomorphisms.*

Proof. Let $\mathfrak{W}$ denote the class of monomorphisms of simplicial sets that are taken to isomorphisms by C. To see that $\mathfrak{W}$ contains the inner anodyne morphisms we apply [JT07, Lemma 3.5], which says that $\mathfrak{W}$ contains the inner anodyne maps if

(i) $\mathfrak{W}$ is weakly saturated, i.e. it contains the isomorphisms and is closed under composition, transfinite composition, cobase change, and codomain retracts,
(ii) $\mathfrak{W}$ has the right cancellation property, i.e. it if fg and g are in $\mathfrak{W}$ then f is in $\mathfrak{W}$,
(iii) $\mathfrak{W}$ contains the inclusions $\mathrm{Sp}^n \hookrightarrow \Delta^n$, where Sp^n denotes the n-spine, i.e. the simplicial set $\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}$.

Here conditions (i) and (ii) follow immediately from the definition of $\mathfrak{W}$, as the functor C preserves colimits. It remains to prove (iii), i.e. to show that $\mathrm{C}(\mathrm{Sp}^n) \to \mathrm{C}(\Delta^n)$ is an isomorphism. Since C preserves colimits, this is the map of categories

$$[1] \amalg_{[0]} \cdots \amalg_{[0]} [1] \to [n].$$

But the category $[n]$ is the free category on the graph with vertices $0, \dots, n$ and edges $i \to (i+1)$, which obviously decomposes as a colimit in this way, and the free category functor on graphs preserves colimits. □

Proposition A.7. *The functor* $\mathrm{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}$ *preserves products.*

Proof. Since C preserves colimits and the Cartesian products in Cat and Set_Δ both commute with colimits in each variable, it suffices to check that the natural map $\mathrm{C}(\Delta^n \times \Delta^m) \to \mathrm{C}(\Delta^n) \times \mathrm{C}(\Delta^m)$ is an isomorphism for all n, m. Since products of inner anodyne maps are inner anodyne by [Lur09a, Corollary 2.3.2.4], the inclusion $\mathrm{Sp}^n \times \mathrm{Sp}^m \to \Delta^n \times \Delta^m$ is inner anodyne. Thus in the diagram

$$\begin{tikzcd} \mathrm{C}(\mathrm{Sp}^n \times \mathrm{Sp}^m) \arrow[r] \arrow[d] & \mathrm{C}(\mathrm{Sp}^n) \times \mathrm{C}(\mathrm{Sp}^m) \arrow[d] \\ \mathrm{C}(\Delta^n \times \Delta^m) \arrow[r] & \mathrm{C}(\Delta^n) \times \mathrm{C}(\Delta^m) \end{tikzcd}$$

the vertical maps are isomorphisms by Lemma A.6. It hence suffices to prove that the upper horizontal map is an isomorphism. Since C preserves colimits and the Cartesian products preserve colimits in each variable, in the commutative diagram

$$\begin{tikzcd} \mathrm{C}(\Delta^1 \times \mathrm{Sp}^m) \amalg_{\mathrm{C}(\Delta^0 \times \mathrm{Sp}^m)} \mathrm{C}(\mathrm{Sp}^n \times \mathrm{Sp}^m) \arrow[r] \arrow[d] & \mathrm{C}(\mathrm{Sp}^{n+1} \times \mathrm{Sp}^m) \arrow[d] \\ \mathrm{C}(\Delta^1) \times \mathrm{C}(\mathrm{Sp}^m) \amalg_{\mathrm{C}(\Delta^0) \times \mathrm{C}(\mathrm{Sp}^m)} \mathrm{C}(\mathrm{Sp}^n) \times \mathrm{C}(\mathrm{Sp}^m) \arrow[r] & \mathrm{C}(\mathrm{Sp}^{n+1}) \times \mathrm{C}(\mathrm{Sp}^m) \end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{C}(\Delta^1 \times \mathrm{Sp}^m) \amalg_{\mathrm{C}(\Delta^0 \times \mathrm{Sp}^m)} \mathrm{C}(\mathrm{Sp}^n \times \mathrm{Sp}^m) \arrow[r] \arrow[d] & \mathrm{C}(\mathrm{Sp}^{n+1} \times \mathrm{Sp}^m) \arrow[d] \\
\mathrm{C}(\Delta^1) \times \mathrm{C}(\mathrm{Sp}^m) \amalg_{\mathrm{C}(\Delta^0) \times \mathrm{C}(\mathrm{Sp}^m)} \mathrm{C}(\mathrm{Sp}^n) \times \mathrm{C}(\mathrm{Sp}^m) \arrow[r] & \mathrm{C}(\mathrm{Sp}^{n+1}) \times \mathrm{C}(\mathrm{Sp}^m)
\end{tikzcd}$$

the vertical morphisms are isomorphisms. By inducting on n and m this implies that the map in question is an isomorphism for all n and m provided

$$\mathrm{C}(\Delta^n \times \Delta^m) \to \mathrm{C}(\Delta^n) \times \mathrm{C}(\Delta^m)$$

is an isomorphism when n and m are both either 0 or 1. The cases where n or m is 0 are trivial, so it only remains to show that $\mathrm{C}(\Delta^1 \times \Delta^1) \to [1] \times [1]$ is an isomorphism. The simplicial set $\Delta^1 \times \Delta^1$ is the pushout $\Delta^2 \amalg_{\Delta^{\{0,2\}}} \Delta^2$, so this amounts to showing that the analogous functor $[2] \amalg_{[1]} [2] \to [1] \times [1]$ is an isomorphism, or equivalently that for any category $\mathbf{C}$, the square

$$\begin{tikzcd}
\mathrm{Hom}([1] \times [1], \mathbf{C}) \arrow[r] \arrow[d] & \mathrm{Hom}([2], \mathbf{C}) \arrow[d] \\
\mathrm{Hom}([2], \mathbf{C}) \arrow[r] & \mathrm{Hom}([1], \mathbf{C})
\end{tikzcd}$$

is Cartesian. But this claim is equivalent to the statement that a commutative square in $\mathbf{C}$ is the same as two compatible commutative triangles, which is obvious. □

Definition A.8. The functor N preserves products, being a right adjoint, and so induces a functor $\mathrm{N}_*\colon \mathrm{Cat}_2 \to \mathrm{Cat}_\Delta$, given by applying N on the mapping spaces; this has a left adjoint $\mathrm{C}_*\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}_2$ given by composition with C, since C preserves products by Proposition A.7.

We now briefly recall the definition of the coherent nerve functor from simplicial categories to simplicial sets, following [Lur09a, §1.1.5]:

Definition A.9. Let $P_{i,j}$ be the partially ordered set of subsets of $\{i, i+1, \ldots, j\}$ containing i and j. Then $\mathfrak{C}(\Delta^n)$ denotes the simplicial category with objects $0, \ldots, n$ and

$$\mathfrak{C}(\Delta^n)(i,j) = \begin{cases} \emptyset, & i > j \\ \mathrm{N}P_{i,j}, & \text{otherwise} \end{cases}$$

Composition of morphisms is induced by union of the subsets in the $P_{i,j}$'s.

Remark A.10. The simplicial set $\mathrm{N}P_{i,j}$ is isomorphic to $(\Delta^1)^{\times(j-i-1)}$ for $j > i$.

Definition A.11. The *coherent nerve* is the functor $\mathfrak{N}\colon \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$ defined by

$$\mathfrak{N}\mathbf{C}_k = \mathrm{Hom}(\mathfrak{C}(\Delta^k), \mathbf{C}).$$

This has a left adjoint $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, which is the unique colimit-preserving functor extending the cosimplicial simplicial category $\mathfrak{C}(\Delta^\bullet)$.

Definition A.12. Let $\mathrm{N}_2\colon \mathrm{Cat}_2 \to \mathrm{Set}_\Delta$ denote the composite

$$\mathrm{Cat}_2 \xrightarrow{\mathrm{N}_*} \mathrm{Cat}_\Delta \xrightarrow{\mathfrak{N}} \mathrm{Set}_\Delta.$$

This functor has a left adjoint C_2, which is the composite

$$\mathrm{Set}_\Delta \xrightarrow{\mathfrak{C}} \mathrm{Cat}_\Delta \xrightarrow{\mathrm{C}_*} \mathrm{Cat}_2.$$

Remark A.13. It is clear from the definitions given in [Dus02, §§6.1–6.7] that the functor N_2 as we have defined it is simply the restriction of Duskin's nerve for bicategories to strict 2-categories. (This nerve also implicitly appeared earlier in [Str87].)

Remark A.14. We can describe the strict 2-category $\mathrm{C}_2(\Delta^n)$ as follows: its objects are $0, \ldots, n$. For $i > j$, the category $\mathrm{C}_2(\Delta^n)(i,j)$ is empty, and for $j > i$ it is the partially ordered set $P_{i,j}$ (which is isomorphic to $[1]^{\times(j-i-1)}$ if $j > i$). We can thus describe the low-dimensional simplices of the nerve $\mathrm{N}_2\mathbf{C}$ of a strict 2-category $\mathbf{C}$ as follows:

- The 0-simplices are the objects of $\mathbf{C}$.
- The 1-simplices are the 1-morphisms of $\mathbf{C}$.
- A 2-simplex in $\mathrm{N}_2\mathbf{C}$ is given by objects x_0, x_1, x_2, 1-morphisms $f_{01}\colon x_0 \to x_1$, $f_{12}\colon x_1 \to x_2$, $f_{02}\colon x_0 \to x_2$, and a 2-morphism $\phi_{012}\colon f_{02} \Rightarrow f_{12} \circ f_{01}$.
- A 3-simplex is given by
  - objects x_0, x_1, x_2, x_3,
  - 1-morphisms $f_{ij}\colon x_i \to x_j$ for $0 \le i < j \le 3$,
  - 2-morphisms $\phi_{012}\colon f_{02} \Rightarrow f_{12} \circ f_{01}$, $\phi_{123}\colon f_{13} \Rightarrow f_{23} \circ f_{12}$, $\phi_{023}\colon f_{03} \Rightarrow f_{23} \circ f_{02}$ and $\phi_{013}\colon f_{03} \Rightarrow f_{13} \circ f_{01}$, such that the square

$$\begin{tikzcd} f_{03} \arrow[r, "\phi_{013}"] \arrow[d, "\phi_{023}"'] & f_{13} \circ f_{01} \arrow[d, "\phi_{123} \circ \mathrm{id}"] \\ f_{23} \circ f_{02} \arrow[r, "\mathrm{id} \circ \phi_{012}"'] & f_{23} \circ f_{12} \circ f_{01} \end{tikzcd}$$

    commutes.

Definition A.15. Let $\mathbf{\Delta}_{\le k}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[n]$ for $n \le k$. The restriction $\mathrm{sk}_k\colon \mathrm{Set}_\Delta \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\le k}, \mathrm{Set})$ has a right adjoint

$$\mathrm{cosk}_k\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\le k}, \mathrm{Set}) \to \mathrm{Set}_\Delta.$$

We say a simplicial set X is *k-coskeletal* if it is in the image of the functor cosk_k. Equivalently, X is k-coskeletal if every map $\partial\Delta^n \to X$ extends to a unique n-simplex $\Delta^n \to X$ when $n > k$.

Proposition A.16. *For every strict 2-category* $\mathbf{C}$, *the simplicial set* $\mathrm{N}_2\mathbf{C}$ *is 3-coskeletal.*

Remark A.17. A more general version of this result in the setting of bicategories appears in [Dus02].

Proof. We must show that every map $\partial\Delta^k \to \mathrm{N}_2\mathbf{C}$ extends to a unique map from Δ^k if $k > 3$. Equivalently, we must show that given a map $\mathfrak{C}(\partial\Delta^k) \to \mathrm{N}_*\mathbf{C}$ it has a unique extension to $\mathfrak{C}(\Delta^k)$ for $k > 3$. We can describe the simplicial category $\mathfrak{C}(\partial\Delta^k)$ and its map to $\mathfrak{C}(\Delta^k)$ as follows:

- the objects of $\mathfrak{C}(\partial\Delta^k)$ are $0, \dots k$,
- the maps $\mathfrak{C}(\partial\Delta^k)(i,j) \to \mathfrak{C}(\Delta^k)(i,j)$ are isomorphisms except when $i = 0$ and $j = k$,
- the simplicial set $\mathfrak{C}(\partial\Delta^k)(0,k)$ is the boundary of the $(k-1)$-cube $\mathfrak{C}(\Delta^k)(0,k) \cong (\Delta^1)^{\times(k-1)}$.

Thus extending a map $F\colon \mathfrak{C}(\partial\Delta^k) \to \mathrm{N}_*\mathbf{C}$ to $\mathfrak{C}(\Delta^k)$ amounts to extending the map

$$\mathfrak{C}(\partial\Delta^k)(0,k) \to \mathrm{N}\mathbf{C}(F(0), F(k))$$

to $\mathfrak{C}(\Delta^k)(0,k)$. But the inclusion $\mathfrak{C}(\partial\Delta^k)(0,k) \to \mathfrak{C}(\Delta^k)(0,k)$ is a composition of pushouts of inner horn inclusions and the inclusion $\partial\Delta^{k-1} \to \Delta^{k-1}$, and if $k-1 > 2$ the nerve of a category has unique extensions along these. $\square$

Theorem A.18 (Duskin, Bullejos–Faro–Blanco). *Suppose $\mathbf{C}$ and $\mathbf{D}$ are strict 2-categories. Then the maps of simplicial sets $\mathrm{N}_2\mathbf{C} \to \mathrm{N}_2\mathbf{D}$ can be identified with the normal oplax functors $\mathbf{C} \to \mathbf{D}$.*

Remark A.19. The more general version of this result for bicategories appears to be an unpublished result of Duskin; for 2-categories it is proved by Bullejos, Faro, and Blanco as [BFB05, Proposition 4.3]. We do not include a complete proof here, but we will now briefly indicate how a map of nerves gives rise to a normal oplax functor. By Proposition A.16, a map $\mathrm{N}_2\mathbf{C} \to \mathrm{N}_2\mathbf{D}$ can be identified with a map $F\colon \mathrm{sk}_3\mathrm{N}_2\mathbf{C} \to \mathrm{sk}_3\mathrm{N}_2\mathbf{D}$. Using Remark A.14 we can identify this with the data of a normal oplax functor as given in Definition A.2:

- The 0-simplices of $\mathrm{N}_2\mathbf{C}$ are the objects of $\mathbf{C}$, so F assigns an object $F(c) \in \mathbf{D}$ to every $c \in \mathbf{C}$, which gives (a)
- The 1-simplices of $\mathrm{N}_2\mathbf{C}$ are the 1-morphisms in $\mathbf{C}$, with sources and targets given by the face maps $[0] \to [1]$, so F assigns a 1-morphism $F(f)\colon F(x) \to F(y)$ to every 1-morphism $f\colon x \to y$ in $\mathbf{C}$, which gives (b).
- Moreover, identity 1-morphisms correspond to degenerate edges in $\mathrm{N}_2\mathbf{C}$, so since these are preserved by any map of simplicial sets we get $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$, i.e. (i).
- The 2-simplices of $\mathrm{N}_2\mathbf{C}$ are given by three 1-morphisms $f\colon x \to y$, $g\colon y \to z$, $h\colon z \to w$ (corresponding to the three face maps), and a 2-morphism $\phi\colon h \Rightarrow g \circ f$. In particular:
  - Considering 2-simplices where the second edge is degenerate, which correspond to 2-morphisms in $\mathbf{C}$, we see that F assigns a 2-morphism $F(\phi)\colon F(h) \Rightarrow F(g)$ to every $\phi\colon h \Rightarrow g$ in $\mathbf{C}$, which gives (c).
  - Considering 2-simplices where the 2-morphism ϕ is the identity, we see (as this condition is not preserved by F) that F assigns a 2-morphism $F(g \circ f) \Rightarrow F(g) \circ F(f)$ to all composable pairs of 1-morphisms, which gives (d).

- Since F preserves degenerate 2-simplices, which correspond to identity 2-morphisms of the form $f \circ \mathrm{id} \Rightarrow f$ and $\mathrm{id} \circ f \Rightarrow f$, we get (ii) and (iv).
- The 3-simplices of $\mathrm{N}_2\mathbf{C}$ are given by
  - objects x_0, x_1, x_2, x_3,
  - 1-morphisms $f_{ij}\colon x_i \to x_j$ for $0 \leq i < j \leq 3$,
  - 2-morphisms $\phi_{012}\colon f_{02} \Rightarrow f_{12} \circ f_{01}$, $\phi_{123}\colon f_{13} \Rightarrow f_{23} \circ f_{12}$, $\phi_{023}\colon f_{03} \Rightarrow f_{23} \circ f_{02}$ and $\phi_{013}\colon f_{03} \Rightarrow f_{13} \circ f_{01}$, such that the square

$$\begin{tikzcd} f_{03} \arrow[r, "\phi_{013}"] \arrow[d, "\phi_{023}"'] & f_{13} \circ f_{01} \arrow[d, "\phi_{123} \circ \mathrm{id}"] \\ f_{23} \circ f_{02} \arrow[r, "\mathrm{id} \circ \phi_{012}"'] & f_{23} \circ f_{12} \circ f_{01} \end{tikzcd}$$

    commutes.

  In particular, we have:
  - If $x_1 = x_2 = x_3$, $f_{12} = f_{13} = f_{23} = \mathrm{id}_{x_1}$, and $\phi_{123} = \mathrm{id}_{\mathrm{id}_{x_1}}$, then this says $\phi_{013} = \phi_{012} \circ \phi_{023}$, and since F preserves identities this gives (iii).
  - In the case where the 2-morphisms are all identities, we get (vi).
  - To get (v), we consider the 3-simplices where $f_{12} = \mathrm{id}$, $\phi_{023} = \mathrm{id}$, and ϕ_{013} is the composite of ϕ_{012} and ϕ_{123}.

Definition A.20. The inclusion $\mathrm{Gpd} \hookrightarrow \mathrm{Cat}$ of the category of groupoids preserves products, and so induces a functor $\mathrm{Cat}_{(2,1)} \to \mathrm{Cat}_2$; we write $\mathrm{N}_{(2,1)}$ for the composite

$$\mathrm{Cat}_{(2,1)} \to \mathrm{Cat}_2 \xrightarrow{\mathrm{N}_2} \mathrm{Set}_\Delta.$$

Corollary A.21. *If $\mathbf{C}$ and $\mathbf{D}$ are strict (2,1)-categories, then a morphism of simplicial sets $\mathrm{N}_{(2,1)}\mathbf{C} \to \mathrm{N}_{(2,1)}\mathbf{D}$ can be identified with a normal pseudofunctor $\mathbf{C} \to \mathbf{D}$.*

Definition A.22. Recall that a *relative category* is a category $\mathbf{C}$ equipped with a subcategory W containing all isomorphisms; see [BK12] for a more extensive discussion. A functor of relative categories $f\colon (\mathbf{C}, W) \to (\mathbf{C}', W')$ is a functor $f\colon \mathbf{C} \to \mathbf{C}'$ that takes W into W'. We write $\mathrm{RelCat}_{(2,1)}$ for the strict (2,1)-category of relative categories, functors of relative categories, and all natural isomorphisms between these.

We now want to prove that a normal pseudofunctor to $\mathrm{RelCat}_{(2,1)}$ determines a map of ∞-categories to Cat_∞ via the following construction:

Definition A.23. If $(\mathbf{C}, W)$ is a relative category, let $\mathrm{L}(\mathbf{C}, W) \in \mathrm{Set}_\Delta^+$ be the marked simplicial set $(\mathrm{N}\mathbf{C}, \mathrm{N}W_1)$. This defines a simplicial functor $\mathrm{N}_*\mathrm{RelCat}_{(2,1)} \to \mathrm{Set}_\Delta^+$.

Definition A.24. If $(\mathbf{C}, W)$ is a relative category, we write $\mathbf{C}[W^{-1}]$ for the ∞-category obtained by taking a fibrant replacement of the marked simplicial set

$\mathrm{L}(\mathbf{C}, W)$. More generally, if $\mathbf{C}$ is a strict (2,1)-category and W is a collection of 1-morphisms in $\mathbf{C}$, we write $\mathbf{C}[W^{-1}]$ for the ∞-category obtained by fibrantly replacing the marked simplicial set $(\mathrm{N}_{(2,1)}\mathbf{C}, W)$.

LEMMA A.25. *Let $\mathbf{C}$ be a strict (2,1)-category, and let F be a normal pseudofunctor $F\colon \mathbf{C} \to \mathrm{RelCat}_{(2,1)}$. If W is a collection of 1-morphisms in $\mathbf{C}$ such that F takes the morphisms in W to weak equivalences of relative categories, then F determines a functor of ∞-categories $\mathbf{C}[W^{-1}] \to \mathrm{Cat}_\infty$, which sends $x \in \mathbf{C}$ to $\mathbf{E}_x[W_x^{-1}]$ where $F(x) = (\mathbf{E}_x, W_x)$.*

Proof. By Proposition A.21 the normal pseudofunctor F corresponds to a map of simplicial sets $\mathrm{N}_{(2,1)}\mathbf{C} \to \mathrm{N}_{(2,1)}\mathrm{RelCat}_{(2,1)}$. Composing this with the map $\mathfrak{N}(L)\colon \mathrm{N}_{(2,1)}\mathrm{RelCat}_{(2,1)} \to \mathfrak{N}\mathrm{Set}_\Delta^+$ we get a map $\mathrm{N}_{(2,1)}\mathbf{C} \to \mathfrak{N}\mathrm{Set}_\Delta^+$. We may regard this as a map of marked (large) simplicial sets

$$(\mathrm{N}_{(2,1)}\mathbf{C}, W) \to (\mathfrak{N}\mathrm{Set}_\Delta^+, W'),$$

where W' is the collection of marked equivalences in Set_Δ^+. Now invoking [Lur14, Theorem 1.3.4.20] we conclude that Cat_∞ is a fibrant replacement for the marked simplicial set $(\mathfrak{N}\mathrm{Set}_\Delta^+, W')$, so this map corresponds to a map $\mathbf{C}[W^{-1}] \to \mathrm{Cat}_\infty$ in the ∞-category $\widehat{\mathrm{Cat}}_\infty$ underlying the model category of (large) marked simplicial sets. □

We will now make use of Grothendieck's description of pseudofunctors to the (2,1)-category of categories to get a way of constructing pseudofunctors to $\mathrm{RelCat}_{(2,1)}$:

THEOREM A.26 (Grothendieck [Gro63]). *Let $\mathbf{C}$ be a category. Then pseudofunctors from $\mathbf{C}^{\mathrm{op}}$ to the strict 2-category* CAT *correspond to Grothendieck fibrations over $\mathbf{C}$.*

REMARK A.27. Let us briefly recall how a pseudofunctor is constructed from a Grothendieck fibration, as this is the part of Grothendieck's theorem we will actually use. A *cleavage* of a Grothendieck fibration $p\colon \mathbf{E} \to \mathbf{B}$ is the choice, for each $(e \in \mathbf{E}, f\colon b \to p(e))$, of a single Cartesian morphism over f with target e; cleavages always exist, by the axiom of choice. Given a choice of cleavage of p, we define the pseudofunctor $\mathbf{C}^{\mathrm{op}} \to \mathrm{CAT}$ by assigning the fibre $\mathbf{E}_b$ to each $b \in \mathbf{B}$, and for each $f\colon b \to b'$ the functor f^* assigns to $e \in \mathbf{E}_b$ the source of the Cartesian morphism over f with target e in the cleavage. Clearly, this pseudofunctor will be normal precisely when the cleavage is *normal* in the sense that the Cartesian morphisms over the identities in $\mathbf{B}$ are all chosen to be identities in $\mathbf{E}$. Every Grothendieck fibration obviously has a normal cleavage, so from any Grothendieck fibration we can construct a normal pseudofunctor.

DEFINITION A.28. A *relative Grothendieck fibration* is a Grothendieck fibration $p\colon \mathbf{E} \to \mathbf{C}$ together with a subcategory $\mathbf{W}$ of $\mathbf{E}$ containing all the p-Cartesian morphisms. In particular, the restricted projection $\mathbf{W} \to \mathbf{C}$ is also a Cartesian fibration. Moreover, for every $x \in \mathbf{C}$ the fibres $(\mathbf{E}_x, \mathbf{W}_x)$ are relative categories, and the functor f^* induced by each f in $\mathbf{C}$ is a functor of relative categories.

If $(\mathbf{C}, U)$ is a relative category, we say that the relative Grothendieck fibration is *compatible with* U if this functor $f^*\colon (\mathbf{E}_q, \mathbf{W}_q) \to (\mathbf{E}_p, \mathbf{W}_p)$ is a weak equivalence of relative categories for every $f\colon p \to q$ in U.

The following is then an obvious consequence of Theorem A.26:

LEMMA A.29. *Relative Grothendieck fibrations over a category* $\mathbf{C}$ *correspond to normal pseudofunctors* $\mathbf{C}^{\mathrm{op}} \to \mathrm{RelCat}_{(2,1)}$.

PROPOSITION A.30. *Let* $(\mathbf{E}, \mathbf{W})$ *be a relative Grothendieck fibration over* $\mathbf{C}$ *compatible with a collection* U *of morphisms in* $\mathbf{C}$*. Then this induces a functor of* ∞*-categories*

$$\mathbf{C}[U^{-1}]^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

that sends $p \in \mathbf{C}$ *to* $\mathbf{E}_p[\mathbf{W}_p^{-1}]$.

Proof. Combine Lemmas A.29 and A.25. □

All the maps whose naturality we are interested in can easily be constructed as relative Grothendieck fibrations. We will explicitly describe this in the case of the unstraightening equivalence, and leave the other cases to the reader.

PROPOSITION A.31. *The unstraightening functors*

$$\mathrm{Un}_S^+\colon \mathrm{Fun}_{\mathbf{\Delta}}(\mathfrak{C}(S)^{\mathrm{op}}, \mathrm{Set}_\Delta^+)^{\mathrm{fib}} \to (\mathrm{Set}_\Delta^+)^{\mathrm{fib}}_{/S}$$

define a relative Grothendieck fibration over $\mathrm{Set}_\Delta \times \Delta^1$ *compatible with the categorical equivalences in* Set_Δ.

Proof. Let $\mathbf{E}$ be the category whose objects are triples (i, S, X) where $i = 0$ or 1, $S \in \mathrm{Set}_\Delta$, and X is a fibrant map $Y \to S^\sharp$ in Set_Δ^+ if $i = 0$ and a fibrant simplicial functor $\mathfrak{C}(S)^{\mathrm{op}} \to \mathrm{Set}_\Delta^+$ if $i = 1$; the morphisms $(i, S, X) \to (j, T, Y)$ consist of a morphism $i \to j$ in $[1]$, a morphism $f\colon S \to T$ in Set_Δ, and the following data:

- if $i = j = 1$, $X\colon \mathfrak{C}(S) \to \mathrm{Set}_\Delta^+$ and $Y\colon \mathfrak{C}(T) \to \mathrm{Set}_\Delta^+$, a simplicial natural transformation $X \to \mathfrak{C}(f) \circ Y$,
- if $i = j = 0$, X is $E \to S$ and Y is $F \to T$, a commutative square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ S^\sharp \arrow[r] & T^\sharp \end{tikzcd}$$

in Set_Δ^+,

- if $i = 1$ and $j = 0$, Y is a functor $\mathfrak{C}(S)^{\mathrm{op}} \to \mathrm{Set}_\Delta^+$ and X is $E \to T$, a commutative square

$$\begin{tikzcd} \mathrm{Un}_S^+(X) \arrow[r] \arrow[d] & E \arrow[d] \\ S \arrow[r] & T. \end{tikzcd}$$

Composition is defined in the obvious way, using the natural maps of [Lur09a, Proposition 3.2.1.4]. We claim that the projection $\mathbf{E} \to \Delta^1 \times \mathrm{Set}_\Delta$ is a Grothendieck fibration. It suffices to check that Cartesian morphisms exist for morphisms of the form (id_i, f) and $(0 \to 1, \mathrm{id}_S)$, which is clear. $\square$

Corollary A.32. *There is a functor of ∞-categories $\mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ that sends $\mathcal{C}$ to the unstraightening equivalence*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}.$$

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