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CONTENTS

Abstract

We give a systematic study of a new equivariant cohomology theory $t(k_G)^*$ that we construct from a given equivariant cohomology theory k_G^*, where G is a compact Lie group. If k_G^* is ordinary cohomology, then $t(k_G)^*$ is classical Tate-Swan cohomology if G is finite and is Jones' version of cyclic cohomology if G is the circle group. As in these cases, $t(k_G)^*$ vanishes on free G-spaces, enjoys useful periodicity properties, and is obtained by splicing the k-homology with the k-cohomology of the Borel construction $EG \times_G X$, where k^* is the nonequivariant cohomology theory that underlies k_G^*.

After establishing the formal properties of our theories, including the fact that $t(k_G)^*$ is multiplicative if k_G^* is multiplicative, we construct an analog of the Atiyah-Hirzebruch spectral sequence for the calculation of $t(k_G)^*(X)$. If G is finite or S^1, then the E_2-term is the Tate-Swan or cyclic cohomology of X with coefficients in k^*. The convergence of our spectral sequences is rather delicate, and we give a careful study that may be of wider interest.

We consider various special cases in detail. If G is cyclic of order 2, then the fixed point spectrum of the G-spectrum $t(k_G)$ is equivalent to $\text{holim}(\mathbb{R}P^{\infty}_{-i} \wedge \Sigma k)$ and our Atiyah-Hirzebruch spectral sequence generalizes Mahowald's root invariant spectral sequence for the stable homotopy groups of spheres. This result, among others, establishes a close relationship between Tate theory and areas of current interest in nonequivariant stable homotopy theory. In particular, we show that there is a web of relations connecting Tate cohomology for general finite groups with nonequivariant stable homotopy groups.

In the case of periodic equivariant K-theory K_G^*, we prove the striking fact that $t(K_G)^*$ is rational if G is finite, and we identify $t(K_G)^*$ explicitly. We include a complete algebraic analysis of the equivariant rational stable homotopy category for this purpose. Here the E_2-term of our Atiyah-Hirzebruch spectral sequence is annihilated by the order of G, and yet it converges strongly to the rational vector space $t(K_G)^*$. The arguments establish an intimate connection between the rationality of the Tate theory and the Atiyah-Segal completion theorem.

These results all have analogs for general families of isotropy groups. We develop these analogs, together with the relevant algebra, in a final part. Here, for finite groups, the role of classical Tate cohomology is played by Amitsur-Dress-Tate cohomology, which first appeared in induction theory and deserves much further study.

Key words and phrases:

Amitsur-Dress cohomology, Atiyah-Hirzebruch spectral sequence, Atiyah-Segal completion theorem, Borel cohomology, Burnside ring, Cyclic cohomology, Eilenberg-MacLane G-spectrum, Equivariant cohomology theory, Equivariant stable homotopy theory, Homotopy limit problem, K-theory, Mackey functor, Root invariant, Segal conjecture, Stable homotopy groups of spheres, Tate cohomology, Transfer

Introduction

Tate cohomology plays a prominent role in finite group theory and its applications. In connection with Smith theory, Swan generalized the purely algebraic theory to a cohomology theory defined on G-spaces. The resulting theory is related to Borel cohomology, $H^*(EG \times_G X)$, by a long exact sequence whose third term we call f-cohomology. Borel cohomology is one of the most basic tools in the theory of transformation groups. Tate cohomology can be thought of as obtained from Borel cohomology by a process of killing the cohomology groups of free G-spaces, and the vanishing of Tate theory on free G-spaces makes it particularly well-suited to the study of fixed point phenomena. Tate theory also enjoys calculationally powerful periodicity properties.

More recent topological work, especially in surgery theory, has led a number of people to consider analogs of Tate theory associated to spectra with G-actions and to consider analogs for compact Lie groups. When specialized to the circle group, Tate theory recovers and generalizes one manifestation of cyclic cohomology theory. However, there has been no systematic study.

The last decade has seen a large-scale development of equivariant stable homotopy theory, with a concomitant understanding of generalized equivariant homology and cohomology theories. We shall redevelop and generalize Tate and Borel cohomology theories within this now well-established framework.

Let G be a compact Lie group, let EG be a free contractible G-space, and let $\widetilde{E}G$ be the unreduced suspension of EG with one of the cone points as basepoint. Let X_+ denote

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the disjoint union of a G-space X and a G-fixed basepoint. We have an evident cofiber sequence

$$(A) \qquad EG_+ \to S^0 \to \widetilde{E}G.$$

Let k_G be a G-spectrum and let $F(EG_+, k_G)$ be the function G-spectrum of maps $EG_+ \to k_G$. The projection $EG_+ \to S^0$ induces a map of G-spectra

$$(B) \qquad \varepsilon : k_G = F(S^0, k_G) \to F(EG_+, k_G).$$

By taking the smash product of the cofibering (A) with the map (B), we obtain the following map of cofiberings of G-spectra:

$$(C) \qquad \begin{tikzcd} k_G \wedge EG_+ \arrow[r] \arrow[d, "\varepsilon \wedge 1"'] & k_G \arrow[r] \arrow[d, "\varepsilon"] & k_G \wedge \widetilde{E}G \arrow[d, "\varepsilon \wedge 1"] \\ F(EG_+, k_G) \wedge EG_+ \arrow[r] & F(EG_+, k_G) \arrow[r] & F(EG_+, k_G) \wedge \widetilde{E}G. \end{tikzcd}$$

We introduce abbreviated notations for these spectra and explain the intuitions. Roughly speaking, smashing a G-spectrum with the cofibering (A) has the effect of breaking the represented homology and cohomology theories into parts that see the free orbits and the singular orbits of G-spaces: this is a modern formulation of an old idea.

Define

$$f(k_G) = k_G \wedge EG_+.$$

We call $f(k_G)$ the free G-spectrum associated to k_G. This construction is the spectrum level analog of the standard way of associating to a based G-space X a new based G-space whose action is free away from the basepoint. We shall see that $f(k_G)$ represents the appropriate generalized version of the Borel homology theory $H_*(EG \times_G X)$, and we shall therefore refer to all of the homology theories represented by G-spectra $f(k_G)$ as Borel homology theories. We shall refer to the cohomology theories represented by the $f(k_G)$ simply as f-cohomology theories.

Define

$$f'(k_G) = F(EG_+, k_G) \wedge EG_+.$$

It will turn out the map $\varepsilon \wedge 1 : f(k_G) \to f'(k_G)$ is always an equivalence, so that the G-spectra $f(k_G)$ and $f'(k_G)$ can be used interchangeably. After proving this equivalence, we will drop the notation f' and just use f.

Define

$$f^{\perp}(k_G) = k_G \wedge \tilde{E}G.$$

We call $f^{\perp}(k_G)$ the singular G-spectrum associated to k_G. This construction gives the spectrum level analog of the standard way of associating to a based G-space X a new based G-space that has the same fixed points as X under the action of any non-trivial subgroup of G but is nonequivariantly contractible.

Define

$$c(k_G) = F(EG_+, k_G).$$

We call $c(k_G)$ the geometric completion of k_G. The map $\varepsilon : k_G \to c(k_G)$ displayed in (B) is the object of study of such results as the Atiyah-Segal completion theorem and the Segal conjecture: quite generally, the question of the behavior of ε on G-fixed point spectra is called the "stable homotopy limit problem". As we shall explain later, one interpretation of this problem is that of comparing the geometric completion $c(k_G)$ with the algebraic completion $(k_G)^{\wedge}_I$ of k_G at the augmentation ideal of the Burnside ring. We shall see that $c(k_G)$ represents the appropriate generalized version of Borel cohomology $H^*(EG \times_G X)$, and we shall therefore refer to all of the cohomology theories represented by G-spectra $c(k_G)$ as Borel cohomology theories. We shall refer to the homology theories represented by the $c(k_G)$ as c-homology theories.

Define

$$t(k_G) = F(EG_+, k_G) \wedge \tilde{E}G = f^{\perp}c(k_G).$$

We call $t(k_G)$ the Tate G-spectrum associated to k_G. It is the singular part of the geometric completion of k_G. Our primary focus will be on the theories represented by the $t(k_G)$. These are our generalized Tate homology and cohomology theories.

With this cast of characters, and with the abbreviation of $\varepsilon \wedge 1$ to ε, diagram (C) can be rewritten in the form

(D)
$$\begin{tikzcd} f(k_G) \arrow[r] \arrow[d, "\varepsilon"'] & k_G \arrow[r] \arrow[d, "\varepsilon"] & f^{\perp}(k_G) \arrow[d, "\varepsilon"] \\ f'(k_G) \arrow[r] & c(k_G) \arrow[r] & t(k_G). \end{tikzcd}$$

We call the bottom row the "norm sequence". It is a generalization of the classical norm sequence in the Tate cohomology of groups.

We shall study general features of the homology and cohomology theories on G-spaces represented by these G-spectra and shall discuss a few important examples in detail. These simple and conceptual definitions include all previous versions of these theories, and they lead to a number of new and unexpected calculations. Diagram (D) relating them encodes an extremely convenient unifying framework for the conceptual study of a variety of phenomena that are central to equivariant homology and cohomology theory. We shall see that it is also closely related to certain areas of current interest in nonequivariant stable homotopy theory.

It is clearly sensible to break the study of the homotopy limit problem into free and singular parts, as is formalized by the diagram. This was a key idea in Carlsson's proof of the Segal conjecture for finite p-groups [11]. The present framework began to emerge in [13]; all ingredients of diagram (C) were explicit there, and its bottom row was exploited with G taken to be a p-group and k_G taken to be the Eilenberg-MacLane G-spectrum $H\underline{Z}_p$ constructed in [32].

For finite groups G and Eilenberg-MacLane G-spectra HM, $t(HM)$ represents classical Tate-Swan homology and cohomology. This is an insight of the first author [19], who was the first to study the representation of Tate theories. The second author wishes to emphasize that the germ of much of the material here is in [19]; in particular, the definitions above are obvious generalizations of those given there. Elsewhere [20], [21], the first author has used the c-homology and f-cohomology theories associated to $H\underline{Z}_p$ to set up and analyze equivariant versions of the Adams spectral sequence.

For the circle group G and Eilenberg-MacLane G-spectra HM, $t(HM)$ represents periodic cyclic homology and cohomology theory. There is an analogous identification for the group of unit quaternions. In his paper [29] on cyclic theory, Jones had noted that "there is a clear and precise analogy with the Tate homology of groups". Adem, Cohen, and Dwyer [4] were the first to make this insight explicit and the first to consider Tate theories associated to compact Lie groups. For finite G-CW complexes X, they constructed spectra whose homotopy groups are the cyclic homology groups of X when $G = S^1$. They did not consider cohomology.

For general compact Lie groups G and Eilenberg-MacLane G-spectra HM, our theories give the appropriate generalization of Tate-Swan and cyclic homology and cohomology, but we no longer have a chain level method to calculate the values of these theories on general G-spaces. However, we do have a reasonable, and quite computable, chain level method for calculating the coefficient groups $t(HM)^*$. Here M is a Mackey functor, and the bottom row of diagram (C) depends only on the $\pi_0(G)$-module $V = M(G/e)$. We therefore write the represented Tate cohomology theory as $\widehat{H}_G^*(X;V)$ in all cases, generalizing the standard notation for finite groups and the circle group. When G is connected of dimension $d > 0$ and $X = S^0$, we have the following explicit calculation in terms of the ordinary (unreduced) homology and cohomology groups of the classifying space BG.

$$\widehat{H}_G^n(V) = t(HM)^n \cong \begin{cases} H^n(BG;V) & \text{if } 0 \leq n \\ 0 & \text{if } -d \leq n < 0 \\ H_{-n-1-d}(BG;V) & \text{if } n \leq -d-1. \end{cases}$$

The dependence on d is a natural consequence of the fact that the Spanier-Whitehead dual of G_+ is $G_+ \wedge S^{-d}$, but it is nevertheless a rather startling phenomenon at first sight.

One of the main contributions of this paper is the construction of spectral sequences of Atiyah-Hirzebruch type that generalize the identifications of the previous paragraphs. In Tate cohomology, the E_2-term is $\widehat{H}^*_G(X;k^*)$, where k is the underlying nonequivariant spectrum of k_G and $k^{-q} = \pi_q(k)$ regarded as a $\pi_0(G)$-module. Although this is a whole plane spectral sequence, it converges strongly to $t(k_G)^*(X)$ provided that there are not too many non-zero higher differentials. Moreover, when k_G is a ring spectrum, this is a spectral sequence of differential algebras. We emphasize that this works for general compact Lie groups G. We have similar spectral sequences for Borel and f-cohomology, and in these cases too the E_2-terms depend only on the graded $\pi_0(G)$-module k^*.

This very weak dependence on k_G makes the bottom row of Diagram (D) far more calculationally accessible than the top row, and another contribution of our paper is to bring the theory down to earth with some explicit calculations in K-theory and some concrete relationships to nonequivariant stable homotopy theory.

Our most interesting calculation shows that, for any finite group G, $t(KU_G)$ is a rational G-spectrum, namely

$$t(KU_G) \simeq \bigvee K(\underline{J}^\wedge \otimes \mathbb{Q}, 2i),$$

where $\underline{J}^\wedge$ is the Mackey functor of completed augmentation ideals of representation rings and i ranges over the integers. In this case, the relevant Atiyah-Hirzebruch spectral sequence is rather amazing. Its E_2-term is torsion, being annihilated by multiplication by the order of G. If G is cyclic, then $E_2 = E_\infty$ and the spectral sequence certainly converges strongly. In general, the E_2-term depends solely on the classical Tate cohomology of G and not at all on its representation ring, whereas $t(KU_G)^*$ depends solely on the representation ring and not at all on the Tate cohomology. As an immediate corollary of the calculation of $t(KU_G)$, we obtain a surprisingly simple and explicit calculation of the nonequivariant K-homology of the classifying space BG:

$$K_0(BG) \cong \mathbb{Z} \quad \text{and} \quad K_1(BG) \cong J(G)^\wedge_{J(G)} \otimes (\mathbb{Q}/\mathbb{Z}).$$

For comparison, if G is the circle group, then $t(KU_G)^G$ is a homotopy inverse limit of wedges of even suspensions of KU and each even homotopy group of $t(KU_G)^G$ is isomorphic to $\mathbb{Z}[[\chi]][\chi^{-1}]$, where $1-\chi$ is the canonical irreducible one-dimensional representation of G.

Connections with current nonequivariant work come from our observation that, if G is cyclic of order 2 and $k_G = i_*k$ is "the G-spectrum associated to a non-equivariant spectrum k", then (with a mnemonic labelling of the statement), we have

$$(\mathbb{R}) \qquad t(k_G)^G \simeq \operatorname{holim}(\mathbb{R}P^{\infty}_{-i} \wedge \Sigma k).$$

Similarly, if G is cyclic of odd prime order and $k_G = i_*k$, we have

$$(\mathbb{L}) \qquad t(k_G)^G \simeq \operatorname{holim}(L^{\infty}_{-i} \wedge \Sigma k),$$

where L^{∞}_{-i} is the lens space analog of $\mathbb{R}P^{\infty}_{-i}$. Again, if G is the circle group and $k_G = i_*k$, then

$$(\mathbb{C}) \qquad t(k_G)^G \simeq \operatorname{holim}(\mathbb{C}P^{\infty}_{-i} \wedge \Sigma^2 k).$$

These are special cases of a phenomenon which occurs whenever G acts freely on the unit sphere of a representation, and this phenomenon is the source of periodic behavior in Tate theory.

These equivalences allow us to apply the reservoir of nonequivariant calculations of spectra on the right sides to study equivariant theories. It also gives new insight into the nonequivariant theories. In particular, if k is a ring spectrum, then $t(k_G)$ is naturally a ring G-spectrum and $t(k_G)^G$ is naturally a ring spectrum. This is not at all apparent from a purely nonequivariant point of view of the spectra on the right sides.

When G is cyclic of order 2 and k is the sphere spectrum S, our Atiyah-Hirzebruch spectral sequence is (up to suspension) exactly the spectral sequence constructed by Mahowald (see [1]) some 20 years ago. Its conjectural behavior led to the form of the Segal conjecture proven by Lin [34], and its structure encodes Mahowald's "root invariants". This spectral sequence and its odd prime analog have been studied more recently by Jones [28], Miller [39], Sadofsky [43], and others, but there is little hint in the literature that these spectral sequences might be multiplicative. Our spectral sequences for general finite groups G and $k = S$ encode a slew of further such symmetry invariants—not usually of a periodic nature—and they give a fascinating and mysterious web of relations among the stable homotopy groups of spheres and classifying spaces.

The theory and calculations described above are only part of the story. The entire theory admits a vast generalization, in which the universal free G-space EG is replaced by the universal $\mathcal{F}$-space $E\mathcal{F}$ for any family $\mathcal{F}$ of subgroups of G. The definitions above deal with the case $\mathcal{F} = \{e\}$, and there are precisely analogous definitions for any other family. The map $\varepsilon : k_G \to F(E\mathcal{F}_+, k_G)$ is the object of study of a generalized homotopy limit problem, special cases of which include the generalized Atiyah-Segal completion theory of [2] and the generalized Segal conjecture of [3].

When G is finite, we shall analyze the $\mathcal{F}$-Tate G-spectra $t_{\mathcal{F}}(KU_G)$, and $t_{\mathcal{F}}(KO_G)$, which turn out to be rational for any family $\mathcal{F}$, and the $\mathcal{F}$-free G-spectra $f_{\mathcal{F}}(KU_G)$ and $f_{\mathcal{F}}(KO_G)$. The generalization to families is interesting in its own right, and it leads to considerably simpler proofs than would be possible if we concentrated solely on the case $\mathcal{F} = \{e\}$.

When G is finite and k_G is an Eilenberg-MacLane G-spectrum HM, the $\mathcal{F}$-Tate G-spectrum $t_{\mathcal{F}}(HM)$ represents the generalization to homology and cohomology theories on G-spaces and G-spectra of the Amitsur-Dress-Tate theories that figure prominently in induction theory. We again obtain general Atiyah-Hirzebruch spectral sequences in the context of families. These vastly extend the web of symmetry relations relating equivariant theory with the stable homotopy groups of spheres.

The phenomena uncovered here deserve much further study. The present paper raises far more questions than it answers, and its later sections are sprinkled with open problems, conjectures, and glimpses of new and unexplored mathematical terrain.

The paper is divided into four parts. In Part I, which comprises Sections 0 through 5, we set up the general framework. We explain our definitions in Section 0. We prove invariance statements that allow us to understand when different G-spectra k_G give the same bottom row in (D) in Section 1. We record the basic general properties of our represented theories and describe the behavior of f, c, and t regarded as functors on G-spectra in Sections 2 and 3. It is to be expected that Diagram (D) is closely related to completion at the augmentation ideal I of the Burnside ring in view of the relevance of completion at I to the homotopy limit problem. In Section 4, we show that, when G is finite, $c(k_G)$ is always I-complete and $f(k_G)$ and $t(k_G)$ are I-complete if k_G is bounded below. The bounded below hypothesis is necessary since we shall see that the completion of $t(KU_G)$ at I is trivial. We also discuss the relationship between completion theorems and suitable homological analogs. In Section 5, we describe Tate homology in terms of transfer and compare our definition with other recent topological definitions.

In Part II, which comprises Sections 6 through 10, we study the "ordinary" f, c, and Tate theories obtained when k_G in (D) is an Eilenberg-MacLane G-spectrum HM and give our generalized Atiyah-Hirzebruch spectral sequences. We take a completely topological point of view in this part. For example, we give an axiomatic proof, independent of the usual chain level description, that $t(HM)^*$, with its products, agrees with classical Tate cohomology when G is finite. The interplay between the topology and algebra is especially interesting here. For example, the topology gives a new construction of a basic algebraic functor from coefficient systems to Mackey functors that was central to Lewis' study [31] of the equivariant Hurewicz theorem. One point of this "no chains" approach is that we don't know how to realize topologically the usual chain level description of products. More importantly, we want to emphasize the inevitability of our definition of the Tate cohomology of general compact Lie groups, despite the unfamiliar calculational behavior noted above.

Part III comprises Sections 11 through 16. It gives specializations and calculations, and the impatient reader may wish to turn to it first. Sections 11 and 14, respectively, give chain level studies of the relation between our represented theories for finite groups and the classical Tate-Swan theories and between our represented theories for the circle group and cyclic theories. Sections 12 and 13 give our calculational results for finite groups, except that the proofs of our results about periodic K-theory are deferred to Part IV. Section 15 gives our calculational results for the circle group. The proofs of ($\mathbb{R}$), ($\mathbb{L}$), and ($\mathbb{C}$), together with a discussion of periodicity phenomena, appear in Section 16.

Part IV, Sections 17 through 25, deals with the generalization to families, about which nothing is said in Parts I–III. Sections 17 and 18 describe the $\mathcal{F}$-version of the material of Part I. Section 19 gives the formulation and proofs of our results about KU_G and KO_G. In Section 20, we specialize to Eilenberg-MacLane G-spectra. Here, for general compact Lie groups, another interesting phenomenon appears: we must use two quite different kinds of Eilenberg-MacLane G-spectra. The cohomology theory satisfying the dimension axiom specified in terms of a Mackey functor M is represented by HM, whereas the homology theory satisfying the dimension axiom specified in terms of a "coMackey functor" N is represented by JN.

In Sections 21 through 25, we focus on finite groups. We use explicit models for the universal $\mathcal{F}$-spaces $E\mathcal{F}$ to define algebraic Amitsur-Dress-Tate cohomology theories, and we then relate them to our topological theories, giving generalized Atiyah-Hirzebruch type spectral sequences. In Section 23, we give some methods for the calculation of Amitsur-Dress cohomology groups, the most interesting of which involves use of a special case of the AHSS, and apply these methods to calculate the Amitsur-Dress cohomology groups for the family of proper subgroups of a nonabelian group of order pq, where $p < q$ are primes.

In Sections 24 and 25, we specialize to stable homotopy theory. We use the family $\mathcal{P}$ of proper subgroups of a finite group G to obtain two related spectral sequences, both of which converge to the completion of the (nonequivariant) stable homotopy groups of spheres at

$n(\mathcal{P})$, where $n(\mathcal{P})$ is the product of those primes p such that $\mathbb{Z}/p\mathbb{Z}$ is a quotient of G. For example, if G is a nonabelian group of order pq, $p < q$, then $n(\mathcal{P}) = p$ and the spectral sequences provide a mechanism for the prime q to affect stable homotopy groups at the prime p. One of the spectral sequences is the Atiyah-Hirzebruch spectral sequence whose E_2-term is the Amitsur-Dress-Tate cohomology of $\mathcal{P}$. The other comes from a filtration of $\widetilde{E}G$ in terms of the reduced regular representation of G. These spectral sequences lead to new equivariant root invariants, and the Jones-Miller root invariant theorem [28], [39] generalizes to the spectral sequence constructed by use of the regular representation.

There are two appendices. The first proves the folklore result that rational G-spectra split as products of Eilenberg-MacLane G-spectra when G is finite and gives a complete algebraic analysis of the rational equivariant stable category.

The second gives an analysis and comparison of the versions of the Atiyah-Hirzebruch spectral sequence with target $[X, Y]^*$ that arise from filtrations of X and of Y. It is written nonequivariantly, but it applies verbatim equivariantly. In Section 10, we use filtrations of X to obtain chain level information and prove convergence, and we use filtrations of Y to study products.

In all of what follows, G-spaces are to be based G-CW complexes, and homology and cohomology are to be understood in the *reduced* sense. As we recall in Section 0, there are two basic kinds of G-spectra, namely ordinary spectra with G-actions, which we refer to as "naive G-spectra", and genuine G spectra indexed on representations. Our G-spectra are of the latter sort unless otherwise specified. There is a forgetful functor i^* from G-spectra to naive G-spectra. This functor has a left adjoint i_* which constructs G-spectra from naive G-spectra by building in nontrivial representations. The algebraic counterparts are coefficient systems and Mackey functors, and the topology will lead to an analogous algebraic functor s_* that constructs Mackey functors out of coefficient systems by building in transfer maps.

Naive G-spectra represent $\mathbb{Z}$-graded cohomology theories; genuine G-spectra represent

$RO(G)$-graded, or $RO(G)$-gradable, theories. In fact, most interesting theories are $RO(G)$-gradable, and such theories come with a great deal of calculationally useful extra structure, whereas theories represented only by naive G-spectra lack some of the most rudimentary properties. For example, they do not admit a Spanier-Whitehead duality theorem (simply because one cannot embed non-fixed G-spaces in trivial representations), and, in fact, interesting homology theories on G-spaces cannot be represented in the form $\pi_*((j_G \wedge X)^G)$ for a naive G-spectrum j_G: such represented theories vanish on X/X^G.

Much of the force of our work comes from the fact that (D) is a diagram of genuine and conveniently explicit G-spectra indexed on representations, so that all of the $\mathbb{Z}$-graded theories mentioned above are $RO(G)$-gradable. The significance of the $RO(G)$-grading will be illustrated in our discussion of periodicity and Euler classes in Section 16, and it will be vital to the proofs of our results on periodic K-theory in Section 19.

While [19] worked with genuine G-spectra, later authors, motivated by nonequivariant applications, worked with naive G-spectra and gave central emphasis to the role of the transfer map. Essentially, their versions of Tate spectra can be obtained from ours by passing to fixed point spectra. With our definitions, the Tate homology of X is

$$t(k_G)_*(X) = \pi_*((t(k_G) \wedge X)^G).$$

As we show in Section 1, we may assume without loss of generality that $k_G = i_* j_G$ for a naive G-spectrum j_G. As we show in Section 5, provided that X is finite, the spectrum $(t(k_G) \wedge X)^G$ is then equivalent to the cofiber of a suitable transfer map

$$(j_G \wedge \Sigma^{\mathrm{Ad}(G)} X)/hG \equiv (j_G \wedge EG_+ \wedge \Sigma^{\mathrm{Ad}(G)} X)/G \to F(EG_+, j_G \wedge X)^G \equiv (j_G \wedge X)^{hG}.$$

When G is finite, j_G is a nonequivariant spectrum k given trivial action by G, and $X = S^0$, this reduces to $k \wedge BG_+ \to F(BG_+, k)$. Detailed comparisons of our definitions with those of Adem, Cohen, and Dwyer [4] and of Weiss and Williams [47] would only require detailed verification that our version of the transfer map agrees with theirs.

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At the insistence of several of those mentioned, and of the referee, we have changed some names and notations from the preprint version, which conformed with the first author's papers [19–21]. In the preprint, the G-spectra $f'(k_G)$ were called coBorel spectra and were denoted $c(k_G)$ and the geometric completions $c(k_G)$ were called Borel spectra and denote $b(k_G)$.

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Part I: General theory

§0. Preamble: definitions, change of universe, and split G-spectra

In the interest of intelligibility, we begin by recalling some of the basic definitional framework of [33]. We especially want to make clear the idea of "change of universe". In any framework, in one guise or another, this notion is central to the mathematics at hand. We also want to explain the technical notion of a split G-spectrum, which plays an important role in comparing equivariant and nonequivariant phenomena. The essential point is to make clear how to go back and forth among genuine G-spectra, naive G-spectra, and nonequivariant spectra. This is crucial to explicit calculation in all of equivariant cohomology theory.

Let U be a "complete G-universe", namely the sum of countably many copies of each irreducible representation of G. If G is finite, we may take U to be the sum of countably many copies of the regular representation. An indexing G-space V is a finite-dimensional sub inner product space of U. If $V \subset W$, let $W - V$ be the orthogonal complement of V in W. A G-spectrum k_G indexed on U consists of based G-spaces $k_G V$ for indexing spaces V and a transitive system of G-homeomorphisms $k_G V \cong \Omega^{W-V}(k_G W)$ for $V \subset W$. A map $k_G \to k'_G$ consists of based G-maps $k_G V \to k'_G V$ that are strictly compatible with the given homeomorphisms. Let $G\mathcal{S}U$ be the category of G-spectra indexed on U. The G-spectra of this paper are to be understood as G-spectra indexed on U unless otherwise specified.

The G-fixed point space U^G is a "trivial G-universe" and may be identified with $\mathbb{R}^\infty$. We define the category $G\mathcal{S}U^G$ of G-spectra indexed on U^G exactly as above, but restricting attention to the G-fixed indexing G-spaces. There is no loss of information if we restrict further to just the indexing spaces $\mathbb{R}^n$. Thus G-spectra indexed on U^G are just ordinary spectra with G-action, and we refer to them as naive G-spectra. We regard nonequivariant spectra as G-trivial naive G-spectra.

Working in either universe, we define G-prespectra just as we defined G-spectra, except that we place no restrictions on the structural G-maps $k_G V \to \Omega^{W-V}(k_G W)$, and we obtain categories $G\mathcal{P}U^G$ and $G\mathcal{P}U$ of G-prespectra. We then construct functors L from G-prespectra to G-spectra that are left adjoint to the evident forgetful functors ℓ from G-spectra to G-prespectra. This allows us to lift any prespectrum level functor F that fails to preserve spectra to the spectrum level, by taking $LF\ell$. In general, spacewise constructions on prespectra that are left adjoints, such as wedges, cofibers, and smash products, fail to preserve spectra while constructions that are right adjoints, such as products, fibers, and function spectra, do preserve spectra.

Let $i : U^G \to U$ be the inclusion. We have the forgetful functor $i^* : G\mathcal{S}U \to G\mathcal{S}U^G$ given by forgetting about those indexing G-spaces with non-trivial G-action. The underlying nonequivariant spectrum k of $k_G \in G\mathcal{S}U$ is $i^* k_G$ with its action by G ignored. The functor i^* has a left adjoint $i_* : G\mathcal{S}U^G \to G\mathcal{S}U$, which builds in non-trivial representations. Explicitly, for a naive G-prespectrum j_G and an indexing G-space V,

$$(i_* j_G)(V) = j_G(V^G) \wedge S^{V-V^G}$$

For a naive G-spectrum j_G, $i_* j_G = Li_* \ell j_G$, as prescribed above. The following intuitively obvious result is proven in [33, II.1.8].

LEMMA 0.1. *For $j_G \in G\mathcal{S}U^G$, the unit G-map $\eta : j_G \to i^* i_* j_G$ of the (i_*, i^*) adjunction is a nonequivariant equivalence. For $k_G \in G\mathcal{S}U$, the counit G-map $\varepsilon : i_* i^* k_G \to k_G$ is a nonequivariant equivalence.*

The composite of i_* and the suspension spectrum functor $\Sigma_G^\infty : G\mathcal{T} \to G\mathcal{S}U^G$ is the suspension spectrum functor $\Sigma_G^\infty : G\mathcal{T} \to G\mathcal{S}U$, where $G\mathcal{T}$ is the category of based G-spaces. The V^{th} space $(\Sigma_G^\infty X)(V)$ is the G-space $Q_G(\Sigma^V X)$, where $Q_G Y$ is the union of the G-spaces $\Omega^W \Sigma^W Y$; $\Sigma_G^\infty : G\mathcal{T} \to G\mathcal{S}U^G$ is defined similarly, but using only G-fixed indexing spaces.

In either category of G-spectra, for a G-space X and a G-spectrum k_G, we have the function G-spectrum $F(X, k_G)$. Its V^{th} space is the space $F(X, k_G V)$ of based maps $X \to k_G V$, with G acting by conjugation. The smash product $X \wedge k_G$ is defined on G-prespectra k_G by

$$(X \wedge k_G)(V) = X \wedge k_G V$$

and on G-spectra k_G by $X \wedge k_G = L(X \wedge \ell k_G)$. It is obvious from the definitions that

$$i^* F(X, k_G) = F(X, i^* k_G) \quad \text{for } k_G \in G\mathcal{S}U.$$

By playing with adjunctions [33, pp. 18–20], this implies that

$$i_*(X \wedge j_G) \cong X \wedge i_* j_G \quad \text{for } j_G \in G\mathcal{S}U^G.$$

Smash products of G-spectra, and the concomitant function G-spectra, are defined and proven to have all of the expected properties in [33, II§3]. The functor i_* commutes with smash products.

For $j_G \in G\mathcal{S}U^G$, we define the fixed point spectrum $(j_G)^G$ simply by passing to fixed points spacewise, $(j_G)^G(V) = (j_G V)^G$. It is essential that G act trivially on V to obtain well-defined structural homeomorphisms here. For $k_G \in G\mathcal{S}U$, we define $(k_G)^G = (i^* k_G)^G$.

Definitions 0.2. *A naive G-spectrum j_G with underlying nonequivariant spectrum j is said to be split if there is a map of spectra $\zeta : j \to (j_G)^G$ whose composite with the inclusion of $(j_G)^G$ in j is homotopic to the identity map. A G-spectrum k_G is said to be split if $i^* k_G$ is split.*

The K-theory G-spectra KU_G and KO_G are split [33, p. 458]. The Eilenberg-Maclane G-spectrum HM associated to a Mackey functor M is split if and only if the canonical map $M(G/G) \to M(G/e)$ is a split epimorphism; this implies that G acts trivially on $M(G/e)$, which is usually not the case. (See Section 6 for definitions.) The suspension G-spectrum $\Sigma_G^\infty X$ of a G-space X is split if and only if X is stably a retract up to homotopy of X^G, which again is usually not the case unless G acts trivially on X. In particular, however, the sphere G-spectrum $S_G = \Sigma_G^\infty S^0$ is split. The following observation gives more examples.

LEMMA 0.3. *If $j_G \in GSU^G$ is split, for example if G acts trivially on j_G, then $i_* j_G \in GSU$ is also split.*

PROOF: Let $\zeta : j_G \to (j_G)^G$ be a splitting map. Then, by Lemma 0.1 and an easy diagram chase, the following composite is a splitting map

$$i^* i_* j_G \xrightarrow{i^* i_* \zeta} i^* i_*((j_G)^G) \xrightarrow{\eta^{-1}} (j_G)^G \xrightarrow{\eta^G} (i^* i_* j_G)^G.$$

The notion of a split G-spectrum is defined in nonequivariant terms, but it admits the following equivariant interpretation.

LEMMA 0.4. *Let k_G be a G-spectrum with underlying nonequivariant spectrum k. Then k_G is split if and only if there is a map of G-spectra $i_* k \to k_G$ which is a nonequivariant equivalence.*

PROOF: Composing a splitting map $\zeta : k \to (i^* k_G)^G$ with the inclusion $(i^* k_G)^G \to i^* k_G$, we obtain a G-map $\nu : k \to i^* k_G$ which is the identity map nonequivariantly. Its adjoint $\widetilde{\nu} : i_* k \to k_G$ is a G-map which is a nonequivariant equivalence since $\nu = i^* \widetilde{\nu} \circ \eta$ and $\eta : k \to i^* i_* k$ is a nonequivariant equivalence. Conversely, given a nonequivariant equivalence $\widetilde{\nu}$, its adjoint ν, regarded as a nonequivariant map, is an equivalence $k \to k$ which factors through a map $\zeta' : k \to (i^* k_G)^G$. The composite of ζ' and an inverse to ν gives a splitting map ζ.

Sphere G-spectra $G/H_+ \wedge S^n$ in GSU are obtained by applying i_* to the corresponding sphere G-spectra in GSU^G. A map of G-spectra is called a weak G-equivalence if it induces an isomorphism on all homotopy groups $\pi_n^H(?) = [G/H_+ \wedge S^n, ?]_G$. Such a map between G-CW spectra is a G-equivalence by the G-Whitehead theorem [33, I.5.10]. The stable category is constructed from the homotopy category of G-spectra by formally inverting the weak G-equivalences [33,I§6], so we make no distinction between weak and actual G-equivalences in what follows.

There are also sphere G-spectra $S^a \in GSU$ for virtual representations a [33, p.34]. For G-spectra X and k_G,

$$(0.5) \qquad k_a^G(X) = [S^a, X \wedge k_G]_G \quad \text{and} \quad k_G^a(X) = [X \wedge S^{-a}, k_G]_G.$$

When we restrict to integer gradings, we may use standard adjunctions to rewrite this definition as follows in terms of the ordinary homotopy groups of nonequivariant fixed point spectra:

$$(0.6) \qquad k_n^G(X) = \pi_n((X \wedge k_G)^G) \quad \text{and} \quad k_G^n(X) = \pi_{-n}(F(X, k_G)^G).$$

Orbit spectra j_G/G of naive G-spectra are constructed by first passing to orbits spacewise on the prespectrum level and then applying the functor L from prespectra to spectra. On free G-spectra X, we can often reduce (0.6) to nonequivariant calculations on orbits. A based G-space is said to be free if it is free away from its G-fixed basepoint. A G-spectrum, in either sense, is said to be free if it is equivalent to a G-CW spectrum built up out of free cells $G_+ \wedge CS^n$. The functors $\Sigma^\infty : \mathcal{T} \to GSU^G$ and $i_* : GSU^G \to GSU$ carry free G-spaces to free naive G-spectra and free naive G-spectra to free G-spectra. In all three categories, X is homotopy equivalent to a free object if and only if the canonical map $EG_+ \wedge X \to X$ is a G-equivalence [33, II.2.12]. A free G-spectrum Y has the form i_*X for a free naive G-spectrum X, which is unique up to equivalence [33, II.2.8]. A useful slogan is that "free G-spectra live in the trivial universe".

When k_G is split and X is a free naive G-spectrum, we have

$$(0.7) \qquad k_n^G(i_*X) \cong k_n((\Sigma^{\mathrm{Ad}(G)}X)/G) \quad \text{and} \quad k_G^n(i_*X) \cong k^n(X/G)$$

by [33, II.8.4]. The second isomorphism is elementary. The first depends on the dimension-shifting transfer isomorphism discussed in Section 5.

The functional (or Spanier-Whitehead) dual of a G-spectrum X is $D(X) = F(X, S_G)$. For $H \subset G$, let $L(H)$ be the tangent representation of H at the identity coset of G/H.

Then

$$D(\Sigma_G^\infty G/H_+) \text{ is equivalent to } G \ltimes_H S^{-L(H)} \tag{0.8}$$

where the functor $G \ltimes_H (?)$ extends H-spectra to G-spectra [33,II.6.3]. In particular, $D(\Sigma_G^\infty G_+)$ is equivalent to $\Sigma_G^\infty G_+ \wedge S^{-d}$ [33,II.4.8].

§1. Invariance properties of the functors f, c, and t

The bottom row of Diagram (D) is a substantial simplification of the top row because it is invariant under G-maps that are nonequivariant equivalences.

PROPOSITION 1.1. *Let $\phi : k_G \to k'_G$ be a map of G-spectra that is a nonequivariant equivalence. Then the induced maps*

$$\phi \wedge 1 : k_G \wedge EG_+ \to k'_G \wedge EG_+ \quad \text{and} \quad F(1, \phi) : F(EG_+, k_G) \to F(EG_+, k'_G)$$

are G-equivalences. Therefore the cofibration sequences

$$f'(k_G) \to c(k_G) \to t(k_G) \quad \text{and} \quad f'(k'_G) \to c(k'_G) \to t(k'_G)$$

are G-equivalent.

PROOF: The smash product of any G-spectrum with EG_+ is a free G-spectrum (see p.20), and the statement about $\phi \wedge 1$ follows from the G-Whitehead theorem of [33,II.2.2]. For G-spectra X

$$[X, F(EG_+, k_G)]_G \cong [X \wedge EG_+, k_G]_G.$$

Using the natural isomorphism $[X \wedge G_+, k_G]_G \cong [i^*X, k]$ of [33, II.4.7 and 4.8] to handle skeletal subquotients, we see by induction up the skeleta of EG_+ and use of the $\lim^1$ exact sequence of $\{[X \wedge EG^q_+, k_G]_G\}$ that $[X, F(1, \phi)]_G$ is an isomorphism for all X. It follows that $F(1, \phi)$ is a G-equivalence.

Since the middle vertical arrow $\varepsilon : k_G \to c(k_G)$ of Diagram (D) is a nonequivariant equivalence, the first statement of the proposition implies the following basic fact about the left vertical arrow $\varepsilon = \varepsilon \wedge 1$.

PROPOSITION 1.2. *For any G-spectrum k_G,*

$$\varepsilon : f(k_G) = k_G \wedge EG_+ \to F(EG_+, k_G) \wedge EG_+ = f'(k_G)$$

is an equivalence of G-spectra .

From now on, we agree to write f instead of f'.

By Lemma 0.1, Proposition 1.1 directly implies the following result. It says that, when studying the bottom row of Diagram (D), there is no loss of generality if we restrict attention to G-spectra of the form i_*j_G.

COROLLARY 1.3. *If $\phi : j_G \to j'_G$ is a map of naive G-spectra that is a nonequivariant equivalence, then the cofibration sequences*

$$f(i_*j_G) \to c(i_*j_G) \to t(i_*j_G) \quad \text{and} \quad f(i_*j'_G) \to c(i_*j'_G) \to t(i_*j'_G)$$

are G-equivalent. If $j_G = i^*k_G$ for a G-spectrum k_G, then the cofibration sequences

$$f(k_G) \to c(k_G) \to t(k_G) \quad \text{and} \quad f(i_*j_G) \to c(i_*j_G) \to t(i_*j_G)$$

are G-equivalent.

REMARK 1.4: Propositions 1.1 and 1.2 are true as stated, with the same proofs, for naive G-spectra. We could naively define $c(j_G) = F(EG_+, j_G)$ since Lemma 0.1 implies an equivalence

$$F(EG_+, j_G) \to F(EG_+, i^*i_*j_G) \cong i^*F(EG_+, i_*j_G),$$

so that $c(j_G) \simeq i^*c(i_*j_G)$. However, because i^* fails to commute with smash products, nothing like this is true for f or t: $j_G \wedge EG_+$ is very different from $i^*(i_*j_G \wedge EG_+)$. For example, if $j = j_G$ is the nonequivariant sphere spectrum, then

$$(j \wedge EG_+)^G \simeq * \quad \text{but} \quad i^*(i_*j \wedge EG_+)^G \simeq \Sigma^\infty(\Sigma^{\mathrm{Ad}(G)} EG_+/G).$$

It is an interesting open question, raised recently by Rognes, to determine which naive G-spectra j_G come from genuine G-spectra. The answer for naive Eilenberg-MacLane G-spectra was determined in [32] and will be recalled in Section 6.

Of course, the action of G on j_G is non-trivial in general. However, by Lemma 0.4, Proposition 1.1 implies that in surprisingly many cases we can reduce further by ignoring this action without changing the bottom row of diagram (C).

COROLLARY 1.5. *If k_G is a split G-spectrum with underlying nonequivariant spectrum k, then the cofibration sequences*

$$f(k_G) \to c(k_G) \to t(k_G) \quad \text{and} \quad f(i_*k) \to c(i_*k) \to t(i_*k)$$

are G-equivalent.

The reader primarily interested in equivariant theory as a tool for the study of nonequivariant phenomena may wish to concentrate on the G-spectra i_*k. When considering only split G-spectra, there is no loss of generality. Note however that there usually will be more than one split G-spectrum having underlying nonequivariant spectrum equivalent to a given spectrum k.

EXAMPLE 1.6: Let KU be the classical periodic K-theory spectrum. Then KU_G and i_*KU are inequivalent split G-spectra both of which have KU as underlying nonequivariant spectrum. They therefore have equivalent f, c, and t G-spectra. The analogous assertions hold in the real and connective cases.

In sum, we may always start with naive G-spectra as input, and we may often start with classical nonequivariant spectra. However, the functors f and t are inconvenient to define or to compute on the naive level, and it is essential to our work that the functors f, c, and t all have genuine G-spectra as output.

§2. Basic properties of the theories represented by $f(k_G)$, $c(k_G)$, and $t(k_G)$

Of course, Borel homology and cohomology theories have long been studied. The following result shows how our theories relate to them.

PROPOSITION 2.1. *If k is the underlying nonequivariant spectrum of a split G-spectrum k_G and X is a naive G-spectrum, such as the naive suspension spectrum of a G-space, then*

$$c(k_G)^n(i_*X) \cong k^n(EG_+ \wedge_G X) \quad \textit{and} \quad f(k_G)_n(i_*X) \cong k_n(EG_+ \wedge_G \Sigma^{\mathrm{Ad}(G)}X),$$

where $\mathrm{Ad}(G)$ denotes the adjoint representation of G.

PROOF: Immediate from (0.7).

We have the following analogous reduction to naive level theories; in view of Corollary 1.3, it applies in complete generality. Recall Remark 1.4.

PROPOSITION 2.2. *If $k_G = i_*j_G$ for a naive G-spectrum j_G and X is a naive G-spectrum, such as the naive suspension spectrum of a G-space, then*

$$c(k_G)^n(i_*X) \cong c(j_G)^n(X) \quad \textit{and} \quad f(k_G)_n(i_*X) \cong \pi_n((j_G \wedge EG_+ \wedge \Sigma^{\mathrm{Ad}(G)}X)/G).$$

PROOF: These isomorphisms are composites

$$c(k_G)^n(i_*X) \cong \pi_{-n}(F(i_*(EG_+ \wedge X), k_G)^G) \cong \pi_{-n}(F(EG_+ \wedge X, j_G)^G) \cong c(j_G)^n(X)$$

and

$$f(k_G)_n(i_*X) \cong \pi_n((i_*(j_G \wedge EG_+ \wedge X))^G) \cong \pi_n((j_G \wedge EG_+ \wedge \Sigma^{\mathrm{Ad}(G)}X)/G).$$

In both cases, the first isomorphism holds because i_* commutes with smash products. For c, the second isomorphism holds by [33, II.2.8]. For f, the second isomorphism holds by [33,II.7.2]; see Theorem 5.3 below.

We will say more about the cited results in Section 5, where we will give an analogous reduction of Tate homology, in terms of transfer, when X is finite. No such reduction is known for infinite G-CW complexes X.

The relationship between the three kinds of theories is immediate from the bottom cofiber sequence of diagram (C).

PROPOSITION 2.3. *For G-spaces or G-spectra X, there are natural long exact sequences*

$$\cdots \to f(k_G)^n(X) \to c(k_G)^n(X) \to t(k_G)^n(X) \to f(k_G)^{n+1}(X) \to \cdots$$

and

$$\cdots \to f(k_G)_n(X) \to c(k_G)_n(X) \to t(k_G)_n(X) \to f(k_G)_{n-1}(X) \to \cdots .$$

We also call these norm sequences. Under appropriate hypotheses, they collapse to give isomorphisms.

PROPOSITION 2.4. *Let X be a free G-space or a free G-spectrum . Then*

$$t(k_G)^*(X) = 0 \quad \text{and} \quad t(k_G)_*(X) = 0.$$

Therefore

$$f(k_G)^*(X) \cong c(k_G)^*(X) \quad \text{and} \quad f(k_G)_*(X) \cong c(k_G)_*(X).$$

If, further, k_G is split with underlying nonequivariant spectrum k, then

$$f(k_G)^*(X) \cong k^*(EG_+ \wedge_G X) \cong k^*(X/G)$$

and

$$c(k_G)_*(X) \cong k_*(EG_+ \wedge_G \Sigma^{\mathrm{Ad}(G)} X) \cong k_*((\Sigma^{\mathrm{Ad}(G)} X)/G).$$

PROOF: X is G-equivalent to $EG_+ \wedge X$ and thus $\widetilde{E}G \wedge X$ is G-contractible (see p.20). Therefore $t(k_G)_*(X) = 0$. For cohomology, note that $t(k_G)$ is contractible as a nonequivariant spectrum since $\widetilde{E}G$ is contractible as a space. Since X is constructed out of spaces or spectra $G_+ \wedge S^n$ and all G-maps from these into $t(k_G)$ are null homotopic, $t(k_G)^*(X) = 0$.

PROPOSITION 2.5. *If X is a nonequivariantly contractible G-space or G-spectrum, then*

$$c(k_G)^*(X) = 0 \quad \textit{and} \quad f(k_G)_*(X) = 0.$$

Therefore

$$t(k_G)^n(X) \cong f(k_G)^{n+1}(X) \quad \textit{and} \quad c(k_G)_n(X) \cong t(k_G)_n(X).$$

PROOF: X is G-equivalent to $\widetilde{E}G \wedge X$ and thus $EG_+ \wedge X$ is G-contractible (see [33,II.9.2]).

By definition, Tate homology is a special case of c-homology,

$$t(k_G)_n(X) = c(k_G)_n(\widetilde{E}G \wedge X).$$

The previous results combine to show that, analogously, Tate cohomology is a special case of f-cohomology.

PROPOSITION 2.6. *The Tate spectrum $t(k_G)$ is equivalent to $F(\widetilde{E}G, \Sigma f(k_G))$. Therefore, for any G-space or G-spectrum X,*

$$t(k_G)^n(X) \cong f(k_G)^{n+1}(\widetilde{E}G \wedge X).$$

PROOF: $F(EG_+, t(k_G))$ and $F(\widetilde{E}G, c(k_G))$ are trivial by the previous two results, hence $S^0 \to \widetilde{E}G$ and $\widetilde{E}G \to \Sigma EG_+$ induce equivalences

$$t(k_G) = F(S^0, t(k_G)) \leftarrow F(\widetilde{E}G, t(k_G)) \to F(\widetilde{E}G, \Sigma f(k_G)).$$

This seems like a formality, but it hides substantial content and we will use it heavily. If we filter $\widetilde{E}G$, then we get a homotopy colimit description of $t(k_G)$ but a homotopy limit description of the equivalent G-spectrum $F(\widetilde{E}G, \Sigma f(k_G))$. This dichotomy will sometimes lead us to two quite different means of algebraic computation (and the equivalence actually admits a purely algebraic analog [50]). This is part of the substance of the equivalences ($\mathbb{R}$), ($\mathbb{L}$), and ($\mathbb{C}$) in the introduction.

§3. Homotopical behavior of the functors f, c and t

In the previous section, the G-spectrum k_G was fixed. However, the fact that f, c, and t are well-behaved functors will be crucial to our work. As a left adjoint (see Proposition 1.2), f preserves wedges, cofibers, and colimits. As a right adjoint, c preserves products, fibers, and limits. The functor t is neither a left nor a right adjoint, but it clearly preserves finite wedges and cofibration sequences since finite wedges and products are equivalent and cofibration and fibration sequences are equivalent. The following result along these lines will be needed in our study of Atiyah-Hirzebruch spectral sequences. Given G-spectra X^p, $p \in \mathbb{Z}$, and maps $X^p \to X^{p+1}$, we write

$$\text{(3.1)} \qquad \operatorname{Tel} X^p = \operatorname{hocolim}_{p\to\infty} X^p \quad \text{and} \quad \operatorname{Mic} X^p = \operatorname{holim}_{p\to-\infty} X^p.$$

Precise spectrum level definitions are recalled in Appendix B.

PROPOSITION 3.2. *If the X^p are bounded below, with a uniform bound, then the following diagram displays a G-equivalence between cofiber sequences:*

$$\begin{tikzcd} f(\operatorname{Mic} X^p) \arrow[r] \arrow[d] & c(\operatorname{Mic} X^p) \arrow[r] \arrow[d] & t(\operatorname{Mic} X^p) \arrow[d] \\ \operatorname{Mic} f(X^p) \arrow[r] & \operatorname{Mic} c(X^p) \arrow[r] & \operatorname{Mic} t(X^p) \end{tikzcd}$$

PROOF: For any G-space or spectrum E, the functor $F(E, ?)$ commutes with arbitrary microscopes, so it suffices to prove that the left vertical arrow is an equivalence. In view of Proposition 1.2, it suffices to show that the natural map $E \wedge (\prod X^p) \to \prod(E \wedge X^p)$ is a G-equivalence when E is a bounded below G-CW spectrum with finite skeleta. Since the X^p are uniformly bounded below, consideration of homotopy groups shows that it suffices to prove this when E is finite. Here, for any K, we have

$$[K, E\wedge\textstyle\prod X^p]_G \cong [K\wedge DE, \prod X^p]_G \cong \prod[K\wedge DE, X^p]_G \cong \prod[K, E\wedge X^p]_G \cong [K, \prod E\wedge X]_G.$$

We shall be interested in applications of this result to the Postnikov tower $\{Y^q\}$ of a G-spectrum Y. To keep track of the indexing of spectral sequences, we shall find it convenient to index Postnikov towers so that $Y \to Y^q = Y(-\infty, -q]$ is obtained from Y by killing its homotopy group systems in dimensions greater than $-q$. There results a map $Y^q \to Y^{q+1}$ under Y, unique up to homotopy, and its fiber is an Eilenberg-MacLane G-spectrum $K(\underline{\pi}_{-q}(Y), -q)$. The induced map $Y \to \mathrm{Mic}(Y^q)$ is an equivalence, where the homotopy limit is taken as $q \to -\infty$.

PROPOSITION 3.3. *If $\{Y^q\}$ is the Postnikov tower of a G-spectrum Y, then the following diagram displays a G-equivalence between cofiber sequences:*

$$\begin{tikzcd}
f(Mic\ Y^q) \arrow[r] \arrow[d] & c(Mic\ Y^q) \arrow[r] \arrow[d] & t(Mic\ Y^q) \arrow[d] \\
Mic\ f(Y^q) \arrow[r] & Mic\ c(Y^q) \arrow[r] & Mic\ t(Y^q).
\end{tikzcd}$$

PROOF: Since we have not assumed that Y is bounded below, there is something to prove. Let $X \to Y$ be the connective cover of Y, so that $X = Y[0, \infty)$. The Postnikov tower $\{X^q\}$ of X maps to that of Y, and the cofiber of $X^q \to Y^q$ is Y^q if $q > 0$ and $Y(-\infty, -1]$ if $q \leq 0$. We may take our homotopy limits over $q \leq 0$, and the previous result applies to the system $\{X^q\}$. The conclusion of the previous result holds trivially for the constant system $\{Y(-\infty, -1]\}$, and the result follows.

Smash products and therefore multiplicative structures also behave well. We have the canonical smash product pairing

$$F(X, Y) \wedge F(X', Y') \to F(X \wedge X', Y \wedge Y').$$

Since EG_+ has a diagonal map and since there are equivalences

$$EG_+ \wedge EG_+ \simeq EG_+ \quad \text{and} \quad \widetilde{E}G \wedge \widetilde{E}G \simeq \widetilde{E}G,$$

unique up to homotopy, we obtain a commutative diagram of associative and commutative natural pairings, which we call a "norm pairing diagram":

$$\begin{tikzcd}
f(k_G) \wedge f(k'_G) \arrow[r] \arrow[d] & c(k_G) \wedge c(k'_G) \arrow[r] \arrow[d] & t(k_G) \wedge t(k'_G) \arrow[d] \\
f(k_G \wedge k'_G) \arrow[r] & c(k_G \wedge k'_G) \arrow[r] & t(k_G \wedge k'_G)
\end{tikzcd} \tag{3.4}$$

This easily implies the following result.

PROPOSITION 3.5. *If k_G is a ring G-spectrum, such as i_*j_G for a naive ring G-spectrum j_G, then $c(k_G)$ and $t(k_G)$ are ring G-spectra and the following part of diagram (C) is a commutative diagram of ring G-spectra:*

$$\begin{array}{ccc} k_G & \longrightarrow & k_G \wedge \widetilde{E}G \\ \downarrow & & \downarrow \\ c(k_G) & \longrightarrow & t(k_G). \end{array}$$

If k_G is commutative, then so are $c(k_G)$ and $t(k_G)$. If m_G is a k_G-module G-spectrum, then $c(m_G)$ is a $c(k_G)$-module G-spectrum and $t(m_G)$ is a $t(k_G)$-module G-spectrum.

The unit of $t(k_G)$ is the smash product of the unit of $c(k_G)$ and the canonical map $S^0 \to \widetilde{E}G$; only the lack of a unit prevents $f(k_G)$ from also being a ring G-spectrum.

The G-fixed point spectra of ring G-spectra are ring spectra. The essential point is that there is an associative and commutative natural pairing [33, II.3.14]

$$\omega : (k_G)^G \wedge (k'_G)^G \to (k_G \wedge k'_G)^G. \tag{3.6}$$

In contrast to the space level, ω is not an equivalence. If j_G is a naive ring G-spectrum, then the natural map $(j_G)^G \to (i_*j_G)^G$ is a ring map. Of course, the case when G acts trivially on j is of particular interest.

We cannot expect $t(k_G)$ to be functorial in G since classical Tate cohomology is not functorial in G. However, we do have restriction and transfer maps satisfying all of the usual properties. That is, the collection $\{t^*(k_H)(X)\}$ for $H \subset G$ specifies a Mackey functor (see Sections 6 and 20). This is a direct consequence of the following easy observation. Recall that EG regarded as an H-space is a model for EH, and similarly for $\widetilde{E}G$.

PROPOSITION 3.7. *Let k_H denote k_G regarded as an H-spectrum for $H \subset G$. When regarded as H-spectra, $f(k_G)$, $c(k_G)$, and $t(k_G)$ are equivalent to $f(k_H)$, $c(k_H)$, and $t(k_H)$, respectively.*

There is a more subtle result along the same lines. The forgetful functor from G-spectra to H-spectra has both a left adjoint $G \ltimes_H (?)$ and a right adjoint $F_H[G, ?)$. These two functors are equivalent when H has finite index in G, but in general they differ by a suitable suspension. The results [33; II.4.3, 4.9, and 6.2] imply the following commutation relations.

PROPOSITION 3.8. *Let k_H be an H-spectrum. Then the following cofiber sequences of G-spectra are canonically equivalent:*

$$G \ltimes_H f(k_H) \to G \ltimes_H c(k_H) \to G \ltimes_H t(k_H)$$

and

$$f(G \ltimes_H k_H) \to c(G \ltimes_H k_H) \to t(G \ltimes_H k_H).$$

Similarly, the following cofiber sequences of G-spectra are canonically equivalent:

$$f(F_H[G, k_H)) \to c(F_H[G, k_H)) \to t(F_H[G, k_H))$$

and

$$F_H[G, f(k_H)) \to F_H[G, c(k_H)) \to F_H[G, t(k_H)).$$

§4. Completion at the augmentation ideal of the Burnside ring

We introduced completions of G-spectra at ideals of the Burnside ring $A(G)$ in [25]. Let $I \subset A(G)$ be the augmentation ideal. The following result, which will be proven shortly, may be viewed as a formalization of the intuition that completion at I is intimately connected with the kind of invariance displayed in Proposition 1.1. Of course, completion at I only makes sense for genuine G-spectra since the starting point of the construction is the natural action of $A(G) = \pi_0^G(S_G)$ on G-spectra. Until Proposition 4.20, we restrict attention to finite groups in this section.

THEOREM 4.1. *Let G be finite. Then $c(k_G)$ is I-complete for any G-spectrum k_G. If k_G is bounded below, then $f(k_G)$ and therefore $t(k_G)$ are also I-complete. If G is a p-group and k_G is bounded below, then $t(k_G)$ is p-complete.*

The last statement follows from the topological fact that $t(k_G)$ is nonequivariantly contractible and the algebraic fact that, for any Mackey functor M, I-adic completion agrees with p-adic completion on the kernel of $\pi^* : M(G/G) \to M(G/e)$, $\pi : G/e \to G/G$.

Since $f(KU_G)$ and $t(KU_G)$ are not I-complete (see Theorem 13.1), the bounded below hypothesis is essential. As explained in [25, §4], the Atiyah-Segal completion theorem and the Segal conjecture imply that $\varepsilon : k_G \to b(k_G)$ is a completion at I when k_G is KU_G, KO_G, or S_G. In the last case, we can conclude from Diagram (D) that $f^{\perp}S_G \to t(S_G)$ is also a completion at I. In general, the map $\varepsilon : k_G \to b(k_G)$ is a completion at I if and only if the cohomology theory represented by $(k_G)_I^{\wedge}$ carries G-maps which are nonequivariant equivalences to isomorphisms, and this holds if and only if the left derived functors L_0^I and L_1^I of I-adic completion vanish on $(k_G)^*(X)$ whenever X is a nonequivariantly contractible G-spectrum.

The last statement holds since we have short exact sequences

$$(4.2) \qquad 0 \to L_1^I(Y^{n+1}(X)) \to (Y_I^{\wedge})^n(X) \to L_0^I(Y^n(X)) \to 0$$

for any G-spectra X and Y and any ideal $I \subset A(G)$ [25, 3.3]. Under finite type hypotheses, these sequences reduce to isomorphisms

$$(4.3) \qquad (Y_I^\wedge)^n(X) \to (Y^n(X))_I^\wedge.$$

There is a simple and explicit construction of $Y_I^\wedge$ for any I. For $\alpha \in A(G)$, let $M(\alpha)$ be the fiber of $S_G \to S_G[\alpha^{-1}]$, where $S_G[\alpha^{-1}]$ is the telescope of iterates of $\alpha : S_G \to S_G$. If $I = (\alpha_1, \ldots, \alpha_k)$, let

$$(4.4) \qquad M(I) = M(\alpha_1) \wedge \cdots \wedge M(\alpha_k).$$

Up to equivalence, $M(I)$ depends only on I and not on the choice of its generators, and $M(I) \wedge M(I) \simeq M(I)$. By construction, we have a canonical map $M(I) \to S_G$, and completions at I are given by the induced maps

$$(4.5) \qquad Y = F(S_G, Y) \to F(M(I), Y) = Y_I^\wedge.$$

In fact, completion at I is just Bousfield localization at $M(I)$ [25, 2.2].

This construction of completions leads to a different way of thinking about completion theorems. Returning to the augmentation ideal I, we note that the map $M(I) \to S_G$ is a nonequivariant equivalence. By the Whitehead theorem, there is thus a unique map $\xi : \Sigma^\infty EG_+ \to M(I)$ over S_G. By (4.5), we conclude that $\varepsilon : k_G \to F(EG_+, k_G)$ is a completion at I if and only if the "completion conjecture map"

$$(4.6) \qquad \xi^* : (k_G)_I^\wedge = F(M(I), k_G) \to F(EG_+, k_G) = c(k_G)$$

is an equivalence. See [25, §4] for discussion. This is a question about the cohomology theory k_G^*, and it virtually demands consideration of the corresponding question about the homology theory k_*^G: when is the "cocompletion conjecture map"

$$(4.7) \qquad \xi_* = 1 \wedge \xi : k_G \wedge EG_+ \to k_G \wedge M(I)$$

an equivalence? We shall prove the following very surprising relationship between these two questions. Although we find this result illuminating, it is something of a digression and will not be used later.

THEOREM 4.8. *Let k_G be a ring G-spectrum . Then the map ξ_* of (4.7) is an equivalence if and only if $t(k_G)$ is a rational G-spectrum and the map ξ^* of (4.6) is an equivalence. When these equivalent conditions hold, $f^{\perp}(k_G)^{\wedge}_I$ and $t(k_G)^{\wedge}_I$ are trivial, hence $c(k_G)$ is the completion at I of both k_G and $f(k_G)$.*

In the case of KU_G and KO_G, ξ^* is an equivalence by the Atiyah-Segal completion theorem, ξ_* was proven to be an equivalence by the first author in [22], and the rationality of the Tate spectra will be proven independently in Section 19.

Clearly these results on K-theory are very special. In particular, if k_G is bounded below and $t(k_G)^{\wedge}_I$ is trivial, then $t(k_G)$ is itself trivial, by Theorem 4.1. Aside from the K-theory spectra, the only examples we know of for which the cocompletion conjecture map is an equivalence are the obvious ones, namely free G-spectra k_G and rational G-spectra k_G such that $k_G \simeq k_G \wedge \widetilde{E}G$.

To begin the proof of Theorem 4.8, we see that ξ^* is an equivalence if ξ_* is an equivalence by the following general observation of Adams.

LEMMA 4.9. *Let k_G be a ring G-spectrum and m_G be a k_G-module G-spectrum. If $\alpha : X \to Y$ is a map such that $1 \wedge \alpha : k_G \wedge X \to k_G \wedge Y$ is an equivalence, then the following maps are also equivalences:*

$$1 \wedge \alpha : m_G \wedge X \to m_G \wedge Y \text{ and } F(\alpha, 1) : F(Y, m_G) \to F(X, m_G).$$

PROOF: Equivalently, if $k_G \wedge Z$ is trivial, then so are $m_G \wedge Z$ and $F(Z, m_G)$. For the first, $m_G \wedge Z$ is a retract of $m_G \wedge k_G \wedge Z$. For the second, the adjoint $\tilde{\phi}$ of a map $\phi : W \to F(Z, m_G)$ factors as the composite

$$\mu(1 \wedge \tilde{\phi})(\eta \wedge 1) : W \wedge Z \to k_G \wedge W \wedge Z \to k_G \wedge m_G \to m_G.$$

In particular, since $c(k_G)$ is a k_G-module G-spectrum, the map

$$\xi_* = 1 \wedge \xi : c(k_G) \wedge EG_+ \to c(k_G) \wedge M(I) \tag{4.10}$$

is an equivalence if the map ξ_* of (4.7) is an equivalence. This leads us to the following formal part of Theorem 4.8.

PROPOSITION 4.11. *The map ξ_* of (4.7) is an equivalence if and only if the maps ξ^* of (4.6) and ξ_* of (4.10) are both equivalences.*

PROOF: By Proposition 1.1, the left arrow in the commutative square

$$\begin{tikzcd} k_G \wedge EG_+ \arrow[r] \arrow[d] & k_G \wedge M(I) \arrow[d] \\ c_G(k_G) \wedge EG_+ \arrow[r] & c_G(k_G) \wedge M(I) \end{tikzcd}$$

is always an equivalence. Since completion at I is Bousfield localization at $M(I)$, the right arrow is an equivalence if and only if $\varepsilon : k_G \to c(k_G)$ is a completion at I, and this holds if and only if the map ξ^* of (4.6) is an equivalence. The conclusion follows.

To tie in rationality considerations, let $\widetilde{M}(I)$ denote the cofiber of $M(I) \to S_G$. We have a map of cofiber sequences

$$\begin{tikzcd} EG_+ \arrow[r] \arrow[d, "\xi"'] & S^0 \arrow[r] \arrow[d, equal] & \widetilde{E}G \arrow[d, "\widetilde{\xi}"] \\ M(I) \arrow[r] & S^0 \arrow[r] & \widetilde{M}(I). \end{tikzcd} \tag{4.12}$$

Therefore the map ξ_* of (4.7) is an equivalence if and only if the map

$$\widetilde{\xi}_* = 1 \wedge \widetilde{\xi} : k_G \wedge \widetilde{E}G \to k_G \wedge \widetilde{M}(I) \tag{4.13}$$

is an equivalence, and the map ξ_* of (4.10) is an equivalence if and only if the map

$$\widetilde{\xi}_* = 1 \wedge \widetilde{\xi} : t(k_G) = c(k_G) \wedge \widetilde{E}G \to c(k_G) \wedge \widetilde{M}(I) \tag{4.14}$$

is an equivalence. The main point of Theorem 4.8 is the following result.

PROPOSITION 4.15. *Let k_G be a ring G-spectrum . Then $t(k_G)$ is rational if and only if the map $\widetilde{\xi}_*$ of (4.14) is an equivalence.*

PROOF: By Remark A.14 below, the vertical arrows in (4.12) are rational equivalences. If $t(k_G)$ is rational, then the map $\widetilde{\xi}_*$ of (4.14) is a ring map and a rational equivalence with rational domain. This implies that its target is also rational and thus that it itself is an equivalence. For the converse, q acts invertibly on $t(k_G)$ for any G-spectrum k_G if q is a prime that does not divide $|G|$, by Corollary 11.5 below. The following three lemmas show that p acts invertibly on $t(k_G)$ if the map $\widetilde{\xi}_*$ of (4.14) is an equivalence and p is a prime that does divide $|G|$. Observe that the assumption about the map (4.14) is inherited on passage to subgroups (as in Proposition 3.7).

Recall the construction of $M(I)$ in (4.4). Since $M(I) \to S_G$ is the smash product of maps $M(\alpha) \to S_G$ with cofibers $S_G[\alpha^{-1}]$, where α runs through a finite set of generators of I, $\widetilde{M}(I)$ has a finite filtration each of whose subquotients is of the form $S_G[\alpha^{-1}] \wedge Y$ for some element $\alpha \in I$ and G-spectrum Y. This has the following consequence.

LEMMA 4.16. *If G is a non-trivial p-group, then p acts invertibly on $X \wedge \widetilde{M}(I)$ for any G-spectrum X.*

PROOF: In $A(G)$, p divides some power of any element $\alpha \in I$. Therefore p acts invertibly on each subquotient of the induced filtration of $X \wedge \widetilde{M}(I)$.

Returning to our general finite group G, let p divide $|G|$ and let P be a p-Sylow subgroup of G. For any G-spectrum k_G, the composite

$$k_G \to k_G \wedge G/P_+ \to k_G$$

of transfer and projection is multiplication by the element $[G/P] \in A(G)$, and $k_G \wedge G/P_+$ is isomorphic to $G \ltimes_P k_P$ by [33, II.4.8]. Therefore, if multiplication by $[G/P]$ is an equivalence on k_G, then k_G is a wedge summand of $G \ltimes_P k_P$.

LEMMA 4.17. *If X is a free G-spectrum and Y is a p-local G-spectrum , then multiplication by $[G/P]$ is an equivalence on $X \wedge Y$ and on $F(X,Y)$.*

PROOF: Since G_+ is self-dual, the two cases are the same when $X = G_+$; the conclusion holds in this case since multiplication by $[G/P]$ agrees with multiplication by $|G/P|$ on $G_+ \wedge Y$. The result holds for a wedge of free G-spectra X if it holds for each wedge summand, and the conclusion follows by induction up the sequential filtration of X [33, I.5.2].

LEMMA 4.18. *If k_G is a G-spectrum such that p acts invertibly on $t(k_P)$, then p acts invertibly on $t(k_G)$.*

PROOF: Let M_p be the mod p Moore G-spectrum, that is, the cofiber of $p : S_G \to S_G$. Then p acts invertibly on k_G if and only if $k_G \wedge M_p$ is trivial. Since $f(k_G)$ is free and $c(k_G) \wedge M_p$ is equivalent to $F(EG_+, k_G \wedge M_p)$, the previous lemma gives that multiplication by $[G/P]$ is an equivalence on both $f(k_G) \wedge M_p$ and $c(k_G) \wedge M_p$ and therefore also on $t(k_G) \wedge M_p$. Since the hypothesis implies that $t(k_G) \wedge M_p$ is trivial when regarded as a P-spectrum, the conclusion follows.

The following observation implies the last statement of Theorem 4.8.

LEMMA 4.19. *If X is a G-spectrum such that $1 \wedge \widetilde{\xi} : X \wedge \widetilde{E}G \to X \wedge \widetilde{M}(I)$ is an equivalence, then $(X \wedge \widetilde{E}G)^\wedge_I$ is trivial.*

PROOF: Since completion at I is Bousfield localization at $M(I)$, $Y^\wedge_I$ is trivial if and only if $Y \wedge M(I)$ is trivial, and $\widetilde{M}(I) \wedge M(I)$ is trivial since $M(I) \wedge M(I) \simeq M(I)$.

We must still prove Theorem 4.1. The following more general result is valid for any compact Lie group G.

PROPOSITION 4.20. *Let J be any finitely generated ideal contained in the augmentation ideal $I \subset A(G)$. Then the following conclusions hold.*

(i) *$c(k_G)$ is J-complete.*

(ii) *Any bounded below free G-CW spectrum Y is J-complete.*

PROOF: If J is obtained from an ideal K by adding a new generator α, then (4.4) and (4.5) imply that $Y_J^\wedge = (Y_K^\wedge)_\alpha^\wedge$ for any Y. Inductively, it suffices to prove the results for $J = (\alpha)$. Since α is trivial as a nonequivariant map, the underlying nonequivariant spectrum of $S_G[\alpha^{-1}]$ is trivial and $M(\alpha) \to S_G$ is a nonequivariant equivalence. Part (i) follows from (4.5), adjunction, and Proposition 1.1. By (4.5) again, (ii) will hold if $F(S[\alpha^{-1}], Y)$ is trivial. For any X, we have the exact sequence

$$0 \to \lim^1[\Sigma X, Y]_G \to [X, F(S[\alpha^{-1}], Y)]_G \to \lim[X, Y]_G \to 0,$$

where lim and $\lim^1$ are taken with respect to countable iteration of $\alpha : X \to X$. If $Y = G_+ \wedge K$, then $[X, Y]_G \cong [X, \Sigma^d K]$ by [33, II.4.8 and 6.5], where X and K on the right are underlying nonequivariant spectra. Since α is nonequivariantly trivial, the lim and $\lim^1$ terms vanish. Taking K to be a wedge of spheres S^n, and applying induction, we see that the conclusion holds if Y is finite dimensional. If Y is bounded below and X is finite, then all maps $X \to Y$ factor through a fixed finite dimensional subcomplex of Y and again the lim and $\lim^1$ terms vanish. Thus $F(S[\alpha^{-1}], Y)$ has trivial homotopy groups.

§5. Transfer and the fixed point spectra of Tate G-spectra

Transfer has played no overt role in setting up our basic definitions. However, it is implicitly present, as we now explain.

Looking back at diagram (C) and recalling from Proposition 1.2 that its left vertical map is a G-equivalence, we see that, up to equivalence, $t(k_G)$ can be viewed as the cofiber of the evident composite.

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G). \tag{5.1}$$

As in (0.6), when we compute homology in integer degrees, we first smash with the G-space X, then take G-fixed point spectra, and finally compute homotopy groups. Smashing with X commutes with passage to cofibers. Taking fixed points commutes with fibers, by inspection of definitions, hence commutes up to equivalence with cofibers. As pointed out in Section 0, to pass to fixed points we must first apply the forgetful functor i^* and then pass to fixed points spacewise. Thus Tate homology is obtained by taking the homotopy groups of the cofiber of the composite

$$(i^*(k_G \wedge EG_+ \wedge X))^G \to (i^*(k_G \wedge X))^G \to (i^*(F(EG_+, k_G) \wedge X))^G. \tag{5.2}$$

Replacing X by i_*X for a finite naive G-CW spectrum X, for generality, we here give an equivalent description in terms of naive G-spectra, indexed on U^G, without reference to the change of universe functors i_* and i^*. However, such a description certainly should not be taken as a definition, since it necessarily encodes far less information. By Corollary 1.3, we may assume without loss of generality that $k_G = i_*(j_G)$ for a naive G-spectrum j_G. Since i_* commutes with smash products, the domain of (5.2) now has the form $(i^*i_*(j_G \wedge EG_+ \wedge X))^G$. The following specialization of [33, II.7.1] gives a purely naive description of this spectrum.

THEOREM 5.3. *For naive G-spectra X, there is a natural equivalence of spectra*

$$\tilde{\tau} : (j_G \wedge EG_+ \wedge \Sigma^{\mathrm{Ad}(G)} X)/G \to (i^*i_*(j_G \wedge EG_+ \wedge X))^G.$$

The equivalence $\widetilde{\tau}$ is given by an appropriate transfer map. We sketch its construction.

To handle equivariance, we work with the semidirect product $\Gamma = G \times_c G$ as the ambient group, where c is the conjugation action of G on itself. Write G' for the subgroup $G \times_c e$ of Γ and write Π for the normal subgroup $e \times_c G$, so that $G \cong \Gamma/\Pi$. Write $'G$ for G regarded as a left Γ-space with action given by $(g, g')h = gg'hg^{-1}$. We obtain a Γ-homeomorphism from $'G$ to the orbit Γ/G' by sending h to the coset of (e, h). Define $\varepsilon : \Gamma \to G$ and $\varphi : \Gamma \to G$ by $\varepsilon(g, g') = g$ and $\varphi(g, g') = gg'$.

If Y is a free G-space and φ^*Y denotes Y regarded as a G-space by pullback along φ, then Y is G-homeomorphic to the orbit space $(\varphi^*Y \times {}'G)/\Pi$. The right way to think of $Y \to Y/G$ as an equivariant bundle is to regard the Π-free Γ-space φ^*Y as its associated principal bundle, the Γ-space $'G \cong \Gamma/G'$ as its fibre, and the subgroup Π of Γ as its structural group. As usual, our definition of the transfer begins with an appropriate definition of the "pretransfer" on the level of fibers.

Embed $'G$ in a Γ-representation V. The tangent Γ-bundle of $'G$ is trivial with fibre $A = \mathrm{Ad}(G)$, where Γ acts on A through ε. The complement $V - A$ of A under the embedding of tangent spaces is also a representation of Γ, the normal Γ-bundle of the embedding is $'G \times (V - A)$, and we obtain a Γ-map $t : S^V \to {}'G_+ \wedge S^{V-A}$ by the Pontryagin-Thom construction. Let the complete G-universe U be $(U')^{\Pi}$, where U' is a complete Γ-universe. Passing to suspension Γ-spectra (but leaving out notation for the suspension spectrum functor) and desuspending by $V - A$, we obtain a pretransfer map $t : S^A \to {}'G_+$ of Γ-spectra indexed on U'.

Now let Y be a free naive G-spectrum, such as $j_G \wedge EG_+ \wedge X$. By analogy with the space level description given above, there is a sensible way of defining the "associated principal Π-free Γ-spectrum" determined by Y [33, II.7.4], and [33, II.7.5] explains how to smash this spectrum with the map t and then pass to orbits over Π to obtain a map

$$(5.4) \qquad \tau : i_*((\Sigma^A Y)/G) \to i_* Y$$

of G-spectra indexed on U, where G acts trivially on $(\Sigma^A Y)/G$. The adjoint of τ is the transfer map

$$\widetilde{\tau} : (\Sigma^A Y)/G \to (i^* i_* Y)^G. \tag{5.5}$$

By [33, II.7.1], this map is a natural equivalence.

For the target in (5.2), we have the following more elementary reduction to the naive level; however, it is only valid for finite X.

THEOREM 5.6. *For finite naive G-CW spectra X, there is a natural equivalence of naive G-spectra*

$$F(EG_+, j_G \wedge X) \simeq i^*(F(EG_+, i_* j_G) \wedge i_* X).$$

PROOF: Since X is finite, the canonical map

$$F(EG_+, i_* j_G) \wedge i_* X \to F(EG_+, i_* j_G \wedge i_* X)$$

is an equivalence [33, III.2.8(ii)], and, by Lemma 0.1 and Proposition 1.1,

$$i^* F(EG_+, i_* j_G \wedge i_* X) \cong F(EG_+, i^* i_* (j_G \wedge X)) \simeq F(EG_+, j_G \wedge X).$$

Composing the equivalences of the previous two results with the maps induced by (5.2), we obtain a natural map

$$\overline{\tau} : (j_G \wedge EG_+ \wedge \Sigma^{\mathrm{Ad}(G)} X)/G \to (F(EG_+, j_G \wedge X))^G. \tag{5.7}$$

Our discussion shows that, with $k_G = i_* j_G$, its cofiber is equivalent to $(t(k_G) \wedge X)^G$, whose homotopy groups are $t(k_G)_*(X)$.

For a naive G-spectrum Y, it is fashionable to call $F(EG_+, Y)^G$ the "homotopy fixed point spectrum" of Y and to denote it by Y^{hG}. The dual notion of the "homotopy orbit spectrum" is $(EG_+ \wedge Y)/G$ which, by analogy, is sometimes denoted Y_{hG}. With these notations, (5.7) can be written as

$$\overline{\tau} : (j_G \wedge \Sigma^{\mathrm{Ad}(G)} X)_{hG} \to (j_G \wedge X)^{hG}. \tag{5.8}$$

For a naive G-spectrum Y, Adem, Cohen, and Dwyer [4] define $\widehat{H}(Y)$ to be the fiber of a suitable transfer map from a spectrum they call $(EG_+ \wedge_G Y)^{\mathrm{ad}}$ to Y^{hG}. They call the homotopy groups of $\widehat{H}(Y)$ the Tate homology groups of G with coefficients in Y. While their description is different, $(EG_+ \wedge_G Y)^{\mathrm{ad}}$ is equivalent to $\Sigma^{\mathrm{Ad}(G)} Y_{hG}$. Taking $Y = H\mathbb{Z} \wedge X$, they claim that $\pi_*(\widehat{H}Y)$ gives the Tate-Swan and cyclic homology of X, modulo the obvious difference in grading resulting from their description of $\widehat{H}(Y)$ as a fiber rather than a cofiber. (Our choice is dictated by the inherent logic of diagram (C) and the fact that $t(k_G)$ should be a ring G-spectrum when k_G is a ring G-spectrum.) However, as the following observation makes clear, their claim is not correct for infinite complexes.

SCHOLIUM 5.9. *Let G act trivially on X and $H\mathbb{Z}$. Then $(H\mathbb{Z} \wedge X)^{hG}$ is equivalent to the product over s of the spectra $F(BG_+, K(\widetilde{H}_s(X), s))$. Its $(-n)^{th}$ homotopy group is the product over s of $H^{n+s}(BG; \widetilde{H}_s(X))$. As a functor of X, this clearly fails to satisfy the wedge axiom, whereas the homotopy groups of $(H\mathbb{Z} \wedge \Sigma^{\mathrm{Ad}(G)} X)_{hG}$ clearly do satisfy this axiom. Therefore the homotopy groups of the fiber $\widehat{H}(H\mathbb{Z} \wedge X)$ fail to satisfy the wedge axiom.*

The definitional framework of Weiss and Williams [47], especially their version of function spectra, seems a bit ad hoc to us, so we will not attempt a precise comparison of their definitions with ours. Rather, we show that our Tate spectra have the essential properties they require in their applications. The following is the main point; compare [47, §2].

THEOREM 5.10. *Let G be finite. Let j_G be a naive G-spectrum and write j for j_G regarded as a nonequivariant spectrum. Then the following diagram commutes. Here $i : G \to EG$ is determined by a choice of basepoint and N denotes the norm map $\Sigma g : j \to j$:*

$$\begin{tikzcd}
(j_G \wedge G_+)/G \cong j \arrow[r, "N"] \arrow[d, "(1\wedge i)/G"'] & j \simeq F(EG_+, j) \\
(j_G \wedge EG_+)/G \arrow[r, "\overline{\tau}"] & F(EG_+, j_G)^G \arrow[u, "\cup"']
\end{tikzcd}$$

Moreover, for any nonequivariant spectrum j,

$$\overline{\tau} : (j \wedge G_+ \wedge EG_+)/G \to F(EG_+, j \wedge G_+)^G$$

is an equivalence.

PROOF: The second statement is an immediate consequence of the case $k_G = i_*j$ and $X = G_+$ of Proposition 2.4. Note that, as on the space level

$$(j_G \wedge G_+)/G \cong (j \wedge G_+)/G \cong j$$

since $j_G \wedge G_+$ with its diagonal action by G is isomorphic to $j \wedge G_+$ with its action by G on the factor G_+ [33, II.4.8 and II.4.15(iv)]. Expanding $\overline{\tau}$ via (5.2), with $X = S^0$, and using the naturality of $\widetilde{\tau}$, we see that the following diagram commutes and that traversal of its periphery gives the composite around the bottom of the diagram of the statement.

$$\begin{tikzcd}
j \cong (j_G \wedge G_+)/G \arrow[rr] \arrow[dr, "\overline{\tau}"] & & (j_G \wedge EG_+)/G \arrow[d, "\widetilde{\tau}"] \\
i^*i_*(j \wedge G_+) \arrow[dr] & i^*i_*(j_G \wedge G_+)^G \arrow[l] \arrow[r] \arrow[dr] & i^*i_*(j_G \wedge EG_+)^G \arrow[d] \\
j \arrow[r] \arrow[ddr, "\simeq"', "\xi"] & i^*i_*j \arrow[d, "\varepsilon"] & (i^*i_*j_G)^G \arrow[l] \arrow[d, "\varepsilon^G"] \\
 & F(EG_+, i^*i_*j) & F(EG_+, i^*i_*j_G)^G \arrow[l] \\
 & F(EG_+, j) \arrow[u, "\simeq"] & F(EG_+, j_G)^G \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

Passing to adjoints, we may view τ as a G-map $i_*j \to (i_*j) \wedge G_+$, and it suffices to show that its composite with the projection $\xi : (i_*j) \wedge G_+ \to i_*j$ is $\Sigma i_*(g)$. Here i_*j acts as a dummy variable, and [33, II.7.6] shows that $\tau = 1 \wedge t$, where $t : S \to \Sigma^\infty G_+$ is the usual transfer map. Explicitly, if V is the regular representation of G and $G \times V \to V$ is a tubular neighborhood of the inclusion of basis elements $G \to V$, then t is the desuspension by V of the Pontryagin-Thom map $S^V \to G_+ \wedge S^V$. The relevant comparison of definitions should be clear from the fact that the map $G \to V$ can be reinterpreted as an allowable choice of the Γ-embedding $'G \to V$ that was the starting point of our construction of $\widetilde{\tau}$. Incidentally, it now follows that $\xi \circ \tau : i_*j \to i_*j$ is multiplication by the element $[G] \in A(G)$. Since the Pontryagin-Thom map just cited is a pinch map suitable for computing sums, it is easily checked (by use of [33, II.4.8 and 4.15(iii)]) that $\xi \circ \tau$ does indeed coincide with $\Sigma i_*(g)$.

Part II: Eilenberg-Maclane G-spectra and the spectral sequences

§6. Eilenberg-MacLane G-spectra and their associated theories

In this and the following two sections, we study the cohomology theories represented by $t(HM)$ for an Eilenberg-MacLane spectrum HM. We are particularly interested in their coefficient groups, which we think of as topologically defined algebraic invariants of the group. In principle, there are at least three ways to get at these theories. In favorable cases, namely finite groups and the circle and unit quaternion groups, these theories can be obtained as hypercohomology groups defined by mixing canonical algebraic complexes and singular or cellular cochain complexes. In the same cases, as we shall prove in Sections 11 and 14, we can realize these mixed algebraic and topological cochain complexes by appropriate topologically defined cellular cochain groups. Our present point of view is that both of these cochain level approaches are calculational devices for the study of the topologically defined, represented, theories. We prefer to understand the latter before restricting attention to special cases.

Let G be a compact Lie group of dimension d and let $\overline{G}$ denote its finite group $\pi_0(G)$ of components. Let $\mathcal{OS}$ be the full subcategory of the stable category of G-spectra whose objects are the suspension spectra of orbits G/H_+. A Mackey functor M is an additive contravariant functor $\mathcal{OS} \to \mathcal{A}b$, written $M(G/H)$ on the object $\Sigma_G^\infty G/H_+$. For finite G, this is equivalent to the standard definition of Dress [17], by [33, V.9.9]. For any G, $M(G/e)$ is a $\overline{G}$-module, which we denote by UM (and think of as the underlying $\overline{G}$-module of M). For a G-spectrum X and an integer n, we have the n^{th} homotopy group Mackey functor $\underline{\pi}_n(X)$. Its value on G/H is

$$\pi_n(X^H) = [S^n, X]_H = [G/H_+ \wedge S^n, X]_G,$$

and its contravariant functorality on $\mathcal{OS}$ is obvious.

By [32], for a Mackey functor M, there is an Eilenberg-MacLane G-spectrum HM, unique up to equivalence, whose only nonvanishing homotopy group Mackey functor is

$\underline{\pi}_0(HM) \cong M$. Maps $HM \to HM'$ correspond bijectively to maps $M \to M'$ of Mackey functors. We will prove shortly that, as suggested by Proposition 1.1, the bottom row of Diagram (D) for HM depends only on UM. The proof is based on an understanding of the relationship among $\overline{G}$-modules, Mackey functors, and G-spectra, and this understanding will be critical to our study of products in Section 8. Let $\mathcal{Z}[\overline{G}]$ and $\mathcal{M}[G]$ denote the categories of $\mathbb{Z}[\overline{G}]$-modules and of Mackey functors over G.

LEMMA 6.1. *The functor $U : \mathcal{M}[G] \to \mathcal{Z}[\overline{G}]$ has a left adjoint F, so that*

$$\mathrm{Hom}_{\mathcal{M}[G]}(FV, M) \cong \mathrm{Hom}_{\overline{G}}(V, UM)$$

for a Mackey functor M and a $\overline{G}$-module V. Moreover, $UFV = V$.

As a left adjoint, the functor F is right exact. It is not left exact, and it has left derived functors L_iF, $i \geq 0$, with $L_0F = F$; see [12, V§§2,3]. The topology realizes this algebraic fact in a rather remarkable fashion.

THEOREM 6.2. *There is a functor L from $\mathcal{Z}[\overline{G}]$ to the stable category of connective G-spectra such that $\underline{\pi}_0(LV) \cong FV$ and L is exact, in the sense that it transforms short exact sequences $0 \to V' \to V \to V'' \to 0$ of $\overline{G}$-modules to cofibration sequences $LV' \to LV \to LV''$ of G-spectra . If G_1 denotes the component of the identity element of G, then*

$$\pi_i^G(L(\mathbb{Z}[\overline{G}] \otimes A)) \cong H_{i-d}(BG_{1+}; A) \text{ for any Abelian group } A \text{ and all } i \geq 0.$$

If G is finite, then

$$\underline{\pi}_i(LV) \cong L_iFV \text{ for any } G\text{-module } V \text{ and all } i \geq 0;$$

moreover, if $V = \mathbb{Z}[G] \otimes A$, then $t(LV)$ is trivial and $LV \simeq f(LV) \simeq c(LV)$.

THEOREM 6.3. *Let M be a Mackey functor and let $V = UM$. Then the following three cofibration sequences are canonically G-equivalent:*

$$f(HM) \longrightarrow c(HM) \longrightarrow t(HM),$$

$$f(HFV) \longrightarrow c(HFV) \longrightarrow t(HFV),$$

and

$$f(LV) \longrightarrow c(LV) \longrightarrow t(LV).$$

Therefore, if M and M' are Mackey functors such that $UM \cong UM'$, then the norm cofibration sequences of HM and HM' are canonically G-equivalent.

Except for the part about LV, this is immediate from Lemma 6.1 and Proposition 1.1: the identity map on V extends uniquely to a map of Mackey functors $FV \to M$, and the resulting map $HFV \to HM$ induces an equivalence of bottom rows of Diagram (D). By Corollary 1.5, the theorem specializes as follows to trivial $\overline{G}$-modules and split G-spectra.

EXAMPLE 6.4: Let M be a Mackey functor such that the natural map $M(G/G) \to M(G/e) = V$ is a split epimorphism; thus $\overline{G}$ acts trivially on V. Then the split G-spectra HM, HFV, LV, and i_*HV all have equivalent f, c, and t G-spectra, where HV is the Eilenberg-MacLane spectrum (regarded as a naive G-spectrum with trivial action). FV and the "constant Mackey functor" $\underline{V}$ with $\underline{V}(G/H) = V$ and with identity restriction maps (see [33, V.9.10]) give two different examples of Mackey functors M with $UM = V$.

The results stated above will be proven in the next section. They allow us to make the following definition. Modulo variant grading conventions, it includes the Tate and cyclic homology and cohomology theories. Our gradings are dictated by the usual definitions of represented homology and cohomology theories and by a shift of dimensions resulting from the fact that the dual of G_+ is $G_+ \wedge S^{-d}$, so that the homology theories represented by Eilenberg-MacLane G-spectra fail to satisfy the dimension axiom unless G is finite.

In fact, this last observation leads to a deeper, and definitive, topological motivation for our choices, but we prefer not to explain this until Section 20. For H_*, H^*, and $\widehat{H}^*$, our gradings agree with the others that we have seen in the literature, but gradings of the other three functors tend to vary from source to source.

DEFINITIONS 6.5. *Let V be a $\overline{G}$-module and let M be any Mackey functor such that $UM \cong V$. For G-spectra X and integers n, define*

$$H^G_{n-d}(X;V) = f(HM)_n(X) \quad \text{and} \quad H^n_G(X;V) = c(HM)^n(X);$$

$$\check{H}^G_{n-d}(X;V) = c(HM)_n(X) \quad \text{and} \quad \check{H}^n_G(X;V) = f(HM)^n(X);$$

$$\widehat{H}^G_{n-d}(X;V) = t(HM)_n(X) \quad \text{and} \quad \widehat{H}^n_G(X;V) = t(HM)^n(X).$$

When $X = S^0$, we delete it from the notation, writing $\widehat{H}^n_G(V)$, etc.

The differing decorations on H for f and c are suggested by Proposition 2.1, which tells us that the first two groups specialize to the appropriate homology and cohomology groups of the Borel construction when $\overline{G}$ acts trivially on V; compare Example 6.4. In this and the next few sections, we concentrate on the case $X = S^0$, thinking of Definitions 6.5 as giving invariants of groups. For finite G, they are the standard invariants by the following consequence of Theorems 6.2 and 6.3. For the moment, let $\widehat{H}^*(G,V)$ denote Tate cohomology as defined in [12, XII§2].

PROPOSITION 6.6. *Let G be finite. Then there are natural identifications*

$$H^G_n(V) = \mathrm{Tor}^{\mathbb{Z}[G]}_n(\mathbb{Z},V),\ H^n_G(V) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},V),\ \text{and}\ \widehat{H}^n_G(V) = \widehat{H}^n(G,V).$$

PROOF: Note first that, by connectivity and obstruction theory, H^G_n and H^n_G are identically zero for $n < 0$. All three functors on the right admit standard axiomatizations, and we need only verify the axioms. By Theorem 6.3, we may view the theories on the left as $f(LV)_*$, $c(LV)^*$, and $t(LV)^*$. Theorem 6.2 then gives the required long exact sequences in all three theories and shows that they agree with the theories on the right when specialized to free modules V. As a matter of algebra, this implies that, for any G-module V, $H_0^G(V) = V_G \equiv V/IV$, where I is the augmentation ideal of $\mathbb{Z}[G]$, and $H_G^0(V) = V^G$. To complete the check of axioms for $\widehat{H}_G^*(V)$, it suffices to compute $\widehat{H}_G^i(V)$ for any one i, and the norm sequence implies that $\widehat{H}_G^i(V) \cong H_G^i(V)$ for all positive i. Alternatively, the norm sequence and Theorem 5.10 imply that $\widehat{H}_G^0(V) \cong \widehat{H}^0(G, V)$.

§7. Mackey functors and coefficient systems

We must prove Lemma 6.1 and Theorems 6.2 and 6.3. The main arguments depend on the relationship between coefficient systems and Mackey functors, which is the algebraic counterpart of the relationship between naive G-spectra and genuine G-spectra .

We first give an algebraic proof of Lemma 6.1, which is actually an instance of a general categorical result. An alternative topological proof will drop out of the construction of the functor L of Theorem 6.2.

ALGEBRAIC PROOF OF LEMMA 6.1: Write G/H_+ for $\Sigma_G^\infty G/H_+$. The essential point is that $[G_+, G_+]_G \cong \mathbb{Z}[\overline{G}]$. If $V = \mathbb{Z}[\overline{G}]$, we let FV be the represented Mackey functor $(FV)(G/H) = [G/H_+, G_+]_G$. The isomorphism of Lemma 6.1 is formal in this case. Extend F to free $\mathbb{Z}[G]$-modules by additivity. For a general V, let $P_1 \to P_0 \to V \to 0$ be the initial segment of a resolution by free $\mathbb{Z}[\overline{G}]$-modules on specified bases and define FV to be the cokernel of the induced map $FP_1 \to FP_0$. By a standard comparison of resolutions argument, FV is independent of the choice of the resolution, and the isomorphism of Lemma 6.1 follows by a comparison of exact sequences.

We next recall the definition of coefficient systems. Let $\mathcal{NOS}$ be the full subcategory of the stable category of naive G-spectra whose objects are the suspension spectra of orbits G/H_+. As is easily checked, the group of morphisms $G/H_+ \to G/K_+$ in $\mathcal{NOS}$ can be identified with the ordinary (reduced) integral homology group $H_0((G/K)_+^H)$. A coefficient system R is an additive contravariant functor $\mathcal{NOS} \to \mathcal{A}b$, written $R(G/H)$ on the object G/H_+. Again, $UR = R(G/e)$ is a $\overline{G}$-module.

Let $\mathcal{C}[G]$ denote the category of coefficient systems. The analog of Lemma 6.1 for coefficient systems is trivial: we identify the category of $\overline{G}$-modules with the full subcategory of coefficient systems V such that $V(G/H) = 0$ for $H \neq e$, and we find immediately that

$$\mathrm{Hom}_{\mathcal{C}[G]}(V, M) \cong \mathrm{Hom}_{\overline{G}}(V, UM). \tag{7.1}$$

In particular, the identity map of V lifts uniquely to a map of coefficient systems $V \to R$

for any coefficient system R such that $UR = V$.

For a coefficient system R, there is a naive Eilenberg-MacLane G-spectrum HR, unique up to equivalence, whose only nonvanishing homotopy group coefficient system is $\underline{\pi}_0(HR) \cong R$. Maps $HR \to HR'$ correspond bijectively to maps $R \to R'$ of coefficient systems. A naive Eilenberg-MacLane G-spectrum HR comes from a genuine Eilenberg-MacLane G-spectrum if and only if R comes from a Mackey functor.

Formally, following Lewis [31], let $s : \mathcal{NOS} \to \mathcal{OS}$ denote the functor given by application of i_* to orbits and let s^* denote the forgetful functor from Mackey functors to coefficient systems that is obtained by precomposition with s. Then $i^*HM = Hs^*M$. When we write HV below, we mean the naive Eilenberg-MacLane G-spectrum associated to the $\overline{G}$-module V regarded as the coefficient system specified by

$$V(G/e) = V \quad \text{and} \quad V(G/H) = 0 \text{ for } H \neq e.$$

DEFINITION 7.2. *For a coefficient system R, let LR be the G-spectrum i_*HR and let s_*R be the Mackey functor $\underline{\pi}_0(i_*HR)$. Write LV and s_*V for these functors on $\overline{G}$-modules V regarded as coefficient systems.*

PROPOSITION 7.3. *The functor $s_* : \mathcal{C}[G] \to \mathcal{M}[G]$ is left adjoint to the forgetful functor $s^* : \mathcal{M}[G] \to \mathcal{C}[G]$.*

PROOF: For a coefficient system R we can attach cells to i_*HR to kill its higher homotopy groups and so obtain a map $\iota : i_*HR \to H\underline{\pi}_0(i_*HR)$ that induces the identity on $\underline{\pi}_0$. For a Mackey functor M, ι induces an isomorphism on $H^0_G(?;M) = [?, HM]_G$. Therefore

$$\begin{aligned} \operatorname{Hom}_{\mathcal{M}[G]}(s_*R, M) &= [H\underline{\pi}_0(i_*HR), HM]_G \cong [i_*HR, HM]_G \\ &\cong [HR, i^*HM]_G = [HR, Hs^*M]_G = \operatorname{Hom}_{\mathcal{C}[G]}(R, s^*M). \end{aligned}$$

REMARK 7.4: Lewis [31, 4.5] proved the existence of the left adjoint s_* by quoting a general categorical criterion for the existence of adjoints. As a matter of category theory, s_*R is a certain coend, and [31, 4.8] gives a direct, but necessarily quite complicated, algebraic description. The functor i_* from naive G-spectra to genuine G-spectra somehow builds in this algebra. In general, there are many ways to furnish a given coefficient system R with transfer maps that make it into a Mackey functor M. The identity map of $R = s^*M$ has an adjoint map $s_*R \to M$ of Mackey functors for every such M. The price of this universality is that it cannot be true that $s^*s_*R = R$ except in the very special case when $R(G/H) = 0$ for $H \neq e$, when there are no non-zero transfer maps to furnish.

PROOF OF THEOREM 6.2: By the uniqueness of adjoints, $s_*V = FV$ since $Us^* = U$. That is, $\underline{\pi}_0(LV) = FV$. The functor H from coefficient systems to naive G-spectra carries short exact sequences to cofibrations and the functor i_* preserves cofibrations, hence L is exact. Now let $V = \mathbb{Z}[\overline{G}] \otimes A$ for an Abelian group A. We may take HV to be the naive G-spectrum $\overline{G}_+ \wedge EG_+ \wedge HA$. In fact, the computation of fixed point spectra of naive G-spectra is easy, and, if $H \neq e$, then $(\overline{G}_+ \wedge EG_+ \wedge HA)^H$ is trivial since EG^H is empty. As a nonequivariant spectrum, $\overline{G}_+ \wedge EG_+ \wedge HA \simeq \overline{G}_+ \wedge HA$, and its zero$^{\text{th}}$ homotopy group is V. By [33, II.6.5 and II.7.2] and a standard Serre spectral sequence argument, we find that

$$\begin{aligned}
\pi_*^G(L(\mathbb{Z}[\overline{G}] \otimes A)) = \pi_*^G(i_*HV) &\cong \pi_*^{G_1}(EG_{1+} \wedge i_*HA) \\
&\cong \pi_*(S^{\mathrm{Ad}(G_1)} \wedge_{G_1} EG_{1+} \wedge HA) \\
&\cong H_*(\Sigma^d BG_{1+}; A).
\end{aligned}$$

Finally, suppose that G is finite. By the axioms for derived functors, to prove that $\underline{\pi}_i(LV) \cong L_iFV$ for all G-modules V it only remains to prove that $\underline{\pi}_i(LV) = 0$ for $i > 0$ when $V = \mathbb{Z}[G] \otimes A$. Here, since $G_+ \wedge EG_+ \simeq G_+$,

$$LV \simeq G_+ \wedge i_*HA \simeq c(LV).$$

By [33, II.6.5], $\pi_*^H(LV) = H_*(G/H_+; A)$. Since G_+ is self-dual and $\widetilde{E}G \wedge G_+$ is trivial, $t(LV)$ is trivial by a standard duality equivalence.

PROOF OF THEOREM 6.3: Let M be any Mackey functor with $UM = V$. The identity map of V lifts uniquely to a map of coefficient systems $V \to s^*M$. We have an induced map of G-spectra

$$LV = i_*HV \to i_*Hs^*M = i_*i^*HM.$$

By Proposition 1.1 and Lemma 0.1, both this map and the natural map $\varepsilon : i_*i^*HM \to HM$ induce G-equivalences of bottom rows in Diagram (D).

§8. Products in the theories associated to Eilenberg-MacLane G-spectra

We need a multiplicative elaboration of Theorem 6.3 to introduce product pairings in all of the theories of Definition 6.5 and, later, to study products in our generalized Atiyah-Hirzebruch spectral sequences. The categories of coefficient systems and of Mackey functors, in common with all such categories of additive functors, have their own internal tensor products; see e.g. [30], [40], [48]. In view of the invariance properties of the theories of interest to us here, we do not need an algebraic understanding of these products, and we therefore simply define them topologically by

$$M \otimes M' = \underline{\pi}_0(HM \wedge HM'). \tag{8.1}$$

By killing the higher homotopy groups of $HM \wedge HM'$, we obtain a canonical map

$$\iota : HM \wedge HM' \to H(M \otimes M').$$

Since ι induces an isomorphism on $H^0_G(?;M'') = [?, HM'']_G$, pairings of G-spectra $HM \wedge HM' \to HM''$ are in bijective correspondence with pairings $M \otimes M' \to M''$.

These tensor products of coefficient systems and of Mackey functors extend the tensor product of $\overline{G}$-modules in the sense that

$$U(M \otimes M') = UM \otimes UM'.$$

To see this, just observe that we are here computing the zero$^{\text{th}}$ homotopy group of the smash product of two ordinary Eilenberg-MacLane spectra and then remembering the actions. Thus a pairing $HM \wedge HM' \to HM''$ of G-spectra or, equivalently, a pairing $M \otimes M' \to M''$ of Mackey functors induces a pairing $UM \otimes UM' \to UM''$ of $\overline{G}$-modules; we say that the former pairing realizes the latter one. Such a pairing gives rise to the following special case of the norm pairing diagram of (3.4):

$$\begin{tikzcd} f(HM) \wedge f(HM') \arrow[r] \arrow[d] & c(HM) \wedge c(HM') \arrow[r] \arrow[d] & t(HM) \wedge t(HM') \arrow[d] \\ f(HM'') \arrow[r] & c(HM'') \arrow[r] & t(HM''). \end{tikzcd} \tag{8.2}$$

This gives compatible pairings in the theories of Definition 6.5, and these are well-defined by the following elaboration of Theorem 6.3.

THEOREM 8.3. *Let $\alpha : V\otimes V' \to V''$ be a pairing of $\overline{G}$-modules. Then there are canonically induced pairings of G-spectra*

$$LV \wedge LV' \to LV'' \quad \text{and} \quad HFV \wedge HFV' \to HFV''$$

that realize α on π_0, and their induced norm pairing diagrams are canonically equivalent. If $HM \wedge HM' \to HM''$ also realizes α, then its norm pairing diagram is canonically equivalent to these norm pairing diagrams, hence any two realizations give rise to canonically equivalent diagrams.

PROOF: Clearly we have $\underline{\pi}_0(HV \wedge HV') \cong V \otimes V'$, so that α induces a map

$$HV \wedge HV' \to H(V \otimes V') \to HV''.$$

Applying i_* and using that it commutes with smash products, we obtain

$$LV \wedge LV' = i_*HV \wedge i_*HV' \to i_*HV'' = LV''.$$

Killing higher homotopy groups and using obstruction theory, this gives

$$HFV \wedge HFV' \to HFV''.$$

Given any other realization $HM \wedge HM' \to HM''$, we obtain the following commutative diagram. Its horizontal arrows are those used to prove Theorem 6.3.

$$\begin{tikzcd}
Hs_*V \wedge Hs_*V' \arrow[d] & i_*HV \wedge i_*HV' \arrow[l] \arrow[r] \arrow[d] & i_*i^*HM \wedge i_*i^*HM' \arrow[r] & HM \wedge HM' \arrow[d] \\
Hs_*V'' & i_*HV'' \arrow[l] \arrow[r] & i_*i^*HM'' \arrow[r] & HM''.
\end{tikzcd}$$

To see that the right rectangle commutes, pass to adjoints and note that the following diagram commutes by inspection on the zeroth homotopy group coefficient system level:

$$\begin{tikzcd}
HV \wedge HV' \arrow[r] \arrow[d] & Hs^*M \wedge Hs^*M' \arrow[d] \\
HV'' \arrow[r] & Hs^*M''.
\end{tikzcd}$$

Similarly, an elaboration of the proof of Proposition 6.6 shows that these topologically defined pairings agree with the usual algebraic pairings when G is finite. This is particularly important to us since, in contrast with Proposition 6.6, we do not have an alternative chain level proof in the case of Tate cohomology.

PROPOSITION 8.4. *If G is finite, then the equivalences*

$$H_n^G(V) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, V), \quad H_G^n(V) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, V), \text{ and } \widehat{H}_G^n(V) = \widehat{H}^n(G, V)$$

carry the pairings just introduced on the left sides to the usual algebraic pairings on the right sides.

PROOF: Since the smash product of a cofiber sequence with a G-spectrum is a cofiber sequence, easy diagram chases show that the pairings on the left commute appropriately with connecting homomorphisms. As a matter of algebra, an elaboration of [12, XII§5] shows that it suffices to check the conclusions on $H_0^G(V) \otimes H_0^G(V')$ and $H_G^0(V) \otimes H_G^0(V')$ when $V = \mathbb{Z}[G] \otimes A$ and $V' = \mathbb{Z}[G] \otimes A'$ and on $\widehat{H}_G^i(V) \otimes \widehat{H}_G^j(V')$ for any one pair (i, j). In all three cases, it suffices to check on the identity pairings to $V \otimes V'$. The first two cases are easily checked by diagram chases from the calculation in degree zero given in the proof of Theorem 6.2. As in the proof of Proposition 6.6, the third case follows directly from the second.

§9. Chain level calculation of the coefficient groups

In this section and the next, we shall be concerned with chain complexes that come from filtered G-spectra with free subquotients, and we must first explain precisely how to pass from the topology to algebra.

Let $W = \cup W^p$, $p \in \mathbb{Z}$, where each $W^p \to W^{p+1}$ and $W^p \to W$ is a cofibration, and let $\overline{W}^p = W^p/W^{p-1}$. If we were just given a sequence of G-spectra and maps, we could use the telescope construction to arrange to have cofibrations as specified. Assume that $\overline{W}^p$ is equivalent to $G_+ \wedge K^p$, where K^p is the wedge of copies of the sphere spectrum S^p. As usual, we have the geometric boundary map

$$\partial : \overline{W}^p \simeq C(W^{p-1} \to W^p) \to \Sigma W^{p-1} \to \Sigma(W^{p-1}/W^{p-2}) = \Sigma \overline{W}^{p-1}.$$

Let k_G be a G-spectrum with underlying nonequivariant spectrum k. Taking the induced filtrations of $k_G \wedge W$ and of $F(W, k_G)$ and passing to G-homotopy groups, we obtain exact couples and spectral sequences. (See Appendix B for details.) They satisfy

$$E^1_{p,q} = k^G_{p+q}(\overline{W}^p) \quad \text{and} \quad E_1^{p,q} = k_G^{p+q}(\overline{W}^p),$$

and d^1 and d_1 are induced by ∂.

We can compute d^1 and d_1 in terms of ordinary integral homology and cohomology. To see this, let G have dimension d and recall that $D(G_+) = G_+ \wedge S^{-d}$. By [33, II.4.7, II.4.8, and II.6.5], we have

$$(9.1) \qquad k^G_*(G_+ \wedge K^p) \cong k_*(\Sigma^d K^p) \quad \text{and} \quad k^*_G(G_+ \wedge K^p) \cong k^*(K^p).$$

As written, these isomorphisms are natural in K^p but not in $G_+ \wedge K^p$. However, a G-map $G_+ \wedge J^p \to G_+ \wedge K^p$, where J^p and K^p are both wedges of p-spheres, is determined by its nonequivariant restriction $J^p \to G_+ \wedge K^p$, which is given by a collection of elements in the homotopy group $\pi_p(G_+ \wedge K^p)$. Remembering that our homology groups are reduced, we see that the Hurewicz and Künneth theorems give isomorphisms

$$\pi_p(G_+ \wedge K^p) \cong H_p(G_+ \wedge K^p) \cong H_0(G_+) \otimes H_p(K^p) = \mathbb{Z}[\overline{G}] \otimes H_p(K^p),$$

where $\overline{G} = \pi_0(G)$. The action of G on k_G induces an action of $\overline{G}$ on k_*, and (9.1) can be rewritten in the following forms:

$$k^G_{p+q}(G_+ \wedge K^p) \cong k_{q-d} \otimes_{\overline{G}} H_p(G_+ \wedge K^p) \tag{9.2}$$

$$k_G^{p+q}(G_+ \wedge K^p) \cong \mathrm{Hom}_{\overline{G}}(H_p(G_+ \wedge K^p), k^q). \tag{9.3}$$

These descriptions correctly encode the naturality in $G_+ \wedge K^p$ and thus tell us how to compute $d^1 = \partial_*$ and $d_1 = \partial^*$ in terms of integral homology and cohomology. Note that the right sides depend only on $\overline{G}$ actions.

These isomorphisms suggest the following algebraic definition. We write $\underline{W}$ for a G-spectrum W with a given filtration as specified above.

Definition 9.4. *Define a chain complex $C_*(\underline{W})$ of $\overline{G}$-modules by letting*

$$C_p(\underline{W}) = H_p(\overline{W}^p) = H_p(G_+ \wedge K^p) \cong \mathbb{Z}[\overline{G}] \otimes H_p(K^p),$$

with differential ∂_*. For a $\overline{G}$-module V, define $H_*(\underline{W}; V)$ and $H^*(\underline{W}; V)$ to be the homology of $C_*(\underline{W}) \otimes_{\overline{G}} V$ and $\mathrm{Hom}_{\overline{G}}(C_*(\underline{W}), V)$, respectively.

The reader is warned that, in this generality, these homology and cohomology groups need not calculate anything of topological interest. The underline on W is meant to emphasize that they may depend on the given filtration and not just on the homotopy type of W.

While the theories displayed in Definition 6.5 often have chain level descriptions of this general sort, we have not succeeded in proving that they always do. The problem arises in filtering the variable X. Taking $X = S^0$, we can give such a chain level calculation of the coefficient groups in all cases. We describe how to do this in the rest of this section.

Choose a model for EG as a free G-CW complex with finite skeleta EG^p. When G is finite, the bar construction $B(G, G, *)$ is a suitable model, but the natural filtration on $B(G, G, *)$ is not the skeletal filtration of a G-CW structure in general. We can embed any G in a unitary group $U(n)$ and take EG to be the infinite Stiefel variety $U(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)/e \times U(\mathbb{C}^\infty)$, regarded as a G-space via the inclusion of $G = G \times e$ in $U(n) \times U(\mathbb{C}^\infty)$. Since $U(2k)/e \times U(k)$ is a smooth compact G-manifold, it is triangulable as a finite G-CW complex. It is convenient to let EG^p be empty for $p < 0$ and to filter EG_+ by the EG_+^p.

We also need a suitable filtration of $\widetilde{E}G$. We have $\widetilde{E}G = S^0 \cup C(EG_+)$, and we give it, or rather its suspension G-spectrum, the "filtration"

$$\widetilde{E}G^p = \begin{cases} S^0 \cup C(EG_+^{p-1}) & \text{if } p > 0 \\ S^0 & \text{if } -d \le p \le 0 \\ D(\widetilde{E}G^{-p-d}) & \text{if } p < -d. \end{cases} \tag{9.5}$$

We use the telescope construction to convert this to an actual filtration of a G-spectrum equivalent to $\widetilde{E}G$ without change of notation. Replacing $\widetilde{E}G^p$ by S^0 for $p > 0$, we obtain a compatible filtration of the zero sphere spectrum S^0. Note that $\Sigma EG_+ = \widetilde{E}G/S^0$; the quotient filtration of ΣEG_+ is that specified explicitly by $(\Sigma EG_+)^p = \Sigma(EG_+^{p-1})$.

This strange looking filtration is arranged so as to ensure that, for all integers p, $\widetilde{E}G^p/\widetilde{E}G^{p-1} = G_+ \wedge K^p$ where K^p is equivalent to a wedge of finitely many p-spheres. For $p > 1$ this quotient is clearly equivalent to $\Sigma EG^{p-1}/EG^{p-2}$. If $p = 1$, it is equivalent to ΣEG_+^0. If $-d < p \le 0$, it is trivial. For $p \le -d$, the implicit maps are duals of inclusions, and the p^{th} subquotient is equivalent to

$$\Sigma D(\widetilde{E}G^{-p-d+1}/\widetilde{E}G^{-p-d}) = \Sigma D(G_+ \wedge K^{-p-d+1}) \simeq \Sigma G_+ \wedge S^{-d} \wedge D(K^{-p-d+1}).$$

Of course, $D(K^{-p-d+1})$ is a finite wedge of spheres S^{p+d-1}.

The following explicit examples will be central to our study of the circle and unit quaternion groups.

EXAMPLE 9.6: Let $\mathbb{T}$ be the group of unit complex numbers and let $V = \mathbb{C}$ regarded as the canonical representation of $\mathbb{T}$. The union $S(\infty V)$ of the unit spheres $S(qV)$ is a model for $E\mathbb{T}$ and the union $S^{\infty V}$ of the S^{qV} is a model for $\widetilde{E}\mathbb{T}$. We give $\widetilde{E}\mathbb{T}$ the filtration

$$\widetilde{E}\mathbb{T}^{2p} = \widetilde{E}\mathbb{T}^{2p-1} = S^{pV} \quad \text{for all integers } p.$$

Here the successive odd filtration quotients are

$$\widetilde{E}\mathbb{T}^{2p+1}/\widetilde{E}\mathbb{T}^{2p} \cong (S^V/S^0) \wedge S^{pV} = (\Sigma\mathbb{T}_+) \wedge S^{pV} \cong \Sigma^{2p+1}\mathbb{T}_+.$$

Analogously, let $\mathbb{U}$ be the group of unit quaternions. Let $V = \mathbb{H}$ regarded as the canonical representation of $\mathbb{U}$ and define

$$\widetilde{E}\mathbb{U}^{4p} = \widetilde{E}\mathbb{U}^{4p-1} = \widetilde{E}\mathbb{U}^{4p-2} = \widetilde{E}\mathbb{U}^{4p-3} = S^{pV} \text{ for all integers } p.$$

Then

$$\widetilde{E}\mathbb{U}^{4p+1}/\widetilde{E}\mathbb{U}^{4p} \cong S^V/S^0 \wedge S^{pV} = \Sigma\mathbb{U}_+ \wedge S^{pV} \cong \Sigma^{4p+1}\mathbb{U}_+.$$

REMARK 9.7: We emphasize that the filtration on $\widetilde{E}G$ is not the skeletal filtration of a structure of free G-CW spectrum. A free G-CW spectrum X is equivalent to $EG_+ \wedge X$, but $EG_+ \wedge \widetilde{E}G$ is trivial. However, each pair $(\widetilde{E}G, \widetilde{E}G^p)$ is equivalent relative to $\widetilde{E}G^p$ to a relative free G-CW spectrum, as is proven by an inductive argument from the form of the filtration subquotients. The point is just the obvious distinction between relative and absolute free G-CW spectra.

Now return to the context of Definition 6.5, with $X = S^0$. Of course, there are only three distinct sequences of invariants of groups here since, as with the coefficient groups of any G-spectra, we have

$$f(HM)_n = f(HM)^{-n}, \; c(HM)_n = c(HM)^{-n}, \text{ and } t(HM)_n = t(HM)^{-n}.$$

Nevertheless, we shall give six chain level prescriptions.

THEOREM 9.8. *Let V be any $\overline{G}$-module and let M be any Mackey functor such that $UM = V$. Then there are canonical isomorphisms*

$$H^G_{n-d}(V) \equiv f(HM)_n \cong H_{n-d}(\underline{EG_+}; V) \quad \text{and} \quad H^n_G(V) \equiv c(HM)^n \cong H^n(\underline{EG_+}; V);$$
$$\check{H}^G_{n-d}(V) \equiv c(HM)_n \cong H_{n-d}(\underline{S}^0; V) \quad \text{and} \quad \check{H}^n_G(V) \equiv f(HM)^n \cong H^n(\underline{S}^0; V);$$
$$\widehat{H}^G_{n-d}(V) \equiv t(HM)_n \cong H_{n-d}(\underline{\widetilde{E}G}; V) \quad \text{and} \quad \widehat{H}^n_G(V) \equiv t(HM)^n \cong H^{n+1}(\underline{\widetilde{E}G}; V).$$

The proof of the theorem will be immediate once we set up and prove the convergence of our spectral sequences in the next section. To see its consistency, note that $C_p(\underline{\Sigma EG_+}) = C_{p-1}(\underline{EG_+})$. We have a short exact sequence of chain complexes

$$0 \to C_*(\underline{S}^0) \to C_*(\underline{\widetilde{E}G}) \to C_*(\underline{\Sigma EG_+}) \to 0, \tag{9.9}$$

and it gives rise to the norm sequences of Proposition 2.3 for $k_G = HM$ and $X = S^0$. Of course, since we started with a free G-CW complex EG and any two choices of EG are cellularly homotopy equivalent, the evident homotopy invariance of the conclusion was only to be expected.

The essential point is that these are eminently computable algebraic invariants of compact Lie groups. Since $EG/G = BG$ is a CW-complex with the obvious quotient cells, we have the following immediate corollary, which gives complete information when G is connected. It could also be derived from Propositions 2.1 and 2.3.

COROLLARY 9.10. *Suppose that $d > 0$ and $\overline{G}$ acts trivially on V. Then*

$$H_n^G(V) = f(HM)_{n+d} \cong H_n(BG_+; V), \quad H_G^n(V) = c(HM)^n \cong H^n(BG_+; V),$$

and

$$\widehat{H}_G^n(V) = t(HM)^n \cong \begin{cases} H^n(BG_+; V) & \text{if } 0 \leq n \\ 0 & \text{if } -d \leq n < 0 \\ H_{-n-1-d}(BG_+; V) & \text{if } n \leq -d-1. \end{cases}$$

Of course, the analog when $d = 0$ and thus G is finite is familiar: the same conclusions hold except that $\widehat{H}_G^{-1}(V)$ and $\widehat{H}_G^0(V)$ are now the kernel and cokernel of $|G| : V \to V$.

REMARK 9.11: We do not know how to compute products in $\widehat{H}_G^*(V)$ in general. However, if G is connected and V is a field, then the methods of Benson and Carlson [6] can be adapted to give complete information. The essential point is that $t(HFV)$ is a $c(HFV)$-module spectrum. Let x and y be elements of degrees m and n. If $m \geq 0$ and $n \geq 0$, xy is the usual cup product. Restricting to skeleta and using a comparison between equivariant and nonequivariant duality [33, III.2.12], we find that xy is the usual cap product if $m \geq 0$

and $n < -m$. If G has p-rank at least 2, where $\operatorname{char} V = p$, then $xy = 0$ in the remaining cases. The p-rank 1 cases are $U(1)$, $SU(2)$, and $SO(3)$ at $p \neq 2$ (where it is equivalent to $SU(2)$). Here we have the evident periodic products; see Section 14.

§10. The f, c, and t Atiyah-Hirzebruch spectral sequences

We will have two variants of our spectral sequences which will coincide when both are defined and well-behaved. One is based on a filtration of the variable X and the other is based on the Postnikov filtration of k_G. The general theory is explained in Appendix B, which includes a discussion of convergence. With the language there, we are only interested in our spectral sequences when they are both relevant and conditionally convergent; we then say that they are *potentially convergent.* Theorem B.6 explains in terms of the behavior of higher differentials what more is needed to ensure strong convergence to the specified target groups.

The following somewhat ad hoc definition describes what is needed to set up our first spectral sequences. Recall the filtrations of EG_+, S^0, and $\widetilde{E}G$ described in the previous section.

DEFINITION 10.1. *Let X be a G-CW spectrum with skeleta X^n. We say that X is calculable if, for W any of EG_+, S^0, and $\widetilde{E}G$, $W \wedge X$ can be given a filtration $\{(W \wedge X)^p \mid p \in \mathbb{Z}\}$, arranged as an increasing sequence of G-cofibrations, such that the following properties are satisfied.*

(i) Each subquotient $(W \wedge X)^p/(W \wedge X)^{p-1}$ is equivalent to a G-spectrum $G_+ \wedge K^p$, where K^p is a wedge of p-sphere G-spectra.

(ii) The maps $EG_+ \wedge X \to S^0 \wedge X \to \widetilde{E}G \wedge X$ are filtration-preserving.

(iii) If X is bounded below, then the filtrations of $EG_+ \wedge X$ is bounded below in the sense that $(EG_+ \wedge X)^p = *$ for p sufficiently small.

(iv) If X is finite, then, for each p, there exist non-negative integers r and s such that $W^p \wedge X \subset (W \wedge X)^{p+r}$ and $(W \wedge X)^p \subset W^{p+s} \wedge X$.

These seem to be the minimal conditions needed to set up spectral sequences with both calculable E_2-terms and reasonable convergence properties. Here (i) will be used to identify E_2-terms, (ii) will ensure the compatibility of the f, c, and t spectral sequences, and (iii) and (iv) will be used in convergence arguments. The obvious product filtrations on $EG_+ \wedge X, S^0 \wedge X$, and $\widetilde{E}G \wedge X$ show that any X is calculable when G is finite. More generally, the product filtrations show that X is calculable if all of its cells have orbit type of the form G/H where H has finite index in G. In Section 14, we shall construct different filtrations which show that any X is calculable when G is S^1. Unfortunately, for general positive dimensional groups, calculability seems to be a stringent condition on X.

PROBLEM 10.2: For general $G-CW$ spectra X, non-trivial positive dimensional orbit types G/H give rise to smash products of spheres with $(G \times G/H)_+$ in the subquotients of the product filtration. While the diagonal action can be replaced by the left action on G and then G/H can be triangulated as a nonequivatiant CW-complex, the resulting description of subquotients is inconvenient for calculation. Is there a sensible way to refine these observations to show that X is calculable?

Via the definitions and Propositions 1.2 and 2.6, we may view our homology and cohomology theories as

$$\begin{aligned} f(k_G)_n(X) &\cong k_n^G(EG_+ \wedge X) &\quad\text{and}\quad c(k_G)^n(X) &\cong k_G^n(EG_+ \wedge X);\\ c(k_G)_n(X) &\cong c(k_G)_n(S^0 \wedge X) &\quad\text{and}\quad f(k_G)^n(X) &\cong f(k_G)^n(S^0 \wedge X);\\ t(k_G)_n(X) &\cong c(k_G)_n(\widetilde{E}G \wedge X) &\quad\text{and}\quad t(k_G)^n(X) &\cong f(k_G)^n(\Sigma^{-1}\widetilde{E}G \wedge X). \end{aligned}$$

When X is calculable, we use the given filtrations of $EG_+ \wedge X, S^0 \wedge X$, and $\widetilde{E}G \wedge X$ (or the desuspension of the filtration of $\widetilde{E}G \wedge X$) to obtain three homology spectral sequences from (B.0) and three cohomology spectral sequences from (B.2). We can read off all six E_2-terms from the discussion in the previous section, in particular formulas (9.2) and (9.3). Remember that $\overline{G} = \pi_0(G)$ and that $f(k_G)$, $c(k_G)$, and k_G have the same underlying nonequivariant spectrum k. Recall Definitions 6.5 and 9.4.

THEOREM 10.3. *Let* $k_q = k^{-q}$ *denote* $\pi_q(k)$ *regarded as a* $\overline{G}$*-module and let* $M_q = \underline{\pi}_q(k_G) = M^{-q}$. *Assume that* X *is calculable. Then the given filtrations give* *rise to spectral sequences with E_2-terms and targets:*

$$E^2_{p,q} = H_p(\underline{EG_+ \wedge X}; k_{q-d}) \Rightarrow f(k_G)_n(X); \quad E_2^{p,q} = H^p(\underline{EG_+ \wedge X}; k^q) \Rightarrow c(k_G)^n(X);$$
$$E^2_{p,q} = H_p(\underline{S^0 \wedge X}; k_{q-d}) \Rightarrow c(k_G)_n(X); \quad E_2^{p,q} = H^p(\underline{S^0 \wedge X}; k^q) \Rightarrow f(k_G)^n(X);$$
$$E^2_{p,q} = H_p(\underline{\widetilde{E}G \wedge X}; k_{q-d}) \Rightarrow t(k_G)_n(X); \quad E_2^{p,q} = H^{p+1}(\underline{\widetilde{E}G \wedge X}; k^q) \Rightarrow t(k_G)^n(X).$$

If X is bounded below, the top two spectral sequences are potentially convergent and their $E^2_{p,q}$ and $E_2^{p,q}$ terms are isomorphic to

$$H_p^G(X; k_{q-d}) \equiv f(HM_{q-d})_{p+d}(X) \quad \text{and} \quad H^p_G(X; k^q) \equiv c(HM^q)^p(X).$$

If X is finite, the remaining four spectral sequences are potentially convergent and their $E^2_{p,q}$ and $E_2^{p,q}$ terms are isomorphic to

$$\check{H}_p^G(X; k_{q-d}) \equiv c(HM_{q-d})_{p+d}(X) \quad \text{and} \quad \check{H}^p_G(X; k^q) \equiv f(HM^q)^p(X);$$
$$\widehat{H}_p^G(X; k_{q-d}) \equiv t(HM_{q-d})_{p+d}(X) \quad \text{and} \quad \widehat{H}^p_G(X; k^q) \equiv t(HM^q)^p(X).$$

When $k_G = HM$, all six spectral sequences collapse at E_2 for any calculable X, hence they converge strongly when they are potentially convergent. Therefore the statements about potential convergence will imply the second descriptions of the E_2-terms. We emphasize that this works independently of any particular choice or construction of the filtrations we start with. Since $X = S^0$ is obviously finite and calculable, this case of Theorem 10.3 implies Theorem 9.8. The following lemma will be proven at the end of the section and will complete the proof of Theorem 10.3. In it, the given filtrations need only satisfy (iii) and (iv) of Definition 10.1, not (i) and (ii). We point this out since there are many variant spectral sequences that fit into the framework above except for the identifications of E_2-terms, and some of them might well prove useful,

LEMMA 10.4. *If X is bounded below, then the spectral sequences obtained from the filtration $\underline{EG_+ \wedge X}$ are potentially convergent. If X is finite, then the spectral sequences obtained from the filtrations $\underline{S^0 \wedge X}$ and $\underline{\widetilde{E}G \wedge X}$ are potentially convergent.*

In general, since we have not assumed that k_G is bounded below, the Tate theory spectral sequences can be non-zero throughout the whole plane even when X is finite, although the f and c homology and cohomology spectral sequences are then restricted to half planes. Theorem B.6 specifies two conditions, (ω) and (ρ), on higher differentials which together suffice to ensure the strong convergence of a potentially convergent spectral sequence; (ω) holds if the spectral sequence lies in any half plane, and (ρ) holds if only finitely many differentials are non-zero on any given bidegree, for example if each $E_2^{p,q}$ is finite.

We don't have calculable filtrations in general, and even when we do have them we don't know how to use them to study products. Moreover, we used $f(k_G)$, which is not a ring G-spectrum, to set up the Tate cohomology spectral sequence above. We get around these problems by filtering $f(k_G)$, $c(k_G)$, and $t(k_G)$, leaving the X variable alone.

By Proposition 3.3 and the discussion above it, application of the functors f, c, and t to the Postnikov tower of k_G gives us compatible "f, c, and t Postnikov towers". That is, $t(k_G) \simeq \operatorname{Mic} t((k_G)^q)$, and similarly for f and c. By (B.4), these Postnikov filtrations give rise to spectral sequences for the calculation of $t(k_G)^*(X)$, etc. We emphasize that, a priori, this construction has nothing to do with any possible filtration on X. The cited results immediately imply the first statement of the following theorem.

THEOREM 10.5. *Let k^{-q} denote $\pi_q(k)$ regarded as a $\overline{G}$-module and let $M^{-q} = \underline{\pi}_q(k_G)$. Then the Postnikov tower of k_G gives rise to conditionally convergent spectral sequences with respective E_2-terms and targets:*

$$E_2^{p,q} = \check{H}_G^p(X; k^p) \equiv f(HM^q)^p(X) \implies f(k_G)^n(X)$$

$$E_2^{p,q} = H_G^p(X; k^q) \equiv c(HM^q)^p(X) \implies c(k_G)^n(X)$$

$$E_2^{p,q} = \widehat{H}_G^p(X; k^q) \equiv t(HM^q)^p(X) \implies t(k_G)^n(X).$$

Moreover, there are natural external pairings of spectral sequences in each case. If k_G is a ring G-spectrum and X is a G-space, the c and t spectral sequences are spectral sequences *of differential algebras.*

PROOF: A pairing $k_G \wedge k'_G \to k''_G$ induces a map $HM^i \wedge HM'^j \to HM''^{i+j}$ to which f, c, and t can be applied, giving canonical induced pairings

$$t(k_G)^*(X) \wedge t(k'_G)^*(X') \to t(k''_G)^*(X \wedge X')$$

$$t(HM^i)^*(X) \wedge t(HM'^j)^*(X') \to t(HM''^{i+j})^*(X \wedge X'),$$

and similarly for f and c. Compare (3.4) and (B.9). The latter makes clear that these pairings induce pairings of exact couples with all of the expected properties. The last statement follows by naturality.

On the E_2-level, for Tate theory, the products

$$\widehat{H}^p_G(X; k^q) \otimes \widehat{H}^{p'}_G(X; k^{q'}) \to \widehat{H}^{p+p'}_G(X; k^{q+q'})$$

are induced from the diagonal map of X. Here the ordinary Tate theories are viewed as represented by the $t(HM^q)$, and these are paired by maps induced by the product on k_G. Compare Proposition 8.3 and the diagram (8.2).

When k_G is bounded below, these spectral sequences are certainly relevant. They are then lower half-plane spectral sequences, so that (ω) of Theorem B.6 is satisfied and the spectral sequences converge strongly if (ρ) holds. When $k_G = i_* k$ for a nonequivariant spectrum k, the Borel cohomology spectral sequence agrees under the isomorphisms of Proposition 2.1 with the classical Atiyah-Hirzebruch spectral sequence for $EG_+ \wedge_G X$. We have the following comparison between the two triples of cohomology spectral sequences given in Theorems 10.3 and 10.5.

THEOREM 10.6. *Let k_G be any G-spectrum and let X be a calculable $G-CW$ spectrum. If X is bounded below, then the two spectral sequences for the calculation of $c(k_G)^*(X)$ are isomorphic. If X is finite, then the two spectral sequences for the calculation of $f(k_G)^*(X)$ and of $t(k_G)^*(X)$ are isomorphic. Under the specified bounded below or finiteness hypothesis, if the isomorphic spectral sequences satisfy condition (ω), then the spectral sequence derived from the Postnikov tower is relevant.*

PROOF: By Proposition 2.6 and its proof, we have natural isomorphisms

$$t(k_G)^*(X) \leftarrow t(k_G)^*(\widetilde{E}G \wedge X) \rightarrow (\Sigma f(k_G))^*(\widetilde{E}G \wedge X) \cong f(k_G)^*(\Sigma^{-1}\widetilde{E}G \wedge X).$$

In view of Proposition 3.3, these induce isomorphisms of Postnikov tower spectral sequences, and we use the rightmost version in our comparison for t. We deal with the three cases simultaneously in the rest of the proof. We need only check the hypotheses (i)–(iii) of Theorem B.8 below, and (i) is part of Lemma 10.4. The underlying nonequivariant spectra of the $f((k_G)^q)$ and $c((k_G)^q)$ both give Postnikov towers for the underlying nonequivariant spectrum k of k_G. For any G-spectrum k_G, we have

$$c(k_G)^*(G_+) \cong f(k_G)^*(G_+) \cong k_G^*(G_+) \cong k^*,$$

and (ii) and (iii) are now clear by standard properties of Postnikov towers.

REMARK 10.7: Let X be finite and calculable. We have a cohomology spectral sequence for the computation of $t(k_G)^*(X)$, described in the two equivalent ways of Theorem 10.6, and a homology spectral sequence for the computation of the isomorphic groups $t(k_G)_*(DX)$. In fact, as one would expect, these spectral sequences are isomorphic. However, this is not obvious from the constructions. A proof can be obtained by the same method in the previous argument, but also using a homological variant of Theorem B.8.

PROOF OF LEMMA 10.4: As observed in Appendix B, the homology spectral sequences are relevant and the cohomology spectral sequences are conditionally convergent for any X. If X is bounded below, then so is $\underline{EG_+ \wedge X}$. As also observed in Appendix B, this implies that the spectral sequence for $f(k_G)_*$ is conditionally convergent and the spectral sequence for $c(k_G)^*$ is relevant. The next lemma says that the other two homology spectral sequences are conditionally convergent and the other two cohomology spectral sequences are relevant when X is finite.

LEMMA 10.8. *Let X be finite. Then, for $\underline{W} = \underline{\widetilde{E}G \wedge X}$ or $\underline{W} = \underline{S^0 \wedge X}$,*

$$\textit{lim}^{\varepsilon}_{p\to -\infty} c(k_G)_*(W^p) = 0,\ \varepsilon = 0 \textit{ and } \varepsilon = 1, \quad \textit{and} \quad \textit{colim}_{p\to -\infty} f(k_G)^*(W^p) = 0.$$

PROOF: Since $\widetilde{E}G^p = (S^0)^p$ for $p \leq 0$, the statements are the same for the two choices of W. For the first statement, it is equivalent to show that Mic $c(k_G) \wedge W^p$ is trivial. For any finite G-CW spectra J and K,

$$[J, \text{holim}_{p \to -\infty} c(k_G) \wedge \widetilde{E}G^p \wedge K]_G = [J, \text{holim}_{p \to \infty} F(EG_+, k_G) \wedge D(\widetilde{E}G^p) \wedge K]_G$$
$$\cong [J \wedge EG_+ \wedge \text{hocolim}_{p \to \infty} \widetilde{E}G^p, k_G \wedge K]_G \cong [J \wedge EG_+ \wedge \widetilde{E}G, k_G \wedge K]_G = 0$$

and

$$\text{colim}_{p \to -\infty} [\widetilde{E}G^p \wedge K, k_G \wedge EG_+]_G = \text{colim}_{p \to \infty} [D(\widetilde{E}G^p) \wedge K, k_G \wedge EG_+]_G$$
$$\cong \text{colim}_{p \to \infty} [K, k_G \wedge EG_+ \wedge \widetilde{E}G^p]_G \cong [K, k_G \wedge EG_+ \wedge \widetilde{E}G]_G = 0$$

by standard duality isomorphisms and the fact that $EG_+ \wedge \widetilde{E}G \simeq *$. A cofinality argument from Definition 10.1 (iv) completes the proof.

Part III: Specializations and calculations

§11. Tate-Swan cohomology and the spectral sequences for finite groups

Let G be a finite group throughout this section. Let X be a CW complex with a cellular action by G, such as a G-CW complex viewed as a CW complex with $|G:H|$ cells for each G-cell of orbit type G/H. (We could work more generally with suitable G-spectra X, but we prefer to leave that generalization to the reader.) Via product filtrations, X is calculable in the sense of Definition 10.1, and it is easy to interpret the ordinary f, c, and t homology and cohomology groups of X in classical algebraic terms. In fact, except that not all of our grading conventions are standard, the answer is dictated notationally by Definition 6.5 and the descriptions of the E_2-terms in Theorem 10.3 (with $d = 0$). The following theorem will be proven after we recall the classical chain level definitions of the groups that appear on the left. For trivial G-modules, the result is originally due to the first author [19] and an equivalent version of the homology half, for finite X, was later proven by Adem, Cohen, and Dwyer [4]; Section 5 explains the equivalence.

THEOREM 11.1. *Let V be a G-module and let M be any Mackey functor such that $UM \equiv M(G/e) = V$. Then there are natural isomorphisms*

$$H_n^G(C_*(X);V) \cong H_n(\underline{EG_+ \wedge X};V) \cong f(HM)_n(X) \equiv H_n^G(X;V)$$

$$\check{H}_n^G(C_*(X);V) \cong H_n(\underline{S^0 \wedge X};V) \cong c(HM)_n(X) \equiv \check{H}_n^G(X;V)$$

$$\widehat{H}_n^G(C_*(X);V) \cong H_n(\underline{\widetilde{E}G \wedge X};V) \cong t(HM)_n(X) \equiv \widehat{H}_n^G(X;V)$$

and

$$H^n_G(C_*(X);V) \cong H^n(\underline{EG_+ \wedge X};V) \cong c(HM)^n(X) \equiv H^n_G(X;V)$$

$$\check{H}^n_G(C_*(X);V) \cong H^n(\underline{S^0 \wedge X};V) \cong f(HM)^n(X) \equiv \check{H}^n_G(X;V)$$

$$\widehat{H}^n_G(C_*(X);V) \cong H^{n+1}(\underline{\widetilde{E}G \wedge X};V) \cong t(HM)^n(X) \equiv \widehat{H}^n_G(X;V).$$

In fact, it will suffice to prove the left-hand isomorphisms. The middle isomorphisms come from the collapsed spectral sequences of Theorem 10.3, except that in four of the six cases we only know that the spectral sequences converge when X is finite. Two of these cases occur in homology, and here the isomorphisms for general X follow by passage to colimits. In the two cohomology cases, the isomorphisms for general X follow by a standard application of Brown's representability theorem and the wedge axiom: the left hand side theories can be represented on all X by some naive G-spectra $f(V)$ and $t(V)$, and the uniqueness clause of Brown's representability theorem on finite complexes implies that $f(V)$ and $t(V)$ must be equivalent to $i^*f(HM)$ and $i^*t(HM)$.

Let P^+ be a resolution of $\mathbb{Z}$ by finitely generated free left $\mathbb{Z}[G]$-modules. Recall from [12, XII§3] that a complete resolution P is a $\mathbb{Z}$-graded exact complex of finitely generated free left $\mathbb{Z}[G]$-modules P_i such that $d_0 : P_0 \to P_{-1}$ factors through a surjection $P_0 \to \mathbb{Z}$ and an injection $\mathbb{Z} \to P_{-1}$. The standard way to construct such a P is to splice P^+ with its dual. Conversely, letting P^- be obtained from P by replacing P_i by 0 for $i \geq 0$, we can identify P^+ with P/P^-.

DEFINITION 11.2. *For a $\mathbb{Z}[G]$-chain complex C and a $\mathbb{Z}[G]$-module V, define the following homology and cohomology groups:*

$$\begin{aligned}
H_n^G(C;V) &= H_n((P^+\otimes C)\otimes_G V) &&\text{and}&& H_G^n(C;V) = H^n(\mathrm{Hom}_G(P^+\otimes C, V));\\
\check{H}_n^G(C;V) &= H_{n-1}((P^-\otimes C)\otimes_G V) &&\text{and}&& \check{H}_G^n(C;V) = H^{n-1}(\mathrm{Hom}_G(P^-\otimes C, V));\\
\widehat{H}_n^G(C;V) &= H_{n-1}((P\otimes C)\otimes_G V) &&\text{and}&& \widehat{H}_G^n(C;V) = H^n(\mathrm{Hom}_G(P\otimes C, V))
\end{aligned}$$

Note that the short exact sequence $0 \to P^- \to P \to P^+ \to 0$ gives rise to long exact sequences connecting these groups.

The classical Tate cohomology groups are obtained by taking $C = \mathbb{Z}$. The generalization to G-complexes and G-spaces is due to Swan [45]. If $Q_{-j} = \mathrm{Hom}(P_{j-1}, \mathbb{Z})$, then, with the induced differential, Q is another complete resolution, and ΣP is isomorphic to $\mathrm{Hom}(Q, \mathbb{Z})$. This accounts for a standard shift of degrees in the comparison of Tate homology and cohomology. We have chosen to regrade Tate homology so as to eliminate this shift. That is, with our definitions we have $\widehat{H}_n^G(V) \cong \widehat{H}_G^{-n}(V)$. Our justification is that the shift is incompatible with our topological point of view, which clearly suggests a grading of these homology and cohomology theories in which Spanier-Whitehead duality takes the usual form.

PROOF OF THEOREM 11.1: For $\underline{W}$ any of $\underline{EG_+}$, $\underline{S^0}$, or $\underline{\widetilde{E}G}$, Definition 9.4 gives $C_*(\underline{W \wedge X}) \cong C_*(\underline{W}) \otimes C_*(X)$. (We drop the underline when we are looking at the chains of a CW-complex but we keep it when we are looking at intrinsically spectrum level filtrations.) Of course, $C_*(EG_+)$ is a resolution P^+ of $\mathbb{Z}$. Looking at the filtration (9.5) of $\widetilde{E}G$, we see that $C_*(\underline{\widetilde{E}G})$ is the suspension of the complete resolution P obtained by splicing P^+ with its dual resolution and that $C_*(\underline{S^0}) = \Sigma P^-$. The left-hand isomorphisms of the theorem follow directly. For consistency, note that the exact sequence (9.9) is the suspension of $0 \to P^- \to P \to P^+ \to 0$.

Theorem 11.1 gives us a good algebraic hold on the E_2-terms of the Atiyah-Hirzebruch spectral sequences of Theorems 10.3 and 10.5. In particular, we have the following generalization of a standard observation about the classical Tate cohomology groups [12, XII.2.5].

PROPOSITION 11.3. *Let G be finite of order n. Then multiplication by n annihilates $\widehat{H}_*^G(C;V)$ and $\widehat{H}_G^*(C;V)$. If C and V are finitely generated Abelian groups (so that each C_j is finitely generated and only finitely many C_j are non-zero), then each $\widehat{H}_s^G(C;V)$ and $\widehat{H}_G^s(C;V)$ is finite.*

PROOF: As in [12, XII.2.4 and 2.5], it suffices to observe that $\widehat{H}_*^G(C;V)$ and $\widehat{H}_G^*(C;V)$ are both zero when $V = Z[G] \otimes A$ for an Abelian group A. The verification uses elementary isomorphisms and change of rings to reduce to the obvious acyclicity of $P \otimes C \otimes A$ and $\mathrm{Hom}(P \otimes C, A)$.

COROLLARY 11.4. *Let k_G be a G-spectrum with underlying nonequivariant spectrum k such that k^* is of finite type and let X be finite. Then the Tate homology and cohomology spectral sequences for k_G are annihilated by n and are finite in each bidegree. If k is* *bounded below, they are strongly convergent.*

COROLLARY 11.5. *if q is prime to the order of G, then q acts invertibly on $t(k_G)$ for any G-spectrum k_G.*

PROOF: Since $t(k_G)$ is a $t(S_G)$-module G-spectrum, by Proposition 3.5, it suffices to prove the result for S_G. Here the conclusion is immediate from the strong convergence of the spectral sequence.

We illustrate the form of the spectral sequences by recording the Tate E^2-terms in some cases of groups with periodic Tate cohomology [12, XII§7]. In fact, the p-groups in the following examples are exactly those which contain a unique subgroup of order p, and these are the only p-groups with periodic Tate cohomology [12, XII.11.6]. For an Abelian group A, let

$$(A)_n = A/nA \quad \text{and} \quad {}_n(A) = \mathrm{Ker}(n : A \to A).$$

Of course, when $X = S^0$, $\{E^r_{p,q}\} = \{E_r^{-p,-q}\}$, and this is a multiplicative spectral sequence if k_G is a ring G-spectrum.

EXAMPLES 11.6: Take $X = S^0$ and assume that G acts trivially on k^*.

(i) Let G be cyclic of order n. Then

$$E_2^{p,q} = \begin{cases} (k^q)_n & \text{if } p \text{ is even} \\ {}_n(k^q) & \text{if } p \text{ is odd.} \end{cases}$$

If k_G is a ring spectrum, the product on E_2 is induced from the product of the ring k^*, except that the product of two elements of odd filtration degree is multiplied by $n/2$ if n is even and is 0 if n is odd.

(ii) Let G be the generalized quaternion group of order $4n$ with generators x and y such that $x^n = y^2$ and $xyx = y$. Then, if n is even or odd, respectively

$$E_2^{p,q} = \begin{cases} (k^q)_{4n} & \text{if } p \equiv 0 \bmod 4 \\ {}_2(k^q) \oplus {}_2(k^q) \text{ or } {}_4(k^q) & \text{if } p \equiv 1 \bmod 4 \\ (k^q)_2 \oplus (k^q)_2 \text{ or } (k^q)_4 & \text{if } p \equiv 2 \bmod 4 \\ {}_{4n}(k^q) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

In fact, these examples are extremely special and highly misleading. For general finite groups G, the E^2-terms cannot display periodicity. Moreover, as was proven by Benson and Carlson [6], all products between negative degree elements of Tate cohomology are zero for "most" finite groups. Precisely, taking coefficient groups in a field A, this holds in $\widehat{H}^*_G(A)$ whenever $H^*_G(A)$ contains a regular sequence of length two. We illustrate this with the simplest possible example.

EXAMPLE 11.7: Let $G = C_m \times C_n$, where C_n is the cyclic group of order n. Let d be the least common divisor of m and n. Let A be an Abelian group with trivial action by G. Then, with

$$A_{m,n} = \mathrm{Ker}(-n+m : A \oplus A \to A)/\,\mathrm{Im}((m,n) : A \to A \oplus A),$$

$$\widehat{H}^p_G(A) = \begin{cases} {}_mA \oplus (A_d)^s \oplus ({}_dA)^{s-1} \oplus {}_nA & \text{if } p = -2s-1, s \geq 1 \\ A_m \oplus (A_{m,n})^{s-1} \oplus A_n & \text{if } p = -2s, s \geq 1 \\ {}_{mn}A & \text{if } p = -1 \\ A_{mn} & \text{if } p = 0 \\ {}_mA \oplus (A_{m,n})^s \oplus {}_nA & \text{if } p = 2s+1, s \geq 0 \\ A_m \oplus ({}_dA)^s \oplus (A_d)^{s-1} \oplus A_n & \text{if } p = 2s, s \geq 1. \end{cases}$$

The lack of periodicity is obvious: the number of summands increases as p either increases from 0 or decreases from -1. Products are difficult to describe in general but, if m and n are divisible by the same primes and A is a field, then all products of negative degree classes are zero.

PROBLEM 11.8: Products are computed in the Tate cohomology of finite groups by use of suitable maps $\psi : P_n \to P_i \otimes P_{n-i}$ [12, XII§4]. Except that a given element $p \in P_n$ will generally map non-trivially to infinitely many $P_i \otimes P_{n-i}$, we may regard ψ as a chain map $P \to P \otimes P$. Although we know by Proposition 8.4 that our topologically defined products agree with those computed in this algebraic fashion, it would still be interesting to determine if such a chain map $P \to P \otimes P$ can be realized by a comparison $\underline{\widetilde{E}G} \to \underline{\widetilde{E}G \wedge \widetilde{E}G}$ of filtered G-spectra . If so, the products that we obtained on the $\widehat{H}^*_G(X;V)$ by naturality from external pairings should agree with the products obtained by chain level use of cellular approximations of diagonal maps.

§12. Some remarks on nonequivariant stable homotopy theory

Still assuming that G is finite, we next consider stable homotopy theory. We have

$$\pi_*^G(X) = (S_G)_*(X) = \bigoplus \pi_*(EWH_+ \wedge_{WH} X^H)$$

for any G-space X, where $WH = NH/H$ and the sum runs over one H in each conjugacy class of subgroups of G (e.g. [16] or [33, V.9.1]). With X replaced by $EG_+ \wedge X$, only the trivial subgroup contributes and we get the summand $\pi_*(EG_+ \wedge_G X)$; with X replaced by $\widetilde{E}G \wedge X$, only the nontrivial subgroups contribute. In view of Section 4, the Segal conjecture has the following immediate consequence.

THEOREM 12.1. *Let G be finite and let X have finite skeleta. Then*

$$t(S_G)_*(X) \cong \Big(\bigoplus_{(H)\neq(e)} \pi_*(EWH_+ \wedge_{WH} X^H)\Big)^\wedge_I.$$

If G is a p-group, then

$$t(S_G)_*(X) \cong \bigoplus_{(H)\neq(e)} \pi_*(EWH_+ \wedge_{WH} X^H)^\wedge_p.$$

The following immediate corollary of Theorems 10.5, 11.1, and 12.1 seems fascinating to us. For simplicity, we state it only for $X = S^0$.

THEOREM 12.2. *Let G be a finite p-group. Then there is an upper half-plane spectral sequence such that $E^2_{*,*} = \widehat{H}^G_*(\pi_*)$ and which converges strongly to the sum over $(H) \neq (e)$ of the groups $\pi_*(BWH_+)^\wedge_p$, the $(G)^{th}$ summand being $(\pi_*)^\wedge_p$. Thought of cohomologically, this is spectral sequence of differential algebras.*

PROBLEM 12.3: How can one compute the products on the target groups? Nothing at all is known about this question. The splitting used to identify the target groups bears no obvious relation to the Tate theoretic source of the multiplicative structure. The question is interesting from both the equivariant and the nonequivariant points of view. For the former, note that, by Proposition 3.4, $t(S_G)^*$ acts on $t(k_G)^*(X)$ for all k_G and X.

The restriction to p-groups is made only to simplify the form of I-adic completion. We actually have such spectral sequences for all finite groups G. In Section 23, we shall obtain a great many more such spectral sequences, some of which seem even more interesting to us than those just described. For example, for a finite p-group G we will obtain a spectral sequence which converges to $(\pi_*)^\wedge_p$ itself, with no additional summands, at the price of requiring a less familiar algebraic cohomology theory for the E_2-term.

The spectral sequence of Theorem 12.2 is constructed in various equivalent ways. With the cohomological construction of Theorem 10.3, (B.2) makes clear that it is obtained by passage to homotopy groups from the inverse sequence of spectra

$$W^s \equiv F(\widetilde{E}G/\widetilde{E}G^{s-1}, \Sigma EG_+)^G.$$

Here

$$W^s/W^{s+1} \simeq F(G_+ \wedge K^s, \Sigma EG_+)^G \simeq F(K^s, S^1),$$

where K^s is a wedge of s-spheres and thus $F(K^s, S^1) \simeq \Sigma D(K^s)$ is a wedge of $(1-s)$-spheres; see (9.5). The spectral sequence gives rise to a version of Mahowald's root invariant. For a given element $\alpha : S^q \to S^0$ of $\pi_* = \pi_*(BWG_+)$, one takes the smallest s such that α maps nontrivially to W^{s+1} and defines $R_G(\alpha)$ to be the set of all lifts of α to a map $S^q \to D(K^s)$, where $D(K^s)$ is viewed as the fiber of $W^{s+1} \to W^s$. Thus $R_G(\alpha)$ is a finite set of elements of π_{q+s} associated to the given element $\alpha \in \pi_q$. (Technically, to avoid dependence on the choice of cells, $R_G(\alpha)$ should be interpreted as an element of E^2.)

In negative total degrees, all elements in E^2 must be wiped out by higher differentials. In total degree zero, there are c copies of the p-adic integers in the target group of the spectral sequence, where c is the number of conjugacy classes of non-trivial subgroups of G. Since $|G|$ annihilates E^2 and therefore E^∞, each of these copies is built up from E^∞ by infinitely many non-trivial extensions involving infinitely many of the higher stable homotopy groups of spheres. In particular, the root invariants of the elements p^j are highly non-trivial. Even these root invariants are different for different choices of G.

If G is cyclic of order $n = p^j$, the E^2-term is given explicitly in Examples 11.6. It depends simply and explicitly on the p-torsion structure of the stable stems, and yet convergence is to the sum of the p-completed stable homotopy groups of the classifying spaces of the cyclic groups of order p^i, $0 \leq i < j$. When $n = 2$, this is precisely the suspension of the spectral sequence of Mahowald discussed by Adams in [1], namely the inverse limit of the stable homotopy Atiyah-Hirzebruch spectral sequences of the spectra $\mathbb{R}P^\infty_{-i}$. For general n, such a spectral sequence is implicit in the work of Ravenel [42]. Even in these cases, the fact that the spectral sequences are multiplicative is new. The Mahowald spectral sequence is already a miracle, and this collection of spectral sequences more of one. The E^2-terms build up only in that more and more of each torsion tower in π_* becomes visible as j increases. It is remarkable that this is sufficient to allow the building up of the full stable homotopy groups of more and more classifying spaces.

The generalized quaternion groups give more of a glimpse into how extraordinary these spectral sequences are. As seen in Examples 11.6, the E^2-terms are scarcely more complicated than for cyclic groups, but the following tabulation of subgroups shows that the target of the spectral sequence involves classifying spaces of dihedral groups and many copies of $\mathbb{R}P^\infty$. The existence of a consistent pattern of differentials must imply constraints on both the stable stems and the stable homotopy groups that appear in the target.

EXAMPLE 12.4: Let $Q(j)$ have generators x and y and relations $x^{2^j} = y^2$ and $xyx = y$. Let $D(j)$ have generators w and z and relations $w^{2^j} = 1$, $z^2 = 1$, and $wzw = z$. These have order 2^{j+2} and 2^{j+1}, respectively; $Q(0)$ and $D(0)$ are cyclic. Let $C(j)$ be cyclic of order 2^j. The distinct conjugacy classes of proper non-trivial subgroups of $Q(j)$ are represented by the cyclic subgroups $C(i)$ generated by $x^{2^{j+1-i}}$ for $0 < i \leq j+1$ and the two copies of $Q(i)$ generated by $x^{2^{j-i}}$ and y and by $x^{2^{j-i}}$ and xy, respectively, for $0 \leq i < j$. The $C(i)$ are normal in $Q(j)$ and have $WC(i) \cong D(j+1-i)$. The normalizer of each copy of $Q(i)$

is the corresponding copy of $Q(i+1)$ and therefore satisfies $WQ(i) \cong C(1)$. According to Mitchell and Priddy [41], there are 2-local stable splittings

$$BD(n) \simeq BPSL_2\mathbb{F}_q \vee 2(\Sigma^{-2}Sp^4S/Sp^2S) \vee 2\mathbb{R}P^\infty,$$

where q is so chosen that $D(n)$ is a 2-Sylow subgroup of $PSL_2\mathbb{F}_q$.

§13. The Tate K-theory of finite groups and related calculations

Turning to periodic K-theory, we have the following identification. Our original proof was quite convoluted. A simpler proof of a much more general result will be given in Section 19. Let $\underline{J}^\wedge$ be the Mackey functor whose value at G/H is the $J(H)$-adic completion of $J(H)$, where $J(H)$ is the augmentation ideal of the complex representation ring $R(H)$. For the real case, let $\underline{JO}^\wedge$, $\underline{JSp}^\wedge$, $\underline{R/RO}^\wedge$, and $\underline{R/RSp}^\wedge$ be the Mackey functors whose values at G/H are the $JO(H)$-adic completions of $JO(H)$, $JSp(H)$, $RU(H)/RO(H)$, and $RU(H)/RSp(H)$, respectively, where $JO(H)$ is the augmentation ideal of the real representation ring $RO(H)$ and $JSp(H)$ is the augmentation group in $RSp(H)$. Where $RO(G)$ acts through $R(G)$, the completions at $J(G)$ and $JO(G)$ are isomorphic. Since G is finite, these completions are also isomorphic to completions at the augmentation ideal I of $A(G)$, by [25, 4.5].

THEOREM 13.1. *With wedges taken over all integers i,*

$$t(KU_G) \simeq t(i_*KU) \simeq \bigvee K(\underline{J}^\wedge \otimes \mathbb{Q}, 2i)$$

and

$$\begin{aligned} t(KO_G) \simeq t(i_*KO) \simeq \bigvee \big(K(\underline{JO}^\wedge \otimes \mathbb{Q}, 8i) &\vee K(\underline{RU/RSp}^\wedge \otimes \mathbb{Q}, 8i+2) \\ &\vee K(\underline{JSp}^\wedge \otimes \mathbb{Q}, 8i+4) \vee K(\underline{RU/RO}^\wedge \otimes \mathbb{Q}, 8i+6)\big). \end{aligned}$$

Moreover, the completions of $t(KU_G)$ and $t(KO_G)$ at I are trivial.

We will explain the multiplicative structure in Section 18. In the real case, we have expressed the answer in a form that will look plausible to the reader familiar with $KO^G_*(pt)$. However, we have

$$\underline{JO}^\wedge \otimes \mathbb{Q} \cong \underline{JSp}^\wedge \otimes \mathbb{Q} \quad \text{and} \quad \underline{RU/RSp}^\wedge \otimes \mathbb{Q} \cong \underline{RU/RO}^\wedge \otimes \mathbb{Q},$$

hence the displayed 8-fold periodicity reduces to 4-fold periodicity.

We have already noted that $t(KU_G) \simeq t(i_*KU)$ and $t(KO_G) \simeq t(i_*KO)$. However, even if we are only interested in i_*KU we must work with KU_G since equivariant Bott periodicity plays a role in the proof. The essential point is that $t(KU_G)$ is rational, in sharp contrast to Theorem 4.1 and Proposition 11.3. By Corollary 11.5, only the behavior of primes dividing $|G|$ is at issue.

REMARK 13.2: If G is cyclic of order n and k_G is any G-spectrum such that k^* is torsion free and concentrated in even degrees, then the Tate cohomology spectral sequence of Theorem 10.2 is concentrated in even degrees when $X = S^0$, by Example 11.6(i). Therefore $E_2 = E_\infty$ and the spectral sequence converges strongly to $t(k_G)^*$. That is, $t(k_G)^*$ is filtered with associated graded group isomorphic to $\widehat{H}^*_G(k^*)$. In particular, this applies to KU_G. Since n annihilates $\widehat{H}^*_G(KU^*)$ and $t(KU_G)^*$ is rational, this may seem incredible at first glance, but there is no contradiction. To see what is going on, it helps to observe that $\mathbb{Z}^\wedge_n[1/n]$ is rational and can be filtered by the subgroups $n^k\mathbb{Z}^\wedge_n$ for integers k. The compatibility of Theorem 13.1 with the existence of the Tate cohomology spectral sequences for general finite groups G is much more mysterious.

Since $KU_n(BG_+) \cong f(KU_G)^{-n}$, by Proposition 2.1, and we know $c(KU_G)^*$ by the Atiyah-Segal completion theorem and $t(KU_G)^*$ by Theorem 13.1, we can read off $KU_*(BG_+)$ from the norm sequence, and similarly for $KO_*(BG_+)$. (More details will be given in Section 19.) Let $\mathrm{Tor}(A)$ denote the torsion subgroup of an Abelian group A.

COROLLARY 13.3. *$K_0(BG_+) \cong \mathbb{Z}$ and $K_1(BG_+) \cong J(G)^\wedge_{J(G)} \otimes \mathbb{Q}/\mathbb{Z}$. As q runs from 0 to 7, the groups $KO_q(BG_+)$ take the following values:*

$$\mathbb{Z} \quad (RU/RSp)^\wedge_{JO} \otimes \mathbb{Q}/\mathbb{Z} \oplus (RO/RU)^\wedge_{JO} \qquad \mathrm{Tor}((RU/RSp)^\wedge_{JO}) \quad JSp^\wedge_{JO} \otimes \mathbb{Q}/\mathbb{Z}$$

$$\mathbb{Z} \quad (RU/RO)^\wedge_{JO} \otimes \mathbb{Q}/\mathbb{Z} \oplus (RSp/RU)^\wedge_{JO} \qquad \mathrm{Tor}((RU/RO)^\wedge_{JO}) \quad JO^\wedge_{JO} \otimes \mathbb{Q}/\mathbb{Z}$$

We conjecture that the connective versions kUG and kOG of equivariant K-theory satisfy the following analog of Theorem 13.1. Note that the J-adic completion of J is certainly local at $|G|$, since $|G|$ annihilates J^n/J^{n+1}.

CONJECTURE 13.4. *With wedges taken over all integers i,*

$$t(kU_G) \simeq t(i_*kU) \simeq \bigvee K(\underline{J}^\wedge, 2i)$$

and

$$t(kO_G) \simeq t(i_*kO) \simeq \bigvee \big(K(\underline{JO}^\wedge, 8i) \vee K(((\underline{RU/RSp)/torsion})^\wedge, 8i+2)$$
$$\vee K(\underline{JSp}^\wedge, 8i+4) \vee K(((\underline{RU/RO)/torsion})^\wedge, 8i+6)\big)$$

This is correct if G is cyclic of order p for any prime p. In the real case of the conjecture, we have quotiented out the torsion from our Mackey functors, which is 2-torsion only, on the perhaps flimsy evidence that this is what happens when G is cyclic of order 2. When G is cyclic of prime order, only calculations on fixed point spectra are required since both sides are trivial as nonequivariant spectra, and I-adic and p-adic completion agree on J and JO. In view of ($\mathbb{R}$) and ($\mathbb{L}$), the following version of the conjecture in these cases is essentially just a restatement of the main results of Davis and Mahowald [14, 1.4 and 1.5] and their odd primary analogs.

THEOREM 13.5 (DAVIS AND MAHOWALD). *Let G be cyclic of order p. Then*

$$t(kU_G)^G \simeq t(i_*kU)^G \simeq \bigvee K((\widehat{\mathbb{Z}}_p)^{p-1}, 2i)$$

and

$$t(kO_G)^G \simeq t(i_*kO)^G \simeq \begin{cases} \bigvee K(\widehat{\mathbb{Z}}_2, 4i) & \text{if } p=2 \\ \bigvee K((\widehat{\mathbb{Z}}_p)^{(p-1)/2}, 2i) & \text{if } p>2. \end{cases}$$

Actually, the direct odd primary analog of the results of [14] concerns $BP\langle 1\rangle$ and gives that

$$t(i_*BP\langle 1\rangle)^G \simeq \bigvee K(\widehat{\mathbb{Z}}_p, 2i).$$

To deduce the theorem, it suffices to work p-locally, where kU is the wedge of $\Sigma^{2j}BP\langle 1\rangle$, $0 \le j \le p-2$, and kO is the wedge of $\Sigma^{4j}BP\langle 1\rangle$, $0 \le j \le (p-3)/2$. Since the functor $t(i_*?)^G$ commutes with finite wedges, the conclusions of the theorem follow. Note that the number of factors $\mathbb{Z}_p^\wedge$ in each Eilenberg-MacLane spectrum is predicted equivariantly by the structure of the relevant representation rings, but appears for seemingly quite different reasons in the nonequivariant calculations. Note too that the 2-torsion in real K-theory makes no contribution in the case $p = 2$.

Conjecture 13.4 seems to be the appropriate substitute for the (false) connective analog of the Atiyah-Segal completion theorem. The known cases put us in a situation analogous to knowing only the theorems of Lin and Gunawardena en route to a proof of the Segal conjecture. The critical step towards a general proof should be a good construction of Chern and Pontryagin classes in the Tate theories of connective K-theories. The construction of such classes is just one of many problems and questions about such classes in these and related theories.

The following remark describes the differentials in the spectral sequence of Theorem 10.1 which allow the calculation of Theorem 13.5 to work in the real case when $p = 2$.

REMARK 13.6: The ring kO_* has generators η of degree 1, α of degree 4 and β of degree 8, with $2\eta = 0$, $\eta^3 = 0$, $\eta\alpha = 0$, and $\alpha^2 = 4\beta$. Write elements of E^2 in the form γi_p with $p \in \mathbb{Z}$ and $\gamma \in kO_*$, using Example 11.6(i). Then the following are all of the non-zero differentials:

$$\begin{aligned} d^2(\beta^n i_p) &= \eta\beta^n i_{p-2} && \text{if } p \equiv 2 \mod 4 \\ d^2(\beta^n \eta i_p) &= \eta^2\beta^n i_{p-2} && \text{if } p \equiv 1 \text{ or } 2 \mod 4 \\ d^3(\beta^n \eta^2 i_p) &= \beta^n \alpha i_{p-3} && \text{if } p \equiv 1 \mod 4. \end{aligned}$$

The survivors to E^∞ are i_{4p}, αi_{4p}, ηi_{4p-1}, $\eta^2 i_{4p-2}$, and their products with β^n for all n. In fact, with $n = 0$, the d^2's are forced by naturality from the case of the sphere spectrum, where they were already known to Adams and Mahowald [1], and the d^3's are forced by the requirement that E^∞ be concentrated in total degrees divisible by 4.

In view of ($\mathbb{R}$) and ($\mathbb{L}$), the main results of [15] can be written in the following form, where $G = \mathbb{Z}/p$ and the products run over all integers k.

$$t(i_*BP\langle 2\rangle)^G \simeq \prod \Sigma^{2k} BP\langle 1\rangle_p^\wedge \quad \text{and} \quad t(i_*BP)^G \simeq \prod \Sigma^{2k} BP_p^\wedge.$$

The proof in [15] is not complete, since [15, 2.3(iii)] is based on the incorrect assertion that completion converts wedges to products, but it seems likely that this hole can be patched. More generally, the authors of [14] and [15] conjecture that

$$t(i_*BP\langle n\rangle)^G \simeq \prod \Sigma^{2k} BP\langle n-1\rangle_p^\wedge \quad \text{for all } n.$$

Remark 13.2 can be used to recheck that the homotopy groups are right, and the ring structure implied by our work may well be helpful, but we don't see an equivariant strategy of proof.

REMARK 13.7: Since $p \cong t(i_*p) : t(i_*KU) \to t(i_*KU)$, Theorem 13.1 implies that p is an equivalence and thus that $t(i_*KU/p) \simeq *$ for any prime p, where KU/p denotes mod p K-theory. More generally, Sadofsky and the first author [51] have proven that $t(i_*K(n)) \simeq *$ for all finite groups G and all $n \geq 1$, where $K(n)$ is the n^{th} Morava K-theory spectrum.

§14. Cyclic cohomology and the spectral sequences for the circle group

Let $\mathbb{T}$ be the group of unit complex numbers. As explained below, there are precise analogs of the results to follow for the group of unit quaternions. Give $\mathbb{T}$ its standard CW-structure with identity element as the unique vertex and a single 1-cell. Let X be a CW-complex with a cellular action by $\mathbb{T}$, so that $\mathbb{T}X^j \subset X^{j+1}$. The following lemma, which will be proven at the end of the section, shows that any $\mathbb{T}$-CW complex can be replaced by an equivalent CW-complex with a cellular $\mathbb{T}$-action.

LEMMA 14.1. *Let Y be a $\mathbb{T}$-CW complex. Then there is a $\mathbb{T}$-CW complex X which is $\mathbb{T}$-homotopy equivalent to Y and has a decomposition as an ordinary CW-complex with a cellular action by $\mathbb{T}$.*

We shall see that X is calculable in the sense of Definition 10.1, and we shall interpret the ordinary f, c and t homology and cohomology groups of X in terms of cyclic theory. Again, except that not all of our grading conventions are standard, the answer is dictated notationally by Definition 6.5 and the descriptions of the E_2-terms in Theorem 10.3 (with $d = 1$). The following exact analog of Theorem 11.1 will be proven after we recall the chain level definitions of the algebraic cyclic homology groups that appear on the left. For finite X, an equivalent version of the homology half of the result was first proven by Adem, Cohen, and Dwyer [4]; Section 5 explains the equivalence.

THEOREM 14.2. *Let V be an Abelian group and let M be any Mackey functor such that $UM \equiv M(\mathbb{T}/e) = V$. Then there are natural isomorphisms*

$$H^{\mathbb{T}}_{n-1}(C_*(X);V) \cong H_{n-1}(\underline{E\mathbb{T}_+ \wedge X};V) \cong f(HM)_n(X) \equiv H^{\mathbb{T}}_{n-1}(X;V)$$

$$\check{H}^{\mathbb{T}}_{n-1}(C_*(X);V) \cong H_{n-1}(\underline{S^0 \wedge X};V) \cong c(HM)_n(X) \equiv \check{H}^{\mathbb{T}}_{n-1}(X;V)$$

$$\widehat{H}^{\mathbb{T}}_{n-1}(C_*(X);V) \cong H_{n-1}(\underline{\widetilde{E\mathbb{T}} \wedge X};V) \cong t(HM)_n(X) \equiv \widehat{H}^{\mathbb{T}}_{n-1}(X;V)$$

and

$$H^n_{\mathbb{T}}(C_*(X);V) \cong H^n(\underline{E\mathbb{T}_+ \wedge X};V) \cong c(HM)^n(X) \equiv H^n_{\mathbb{T}}(X;V)$$

$$\check{H}^n_{\mathbb{T}}(C_*(X);V) \cong H^n(\underline{S^0 \wedge X};V) \cong f(HM)^n(X) \equiv \check{H}^n_{\mathbb{T}}(X;V)$$

$$\widehat{H}^n_{\mathbb{T}}(C_*(X);V) \cong H^{n+1}(\underline{\widetilde{E}\mathbb{T} \wedge X};V) \cong t(HM)^n(X) \equiv \widehat{H}^n_{\mathbb{T}}(X;V).$$

Again, by precisely the same arguments as in the paragraph after Theorem 11.1, it will suffice to prove the left hand isomorphisms. The left hand groups are variants of the cyclic homology and cohomology groups that were defined by Jones in [29]. Let $P = \mathbb{Z}[u, u^{-1}]$, $\deg(u) = -2$. Let P^- be the negative degree part of P and let $P^+ = \mathbb{Z}[u^{-1}]$; thus $P^+ = P/P^-$.

DEFINITIONS 14.3. *Let V be an abelian group. Let C be a chain complex together with a degree one operator J such that $dJ = -Jd$ and $J^2 = 0$ and define a differential on $P \otimes C$ by the formula*

$$d(p \otimes c) = up \otimes J(c) + p \otimes d(c).$$

Observe that $P^- \otimes C$ is a subcomplex, and regard $P^+ \otimes C$ as a quotient complex. Define the following homology and cohomology groups:

$$H^{\mathbb{T}}_n(C;V) = H_n((P^+ \otimes C) \otimes V) \quad \text{and} \quad H^n_{\mathbb{T}}(C;V) = H^n(\mathrm{Hom}(P^+ \otimes C, V));$$

$$\check{H}^{\mathbb{T}}_n(C;V) = H_{n-1}((P^- \otimes C) \otimes V) \quad \text{and} \quad \check{H}^n_{\mathbb{T}}(C;V) = H^{n-1}(\mathrm{Hom}(P^- \otimes C, V));$$

$$\widehat{H}^{\mathbb{T}}_n(C;V) = H_{n-1}((P \otimes C) \otimes V) \quad \text{and} \quad \widehat{H}^n_{\mathbb{T}}(C;V) = H^n(\mathrm{Hom}(P \otimes C, V)).$$

Note that the short exact sequence $0 \to P^- \to P \to P^+ \to 0$ gives rise to long exact sequences connecting these groups.

The product of $\mathbb{T}$ is obviously a cellular map which, on the chain level, carries $z \otimes z$ to zero, where z is the 1-cell. Since $\mathbb{T}X^j \subset X^{j+1}$, we can define $J : C_j(X) \to C_{j+1}(X)$ by $J(x) = f_*(z \otimes X)$, where $f : \mathbb{T} \times X \to X$ is the action. (Reduced chains are understood.) Then $dJ = -Jd$ and $J^2 = 0$. Thus the left hand groups in Theorem 14.2 are now defined.

These definitions vary in two substantive ways (beyond differing signs and grading) from those of Jones [29, §5]. First, he uses singular rather than cellular chains, with the intervention of the shuffle map to pass from the tensor product of chains to the chains of Cartesian products. It is technical, but not difficult, to use the techniques of [36, §13] to obtain an isomorphism between the cellular and singular versions of these cyclic homology and cohomology groups.

Second, he uses $\mathrm{Hom}(C, V) \otimes P$ with the differential

$$\delta(\varphi \otimes p) = \varphi d \otimes p + \varphi J \otimes up$$

rather than $\mathrm{Hom}(P \otimes C, V)$ in his definition of cohomology. These two complexes are isomorphic if C is a free and finite dimensional chain complex, such as the chains on a finite CW-complex. Our variant yields a cohomology theory that satisfies the wedge axiom, and there are analogous variants of many of Jones' other definitions and theorems.

Actually, Jones is only concerned with $V = \mathbb{Z}$, so that there is no coefficient slot in his notation. We offer the following reconciliation of grading conventions.

NOTATIONS 14.4: Jones' grading on $H_*^{\mathbb{T}}(X)$, $H_{\mathbb{T}}^*(X)$, and $\widehat{H}_{\mathbb{T}}^*(X)$ agrees with ours. However, his $\widehat{H}_n^{\mathbb{T}}(X)$ is our $\widehat{H}_{n-1}^{\mathbb{T}}(X)$, his $G_n^{\mathbb{T}}(X)$ is our $\check{H}_{n-1}^{\mathbb{T}}(X)$, and his $G_{\mathbb{T}}^n(X)$ is our $\check{H}_{\mathbb{T}}^{n-1}(X)$.

Jones proves in [29,3.3] that the three sequences of homology groups are isomorphic to the corresponding sequences of cyclic homology groups derived from the cyclic structure on the singular chains of X. Similar identifications can be proven in cohomology, and these apply to all spaces with our modified definitions. Closely related earlier results, due to Goodwillie [18] and Burghelea and Fiedorowicz [10] and also proven in [29], calculate the homology of free loop spaces in terms of cyclic homology.

Turning to the proof of Theorem 14.2, notice first that, by a glance at the explicit filtration of $\widetilde{E}\mathbb{T}$ in Example 9.6, the exact sequence

$$0 \to C_*(\underline{S}^0) \to C_*(\underline{\widetilde{E}\mathbb{T}}) \to C_*(\underline{\Sigma E\mathbb{T}_+}) \to 0$$

of (9.9) can be identified with the suspension of the exact sequence

$$0 \to P^- \to P \to P^+ \to 0.$$

Roughly speaking, we will construct a filtration on $\widetilde{E}\mathbb{T} \wedge X$ (or rather its suspension spectrum) that behaves algebraically as if it were the skeletal filtration of a free $\mathbb{T}$-CW spectrum with chains isomorphic to $\Sigma P \otimes C_*(X)$. The filtration twists the nonequivariant skeletal filtration of X with the filtration of $\widetilde{E}\mathbb{T}$ specified in Example 9.6. We explain the idea in a more general context.

Intuitively, the idea is to construct a twisted G-CW structure on $\widetilde{E}G \wedge X$ using cell by cell application of the standard G-homeomorphism from $G_+ \wedge X$ with G acting on the left of G (and not on X) to $G_+ \wedge X$ with G acting diagonally. Note that [33, II.4.8] gives a spectrum level version of this G-homeomorphism. In fact, this G-homeomorphism was used in the identification of the subquotients $\widetilde{E}\mathbb{T}^p/\widetilde{E}\mathbb{T}^{p-1}$ in Example 9.6. Working concretely on the space level, we have the following observation, in which G can be any topological group.

PROPOSITION 14.5. *Let W be a free G-CW complex and let X be a CW complex with an action by G (not assumed to be related to the cell decomposition). Then $W \times X$ is a free G-cell complex with an n-cell $\omega * \chi$ for each i-cell $\omega : G \times e^i \to W$ and j-cell $\chi : e^j \to X$, $i + j = n$; as a map $G \times e^i \times e^j \to W \times X$, $(\omega * \chi)(g, a, b) = (g\omega(a), g\chi(b))$, where $\omega(a) = \omega(e, a)$. If, for some fixed positive integer d, $GX^k \subset X^{k+d}$ for all k and W has cells only in dimensions at least $d+1$ apart, then $W \times X$ is a free G-CW complex with this cell structure.*

PROOF: It is elementary to check that, with the specified cells, $W \times X$ has all the properties required of a G-CW complex *except that n-cells may be attached to cells of higher dimension.* To determine the boundary of a cell $\omega * \chi$ of $W \times X$, it suffices to determine the restriction of $\omega * \chi$ to the boundary of $e^i \times e^j$. If $b \in \partial e^j$ and $\chi(b) = \chi'(b')$, where b'

is in the interior of a cell χ' of X, $\dim \chi' < j$, then

$$(14.6) \qquad (\omega * \chi)(g, a, b) = (g\omega(a), g\chi'(b')) = (\omega * \chi')(g, a, b').$$

More interestingly, if $a \in \partial e^i$, if $\omega(a) = \omega'(g', a')$, where a' is in the interior of a cell ω' of W, $\dim \omega' < i$, and if $(g')^{-1}\chi(b) = \chi'(b')$, where b' is in the interior of a cell χ' of X, then

$$(14.7) \qquad (\omega * \chi)(g, a, b) = (gg'\omega'(a'), gg'\chi'(b')) = (\omega' * \chi')(gg', a', b').$$

If $GX^k \subset X^{k+d}$ for all k, then $\dim \chi' \leq j + d$. If W has cells at least $d + 1$ dimensions apart, then $\dim \omega' < i - d$.

There is a relative generalization in which W is obtained from a sub G-space W' by attaching free G-cells. It is a pleasant fact that the cases of immediate interest to us are among the very few in which cells are attached only to cells of lower dimension.

When G is a compact Lie group, it is technical, but not difficult, to extend Proposition 14.5 and its relative generalization to the spectrum level, with W a free G-CW spectrum and X a based CW-complex and a G-space with a G-fixed basepoint.

We apply this to $W = \underline{E\mathbb{T}}_+$, $\underline{\widetilde{E}\mathbb{T}}$, or $\underline{S}^0$. In view of Remark 9.7, we must apply the relative spectrum level version in the latter two cases. The two parts (14.6) and (14.7) of the geometric boundary give rise to the two summands of the differential displayed in Definition 14.3. The verification of this boils down to a check of signs and can be seen most clearly by mapping in cells and so using twisted pairs of cells and their boundaries as universal examples. Thus the cellular chain groups used to compute the second column of homology and cohomology groups in Theorem 14.2 are isomorphic to the chain groups specified in Definition 14.3 that compute the first column.

PROBLEM 14.8: Let C be a differential coalgebra with coproduct ψ such that $\psi J = (J \otimes 1 + 1 \otimes J)\psi$, for example $C_*(X)$ with the coproduct induced by any $\mathbb{T}$-equivariant cellular approximation to the diagonal map. Let V be a commutative ring. Define $\psi : P_{-2n} \to P_{-2i} \otimes P_{2i-2n}$ by sending u^n to $u^i \otimes u^{n-i}$. If C is finitely generated,

$\mathrm{Hom}(P \otimes C, V)$ is a differential algebra with respect to the evident cup product. This raises the obvious analog of Problem 11.8. Can one prove that the products that we have defined topologically agree with these algebraically defined products?

Let $\mathbb{U}$ be the group of unit quaternions. Nearly everything done above works equally well for $\mathbb{U}$. We quickly run through the analogous definitions and results. Give $\mathbb{U}$ its usual CW-structure, with identity element as the unique vertex and a single 3-cell. The product of $\mathbb{U}$ is a cellular map which, on the chain level, carries $z \otimes z$ to zero, where z is the 3-cell. Let X be a CW complex with a cellular action by $\mathbb{U}$, so that $\mathbb{U}X^j \subset X^{j+3}$, and define $J : C_j(X) \to C_{j+3}(X)$ by $J(x) = f_*(z \otimes x)$, where $f : \mathbb{U} \times X \to X$ is the action. Then $dJ = -Jd$ and $J^2 = 0$.

Let $P = \mathbb{Z}[u, u^{-1}]$, $\deg(u) = -4$, let P^- be its negative degree elements, and let $P^+ = \mathbb{Z}[u^{-1}]$. For an abelian group V and a chain complex C together with a degree three operator J such that $dJ = -Jd$ and $J^2 = 0$, we define six homology and cohomology groups precisely as in Definition 14.3; even the gradings work identically.

THEOREM 14.9. *Let V be an Abelian group and let M be any Mackey functor such that $LM = M(\mathbb{U}/e) = V$. Then there are natural isomorphisms*

$$H^{\mathbb{U}}_{n-3}(C_*(X); V) \cong H_{n-3}(\underline{E\mathbb{U}_+ \wedge X}; V) \cong f(HM)_n(X) \equiv H^{\mathbb{U}}_{n-3}(X; V)$$
$$\check{H}^{\mathbb{U}}_{n-3}(C_*(X); V) \cong H_{n-3}(\underline{S^0 \wedge X}; V) \cong c(HM)_n(X) \equiv \check{H}^{\mathbb{U}}_{n-3}(X; V)$$
$$\widehat{H}^{\mathbb{U}}_{n-3}(C_*(X); V) \cong H_{n-3}(\underline{\widetilde{E}\mathbb{U} \wedge X}; V) \cong t(HM)_n(X) \equiv \widehat{H}^{\mathbb{U}}_{n-3}(X; V)$$

and

$$H_{\mathbb{U}}^n(C_*(X); V) \cong H^n(\underline{E\mathbb{U}_+ \wedge X}; V) \cong c(HM)^n(X) \equiv H_{\mathbb{U}}^n(X; V)$$
$$\check{H}_{\mathbb{U}}^n(C_*(X); V) \cong H^n(\underline{S^0 \wedge X}; V) \cong f(HM)^n(X) \equiv \check{H}_{\mathbb{U}}^n(X; V)$$
$$\widehat{H}_{\mathbb{U}}^n(C_*(X); V) \cong H^{n+1}(\underline{\widetilde{E}\mathbb{U} \wedge X}; V) \cong t(HM)^n(X) \equiv \widehat{H}_{\mathbb{U}}^n(X; V).$$

Proposition 14.5 and its relative generalization apply to prove the result. Of course, we use the filtration on $\widetilde{E}\mathbb{U}$ specified in Example 9.6. The one thing that does not work for

$\mathbb{U}$ (except under restrictive isotropy type hypotheses) is the proof of the analog of Lemma 14.1.

PROOF OF LEMMA 14.1: We shall construct X so as to have $\mathbb{T}$-cells in bijective correspondence with those of our given $\mathbb{T}$-CW complex Y. Since any proper closed subgroup $H \subset \mathbb{T}$ is finite, we can give $\mathbb{T}/H \cong S^1$ a CW-structure with eH as vertex and a single 1-cell. The action of $\mathbb{T}$ on $\mathbb{T}/H$ is cellular. Take the 0-skeleton of X to be the 0-skeleton of Y (as a $\mathbb{T}$-complex) with its CW structure as a disjoint union of orbits.

Assume inductively that the (n-1)-skeleton of X as a $\mathbb{T}$-CW complex has been constructed together with a $\mathbb{T}$-homotopy equivalence $\xi_{n-1} : Y^{n-1} \to X^{n-1}$, where X^{n-1} is a CW-complex of dimension n with cellular action by $\mathbb{T}$. Note that $(X^{n-1})^H$ is the union of the $\mathbb{T}$-cells of X^{n-1} of type $\mathbb{T}/K$ with $H \subset K$ and is a CW-complex. For an n-cell of Y with domain $\mathbb{T}/H \times e^n$, choose a cellular approximation γ to the composite of ξ^H_{n-1} and the restriction $S^{n-1} \to (Y^{n-1})^H$ of the attaching map, where S^{n-1} has its usual cell structure. The extension of γ to a $\mathbb{T}$-map $\mathbb{T}/H \times S^{n-1} \to X^{n-1}$ is cellular, and we take it as a typical attaching map for the construction of X^n. We give X^n the evident structure of a CW-complex, and the action of $\mathbb{T}$ is clearly cellular. Comparisons of cofiber sequences show that ξ_{n-1} extends cell by cell to a $\mathbb{T}$-map $\xi_n : Y^n \to X^n$ and that a homotopy inverse of ξ_{n-1} extends cell by cell to a homotopy inverse of ξ_n.

§15. Calculations in homotopy and K-theory for the circle group

We first consider the Tate theory spectral sequence of Section 10 in the case of the circle group $\mathbb{T}$. Here, since we no longer have the finiteness assertion of Proposition 11.3, we must assume (ρ) as well as (ω) of Theorem B.6 to deduce strong convergence from potential convergence. However, both conditions hold automatically when the spectral sequences collapse, and this often happens for dimensional reasons. Let $k_{\mathbb{T}}$ be a $\mathbb{T}$-spectrum with underlyng nonequivariant spectrum k.

PROPOSITION 15.1. *Let $X = S^0$. Then $E^2_{**} = \mathbb{Z}[u, u^{-1}] \otimes k_*$ If k_* is concentrated in even degrees, then $E^2_{**} = E^\infty_{**}$. If, further, $k_{\mathbb{T}} = i_*k$ for a ring spectrum k, then*

$$t(k_{\mathbb{T}})^{\mathbb{T}} = \operatorname*{holim}_i \Big(\bigvee_{j > -i} \Sigma^{2j} k \Big).$$

If k is connective, then this homotopy limit reduces to a product, so that

$$t(k_{\mathbb{T}})^{\mathbb{T}} \simeq \prod_j \Sigma^{2j} k.$$

PROOF: The first two statements are clear. There are truncated version of the spectral sequence for which they remain true. When $k_{\mathbb{T}} = i_*k$ for a ring spectrum k, these truncated spectral sequences converge to the homotopy groups of the k-module spectra $\mathbb{C}P^\infty_{-i} \wedge \Sigma^2 k$ that appear as the terms of the homotopy limit on the right side of $(\mathbb{C})$ of the introduction. The collapse shows that these homotopy groups are k_*-free. Therefore the cited terms must be equivalent as k-module spectra to wedges of suspensions of k. If k is connective, then the composite

$$\Sigma^{-2i} k \to \mathbb{C}P^\infty_{-i-1} \wedge \Sigma^2 k \simeq \bigvee_{j \geq -i} \Sigma^{2j} k \to \Sigma^{-2i} k$$

of the map induced by the inclusion of the bottom cell of $\mathbb{C}P^\infty_{-i-1}$, the constructed equivalence, and the projection to the wedge summand is a map of k-module spectra which induces an isomorphism on π_{-2i} and is therefore an equivalence. This implies that the maps of the inverse system $\{\mathbb{C}P^\infty_{-i} \wedge \Sigma^2 k\}$ are equivalent to the obvious projections on wedge summands.

REMARK 15.2: The previous result has an obvious analog for $\mathbb{U}$-spectra. Rather amazingly, by Corollary 9.10, we actually have that $E^2 = E^\infty$ for dimensional reasons when k_* is concentrated in even degrees for any compact connected Lie group G of odd dimension whose classifying space has homology concentrated in even degrees. It is possible dimensionally to have nontrivial higher differentials if G has even dimension.

From the nonequivariant point of view, Proposition 15.1 says that the spectra on the right side of ($\mathbb{C}$) are too simple to be of much interest. From the equivariant point of view, however, the simplicity is welcome.

For a general compact Lie group G and any G-space X, we have

$$\pi_*^G(X) = (S_G)_*(X) = \bigoplus \pi_*(EWH_+ \wedge_{WH} \Sigma^{\mathrm{Ad}(WH)} X^H),$$

where $WH = NH/H$ and the sum runs over one H in each conjugacy class of (closed) subgroups of G ([33, V.9.1]). The summand corresponding to the trivial group comes from $(S_G \wedge EG_+)_*(X)$ and the other summands map isomorphically to $(S_G \wedge \widetilde{E}G)_*(X)$. When $G = \mathbb{T}$, $WH \cong \mathbb{T}$ for every proper subgroup H. This compares plausibly with the following result, which, in view of ($\mathbb{C}$), is just a reinterpretation of part of Ravenel's result [42, Cor 1.15].

THEOREM 15.3 (RAVENEL). *Complete all spectra at p. Then $t(S_\mathbb{T})^\mathbb{T}$ is equivalent to the wedge of S with the product of countably many copies of the suspension spectrum of $\Sigma\mathbb{C}P_+^\infty$, where the copies of $\mathbb{C}P^\infty$ are thought of as the classifying spaces $B(\mathbb{T}/H)$ for H of p-power order.*

As Ravenel noted [42, 1.16], this implies another variant, $R_\mathbb{T}$ say, of Mahowald's root invariant. The first author has recently obtained a more precise result [24] which gives a global rather than just a p-adic description of $t(S_\mathbb{T})^\mathbb{T}$.

Of course, Proposition 15.1 includes a computation of $t(kU_\mathbb{T})^\mathbb{T}$ for connective complex K-theory. The following companion computation for connective real K-theory is due to

Hal Sadofsky. It may be illuminating to see two sketch proofs, Sadofsky's using the Adams spectral sequence and another based on the present theory.

PROPOSITION 15.4 (SADOFSKY). $t(kO_{\mathbb{T}})^{\mathbb{T}} \simeq t(i_*kO)^{\mathbb{T}} \simeq \prod_j \Sigma^{4j}kU.$

FIRST SKETCH PROOF: Modulo putting things together to get a global answer, it is mainly behavior at the prime 2 that is at issue. Since $H^*(kO) = A \otimes_{A(1)} \mathbb{Z}_2$ and $H^*(\mathbb{C}P^\infty)$ breaks up as an $A(1)$-module into a sum of suspensions of the $A(1)$-module M which is $\mathbb{Z}_2$ in degrees 0 and 2, connected by Sq^2, change of rings gives the isomorphisms

$$\begin{aligned}\mathrm{Ext}_A(H^*(\mathbb{C}P^\infty_{-2i+1} \wedge \Sigma^2 kO), \mathbb{Z}_2) &\cong \mathrm{Ext}_{A(1)}(H^*(\Sigma^2\mathbb{C}P^\infty_{-2i+1}), \mathbb{Z}_2)\\ &\cong \bigoplus_{j>-i} \mathrm{Ext}_{A(1)}(\Sigma^{4j}M, \mathbb{Z}_2).\end{aligned}$$

This is the same as E_2 for the wedge of copies of $\Sigma^{4j}kU$, and $E_2 = E_\infty$ since E_2 lies in even bidegrees. Thus $\pi_*(\mathbb{C}P^\infty_{-2i+1} \wedge \Sigma^2 kO) \cong \oplus \Sigma^{4j}\pi_*(kU)$. Since there are no homotopy groups in odd degrees, the generator in degree $4j$ extends over the cofiber $\Sigma^{4j-2}\mathbb{C}P^2$ of $\eta : S^{4j+1} \to S^{4j}$. There results a composite map

$$\Sigma^{4j}kU \simeq \Sigma^{4j-2}\mathbb{C}P^2 \wedge kO \to \mathbb{C}P^\infty_{-2i+1} \wedge \Sigma^2 kO \wedge kO \to \mathbb{C}P^\infty_{-2i+1} \wedge \Sigma^2 kO$$

that induces an inclusion on π_*. The wedge of these maps is an equivalence. The conclusion follows via ($\mathbb{C}$) and passage to limits, as in the last part of the proof of Proposition 15.1.

SECOND SKETCH PROOF: By naturality (compare Remarks 13.6) or otherwise,

$$d^2(\beta^n \eta u^t) = \beta^n \eta^2 u^{t+1}$$

in the spectral sequence of Theorem 10.1, and $E^3 = E^\infty$ since it is concentrated in even degrees. Therefore, $\eta \in \pi_1(kO)$ maps trivially to $\pi_1(t(i_*kO)^{\mathbb{T}})$. It follows by use of Proposition 3.4 that the map $t(i_*\eta)^{\mathbb{T}}$ is trivial. The functor $t(i_*?)^{\mathbb{T}}$ preserves cofiber sequences, and we conclude that the evident cofiber sequence

$$t(i_*kO)^{\mathbb{T}} \to t(i_*kU)^{\mathbb{T}} \to t(i_*\Sigma^2 kO)^{\mathbb{T}}$$

splits. We deduce the conclusion from its already known analog for kU.

Finally, let us consider periodic K-theory. By Proposition 15.1 we know that

$$t(KU_{\mathbb{T}})^{\mathbb{T}} \simeq t(i_*KU)^{\mathbb{T}} \simeq \operatorname{holim}_i \bigvee_{j>-i} \Sigma^{2j}KU.$$

We have the following explicit computation of the homotopy groups of this ring spectrum.

PROPOSITION 15.5. *Let $\rho \in R(\mathbb{T})$ be the standard irreducible representation and let $\chi = 1 - \rho$. Then*

$$t(KU_{\mathbb{T}})^* = \mathbb{Z}[[\chi]][\chi^{-1}] \otimes \mathbb{Z}[u, u^{-1}].$$

PROOF: In a more general context, Proposition 19.12 below gives that $t(KU_{\mathbb{T}})^*$ is the localization $c(KU_{\mathbb{T}})^*[\chi^{-1}]$. By the Atiyah-Segal completion theorem, $c(KU_{\mathbb{T}})^* = R(\mathbb{T})^\wedge_J \otimes \mathbb{Z}[u, u^{-1}]$. As a matter of algebra, the inclusion

$$\mathbb{Z}[\chi] \subset \mathbb{Z}[\chi, \rho^{-1}] = \mathbb{Z}[\rho, \rho^{-1}] = R(\mathbb{T})$$

induces an isomorphism when we complete at $J = (\chi)$, and the conclusion follows.

The ring $R(\mathbb{T})^\wedge_J[\chi^{-1}]$ is also isomorphic to

$$\lim(R(\mathbb{T})[\chi^{-1}]/R(\mathbb{T})),$$

where the inverse system is taken over multiplication by χ. This can be seen algebraically, where it is analogous to $(\mathbb{Z}_p^\wedge)[1/p] \cong \lim(\mathbb{Z}[1/p]/\mathbb{Z})$, and it also follows by inspection of $F(\widetilde{E}\mathbb{T}, \Sigma KU_G \wedge EG_+)$ via Proposition 2.6.

We owe the following analog of Proposition 15.1 for KO to Sadofsky.

PROPOSITION 15.6 (SADOFSKY). $t(KO_{\mathbb{T}})^{\mathbb{T}} \simeq t(i_*KO)^{\mathbb{T}} \simeq \operatorname{holim}_i \bigvee_{j>-i} \Sigma^{4j}KU.$

PROOF: Writing KO as the telescope of iterates of $\beta : kO \to \Sigma^{-8}kO$ and noting that, after smashing with $\Sigma^{-2}\mathbb{C}P^2$, the real Bott map becomes the fourth power of the complex Bott map $\beta : kU \to \Sigma^{-2}kU$, we easily deduce the required splitting of the terms $\mathbb{C}P^{\infty}_{-2i+1} \wedge KO$ in $(\mathbb{C})$.

§16. Free G-spheres and periodicity phenomena

After proving a generalization of the identifications $(\mathbb{R}), (\mathbb{L})$, and $(\mathbb{C})$ stated in the introduction, we shall use the $RO(G)$ grading of our theories to discuss periodicity phenomena in generalized Tate cohomology.

Assume given a representation V of G whose unit sphere $S(V)$ is a free G-space. Necessary and sufficient conditions on G for there to exist such a V are well known [49]. The union $S(\infty V)$ of the spheres $S(qV)$ is a model for EG, the union $D(\infty V)$ of the unit discs $D(qV)$ is G-contractible, and the quotient space $D(\infty V)/S(\infty V) \cong S^{\infty V}$ is a model for $\widetilde{E}G$. Thus we may view $\widetilde{E}G$ as equipped with a filtration by the subspheres S^{qV}. This leads to the following result. Let BG^α denote the Thom spectrum associated to a virtual representation α (e.g., [33], [37]).

THEOREM 16.1. *Suppose that G acts freely on the unit sphere of a representation V. Then, for any nonequivariant spectrum k, $t(i_*k)^G$ is equivalent to* $\mathrm{Mic}(BG^{\mathrm{Ad}(G)-iV} \wedge \Sigma k)$.

PROOF: Propositions 2.6 and 1.2 and the equivalence $\widetilde{E}G \simeq S^{\infty V}$ give

$$\begin{aligned} t(k_G) &\simeq F(\widetilde{E}G, \Sigma c(k_G)) \simeq F(S^{\infty V}, \Sigma(k_G \wedge EG_+)) \\ &\simeq F(\mathrm{Tel}\, S^{iV}, \Sigma(k_G \wedge EG_+)) \simeq \mathrm{Mic}(EG_+ \wedge S^{-iV} \wedge \Sigma k_G) \end{aligned}$$

for any G-spectrum k_G. Now let $k_G = i_*k$. By [33, VI.1.17],

$$i_*(EG \ltimes S^{-iV}) \simeq EG_+ \wedge S^{-iV},$$

where $EG \ltimes ?$ is a twisted half smash functor from G-spectra to free naive G-spectra . Since i_* commutes with smash products, we have

$$EG_+ \wedge S^{-iV} \wedge \Sigma k_G \simeq i_*(EG \ltimes S^{-iV} \wedge \Sigma k).$$

Applying the equivalence (5.5) to $Y = EG \ltimes S^{-iV} \wedge \Sigma k$ and using that smash products commute appropriately with twisted half smash products and Thom spectra [33, VI.1.5 and X.3.9], we find that

$$(EG_+ \wedge S^{-iV} \wedge \Sigma k_G)^G \simeq (EG \ltimes_G S^{\mathrm{Ad}(G)-iV}) \wedge \Sigma k.$$

By [33, X.6.3] and [37, pp 362–363], $EG \ltimes_G S^{\mathrm{Ad}(G)-iV}$ is equivalent to $BG^{\mathrm{Ad}(G)-iV}$. These equivalences are natural, and the conclusion follows on passage to homotopy limits.

With the usual choices of V, this gives ($\mathbb{R}$), ($\mathbb{L}$) and ($\mathbb{C}$); $\mathrm{Ad}(S^1) = \mathbb{R}$ introduces the second suspension coordinate that appears in ($\mathbb{C}$).

In homology and cohomology, the description of $\widetilde{E}G$ as $S^{\infty V}$ implies the following canonical isomorphisms for a G-spectrum m_G, an integer n (or more general element of $RO(G)$), and a G-space X, where X is finite in the case of cohomology:

$$(m_G \wedge \widetilde{E}G)_n(X) \cong \mathrm{colim}(m_G \wedge S^{qV})_n(X) \cong \mathrm{colim}(m_G)_{n-qV}(X)$$

and

$$(m_G \wedge \widetilde{E}G)^n(X) \cong \mathrm{colim}(m_G \wedge S^{qV})^n(X) \cong \mathrm{colim}(m_G)^{n+qV}(X).$$

To interpret the colimits algebraically, assume that m_G is a module G-spectrum over a ring G-spectrum k_G. Let V be any representation of G, let $e : S^0 \to S^V$ send the non-basepoint to 0, and let $\alpha_V \in k_G^V(S^0)$ be the image of the identity element of $k_G^0(S^0)$ under the map

$$e^* : k_G^0(S^0) \cong k_G^V(S^V) \to k_G^V(S^0).$$

Equivalently, α_V is represented by $e \wedge \eta : S^0 \wedge S_G \to S^V \wedge k_G$. The theories m_G^* and m_*^G are module-valued over $k_G^*(S^0)$, where everything is interpreted in the $RO(G)$-graded sense, and the colimits above are both taken over iterated action by the element α_V. The algebraic theory of localization works perfectly well for general types of gradings, and the observations above imply the following conclusion.

PROPOSITION 16.2. *Let m_G be a module G-spectrum over a ring G–spectrum k_G. For any representation V of G, $(m_G \wedge S^{\infty V})_*(X)$ and, if X is finite, $(m_G \wedge S^{\infty V})^*(X)$ are the localizations of $m_*^G(X)$ and $m_G^*(X)$ away from α_V. Thus multiplication by α_V provides a periodicity isomorphism with period V on $(m_G \wedge S^{\infty V})_*(X)$ and $(m_G \wedge S^{\infty V})^*(X)$.*

By Proposition 3.5, the proposition has the following implication.

COROLLARY 16.3. *Let k_G be a ring G-spectrum. If G acts freely on the unit sphere $S(V)$, then $t(k_G)_*(X)$ and, if X is finite, $t(k_G)^*(X)$ are the localizations of $c(k_G)_*(X)$ and $c(k_G)^*(X)$ away from α_V. Therefore, multiplication by α_V provides a periodicity isomorphism with period V on $t(k_G)_*(X)$ and $t(k_G)^*(X)$.*

In the presence of a suitable Thom isomorphism, we can sometimes deduce an integral period. We say that k_G is a split ring G-spectrum if it has a splitting $\zeta : k \to (k_G)^G$ which is a map of ring spectra. This holds if $k_G = i_*k$ for a nonequivariant ring spectrum k. The isomorphism $c(k_G)^*(X) \cong k^*(EG_+ \wedge_G X)$ of Proposition 2.1 is then an isomorphism of $\mathbb{Z}$-graded rings and of modules over $c(k_G)^*(S^0) \cong k^*(BG_+)$. Moreover, for any representation V of G, we also have isomorphisms

$$c(k_G)^V(S^0) \cong k^0_G(EG_+ \wedge S^{-V}) \cong k^0(BG^{-V}). \tag{16.4}$$

The first follows from the definitions. The second follows from (0.7) and the relationship between the Thom spectrum BG^{-V} and the G-spectrum $EG_+ \wedge S^{-V}$ (as recorded in the proof of Proposition 16.2).

Now let $u = \dim V$. In favorable cases, we will have a Thom isomorphism $\mu : k^n(BG_+) \to k^{n-u}(BG^{-V})$. In particular, we will have

$$\mu : k^u(BG_+) \cong k^0(BG^{-V}). \tag{16.5}$$

Combining (16.4) and (16.5), applied both to V and to the trivial representation $\mathbb{R}^u$ of dimension u, we obtain

$$c(k_G)^V(S^0) \cong k^u(BG_+) \cong c(k_G)^u(S^0). \tag{16.6}$$

If k is an Eilenberg-MacLane spectrum HA for a commutative ring A, then we have a Thom isomorphism (16.5) if $H^{-u}(BG^{-V}; A)$ contains a Thom class, for example if V is a complex representation or if A has characteristic two ([37, Thm B]). In these cases, a check of definitions shows that the image in $H^u(BG; A)$ of $\alpha_V \in b(k_G)^V(S^0)$ is just the classical Euler class χ_V of V. This gives the following result.

COROLLARY 16.7. *Let G act freely on the unit sphere $S(V)$ where $\dim V = u$. If $H^{-u}(BG^{-V}; A)$ contains a Thom class and X is finite, then, in integer gradings, $\widehat{H}_G^*(X; A)$ is isomorphic to the localization $H^*(EG_+ \wedge_G X; A)[\chi_V^{-1}]$ and is therefore periodic with period u.*

Part IV: The generalization to families

§17. Families and their f, c, and t G-spectra

Many of our results directly generalize to families of subgroups of G, namely collections of subgroups closed under subconjugacy. We give a recapitulation of the relevant portions of part I in this section, and we let $\mathcal{F}$ be a fixed given family. We agree to exclude the empty family from consideration. Some of the many important examples to keep in mind are the following:

- The family $\mathcal{F}\langle p\rangle$ of p-subgroups of a finite group G.
- The family $\mathcal{F}$ of finite subgroups of G (when G is not itself finite).
- The family $\mathcal{C}$ of (topologically) cyclic subgroups of G.
- The family $\mathcal{F}(H)$ of subconjugates of a given subgroup H.
- The family $\mathcal{F}[N]$ of subgroups of G which do not contain a given normal subgroup N.
- The family $\mathcal{F}(N,G)$ of subgroups whose intersection with a given normal subgroup N is the trivial subgroup.
- The family $\mathcal{P}$ of proper subgroups of G.
- The family $\mathcal{F}(V)$ of subgroups H of G such that $V^H \neq 0$ for a given representation V of G.

There is a universal $\mathcal{F}$-space $E\mathcal{F}$ characterized up to homotopy by the requirement that $E\mathcal{F}^H$ be contractible if $H \in \mathcal{F}$ and empty if $H \notin \mathcal{F}$. We shall give an explicit model in Section 20. If $\mathcal{F} = \{e\}$, then $E\mathcal{F} = EG$. If $\mathcal{F} = \mathcal{F}(N,G)$, then $E\mathcal{F}$ is the universal N-free G-space. Let $\widetilde{E}\mathcal{F}$ be the cofiber of the projection $E\mathcal{F}_+ \to S^0$. We have the cofibering

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}. \tag{A}$$

Let k_G be a G-spectrum and let $F(E\mathcal{F}_+, k_G)$ be the function G-spectrum of maps $E\mathcal{F}_+ \to k_G$. The projection $E\mathcal{F}_+ \to S^0$ induces a G-map

$$\varepsilon : k_G = F(S^0, k_G) \to F(E\mathcal{F}_+, k_G). \tag{B}$$

By taking the smash product of the cofibering (A) with the map (B), we obtain the following map of cofiberings of G-spectra :

$$\text{(C)} \qquad \begin{tikzcd} k_G \wedge E\mathcal{F}_+ \arrow[r] \arrow[d, "\varepsilon \wedge 1"'] & k_G \arrow[r] \arrow[d, "\varepsilon"] & k_G \wedge \widetilde{E}\mathcal{F} \arrow[d, "\varepsilon \wedge 1"] \\ F(E\mathcal{F}_+, k_G) \wedge E\mathcal{F}_+ \arrow[r] & F(E\mathcal{F}_+, k_G) \arrow[r] & F(E\mathcal{F}_+, k_G) \wedge \widetilde{E}\mathcal{F}. \end{tikzcd}$$

Roughly, smashing a G-spectrum with the cofibering (A) has the effect of breaking the represented homology and cohomology theories into parts that see orbit types in $\mathcal{F}$ and not in $\mathcal{F}$. We introduce abbreviated notations for these spectra.

Define the $\mathcal{F}$-free G-spectrum associated to k_G to be

$$f_{\mathcal{F}}(k_G) = k_G \wedge E\mathcal{F}_+.$$

We refer to the homology theories represented by G-spectra $f_{\mathcal{F}}(k_G)$ as $\mathcal{F}$-Borel homology theories. We refer to the cohomology theories they represent as $f_{\mathcal{F}}$-cohomology theories.

Define

$$f'_{\mathcal{F}}(k_G) = F(E\mathcal{F}_+, k_G) \wedge E\mathcal{F}_+.$$

It will turn out that $\varepsilon \wedge 1 : f_{\mathcal{F}}(k_G) \to f'_{\mathcal{F}}(k_G)$ is always an equivalence. After proving this, we will drop the notation $f'_{\mathcal{F}}$ and just use $f_{\mathcal{F}}$. Define

$$f^{\perp}_{\mathcal{F}}(k_G) = k_G \wedge \tilde{E}\mathcal{F}.$$

Define

$$c_{\mathcal{F}}(k_G) = F(E\mathcal{F}_+, k_G).$$

We call $c_{\mathcal{F}}(k_G)$ the geometric $\mathcal{F}$-completion of k_G. The map $\varepsilon : k_G \to c_{\mathcal{F}}(k_G)$ of (B) is the object of study of such results as the generalized Atiyah-Segal completion theorem and the generalized Segal conjecture of [2,3,38]. The general $\mathcal{F}$-homotopy limit problem can be interpreted as the problem of comparing the geometric $\mathcal{F}$-completion $c_{\mathcal{F}}(k_G)$ with the algebraic completion $(k_G)^{\wedge}_{I\mathcal{F}}$ of k_G at a certain ideal of the Burnside ring. We refer to the cohomology theories represented by G-spectra $c_{\mathcal{F}}(k_G)$ as $\mathcal{F}$-Borel cohomology theories. We refer to the homology theories they represent as $c_{\mathcal{F}}$-homology theories.

Define

$$t_{\mathcal{F}}(k_G) = F(E\mathcal{F}_+, k_G) \wedge \tilde{E}\mathcal{F} = f_{\mathcal{F}}^{\perp} c_{\mathcal{F}}(k_G).$$

We call $t_{\mathcal{F}}(k_G)$ the $\mathcal{F}$-Tate G-spectrum associated to k_G. These G-spectra represent $\mathcal{F}$-Tate homology and cohomology theories.

With this cast, and with the abbreviation of $\varepsilon \wedge 1$ to ε, Diagram (C) can be rewritten in the form

(D)
$$\begin{tikzcd}
f_{\mathcal{F}}(k_G) \arrow[r] \arrow[d, "\varepsilon"'] & k_G \arrow[r] \arrow[d, "\varepsilon"] & f_{\mathcal{F}}^{\perp}(k_G) \arrow[d, "\varepsilon"] \\
f'_{\mathcal{F}}(k_G) \arrow[r] & c_{\mathcal{F}}(k_G) \arrow[r] & t_{\mathcal{F}}(k_G).
\end{tikzcd}$$

We call the bottom row the "$\mathcal{F}$-norm sequence".

A map $\phi : k_G \to k'_G$ of G-spectra is said to be an $\mathcal{F}$-equivalence if ϕ^H is an equivalence for $H \in \mathcal{F}$, or, equivalently by the Whitehead theorem, if ϕ is an H-equivalence for $H \in \mathcal{F}$. The proof of Proposition 1.1 applies verbatim to give the following $\mathcal{F}$-invariance statement.

PROPOSITION 17.1. *Let $\phi : k_G \to k'_G$ be an $\mathcal{F}$-equivalence. Then the maps*

$$\phi \wedge 1 : k_G \wedge E\mathcal{F}_+ \to k'_G \wedge E\mathcal{F}_+ \quad \text{and} \quad F(1, \phi) : F(E\mathcal{F}_+, k_G) \to F(E\mathcal{F}_+, k'_G)$$

are G-equivalences. Therefore the cofibration sequences

$$f'_{\mathcal{F}}(k_G) \to c_{\mathcal{F}}(k_G) \to t_{\mathcal{F}}(k_G) \quad \text{and} \quad f'_{\mathcal{F}}(k'_G) \to c_{\mathcal{F}}(k'_G) \to t_{\mathcal{F}}(k'_G)$$

are G-equivalent.

Since the middle vertical arrow $\varepsilon : k_G \to c_{\mathcal{F}}(k_G)$ of Diagram (D) is an $\mathcal{F}$-equivalence, the first statement of the proposition implies the following promised result.

PROPOSITION 17.2. *For any G-spectrum k_G,*

$$\varepsilon : f_{\mathcal{F}}(k_G) = k_G \wedge E\mathcal{F}_+ \to F(E\mathcal{F}_+, k_G) \wedge E\mathcal{F}_+ = f'_{\mathcal{F}}(k_G)$$

is an equivalence of G-spectra .

The relationship between the three kinds of theories is immediate from the bottom cofiber sequence of (D), which gives $\mathcal{F}$-norm sequences.

PROPOSITION 17.3. *For G-spectra X, there are long exact sequences*

$$\cdots \to f_{\mathcal{F}}(k_G)^n(X) \to c_{\mathcal{F}}(k_G)^n(X) \to t_{\mathcal{F}}(k_G)^n(X) \to f_{\mathcal{F}}(k_G)^{n+1}(X) \to \cdots$$

and

$$\cdots \to f_{\mathcal{F}}(k_G)_n(X) \to c_{\mathcal{F}}(k_G)_n(X) \to t_{\mathcal{F}}(k_G)_n(X) \to f_{\mathcal{F}}(k_G)_{n-1}(X) \to \cdots .$$

Under appropriate hypotheses, these collapse to give isomorphisms. A G-space is said to be an $\mathcal{F}$-space if all of its isotropy groups away from its G-fixed basepoint are in $\mathcal{F}$. A (naive or genuine) G-spectrum is said to be an $\mathcal{F}$-spectrum if it is equivalent to a G-CW spectrum built up from cells of orbit type G/H with $H \in \mathcal{F}$. In all three cases, an object X is equivalent to an $\mathcal{F}$-object if and only if the natural map $E\mathcal{F}_+ \wedge X \to X$ is a G-equivalence. A G-space or G-spectrum X is said to be $\mathcal{F}$-trivial or $\mathcal{F}$-contractible if X^H is contractible for all $H \in \mathcal{F}$. The proofs of the next three results are identical to those of Propositions 2.4–2.6.

PROPOSITION 17.4. *Let X be an $\mathcal{F}$-space or an $\mathcal{F}$-spectrum. Then*

$$t_{\mathcal{F}}(k_G)^*(X) = 0 \quad \text{and} \quad t_{\mathcal{F}}(k_G)_*(X) = 0.$$

Therefore

$$f_{\mathcal{F}}(k_G)^*(X) \cong c_{\mathcal{F}}(k_G)^*(X) \quad \text{and} \quad f_{\mathcal{F}}(k_G)_*(X) \cong c_{\mathcal{F}}(k_G)_*(X).$$

PROPOSITION 17.5. *If X is an $\mathcal{F}$-contractible G-space or G-spectrum, then*

$$c_{\mathcal{F}}(k_G)^*(X) = 0 \quad \text{and} \quad f_{\mathcal{F}}(k_G)_*(X) = 0.$$

Therefore

$$t_{\mathcal{F}}(k_G)^n(X) \cong f_{\mathcal{F}}(k_G)^{n+1}(X) \quad \text{and} \quad c_{\mathcal{F}}(k_G)_n(X) \cong t_{\mathcal{F}}(k_G)_n(X).$$

By definition, Tate homology is a special case of c-homology,

$$t_{\mathcal{F}}(k_G)_n(X) = c_{\mathcal{F}}(k_G)_n(\widetilde{E}\mathcal{F} \wedge X).$$

Analogously, Tate cohomology is a special case of f-cohomology.

PROPOSITION 17.6. *The Tate spectrum $t_{\mathcal{F}}(k_G)$ is equivalent to $F(\widetilde{E}\mathcal{F}, \Sigma f_{\mathcal{F}}(k_G))$. Therefore, for any G-space or G-spectrum X,*

$$t_{\mathcal{F}}(k_G)^n(X) \cong f_{\mathcal{F}}(k_G)^{n+1}(\widetilde{E}\mathcal{F} \wedge X).$$

Formally, the functor $f_{\mathcal{F}}$ preserves wedges, cofibers, and colimits while the functor $c_{\mathcal{F}}$ preserves products, fibers, and limits. Therefore the functor $t_{\mathcal{F}}$ preserves finite wedges and cofibration sequences. The proofs of the following results are identical with those of Propositions 3.2 and 3.3, but the hypothesis on $\widetilde{E}\mathcal{F}$ can only be expected to be satisfied when the family $\mathcal{F}$ contains only finitely many maximal conjugacy classes (H).

PROPOSITION 17.7. *The functor $c_{\mathcal{F}}$ preserves microscopes. If $\widetilde{E}\mathcal{F}$ has a model as an $\mathcal{F}$-CW complex with finite skeleta, then the functors $f_{\mathcal{F}}$ and $t_{\mathcal{F}}$ preserve microscopes of inverse sequences of uniformly bounded below G-spectra.*

PROPOSITION 17.8. *If $\{Y^q\}$ is the Postnikov tower of a G-spectrum $Y, Y^q = Y(-\infty, q]$, and $\widetilde{E}\mathcal{F}$ has a model as an $\mathcal{F}$-CW complex with finite skeleta, then the following diagram displays a G-equivalence between cofiber sequences:*

$$\begin{array}{ccccc}
f_{\mathcal{F}}(\mathit{Mic}\ Y^q) & \longrightarrow & c_{\mathcal{F}}(\mathit{Mic}\ Y^q) & \longrightarrow & t_{\mathcal{F}}(\mathit{Mic}\ Y^q) \\
\downarrow & & \downarrow & & \downarrow \\
\mathit{Mic}\ f_{\mathcal{F}}(Y^q) & \longrightarrow & \mathit{Mic}\ c_{\mathcal{F}}(Y^q) & \longrightarrow & \mathit{Mic}\ t_{\mathcal{F}}(Y^q).
\end{array}$$

Since $E\mathcal{F}_+$ has a diagonal map and since there are equivalences

$$E\mathcal{F}_+ \wedge E\mathcal{F}_+ \simeq E\mathcal{F}_+ \quad \text{and} \quad \widetilde{E}\mathcal{F} \wedge \widetilde{E}\mathcal{F} \simeq \widetilde{E}\mathcal{F},$$

unique up to homotopy, we obtain a commutative diagram of associative and commutative pairings exactly as in (3.4); we shall call it the $\mathcal{F}$-norm pairing diagram.

PROPOSITION 17.9. *If k_G is a ring G-spectrum, then $c_{\mathcal{F}}(k_G)$ and $t_{\mathcal{F}}(k_G)$ are ring G-spectra, commutative if k_G is, and the right square of (D) is a diagram of ring G-spectra. If m_G is a k_G-module G-spectrum, then $c_{\mathcal{F}}(m_G)$ is a $c_{\mathcal{F}}(k_G)$-module G-spectrum and $t_{\mathcal{F}}(m_G)$ is a $t_{\mathcal{F}}(k_G)$-module G-spectrum.*

The change of groups results of Propositions 3.7 and 3.8 also generalize to the present context.

PROPOSITION 17.10. *Regarded as an H-spectrum, $t_{\mathcal{F}}(k_G)$ is equivalent to $t_{\mathcal{F}|H}(k_H)$, where $\mathcal{F}|H = \{K \mid K \subset H \text{ and } K \in \mathcal{F}\}$, and similarly for f and c.*

PROPOSITION 17.11. *Let k_H be an H-spectrum. Then $t_{\mathcal{F}}(G \ltimes_H k_H)$ is equivalent to $G \ltimes_H t_{\mathcal{F}|H}(k_H)$ and $t_{\mathcal{F}}(F_H[G, k_H])$ is equivalent to $F_H[G, t_{\mathcal{F}|H}(k_H))$, and similarly for f and c.*

§18. Cohomological and homological completion phenomena

Let $I\mathcal{F} \subset A(G)$ be the intersection of the kernels of the restrictions $A(G) \to A(H)$ for $H \in \mathcal{F}$. The following result generalizes Theorem 4.1.

THEOREM 18.1. *Let G be finite. Then $c_{\mathcal{F}}(k_G)$ is $I\mathcal{F}$-complete for any G-spectrum k_G. If k_G is bounded below, then $f_{\mathcal{F}}(k_G)$ and therefore $t_{\mathcal{F}}(k_G)$ are also $I\mathcal{F}$-complete.*

Exactly as in Section 4, this is a consequence of the following more general result, which is valid for any compact Lie group G.

PROPOSITION 18.2. *Let J be any finitely generated ideal contained in $I\mathcal{F}$. Then the following conclusions hold.*

(i) *$c_{\mathcal{F}}(k_G)$ is J-complete.*

(ii) *Any bounded below $\mathcal{F}$-spectrum Y is J-complete.*

We take G to be finite in the rest of this section. As explained in [25,§4], the generalized Atiyah-Segal completion theorem of [2] and the generalized Segal conjecture of [3] imply that the map $\varepsilon : k_G \to c_{\mathcal{F}}(k_G)$ is a completion at $I\mathcal{F}$ when k_G is KU_G, KO_G, or S_G. In the last case, $f_{\mathcal{F}}^{\perp} S_G \to t_{\mathcal{F}}(S_G)$ is also a completion at $I\mathcal{F}$.

In general, $\varepsilon : k_G \to c_{\mathcal{F}}(k_G)$ is a completion at $I\mathcal{F}$ if and only if the cohomology theory represented by $(k_G)^{\wedge}_{I\mathcal{F}}$ carries $\mathcal{F}$-equivalences to isomorphisms. By (4.2), this holds if and only if the left derived functors $L_0^{I\mathcal{F}}$ and $L_1^{I\mathcal{F}}$ of $I\mathcal{F}$-adic completion vanish on $(k_G)^*(X)$ whenever X is an $\mathcal{F}$-trivial G-spectrum..

As in Section 4, there is another way to think about this. Defining $M(I\mathcal{F})$ as in (4.4), we see that the canonical map $M(I\mathcal{F}) \to S_G$ is an $\mathcal{F}$-equivalence. Therefore, by the Whitehead theorem, there is a unique map $\xi : \Sigma^\infty E\mathcal{F}_+ \to M(I\mathcal{F})$ over S_G. By (4.5), we conclude that ε is a completion at $I\mathcal{F}$ if and only if the "completion conjecture map"

$$\xi^* : (k_G)^{\wedge}_I = F(M(I\mathcal{F}), k_G) \to F(E\mathcal{F}_+, k_G) = c_{\mathcal{F}}(k_G) \tag{18.3}$$

is an equivalence. We can also ask when the "cocompletion conjecture map"

$$\xi_* = 1 \wedge \xi : k_G \wedge E\mathcal{F}_+ \to k_G \wedge M(I\mathcal{F}) \tag{18.4}$$

is an equivalence. By Lemma 4.9, if k_G is a ring G-spectrum and ξ_* is an equivalence, then so is

$$\xi_* = 1 \wedge \xi : c_{\mathcal{F}}(k_G) \wedge E\mathcal{F}_+ \to c_{\mathcal{F}}(k_G) \wedge M(I\mathcal{F}). \tag{18.5}$$

Exactly as in Proposition 4.11, we have the following formal comparison.

PROPOSITION 18.6. *The map ξ_* of (18.4) is an equivalence if and only if the maps ξ^* of (18.3) and ξ_* of (18.5) are both equivalences.*

Let $\widetilde{M}(I\mathcal{F})$ be the cofiber of $M(I\mathcal{F}) \to S_G$. We have a map of cofibrations

$$\begin{tikzcd} E\mathcal{F}_+ \arrow[r] \arrow[d, "\xi"'] & S^0 \arrow[r] \arrow[d, equal] & \widetilde{E}\mathcal{F} \arrow[d, "\widetilde{\xi}"] \\ M(I\mathcal{F}) \arrow[r] & S^0 \arrow[r] & \widetilde{M}(I\mathcal{F}). \end{tikzcd} \tag{18.7}$$

Therefore the map ξ_* of (18.4) is an equivalence if and only if the map

$$\widetilde{\xi}_* = 1 \wedge \widetilde{\xi} : k_G \wedge \widetilde{E}\mathcal{F} \to k_G \wedge \widetilde{M}(I\mathcal{F}) \tag{18.8}$$

is an equivalence and the map ξ_* of (18.5) is an equivalence if and only if the map

$$\widetilde{\xi}_* = 1 \wedge \widetilde{\xi} : t_{\mathcal{F}}(k_G) = c_{\mathcal{F}}(k_G) \wedge \widetilde{E}\mathcal{F} \to c_{\mathcal{F}}(k_G) \wedge \widetilde{M}(I\mathcal{F}) \tag{18.9}$$

is an equivalence. The easier implication of Proposition 4.15 generalizes readily to the present context.

PROPOSITION 18.10. *Let k_G be a ring G-spectrum. If $t_{\mathcal{F}}(k_G)$ is rational, then the map $\widetilde{\xi}_*$ of (18.9) is an equivalence.*

We do not see how to generalize the transfer theoretic proof of the converse given in Lemmas 4.16 through 4.18. However, the first author has recently obtained a quite different proof. It involves a ring theoretic analog of Tate cohomology and will be presented elsewhere [50]. Lemma 4.19 generalizes directly to give the following result.

LEMMA 18.11. *If X is a G-spectrum such that $1 \wedge \widetilde{\xi} : X \wedge \widetilde{E}\mathcal{F} \to X \wedge \widetilde{M}(I\mathcal{F})$ is an equivalence, then $(X \wedge \widetilde{E}\mathcal{F})^{\wedge}_{I\mathcal{F}}$ is trivial.*

We combine the previous results and the cited result from [50] to obtain the following generalization of Theorem 4.8.

THEOREM 18.12. *Let k_G be a ring G-spectrum. Then the map ξ_* of (18.4) is an equivalence if and only if $t_{\mathcal{F}}(k_G)$ is a rational G-spectrum and the map ξ^* of (18.3) is an equivalence. When these equivalent conditions hold, $f^{\perp}_{\mathcal{F}}(k_G)^{\wedge}_{I\mathcal{F}}$ and $t_{\mathcal{F}}(k_G)^{\wedge}_{I\mathcal{F}}$ are trivial, hence $c_{\mathcal{F}}(k_G)$ is the completion at $I\mathcal{F}$ of both k_G and $f_{\mathcal{F}}(k_G)$.*

In the case of KU_G and KO_G, the map ξ^* of (18.3) was proven to be an equivalence in [2], the map ξ_* of (18.4) was recently proven to be an equivalence by the first author in [22], and the rationality of the Tate spectra will be proven in the next section. Thus the results of [2] and [22] are in fact equivalent. This is not surprising since all three proofs proceed by direct inductive reduction to quotations of equivariant Bott periodicity.

§19. The generalized Tate G-spectra of periodic K-theory

Let G be finite in this section. We shall formulate and prove a generalization of Theorem 13.1. In fact, for an arbitrary family $\mathcal{F}$ in G, we shall give a complete analysis of the cofiber sequence

$$f_{\mathcal{F}}(KU_G) \to c_{\mathcal{F}}(KU_G) \to t_{\mathcal{F}}(KU_G).$$

Although we shall not work out full details, we shall also explain the corresponding analysis for KO_G.

Let $J\mathcal{F} \subset R(G)$ be the intersection over $H \in \mathcal{F}$ of the kernels of the restriction homomorphisms $R(G) \to R(H)$. Define $JO\mathcal{F} \subset RO(G)$ similarly. The completions of an $R(G)$-module at $I\mathcal{F}$ and at $J\mathcal{F}$ are isomorphic, by [25,4.5], and similarly in the real case. By the generalized Atiyah-Segal completion theorem cited in the previous section, $c_{\mathcal{F}}(KU_G)$ and $c_{\mathcal{F}}(KO_G)$ are the completions of KU_G and KO_G at $I\mathcal{F}$. Therefore

$$c_{\mathcal{F}}(KU_G)_{2i} \cong R(G)^{\wedge}_{J\mathcal{F}} \quad \text{and} \quad c_{\mathcal{F}}(KU_G)_{2i+1} = 0;$$

similarly, $c_{\mathcal{F}}(KO_G)_* \cong ((KO_G)_*)^{\wedge}_{JO(G)}$. The following theorem is the key to our analysis.

THEOREM 19.1. *$t_{\mathcal{F}}(KU_G)$ and $t_{\mathcal{F}}(KO_G)$ are rational G-spectra .*

We explain the implications before giving the proof. We shall first determine the rationalization of $f_{\mathcal{F}}(KU_G)_*$, next use the rationalization of the $\mathcal{F}$-norm sequence to compute $t_{\mathcal{F}}(KU_G)_*$, and then go back to the unrationalized $\mathcal{F}$-norm sequence to determine $f_{\mathcal{F}}(KU_G)_*$. In principle, this procedure applies equally well to KO_G. The analysis will simultaneously deal with subgroups in view of Proposition 17.10. In Appendix A, we shall prove the folklore fact that rational G-spectra split as products of Eilenberg-MacLane G-spectra (since G is finite). Therefore the rationalizations of the homotopy groups of G-spectra determine their rational homotopy types.

Let $\langle g \rangle$ denote the subgroup of G generated by the element g. It is clear by character theory that

$$J\mathcal{F} = \{\chi \mid \chi(g) = 0 \text{ if } \langle g \rangle \in \mathcal{F}\} \subset R(G).$$

Define the "complementary ideal" $J'\mathcal{F}$ by

$$J'\mathcal{F} = \{\chi \mid \chi(g) = 0 \text{ if } \langle g\rangle \notin \mathcal{F}\} \subset R(G).$$

Since $J\mathcal{F}\cdot J'\mathcal{F} = 0$, $J'\mathcal{F}^{\wedge}_{J\mathcal{F}} = J'\mathcal{F}$ and $(R(G)/J'\mathcal{F})^{\wedge}_{J\mathcal{F}} \cong R(G)^{\wedge}_{J\mathcal{F}}/J'\mathcal{F}$.

LEMMA 19.2. *$f_{\mathcal{F}}(KU_G)_{2i}\otimes\mathbb{Q} = J'\mathcal{F}\otimes\mathbb{Q}$ and $f_{\mathcal{F}}(KU_G)_{2i+1}\otimes\mathbb{Q} = 0$.*

PROOF: Rationalize everything in this proof without change of notation. As explained in Appendix A, the rational equivariant stable category decomposes in terms of idempotents e_H of the rationalized Burnside $A(G)$. By Remark A.13, if $e_{\mathcal{F}}$ is the sum over $H\in\mathcal{F}$ of the idempotents e_H and $\widetilde{e}_{\mathcal{F}} = 1-e_{\mathcal{F}}$, then $I\mathcal{F} = \widetilde{e}_{\mathcal{F}}A(G)$, $e_{\mathcal{F}}S^0 \simeq E\mathcal{F}_+$, and $\widetilde{e}_{\mathcal{F}}S^0\simeq \widetilde{E}\mathcal{F}$. Let $f_{\mathcal{F}}$ and $\widetilde{f}_{\mathcal{F}}$ be the images of $e_{\mathcal{F}}$ and $\widetilde{e}_{\mathcal{F}}$ in the rationalized representation ring $R(G)$. Let $\mathcal{H}$ and $\mathcal{E}$ denote the sets of conjugacy classes of subgroups of G and of elements of G, respectively. Passage from elements g to cyclic groups $\langle g\rangle$ gives a function $\mathcal{E}\to\mathcal{H}$. Let $\mathbb{Q}^{\mathcal{H}}$ and $\mathbb{C}^{\mathcal{E}}$ be the rings of functions $\mathcal{H}\to\mathbb{Q}$ and $\mathcal{E}\to\mathbb{C}$, and let $\pi:\mathbb{Q}^{\mathcal{H}}\to\mathbb{C}^{\mathcal{E}}$ and $\rho: A(G)\to R(G)$ be the evident maps of rings. Then the following diagram commutes, where φ is the isomorphism described in Appendix A and χ is the trace map.

$$\begin{array}{ccc} A(G) & \xrightarrow{\rho} & R(G) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\chi} \\ \mathbb{Q}^{\mathcal{H}} & \xrightarrow{\pi} & \mathbb{C}^{\mathcal{E}} \end{array}$$

This easily implies that, rationally,

$$(19.3)\qquad J\mathcal{F} = \widetilde{f}_{\mathcal{F}}R(G),\ J'\mathcal{F} = f_{\mathcal{F}}R(G),\quad\text{and}\quad J\mathcal{F}+J'\mathcal{F} = R(G).$$

Therefore

$$(19.4)\qquad f_{\mathcal{F}}(KU_G)_*\otimes\mathbb{Q}\cong J'\mathcal{F}\cdot KU_*^G\otimes\mathbb{Q}.$$

As a matter of algebra, the proof just given implies the following result, which is needed to reconcile the statements of Theorem 13.1 and Corollary 13.3 with the corollaries which follow.

LEMMA 19.5. *The map* $J\mathcal{F}^{\wedge}_{J\mathcal{F}} \otimes \mathbb{Q} \to R(G)^{\wedge}_{J\mathcal{F}}/J'\mathcal{F} \otimes \mathbb{Q}$ *induced by the inclusion* $J\mathcal{F} \to R(G)$ *is an isomorphism.*

COROLLARY 19.6. $t_{\mathcal{F}}(KU_G)_{2i} = (R(G)/J'\mathcal{F})^{\wedge}_{J\mathcal{F}} \otimes \mathbb{Q}$ *and* $t_{\mathcal{F}}(KU_G)_{2i+1} = 0$.

PROOF: We deduce this from the rationalized $\mathcal{F}$-norm sequence, noting that the map $f_{\mathcal{F}}(KU_G) \to c_{\mathcal{F}}(KU_G)$ factors as the composite

$$f_{\mathcal{F}}(KU_G) = KU_G \wedge E\mathcal{F}_+ \to KU_G \to c_{\mathcal{F}}(KU_G).$$

In homotopy, the observation (19.4) identifies the rationalization of the first map, and the generalized Atiyah-Segal completion theorem identifies the second. We see that $f_{\mathcal{F}}(KU_G)_{2i} \otimes \mathbb{Q} \to c_{\mathcal{F}}(KU_G)_{2i} \otimes \mathbb{Q}$ is the evident inclusion.

COROLLARY 19.7. *Let* $\underline{R/J'\mathcal{F}}^{\wedge}_{J\mathcal{F}}$ *be the Mackey functor whose value on* G/H *is* $(R(H)/J'\mathcal{F}|H)^{\wedge}_{J\mathcal{F}|H}$. *Then* $t_{\mathcal{F}}(KU_G)$ *is the product over integers* i *of the Eilenberg-MacLane* G*-spectra* $K(\underline{R/J'\mathcal{F}}^{\wedge}_{J\mathcal{F}} \otimes \mathbb{Q}, 2i)$. *As a ring,*

$$(t_{\mathcal{F}}(KU_G)^H)_* \cong (R(H)/J'\mathcal{F}|H)^{\wedge}_{J\mathcal{F}|H} \otimes \mathbb{Q}[v, v^{-1}], \quad \deg v = 2.$$

PROOF: Only the multiplicative structure requires comment. We know the ring structure of $((KU_G)^H)_*$ from Bott periodicity, and the map of the generalized Atiyah-Segal completion theorem induces a completion of graded rings. This determines the ring structure on $(c_{\mathcal{F}}(KU_G)^H)_* \otimes \mathbb{Q}$ and, by Corollary 19.6 (together with the remark above Lemma 19.2), we see that $(t_{\mathcal{F}}(KU_G)^H)_*$ is the specified quotient ring of $(c_{\mathcal{F}}(KU_G)^H)_* \otimes \mathbb{Q}$.

COROLLARY 19.8. $f_{\mathcal{F}}(KU_G)_{2i} = J'\mathcal{F}$ *and* $f_{\mathcal{F}}(KU_G)_{2i+1} = (R(G)/J'\mathcal{F})^{\wedge}_{J\mathcal{F}} \otimes (\mathbb{Q}/\mathbb{Z})$.

PROOF: The map $c_{\mathcal{F}}(KU_G)_{2i} \to t_{\mathcal{F}}(KU_G)_{2i}$ is induced by $R(G) \to R(G)/J'\mathcal{F}$.

As explained in [22], the equivalence $KU_G \wedge E\mathcal{F}_+ \to KU_G \wedge M(I\mathcal{F})$ implies an alternative algebraic description of $f_{\mathcal{F}}(KU_G)_*$ in terms of Grothendieck's local cohomology groups.

Now consider the real case. The groups KO_q^G are periodic of period 8 with the following values as q runs from 0 to 7:

$$(19.9)\qquad \begin{array}{cccc} RO(G) & RO(G)/rRU(G) & RU(G)/c'RSp(G) & 0 \\ RSp(G) & RSp(G)/qRU(G) & RU(G)/cRO(G) & 0 \end{array}$$

Here r, c, q, c' are the evident homomorphisms. They and conjugation t on $RU(G)$ are related by the equations

$$\begin{array}{llll} rc = 2, & cr = 1 + t, & qc' = 2, & c'q = 1 + t, \\ tc = c, & rt = r, & tc' = c', & qt = q, \quad \text{and} \quad t^2 = 1. \end{array}$$

For well chosen sets $\{U_i\}$, $\{V_j\}$, and $\{W_k\}$ of irreducible real, complex, and quaternionic representations, $RO(G)$ is spanned by $\{U_i, rV_j, rc'W_k\}$, $RU(G)$ is spanned by $\{cU_i, V_j, tV_j, c'W_k\}$, and $RSp(G)$ is spanned by $\{qcU_i, qV_j, W_k\}$. This completely determines r, c, q, and c'. We see in particular that $RO(G)/rRU(G)$ and $RSp(G)/qRU(G)$ are 2-torsion and that, modulo their 2-torsion, $RU(G)/c'RSp(G)$ and $RU(G)/cRO(G)$ are isomorphic. We also see that $qc : RO(G) \to RSp(G)$ is a monomorphism with cokernel of exponent 4.

We can define a "complementary ideal" $J'O\mathcal{F} \subset RO(G)$ such that $JO\mathcal{F} \cdot J'O\mathcal{F} = 0$. With $f_{\mathcal{F}}$ and $\widetilde{f}_{\mathcal{F}}$ replaced by the images of $e_{\mathcal{F}}$ and $\widetilde{e}_{\mathcal{F}}$ in $RO(G) \otimes \mathbb{Q}$, the proof of Lemma 19.2 gives the evident real analogs of (19.3), (19.4), and (19.5). However, the computation of $J'O\mathcal{F} \cdot KO_q^G \otimes \mathbb{Q}$ from (19.9) is not immediate. These groups are zero for odd q and are isomorphic to $J'O\mathcal{F} \otimes \mathbb{Q}$ for $q \equiv 0 \mod 4$. They are isomorphic for all $q \equiv 2 \mod 4$, but whether or not they are zero in these degrees depends on $\mathcal{F}$. They are zero in these degrees when $\mathcal{F} = \{e\}$. One can deduce this algebraically from the fact that $J'O\{e\}$ is one dimensional with basis the regular representation. Alternatively, one can note that the rational splitting $KU \simeq KO \vee \Sigma^2 KO$ induces a rational splitting

$$f(i_*KU) \simeq f(i_*KO) \vee f(i_*\Sigma^2 KO),$$

so that $f(KO_G)_{4q+2} \otimes \mathbb{Q}$ must be zero by a comparison of dimensions. In principle, we can now read off the real analogs of Corollaries 19.6–19.8. We have recorded the answers in the case $\mathcal{F} = \{e\}$ in Theorem 13.1 and Corollary 13.3. In the latter result, the non-zero groups in odd degrees are direct sums because the kernels in the extensions that appear in the norm sequence are divisible and hence injective Abelian groups.

PROOF OF THEOREM 19.1: Assume inductively that the conclusion is true for all families in any proper subgroup of G. The induction starts since Tate spectra for the trivial group are trivial. If $\mathcal{F}$ is the family of all subgroups of G, then $t_{\mathcal{F}}(k_G)$ is trivial for any k_G. Thus we may as well assume that $\mathcal{F} \subset \mathcal{P}$, where $\mathcal{P}$ is the family of proper subgroups of G. Taking the smash product of the cofibering $E\mathcal{P}_+ \to S^0 \to \widetilde{E}\mathcal{P}$ with $t_{\mathcal{F}}(KU_G)$, we see that it suffices to prove that $t_{\mathcal{F}}(KU_G) \wedge E\mathcal{P}_+$ and $t_{\mathcal{F}}(KU_G) \wedge \widetilde{E}\mathcal{P}$ are both rational. It is clear by inspection of fixed-point subspaces that $\widetilde{E}\mathcal{F} \wedge \widetilde{E}\mathcal{P} \simeq \widetilde{E}\mathcal{P}$. Therefore the following two lemmas complete the proof.

LEMMA 19.10. *$t_{\mathcal{F}}(KU_G) \wedge X$ is rational for any family $\mathcal{F}$ and any $\mathcal{P}$-spectrum X.*

PROOF: The colimit of rational G-spectra is rational, and if two terms in a cofibering are rational, then so is the third term. Thus it suffices to prove the result when $X = G/H_+$ for a proper subgroup H. Here the conclusion is immediate from the induction hypothesis and Propositions 17.10 and 17.11.

LEMMA 19.11. *$t_{\mathcal{F}}(KU_G) \wedge \widetilde{E}\mathcal{P}$ is rational for any family $\mathcal{F}$.*

We deduce this from the following consequence of Proposition 16.2, in which G need not be finite. For a complex representation V, let $\lambda_V \in KU^0_G(S^V)$ be the Bott class. By equivariant Bott periodicity, multiplication by λ_V gives an isomorphism

$$KU^V_G(S^0) \to KU^V_G(S^V) \cong KU^0_G(S^0) = R(G).$$

Recall from Section 16 that $\alpha_V = e^*(1) \in KU^V_G(S^0)$, $e : S^0 \to S^V$. The image of α_V under the displayed isomorphism is the K-theory Euler class, which we denote by χ_V; explicitly, it is just the alternating sum of the exterior powers of V. Recall that $\mathcal{F}(V)$ is the family $\{H | V^H \neq 0\}$.

PROPOSITION 19.12. *If m_G is a KU_G-module spectrum, such as $F(Y, KU_G)$ for any Y, then $(m_G \wedge \widetilde{E}\mathcal{F}(V))_*(X)$ and, if X is finite, $(m_G \wedge \widetilde{E}\mathcal{F}(V))^*(X)$ are the localizations of $m_*^G(X)$ and $m_G^*(X)$ away from χ_V.*

PROOF OF LEMMA 19.11: Let V be the reduced regular representation of G. Then $V^G = 0$ but $V^H \neq 0$ for all proper subgroups H. Thus $\mathcal{F}(V) = \mathcal{P}$. It also follows that e is H-trivial and χ_V restricts to zero in $R(H)$ for any proper subgroup H. That is, $\chi_V \in J\mathcal{P}(G)$. Since the product over cyclic subgroups C of the restrictions $R(G) \to R(C)$ is an injection, $J\mathcal{P}(G) = 0$ unless G is cyclic. Since localization away from zero is zero, the conclusion holds trivially unless G is cyclic. Thus let B be cyclic of order n. The primes that do not divide n act invertibly on $t_{\mathcal{F}}(k_G)$ for any G-spectrum k_G and any family $\mathcal{F}$ by Corollary 22.10 below, which generalizes Corollary 11.5. Let p divide n. It suffices to show that p acts invertibly on $m_G \wedge \widetilde{E}\mathcal{P}$ for any KU_G-module spectrum m_G, and this certainly holds if p divides some power of χ_V in $R(G)$. Choose an epimorphism $G \to \mathbb{Z}_p$ and a nontrivial one-dimensional irreducible representation W of $\mathbb{Z}_p$. Regard W as a representation of G, by pullback. Then W is a summand of V and χ_W divides χ_V. Thus it suffices just to observe that $(\chi_W)^p$ is divisible by p in $R(\mathbb{Z}_p)$ since $\chi_W = 1 - W$ and $W^p = 1$.

Applying real equivariant Bott periodicity to Spin representations of dimension divisible by 8, and using 8 times the reduced real regular representation to model $E\mathcal{P}$, we find that a slight modification of the argument just given works to prove the real case of Theorem 19.1.

§20. Theories associated to Mackey and coMackey functors

We first recall some general categorical algebra that was implicitly used earlier but must now be made explicit for intelligibility. Let $\mathcal{C}$ be any small additive category and let $\mathcal{C}^*$ and $\mathcal{C}_*$ be the categories of additive contravariant and covariant functors $\mathcal{C} \to \mathcal{A}b$. We think of $\mathcal{C}$ as a "ring with many objects", and we think of objects of $\mathcal{C}^*$ and $\mathcal{C}_*$ as right and left $\mathcal{C}$-modules (compare Mitchell [40]). For an object C of $\mathcal{C}$, define the represented contravariant functor C^* by $C^*(?) = \mathrm{Hom}_{\mathcal{C}}(?, C)$. It is then a categorical triviality called the Yoneda lemma that

$$(20.1) \qquad \mathrm{Hom}_{\mathcal{C}^*}(C^*, M) \cong M(C) \quad \text{for any } M \in \mathcal{C}^*.$$

We have a tensor product over $\mathcal{C}$. For $M \in \mathcal{C}^*$ and $N \in \mathcal{C}_*$,

$$(20.2) \qquad M \otimes_{\mathcal{C}} N = \sum M(C) \otimes N(C)/(m\varphi^* \otimes n - m \otimes \varphi_* n),$$

where ϕ, m, and n run over the maps $\phi : C \to C'$ in $\mathcal{C}$ and the elements $m \in M(C')$ and $n \in N(C)$. This is just an Abelian group, and this tensor product over $\mathcal{C}$ must not be confused with the more sophisticated tensor product $\otimes : \mathcal{C}^* \times \mathcal{C}^* \to \mathcal{C}^*$ that we defined topologically in certain special cases in Section 8. For an Abelian group A, we have the adjunction isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_*}(N, \mathrm{Hom}(M, A)) \cong \mathrm{Hom}(M \otimes_{\mathcal{C}} N, A) \cong \mathrm{Hom}_{\mathcal{C}^*}(M, \mathrm{Hom}(N, A)).$$

Another manifestation of the Yoneda lemma gives the isomorphism

$$(20.3) \qquad C^* \otimes_{\mathcal{C}} N \cong N(C) \text{ for any } N \in \mathcal{C}_*.$$

We define a free object of $\mathcal{C}^*$ to be any sum of objects of the form C^*.

The categories $\mathcal{C}[G]$ and $\mathcal{M}[G]$ of coefficient systems and Mackey functors are examples of such generalized categories of modules. In the context of families, one might be tempted to define an $\mathcal{F}$-module to be a coefficient system R such that $R(G/H) = 0$ for $H \notin \mathcal{F}$ and to try to prove that $t_{\mathcal{F}}(HM)$ depends only on the underlying $\mathcal{F}$-module of M, generalizing what we did in the case $\mathcal{F} = \{e\}$ in Part II. While much of the analysis does go through, the program fails because $s^*s_*R \neq R$, as we observed in Remark 7.4. For this reason, we shall make little use of coefficient systems in this part.

The theory for families in positive dimensional compact Lie groups diverges from the theory for $\mathcal{F} = \{e\}$ in a still more fundamental way: we cannot use Eilenberg-MacLane G-spectra HM to study homology theories. Define a "coMackey functor" to be a covariant functor $\mathcal{OS} \to \mathcal{A}b$. Let k_G be a G-spectrum . The Mackey functor $\underline{\pi}_{-n}(k_G)$ may be viewed as the restriction $\underline{k}^n_G$ of the cohomology functor k^n_G to orbit G-spectra,

$$\underline{k}^n_G(G/H) = k^n_G(\Sigma^\infty G/H_+) = [G/H_+, k_G]^n_G.$$

Dually, we have the homology coMackey functors $\underline{k}^G_n$ specified by

$$\underline{k}^G_n(G/H) = k^G_n(\Sigma^\infty G/H_+) = [S^0, k_G \wedge G/H_+]^G_n.$$

As was pointed out in [32, p.211], for a coMackey functor N, there is a G-spectrum JN such that $\underline{JN}^G_0 = N$ and $\underline{JN}^G_n = 0$ for $n \neq 0$. This homological Eilenberg-MacLane G-spectrum represents the ordinary homology theory on G-spectra characterized by the dimension axiom specified by N, and JN is uniquely determined up to homotopy.

The price of arranging the dimension axiom in homology is that JN is not connective and does not have obvious homotopy groups. Quite generally, the homotopy groups of a G-spectrum X are

$$\pi^H_n(X) = [\Sigma^n G/H_+, X] = [S^n, D(G/H_+) \wedge X]_G.$$

Here $D(G/H_+) = G \ltimes_H S^{-L(H)}$, where $L(H)$ is the tangent H-representation of G/H at the identity coset eH. This representation contains a trivial representation of dimension $\dim WH$, and the complementary representation can be triangulated as an H-CW complex. We see in particular that $\pi^H_n(JN)$ can be non-zero for $-\dim G/H \leq n \leq -\dim WH$.

The ordinary homology theory represented by JN and the ordinary cohomology theory represented by HM have chain level descriptions precisely similar to those used in the definition of Bredon homology and cohomology [8]. For a G-CW spectrum X, we have the chain complex $\underline{C}_*(X)$ in the Abelian category $\mathcal{M}[G]$ given by $\underline{C}_n(X) = \underline{\pi}_n(X^n/X^{n-1})$; d_n is the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}). Here $\underline{C}_n(X)$ is free with one generator for each n-cell of X. For a coMackey functor N and a Mackey functor M, we define

$$(20.4) \qquad C_*^G(X;N) = \underline{C}_*(X) \otimes_{\mathcal{O}S} N \quad \text{and} \quad C_G^*(X;M) = \operatorname{Hom}_{\mathcal{O}S}(\underline{C}_*(X), M).$$

These are ordinary chain and cochain complexes of Abelian groups, and we obtain homology and cohomology theories $H_*^G(X;N)$ and $H_G^*(X;M)$ on passage to homology and cohomology groups. The Eilenberg-MacLane G-spectra JN and HM are obtained by application of Brown's representability theorem to these theories [32], and there result isomorphisms

$$(20.5) \qquad H_*^G(X;N) \cong (JN)_*^G(X) \quad \text{and} \quad H_G^*(X;M) \cong (HM)_G^*(X).$$

Remark 20.6: We can make precisely analogous chain level definitions of $H_*^G(X;S)$ and $H_G^*(X;R)$ for naive G-CW spectra X and covariant and contravariant coefficient systems S and R; these are the spectrum level versions of Bredon homology and cohomology [8]. Using freeness and (20.1) and (20.3), we see that, for a coMackey functor N and a Mackey functor M,

$$H_*^G(i_*X;N) \cong H_*^G(X;s^*N) \quad \text{and} \quad H_G^*(i_*X;M) \cong H_G^*(X;s^*M)$$

since these are computed from isomorphic chain and cochain complexes. There is a naive Eilenberg-MacLane G-spectrum HR such that $H_G^*(X;R) \cong [X, HR]_G^*$ and, in cohomology, the isomorphism above also follows from the relation $i^*HM = Hs^*M$. However, the naive level homology theory is not represented in any usual sense since, in the world of naive G-spectra, $(j_G \wedge G/H_+)^G = *$ whenever $H \neq G$.

We relate these theories to families in the following definition. The theories of (20.5) correspond to the family "$\mathcal{A}\ell\ell$" of all subgroups of G, for which $E\mathcal{A}\ell\ell$ and $\widetilde{E}\mathcal{A}\ell\ell$ can each be taken to be a single point.

DEFINITION 20.7. *For G-spectra X, a family $\mathcal{F}$, and integers n, define*

$$H_n^{\mathcal{F}}(X;N) = f_{\mathcal{F}}(JN)_n(X) \qquad \text{and} \qquad H^n_{\mathcal{F}}(X;N) = c_{\mathcal{F}}(HM)^n(X);$$

$$\check{H}_n^{\mathcal{F}}(X;N) = c_{\mathcal{F}}(JN)_n(X) \qquad \text{and} \qquad \check{H}^n_{\mathcal{F}}(X;M) = f_{\mathcal{F}}(HM)^n(X);$$

$$\widehat{H}_n^{\mathcal{F}}(X;N) = t_{\mathcal{F}}(JN)_n(X) \qquad \text{and} \qquad \widehat{H}^n_{\mathcal{F}}(X;M) = t_{\mathcal{F}}(HM)^n(X).$$

When $X = S^0$, we delete it from the notation, writing $\widehat{H}^n_{\mathcal{F}}(M)$, etc.

There are no dimension shifts here. The dimension shifts that occur in homology in Definition 6.5 are entirely an artifact of the fact that we were there using cohomological Eilenberg-MacLane G-spectra to study homology theories, which is an entirely unnatural thing to do. Of course, since $E\mathcal{F}$ only has cells in non-negative dimensions,

$$H_n^{\mathcal{F}}(N) = 0 \qquad \text{and} \qquad H^n_{\mathcal{F}}(M) = 0 \qquad \text{for} \qquad n < 0.$$

More interestingly, if $d(\mathcal{F})$ is the minimum dimension of WH for $H \in \mathcal{F}$, then

$$\check{H}_n^{\mathcal{F}}(N) = 0 \qquad \text{and} \qquad \check{H}^n_{\mathcal{F}}(M) = 0 \qquad \text{for} \qquad n > -d(\mathcal{F}).$$

This also follows from the cell decomposition of $E\mathcal{F}$, but now it is the duals of cells that are relevant, as in our discussion of the homotopy groups of JN. The $\mathcal{F}$-norm sequences give consequences for the Tate theories.

The following complement to Theorem 6.3 will imply that the case $\mathcal{F} = \{e\}$ of the new definition is consistent with the old definition. For a coMackey functor N, let $UN = N(G/e)$. The opposite group of $\pi_0(G) = \overline{G}$ embeds in $[G/e_+, G/e_+]_G$ via right action by its elements. Therefore, for a Mackey functor M, $UM = M(G/e)$ is a left $\overline{G}$-module whereas, for a coMackey functor N, $UN = N(G/e)$ is a right $\overline{G}$-module. Write V^{op} for a left $\overline{G}$-module V regarded as a right $\overline{G}$-module via $vg = g^{-1}v$.

PROPOSITION 20.8. *Let N be any coMackey functor such that $UN = V^{\mathrm{op}}$. Then there is a Mackey functor M with $UM = V$ and a canonical G-map $\Sigma^d JN \to HM$ which is a nonequivariant equivalence. Therefore the cofiber sequences*

$$f(\Sigma^d JN) \to c(\Sigma^d JN) \to t(\Sigma^d JN) \quad \text{and} \quad f(HM) \to c(HM) \to t(HM)$$

are canonically G-equivalent and, if N and N' are coMackey functors such that $UN \cong UN'$, then the norm cofibration sequences of HN and HN' are canonically G-equivalent.

PROOF: By the discussion above, we find that $\pi_n(JN) = 0$ if $n \neq -d$ and $\pi_{-d}(JN) = V^{op}$. Define $M = \underline{\pi}_{-d}(JN)$, note that JN is $(-d-1)$-connected, and construct a G-map $JN \to \Sigma^{-d}HM$ that induces an isomorphism on $\underline{\pi}_{-d}$ by killing the higher homotopy groups of JN. This map is a nonequivariant equivalence, and the rest follows from Proposition 1.1 and Theorem 6.3.

Now a comparison of Definitions 6.5 and 20.7 gives the following consistency statement.

COROLLARY 20.9. *If N is a coMackey functor such that $UN = V^{\mathrm{op}}$, then*

$$H_n^{\{e\}}(X, N) \cong H_n^G(X; V), \ \check{H}_n^{\{e\}}(X, N) \cong \check{H}_n^G(X; V), \ \text{and } \widehat{H}_n^{\{e\}}(X, N) \cong \widehat{H}_n^G(X; V).$$

If M is a Mackey functor such that $UM = V$, then

$$H^n_{\{e\}}(X; M) \cong H^n_G(X; V), \ \check{H}^n_{\{e\}}(X; M) \cong \check{H}^n_G(X; V), \ \text{and } \widehat{H}^n_{\{e\}}(X; M) \cong \widehat{H}^n_G(X; V).$$

REMARK 20.10: When G is finite, Spanier-Whitehead duality specifies an equivalence of categories $D : \mathcal{OS}^{\mathrm{op}} \to \mathcal{OS}$ that is the identity on objects and satisfies $D \circ D = \mathrm{Id}$. Here a Mackey functor M determines a coMackey functor $M^{\mathrm{op}} = M \circ D$ and a coMackey functor N determines a Mackey functor $N^{\mathrm{op}} = N \circ D$. This bijective correspondence between Mackey functors and coMackey functors is realized topologically as a bijective correspondence between the two kinds of Eilenberg-MacLane G-spectra : $JN = HN^{\mathrm{op}}$ and

$HM = JN^{\mathrm{op}}$. That is, there is really only one kind of Eilenberg-MacLane G-spectrum, but it should be thought of differently when considered as representing a homology or a cohomology theory.

Just as we defined the chains of suitably filtered G-spectra, not necessarily G-CW spectra, in Section 9, so we can generalize the definition of cellular chains given in (20.4). As in Section 10, this generalization will lead via "$\mathcal{F}$-calculable" filtrations to generalized Atiyah-Hirzebruch spectral sequences. Let $W = \cup W^p$, $p \in \mathbb{Z}$, be a filtered G-spectrum such that each map $W^p \to W^{p+1}$ and $W^p \to W$ is a cofibration. Let $\overline{W}^p = W^p/W^{p-1}$, and assume that each $\overline{W}^p$ is equivalent to a wedge of G-spectra of the form $G/H_+ \wedge S^p$ for varying H in $\mathcal{F}$. Write $\underline{W}$ for a G-spectrum W with a given filtration of this sort. Define a chain complex $\underline{C}_*(\underline{W})$ of Mackey functors by letting $\underline{C}_p(\underline{W}) = \underline{\pi}_p(\overline{W}^p)$, with differential ∂_* induced by the evident geometric boundary map. For a coMackey functor N and a Mackey functor M, define $H_*^G(\underline{W}; N)$ and $H_G^*(\underline{W}; M)$ to be the homology of the respective chain and cochain complexes of Abelian groups.

(20.11) $\qquad C_*^G(\underline{W}; N) = \underline{C}_*(\underline{W}) \otimes_{\mathcal{O}S} N \quad \text{and} \quad C_G^*(\underline{W}; M) = \mathrm{Hom}_{\mathcal{O}S}(\underline{C}_*(\underline{W}), M).$

Given such a filtered G-spectrum $\underline{W}$ and a G-spectrum k_G, we obtain exact couples by giving $k_G \wedge W$ and $F(W, k_G)$ the filtrations induced by that of W and passing to G-homotopy groups; see Appendix B.

LEMMA 20.12. *In the resulting spectral sequences,*

$$E^1_{p,q} = k^G_{p+q}(\overline{W}^p) \cong C^G_p(\underline{W}; \underline{k}^G_q)$$

and

$$E_1^{p,q} = k_G^{p+q}(\overline{W}^p) \cong C^p_G(\underline{W}; \underline{k}^q_G),$$

and these are isomorphisms of chain and cochain complexes.

PROOF: Observe that $\underline{C}_*(\underline{W})$ is a free Mackey functor, with one summand for each wedge summand of each $\overline{W}^p$ and apply (20.1) and (20.3) cellwise.

§21. Amitsur-Dress-Tate cohomology theories

We begin by constructing an explicit model for $E\mathcal{F}$. For this, G can be any topological group and $\mathcal{F}$ can be any family of closed subgroups.

There is a product-preserving functor E_* from spaces to simplicial spaces. The space E_qX of q-simplices is X^{q+1}, the i^{th} face map is the projection away from the $(\text{i+1})^{\text{st}}$ coordinate, the i^{th} degeneracy map is the diagonal map applied to the $(\text{i+1})^{\text{st}}$ coordinate, and E_*X has a functorial simplicial contracting homotopy. Let EX be the geometric realization of E_*X. Since realization preserves products and homotopies, E is a product-preserving functor from spaces to contractible spaces. (See [35, pp 97–99], where E_* is denoted D_*.)

If X is a G space, then E_*X is a simplicial G-space under diagonal actions, and $(E_*X)^H = E_*(X^H)$. Therefore E restricts to a functor from G-spaces to G-spaces such that $(EX)^H = E(X^H)$. Obviously $(EX)^H$ is contractible if X^H is non-empty and is empty if X^H is empty. Thus, if $\mathcal{F}(X)$ is the family of subconjugates of isotropy groups of points of X, then EX is a model for $E\mathcal{F}(X)$. If $X = G$, then EG is G-homeomorphic to the usual bar construction model, by [35, 10.3]. If we start with a family $\mathcal{F}$ and define $X(\mathcal{F})$ to be the disjoint union of orbits G/H, one for each maximal conjugacy class $(H) \subset \mathcal{F}$, then $\mathcal{F}(X(\mathcal{F})) = \mathcal{F}$. Any larger disjoint union of orbits G/H with $H \in \mathcal{F}$ would serve equally well.

Now assume that G is finite and restrict attention to finite G-sets X. Recall that, viewed purely algebraically, a Mackey functor M consists of a contravariant functor M^* and a covariant functor M_* from finite G-sets to Abelian groups. These functors have the same object function, denoted M, and M converts disjoint unions to direct sums. It is required that $\alpha^* \circ \beta_* = \delta_* \circ \gamma^*$ for pullback diagrams of finite G-sets

$$\begin{array}{ccc} P & \xrightarrow{\delta} & S \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ R & \xrightarrow[\beta]{} & T. \end{array}$$

A Mackey functor in our original sense of a contravariant functor $M : \mathcal{OS} \to \mathcal{A}b$ extends by additivity to the full subcategory of the stable homotopy category consisting of the G-spectra of the form $\Sigma^\infty X_+$. Maps of G-sets and the corresponding transfer maps of G-spectra determine the contravariant and covariant parts of the associated algebraic Mackey functor. We shall use these two ways of looking at Mackey functors interchangeably. While our notations and framework are different, the following definitions agree with some of those used by Dress [17] in his homological study of induction theory. (Our notations partly duplicate those in Definition 20.7, in anticipation of consistency statements to be proven in the next section.)

DEFINITION 21.1. *Let M be a Mackey functor, let X be a finite G-set, and let $\mathcal{F} = \mathcal{F}(X)$. Define the Amitsur-Dress homology, cohomology, and Tate cohomology of $\mathcal{F}$ with coefficients in M as follows. For homology, denoted $H_*^{\mathcal{F}}(M)$, apply M_* to the simplicial G-set $E_*(X)$, and take the homology of the resulting simplicial Abelian group. For cohomology, denoted $H^*_{\mathcal{F}}(M)$, apply M^* to $E_*(X)$, regard the result as a cochain complex of Abelian groups, and take its cohomology. For Tate cohomology, denoted $\widehat{H}^*_{\mathcal{F}}(M)$, splice the desuspension of the homology complex, graded negatively, to the cohomology complex by use of the composite*

$$\pi^* \circ \pi_* : M(X) \to M(*) \to M(X);$$

then take the cohomology of the resulting $\mathbb{Z}$-graded cochain complex

$$\begin{aligned} \cdots \to M(X^n) \to M(X^{n-1}) \to \cdots \to M(X) \\ \to M(X) \to \cdots \to M(X^n) \to M(X^{n+1}) \to \cdots . \end{aligned}$$

(The zero$^{\text{th}}$ group of this cochain complex is the second, target, copy of $M(X)$.)

This is precisely analogous to defining the homology, cohomology and Tate cohomology of the finite group G explicitly in terms of the canonical resolution of $\mathbb{Z}$. It is routine homological algebra to interpret the result in terms of more general resolutions and to axiomatize these functors on Mackey functors. The homology groups are the left derived functors of the right exact functor $H_0^{\mathcal{F}}$ and the cohomology groups are the right derived functors of the left exact functor $H^0_{\mathcal{F}}$ (both relative to the class of $X(\mathcal{F})$-projective functors). The groups $H^0_{\mathcal{F}}(M)$ and $H_0^{\mathcal{F}}(M)$ have interpretations as suitable invariants and coinvariants, the displayed composite $\pi^* \circ \pi_*$ induces a norm map

$$N : H_0^{\mathcal{F}}(M) \to H^0_{\mathcal{F}}(M),$$

and

$$\widehat{H}_{\mathcal{F}}^{-1}(M) = \mathrm{Ker}(N) \quad \text{and} \quad \widehat{H}^0_{\mathcal{F}}(M) = \mathrm{Coker}(N).$$

The Tate cohomology functors comprise the complete derived sequence of the natural transformation N; compare [12, V§10 and XII§2].

The following generalization of part of Proposition 11.3 is due to Dress [17, p. 204]. We explain the proof since its implications are as important to our work as the result itself.

PROPOSITION 21.3. *Let G have order n. Then multiplication by n annihilates $\widehat{H}^*_{\mathcal{F}}(M)$ for all Mackey functors M.*

PROOF: Let $\underline{A}$ be the Burnside Mackey functor. For a finite G-set Y, $\underline{A}(Y)$ is the Grothendieck ring of the set of finite G-sets over Y. We have a pairing $\underline{A} \times M \to M$ given by the maps $\underline{A}(Y) \times M(Y) \to M(Y)$ that send a pair $(\beta : Z \to Y, m)$ to $\beta_*(\beta^*(m))$. By [17, Prop. 2.3], this pairing induces pairings $\widehat{H}^*_{\mathcal{F}}(\underline{A}) \otimes \widehat{H}^*_{\mathcal{F}}(M) \to \widehat{H}^*_{\mathcal{F}}(M)$ that satisfy all of the formal properties familiar from the classical case. In particular, $\widehat{H}^*_{\mathcal{F}}(M)$ is a module over the ring $\widehat{H}^*_{\mathcal{F}}(\underline{A})$. It suffices to show that $n \cdot 1 = 0$, where $1 \in \widehat{H}^0_{\mathcal{F}}(\underline{A})$ is the identity element. Here $1 \in \widehat{H}^0_{\mathcal{F}}(\underline{A})$ is the image of $1 \in A(G)$ under

$$\pi^* : A(G) = \underline{A}(*) \to \underline{A}(X) = \bigoplus_{H \in \mathcal{F}} A(H),$$

and the kernel of π^* is the ideal $I\mathcal{F}$. We see from the definitions that $q \cdot 1 = 0$ in $\widehat{H}^0_{\mathcal{F}}(\underline{A})$ if and only if $q \in I\mathcal{F} + I'\mathcal{F}$ in $A(G)$, where

$$I'\mathcal{F} = \mathrm{Im}(\pi_* : \bigoplus_{H \in \mathcal{F}} A(H) \to A(G)).$$

In terms of the homomorphisms $\varphi_H : A(G) \to \mathbb{Z}$ given by the cardinality of H-fixed point sets, we have

$$I\mathcal{F} = \{\alpha \mid \varphi_H(\alpha) = 0 \text{ for all } H \in \mathcal{F}\}$$

and

$$I'\mathcal{F} = \{\alpha \mid \varphi_K(\alpha) = 0 \text{ for all } K \notin \mathcal{F}\}.$$

For the last equality, [33, V.2.15 and 2.16(ii)] immediately give that, in terms of the standard basis $\{[G/H]\}$ for $A(G)$, $I'\mathcal{F}$ is the span of the $[G/H]$ for $H \in \mathcal{F}$ and is contained in the right side. For the opposite inclusion, let $\alpha = \Sigma n_J[G/J]$ and suppose that $\varphi_K(\alpha) = 0$ for $K \notin \mathcal{F}$. If (H) is maximal such that $n_H \neq 0$, then $\varphi_H(\alpha) = n_H|WH| \neq 0$ and thus $H \in \mathcal{F}$. Therefore $n_K = 0$ for $K \notin \mathcal{F}$.

Let $C(G)$ be the product over conjugacy classes (H) of copies of $\mathbb{Z}$. As is well known (e.g.[16,17, or 33, V.2.11]), the product of the φ_H is an inclusion $\varphi : A(G) \to C(G)$ such that $nC(G) \subset A(G)$. In $C(G)$, let $e_{\mathcal{F}}$ be the idempotent whose H^{th} coordinate is 1 if $H \in \mathcal{F}$ and 0 if $H \notin \mathcal{F}$ and let $\widetilde{e}_{\mathcal{F}} = 1 - e_{\mathcal{F}}$. Then $n = n\widetilde{e}_{\mathcal{F}} + ne_{\mathcal{F}}$ is in $I\mathcal{F} + I'\mathcal{F}$ for any $\mathcal{F}$.

Corollary 21.4. *Define $n(\mathcal{F})$ to be the smallest positive integer such that multiplication by $n(\mathcal{F})$ annihilates $\widehat{H}^*_{\mathcal{F}}(M)$ for all Mackey functors M. Then $n(\mathcal{F})$ is the smallest positive integer such that $n(\mathcal{F})e_{\mathcal{F}} \in A(G)$, or equivalently, the smallest positive integer such that $n(\mathcal{F})\widetilde{e}_{\mathcal{F}} \in A(G)$.*

In principle, $n(\mathcal{F})$ is computable from the standard congruences that characterize the image of φ, namely $(n_K) \in \operatorname{Im}(\varphi)$ if and only if

$$\sum [NH : NH \cap NK]\mu(|K/H|)n_K \equiv 0 \mod |WH|$$

for all H, where the sum runs over the NH-conjugacy classes of groups K such that $H \subset K \subset NH$ and K/H is cyclic and where $\mu(|K/H|)$ is the number of generators of

K/H (e.g. [17 or 33, V.2.11]). Note that the Möbius function μ has the properties that $\mu(p^k)$ is not divisible by p^k and $\mu(rs) = \mu(r)\mu(s)$ if $(r,s) = 1$. We display a few examples.

EXAMPLES 21.5: (i) If G is a p-group and $\mathcal{F} \subset \mathcal{P}$, then p divides $n(\mathcal{F})$.

(ii) If $\mathcal{F}\langle p\rangle$ is the family of p-groups in a group which is not a p-group, then p need not divide $n(\mathcal{F}\langle p\rangle)$.

(iii) Let $\mathcal{P}$ be the family of proper subgroups of G. Then $q\widetilde{e}_{\mathcal{P}} \in A(G)$ if and only if $\mu(|G/H|)q \equiv 0 \mod |G/H|$ for all normal subgroups H such that G/H is cylic. It follows that a prime p divides $n(\mathcal{P})$ if and only if the cyclic group of order p is a quotient of G. If G is cyclic, then p divides $n(\mathcal{P})$ if and only if p divides $|G|$. In contrast, if G is perfect, then no primes divide $n(\mathcal{P})$; that is, $n(\mathcal{P}) = 1$. If G is the symmetric group on three letters then $n(\mathcal{P}) = 2$. More generally, if G is a non-Abelian group of order pq with $p < q$, then $n(\mathcal{P}) = p$.

§22. The generalized Tate Atiyah-Hirzebruch spectral sequences

We begin by explaining how to realize topologically the chain level algebraic description of Amitsur-Dress-Tate cohomology theories that we gave in the previous section. We then obtain generalized Tate, Borel, and free Atiyah-Hirzebruch spectral sequences. The discussion is precisely parallel to that in Sections 9, 10, and 11. Unless otherwise specified, we assume that G is finite.

Fix a model for $E\mathcal{F}$ as an $\mathcal{F}$-CW complex with finite skeleta $E\mathcal{F}^p$, such as that specified in the previous section. Let $E\mathcal{F}^p$ be empty for $p < 0$ and filter $E\mathcal{F}_+$ by the $E\mathcal{F}^p_+$. Recall that $\widetilde{E}\mathcal{F} = S^0 \cup C(E\mathcal{F}_+)$ and, as in (9.5), give it, or rather its suspension G-spectrum , the "filtration"

$$\widetilde{E}\mathcal{F}^p = \begin{cases} S^0 \cup C(E\mathcal{F}^{p-1}_+) & \text{if } p > 0 \\ S^0 & \text{if } p = 0 \\ D(\widetilde{E}\mathcal{F}^{-p}) & \text{if } p < 0. \end{cases} \tag{22.1}$$

For $p \leq 0$, the implicit maps are duals of inclusions. We use the telescope construction to convert this to an actual filtration of a G-spectrum equivalent to $\widetilde{E}\mathcal{F}$ without change of notation. Replacing $\widetilde{E}\mathcal{F}^p$ by S^0 for $p > 0$, we obtain a compatible filtration of the sphere spectrum S^0.

For each integer p, $\widetilde{E}\mathcal{F}^p/\widetilde{E}\mathcal{F}^{p-1}$ is a wedge of G-spectra of the form $G/H_+ \wedge S^p$. This would fail if we attempted to generalize to infinite compact Lie groups as in (9.5), and for this reason it seems unreasonable to seek chain level descriptions of the Tate theories of Definition 20.7 in that generality. For $p > 1$, this quotient is $\Sigma E\mathcal{F}^{p-1}/E\mathcal{F}^{p-2}$. For $p = 1$, it is $\Sigma E\mathcal{F}^0_+$. For $p \leq 0$, it is $\Sigma D(\widetilde{E}\mathcal{F}^{1-p}/\widetilde{E}\mathcal{F}^{-p})$. When interpreting chains as in Definition 21.1, it is important to notice that applying $\underline{\pi}_*$ in non-positive degrees is the same as applying $\underline{\pi}_* \circ D$ in positive degrees.

REMARK 22.2: Exactly as in Remark 9.7, the filtration on $\widetilde{E}\mathcal{F}$ is not the skeletal filtration of a structure of $\mathcal{F}$-CW spectrum. An $\mathcal{F}$-CW spectrum X is equivalent to $E\mathcal{F}_+ \wedge X$, but $E\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}$ is trivial.

Of course, the filtration is cooked up so as to realize the $\mathbb{Z}$-graded cochain complex displayed in Definitions 21.1. Formally, if we define chain complexes of Mackey functors by

(22.3) $$P^+(\mathcal{F}) \equiv \underline{C}_*(\underline{E\mathcal{F}_+}),\ P^-(\mathcal{F}) \equiv \Sigma^{-1}\underline{C}_*(\underline{S}^0),\ \text{and } P(\mathcal{F}) \equiv \Sigma^{-1}\underline{C}_*(\underline{\widetilde{E}\mathcal{F}}),$$

then we have a geometrically realized short exact sequence

$$0 \to P^-(\mathcal{F}) \to P(\mathcal{F}) \to P^+(\mathcal{F}) \to 0.$$

Application of (20.1) and (20.3) to (20.4) and (20.11) shows that the complexes used in Definitions 21.1 are realized in terms of the complexes in (22.3). Think of M_* as $M \circ D$ and recall Remarks 20.6 and 20.10.

PROPOSITION 22.4. *Let M be a Mackey functor. Then the Amitsur-Dress homology, cohomology, and Tate groups are given by*

$$H_*^{\mathcal{F}}(M) \cong H_*(P^+(\mathcal{F}) \otimes_{\mathcal{O}\mathcal{S}} M) = H_*^G(E\mathcal{F}_+; M),$$

$$H^*_{\mathcal{F}}(M) \cong H^*(\mathrm{Hom}_{\mathcal{O}\mathcal{S}}(P^+(\mathcal{F}), M)) = H^*_G(E\mathcal{F}_+; M),$$

and

$$\widehat{H}^*_{\mathcal{F}}(M) \cong H^*(\mathrm{Hom}_{\mathcal{O}\mathcal{S}}(P(\mathcal{F}), M)).$$

In turn, the following analog of Theorem 9.8 shows that the right-hand groups in the proposition are among the represented groups that we specified in Definition 20.7. (The duplicative notations there anticipated this consistency result.) The proof will be immediate from the spectral sequences below.

THEOREM 22.5. *Let G be finite and let M be a Mackey functor and N be a coMackey functor. Then there are canonical isomorphisms*

$$H_n^{\mathcal{F}}(N) \equiv f_{\mathcal{F}}(HN)_n \cong H_n(\underline{E\mathcal{F}_+}; N) \quad \text{and} \quad H^n_{\mathcal{F}}(M) \equiv c_{\mathcal{F}}(HM)^n \cong H^n(\underline{E\mathcal{F}_+}; M);$$

$$\check{H}_n^{\mathcal{F}}(N) \equiv c_{\mathcal{F}}(HN)_n \cong H_n(\underline{S}^0; N) \quad \text{and} \quad \check{H}^n_{\mathcal{F}}(M) \equiv f_{\mathcal{F}}(HM)^n \cong H^n(\underline{S}^0; M);$$

$$\widehat{H}_n^{\mathcal{F}}(N) \equiv t_{\mathcal{F}}(HN)_n \cong H_n(\underline{\widetilde{E}\mathcal{F}}; N) \quad \text{and} \quad \widehat{H}^n_{\mathcal{F}}(M) \equiv t_{\mathcal{F}}(HM)^n \cong H^{n+1}(\underline{\widetilde{E}\mathcal{F}}; M).$$

For chain complexes $\underline{C}$ of Mackey functors, we can define $\mathcal{F}$-Tate-Swan homology theories with coefficients in coMackey functors and $\mathcal{F}$-Tate-Swan cohomology theories with coefficients in Mackey functors exactly as in Definition 11.2. The tensor products $P^{\pm} \otimes C$ with diagonal G-action used there must here be replaced by (graded) tensor products of Mackey functors $P^{\pm} \otimes \underline{C}$, as specified in (8.1). The functors $\otimes_G$ and Hom_G used there must be replaced by the functors $\otimes_{\mathcal{O}S}$ and $\mathrm{Hom}_{\mathcal{O}S}$. We prefer not to go into detail. In order to tie this to the purely cellular approach discussed below, one need only verify the equivariant Künneth isomorphism theorem showing that the chain complex Mackey functor of the smash product of suitably filtered G-spectra is isomorphic to the tensor product of their chain complexes, as specified in (8.1).

Recall the discussion at the start of Section 10. When filtering X, we continue to restrict to finite groups G. Here the skeletal filtration of a G-CW structure is "$\mathcal{F}$-calculable" in the sense that the product filtrations on $E\mathcal{F}_+ \wedge X$, $S^0 \wedge X$, and $\widetilde{E}\mathcal{F} \wedge X$ satisfy the conditions specified on W at the end of Section 20. The spectral sequences of the following result are constructed from these filtrations. The dimension shift for $t_{\mathcal{F}}(k_G)^*$ comes from Proposition 17.6.

THEOREM 22.6. *Let G be finite and let X be a G-CW spectrum. Then there are spectral sequences with E_2-terms and targets:*

$$E^2_{p,q} = H_p(\underline{E\mathcal{F}_+ \wedge X}; \underline{k}^G_q) \Longrightarrow f_{\mathcal{F}}(k_G)_n(X); \quad E_2^{p,q} = H^p(\underline{E\mathcal{F}_+ \wedge X}; \underline{k}^q_G) \Longrightarrow c_{\mathcal{F}}(k_G)^n(X);$$
$$E^2_{p,q} = H_p(\underline{S^0 \wedge X}; \underline{k}^G_q) \Longrightarrow c_{\mathcal{F}}(k_G)_n(X); \qquad E_2^{p,q} = H^p(\underline{S^0 \wedge X}; \underline{k}^q_G) \Longrightarrow f_{\mathcal{F}}(k_G)^n(X);$$
$$E^2_{p,q} = H_p(\underline{\tilde{E}\mathcal{F} \wedge X}; \underline{k}^G_q) \Longrightarrow t_{\mathcal{F}}(k_G)_n(X); \quad E_2^{p,q} = H^{p+1}(\underline{\tilde{E}\mathcal{F} \wedge X}; \underline{k}^q_G) \Longrightarrow t_{\mathcal{F}}(k_G)^n(X).$$

If X is bounded below, the top two spectral sequences are potentially convergent and their $E^2_{p,q}$ and $E_2^{p,q}$ terms are isomorphic to

$$H_p^{\mathcal{F}}(X; \underline{k}^G_q) \equiv f_{\mathcal{F}}(H\underline{k}^G_q)_p(X) \quad \textit{and} \quad H^p_{\mathcal{F}}(X; \underline{k}^q_G) \equiv c_{\mathcal{F}}(H\underline{k}^q_G)^p(X).$$

If X is finite, the remaining four spectral sequences are potentially convergent and their $E^2_{p,q}$ and $E_2^{p,q}$ terms are isomorphic to

$$H_p^{\mathcal{F}}(X;\underline{k}_q^G) \equiv c_{\mathcal{F}}(H\underline{k}_q^G)_p(X) \qquad \text{and} \qquad \check{H}^p_{\mathcal{F}}(X;\underline{k}^q_G) \equiv f_{\mathcal{F}}(H\underline{k}^q_G)^p(X);$$
$$\widehat{H}_p^{\mathcal{F}}(X;\underline{k}_q^G) \equiv t_{\mathcal{F}}(H\underline{k}_q^G)_p(X) \qquad \text{and} \qquad \widehat{H}^p_{\mathcal{F}}(X;\underline{k}^q_G) \equiv t_{\mathcal{F}}(H\underline{k}^q_G)^p(X).$$

The proof is identical to that of Theorem 10.3. In particular, the evident generalizations of Lemmas 10.4 and 10.8 are valid. In view of Proposition 17.7, we also have the following analog of Theorem 10.5, in which G can be any compact Lie group.

THEOREM 22.7. *In the cases of $f_{\mathcal{F}}$ and $t_{\mathcal{F}}$, assume that $\mathcal{F}$ contains only finitely many maximal conjugacy classes (H). Then the Postnikov tower of k_G gives rise to conditionally convergent spectral sequences with respective E_2-terms and targets:*

$$E_2^{p,q} = \check{H}^p_{\mathcal{F}}(X;\underline{k}^q_G) \equiv f_{\mathcal{F}}(H\underline{k}^q_G)^p(X) \implies f_{\mathcal{F}}(k_G)^n(X)$$
$$E_2^{p,q} = H^p_{\mathcal{F}}(X;\underline{k}^q_G) \equiv c_{\mathcal{F}}(H\underline{k}^q_G)^p(X) \implies c_{\mathcal{F}}(k_G)^n(X)$$
$$E_2^{p,q} = \widehat{H}^p_{\mathcal{F}}(X;\underline{k}^q_G) \equiv t_{\mathcal{F}}(H\underline{k}^q_G)^p(X) \implies t_{\mathcal{F}}(k_G)^n(X)$$

Moreover, there are natural external pairings of spectral sequences in each case. If k_G is a ring G-spectrum and X is a G-space, then the $c_{\mathcal{F}}$ and $t_{\mathcal{F}}$ spectral sequences are spectral sequences of differential algebras.

When k_G is bounded below, these spectral sequences are certainly relevant. They are then lower half-plane spectral sequences and thus converge strongly if only finitely many higher differentials defined on any given bidegree are non-zero. The proof of Theorem 10.6 applies to give the following comparison of our two triples of cohomology spectral sequences.

THEOREM 22.8. *Let G be finite and let X be a G-CW spectrum. If X is bounded below, then the two spectral sequences for the calculation of $c_{\mathcal{F}}(k_G)^*(X)$ are isomorphic. If X is finite, then the two spectral sequences for the calculation of $f_{\mathcal{F}}(k_G)^*(X)$ and of*

$t_{\mathcal{F}}(k_G)^*(X)$ are isomorphic. Under the specified bounded below or finiteness hypothesis, if the isomorphic spectral sequences satisfy condition (ω), then the spectral sequence derived from the Postnikov tower is relevant.

Finally, we have analogs of Proposition 11.3 and its corollaries.

PROPOSITION 22.9. *Let G be finite. Then multiplication by $n(\mathcal{F})$ annihilates the E_2-terms of the $\mathcal{F}$-Tate homology and cohomology spectral sequences. If X is finite and $\underline{k}_G^*$ is of finite type, then these E_2 terms are finite in each bidegree, hence the spectral sequences converge strongly if k_G is also bounded below.*

PROOF: One can do enough algebra to mimic the proof of Proposition 11.3, using the Künneth theorem that we cited above. Note that Corollary 21.4 gives the result directly when $X = S^0$ and, at least when k_G is bounded below, the result follows in general by pairing the spectral sequence for k_G and X with the spectral sequence for S_G and S^0.

COROLLARY 22.10. *If G is finite and q is prime to $n(\mathcal{F})$, then q acts invertibly on $t_{\mathcal{F}}(k_G)$ for any G-spectrum k_G.*

§23. Some calculational methods and examples: groups of order pq

Let G be finite until otherwise specified. By Proposition 22.4, the Amitsur-Dress homology and cohomology groups of a family $\mathcal{F}$ with coefficients in a Mackey functor M can be described as follows in terms of Bredon homology and cohomology:

$$H_*^{\mathcal{F}}(M) \cong H_*^G(E\mathcal{F}_+; M) \quad \text{and} \quad H^*_{\mathcal{F}}(M) \cong H^*_G(E\mathcal{F}_+; M). \tag{23.1}$$

Of course, Bredon homology and cohomology are themselves quite difficult to compute in general. However, calculations can sometimes be reduced to calculations in the classical homology and cohomology of groups.

As a first example, let N be a normal subgroup of G, let $J = G/N$, and let $\mathcal{F}(N)$ be the family of subgroups of N. Regarded as a G-space by pullback along the quotient map $G \to J$, EJ is a model for $E\mathcal{F}(N)$.

Proposition 23.2. *For any Mackey functor M*

$$H_*^G(EJ_+; M) \cong H_*^J(M(G/N)), \quad H^*_G(EJ_+; M) \cong H^*_J(M(G/N)),$$

and

$$\widehat{H}^*_{\mathcal{F}(N)}(M) \cong \widehat{H}^*_J(M(G/N)),$$

where the right sides are the classical homology, cohomology, and Tate cohomology groups of the J-module $M(G/N)$.

Proof: This is immediate by inspection on the chain and cochain level, starting from (20.4). The point is that, because the restriction maps relating the $C_*((EJ_+)^H)$ for $H \subset N$ are identity maps,

$$\underline{C}_*(EJ_+) \otimes_{\mathcal{OS}} M_* \cong C_*(EJ_+) \otimes_J M(G/N)$$

and

$$\mathrm{Hom}_{\mathcal{OS}}(\underline{C}_*(EJ_+), M^*) \cong \mathrm{Hom}_J(C_*(EJ_+), M(G/N)).$$

For the Tate isomorphism, we simply check that these chain and cochain isomorphisms splice together properly.

This observation can sometimes serve as the starting point for an induction up orbit types. For families $\mathcal{F}' \subset \mathcal{F}$, write $E(\mathcal{F},\mathcal{F}') = E\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}'$. Then the fixed point space $(E(\mathcal{F},\mathcal{F}'))^H$ is equivalent to S^0 if $H \in \mathcal{F} - \mathcal{F}'$ and is contractible otherwise. Moreover, $E(\mathcal{F},\mathcal{F}')$ is characterized up to G-homotopy type by this behavior on fixed point spaces. Smashing the space $E\mathcal{F}_+$ with the cofibration sequence $E\mathcal{F}'_+ \to S^0 \to \widetilde{E}\mathcal{F}'$ and noting that $E\mathcal{F}_+ \wedge E\mathcal{F}'_+ \simeq E\mathcal{F}'_+$, we obtain a cofiber sequence

$$E\mathcal{F}'_+ \to E\mathcal{F}_+ \to E(\mathcal{F},\mathcal{F}').$$

Of course, the displayed cofibration sequence gives rise to long exact sequences of Bredon homology and cohomology groups. In view of (23.1), these sequences relate the Amitsur-Dress homology and cohomology groups of $\mathcal{F}'$ and $\mathcal{F}$. (We will record some general diagrams relating the norm sequences for $\mathcal{F}$ and $\mathcal{F}'$ at the end of the section.)

In order to exploit these sequences, we need a method for computing the Bredon homology and cohomology of $E(\mathcal{F},\mathcal{F}')$, and this is provided by special cases of our AHSS. In cohomology, Theorem 22.6 gives an AHSS converging from $H^*_{\mathcal{F}}(\underline{k}^*_G) = H^*_G(E\mathcal{F}_+;\underline{k}^*_G)$ to $c_{\mathcal{F}}(k_G)^*$, and we apply it to $k_G = F(\widetilde{E}\mathcal{F}', HM)$. The target of the resulting spectral sequence is

$$c_{\mathcal{F}}(k_G)^* \cong H^*_G(E(\mathcal{F},\mathcal{F}');M),$$

and the Mackey functor $\underline{k}^*_G$ used to compute its E_2-term is given by

$$\underline{k}^*_G(G/H) = \underline{k}^*_H = H^*_H(\widetilde{E}(\mathcal{F}'|H);M|H),$$

where $(M|H)(H/K) = M(G/K)$ for $K \subset H$. Clearly $\underline{k}^*_H = 0$ if $H \in \mathcal{F}'$, since $\widetilde{E}(\mathcal{F}'|H)$ is then contractible, hence E_2 depends only on the value of M on G/H for H in $\mathcal{F} - \mathcal{F}'$. It is natural to apply this with $\mathcal{F} - \mathcal{F}' = (H)$ for some conjugacy class (H) in $\mathcal{F}$, necessarily maximal. Here $\mathcal{F}'|H$ is the family of proper subgroups of H, and we have the following result.

PROPOSITION 23.3. *Let $\mathcal{F} - \mathcal{F}' = (H)$ and let $\mathcal{P}$ be the family of proper subgroups of H. Assume that $WH = e$. Then, for any Mackey functor M,*

$$H_*^G(E(\mathcal{F},\mathcal{F}');M) \cong H_*^H(\widetilde{E}\mathcal{P};M|H) \text{ and } H_G^*(E(\mathcal{F},\mathcal{F}');M) \cong H_H^*(\widetilde{E}\mathcal{P};M|H).$$

PROOF: Let $X = \coprod G/K$, where the union is taken over one K from each maximal conjugacy class $(K) \subset \mathcal{F}$; we may as well choose H as our representative in (H). Since $WH = e$, X^H is a point. This implies that $(EX)^H = E(X^H)$ is also a point. For any Mackey functor L such that $L(K) = 0$ for $K \in \mathcal{F}'$, we see by inspection on the chain level that

$$H_*^G(E\mathcal{F}_+;L) = H_0^G(E\mathcal{F}_+;L) = L(G/H) \text{ and } H_G^*(E\mathcal{F}_+;L) = H_G^0(E\mathcal{F}_+;L) = L(G/H).$$

The conclusion in cohomology follows immediately from the collapse of the AHSS just discussed, and the conclusion in homology follows similarly from the AHSS for the calculation of $f_{\mathcal{F}}(k_G)_*$, where $k_G = \widetilde{E}\mathcal{F}' \wedge HM$.

To calculate the right-hand homology and cohomology groups in Proposition 23.3, we can apply $H_*^H(?;M|H)$ and $H_H^*(?;M|H)$ to the cofibration sequence $E\mathcal{P}_+ \to S^0 \to \widetilde{E}\mathcal{P}$. By the dimension axiom in Bredon theory, these homology and cohomology groups on S^0 are $M(G/H)$ in degree 0 and zero in other degrees. By (23.1), these groups on $E\mathcal{P}_+$ are Amitsur-Dress homology and cohomology groups for H. Were it not for the very restrictive assumption $WH = e$ in the last proposition, these methods would in principle give a general procedure for computing the Amitsur-Dress homology and cohomology groups of families by a double induction on the order of groups and on orbit types.

We collate these results to describe the Amitsur-Dress homology and cohomology groups of the family $\mathcal{P}$ of proper subgroups in a non-Abelian group G of order pq, where p and q are primes with $p < q$. Let N be the unique (hence normal) subgroup of G of order q, let $J = G/N$, and let H be one of the q conjugate subgroups of order p. Let M be any Mackey functor. We take $\mathcal{F}' = \mathcal{F}(N)$ and $\mathcal{F} = \mathcal{P}$ in the discussion above. Proposition

23.2 applies to the family $\mathcal{F}(N)$. Certainly $WH = e$, hence Proposition 23.3 applies to the space $E(\mathcal{P}, \mathcal{F}(N))$. Moreover, since the only proper subgroup of H is the trivial group, the Bredon homology and cohomology groups of $E(\mathcal{P}, \mathcal{F}(N))$ reduce to classical H-homology and cohomology groups. Therefore we can calculate the Amitsur-Dress groups $H_*^{\mathcal{P}}(M)$ and $H_{\mathcal{P}}^*(M)$ in terms of the classical homology and cohomology groups of the J-module $M(G/N)$ and of the H-module $M(G/e)$.

PROPOSITION 23.4. *Retain the notations of the previous paragraph and define A_1, A_0, A^0 and A^1 by the exact sequences*

$$0 \longrightarrow A_1 \longrightarrow H_0^H(M(G/e)) \longrightarrow M(G/H) \longrightarrow A_0 \longrightarrow 0$$

and

$$0 \longrightarrow A^0 \longrightarrow M(G/H) \longrightarrow H_H^0(M(G/e)) \longrightarrow A^1 \longrightarrow 0.$$

Here the middle arrows are factors of the homomorphisms

$$\rho_* : M(G/e) \longrightarrow M(G/H) \text{ and } \rho^* : M(G/H) \longrightarrow M(G/e),$$

where ρ is the quotient G-map $G/e \to G/H$. Then there are long exact sequences

$$\begin{aligned}\cdots \to H_{n+1}^{\mathcal{P}}(M) \to H_n^H(M(G/e)) \to H_n^J(M(G/N)) \to H_n^{\mathcal{P}}(M) \to \cdots \\ \to H_1^{\mathcal{P}}(M) \to A_1 \to H_0^J(M(G/N)) \to H_0^{\mathcal{P}}(M) \to A_0 \to 0\end{aligned}$$

and

$$\begin{aligned}0 \to A^0 \to H_{\mathcal{P}}^0(M) \to H_J^0(M(G/N)) \to A^1 \to H_{\mathcal{P}}^1(M) \to \cdots \\ \to H_{\mathcal{P}}^n(M) \to H_J^n(M(G/N)) \to H_H^n(M(G/e)) \to H_{\mathcal{P}}^{n+1}(M) \to \cdots .\end{aligned}$$

Of course, for $n > 0$, $\widehat{H}_{\mathcal{P}}^{-n-1}(M) = H_n^{\mathcal{P}}(M)$ and $\widehat{H}_{\mathcal{P}}^n(M) = H_{\mathcal{P}}^n(M)$. Because the groups $H_0^{\mathcal{F}}(M)$ vary covariantly with $\mathcal{F}$ while the groups $H_{\mathcal{F}}^0(M)$ vary contravariantly, it is not obvious how to describe the groups $\widehat{H}_{\mathcal{P}}^n(M)$ for $n = -1$ and $n = 0$ in terms of other groups, but of course they are easily computed when M is given explicitly. It is relevant that the norm map for the J-theory of $M(G/N)$ factors as follows through the norm map for the $\mathcal{P}$-theory of M:

$$H_0^J(M(G/N)) \to H_0^{\mathcal{P}}(M) \to H^0_{\mathcal{P}}(M) \to H^0_J(M(G/N)).$$

In fact, it is clear from Definition 21.1 that there is a similar factorization of the norm map of $\mathcal{F}'$ through the norm map of $\mathcal{F}$ for any inclusion $\mathcal{F}' \subset \mathcal{F}$ of families in any finite group G.

We end this section with some general change of families diagrams. Here G can be any compact Lie group, and we assume given families $\mathcal{F}' \subset \mathcal{F}$ in G. Recall (A)–(D) in Section 17. We have a canonical map $E\mathcal{F}'_+ \to E\mathcal{F}_+$ over S^0, where S^0 should be thought of as $E\mathcal{A}\ell\ell_+$, and this gives a factorization $k_G \to c_{\mathcal{F}}(k_G) \to c_{\mathcal{F}'}(k_G)$ of the $\mathcal{F}'$-completion map. This implies the following factorization of Diagram (D) for $\mathcal{F}'$:

$$\begin{array}{ccccc}
f_{\mathcal{F}'}(k_G) & \longrightarrow & k_G & \longrightarrow & f^{\perp}_{\mathcal{F}'}(k_G) \\
\downarrow & & \downarrow & & \downarrow \\
f_{\mathcal{F}'}(c_{\mathcal{F}}(k_G)) & \longrightarrow & c_{\mathcal{F}}(k_G) & \longrightarrow & f^{\perp}_{\mathcal{F}'}(c_{\mathcal{F}}(k_G)) \\
\downarrow & & \downarrow & & \downarrow \\
f_{\mathcal{F}'}(k_G) & \longrightarrow & c_{\mathcal{F}'}(k_G) & \longrightarrow & t_{\mathcal{F}'}(k_G).
\end{array}$$

The map $E\mathcal{F}' \to E\mathcal{F}$ also induces a map from the cofibering (A) for $\mathcal{F}'$ to the cofibering (A) for $\mathcal{F}$. In turn, this induces a map from the entire diagram just displayed to the diagram

$$\begin{array}{ccccc}
f_{\mathcal{F}}(k_G) & \longrightarrow & k_G & \longrightarrow & f^{\perp}_{\mathcal{F}}(k_G) \\
\downarrow & & \downarrow & & \downarrow \\
f'_{\mathcal{F}}(k_G) & \longrightarrow & c_{\mathcal{F}}(k_G) & \longrightarrow & t_{\mathcal{F}}(k_G) \\
\downarrow & & \downarrow & & \downarrow \\
f_{\mathcal{F}}(c_{\mathcal{F}'}(k_G)) & \longrightarrow & c_{\mathcal{F}'}(k_G) & \longrightarrow & f^{\perp}_{\mathcal{F}}(c_{\mathcal{F}'}(k_G)).
\end{array}$$

These diagrams substitute for the evident lack of functoriality of $t_{\mathcal{F}}(k_G)$ in $\mathcal{F}$ that is caused by its simultaneous covariant and contravariant dependence on $\mathcal{F}$.

These diagrams are particularly interesting when $\mathcal{F}' = \{e\}$ and thus $c_{\mathcal{F}'}(k_G) = F(EG_+, k_G)$. The first diagram then shows that every family $\mathcal{F}$ gives rise to a factorization of our original Diagram (D). The bottom row of the second diagram suggests the use of particular families as a tool for the analysis of $F(EG_+, k_G)$. For example, with $\mathcal{F}$ taken to be the family of finite subgroups of G, this was the starting point of [24].

§24. Equivariant root invariants of stable homotopy groups of spheres

We specialize to consideration of $t_{\mathcal{F}}(S_G)$, where G is finite. As in Section 12, the generalized Segal conjecture and the results of the previous section give the following conclusion.

THEOREM 24.1. *Let G be finite and let X be a CW-complex with finite skeleta. Then*

$$t_{\mathcal{F}}(S_G)_*(X) \cong \left(\bigoplus_{(H)\not\subset\mathcal{F}} \pi_*(EWH_+ \wedge_{WH} X^H) \right)^{\wedge}_{I\mathcal{F}}$$

If X is finite, then there is an upper half-plane spectral sequence which converges strongly from $E^2_{s,t} = \widehat{H}^{\mathcal{F}}_s(X; \underline{\pi}_t(S_G))$ to $t_{\mathcal{F}}(S_G)_*(X)$ and a lower half-plane spectral sequence of differential algebras which converges strongly from $E_2^{s,t} = \widehat{H}^s_{\mathcal{F}}(X; \underline{\pi}^t(S_G))$ to $t_{\mathcal{F}}(S_G)^*(X)$.

We focus attention on the family $\mathcal{F} = \mathcal{P}$ of all proper subgroups of G since that seems the most interesting to us. Here, by Corollary 21.4, $E^2_{*,*}$ is annihilated by $n(\mathcal{P})$. By Examples 21.5(iii), the prime divisors of $n(\mathcal{P})$ are those primes p such that the cyclic group of order p is a quotient of G. We have $t_{\mathcal{P}}(S_G)_*(X) = \pi_*(X^G)^{\wedge}_{I\mathcal{P}}$, and the following observation shows that this completion is the product of the integral completions at the cited primes.

LEMMA 24.2. *Let M be a Mackey functor such that $M(G/H) = 0$ for all proper subgroups H. Then $M(G/G)^{\wedge}_{I\mathcal{P}} \cong M(G/G)^{\wedge}_{n(\mathcal{P})}$.*

PROOF: $A(G)$ acts on $M(G/G)$ through φ_G and the proof of Proposition 21.3 shows that the ideal of $\mathbb{Z}$ generated by the image of φ_G is $(n(\mathcal{P}))$.

Now restrict attention to the case $X = S^0$. Here the cohomology spectral sequence is a reindexing of the homology spectral sequence. We shall use homological indexing to conform with the literature. Thus, in the spectral sequence converging from $\widehat{H}^{\mathcal{P}}_s(\underline{\pi}_t(S_G))$ to $t_{\mathcal{P}}(S_G)_*$, E^2 is annihilated by $n(\mathcal{P})$ and the target is $(\pi_*)^{\wedge}_{n(\mathcal{P})}$. In comparison with the spectral sequences of Section 12, the homotopy groups of the classifying spaces BWH for non-trivial proper subgroups H have been shifted from the target to ingredients in the computation of the E^2-term. The previous section describes algebraic methods for the calculation of E^2.

However, what seems most interesting about the family $\mathcal{P}$ is the periodicity that is implicit in the structure of $\widetilde{E}\mathcal{P}$. We have already exploited this in our study of K-theory. Let V be the reduced regular representation of G. As noted in the proof of Lemma 19.11, $\mathcal{F}(V) = \mathcal{P}$. Much of what we will say applies to the family $\mathcal{F}(V)$ associated to any representation V, but we prefer to be specific. The cofibration sequence

$$S(\infty V)_+ \to D(\infty V)_+ \to S^{\infty V}$$

is a model for $E\mathcal{P}_+ \to S^0 \to \widetilde{E}\mathcal{P}$. Let $\dim V = d$ and let $n = d+1 = |G|$.

In analogy with Example 9.6, we give $\widetilde{E}\mathcal{P}$ the filtration

$$S^{sV} = \widetilde{E}\mathcal{P}^{sd} = \widetilde{E}\mathcal{P}^{sd+1} = \cdots = \widetilde{E}\mathcal{P}^{sd+(d-1)} \text{ for all integers } s. \tag{24.3}$$

We call this the "V-filtration" of $\widetilde{E}\mathcal{P}$. The subquotients are trivial except in degrees congruent to $0 \mod d$, where

$$\widetilde{E}\mathcal{P}^{(s+1)d}/\widetilde{E}\mathcal{P}^{(s+1)d-1} \cong (S^V/S^0) \wedge S^{sV} \cong \Sigma(S(V)_+) \wedge S^{sV}. \tag{24.4}$$

The filtration (24.3) gives rise to another spectral sequence, quite different from that of Theorem 24.1, that converges to $(\pi_*)^\wedge_{n(\mathcal{P})}$. We shall say something about the comparison between the two at the end of the section. By Propositions 17.2 and 17.6 and duality, we have

$$t_{\mathcal{P}}(S_G) \simeq F(\widetilde{E}\mathcal{P}, \Sigma E\mathcal{P}_+) \xrightarrow{\simeq} \operatorname{holim} F(S^{sV}, \Sigma E\mathcal{P}_+) \simeq \operatorname{holim} \Sigma E\mathcal{P}_+ \wedge S^{-sV}.$$

Note that $E\mathcal{P}_+$ may be thought of as $\Sigma^{-1}(S^{\infty V}/S^0)$. Define

$$Y_{-sd} = E\mathcal{P}_+ \wedge S^{-sV} \tag{24.5}$$

and let K_{-sd} be the fiber of the evident map $Y_{-sd} \to Y_{-(s-1)d}$. that is,

$$K_{-sd} = E\mathcal{P}_+ \wedge S^{-sV} \wedge \Sigma^{-1}(S^V/S^0) \simeq \Sigma^{-sV-1}(S^V/S^0). \tag{24.6}$$

The last equivalence holds because S^V/S^0 is a $\mathcal{P}$-space. Both Y_{-sd} and K_{-sd} are $(-sd-1)$-connected. The tower of ΣY's gives rise to our new spectral sequence, and $E^1_{s,t} = 0$ unless $s \equiv 0 \mod d$ when

$$E^1_{-sd,t} = \pi^G_{t-sd}(\Sigma K_{-sd}). \tag{24.7}$$

For brevity of notation, write $\hat{\pi}_*$ for

$$(\pi_*)^\wedge_{n(\mathcal{P})} \cong t_{\mathcal{P}}(s_G)_* \cong \lim \pi^G_*(\Sigma Y_{-sd}).$$

It is important to keep in mind that this isomorphism relates ordinary nonequivariant homotopy groups on the left with equivariant homotopy groups on the right. Similarly, write $\widehat{X}$ for the completion of a spectrum X at $n(\mathcal{P})$.

Define the Mahowald V-filtration on $\hat{\pi}_*$ by letting

$$M^{sd}_V(\hat{\pi}_q) = \mathrm{Ker}(\hat{\pi}_q \to \hat{\pi}_q(\Sigma Y_{-sd})). \tag{24.8}$$

We let $M^{sd+i}_V(\widehat{\pi}_q) = M^{sd}_V(\widehat{\pi}_q)$ for $0 < i < d$, in line with the filtration (24.3). By convergence, for any $\alpha \in \widehat{\pi}_q$, there is a unique s such that

$$\alpha \in M^{sd}_V(\widehat{\pi}_q) - M^{(s+1)d}_V(\widehat{\pi}_q). \tag{24.9}$$

We define the root invariant $R_V(\alpha)$ of α to be the set of all maps $\beta : S^{q-1} \to (K_{-(s+1)d})^G$ such that the following diagram commutes:

$$\begin{tikzcd}
 & (K_{-(s+1)d})^G \arrow[dr] & & \\
S^{q-1} \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & & (Y_{-(s+1)d})^G \arrow[r] & (Y_{-sd})^G \\
 & (S^{-1})^\wedge \arrow[ur] & &
\end{tikzcd} \tag{24.10}$$

Here $R_V(\alpha)$ should be regarded as a subset of $E^1_{-(s+1)d,q+(s+1)d}$. We write $|R_V(\alpha)| = sd$ when (24.9) holds. We have the following estimate on s, which is a generalized analog of a theorem of Jones [28] and Miller [39].

THEOREM 24.11. *For every $\alpha \in \hat{\pi}_q$, $|R_V(\alpha)|$ is at least qd. That is, $M_V^{sd}(\hat{\pi}_q) = \hat{\pi}_q$ for $s \leq q$. Therefore $E^\infty_{-sd,q+sd} = 0$ unless $s \geq q$.*

Thus the q-stem is generating equivariant homotopy classes of dimensions at least qd as its root invariants. It follows that E^∞ is concentrated in the wedge in the third octant of the (x, y)-plane specified by $-x \leq y \leq -x - (1/d)x$; this wedge is on or above the anti-diagonal $-x = y$ and to the left of the line through the origin with slope $-(d+1)/d$. Calculational exploration should be interesting but will be difficult. The root invariants lie in groups $\pi_*^G(\Sigma K_{-sd})$. In view of the cofibration $S^V \to S^V/S^0 \to S^1$, these groups lie in exact sequences whose other terms are of the form $\pi_t^G(S^{rV})$ for integers r and t. These are part of the $RO(G)$-graded stable homotopy groups of the zero sphere. Unfortunately, we have few calculations and little understanding of these groups.

We next relate the V-filtration spectral sequence that we have been discussing to the skeletal filtration Atiyah-Hirzebruch spectral sequence of Theorem 24.1. Of course, generalizing (22.1) and using the cellular approximation theorem, we can check that the skeletal filtrations of any two G-CW complexes that are G-homotopy equivalent to $S^{\infty V}$ give rise to the same spectral sequence, from E_2 on, for the computation of $(\pi_*)^\wedge_{n(\mathcal{P})}$.

Starting from a triangulation of $S(V)$ as a G-CW complex, we can obtain a triangulation of S^V as a G-CW complex with two vertices and with its remaining cells of the form $G/H \times e^i$ with H proper and $1 \leq i \leq d$ (think of e^i as $e^{i-1} \times e^1$). Inductively, we can triangulate S^{sV} as a G-CW complex that contains $S^{(s-1)V}$ as a subcomplex. Then S^{sV} is contained in the sd-skeleton of the resulting skeletal filtration of $S^{\infty V}$.

Clearly $\dim(V^H) = [G : H] - 1$. Let $c + 1$ be the minimum over $H \in \mathcal{P}$ of the indices $[G : H]$. Since $(S^V/S^0)^H$ is connected for all H and is a point for $H = G$ while S^{kV^H} is at least $(kc-1)$-connected for $H \in \mathcal{P}$,

$$S^{(k+1)V}/S^{kV} = S^{kV} \wedge (S^V/S^0)$$

is kc-connected in the sense that all of its fixed point spaces are kc-connected. Therefore we can start from $T(0) = S^0$ and inductively construct a G-CW complex $T(\infty) = \cup T(k)$ and compatible G-homotopy equivalences $T(k) \to S^{kV}$ such that $T(k+1)$ is obtained from $T(k)$ by attaching equivariant cells of dimension greater than kc. The resulting equivalence $T(\infty) \to S^{\infty V}$ maps the kc-skeleton of $T(\infty)$ to S^{kV}.

Thus we have two models, $S^{\infty V}$ and $T(\infty)$, for $\widetilde{E}\mathcal{P}$ with its skeletal filtration, together with maps

$$T(\infty)^{kc} \to S^{kV} \to (S^{\infty V})^{kd}$$

that are compatible as k varies. We also have inverse cellular G-homotopy equivalences between $S^{\infty V}$ and $T(\infty)$.

Starting from any model of $\widetilde{E}\mathcal{P}$ as a G-CW complex, we obtain analogs of the definitions in (24.8)–(24.10). (Indeed, we can obtain such analogs for any family $\mathcal{F}$.) Explicitly, let $W_{-s} = E\mathcal{P}_+ \wedge \widetilde{E}\mathcal{P}^{-s}$ and define the Mahowald $\mathcal{P}$-filtration on $\widehat{\pi}_*$ by

$$M^s_{\mathcal{P}}(\widehat{\pi}_q) = \mathrm{Ker}(\widehat{\pi}_q \to \widehat{\pi}_q(\Sigma W_{-s})). \tag{24.12}$$

By convergence, for any $\alpha \in \widehat{\pi}_q$ there is a unique s such that

$$\alpha \in M^s_{\mathcal{P}}(\widehat{\pi}_q) - M^{s+1}_{\mathcal{P}}(\widehat{\pi}_q). \tag{24.13}$$

Let J_{-s} be the fiber of the evident map $W_{-s} \to W_{-s-1}$. Then $E^1_{-s,t} = \pi^G_{t-s}(\Sigma J_{-s})$ in the spectral sequence of Theorem 24.1. Define the root invariant $R_{\mathcal{P}}(\alpha)$ of α to be the set of all maps $\beta : S^{q-1} \to (J_{-s-1})^G$ such that the following diagram commutes:

$$\begin{tikzcd}
 & (J_{-s-1})^G \arrow[dr] & & \\
S^{q-1} \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & & (W_{-(s-1)})^G \arrow[r] & (W_{-s})^G \\
 & (S^{-1})^\wedge \arrow[ur] & &
\end{tikzcd} \tag{24.14}$$

Here, to ensure independence of the choice of the skeletal filtration, $R_{\mathcal{P}}(\alpha)$ should be regarded as a subset of $E^2_{-s-1,q+s+1}$ (rather than E^1). Of course, as discussed at the beginning of the section, this E^2 term is an algebraically computable Amitsur-Dress-Tate cohomology group. Write $|R_{\mathcal{P}}(\alpha)| = s$ when (24.13) holds. Then Theorem 24.11 and our comparison of filtrations gives the following result.

COROLLARY 24.15. *If $|R_{\mathcal{P}}(\alpha)| = s$, then $rd \le |R_V(\alpha)| \le td$, where r is maximal such that $rd \le s$ and t is minimal such that $s \le tc$. If $|R_V(\alpha)| = sd$, then $sc \le |R_{\mathcal{P}}(\alpha)| < (s+1)d$. Therefore, for every $\alpha \in \widehat{\pi}_q$, $|R_V(\alpha)| \ge qc$. That is, $M^s_{\mathcal{P}}(\widehat{\pi}_q) = \widehat{\pi}_q$ for $s \le qc$. Thus, in the skeletal filtration spectral sequence, $E^\infty_{-s,q+s} = 0$ unless $s \ge qc$.*

Here E^∞ is concentrated in the wedge in the third octant of the (x, y)-plane specified by $-x \le y \le -x - (1/c)x$; that is, the wedge is on or above the anti-diagonal $-x = y$ and to the left of the line through the origin with slope $-(c+1)/c$.

EXAMPLES 24.16: Obviously $c = d$ if and only if $\mathcal{P} = \{e\}$, that is, if and only if $G = \mathbb{Z}/p\mathbb{Z}$ for some prime p. Here $c = d = p - 1$. If $G = \mathbb{Z}/2\mathbb{Z}$, then we have only one filtration in sight (since $S(V) = G$) and we recover exactly the root invariant theorem of Jones and Miller. However, our proof is technically different since we shall not use the quadratic construction. (We have not explored the version of cup squares that is implicit in our argument.) If $G = \mathbb{Z}/p\mathbb{Z}$, p odd, then we have four filtrations and concomitant spectral sequences in sight, namely:

(a) The V-filtration of (24.3), with $d = p - 1$.

(b) The analogous V'-filtration, with $d = 2$, where V' is any nontrivial irreducible real representation of G; the evident analog of Theorem 24.11 is valid, by essentially the same proof.

(c) The skeletal filtration of (22.1), or any of its equivalents.

(d) The L filtration studied by Miller [39] and Sadofsky [43].

By construction and by ($\mathbb{L}$) in the introduction, the L-filtration is a p-local wedge summand of the G-fixed points of the skeletal filtration. The estimate of Corollary 24.15 is in essential agreement with that of Miller, although his refers a priori to a different filtration. We verified above that the V-filtration is equivalent to the restriction of the skeletal filtration to levels congruent to zero mod $p-1$, and the same argument shows that the V'-filtration is equivalent to the restriction of the skeletal filtration to levels congruent to zero mod 2. The V'-filtration refines the V-filtration since V is the direct sum of the $(p-1)/2$ distinct nontrivial irreducible real representations V' and $S^{V'}$ is p-locally G-homotopy equivalent to $S^{V''}$ for any two such representations V' and V'' (by a standard consideration of r^{th} power maps relating distinct irreducible 1-dimensional complex representations).

EXAMPLES 24.17: (i) For any p-group G of order p^r, $d = p^r - 1$ and $c = p - 1$. The sequences $G_r = \mathbb{Z}/p^r\mathbb{Z}$ and $H_r = (\mathbb{Z}/p\mathbb{Z})^r$ give rise to two sequences of spectral sequences all of which converge to $(\pi_*)^{\wedge}_p$. Perhaps a sharper version of Corollary 24.15 holds. If not, it would seem that the V-filtration and skeletal filtration diverge more and more as r increases.

(ii) For a non-abelian group G of order pq, where p and q are primes with $p < q$, $d = pq-1$ and $c = p-1$. We again obtain two spectral sequences that converge to $(\pi_*)^{\wedge}_p$. The E_2-term of the skeletal filtration spectral sequence was studied in the previous section. Both spectral sequences seem to retain some dependence on q. This raises the possibility that the p-primary stable homotopy groups of spheres may be influenced calculationally by interaction with other primes through information encoded in the structure of finite groups.

§25. Proof of the root invariant theorem

The proof of Theorem 24.11 uses duality and an elementary smash power construction. To set up the duality, define

$$Y_{-sd}^{(n-s)d-1} = \Sigma^{-1}(S^{nV}/S^0) \wedge S^{-sV} \tag{25.1}$$

for any integers n and s; this is a G-spectrum that is triangulable as a finite G-CW spectrum with cells in dimensions from $-sd$ to $(n-s)d-1$. For $n > 0$, the G-spectrum $\Sigma^{-1}(S^{nV}/S^0)$ can be replaced by the G-space $S(nV)_+ \subset E\mathcal{P}_+$, and the canonical map

$$S^0 \to t_{\mathcal{P}}(S_G) \simeq F(S^{\infty V}, \Sigma E\mathcal{P}_+) \to \Sigma Y_{-sd}$$

factors through $\Sigma Y_{-sd}^{(n-s)d-1}$ for n sufficiently large.

LEMMA 25.2. *The Spanier-Whitehead dual of* $Y_{-sd}^{(n-s)d-1}$ *is* $\Sigma Y_{(s-n)d}^{sd-1}$.

PROOF: This is an exercise from the facts that the dual of a cofibration sequence is a cofibration sequence, the dual of a smash product is the smash product of the duals, and the dual of S^W is S^{-W} for any W.

The following definition sets up the smash power construction.

DEFINITION 25.3. *Define an injection $\tau : G \to \Sigma_n$, where $n = |G|$, by fixing an ordering of the elements of G. Define $\Delta_\tau = (1 \times \tau)\Delta : G \to G \times \Sigma_n$. Let X be a naive G-spectrum. The n-fold external smash power $X^{(n)}$ is a $(G \times \Sigma_n)$-spectrum indexed on $(\mathbb{R}^\infty)^n$, where $(\mathbb{R}^\infty)^n$ is regarded as a G-trivial $G \times \Sigma_n$-universe [33, p.344]. By pullback along Δ_τ, we regard $X^{(n)}$ as a genuine G-spectrum indexed on the complete G-universe W^∞, where $W \cong V \oplus \mathbb{R}$ is the regular representation of G.*

LEMMA 25.4. *Let K be a based G-space and X be a naive G-spectrum.*

(i) $(\Sigma_G^\infty K)^{(n)} \cong \Sigma_G^\infty(K^{(n)})$, *where Σ_G^∞ denotes the naive suspension G-spectrum functor on the left and the genuine suspension G-spectrum functor on the right and where G acts through Δ_τ on $K^{(n)}$.*

(ii) $(S^r)^{(n)} \cong S^{rW} \cong \Sigma^r S^{rV}$ *for any integer* r.

(iii) $K^{(n)} \wedge X^{(n)} \cong (K \wedge X)^{(n)}$.

(iv) *Via (iii), the reduced diagonal* $\Delta : K \to K^{(n)}$ *induces a* G*-map*

$$\psi : K \wedge X^{(n)} \to (K \wedge X)^{(n)}.$$

PROOF: All but (ii) are easy to check from the explicit prespectrum level definitions and the general procedures described in Section 0. The case $r \geq 0$ of (ii) follows from (i) and the case $-r$, $r > 0$, follows:

$$\Sigma^{rW}(S^{-r})^{(n)} = S^{rW} \wedge (S^{-r})^{(n)} \cong (S^r)^{(n)} \wedge (S^{-r})^{(n)} \cong S^0.$$

Now we have the following analog of Miller's version [39] of a key observation of Jones [28]. Our version is actually simpler than theirs since the construction just given is considerably simpler than the extended power construction of [33]. The reader is asked to bear with our unwillingness to introduce different notations for space level, spectrum level (= naive G-spectrum level), and genuine G-spectrum level spheres S^q; we let the names of maps dictate the intended sense of the notation.

LEMMA 25.5. *Let* $\alpha : S^q \to S^0$ *be a map of nonequivariant spectra and let* $\widetilde{\alpha} : S^q \to S^0$ *be the map of genuine* G*-spectra that it induces. Then for any integer* s *and for* r *sufficiently large (depending on* α *and* s*), there is a commutative diagram of* G*-spectra*

$$\begin{tikzcd} \Sigma^q Y_{rd}^{sd-1} \arrow[r] \arrow[d, "\delta"'] & \Sigma^q Y_{qd}^{(r+s+q)d-1} \arrow[d] \\ S^q \arrow[r, "\widetilde{\alpha}"] & S^0, \end{tikzcd}$$

in which δ *is the* $(q-1)^{st}$ *suspension of the dual of the canonical map* $S^{-1} \to Y_{-sd}^{rd-1}$.

PROOF: Choose r such that $r + s \geq 0$ and there is a map $\overline{\alpha} : S^{q+r} \to S^r$ of spaces which represents α. We can take $\widetilde{\alpha}$ to be the map

$$\overline{\alpha} \wedge 1 : S^q \cong S^{q+r} \wedge S^{-r} \to S^r \wedge S^{-r} \cong S^0,$$

where S^{-r} is the genuine $(-r)$-sphere G-spectrum. More conceptually, and equivalently, $\widetilde{\alpha} = i_*\alpha$. By (ii) of the previous lemma, the apex of the following naturality diagram is equivalent to $\Sigma^q S^{-rV}$ and its middle right term is equivalent to $\Sigma^q S^{qV}$.

$$\begin{tikzcd}
 & & S^{q+r} \wedge (S^{-r})^{(n)} \arrow[dll, "1\wedge\psi"'] \arrow[dd, "\tilde{\alpha}\wedge 1"] \arrow[drr, "\psi"] & & \\
S^q \wedge (S^r \wedge S^{-r})^{(n)} \arrow[d, "\cong"'] \arrow[ddrr, "\alpha\wedge 1"] & & & & (S^{q+r} \wedge S^{-r})^{(n)} \arrow[d, "\cong"] \arrow[ddll, "(\tilde{\alpha}\wedge 1)^{(n)}"'] \\
S^q \arrow[d, "\tilde{\alpha}"'] & & S^r \wedge (S^{-r})^{(n)} \arrow[d, "\psi"] & & (S^q)^{(n)} \arrow[d, "(\alpha)^{(n)}"] \\
S^0 \arrow[rr, "\cong"] & & (S^r \wedge S^{-r})^{(n)} & & (S^0)^{(n)} \arrow[ll, "\cong"']
\end{tikzcd}$$

Take the smash product of this diagram with $S((r+s)V)_+$. The apex then becomes $\Sigma^q Y_{-rd}^{sd-1}$ and the middle right term becomes $\Sigma^q Y_{qd}^{(r+s+q)d-1}$. The map from the resulting diagram into the displayed diagram that is induced by $\varepsilon : S((r+s)V)_+ \to S^0$ gives the diagram we want. The identification of δ is a diagram chase (and is a special case of a precise form of equivariant Atiyah duality [33, III.5.1]).

Proof of Theorem 24.11: Let $s \leq q$. We must show that the composite of $\alpha : S^{q-1} \to S^{-1}$ and the canonical map $S^{-1} \to (Y_{-sd})^G$ is null homotopic. Let $\widetilde{\alpha} : S^{q-1} \to S^{-1}$ be the map of sphere G-spectra induced by α. It is equivalent to show that the composite of $\widetilde{\alpha}$ and the canonical map of G-spectra $S^{-1} \to Y_{-sd}$ is null homotopic, and this will hold if and only if the composite

$$S^{q-1} \to S^{-1} \to Y_{-sd}^{(n-s)d-1}$$

is null homotopic for n sufficiently large. Since maps between spheres are self-dual, we see by dualizing and applying Lemma 24.2 that it is equivalent to show that the dual composite

$$\Sigma Y_{(s-n)d}^{sd-1} \to S^1 \to S^{1-q}$$

is null homotopic. In turn, Lemma 24.5 (desuspended by $q-1$) shows that this composite is equal to the composite

$$\Sigma Y^{sd-1}_{(s-n)d} \to \Sigma Y^{(n+q)d-1}_{qd} \to S^{1-q}.$$

Recalling (24.1), we see that the first of these maps is just

$$S^{nV}/S^0 \wedge S^{(s-n)V} \to S^{nV}/S^0 \wedge S^{qV}.$$

We may replace S^{nV}/S^0 by $\Sigma S(nV)_+$. Smashing with $S^{-1} \wedge S^{(n-s)V}$, we find that it suffices to prove that

$$[S(nV)_+, S(nV)_+ \wedge S^{(n+q-s)V}]_G = 0 \quad \text{for } s \leq q.$$

This follows trivially from the exact sequence obtained by use of the following cofibration in the domain variable:

$$\Sigma^{-1} S^{nV} \to S(nV)_+ \to S^0.$$

Appendix A: Splittings of rational G-spectra for finite groups G

We here give an algebraic analysis of the rational equivariant stable category for finite groups G, including the following analog of a standard nonequivariant fact. Write $K(M,n) = \Sigma^n HM$ for a Mackey functor M.

THEOREM A.1. *Let G be finite. Then, for rational G-spectra X, there is a natural equivalence* $X \to \prod K(\underline{\pi}_n(X), n)$.

There is something to prove here. Counterexamples of Triantafillou [46] show that, unless G is cyclic of prime power order, the conclusion is false for naive G-spectra . A counterexample of Haeberly [27] shows that the conclusion is also false for genuine G-spectra when G is the circle group, with the rationalization of KU_G furnishing a counterexample.

The proof depends on two facts, one purely algebraic and the other topological. The first is implicit in Slominska [44] and will be proven shortly. We tacitly assume that G is finite in the rest of this appendix.

PROPOSITION A.2. *In the Abelian category of rational Mackey functors, all objects are projective and injective.*

In contrast, the global projective dimension of rational coefficient systems is one if G is cyclic of prime power order and at least two otherwise [46]. A counterexample of Haeberly [27] shows that the conclusion is also false when G is the circle group, but the first author has shown that in this case all Mackey functors have injective dimension either zero or one. The following result is also false for general compact Lie groups.

PROPOSITION A.3. *For* $H \subset G$ *and* $n \neq 0$, $\underline{\pi}_n(G/H_+) \otimes \mathbb{Q} = 0$.

PROOF: Working rationally, we have the following standard isomorphisms:

$$\begin{aligned}\underline{\pi}_n(G/H_+)(G/K) &= \pi_n((\Sigma^\infty_G G/H_+)^K)\\ &\cong \oplus \pi_n(EWL \times_{WL} (G/H)^L)_+\\ &\cong \oplus H_n(WL; \mathbb{Z}[(G/H)^L]),\end{aligned}$$

where the sum runs over the conjugacy classes of subgroups L of K. For the first, see e.g. [16] or [33, V.9.1]. The second holds since G/H is discrete and the evident spectral sequences collapse. For $n > 0$, these groups are annihilated by the orders of the $WL = NL/L$, and the conclusion follows.

Let $\mathcal{M}[G]$ denote the Abelian category of Mackey functors over G. For G-spectra X and Y, there is an evident natural map

$$\vartheta : [X, Y]_G \to \prod \operatorname{Hom}_{\mathcal{M}[G]}(\underline{\pi}_n(X), \underline{\pi}_n(Y)).$$

Since a Mackey functor is an additive contravariant functor $\mathcal{OS} \to \mathcal{A}b$, the previous result and the Yoneda lemma give that ϑ is an isomorphism when $X = \Sigma_G^\infty G/H_+$ for any H. Throwing in suspensions, we can extend ϑ to a graded map

$$\vartheta : Y_G^q(X) = [X, Y]_G^q = [\Sigma^{-q} X, Y]_G \to \prod \operatorname{Hom}_{\mathcal{M}[G]}(\underline{\pi}_n(\Sigma^{-q} X), \underline{\pi}_n(Y)).$$

It is still an isomorphism when X is an orbit. Now let Y be rational. We obtain the same groups if we replace X and the $\underline{\pi}_n(\Sigma^{-q} X)$ by their rationalizations. Since the Mackey functors $\underline{\pi}_n(Y)$ are injective, the right-hand side is a cohomology theory on G-spectra X. Clearly ϑ is a map of cohomology theories, and this already proves the following result.

Theorem A.4. *If Y is rational, then ϑ is a natural isomorphism.*

Proof of Theorem A.1: Take $Y = \prod K(\underline{\pi}_n(X), n)$. The inverse image under ϑ of the identity maps on the homotopy group Mackey functors of X give the required natural equivalence $X \to Y$.

Applied to Eilenberg-MacLane G-spectra, Theorem A.4 has the following immediate consequence.

Corollary A.5. *For a Mackey functor M and a rational Mackey functor N, $H_G^n(HM; N)$ is zero if $n \neq 0$, and is $\operatorname{Hom}_{\mathcal{M}[G]}(M, N)$ if $n = 0$.*

Let $\underline{A} \cong \underline{\pi}_0(S_G)$ be the Burnside Mackey functor; its value on the orbit G/H is $A(H)$. It plays the same role equivariantly that $\mathbb{Z}$ plays nonequivariantly, and Proposition A.3 gives the following result.

COROLLARY A.6. *The rationalization of S_G is $H(\underline{A} \otimes \mathbb{Q})$.*

Therefore $\underline{\pi}_*(X) \otimes \mathbb{Q} \cong \underline{H}_*^G(X; \underline{A} \otimes \mathbb{Q})$. In particular, rational Moore G-spectra and rational Eilenberg-Maclane G-spectra are the same things.

In the rest of this appendix, we abbreviate $A(G) = A(G) \otimes \mathbb{Q}$. Thus $A(G)$ is the rationalization of the Grothendieck ring of finite G-sets. We have a map of rings $\varphi_H : A(G) \to \mathbb{Q}$ which sends a G-set S to $|S^H|$. The φ_H are the components of an isomorphism of rings from $A(G)$ to the product of copies of $\mathbb{Q}$, one for each conjugacy class of subgroups H (e.g. [16] or [33, V§2]). This gives the complete set of orthogonal idempotents $e_H = e_H^G$ in $A(G)$ specified by $\varphi_H(e_H) = 1$ and $\varphi_J(e_H) = 0$ if J is not conjugate to H. Multiplication by the e_H induces compatible natural splittings of $A(G)$-modules, rational Mackey functors, and rational G-spectra. (For the spectrum level, see [33, p.267].) In all three settings, there are no non-zero maps $e_H X \to e_J Y$ unless H is conjugate to J. This gives the following refinements of Theorems A.1 and A.4.

THEOREM A.7. *For rational G-spectra X, there are natural equivalences*

$$X \simeq \bigvee_{(H)} e_H X \simeq \bigvee_{(H)} \prod_n K(e_H \underline{\pi}_n(X), n).$$

THEOREM A.8. *For rational G-spectra X and Y, there are natural isomorphisms*

$$[X, Y]_G \cong \bigoplus_{(H)} [e_H X, e_H Y]_G \cong \bigoplus_{(H)} \prod_n \operatorname{Hom}_{\mathcal{M}[G]}(e_H \underline{\pi}_n(X), e_H \underline{\pi}_n(Y)).$$

Moreover, if $V_{n,H}(X) = (e_H \underline{\pi}_n(X))(G/H) \subset \pi_n(X^H)$, then

$$\operatorname{Hom}_{\mathcal{M}[G]}(e_H \underline{\pi}_n(X), e_H \underline{\pi}_n(Y)) \cong \operatorname{Hom}_{WH}(V_{n,H}(X), V_{n,H}(Y)).$$

Thus the computation of maps between rational G-spectra reduces to the computation of maps between functorially associated modules over subquotient groups. The last statement of the theorem is a special case of the following purely algebraic result.

THEOREM A.9. *For rational Mackey functors M and N, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{M}[G]}(e_H M, e_H N) \cong \mathrm{Hom}_{WH}(V_H(M), V_H(N)),$$

where $V_H(M)$ is the $\mathbb{Q}[WH]$-module $(e_H M)(G/H) \subset M(G/H)$.

Our proof of Proposition A.2 is based on the following observation.

LEMMA A.10. *For a given M and H, $e_H M$ is projective if the conclusion of Theorem A.9 holds for all N. Similarly, for a given N and H, $e_H N$ is injective if the conclusion of Theorem A.9 holds for all M.*

PROOF: $V_H(N)$ is a projective and injective $\mathbb{Q}[WH]$-module, by Maschke's theorem.

We shall prove Theorem A.9 by using certain adjunctions that were constructed in [26] and sharpened in [23, §§3,6]. Henceforward, let $\mathcal{M}[G]$ be the category of rational Mackey functors over G and let $Q[G]$ be the category of $\mathbb{Q}[G]$-modules. Fix H and let $\iota : NH \to G$ and $\varepsilon : NH \to WH$ be the inclusion and quotient homomorphisms. Then there are functors

$$\mathcal{M}[G] \xrightarrow{\iota_*} \mathcal{M}[NH] \xrightarrow{\varepsilon^*} \mathcal{M}[WH] \xrightarrow{U} Q[WH]$$

and

$$\mathcal{M}[G] \xleftarrow{\iota^*} \mathcal{M}[NH] \xleftarrow{\varepsilon_*} \mathcal{M}[WH] \xleftarrow{F} Q[WH].$$

(The functors U and F were denoted L and R in [26] and e and H^0 in [23].) The functors ι_* and ι^* are both left and right adjoint to each other. The functors U and F are also both left and right adjoint to each other; here F would be right but not left adjoint to U if we were working integrally, but coincides with the left adjoint since we are working rationally. The functors ε^* and ε_* are left and right adjoint to each other if we replace $\mathcal{M}[NH]$ by its full subcategory consisting of those Mackey functors M whose transfer maps $\alpha_* : M(NH/J) \to M(NH/K)$ are zero for all maps $\alpha : NH/J \to NH/K$ such that $J \not\supset H$ and $K \supset H$. They are right and left adjoint to each other if we replace $\mathcal{M}[NH]$ by its full subcategory of Mackey functors M such that $\alpha^* = 0$ for these maps α.

Write U_H and F_H for the displayed composites $U\varepsilon^*\iota_*$ and $\iota^*\varepsilon_*F$. By [23,3.9 and 6.3], these functors are left and right adjoint if we replace $\mathcal{M}[G]$ by its full subcategory of Mackey functors M such that $\alpha_* = 0$ for all maps $\alpha : G/J \to G/H$ such that J is a proper subconjugate of H; they are right and left adjoint if we replace $\mathcal{M}[G]$ by its full subcategory of those Mackey functors M such that $\alpha^* = 0$ for these maps α. In particular, they are both left and right adjoint to each other if we replace $\mathcal{M}[G]$ by its full subcategory $\mathcal{M}[G]/H$ of those Mackey functors M such that $M(G/J) = 0$ for all proper subconjugates J of H. These conditions are less restrictive than would be predicted from the previous paragraph. (This sharpening is implicitly used in the proof of [26, Thm. 12], which we shall recall shortly.) Explicitly, for a G-Mackey functor M and a WH-representation V,

$$U_H M = M(G/H) \quad \text{and} \quad (F_H V)(G/K) = (\mathbb{Q}[(G/K)^H] \otimes V)^{WH}.$$

Since $(F_H V)(G/K) = 0$ unless H is subconjugate to K, $F_H V$ is in $\mathcal{M}[G]/H$.

Proofs of Proposition A.2 and Theorem A.9: For a proper subconjugate J of H, the idempotent e_H restricts to zero in $A(J)$ and $(e_H M)(G/J) = 0$. Thus $e_H M$ is in $\mathcal{M}[G]/H$. Since $U_H(e_H M) = V_H(M)$, one of the adjunctions of the previous paragraph specializes to give

$$\mathrm{Hom}_{\mathcal{M}[G]}(F_H U_H e_H M, e_H N) \cong \mathrm{Hom}_{WH}(V_H(M), V_H(N)).$$

Thus Theorem A.9 will hold for H and M provided that the counit

$$\varepsilon : F_H U_H e_H M \to e_H M$$

of the adjunction is an isomorphism, and this will also imply that $e_H M$ is projective. For $V \in Q[WH]$, $U_H F_H V = V$ and $\varepsilon : F_H V = F_H U_H F_H V \to F_H V$ is an isomorphism. Therefore the following observation completes the proof for Mackey functors of the form $M = F_H V$.

LEMMA A.11. *Let $M = F_H V$ for a $\mathbb{Q}[WH]$-module V. Then $e_H M = M$, hence $e_J M = 0$ for J not conjugate to H.*

PROOF: By [26, Prop.8], $A(G)$ acts through φ_H on $F_H V$.

Now the following result completes the proofs for general M.

PROPOSITION A.12. *Any rational Mackey functor M is a finite direct sum of Mackey functors of the form $F_H V$ for varying H and V.*

PROOF: By [26, Thm. 12], every M is built up by successive extensions from the $F_H V$, and the conclusion follows. We run through the details. Partition the set of subgroups of G as follows. Let $S_0 = \{e\}$. Inductively, let S_j consist of those subgroups which are not in S_{j-1} but each of whose subgroups is in S_i for some $i < j$. Each S_j is closed under conjugation, say with n_j conjugacy classes. Choose one $H_{j,k}$ in each conjugacy class, $1 \leq k \leq n_j$. Say that M is of type (j,k) if $M(G/H_{j,k}) \neq 0$ but $M(G/H_{j',k'}) = 0$ for $j' < j$ and for $j' = j$ and $k' < k$. Of course, if $M(G/H) = 0$ for all proper subgroups H, then $M = F_G M(G/G)$. Inductively, fix (j,k) and assume that all Mackey functors of type (j',k') with $j' > j$ or $j' = j$ and $k' > k$ are finite direct sums of Mackey functors $F_J V_J$, where V_J is a WJ-module and

$$J \in \{H_{j'',k''} | j'' > j' \quad \text{or} \quad j'' = j' \text{ and } k'' \geq k'\}.$$

Let M be a Mackey functor of type (j,k) and write $H = H_{j,k}$. We have a map of Mackey functors $\eta : M \to F_H M(G/H)$ which is the identity on $M(G/H)$; this is where we use the sharpened adjunction cited above. The induction hypothesis applies to $\mathrm{Coker}(\eta)$ and $\mathrm{Ker}(\eta)$, hence these are both annihilated by e_H. Therefore

$$e_H M = e_H F_H M(G/H) = F_H M(G/H)$$

and η is an epimorphism. It is split since $F_H M(G/H)$ is projective.

REMARK A.13: Let $\mathcal{F}$ be a family in G. Define $e_{\mathcal{F}}$ to be the sum over $H \in \mathcal{F}$ of the idempotents e_H and let $\widetilde{e}_{\mathcal{F}} = 1 - e_{\mathcal{F}}$. Since $I\mathcal{F}$ is the intersection over $H \in \mathcal{F}$ of the kernels of the φ_H, $\widetilde{e}_{\mathcal{F}}A(G) = I\mathcal{F}$. For any $\mathcal{F}$-spectrum X, the map $\widetilde{e}_{\mathcal{F}} : X \to X$ is trivial and thus $\widetilde{e}_{\mathcal{F}}X$ is trivial. In particular, $e_{\mathcal{F}}S^0 \simeq E\mathcal{F}_+$, $\widetilde{e}_{\mathcal{F}}S^0 \simeq \widetilde{E}\mathcal{F}$, and the cofibration sequence $E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}$ rationalizes to $S^0 \simeq e_{\mathcal{F}}S^0 \vee \widetilde{e}_{\mathcal{F}}S^0$.

REMARK A.14: Recall from (4.4) and Section 18 that we have a G-spectrum $M(I\mathcal{F})$, an $\mathcal{F}$-equivalence $M(I\mathcal{F}) \to S_G$, and a G-map $\xi : \Sigma^\infty_G E\mathcal{F}_+ \to M(I\mathcal{F})$ over S_G. It is clear by inspection that the rationalization of $M(I\mathcal{F})$ is equivalent to the construction M applied to the rationalization of $I\mathcal{F}$. Since $I\mathcal{F}$ is generated rationally by the idempotent $\widetilde{e}_{\mathcal{F}}$, it follows that the rationalization of ξ is an equivalence.

There is an earlier, and more difficult, topological analysis leading to some of the conclusions that we have obtained. There is a functor Φ^H from G-spectra to WH-spectra that generalizes the fixed point functor on G-spaces, in the sense that $\Phi^H\Sigma^\infty_G X \simeq \Sigma^\infty_{WH}X^H$ for based G-spaces X. On G-spectra X, one regards X as an NH-spectrum and then defines

$$\Phi^H X = (\widetilde{E}\mathcal{F}[H] \wedge X)^H,$$

where $\mathcal{F}[H]$ is the family of subgroups of NH which do not contain H. (See [33, II.9.8 and 9.9].) By [5] or [33, V.6.4 and 6.5], there are natural isomorphisms

$$\text{(A.15)} \qquad \begin{aligned}[X, e^G_H Y]_G &\cong [X, e^{NH}_H Y]_{NH} \cong [\Phi^H X, e^{WH}_1 \Phi^H Y]_{WH} \\ &\cong \{[i^*\Phi^H X, i^*\Phi^H Y]\}^{WH}\end{aligned}$$

for G-spectra X and rational G-spectra Y. In the last group, we are passing to underlying naive WH-spectra, computing nonequivariant homotopy classes of maps, and then taking WH-fixed points. Of course, we can replace the domain G-spectra in the first three groups with $e^G_H X$, $e^{NH}_G X$, and $e^{WH}_1\Phi^H X$, respectively, without changing their values.

In view of Theorem A.1 and the splitting of Y by idempotents, this implies the following topological analog of the algebraic analysis above.

THEOREM A.16. *For rational G-spectra X and Y, there are natural isomorphisms*

$$[X, Y]_G \to \bigoplus_{(H)} \{[i^*\Phi^H X, i^*\Phi^H Y]\}^{WH}$$

and

$$[X,Y]_G \to \bigoplus_{(H)} \prod_n \{[i^*\Phi^H H\underline{\pi}_n(X), i^*\Phi^H H\underline{\pi}_n(Y)]\}^{WH}.$$

We leave it as an exercise for the interested reader to reconcile this result with the more intelligible algebraic description given by Theorem A.8.

Appendix B: Generalized Atiyah-Hirzebruch Spectral Sequences

We need a better understanding of the Atiyah-Hirzebruch spectral sequence than exists in the literature. It is folklore that the AHSS for the calculation of $[X, Y]^*$ can be constructed by use of either the cellular filtration of X or the cocellular (Postnikov) filtration of Y. However, we know of no proof in the literature that the two resulting spectral sequences agree. Moreover there are two different ways of constructing an exact couple from either the filtration of X or the filtration of Y, both of which yields the same spectral sequence but only one of which is convenient for the analysis of multiplicative structure. More important, in our applications the relevant filtrations are more complicated than the usual cellular and cocellular ones, and various groups which vanish in the classical situation do not vanish in our new situations.

In view of this, we shall here give a general discussion of spectral sequences of AHSS type. We shall use a nonequivariant notation for brevity, but all of our arguments and results will apply verbatim in the equivariant context. We fix spectra X and Y throughout the discussion. We obtain spectral sequences by filtering X as a colimit or by filtering Y as a limit.

We assume given spectra X^p together with maps $\iota^p : X^p \to X^{p+1}$ and $\alpha^p : X^p \to X$ such that $\alpha^{p+1} \circ \iota^p = \alpha^p$ for all integers p. We form

$$\text{Tel } X^p = \text{hocolim}_{p\to\infty} X^p \equiv \bigvee (X^p \wedge [p, p+1]_+)/(\approx),$$

where $x \wedge (p+1) \approx \iota^p(x) \wedge (p+1)$ for $x \in X^p$. More precisely, this point-set level description prescribes the prespectrum level construction spacewise, and we apply the spectrification functor L to the result (as in §0). We assume that the evident induced map $\text{Tel } X^p \to X$ is an equivalence. Replacing X by $\text{Tel } X^p$ and X^p by the equivalent subspectrum of $\text{Tel } X^p$ consisting of points with real coordinate $\leq p$, we may assume without loss of generality that we are given an increasing sequence $\{X^p \mid p \in \mathbb{Z}\}$ of subspectra of X such that $X = \bigcup X^p$ and each map $X^p \to X^{p+1}$ and $X^p \to X$ is a cofibration.

We are interested in only one homological spectral sequence, and we describe it first. Let $\overline{X}^p = X^p/X^{p-1}$. We then have the following sequence of cofiberings and associated exact couple.

(B.0) $$X^{p-1} \longrightarrow X^p \longrightarrow \overline{X}^p \longrightarrow \Sigma X^{p-1};$$

$$\text{Exact couple:} \quad D^1_{p,q} = \pi_{p+q}(X^p \wedge Y) \quad \text{and} \quad E^1_{p,q} = \pi_{p+q}(\overline{X}^p \wedge Y).$$

The resulting spectral sequence is *relevant* to the calculation of $\pi_*(X \wedge Y)$ since the natural map colim $\pi_*(X^p \wedge Y) \to \pi_*(X \wedge Y)$ is an isomorphism by our assumption that $X = \bigcup X^p$. It is *conditionally convergent* if $\mathrm{holim}_{p\to -\infty}(X^p \wedge Y)$ is trivial. This obviously holds if $X^p = *$ for p sufficiently small, but the special argument of Lemma 10.8 was required to handle the equivariant situation encountered in the text. It is then *strongly convergent* to $\pi_*(X \wedge Y)$ if certain obstruction groups W and RE^∞ both vanish, by [7, 10.1]. The precise meaning of the italicized convergence statements, together with conditions sufficient to ensure the vanishing of the cited obstruction groups, will be recalled shortly (in a cohomologically graded context, to which we can convert by setting $E_r^{p,q} = E^r_{-p,-q}$, as usual).

We consider cohomology type spectral sequences in the rest of the appendix. In addition to our filtration of X, we assume given a sequence of spectra $\{Y^q \mid q \in \mathbb{Z}\}$ together with maps $\pi^q : Y^q \to Y^{q+1}$ and $\beta^q : Y \to Y^q$ such that $\pi^q \circ \beta^q \simeq \beta^{q+1}$ for all integers q. We form

$$\text{Mic } Y^q = \mathrm{holim}_{q\to -\infty} Y^q \subset \prod F([q, q+1]_+, Y^q),$$

namely the subspectrum of those tuples of maps (f^q) such that $\pi^q f^q(q+1) = f^{q+1}(q+1)$ for all q; this makes perfect sense with the explicit definition of function spectra given in Section 0. We have an induced map $Y \to \text{Mic } Y^q$, and we assume that this map is an equivalence.

In the classical situation, X^p is the p-skeleton of a CW-spectrum X with its skeletal filtration and $Y^q = Y(-\infty, -q]$ is the $(-q)^{\text{th}}$ term in the Postnikov tower of a spectrum

Y; thus $Y^q = *$ for q large if Y is bounded below. The unusual indexing of the Y^q is chosen for convenience in our present cohomological framework.

We let $\overline{Y}^q$ be the fiber of $\pi^q : Y^q \to Y^{q+1}$ and $Y\backslash Y^q$ be the fiber of $\beta^q : Y \to Y^q$. We need no further assumptions on the filtrations of X and Y to set up four spectral sequences related to the computation of $[X,Y]^*$. All result in a standard fashion from sequences of cofiber or fiber sequences and their associated exact couples. In each case, the three arrows, in the order written, will be denoted i, k, and j. They induce the structure maps of the specified exact couples.

(B.1) $$X^{p-1} \to X^p \to \overline{X}^p \to \Sigma X^{p-1};$$

Exact couple: $D_1^{p,q} = [X^{p-1}, Y]^{p+q-1}$ and $E_1^{p,q} = [\overline{X}^p, Y]^{p+q}$

(B.2) $$X/X^{p-1} \to X/X^p \to \Sigma\overline{X}^p \to \Sigma X/X^{p-1};$$

Exact couple: $D_1^{p,q} = [X/X^{p-1}, Y]^{p+q}$ and $E_1^{p,q} = [\overline{X}^p, Y]^{p+q}$.

(B.3) $$Y^{q+1} \leftarrow Y^q \leftarrow \overline{Y}^q \leftarrow \Sigma^{-1}Y^{q+1};$$

Exact couple: $D_2^{p,q} = [X, Y^q]^{p+q-1}$ and $E_2^{p,q} = [X, \overline{Y}^q]^{p+q}$.

(B.4) $$Y\backslash Y^{q+1} \leftarrow Y\backslash Y^q \leftarrow \Sigma^{-1}\overline{Y}^q \leftarrow \Sigma^{-1}Y\backslash Y^{q+1};$$

Exact couple: $D_2^{p,q} = [X, Y\backslash Y^q]^{p+q}$ and $E_2^{p,q} = [X, \overline{Y}^q]^{p+q}$.

The maps $D \to D$ induced by maps i all have bidegree $(-1,1)$, the maps $E \to D$ induced by maps k all have bidegree $(1,0)$, and the maps $D \to E$ induced by maps j have bidegree $(0,0)$ in cases (B.1) and (B.2) and bidegree $(1,-1)$ in cases (B.3) and (B.4). The reader should convince himself that this is correct: the spectral sequences really do begin with their E_2-terms in cases (B.3) and (B.4).

The cofiber sequence (B.2) maps naturally to the suspension of the cofiber sequence (B.1), with the map $\Sigma\overline{X}^p \to \Sigma\overline{X}^p$ being the identity and the other three maps being boundary maps. There results a map on exact couples which is the identity on E_1-terms and thus induces the identity map of spectral sequences. The reader should convince himself that this too is correct: the spectral sequences really are identical despite their different D_1-terms (as Boardman first noted [7, Example in §9]). When studying convergence, one looks at the maps

$$[X/X^p, Y]^n \to [X,Y]^n \to [X^p, Y]^n.$$

In favorable cases, the first induces an isomorphism on passage to colimits as $p \to -\infty$ and the second induces an isomorphism on passage to limits as $p \to +\infty$. The usual construction of the AHSS fits into the context of (B.2), and we agree to write this spectral sequence as $\{E_r(\underline{X}, Y)\}$; the underline is meant to indicate that the filtration of X is used in the construction.

Dually, the fiber sequence (B.3) maps naturally to the suspension of the fiber sequence (B.4), with the map $\overline{Y}^q \to \overline{Y}^q$ being the identity and the other three maps being boundary maps. There results a map of exact couples which is the identity on E_2-terms and thus induces the identity map of spectral sequences. Here, when studying convergence, one looks at the maps

$$[X, Y\backslash Y^q]^n \to [X,Y]^n \to [X, Y^q]^n.$$

In favorable cases, the first induces an isomorphism on passage to colimits as $q \to +\infty$ and the second induces an isomorphism on passage to limits as $q \to -\infty$. The spectral sequence of (B.4) is the one of greatest interest to us, and we agree to denote it by $\{E_r(X, \underline{Y})\}$.

Boardman's discussion of convergence [7] concentrates on the context applicable to (B.2) and (B.4), but [7,§9] indicates how one can handle (B.1) and (B.3) compatibly. We have introduced (B.1) and (B.3) because we shall find it convenient to use them to compare our two genuinely different spectral sequences. That is, we shall use the exact couples of (B.1) and (B.3) to compare spectral sequences, and we shall interpret this as a comparison of the spectral sequences $\{E_r(\underline{X}, Y)\}$ and $\{E_r(X, \underline{Y})\}$ of (B.2) and (B.4).

To be precise about convergence, we set

$$D^n_\infty = \lim_{p\to\infty} D^{p,n-p} \text{ and } D^n_{-\infty} = \operatorname{colim}_{p\to-\infty} D^{p,n-p}$$

for any given cohomologically graded exact couple (D, E). These groups are decreasingly filtered by

$$F^p D_\infty = \ker(D_\infty \to D^p) \text{ and } F^p D_{-\infty} = \mathrm{Im}(D^p \to D_{-\infty}).$$

DEFINITIONS B.5 (BOARDMAN [7]). *The derived spectral sequence is said to be conditionally convergent if D^n_∞ and $\lim^1_{p\to\infty} D^{p,n-p}$ are both zero for all n. It is said to be strongly convergent if the evident natural map from the associated graded $E^0 D_{-\infty}$ to E_∞ is an isomorphism.*

We recall the main features of Boardman's general study of convergence before returning to our homotopical context.

THEOREM B.6 (BOARDMAN [7, 10.1]). *Consider a conditionally convergent spectral sequence derived from a cohomologically graded exact couple (D, E). Assume that the following condition (ω) holds and define*

$$RE^{p,q}_\infty = \lim{}^1_{r\to\infty} Z^{p,q}_r \cong \lim{}^1_{r\to\infty} (Z^{p,q}_r / B^{p,q}_\infty).$$

Then $\{E_r\}$ is strongly convergent if and only if $RE_\infty = 0$. Moreover, the following condition (ρ) is sufficient to ensure that $RE_\infty = 0$.

(ω) *For each fixed n, there exist numbers $u(k)$ and $v(k)$ such that*

$$d_{u+v} : E^{-u,n+u}_{u+v} \to E^{v,n-v+1}_{u+v}$$

is zero for all $u \geq u(k)$ and $v \geq v(k)$.

(ρ) *For each pair of integers (p, q), only finitely many differentials*

$$d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

are non-zero.

Condition (ρ) holds if each $E^{p,q}_r$ is finite, or if $E_r = E_\infty$ for any r, or if the spectral sequence lies in a left or upper half plane. Condition (ω) holds if $\{E_r\}$ lies in any half plane (left, right, upper, or lower); it ensures that Boardman's whole plane obstruction groups W vanish. We shall also have occasion to use the following remarkable comparison theorem of Boardman. Its point is that a conditionally convergent spectral sequence *determines* $D_{-\infty}$ even though it may fail to *calculate* it.

THEOREM B.7 (BOARDMAN [7, 10.2]). *Let $f : (D,E) \to (D',E')$ be a map of exact couples, where both spectral sequences are conditionally convergent and satisfy condition (ω). Suppose that $f : E_\infty \to E'_\infty$ and $f : RE_\infty \to RE'_\infty$ are isomorphisms, for example if $f : E_r \to E'_r$ is an isomorphism for some r. Then $f : D_{-\infty} \to D'_{-\infty}$ is an isomorphism of filtered graded groups.*

Returning to homotopy theory, we next consider the convergence properties of the spectral sequences $\{E_r(\underline{X}, Y)\}$ and $\{E_r(X, \underline{Y})\}$ of (B.2) and (B.4). In either case, we say that the spectral sequence is *relevant* if the natural map $D^*_{-\infty} \to [X,Y]^*$ is an isomorphism.

Clearly $\{E_r(\underline{X}, Y)\}$ is relevant if and only if $\operatorname{colim}_{p\to-\infty}[X^p, Y]^* = 0$. This holds trivially if $X^p = *$ for p sufficiently small, but the special argument of Lemma 10.8 was required to handle the equivariant situation encountered in the text. By our standing assumption that $X = \cup X^p$, we have $\mathrm{Tel}(X/X^p) \simeq *$, and this is easily seen to imply that $\{E_r(\underline{X}, Y)\}$ is conditionally convergent.

Dually, $\{E_r(X, \underline{Y})\}$ is relevant if and only if $\operatorname{colim}_{q\to\infty}[X, Y^q]^* = 0$, and this holds trivially if $Y^q = *$ for q sufficiently large. It is conditionally convergent if $\mathrm{Mic}(Y\backslash Y^q) \simeq *$, and this holds by our assumption that $Y \to \mathrm{Mic}\ Y^q$ is an equivalence. The assumption obviously holds for Postnikov towers, and Proposition 3.3 verifies it in our applications.

So far, everything has been so general that it applies to a plethora of spectral sequences in stable homotopy theory. To identify E_2-terms and compare spectral sequences, we must specialize. In the classical situation, X is a CW-spectrum with skeleta $\{X^p\}$ and $\overline{X}^p$ is a wedge of p-sphere spectra, while Y is a spectrum with Postnikov tower $\{Y^q\}$ and $\overline{Y}^q$ is an Eilenberg-MacLane spectrum $K(\pi_{-q}(Y), -q)$. The hypotheses of the following theorem abstract the key features of this classical situation.

THEOREM B.8. *In addition to our standing hypotheses $X = \cup X^p$ and $Y \simeq Mic(Y^q)$, assume that the following three conditions hold.*

(i) *The spectral sequence $\{E_r(\underline{X}, Y)\}$ is relevant, and similarly with Y replaced by Y^q or $\overline{Y}^q$ for any q.*

(ii) *$[\overline{X}^p, \overline{Y}^q]^n = 0$ if $n \neq p + q$ and the map $\overline{Y}^q \to Y^q$ induces an isomorphism on $[\overline{X}^p, ?]^{p+q}$.*

(iii) *$[\overline{X}^p, Y^q]^n = 0$ if $n < p + q$ and the map $Y \to Y^q$ induces an isomorphism on $[\overline{X}^p, ?]^n$ for $n \geq p + q$.*

Then there is a map from the derived exact couple of (B.2) to the exact couple of (B.4) which is an isomorphism on E_2-terms and therefore induces an isomorphism of spectral sequences $\{E_r(\underline{X}, Y)\} \to \{E_r(X, \underline{Y})\}$. Moreover, if these spectral sequences satisfy condition (ω), then the spectral sequence $\{E_r(X, \underline{Y})\}$ is relevant.

PROOF: Since (i) asserts that $\mathrm{colim}_{p \to -\infty}[X^p, Y]^* = 0$, it also holds with X replaced by any of its filtered subspectra X^p. By (ii) and (iii), $\{E_r(\underline{X}, Y^q)\}$ is an upper half plane spectral sequence and $\{E_r(\underline{X}, \overline{Y}^q)\}$ has a single row. Therefore these spectral sequences are strongly convergent, and similarly with X replaced by any X^p. Of course (ii) and (iii) also give that the maps $Y \to Y^q$ and $\overline{Y}^q \to Y^q$ induce isomorphisms on $E_1^{*,q}$-terms and therefore on $E_2^{*,q}$-terms.

In the classical case, the proof that $E_2(\underline{X}, Y) \cong E_2(X, \underline{Y})$ is just the standard concrete proof of the representability of ordinary cellular cohomology, and an elaboration of that proof gives a comparison of exact couples and thus of spectral sequences. We run through the details in order to show that our hypotheses give sufficient information.

We deduce from $\{E_r(\underline{X}^p, \overline{Y}^q)\}$ that $[X^p, \overline{Y}^q]^n = 0$ for $n > p + q$. This implies that the two maps k in the following diagram are epimorphisms, and by (ii) the map i is clearly a monomorphism.

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
{[\overline{X}^{p-1}, \overline{Y}^q]^{p+q-1}} \arrow[d, "k"'] \arrow[dr, "\delta"] & & {[X^{p+1}, \overline{Y}^q]^{p+q}} \arrow[d, "i"'] & \\
{[X^{p-1}, \overline{Y}^q]^{p+q-1}} \arrow[r, "j"] \arrow[d] & {[\overline{X}^p, \overline{Y}^q]^{p+q}} \arrow[r, "k"] \arrow[dr, "\delta"] & {[X^p, \overline{Y}^q]^{p+q}} \arrow[r] \arrow[d, "j"'] & 0 \\
0 & & {[\overline{X}^{p+1}, \overline{Y}^q]^{p+q+1}.} &
\end{tikzcd}$$

An easy diagram chase shows that we obtain inverse isomorphisms $k^{-1}i$ and $i^{-1}k$ between $[X^{p+1}, \overline{Y}^q]^{p+q}$ and $E_2^{p,q}(\underline{X}, \overline{Y}^q)$, and the latter is isomorphic to $E_2^{p,q}(\underline{X}, Y^q)$ and $E_2^{p,q}(\underline{X}, Y)$. By inspection of the map of spectral sequences

$$\{E_r(\underline{X}, \overline{Y}^q)\} \to \{E_r(\underline{X}^{p+1}, \overline{Y}^q)\},$$

we see that

$$E_2^{p,q}(X, \underline{Y}) = [X, \overline{Y}^q]^{p+q} \to [X^{p+1}, \overline{Y}^q]^{p+q}$$

is an isomorphism. Thus we have an isomorphism $E_2^{p,q}(\underline{X}, Y) \cong E_2^{p,q}(X, \underline{Y})$.

To compare D_2-terms in (B.1) and (B.3), consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
{D_1^{p+1,q-1} = [X^p, Y]^{p+q-1}} \arrow[r] \arrow[d, "i"] & {[X^p, Y^q]^{p+q-1}} \arrow[d, "i"] & {[X, Y^q]^{p+q-1} = D_2^{p,q}} \arrow[l, "\cong"'] \arrow[dl] \\
{D_1^{p,q} = [X^{p-1}, Y]^{p+q-1}} \arrow[r] & {[X^{p-1}, Y^q]^{p+q-1}} &
\end{tikzcd}$$

The right map i is a monomorphism since $[\overline{X}^p, Y^q]^{p+q-1} = 0$, by (iii). By inspection of the map of spectral sequences $\{E_r(\underline{X}, Y^q)\} \to \{E_r(\underline{X}^p, Y^q)\}$, we see that the arrow labeled $\cong$ is an isomorphism. Therefore the diagram displays a map f from $D_2^{p,q} = i(D_1^{p+1,q-1})$

of (B.1) to $D_2^{p,q}$ of (B.3). Explicitly, for $x \in [X^p, Y]^{p+q-1}$, $fi(x)$ is the unique element of $[X, Y^q]^{p+q-1}$ with the same image as $i(x)$ in $[X^{p-1}, Y^q]^{p+q-1}$. Diagram chases verify the compatibility of these maps with the structural maps i, j, and k of the (D_2, E_2) exact couples of (B.1) and (B.3).

We have observed that there is a map from the exact couple of (B.2) to the exact couple of (B.1) which is an isomorphism on E_1 and a map from the exact couple of (B.3) to the exact couple of (B.4) which is an isomorphism on E_2. Thus, on the (D_2, E_2) level, we have maps of exact couples from (B.2) to (B.1) to (B.3) to (B.4), all of which are isomorphisms on E_2. This gives the desired isomorphism $\{E_r(\underline{X}, Y)\} \to \{E_r(X, \underline{Y})\}$. Since our standing hypotheses give that the spectral sequences of (B.2) and (B.4) are conditionally convergent, Theorem B.7 gives that the induced map from D_∞^* of (B.2) to D_∞^* of (B.4) is an isomorphism when condition (ω) holds. Another diagram chase shows that this map is compatible with the maps to $[X, Y]^*$, hence the relevance of $\{E_r(X, \underline{Y})\}$ follows from the relevance of $\{E_r(\underline{X}, Y)\}$.

We must still consider multiplicative structures. The standard procedure is to compose external maps with maps induced by a given diagonal map $\Delta : X \to X \wedge X$ and product $\varphi : Y \wedge Y \to Y$. If we use $\{E_r(\underline{X}, Y)\}$, then we must first approximate Δ by a map that respects filtrations. We don't know how to do this in our equivariant context. We get around this by using $\{E_r(X, \underline{Y})\}$ instead. Here we need φ to be appropriately filtration preserving, but we can just apply naturality in the unfiltered X variable.

In the study of $[X, Y]^*$ for a ring spectrum Y, the essential point is that if $\{Y^q\}$ is a Postnikov tower for Y, then $Y \backslash Y^{q+1}$ is the connected cover $Y[-q, \infty)$ of Y and, by the connectivity of smash products of such covers and obstruction theory, we find that a pairing $Y \wedge Y' \to Y''$ lifts uniquely to give a compatible system of diagrams

(B.9)
$$\begin{tikzcd}
Y \backslash Y^{i+1} \wedge Y' \backslash Y'^{j+1} \arrow[r] \arrow[d] & Y'' \backslash Y''^{i+j+1} \arrow[d] \\
\overline{Y}^i \wedge \overline{Y}'^j \arrow[r] & \overline{Y}''^{i+j}.
\end{tikzcd}$$

From here, the introduction of products into the exact couple of (B.4) is routine. Of course, like everything else above, this applies equally well equivariantly.

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Index

- **Standard notational conventions.**

Homology and Cohomology are reduced.
Underlines indicate either a Mackey functor or a filtered object.
Subscript + indicates the addition of a disjoint basepoint.
Superscript G denotes either G-equivariance or G-fixed points in some sense.
Subscript G denotes either G-equivariance or quotient by G in some sense.

- **Alphabetical index of definitions.**

Reference is given to the last number preceding the definition; the section title is deemed to be numbered 0. An entry "I" refers to the introduction; "A" and "B" refer to the appendices.

Alphabetical index of notation.

An entry "*" indicates that no specific reference is appropriate.

$\mathcal{A}\ell\ell$	family of all subgroups	*
$\mathcal{A}b$	category of abelian groups	*
$Ad(G)$	adjoint representation of G	*
$A(G)$	Burnside ring	4.0
α_V	Euler class of V in k_G^V	16.1
$B(X,G,Y)$	bar construction	9.4
BP	Brown-Peterson spectrum	*
$BP < n >$	Johnson-Wilson spectrum with coefficients $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$	*
$c(k_G)$	geometric completion spectrum for k_G	I
$C(G)$	continuous functions from space of conjugacy classes of closed subgroups to $\mathbb{Z}$	21.3
$\mathcal{C}[G]$	category of coefficient systems	7.0
$C_*(X)$	reduced cellular chain complex of X	*
$C_*(\underline{W})$	chain complex of the "skeletally" filtered spectrum W	9.4, 20.11
$\underline{C}_*(\underline{W})$	Mackey functor version of $C_*(\underline{W})$	20.11
$C_*^G(\underline{W}; N)$	equivariant chain complex of the filtered spectrum W	20.11
$C_G^*(\underline{W}; M)$	equivariant cochain complex of the filtered spectrum W	20.11
χ_V	(i) classical Euler class of an orientable representation	16.6
	(ii) K-theory Euler class shifted into degree zero by Bott periodicity	19.11
χ	special case of (ii) in which $G = \mathbb{T}$ and V is the natural representation	15.5
$\chi * \psi$	twisted G-cell formed from a free G-cell χ and a non-equivariant cell ψ	14.5
d	dimension of G	*
$D(V)$	The unit ball in the orthogonal representation V	*
DX	functional dual $F(X, S_G)$ of a G-spectrum X	0.7
e	(i) neutral element of a group	*
	(ii) inclusion $S^0 \longrightarrow S^V$	16.1
	(iii) an idempotent	*
e_H	idempotent in $A(G) \otimes \mathbb{Q}$ (or $C(G)$) with support H	A.6
$e_{\mathcal{F}}$	idempotent in $A(G) \otimes \mathbb{Q}$ (or $C(G)$) with support $\mathcal{F}$	21.3, A.13
$\tilde{e}_{\mathcal{F}}$	idempotent in $A(G) \otimes \mathbb{Q}$ (or $C(G)$) with support complementary to $\mathcal{F}$	21.3, A.13
EX	geometric realisation of the Amitsur complex for the G-set X	21.0
$E\mathcal{F}$	universal space for the family $\mathcal{F}$	17.0, 21.0
$\tilde{E}\mathcal{F}$	unreduced suspension of $E\mathcal{F}$	17.0
$\tilde{E}\mathcal{F}^{(n)}$	term in the complete filtration	22.1
$E(\mathcal{F}, F')$	isotropy segment space $E\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}'$	23.2
EG	universal free G-space	I
EG_+	universal free G-space with a disjoint basepoint	I
$\tilde{E}G$	unreduced suspension of EG	I

$\tilde{E}G^{(n)}$	term in the complete filtration	9.5
ϵ	counit of an adjunction	*
η	unit of an adjunction	*
$\mathcal{E}$	conjugacy classes of elements of G	19.2
$f_{\mathcal{F}}$	image of $e_{\mathcal{F}}$ in $R(G)\otimes\mathbb{Q}$	19.2
$\tilde{f}_{\mathcal{F}}$	image of $\tilde{e}_{\mathcal{F}}$ in $R(G)\otimes\mathbb{Q}$	19.2
$f(k_G)$	free spectrum of k_G	I
F	$\mathcal{Z}[\overline{G}] \longrightarrow \mathcal{M}[G]$ right adjoint to the forgetful functor; $FV = \underline{\pi}_0(LV)$	6.1 , 6.2
$F_H[G, Y)$	coinduced G-spectrum of an H-spectrum Y	3.7
$\mathcal{F}$	a family of subgroups (i.e. closed under conjugacy and passage to subgroups).	
$\mathcal{F}(X)$	family generated by the isotropy groups of X	21.0
$\mathcal{F}(H)$	family of subconjugates of H in G	17.0
$\mathcal{F}[H]$	family of subgroups not containing a conjugate of H	17.0
$\mathcal{F}(N, G)$	family of subgroups intersecting the normal subgroup N trivially	17.0
$\mathcal{F} < p >$	family of p-subgroups	17.0
$\mathcal{F}(V)$	family of subgroups fixing a nonzero vector in the representation V	17.0
G	a compact Lie group of dimension d	*
$\overline{G}$	the group of path components $\pi_0(G)$ of G	*
G_1	the path component of the identity element	*
$G \ltimes_H Y$	induced G-spectrum of an H-spectrum Y	0.8
$G\mathcal{P}U$	category of G-prespectra indexed on U	0.0
GSU	category of G-spectra indexed on U	0.0
GSU^G	category of naive G-spectra (= G-spectra indexed on U^G)	0.0
HM	Eilenberg-MacLane G-spectrum representing ordinary cohomology with coefficients in the Mackey functor M	6.0
$H_*(\underline{W}; V)$	homology of the "skeletally" filtered spectrum W	9.4
$H^*(\underline{W}; V)$	cohomology of the "skeletally" filtered spectrum W	9.4
$H_*^G(\underline{W}; N)$	coMackey coefficients version of $H_*(\underline{W}; V)$	20.11
$H_G^*(\underline{W}; M)$	Mackey coefficients version of $H^*(\underline{W}; V)$	20.11
$\mathcal{H}$	conjugacy classes of subgroups of G	19.2
i	The inclusion $U^G \longrightarrow U$	0.0
i^*	forgetful functor $GSU \longrightarrow GSU^G$	0.0
i_*	functor $GSU^G \longrightarrow GSU$ building in nontrivial representations; left adjoint to i^*	0.0
$I(G)$	augmentation ideal $ker\{res : A(G) \longrightarrow A(1)\}$	4.0
$I\mathcal{F}(G)$	$\bigcap_{H\in\mathcal{F}} ker\{res : A(G) \longrightarrow A(H)\}$	18.0
$I'\mathcal{F}(G)$	$\sum_{H\in\mathcal{F}} im(ind : A(H) \longrightarrow A(G))$	21.3
j_G	generic notation for a naive G-spectrum	*
J	chain map of degree $d+1$ like that induced by action of $G = \mathbb{T}$ or $\mathbb{U}$	14.3
$J(G)$	augmentation ideal $ker\{res : R(G) \longrightarrow R(1)\}$	13.0
$J\mathcal{F}(G)$	$\bigcap_{H\in\mathcal{F}} ker\{res : R(G) \longrightarrow R(H)\}$	19.0

$J'\mathcal{F}(G)$	characters vanishing off the family $\mathcal{F}$	19.1
$JO(G)$	augmentation ideal $ker\{res : RO(G) \longrightarrow RO(1)\}$	13.0
$JO\mathcal{F}(G)$	$\bigcap_{H\in\mathcal{F}} ker\{res : RO(G) \longrightarrow RO(H)\}$	19.1
$J'O\mathcal{F}(G)$	characters vanishing off the family $\mathcal{F}$	19.9
$JSp(G)$	augmentation ideal $ker\{res : RSp(G) \longrightarrow RSp(1)\}$	13.0
JN	Homological Eilenberg-MacLane G-spectrum representing ordinary homology with coefficients in the coMackey functor N	20.3
k_G	generic notation for a G-spectrum	*
$k_G^n(X)$	k_G cohomology in degree n, $[X \wedge S^{-n}, k_G]_G$	0.5
$k_n^G(X)$	k_G homology in degree n, $[S^n, X \wedge k_G]_G$	0.5
K	nonequivariant periodic complex K-theory	*
K_G	equivariant periodic complex K-theory	*
kU_G	equivariant connective complex K-theory	*
KO_G	equivariant periodic real K-theory	*
kO_G	equivariant connective real K-theory	*
$K(n)$	periodic mod p Morava K-theory	*
$K(M,n)$	Eilenberg-MacLane spectrum for the Mackey functor M in dimension n	*
ℓ	forgetful functor $GSU \longrightarrow G\mathcal{P}U$	0.0
L	(i) spectrification functor $G\mathcal{P}U \longrightarrow GSU$; left adjoint to ℓ	0.0
	(ii) functor $\mathcal{Z}[\overline{G}] \longrightarrow GSU$	6.2
$L(H)$	the tangent space at H in G/H as a representation of H	0.7
$L_*^I(\bullet)$	the left derived functors of I-completion	*
L_a^∞	Thom spectrum of $-a$ times the standard bundle on the infinite lens space	
λ_V	Bott class in $K_G^0(S^V)$	19.11
M	generic letter for a Mackey functor	*
$M(I)$	local cohomology spectrum for the ideal I of the Burnside ring	4.4
$\tilde{M}(I)$	cofiber of $M(I) \longrightarrow S^0$	4.11
mic	mapping microscope: holim of a sequence	3.1
M_p	mod p Moore spectrum $S^0 \cup_p e^1$	*
$M_V^{sd}(\hat{\pi}_q)$	Mahowald V-filtration	24.8
$M_{\mathcal{P}}^{sd}(\hat{\pi}_q)$	Mahowald $\mathcal{P}$-filtration	24.12
$\mu(n)$	number of generators of the cyclic group of order n	21.4
$\mathcal{M}[G]$	category of Mackey functors	6.0
$n(\mathcal{F})$	order of the identity in $\mathcal{F}$ Amitsur-Dress-Tate cohomology	21.4
N	(i) norm element $\Sigma_{g\in G}\, g$	*
	(ii) generic letter for a coMackey functor	*
$\mathcal{N}\mathcal{O}\mathcal{S}$	full subcategory of G-orbits G/H_+ in GSU^G	7.0
$\mathcal{O}\mathcal{S}$	full subcategory of G-orbits G/H_+ in GSU	6.0
Ω^V	Vth loop space functor $F(S^V, \cdot)$	0.0
P	complete resolution of a finite group; analog for S^1 or S^3	11.1, 14.2
P^+	positively graded quotient of P	11.1, 14.2

P^-	negatively graded subcomplex of P	11.1, 14.2
$P(\mathcal{F})$	complete complex of $\mathcal{F}$-projective Mackey functors	22.3
$P(\mathcal{F})^+$	positively graded quotient of $P(\mathcal{F})$	22.3
$P(\mathcal{F})^-$	negatively graded subcomplex of $P(\mathcal{F})$	22.3
$\mathcal{P}$	family of proper subgroups	17.0
ϕ_H	mark homomorphism $A(G) \longrightarrow \mathbb{Z}$ defined on G-sets by $\phi_H(X) = \lvert X^H \rvert$	A.6
$\Phi^G X$	Φ-fixed point spectrum of a G-spectrum X	A.14
$Q_G(X)$	equivariant Q-construction	0.1
$R(G)$	complex representation ring	*
$RO(G)$	real representation ring	*
$RSp(G)$	symplectic representation group	*
$RU(G)$	alternative notation for complex representation ring	*
$R_V(\alpha)$	V-root invariant of α	24.9
$R_{\mathcal{P}}(\alpha)$	$\mathcal{P}$-root invariant of α	24.13
S_G	zero-sphere G-spectrum	0.2
S^V	one point compactification of the representation V	*
$S(V)$	unit sphere of the orthogonal representation V	*
s^*	forgetful functor $\mathcal{M}[G] \longrightarrow \mathcal{C}[G]$	7.1
s_*	functor $\mathcal{C}[G] \longrightarrow \mathcal{M}[G]$; left adjoint of s^*	7.2
Σ_G^∞	suspension spectrum functor $G\mathcal{T} \longrightarrow G\mathcal{S}U$ or $G\mathcal{T} \longrightarrow G\mathcal{S}U^G$	0.1
$t(k_G)$	Tate spectrum of k_G	I
tel	mapping telescope: hocolim of a sequence	3.1
$\mathbb{T}$	unit circle group	*
$\mathcal{T}$	category of based topological spaces	*
u	generator of $H^{d+1}(BG)$ for $G = \mathbb{T}$ or $\mathbb{U}$	14.2
U	(i) a complete G-universe	0.0
	(ii) the underlying module functor $\mathcal{M}[G] \longrightarrow \mathcal{Z}[\overline{G}]$ or $\mathcal{C}[G] \longrightarrow \mathcal{Z}[\overline{G}]$	6.0
$\mathbb{U}$	group of unit quaternions	*
V	generic letter for a representation of G	*
$V_H(M)$	$(e_H M)(G/H)$ for a rational Mackey functor M	A.9
$\underline{W}$	filtered G-spectrum whose pth subquotient is a wedge of $G-p$-cells	9.3
$\overline{W}^p$	pth subquotient of $\underline{W}$	9.3, 20.10
$W_G(H)$	Weyl group $N_G(H)/H$	*
$X(\mathcal{F})$	G-set $\coprod_{H\in F} G/H$	21.0
X^G	(i) set of G-fixed points of a G-space X	*
	(ii) fixed point spectrum of a G-spectrum X	0.1
X^{hG}	homotopy fixed points $F(EG_+, X)^G$ of a space or spectrum	5.7
X/G	quotient of a G-space or spectrum	0.6
X_{hG}	homotopy quotient $(EG_+ \wedge X)/G$ of a G-space or spectrum	5.7

$X^{(n)}$	smash power G-spectrum of a naive G-spectrum X	25.3
$Y_{-sd}^{(n-s)d-1}$	$\Sigma^{-1}(S^{nV}/S^0) \wedge S^{-sV}$ for the reduced regular representation V	25.1
$Y(-\infty, k]$	Y with homotopy groups killed above dimension k	*
Y^q	abbreviation or analog of $Y(-\infty, -q]$	B.0
$\mathcal{Z}[\overline{G}]$	category of $\overline{G}$-modules	*

f, c, and t theories.

1. Associated to a G-spectrum k_G.

Representing spectrum (I)

free	geometric completion	Tate
$f(k_G) := k_G \wedge EG_+$	$c(k_G) := F(EG_+, k_G)$	$t(k_G) := c(k_G) \wedge \tilde{E}G$

Representing spectrum relative to $\mathcal{F}$ (17)

$f_{\mathcal{F}}(k_G) := k_G \wedge E\mathcal{F}_+$	$c_{\mathcal{F}}(k_G) := F(E\mathcal{F}_+, k_G)$	$t_{\mathcal{F}}(k_G) := c_{\mathcal{F}}(k_G) \wedge \tilde{E}\mathcal{F}$

Notes: (i) $f(k_G) \simeq f'(k_G) := c(k_G) \wedge EG_+$ and $f_{\mathcal{F}}(k_G) \simeq f'_{\mathcal{F}}(k_G) := c_{\mathcal{F}}(k_G) \wedge E\mathcal{F}_+$.
(ii) The columns in all subsequent tables are arranged by representing spectrum.

2. Algebraic versions.

Finite groups (11.2)

f-cohomology/Borel homology	Borel cohomology/c-homology	Tate cohomology and homology
$\check{H}^*_G(C;V) = H^*(Hom_G(\Sigma P^- \otimes C, V))$	$H^*_G(C;V) = H^*(Hom_G(P^+ \otimes C, V))$	$\hat{H}^*_G(C;V) = H^*(Hom_G(P \otimes C, V))$
$H^G_*(C;V) = H_*((P^+ \otimes C) \otimes_G V)$	$\check{H}^G_*(C;V) = H_*((\Sigma P^- \otimes C) \otimes_G V)$	$\hat{H}^G_*(C;V) = H_*((\Sigma P \otimes C) \otimes_G V)$

Finite groups relative to $\mathcal{F}$ (22.4)

$\check{H}^*_{\mathcal{F}}(D;M) = H^*(Hom_{\mathcal{O}S}(\Sigma P^-_{\mathcal{F}} \otimes D, M))$	$H^*_{\mathcal{F}}(D;M) = H^*(Hom_{\mathcal{O}S}(P^+_{\mathcal{F}} \otimes D, M))$	$\hat{H}^*_{\mathcal{F}}(D;M) = H^*(Hom_{\mathcal{O}S}(P_{\mathcal{F}} \otimes D, M))$
$H^{\mathcal{F}}_*(D;M) = H_*((P^+_{\mathcal{F}} \otimes D) \otimes_{\mathcal{O}S} M)$	$\check{H}^{\mathcal{F}}_*(D;M) = H_*((\Sigma P^-_{\mathcal{F}} \otimes D) \otimes_{\mathcal{O}S} M)$	$\hat{H}^{\mathcal{F}}_*(D;M) = H_*((\Sigma P_{\mathcal{F}} \otimes D) \otimes_{\mathcal{O}S} M)$

Circle and unit quaternion groups (14.3)

$\check{H}^*_G(C;W) = H^*(Hom(\Sigma P^- \otimes C, W), \delta)$	$H^*_G(C;W) = H^*(Hom(P^+ \otimes C, W), \delta)$	$\hat{H}^*_G(C;W) = H^*(Hom(P \otimes C, W), \delta)$
$H^G_*(C;W) = H_*((P^+ \otimes C) \otimes W, \delta)$	$\check{H}^G_*(C;W) = H_*((\Sigma P^- \otimes C) \otimes W), \delta)$	$\hat{H}^G_*(C;W) = H_*((\Sigma P \otimes C) \otimes W, \delta)$

Notes: (i) For finite groups, V is a $\mathbb{Z}G$ module and C is a complex of $\mathbb{Z}G$-modules; P is a $\mathbb{Z}$-graded complex of free $\mathbb{Z}$G -modules where $P_0 \longrightarrow P_{-1}$ factors through $\mathbb{Z}$. Then P_- is the subcomplex of P which is concentrated in negative degrees and P^+ is the quotient of P which is concentrated in non-negative degrees.
(ii) Similarly, for a family $\mathcal{F}$, M is a Mackey functor and D is a complex of Mackey functors; $P_{\mathcal{F}} = P(\mathcal{F})$ is the complex of Mackey functors obtained by applying the Burnside functor to the Amitsur complex for the G-set $X(\mathcal{F}) = \coprod_{H \in \mathcal{F}} G/H$. The positive and negative parts are defined as before.
(iii) For the circle and quaternion groups W is a $\mathbb{Z}$-module and C is a chain complex equipped with a chain map J of degree d so that $J^2 = 0$ and P is the graded group $\mathbb{Z}[u, u^{-1}]$ with u of degree $-(d+1)$. The differential δ on $P \otimes C$ is then defined by $\delta = u \otimes J + 1 \otimes d$.

3. Filtered versions (calculable X).

(9.4, 9.5, 14.5)

f-cohomology/Borel homology	Borel cohomology/c-homology	Tate cohomology and homology
$H^*(\underline{S^0 \wedge X}; V)$	$H^*(\underline{EG_+ \wedge X}; V)$	$H^*(\underline{\Sigma^{-1}\tilde{E}G \wedge X}; V)$
$H_*(\underline{\Sigma^d EG_+ \wedge X}; V)$	$H_*(\underline{\Sigma^d S^0 \wedge X}; V)$	$H_*(\underline{\Sigma^d \tilde{E}G \wedge X}; V)$

Relative to $\mathcal{F}$ (finite groups only) (9.4, 22.1)

$H^*(\underline{S^0 \wedge X}; V)$	$H^*(\underline{E\mathcal{F}_+ \wedge X}; V)$	$H^*(\underline{\Sigma^{-1}\tilde{E}\mathcal{F} \wedge X}; V)$
$H_*(\underline{\Sigma^{d(\mathcal{F})} E\mathcal{F}_+ \wedge X}; V)$	$H_*(\underline{\Sigma^{d(\mathcal{F})} S^0 \wedge X}; V)$	$H_*(\underline{\Sigma^{d(\mathcal{F})} \tilde{E}\mathcal{F} \wedge X}; V)$

4. Theories for ordinary cohomology.

Represented theory (6.5)

f-cohomology/Borel homology	Borel cohomology/c-homology	Tate cohomology and homology
$\check{H}^*_G(X;V)$ $(\leq -d)$	$H^*_G(X;V)$ (≥ 0)	$\hat{H}^G_*(X;V)$
$H^G_*(X;V)$ (≥ 0)	$\check{H}^G_*(X;V)$ $(\leq -d)$	$\hat{H}^G_*(X;V)$

Represented theory relative to $\mathcal{F}$ (20.7)

$\check{H}^*_{\mathcal{F}}(X;M)$ $(\leq -d(\mathcal{F}))$	$H^*_{\mathcal{F}}(X;M)$ (≥ 0)	$\hat{H}^{\mathcal{F}}_*(X;M)$
$H^{\mathcal{F}}_*(X;N)$ (≥ 0)	$\check{H}^{\mathcal{F}}_*(X;N)$ $(\leq -d(\mathcal{F}))$	$\hat{H}^{\mathcal{F}}_*(X;N)$

Remarks: (i) These are the represented theories for $k_G = HM$ in all cases with cohomology and for $k_G = JN$ in all cases with homology. For the homology theories with $\mathcal{F} = \{e\}$ and $k_G = HM$ there is a shift in dimension by d as recorded in (9.8) and proved in (20.8).
(ii) The entries following the notation summarise the degrees in which the coefficients of the theories are typically nonzero. The integer $d(\mathcal{F})$ is the minimum of $dimW_G(H)$ over $H \in \mathcal{F}$. The picture for the Tate theories is easily deduced from the norm sequence.
(iii) In the case of the circle group the notation of J.D.S.Jones [29] agrees with ours except that his $\hat{H}^{\mathbb{T}}_n(X)$ is our $\hat{H}^{\mathbb{T}}_{n-1}(X)$, his $G^{\mathbb{T}}_n(X)$ is our $\check{H}^{\mathbb{T}}_{n-1}(X)$ and his $G^n_{\mathbb{T}}(X)$ is our $\check{H}^{n-1}_{\mathbb{T}}(X)$ (see (14.4)).

5. Concordance.

The equivalence of the various definitions 1.-4. for ordinary cohomology is proved in the following results. Recall that 1. and 4. are alternative notations for the represented theory, 2. is the algebraic version and 3. is the version obtained from a topological filtration.

Finite groups

$$1 = 4 \overset{10.3}{=} 3 \overset{11.1}{=} 2$$

Circle group

$$1 = 4 \overset{10.3}{=} 3 \overset{14.2}{=} 2$$

Unit quaternion group

$$1 = 4 \overset{10.3}{=} 3 \overset{14.9}{=} 2$$

Finite groups relative to $\mathcal{F}$

$$1 = 4 \overset{22.6}{=} 3 \overset{22.5}{=} 2$$

Note: In all cases the complex C is the complex of cellular chains of X. If G is finite it acts on this, and for $\mathbb{T}$ and $\mathbb{U}$ we assume that the action is cellular so that it provides the operator J.

The equivalence of the f, c, and t theories for $\Sigma^d JN$ and HM with $M(G/e) \cong N(G/e)^{op}$ is proved in (20.8). The equivalence of the theories of Part II and the theories relative to the family $\{e\}$ is then clear; further discussion surrounds (20.9).

6. Historical Dictionary

At the prompting of the referee and almost all other readers we have adopted notation and terminology different from that used in the first author's earlier work and preprint versions of the present volume. We include a dictionary for the use of those who wish to read this earlier work.

Present		Old	
Notation	Name	Notation	Name
$c(k_G)$	geometric completion of k_G	$b(k_G)$	Borel spectrum
$c(k_G)^*(\cdot)$	Borel cohomology	$b(k_G)^*(\cdot)$	Borel cohomology
$c(k_G)_*(\cdot)$	c-homology	$b(k_G)_*(\cdot)$	Borel homology
$f(k_G)$	free spectrum of k_G	$c(k_G)$	coBorel spectrum
$f(k_G)^*(\cdot)$	f-cohomology	$c(k_G)^*(\cdot)$	coBorel cohomology
$f(k_G)_*(\cdot)$	Borel homology	$c(k_G)_*(\cdot)$	coBorel homology

Notes: (i) There is no change in notation for Tate spectra or their associated theories.
(ii) Familiar theories now have familiar names, and the representing spectra have descriptive names.

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Generalized Tate Cohomology
J. P. C. Greenlees and J. P. May

This book presents a systematic study of a new equivariant cohomology theory $t(k_G)^*$ constructed from any given equivariant cohomology theory k^*_G, where G is a compact Lie group. Special cases include Tate-Swan cohomology when G is finite and a version of cyclic cohomology when $G = S^1$. The groups $t(k_G)^*(X)$ are obtained by suitably splicing the k-homology with the k-cohomology of the Borel construction $EG \times_G X$, where k^* is the nonequivariant cohomology theory that underlies k^*_G. The new theories play a central role in relating equivariant algebraic topology with current areas of interest in nonequivariant algebraic topology. Their study is essential to a full understanding of such "completion theorems" as the Atiyah-Segal completion theorem in K-theory and the Segal conjecture in cohomotopy. When G is finite, the Tate theory associated to equivariant K-theory is calculated completely, and the Tate theory associated to equivariant cohomotopy is shown to encode a mysterious web of connections between the Tate cohomology of finite groups and the stable homotopy groups of spheres.

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