

REAL WILSON SPACES I

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INTRODUCTION

In [31] Steve Wilson established the remarkable fact that the even spaces

$$\underline{MU}^{2n} = \Omega^\infty \Sigma^{2n} MU$$

in the complex cobordism spectrum have cell decompositions with only even dimensional cells. One consequence of this is that the unstable MU Adams resolution for a CW complex X with only even cells takes the form

$$X \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$$

in which I_n is the zeroth space of a spectrum of the form

$$\bigvee S^{2n_\alpha} \wedge MU.$$

This leads to an unexpected resolution of X in terms of integer Eilenberg-MacLane spaces.

In joint work with Aravind Asok and Jean Fasel, the second author has explored the hypothesis that the analogue of this holds in motivic homotopy theory. This “Wilson space hypothesis” leads to resolutions of certain \mathbb{A}^1 -homotopy types into motivic complexes. These (conjectural) resolutions have interesting consequences in algebraic geometry.

Once placed in motivic homotopy theory the question of generality of the ground ring arises. If the Wilson Space Hypothesis holds over the real numbers then the topological realization would be a refinement of Wilson’s theorem to the case of *real* homotopy theory. In fact this is the case, and part of the purpose of this series of papers is to establish this refinement. The methods employed actually lead to an analogous result for the spectra $MU^{(C_{2^n})}$ introduced in [9] and in the third paper in this series we formulate and establish this result. Taken together these theorems

suggest that the Wilson Space Hypothesis is something very general, and that one should seek a less computational approach.

In this paper we consider spaces with a C_2 -action, with examples in mind being the complex points of a variety defined over \mathbb{R} . In this spirit we regard \mathbb{C}^n as a C_2 space under complex conjugation, and write

$$S(\mathbb{C}^n) \subset D(\mathbb{C}^n) \subset \mathbb{C}^n$$

for the unit spheres and disks. The natural analogue of a CW complex consisting only of even dimensional cells is a C_2 -space which decomposes into cells of the form $D(\mathbb{C}^n)$

$$X = \coprod D(\mathbb{C}^n) / \sim$$

(with the usual closure finite and weak topology assumptions). We will say that a C_2 -space is ρ -cellular if it is equivariantly weakly equivalent to a space admitting such a decomposition. The Schubert cell decompositions show that the complex points of the real Grassmannians are ρ cellular, and that the classifying space $BU_{\mathbb{R}}$ is ρ cellular. There is an evident analogue of being ρ -cellular in the category of C_2 -spectra, and from the Schubert cell decomposition of Grassmannians one can conclude that the spectrum $MU_{\mathbb{R}}$ of real bordism is ρ cellular.

For a real representation V of C_2 let S^V be its one point compactification. Following [9] the sign representation of C_2 will be denoted by σ and the real regular representation by $\rho = 1 + \sigma$. The representation ρ is realized as the complex numbers with C_2 -action given by complex conjugation, and S^ρ with \mathbf{CP}^1 . This is the reason for the term ρ -cellular.

Given real representations V and W we denote by S^{V-W} the spectrum constructed as the smash product of S^V with the Spanier-Whitehead dual of S^W . The correspondence $V - W \mapsto S^{V-W}$ sends sums of representations to smash products of C_2 -spectra. With this convention the symbol S^V is being used for both a space and its suspension spectrum. Which meaning is intended will be clear from context.

The real analogue of Wilson's result is given by the following

Theorem 1. *For every $n \in \mathbb{Z}$ the C_2 -space*

$$\underline{MU}_{\mathbb{R}}^{n\rho} = \Omega^\infty S^{n\rho} \wedge MU_{\mathbb{R}}$$

is ρ -cellular.

Theorem 1 is proved in [10], and is a consequence of the following apparently weaker result.

Theorem 2. *For any $n \in \mathbb{Z}$ there is a weak equivalence*

$$H\mathbb{Z} \wedge \bigvee S^{n\alpha\rho} \approx H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}^{n\rho}.$$

The implication that Theorem 2 implies Theorem 1 is not very direct. Theorem 2 actually yields the consequences about unstable Adams resolutions one would wish to derive from knowing that $\underline{MU}_{\mathbb{R}}^{n\rho}$ is ρ -cellular, so it's not clear which of these two results is most natural to seek in general.

Most of the work of this paper involves setting up a useful description of the map of Theorem 2, with the more refined statement appearing as Theorem 3.23. This is carried out using the language of *Hopf rings*. Hopf rings were introduced by Ravenel and Wilson in [25], where the homology of the spaces constituting the complex cobordism spectrum were characterized by a universal property. For

equivariant and motivic homotopy theory a refinement of this universal property is needed and in [11] the theory of Hopf rings is developed and this refined universal property is established. A summary of the necessary results is given in §1.

Here is a more detailed summary of the contents of this paper. The first section simply collects the notation and conventions used throughout the paper. In §1 we recall the Ravenel-Wilson theory of “Hopf rings” and describe our refinement of the Ravenel-Wilson universal property. Section 2 contains some mildly technical results about the Hopf rings of interest in this paper. In §3.1 we turn to C_2 equivariant homotopy theory, recall the theory of *real oriented* spectra, and state in precise terms our equivariant refinement of the result of Ravenel-Wilson (Theorem 3.23). The proof of Theorem 3.23 reduces to a question about the mod 2 homology of certain fixed point spaces X^{2k} appearing in the real bordism spectrum. The basic facts about the X^{2k} are established in §4. Finally, everything is put together in §5 where the proof of Theorem 3.23 is completed. The first part of §5 recollects the entirety of the argument, so, after becoming acquainted with the basic setup of this paper, the reader may wish to look there for a further overview.

We prove results about the homology of the spaces X^{2k} by following the inductive argument of Chan [5]. The homology of these spaces has been computed by Kitchloo and Wilson as part of a much more general result [15, Theorem 1.5], and our main theorem about them (Proposition 5.6) can also be deduced from the Kitchloo-Wilson theorem.

Acknowledgements. The authors would like to thank John Francis for the very helpful suggestion of working in the homotopy category of $H\mathbb{Z}$ -modules. They would also like to thank Danny Shi and Dylan Wilson for their interest in this work and for insisting on hearing all of the details.

During the preparation of this manuscript the first author was supported by NSF grant DMS-1509652 and the second author by NSF grant DMS-1510417.

1. UNIVERSAL HOPF RINGS

In this section we recall the Ravenel-Wilson theory of Hopf rings and describe the universal property characterizing the Hopf ring for complex cobordism. Our main purpose is to state a generalization of this universal property (Theorem 1.29). This generalization is proved in [11], and used here to construct the map (3.24) appearing in the statement of the main theorem of this paper (Theorem 3.23).

1.1. Hopf rings and the Ravenel-Wilson theorem. In this section we define the notion of a *Hopf ring* in a symmetric monoidal category. More complete details appear in [11].

1.1.1. *Hopf Rings.* Suppose that $\mathcal{C} = (\mathcal{C}, \otimes)$ is a symmetric monoidal category. Let $\mathbf{coalg} \mathcal{C}$ be the category of counital, cocommutative, coassociative coalgebras in \mathcal{C} . The category $\mathbf{coalg} \mathcal{C}$ has finite products, given on underlying objects in \mathcal{C} by the monoidal product \otimes .

Definition 1.1. A *Hopf ring* in \mathcal{C} is a commutative ring object in $\mathbf{coalg} \mathcal{C}$.

Thus a Hopf ring consists of an object $\mathcal{H} \in \mathbf{coalg} \mathcal{C}$ equipped with addition and multiplication laws

$$(1.2) \quad \begin{aligned} + : \mathcal{H} \times \mathcal{H} &\rightarrow \mathcal{H} \\ \times : \mathcal{H} \times \mathcal{H} &\rightarrow \mathcal{H} \end{aligned}$$

and additive and multiplicative units

$$(1.3) \quad [0], [1] : \mathbf{1} \rightarrow \mathcal{H}$$

satisfying the axioms of a commutative ring. Note that the products in (1.2) are formed in **coalg** \mathcal{C} , so correspond to the monoidal product \otimes in \mathcal{C} , and the maps (1.3) and (1.2) are coalgebra maps.

By the Yoneda lemma, giving a Hopf ring structure on \mathcal{H} is equivalent to lifting the functor

$$\mathcal{H}(-) = \mathbf{coalg} \mathcal{C}(-, \mathcal{H})$$

to one taking values in commutative rings (see [11]).

Example 1.4. The most basic example is when $\mathcal{C} = \mathfrak{S}\mathbf{ets}$, with the symmetric monoidal structure given by the Cartesian product. Since every set has a unique counital coalgebra structure, the forgetful functor $\mathbf{coalg} \mathcal{C} \rightarrow \mathfrak{S}\mathbf{ets}$ is an equivalence of categories. A Hopf ring in $\mathfrak{S}\mathbf{ets}$ is just a commutative ring.

Example 1.5. Similarly, a Hopf ring in the homotopy category $\mathbf{ho} \mathfrak{S}\mathbf{paces}$ of spaces and cartesian products is a ring object in $\mathbf{ho} \mathfrak{S}\mathbf{paces}$. For instance if E is a homotopy commutative ring spectrum then the space $\underline{E}^0 = \Omega^\infty E$ inherits the structure of a Hopf ring in $\mathbf{ho} \mathfrak{S}\mathbf{paces}$.

Example 1.6. Suppose that k is a commutative ring and let \mathbf{Mod}_k^* be the symmetric monoidal category of graded left k -modules, and tensor products with the symmetry given by the usual Koszul sign convention. In concrete terms, a Hopf ring \mathcal{H} over \mathbf{Mod}_k^* is a graded k -module equipped with maps

$$\begin{aligned} \psi : \mathcal{H} &\rightarrow \mathcal{H} \otimes \mathcal{H} \\ \epsilon : \mathcal{H} &\rightarrow k, \end{aligned}$$

making it into a coalgebra, coalgebra maps

$$\begin{aligned} * : \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \\ \circ : \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \end{aligned}$$

corresponding to addition and multiplication, and constants (1.3). The constants may be identified with elements

$$\begin{aligned} [0] &\in \mathcal{H} \\ [1] &\in \mathcal{H}. \end{aligned}$$

The product $*$ makes each \mathcal{H} into a commutative, cocommutative Hopf algebra over k . To keep the notation simple we will denote it by ordinary multiplication. The multiplicative unit for this ring structure is the element $[0]$, so one has

$$1 = [0] \in \mathcal{H}.$$

If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric monoidal functor and \mathcal{H} is a Hopf ring in \mathcal{C} then $F(\mathcal{H})$ is a Hopf ring in \mathcal{D} .

Example 1.7. Suppose that R is an E_∞ ring spectrum and let \mathbf{Mod}_R be the homotopy category of left R -modules, regarded as an additive category. Equipped with the monoidal product

$$M \wedge_R N$$

the category \mathbf{Mod}_R becomes a symmetric monoidal category. The functor

$$\mathcal{F}_R^{\mathrm{mod}}(X) = R \wedge X_+$$

is a symmetric monoidal functor from $\mathrm{ho}\mathfrak{S}\mathrm{paces}$ to \mathbf{Mod}_R . If E is a homotopy commutative ring spectrum then $\mathcal{F}_R^{\mathrm{mod}}(\Omega^\infty E) = R \wedge \Omega^\infty E_+$ is a Hopf ring in \mathbf{Mod}_R .

Example 1.8. If k is a ring, E is a homotopy commutative ring spectrum, and $\underline{E}^0 = \Omega^\infty E$ has the property that

$$H_*(\underline{E}^0; k)$$

is flat over k then $H_*(\underline{E}^0; k)$ is a Hopf ring in \mathbf{Mod}_k^* .

If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric monoidal functor, \mathcal{H} is a Hopf ring in \mathcal{C} , and $X \in \mathbf{coalg}\mathcal{C}$ is a coalgebra, then F provides a ring homomorphism

$$\mathcal{H}(X) \rightarrow (F\mathcal{H})(FX)$$

where, as in the discussion just before Example 1.4, $\mathcal{H}(X)$ denotes the functor $\mathbf{coalg}\mathcal{C}(X, \mathcal{H})$ and $(F\mathcal{H})(FX) = \mathbf{coalg}\mathcal{D}(F\mathcal{H}, FX)$.

Example 1.9. In the situation of Example 1.7, if X is a space and E is a homotopy commutative ring spectrum then the functor $\mathcal{F}_R^{\mathrm{mod}}(-)$ gives a ring homomorphism

$$\underline{E}^0(X) = E^0(X) \rightarrow \mathcal{F}_R^{\mathrm{mod}}(\underline{E}^0)(\mathcal{F}_R^{\mathrm{mod}}(X)) = \mathbf{coalg}\mathbf{Mod}_R(R \wedge X_+, R \wedge \underline{E}_+^0),$$

which may be interpreted as a natural transformation of ring valued functors of X .

1.1.2. *Hopf rings over a ring.* Suppose that A is an ordinary commutative ring.

Definition 1.10. A *Hopf ring over A* in \mathcal{C} is a commutative A -algebra in $\mathbf{coalg}\mathcal{C}$.

Equivalently, a Hopf ring over A in \mathcal{C} is a representable functor from $\mathbf{coalg}\mathcal{C}$ to the category \mathbf{Alg}_A of commutative A -algebras. In addition to the ring structure, a Hopf ring \mathcal{H} over A has constants

$$[a] : \mathbf{1} \rightarrow \mathcal{H} \quad a \in A$$

which, together with the addition, multiplication and units satisfy the axioms of an A -algebra.

If $F : \mathcal{C} \rightarrow \mathcal{D}$ is symmetric monoidal, and \mathcal{H} is a Hopf ring over A in \mathcal{C} then $F\mathcal{H}$ is a Hopf ring over A in \mathcal{D} .

Example 1.11. If E is a ring spectrum then $\underline{E}^0 = \Omega^\infty E$ is a Hopf ring over the ring $E^0 = E^0(\mathrm{pt})$ in the homotopy category of spaces. By functoriality, the free R -module $\mathcal{F}_R^{\mathrm{mod}}(\underline{E}^0)$ is a Hopf ring over E^0 in \mathbf{Mod}_R .

1.1.3. *Evenly graded Hopf rings.* For a category \mathcal{D} let \mathcal{D}^\bullet be the category of evenly graded objects of \mathcal{D} . Thus an object $X^\bullet \in \mathcal{D}^\bullet$ consists of objects $X^m \in \mathcal{D}$ for every even integer $m \in \mathbb{Z}$. When $\mathcal{C} = (\mathcal{C}, \otimes)$ is a symmetric monoidal category we will write

$$\mathbf{coalg}\mathcal{C}^\bullet = (\mathbf{coalg}\mathcal{C})^\bullet$$

for the category of evenly graded objects in $\mathbf{coalg}\mathcal{C}$. We will also employ the notation X^\bullet for an evenly graded object of \mathcal{D} , and use the convention $X^m = X_{-m}$.

Suppose that A^\bullet is an evenly graded commutative ring.

Definition 1.12. A *Hopf ring over A^\bullet* in \mathcal{C} is an evenly graded commutative A^\bullet -algebra in $\mathbf{coalg}\mathcal{C}^\bullet$.

Thus a Hopf ring over A^\bullet in \mathcal{C} is an evenly graded object $\mathcal{H}^\bullet \in \mathbf{coalg} \mathcal{C}^\bullet$, equipped with addition and multiplication maps

$$\begin{aligned} + : \mathcal{H}^{2n} \times \mathcal{H}^{2n} &\rightarrow \mathcal{H}^{2n} \\ \times : \mathcal{H}^n \times \mathcal{H}^{2m} &\rightarrow \mathcal{H}^{2n+2m}, \end{aligned}$$

constants

$$[a] : \mathbf{1} \rightarrow \mathcal{H}^{2n}$$

for every $a \in A^{2n}$, including the additive units

$$[0] : \mathbf{1} \rightarrow \mathcal{H}^{2n},$$

satisfying the axioms of a graded commutative A^\bullet -algebra. We have restricted to *evenly* graded objects in order that there be no confusion with the Koszul sign convention. On evenly graded objects, commutativity is just ordinary commutativity. See [25, §1] and [11] for more explicated details.

Example 1.13. For a spectrum E set

$$\underline{E}^m = \Omega^\infty \Sigma^m E,$$

and, ignoring the odd values of m , consider the evenly graded space \underline{E}^\bullet . If E is a homotopy commutative ring spectrum then \underline{E}^\bullet is a Hopf ring in spaces over the evenly graded ring E^\bullet , in which $E^{2n} = E^{2n}(\text{pt})$.

Example 1.14. By functoriality, if R is an E_∞ ring then $\mathcal{F}_R^{\text{mod}}(\underline{E}^\bullet) = R \wedge \underline{E}_+^\bullet$ is an evenly graded Hopf ring over E^\bullet in \mathbf{Mod}_R .

Example 1.15. If k is a ring, and for each $n \in \mathbb{Z}$, the graded k -module

$$H_*(\underline{E}^{2n}; k)$$

is flat, then $H_*(\underline{E}^\bullet; k)$ is a Hopf ring over E^\bullet in \mathbf{Mod}_k^* .

Notation 1.16. The category $\mathbf{HopfRings}_{R^\bullet}(\mathcal{C})$ is the category of evenly graded Hopf rings over R^\bullet in \mathcal{C} , and Hopf ring homomorphisms over R^\bullet .

1.2. The Ravenel-Wilson theory.

1.2.1. *Examples from complex oriented cohomology theories.* We are especially interested in the evenly graded Hopf rings over MU^\bullet with $MU^{2n} = MU^{2n}(\text{pt})$, and the additional structures present in those that arise from a complex oriented cohomology theory. Suppose that (E, x) is a complex oriented homotopy commutative ring spectrum and let

$$(1.17) \quad MU^\bullet \rightarrow E^\bullet$$

be the map classifying the associated formal group law. As in §1.1.3, the evenly graded space \underline{E}^\bullet is an evenly graded Hopf ring over E^\bullet . Restricting scalars along (1.17) makes \underline{E}^\bullet into a Hopf ring over MU^\bullet . If R is an E_∞ ring then, by Example 1.14, the R -module

$$\mathcal{F}_R^{\text{mod}}(\underline{E}^\bullet)$$

is a Hopf ring over MU^\bullet in the homotopy category \mathbf{Mod}_R of left R -modules.

By definition of the formal group law, the complex orientation $x \in E^2(\mathbf{CP}^\infty)$ satisfies

$$(1.18) \quad \begin{aligned} \beta_0^* x &= 0 \\ \mu^* x &= \sum a_{ij} x^i y^j, \end{aligned}$$

where

$$(1.19) \quad \mu : \mathbf{CP}^\infty \times \mathbf{CP}^\infty \rightarrow \mathbf{CP}^\infty$$

is the map classifying the tensor product of the two universal line bundles, the $a_{ij} \in E^{-2(i+j)}(\text{pt})$ are the coefficients of the formal group law, and

$$\beta_0 : \{\text{pt}\} \rightarrow \mathbf{CP}^\infty$$

is the homotopy class of the inclusion of a point. The identity (1.18) may be interpreted in the degree 2 component of the ring $\underline{E}^\bullet(\mathbf{CP}^\infty \times \mathbf{CP}^\infty)$ associated to the Hopf ring \underline{E}^\bullet over MU^\bullet in $\text{ho}\mathfrak{S}\text{paces}$. By functoriality (Example 1.9) the coalgebra map

$$\tilde{x} = \mathcal{F}_R^{\text{mod}}(x) : \mathcal{F}_R^{\text{mod}}(\mathbf{CP}^\infty) \rightarrow \mathcal{F}_R^{\text{mod}}(\underline{E}^2)$$

satisfies

$$(1.20) \quad \begin{aligned} \beta_0^*(\tilde{x}) &= [0] \\ \mu^* \tilde{x} &= \sum a_{ij} \tilde{x}^i \tilde{y}^j \end{aligned}$$

in which we have written $\beta_0 = \mathcal{F}_R^{\text{mod}}(\beta_0)$ and $\mu = \mathcal{F}_R^{\text{mod}}(\mu)$. To further simplify the notation we will drop the tilde and write $x = \mathcal{F}_R^{\text{mod}}(x)$, etc in the above identity.

If k is a ring and for each $n \in \mathbb{Z}$, $H_*(\underline{E}^{2n}; k)$ is a flat k -module, then $H_*(\underline{E}^\bullet; k)$ is a Hopf ring in \mathbf{Mod}_k^* over MU^\bullet , equipped with an element

$$x : H_*(\mathbf{CP}^\infty; k) \rightarrow H_*(\underline{E}^2; k)$$

satisfying (1.20).

1.2.2. *The Ravenel-Wilson Hopf ring.* In [25] Ravenel and Wilson construct a universal Hopf ring over MU^\bullet equipped with an element x satisfying the relation (1.20).

To make this precise, let $\mathbf{Ab}_* = \mathbf{Mod}_{\mathbb{Z}}^*$ be the symmetric monoidal category of graded abelian groups (with the Koszul sign convention). Denote by $\mathbf{HopfRings}_{MU^\bullet}$ the category of Hopf rings over MU^\bullet in \mathbf{Ab}_* . For brevity, we will refer to the objects of $\mathbf{HopfRings}_{MU^\bullet}$ as *evenly graded Hopf rings over MU^\bullet* , or sometimes as just Hopf rings.

Let \mathbb{A}^1 be the abelian group object in $\mathbf{coalg}\mathbf{Ab}_*$ given by the Hopf algebra $H_*(\mathbf{CP}^\infty; \mathbb{Z})$. Thus \mathbb{A}^1 is the coalgebra

$$\mathbb{Z}\{\beta_0, \beta_1, \dots\} \quad |\beta_i| = 2i,$$

with coproduct given by

$$\beta_n \mapsto \sum_{i+j=n} \beta_i \otimes \beta_j.$$

The product is induced by the map (1.19), and given explicitly by

$$\beta_i \beta_j = \binom{i+j}{i} \beta_{i+j}.$$

For $n \geq 0$ set

$$\mathbb{A}^n = (\mathbb{A}^1)^n = H_*((\mathbf{CP}^\infty)^n; \mathbb{Z}).$$

Write

$$\pi_i : \mathbb{A}^n \rightarrow \mathbb{A}^1$$

for projection to the i^{th} factor, and

$$\mu : \mathbb{A}^2 \rightarrow \mathbb{A}^1$$

for the group law described above. Since our coalgebras are all counital, the coalgebra map

$$\beta_0 : \mathbb{A}^0 \rightarrow \mathbb{A}^1$$

sending $1 \in \mathbb{Z}$ to β_0 is the unique coalgebra map from \mathbb{A}^0 to \mathbb{A}^1 .

Definition 1.21. Suppose that \mathcal{H}^\bullet is an evenly graded Hopf ring over MU^\bullet in $\mathbf{coalg Ab}_*$. An *additive curve* of \mathcal{H}^\bullet is an element

$$x \in \mathcal{H}^2(\mathbb{A}^1)$$

satisfying

$$\begin{aligned} \beta_0^* x &= [0] \in \mathcal{H}^2(\mathbf{1}) \\ \mu^* x &= \sum a_{ij} x^i y^j. \end{aligned}$$

Definition 1.22. The *Ravenel-Wilson Hopf ring* is the evenly graded Hopf ring $\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet$ over MU^\bullet characterized by the following universal property: to give a map

$$\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet \rightarrow \mathcal{H}^\bullet$$

in $\mathbf{HopfRings}_{MU^\bullet}$ is equivalent to giving an element $x \in \mathcal{H}^2(\mathbb{A}^1)$ having the properties

$$(1.23) \quad \begin{aligned} \beta_0^* x &= [0] \in \mathcal{H}^2(\mathbb{Z}) \\ \mu^* x &= \sum a_{ij} x^i y^j \end{aligned}$$

in which the $a_{ij} \in MU^{-2(i+j)}$ are the coefficients of the universal formal group law.

Put differently, the Hopf ring $\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet$ corepresents the functor on $\mathbf{HopfRings}_{MU^\bullet}$ sending \mathcal{H}^\bullet to the set of additive curves of \mathcal{H}^\bullet .

A proof of the existence of $\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet$ is sketched in [25], and given in detail in [13], [28] and [11]. As for the structure of $\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet$, Ravenel and Wilson showed

Theorem 1.24. *For each n the algebra $\mathcal{M}\mathcal{U}_{\text{RW}}^{2n}$ is a polynomial algebra over the group ring $\mathbb{Z}[MU^{2n}]$ on generators of positive even degree. In particular, $\mathcal{M}\mathcal{U}_{\text{RW}}^n$ is free when regarded as a graded abelian group.*

As mentioned in the introduction, the following theorem of Wilson started this whole investigation.

Theorem 1.25 ([31]). *For every integer n , the graded abelian group $H_*(\underline{MU}^n; \mathbb{Z})$ is free. It is the zero group when n is even and $*$ is odd.*

By the discussion at the end of §1.2.1, Theorem 1.25 implies that the graded abelian groups $H_*(\underline{MU}^\bullet; \mathbb{Z})$ form a Hopf ring over MU^\bullet in \mathbf{Ab}_* , equipped with an additive curve $x \in H_*(\underline{MU}^2)(\mathbb{A}^1)$.

Theorem 1.26 ([25], Corollary 4.7). *The map of Hopf rings over MU^\bullet*

$$\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet \rightarrow H_*(\underline{MU}^\bullet)$$

classifying the additive curve x is an isomorphism.

Remark 1.27. In fact, in [31] Wilson also determined the ring structure of $H_*\underline{BP}^n$ and hence $H_*\underline{MU}^n$ for all n . One purpose of [25] was to give new proof of this result organized by the language of Hopf rings.

In [25] the proofs of Theorems 1.24, 1.25 and 1.26 are interleaved, and in fact the result is first proved for homology with coefficients in a field and then the general result is deduced. Without either working over a field, or first assuming Wilson's Theorem 1.25, one does not know in advance that the abelian groups $H_*\underline{MU}^\bullet$ are even coalgebras, much less Hopf rings. In [11] the existence of $\mathcal{MU}_{\text{RW}}^\bullet$ is established purely algebraically, as is Theorem 1.24. Using Theorem 1.24 one can establish a stronger universal property of the Ravenel-Wilson Hopf ring and use it to make the desired map above. This is the subject of the next section.

1.2.3. Generalizing the Ravenel-Wilson universal property. Let $\mathbf{Ab}_\bullet^{\text{fr}}$ be the symmetric monoidal category of evenly graded free abelian groups and tensor products. By Theorem 1.24 the Hopf ring $\mathcal{MU}_{\text{RW}}^\bullet$ can be regarded as a Hopf ring over MU^\bullet in $\mathbf{Ab}_\bullet^{\text{fr}}$. Suppose that (\mathcal{C}, \otimes) is a strongly additive symmetric monoidal category in the sense of Appendix §A, and write $\mathbf{HopfRings}_{MU^\bullet}(\mathcal{C})$ for the category of Hopf rings over MU^\bullet in \mathcal{C} . If

$$\mathbf{G} : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathcal{C}$$

is an additive symmetric monoidal functor then $\mathbf{G}(\mathcal{MU}_{\text{RW}}^\bullet)$ is a Hopf ring over MU^\bullet in \mathcal{C} . Using \mathbf{G} we make the following definition.

Definition 1.28. Suppose that \mathcal{H}^\bullet is an evenly graded Hopf ring over MU^\bullet in \mathcal{C} . An *additive curve* in \mathcal{H}^\bullet is a coalgebra map

$$x : \mathbf{G}(\mathbb{A}^1) \rightarrow \mathcal{H}^\bullet$$

satisfying

$$\begin{aligned} \mathbf{G}(\beta_0)^* x &= [0] \\ \mathbf{G}(\mu)^* x &= \sum a_{ij} x^i y^j. \end{aligned}$$

Let

$$\mathbf{T} : \mathbf{HopfRings}_{MU^\bullet}(\mathcal{C}) \rightarrow \mathfrak{Sets}$$

be the functor sending \mathcal{H}^\bullet to the set of additive curves in \mathcal{H}^\bullet . Since \mathbf{G} is additive, $\mathbf{G}(x_{\text{univ}})$ is an element of $\mathbf{T}(\mathbf{G}(\mathcal{MU}_{\text{RW}}^\bullet))$. The main result of [11] is

Theorem 1.29. *With the above notation, the functor*

$$\mathbf{T} : \mathbf{HopfRings}_{MU^\bullet}(\mathcal{C}) \rightarrow \mathfrak{Sets}$$

is represented by $(\mathbf{G}(\mathcal{MU}_{\text{RW}}^\bullet), \mathbf{G}(x_{\text{univ}}))$.

Phrased more succinctly, the image of the Ravenel-Wilson Hopf ring under a strongly additive symmetric monoidal functor defined on free abelian groups enjoys the same universal property as the Ravenel-Wilson Hopf ring.

Here is a simple application of Theorem 1.29. There is a unique strongly additive symmetric monoidal functor

$$(1.30) \quad \mathbf{C} : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Mod}_{H\mathbb{Z}}$$

satisfying

$$\mathbf{C}(\mathbb{Z}[2n]) = H\mathbb{Z} \wedge S^{2n}.$$

In fact \mathbf{C} is an equivalence between $\mathbf{Ab}_\bullet^{\text{fr}}$ and the smallest full subcategory of $\mathbf{Mod}_{H\mathbb{Z}}^T \subset \mathbf{Mod}_{H\mathbb{Z}}$ containing the $S^{2n} \wedge H\mathbb{Z}$ and closed under arbitrary coproducts. If (E, x) is a homotopy commutative complex oriented cohomology theory then by §1.2.1, the evenly graded coalgebra $\mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(\underline{E}^\bullet) = H\mathbb{Z} \wedge (\underline{E}^\bullet)_+$ becomes a Hopf ring

over MU^\bullet in $\mathbf{coalg} \mathbf{Mod}_{HZ}$ equipped with an additive curve. By Theorem 1.29 there is a unique Hopf ring map

$$\mathbf{C}(\mathcal{M}\mathcal{U}_{\text{RW}}) \rightarrow \mathcal{F}_{HZ}^{\text{mod}}(\underline{E}^\bullet)$$

classifying the additive curve. When $E = MU$ this can be used to define the map of Theorem 1.26 directly.

Remark 1.31. The inverse of the equivalence $\mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Mod}_{HZ}^T$ sends M to $\pi_\bullet M$. By definition then, one has

$$\mathbf{C}(\mathbb{A}^k) = \mathcal{F}_{HZ}^{\text{mod}}((\mathbf{CP}^\infty)^k) = HZ \wedge (\mathbf{CP}^\infty)_+^k.$$

While this application of Theorem 1.29 is relatively minor, in equivariant and motivic homotopy theory it helps significantly.

2. VERSCHIEBUNG AND DECOMPOSITION

Our aim in this section is to set up a certain tensor product decomposition (Proposition 2.33) that will aid the proof of our main result (Theorem 3.23). We also recall an important technical result of Ravenel and Wilson (Proposition 2.36).

2.1. Connected components. Suppose that $\mathcal{C} = \mathbf{Mod}_k^*$ is the category of graded modules over a commutative ring k , made into a symmetric monoidal category with the tensor product and the Koszul sign rule.

Definition 2.1. A coalgebra $C \in \mathbf{coalg} \mathcal{C}$ is *(-1)-connected* if $C_i = 0$ for $i < 0$. A coalgebra is *connected* if it is (-1)-connected and if the counit map $\epsilon : C \rightarrow k$ is an isomorphism.

When C is (-1) connected the projection map $p : C \rightarrow C_0$ is a coalgebra map.

Definition 2.2. A *grouplike element* of C is a coalgebra map $\mathbf{1} \rightarrow C$.

Remark 2.3. Since our maps of coalgebras are counital, the composition $\mathbf{1} \rightarrow C \xrightarrow{\epsilon} \mathbf{1}$ is the identity map.

Definition 2.4. The set $\pi_0 C$ is the set

$$C(\mathbf{1}) = \mathbf{coalg} \mathcal{C}(\mathbf{1}, C)$$

of grouplike elements of C .

A grouplike element $\mathbf{1} \rightarrow C$ is uniquely determined by the image of $1 \in k$ which can be any element $x \in C_0$ with $\epsilon(x) = 1$ and $\psi(x) = x \otimes x$ where $\psi : C \rightarrow C \otimes C$ is the coalgebra structure map.

Example 2.5. The functor sending a set T to the free k -module $k\{T\}$ is symmetric monoidal, giving $k\{T\}$ the structure of a coalgebra. When k has no idempotents other than 0 and 1, the inclusion of the generators $T \rightarrow k\{T\}$ is an isomorphism of T with the set of grouplike elements of $k\{T\}$.

There is a canonical coalgebra map

$$(2.6) \quad k\{\pi_0 C\} = \bigoplus_{\pi_0 C} k \rightarrow C_0.$$

Definition 2.7. A coalgebra $C \in \mathbf{Mod}_k^*$ is *spacelike* if C is (-1)-connected, and the map (2.6) is an isomorphism.

Example 2.8. Let k be a commutative ring with no idempotents other than 0 and 1. If X is a space and $H_*(X; k)$ is flat over k , then $H_*(X; k)$ is a coalgebra in \mathbf{Mod}_k^* . The coalgebra map induced by the inclusion of a point $x \in X$ defines a grouplike element of $H_*(X; k)$, depending only on the path component of x . The resulting map $\pi_0 X \rightarrow \pi_0 H_*(X; k)$ is a bijection and $H_*(X; k)$ is spacelike.

Suppose that C is a (-1) -connected coalgebra and $b \in \pi_0 C$ is a grouplike element.

Definition 2.9. The *connected component of C containing b* is the sub coalgebra $C' = C'_b \subset C$ consisting of elements $x \in C$ whose image under

$$C \rightarrow C \otimes C \xrightarrow{\text{Id} \otimes p} C \otimes C_0$$

is $x \otimes b$.

That C' actually is a subcoalgebra follows easily from the coassociativity law, and the fact that the map $b : \mathbf{1} \rightarrow C$ is split by the counit and so the inclusion of a k -module summand.

Remark 2.10. The coalgebra C'_b fits into a pullback diagram

$$(2.11) \quad \begin{array}{ccc} C'_b & \longrightarrow & C \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & C_0 \end{array}$$

in $\mathbf{coalg Mod}_k^*$.

Example 2.12. Suppose X is a pointed space for which $H_*(X; k)$ is flat over k . As in Example 2.8, the homology class b of the base point is a grouplike element of $H_0 X$, and the coalgebra connected component

$$H_*(X; k)' = H_*(X; k)'_b$$

is the homology of the connected component $X' \subset X$ of X containing the base point.

We highlight one piece of notation used in the above example.

Notation 2.13. If X is a pointed space, with base point $x \in X$, then $X' \subset X$ is the component of X containing the base point.

2.2. Hopf algebras.

Definition 2.14. Let \mathcal{C} be a symmetric monoidal category. A *Hopf algebra in \mathcal{C}* is an abelian group object $\mathcal{A} \in \mathbf{coalg } \mathcal{C}$.

When $\mathcal{C} = \mathbf{Mod}_k^*$ a Hopf algebra \mathcal{A} in \mathcal{C} can be described as a coalgebra \mathcal{A} , equipped with a coalgebra map

$$\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$$

and an additive unit $[0] : \mathbf{1} \rightarrow \mathcal{A}$ making \mathcal{A} into a commutative ring. This data gives $\mathbf{coalg Mod}_k^*(\mathcal{A}, \mathcal{A})$ the structure of an abelian monoid and is subject to the further condition that this abelian monoid structure forms an abelian group. This

condition is equivalent to the existence of a (necessarily unique) *antipode* $\chi : A \rightarrow A$, which is both a coalgebra and an algebra map, making the diagram

$$\begin{array}{ccc} A & \longrightarrow & A \otimes A \\ [0] \circ \epsilon \downarrow & & \downarrow 1 \otimes \chi \\ A & \longleftarrow & A \otimes A \end{array}$$

commute.

Notation 2.15. If \mathcal{A} is a (-1) -connected Hopf algebra in \mathbf{Mod}_k^* then $\mathcal{A}' \subset \mathcal{A}$ is the connected component of the additive unit $[0]$.

Definition 2.16. A Hopf algebra \mathcal{A} in \mathbf{Mod}_k^* is *spacelike* if the underlying coalgebra is spacelike.

If \mathcal{A} is a Hopf algebra in \mathcal{C} then, by definition, the set $\pi_0 \mathcal{A} = \mathcal{A}(\mathbf{1})$ is an abelian group.

Example 2.17. In the situation of Example 2.5, if T is an abelian group then the group algebra $k[T]$ is a Hopf algebra in \mathbf{Mod}_k^* , and the isomorphism

$$T \rightarrow \pi_0 k[T]$$

is an isomorphism of abelian groups.

Note that if \mathcal{A} is a Hopf algebra in \mathbf{Mod}_k^* then the inclusion $\mathcal{A}_0 \rightarrow \mathcal{A}$ is a map of Hopf algebras. If \mathcal{A} is (-1) -connected this map is split by the Hopf algebra map $\mathcal{A} \rightarrow \mathcal{A}_0$.

Proposition 2.18. *If \mathcal{A} is a (-1) -connected Hopf algebra in \mathbf{Mod}_k^* then the map*

$$\mathcal{A}_0 \otimes \mathcal{A}' \rightarrow \mathcal{A}$$

is an isomorphism of Hopf algebras. In particular if \mathcal{A} is spacelike then the map

$$k[\pi_0 \mathcal{A}] \otimes \mathcal{A}' \rightarrow \mathcal{A}$$

is an isomorphism.

Proof: Since the forgetful functor from Hopf algebras to coalgebras creates limits Remark 2.10 implies that the square

$$\begin{array}{ccc} \mathcal{A}'_b & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathcal{A}_0 \end{array}$$

is a pullback square of Hopf algebras. This implies that if \mathcal{B} is any Hopf algebra then

$$0 \rightarrow \mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}') \rightarrow \mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}) \rightarrow \mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}_0) \rightarrow 0$$

is a split short exact sequence in the category $\mathbf{HopfAlg} = \mathbf{HopfAlg} \mathbf{Mod}_k^*$ of abelian group objects in $\mathbf{coalg} \mathbf{Mod}_k^*$, and so gives an isomorphism

$$\mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}) \approx \mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}') \times \mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}_0) = \mathbf{HopfAlg}(\mathcal{B}, \mathcal{A}' \otimes \mathcal{A}_0).$$

The claim follows. \square

2.3. Weight k curves. Let \mathbf{Vect}_* be the symmetric monoidal category of \mathbb{Z} -graded vector spaces over \mathbb{F}_2 . There is a unique coproduct preserving functor

$$\mathbf{V} : \mathbf{Vect}_* \rightarrow \mathbf{Mod}_{H\mathbb{F}_2}$$

sending $\mathbb{F}_2[k]$ to $H\mathbb{F}_2 \wedge S^{2k}$. The following is straightforward

Proposition 2.19. *The functor*

$$\mathbf{V} : \mathbf{Vect}_* \rightarrow \mathbf{Mod}_{H\mathbb{F}_2}$$

is an equivalence of strongly additive symmetric monoidal categories, with inverse given by π_ .* \square

For $k \in \mathbb{Z}$ let $\Phi^k : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Vect}_*$ be the functor given by

$$\Phi^k(A)_m = \begin{cases} A_{2n} \otimes \mathbb{F}_2 & \text{if } m = kn \\ 0 & \text{otherwise.} \end{cases}$$

For each k , the functor Φ^k is symmetric monoidal and so induces functors

$$\Phi^k : \mathbf{coalg Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{coalg Vect}_*$$

$$\Phi^k : \mathbf{HopfRings}_{MU_\bullet}(\mathbf{Ab}_\bullet^{\text{fr}}) \rightarrow \mathbf{HopfRings}_{MU_\bullet}(\mathbf{coalg Vect}_*).$$

For $\mathcal{H}^\bullet \in \mathbf{HopfRings}_{MU_\bullet}(\mathbf{Ab}_\bullet^{\text{fr}})$ and $a \in MU^{2n}$, the scalar $[a] : \mathbb{F}_2 \rightarrow \Phi^k \mathcal{H}^{2n}$ is obtained by applying Φ^k to the scalar $[a] : \mathbb{Z} \rightarrow \mathcal{H}^{2n}$.

For $n \geq 0$ let

$$\mathbb{A}^n(k) = \Phi^k(\mathbb{A}^n) \in \mathbf{coalg Vect}_*.$$

Since $\mathbb{A}^0(k) = \mathbb{A}^0(0) = \mathbb{F}_2$ we will just write \mathbb{A}^0 for this object.

Definition 2.20. An *additive curve of weight k* in a Hopf ring $\mathcal{H}^\bullet \in \mathbf{HopfRings}_{MU_\bullet}(\mathbf{Vect}_*)$ is an element

$$x \in \mathcal{H}^2(\mathbb{A}^1(k)).$$

satisfying

$$\begin{aligned} \beta_0^* x &= [0] \\ \mu^* x &= \sum a_{ij} x^i y^j. \end{aligned}$$

The following is immediate from the definition of $\mathcal{MU}_{RW}^\bullet$.

Proposition 2.21. *The Hopf ring $\Phi^k \mathcal{MU}_{RW}^\bullet$ corepresents the functor sending a Hopf ring $\mathcal{H}^\bullet \in \mathbf{HopfRings}_{MU_\bullet}(\mathbf{Vect}_*)$ to the set of additive curves of weight k in \mathcal{H}^\bullet .* \square

2.4. Verschiebung. The functor sending a graded vector space $f : V \in \mathbf{Vect}_*$ to the graded vector space V^ϕ with

$$\begin{aligned} V_{2n}^\phi &= V_n \\ V_{2n+1}^\phi &= 0, \end{aligned}$$

and a map $f : V \rightarrow W$ to the map $f_{2n}^\phi = f_n$ is symmetric monoidal. It therefore induces a product preserving functor

$$(-)^\phi : \mathbf{coalg Vect}_* \rightarrow \mathbf{coalg Vect}_*.$$

The *Verschiebung* is a natural transformation

$$\mathbf{v} : C \rightarrow C^\phi$$

of functors on $\mathbf{coalg\ Vect}_*$. To define it, note that the map

$$v \rightarrow v \otimes v$$

gives an isomorphism of C_n with the Tate cohomology

$$(2.22) \quad (\ker(1 - \tau)) / (\text{image}(1 + \tau)),$$

where $\tau : (C \otimes C)_{2n} \rightarrow (C \otimes C)_{2n}$ is defined by

$$\tau(x \otimes y) = y \otimes x.$$

For $x \in C_{2n}$, The Verschiebung $\mathbf{v}(x)$ is the element of C_n corresponding to the image of the coproduct $\Delta(x)$ in (2.22).

While the Verschiebung \mathbf{v} is always additive in the sense that

$$\mathbf{v}(x + y) = \mathbf{v}(x) + \mathbf{v}(y),$$

it is not linear over a general ground field k , but rather satisfies

$$\mathbf{v}(\lambda^2 x) = \lambda \mathbf{v}(x).$$

In this paper we are working over \mathbb{F}_2 , where \mathbf{v} is in fact linear.

Remark 2.23. There is a Verschiebung map in any characteristic $p > 0$ given by the modification of the above in which τ is replaced by the cyclic permutation $C^{\otimes p} \rightarrow C^{\otimes p}$, and with the evident modification of the functor $(-)^{\phi}$.

Example 2.24. The coalgebra $\mathbb{A}^1(k) \in \mathbf{coalg\ Vect}_*$ has basis

$$\{\beta_0(k), \beta_1(k), \dots \mid |\beta_i(k)| = i k\}$$

and coproduct given by

$$\Delta(\beta_n(k)) = \sum_{i+j=n} \beta_i(k) \otimes \beta_j(k).$$

One has $\mathbb{A}^1(k)^{\phi} = \mathbb{A}^1(2k)$ and the Verschiebung map is given by

$$\begin{aligned} \beta_{2n}(k) &\mapsto \beta_n(2k) \\ \beta_{2n+1}(2k) &\mapsto 0. \end{aligned}$$

If \mathcal{H}^{\bullet} is an evenly graded Hopf ring over an evenly graded ring A^{\bullet} then $(\mathcal{H}^{\bullet})^{\phi}$ is an evenly graded Hopf ring over A^{\bullet} and the Verschiebung map

$$\mathcal{H}^{\bullet} \rightarrow (\mathcal{H}^{\bullet})^{\phi}$$

is a map of Hopf rings over A^{\bullet} . For a coalgebra C , the Verschiebung provides two maps map of evenly graded A^{\bullet} -algebras

$$\begin{aligned} \mathcal{H}^{\bullet}(C^{\phi}) &\rightarrow \mathcal{H}^{\bullet}(C) \\ \mathcal{H}^{\bullet}(C) &\rightarrow (\mathcal{H}^{\bullet})^{\phi}(C) \end{aligned}$$

related by the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{\bullet}(C^{\phi}) & \longrightarrow & \mathcal{H}^{\bullet}(C) \\ \downarrow & & \downarrow \\ (\mathcal{H}^{\bullet})^{\phi}(C^{\phi}) & \longrightarrow & (\mathcal{H}^{\bullet})^{\phi}(C) . \end{array}$$

Remark 2.25. The grading that is doubled by ϕ is the internal grading in \mathbf{Vect}_* and is independent of the ‘‘even grading’’ $(-)^{\bullet}$.

Proposition 2.26. *The Verschiebung map*

$$(2.27) \quad \Phi^k(\mathcal{M}\mathcal{U}_{RW}^\bullet) \rightarrow \Phi^{2k}\mathcal{M}\mathcal{U}_{RW}^\bullet$$

is a surjective map of evenly graded objects of \mathbf{Vect}_* .

Proof: Here is the idea of the proof. In the framework used in [25] the Hopf ring $\Phi^k(\mathcal{M}\mathcal{U}_{RW}^\bullet)$ is “generated” by the $b_i(k)$ and the constants $[a] \in MU^\bullet$. Since, by Example 2.24, the generators are all in the image of (2.27), the map must be surjective.

To make this precise we will make use of material in [11]. Since the structure of an evenly graded Hopf ring over MU^\bullet is given by a collection of maps between products, for any symmetric monoidal category \mathcal{C} , the forgetful functor

$$(2.28) \quad \mathbf{HopfRings}_{MU^\bullet}(\mathcal{C}) \rightarrow (\mathbf{coalg} \mathcal{C})^\bullet$$

commutes with reflexive coequalizers and therefore sends effective epimorphism to effective epimorphisms. So while the map of coalgebras underlying a surjective map in $\mathbf{HopfRings}_{MU^\bullet}(\mathbf{Vect}_*)$ need not be surjective in general, it is if it is part of a reflexive coequalizer diagram. If \mathcal{C} has enough colimits then the forgetful functor (2.28) has a left adjoint which we denote \mathcal{F} . The Hopf ring $\Phi^k\mathcal{M}\mathcal{U}_{RW}^\bullet$ is constructed from the (reflexive) coequalizer diagram

$$\mathcal{F}(\mathbb{A}^2(k) \amalg \mathbb{A}^1(k) \amalg \mathbb{A}^0) \rightrightarrows \mathcal{F}(\mathbb{A}^1(k)) \rightarrow \Phi^k\mathcal{M}\mathcal{U}_{RW}^\bullet$$

in which $\mathbb{A}^i(k)$ is regarded as an evenly graded object concentrated entirely in degree $2i$, and the parallel arrows express the identities of Definition 1.21 on the summand corresponding to $\mathbb{A}^2(k) \amalg \mathbb{A}^0$ and are the identity on $\mathbb{A}^1(k)$. This implies that the map

$$\mathcal{F}(\mathbb{A}^1(k)) \rightarrow \Phi^k\mathcal{M}\mathcal{U}_{RW}^\bullet$$

is an effective epimorphism, so to show that the Verschiebung

$$\Phi^k\mathcal{M}\mathcal{U}_{RW}^\bullet \rightarrow \Phi^{2k}\mathcal{M}\mathcal{U}_{RW}^\bullet$$

is an effective epimorphism it suffices to show that

$$\mathcal{F}(\mathbf{v}) : \mathcal{F}(\mathbb{A}^1(k)) \rightarrow \mathcal{F}(\mathbb{A}^1(2k))$$

is one. Since \mathcal{F} is a left adjoint this reduces to showing that the Verschiebung map

$$\mathbb{A}^1(k) \rightarrow \mathbb{A}^1(2k)$$

is an effective epimorphism. This is easily checked directly and a formula can be derived by taking the linear dual of the reflexive equalizer of graded \mathbb{F}_2 algebras

$$\mathbb{F}_2[t] \rightrightarrows \mathbb{F}_2[x, y] \rightarrow \mathbb{F}_2[x, y]/(x^2 - y^2),$$

in which the horizontal arrows send t to x^2 and y^2 respectively. \square

2.5. The Verschiebung ideal. Suppose that $C \in \mathbf{coalg\ Vect}_*$ is a coalgebra equipped with a point $x : \mathbf{1} \rightarrow C$. From this data one can construct a coalgebra K by the pullback square

$$\begin{array}{ccc} K & \longrightarrow & C \\ \downarrow & & \downarrow \mathbf{v} \\ \mathbf{1} & \xrightarrow{x} & C^\phi . \end{array}$$

When C is a Hopf algebra and $x = [0]$ is the additive unit, then the diagram above is a pullback diagram of abelian groups in $\mathbf{coalg\ Vect}_*$ and K is the *Verschiebung kernel*. When C is a Hopf ring then K is an ideal we will call the *Verschiebung ideal*. If $C = \mathcal{H}^\bullet$ is an evenly graded Hopf ring over an evenly graded ring A^\bullet then the Verschiebung ideal $K = K^\bullet$ is an evenly graded module over \mathcal{H}^\bullet and the map $K^\bullet \rightarrow \mathcal{H}^\bullet$ is a module map. We are interested in one particular example of this construction.

Definition 2.29. The (*Ravenel-Wilson*) *Verschiebung ideal* K_{RW}^\bullet is defined by the pullback square

$$\begin{array}{ccc} K_{\text{RW}}^\bullet & \longrightarrow & \Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^\bullet \\ \downarrow & & \downarrow \\ \mathbf{1} & \xrightarrow{[0]} & \Phi^2 \mathcal{M}\mathcal{U}_{\text{RW}}^\bullet . \end{array}$$

We now turn to our main decomposition.

Proposition 2.30. *For each k the Hopf algebra $\mathcal{M}\mathcal{U}_{\text{RW}}^{2k}$ is spacelike.*

Proof: This is part of Proposition 1.24. □

Combined with Proposition 2.30, Proposition 2.18 gives the decomposition

$$(2.31) \quad \mathcal{M}\mathcal{U}_{\text{RW}}^{2k} \approx (\mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \otimes \mathbb{Z}[MU^{2k}].$$

Since the Verschiebung is a bijection on group like elements, the Ravenel-Wilson Verschiebung ideal could just as well be defined by the pullback square

$$(2.32) \quad \begin{array}{ccc} K_{\text{RW}}^{2k} & \longrightarrow & \Phi^1 (\mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \\ \downarrow & & \downarrow \\ \mathbf{1} & \xrightarrow{[0]} & \Phi^2 (\mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' . \end{array}$$

Now the two right terms in (2.32) are free commutative algebras by Theorem 1.24 and the right vertical map is surjective by Proposition 2.26. This implies

Proposition 2.33. *For each k there exists a commutative algebra section of the Verschiebung map in (2.32). Associated to a choice of section is an isomorphism of commutative rings*

$$(2.34) \quad \Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k} \approx K^{2k} \otimes (\Phi^2 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \otimes \mathbb{F}_2[MU^{2k}].$$

□

2.6. A further technical result. Our main computation requires Proposition 2.36 below, which is an additional technical result of Ravenel-Wilson [25]. It is not stated explicitly in [25] as a theorem, but appears in an inductive argument as [25, (4.19)] where it is a composition of isomorphisms [25, (4.19)] and [25, (4.20)], and the statement immediately following [25, (4.20)].

Suppose that k is a commutative ring and that \mathcal{H}^\bullet an evenly graded Hopf ring in \mathbf{Mod}_k^* . The graded abelian group structure makes each \mathcal{H}^{2n} into a Hopf algebra. As in Example 1.6 we denote the coproduct and unit by

$$\begin{aligned}\psi : \mathcal{H}^{2n} &\rightarrow \mathcal{H}^{2n} \otimes \mathcal{H}^{2n} \\ \epsilon : \mathcal{H}^{2n} &\rightarrow k,\end{aligned}$$

There is a map

$$\circ : \mathcal{H}^{2n} \otimes \mathcal{H}^{2m} \rightarrow \mathcal{H}^{2n+2m}$$

corresponding to multiplication, and there are constants

$$\begin{aligned}[0] &= [0]_{2n} \in \mathcal{H}^{2n} \\ [1] &\in \mathcal{H}^0.\end{aligned}$$

The multiplicative unit for the Hopf algebra structure is the constant $[0]$, so we have

$$1 = [0] \in \mathcal{H}^{2n}.$$

The kernel of ϵ is the *augmentation ideal* $I^{2n} \subset \mathcal{H}^{2n}$ and the k -module

$$(2.35) \quad Q\mathcal{H}^{2n} = I^{2n} / (I^{2n})^2$$

is the module of indecomposables. The \circ product makes $Q\mathcal{H}^\bullet$ into an evenly graded commutative ring.

The fact that multiplication by 0 in a ring is 0 is expressed in a Hopf ring by the identity

$$1 \circ r = [0] \circ r = \epsilon(r).$$

If $e \in \mathcal{H}^{2m}$ is primitive and $a, b \in I^{2n}$ then

$$e \circ (ab) = \epsilon(a)(e \circ b) + (e \circ a)\epsilon(b) = 0$$

so that “circle product with e ” gives a map

$$e \circ (-) : Q\mathcal{H}^{2n} \rightarrow \mathcal{H}^{2m+2n},$$

whose image lies in the group of primitive elements of \mathcal{H}^{2m+2n} .

Let

$$b_i = x_*(\beta_i)$$

be the image of β_i under the additive curve

$$x : \mathbb{A}^1 \rightarrow H_*(\underline{MU}^2; \mathbb{F}_2).$$

From the assumption $\beta_0^*(x) = [0]$ we have $b_0 = 1$, and the element b_1 is primitive.

Proposition 2.36 ([25]). *The map*

$$b_1 \circ (-) : QH_*(\underline{MU}^{2k}; \mathbb{F}_2) \rightarrow H_*((\underline{MU}^{2k+2})'; \mathbb{F}_2)$$

is an isomorphism of $QH_(\underline{MU}^{2k}; \mathbb{F}_2)$ with the primitives in*

$$H_*((\underline{MU}^{2k+2})'; \mathbb{F}_2),$$

where $(\underline{MU}^{2k+2})' \subset \underline{MU}^{2k+2}$ is the connected component containing the base point. In particular it is a monomorphism.

By Theorem 1.26, the above has a purely algebraic counterpart.

Corollary 2.37. *The map*

$$b_1 \circ : Q(\mathrm{MUR}^{2k} \otimes \mathbb{F}_2) \rightarrow \mathrm{MU}_{RW}^{2k+2} \otimes \mathbb{F}_2$$

induces an isomorphism of $Q(\mathrm{MUR} \otimes \mathbb{F}_2)^{2k}$ with the primitives in

$$(\mathrm{MU}_{RW}^{2k+2} \otimes \mathbb{F}_2)'.$$

In particular it is a monomorphism.

Remark 2.38. Since primitive elements are annihilated by the Verschiebung, the image of $b_1 \circ (-)$ is contained in the Verschiebung ideal. This also follows from the fact that $\mathbf{v}(b_1(1)) = 0$ (Example 2.24) since the Verschiebung ideal is an ideal.

For more on the definition and summary of the useful formulae for graded Hopf rings in the category of graded abelian groups the reader is referred to [25, Lemma 1.12].

3. THE EQUIVARIANT THEORY

In this section we turn to equivariant stable homotopy theory and the Hopf rings arising from real oriented cohomology theories. Our main purpose is to set up the equivalence \mathbf{M} between the category of evenly graded free abelian groups and the category of *pure modules* over the equivariant Eilenberg-MacLane spectrum $H\mathbb{Z}$. Using this we transport the Ravenel-Wilson Hopf ring to equivariant homotopy theory and, appealing to Theorem 1.29, construct the map (3.24) appearing in the statement of Theorem 3.23, our main result.

General references for equivariant stable homotopy theory [22, 8, 17, 21, 26], and [9, §2 and Appendix B]

3.1. Equivariant R -modules. For a G -equivariant E_∞ ring R let \mathbf{Mod}_R denote the homotopy category of equivariant left R -modules and equivariant R -module maps, regarded as an additive category. The category \mathbf{Mod}_R has arbitrary coproducts and products. It becomes an strongly additive symmetric monoidal category under

$$M \otimes N = M \wedge_R N,$$

with R as the tensor unit. The functor

$$\mathcal{F}_R^{\mathrm{mod}}(X) = R \wedge X_+$$

from the homotopy category $\mathrm{ho}\mathfrak{Spaces}^G$ of G -spaces and equivariant maps to \mathbf{Mod}_R is symmetric monoidal when the category of pointed G -spaces is equipped with the symmetric monoidal structure given by the Cartesian product.

3.2. Restriction and geometric fixed points. Associated to an inclusion $i : H \subset G$ there is a strongly additive symmetric monoidal restriction functor

$$\mathrm{res}_i = \mathrm{res} : \mathbf{Mod}_R \rightarrow \mathbf{Mod}_{\mathrm{res} R}.$$

The functor

$$\begin{aligned} \mathbf{Mod}_{\mathrm{res} R} &\rightarrow \mathbf{Mod}_R \\ X &\mapsto G_+ \wedge_H X \end{aligned}$$

is both a left and right adjoint to res (the Wirthmüller isomorphism). There is also a symmetric monoidal *geometric fixed point* functor [9, §B.10.6]

$$\Phi^G : \mathbf{Mod}_R \rightarrow \mathbf{Mod}_{\Phi^G R}.$$

Proposition 3.1. *A map $X \rightarrow Y$ in \mathbf{Mod}_R is an isomorphism if and only if for every $H \subset G$ the map*

$$\Phi^H X \rightarrow \Phi^H Y$$

is an isomorphism.

Proof: This follows directly from, say, [9, Proposition 2.52]. \square

The following result is useful for analyzing the symmetric monoidal structure on \mathbf{Mod}_R .

Lemma 3.2. *Suppose that G is a group and E is a G -equivariant homotopy commutative ring with the property that the map*

$$\pi_0^G(S^0) \rightarrow \pi_0^G E$$

induced by the unit $S^0 \rightarrow E$ factors through the restriction

$$\pi_0^G S^0 \rightarrow \pi_0 S^0 = \mathbb{Z}.$$

If V is a representation of G then the symmetry map $S^V \wedge S^V \rightarrow S^V \wedge S^V$ induces

$$(-1)^{\dim V} : E \wedge S^V \wedge S^V \rightarrow E \wedge S^V \wedge S^V.$$

Proof: The symmetry map

$$S^V \wedge S^V \rightarrow S^V \wedge S^V$$

defines an element

$$\epsilon \in \pi_{2V}^G(S^{2V}) \approx \pi_0^G(S^0),$$

and the assertion is that, under the map induced by the unit, the image of this element in $\pi_0^G E$ is $(-1)^{\dim V}$. By assumption, the map

$$\pi_0^G S^0 \rightarrow \pi_0^G E$$

factors through the restriction

$$\pi_0^G S^0 \rightarrow \pi_0 S^0 = \mathbb{Z}.$$

The result follows from the fact that the degree of the underlying non-equivariant map of spheres is $(-1)^{\dim V}$. \square

Remark 3.3. Lemma 3.2 does not hold without smashing with E . The symmetry map corresponds to the element of the Burnside ring mapping to $(-1)^{\dim V^H}$ under the character sending a G -set to its H fixed points. For instance when $G = C_2$ and $V = \rho$ this element is $(-1 + [C_2])$.

3.3. Pure modules. We now specialize to the case $G = C_2$. Following [9] we let σ be the sign representation and ρ the real regular representation of G . Denote by $\underline{\mathbb{Z}}$ the constant Mackey functor \mathbb{Z} , and $H\underline{\mathbb{Z}}$ the corresponding Eilenberg-MacLane spectrum.

3.3.1. *The category of pure modules.* Let $\mathbf{Mod}_{H\mathbb{Z}}^T \subset \mathbf{Mod}_{H\mathbb{Z}}$ be the smallest full subcategory of $\mathbf{Mod}_{H\mathbb{Z}}$ containing the objects $H\mathbb{Z} \wedge S^{2k\rho}$ for all $2k$, and closed under arbitrary coproducts. The subcategory $\mathbf{Mod}_{H\mathbb{Z}}^T$ is a strongly additive symmetric monoidal category. We will call objects of $\mathbf{Mod}_{H\mathbb{Z}}^T$ *pure*.

Lemma 3.4. *For integers $k, \ell \in \mathbb{Z}$, the restriction mappings*

$$\begin{aligned} \mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z} \wedge S^{k\rho}, H\mathbb{Z} \wedge S^{\ell\rho}) &\rightarrow \mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z} \wedge S^{2k}, H\mathbb{Z} \wedge S^{2\ell}) \\ \mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z} \wedge S^{k\rho-1}, H\mathbb{Z} \wedge S^{\ell\rho}) &\rightarrow \mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z} \wedge S^{2k-1}, H\mathbb{Z} \wedge S^{2\ell}) \end{aligned}$$

are isomorphisms, and so

$$\mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z} \wedge S^{k\rho}, H\mathbb{Z} \wedge S^{\ell\rho}) = \begin{cases} \mathbb{Z} & k = \ell \\ 0 & k \neq \ell, \end{cases}$$

and for all k and ℓ

$$\mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z} \wedge S^{k\rho-1}, H\mathbb{Z} \wedge S^{\ell\rho}) = 0.$$

Proof: In the language of [9] this is just the assertion that $H\mathbb{Z} \wedge S^{k\rho}$ is a $2k$ -slice and $H\mathbb{Z} \wedge S^{k\rho-1}$ is a $(2k-1)$ -slice. It is easily checked directly. By tensoring with the inverse of $H\mathbb{Z} \wedge S^{k\rho}$ and using the isomorphism

$$\mathbf{Mod}_{H\mathbb{Z}}(H\mathbb{Z}, H\mathbb{Z} \wedge X) = H_0(X; \mathbb{Z}) = H^0(DX; \mathbb{Z})$$

one reduces to checking that the restriction mapping

$$\tilde{H}_{C_2}^0(S^0, \mathbb{Z}) \rightarrow \tilde{H}^0(S^0; \mathbb{Z})$$

is an isomorphism and that for all $\ell > 0$

$$\tilde{H}_0^{C_2}(S^{\ell\rho}; \mathbb{Z}) = \tilde{H}^0(S^{\ell\rho}; \mathbb{Z}) = 0.$$

The first is the definition of the constant Mackey functor and the second follows from the fact that $S^{\ell\rho}$ is obtained from S^ℓ by attaching free cells of dimension $(\ell+1)$ and higher (see [9, §3.3]). \square

Corollary 3.5. *If X and Y are pure $H\mathbb{Z}$ -modules then the restriction mapping*

$$\mathbf{Mod}_{H\mathbb{Z}}(X, Y) \rightarrow \mathbf{Mod}_{H\mathbb{Z}}(\text{res } X, \text{res } Y)$$

is an isomorphism. \square

3.3.2. *Pure modules and graded free abelian groups.* There is a unique coproduct preserving functor

$$(3.6) \quad \mathbf{M} : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Mod}_{H\mathbb{Z}}^T$$

from the category of evenly graded free abelian groups, satisfying

$$\mathbf{M}(\mathbb{Z}[2n]) = S^{n\rho} \wedge H\mathbb{Z}.$$

Propositions 3.2 and 3.4 imply

Proposition 3.7. *The functor (3.6) is an equivalence of strongly additive symmetric monoidal categories, with inverse given by $\pi_\bullet \text{res}$. \square*

3.3.3. *Projective space.* For $n \leq \infty$ we make \mathbf{CP}^n into a C_2 -space by identifying it with the space of complex lines in \mathbb{C}^{n+1} and letting C_2 act by complex conjugation. The quotient space

$$\mathbf{CP}^n / \mathbf{CP}^{n-1}$$

is identified with the one point compactification of \mathbb{C}^n which, in turn is canonically isomorphic to $S^{n\rho}$. There is thus a cofibration sequence

$$(3.8) \quad \mathbf{CP}^{n-1} \rightarrow \mathbf{CP}^n \rightarrow S^{n\rho}.$$

Using this and Lemma 3.4, one can easily establish the following result

Proposition 3.9. *For all $0 \leq n_1, \dots, n_k \leq \infty$ the $H\mathbb{Z}$ module*

$$\mathcal{F}_{H\mathbb{Z}}^{mod}(\mathbf{CP}^{n_1} \times \dots \times \mathbf{CP}^{n_k})$$

is pure. □

Corollary 3.10. *For each $k \geq 0$ there is a unique coalgebra isomorphism*

$$(3.11) \quad \mathbf{M}(\mathbb{A}^k) \approx \mathcal{F}_{H\mathbb{Z}}^{mod}((\mathbf{CP}^\infty)^k)$$

restricting to the isomorphism

$$\mathbf{C}(\mathbb{A}^k) = \mathcal{F}_{H\mathbb{Z}}^{mod}((\mathbf{CP}^\infty)^k) = H\mathbb{Z} \wedge (\mathbf{CP}^\infty)_+^k$$

of Remark 1.31. Under this equivalence, the isomorphism (3.11) is an isomorphism of Hopf algebras. □

3.4. **Grading.** In equivariant homotopy theory it is useful to grade things over the real representation ring $RO(G)$. We will follow the convention, introduced by Hu and Kriz in [12], of using the symbol \star as a wildcard matching a real virtual representation, and using M_\star and M^\star to denote $RO(G)$ -graded objects.

Associated to an $RO(G)$ -graded object M^\star and virtual representation $V \in RO(G)$ of virtual dimension d is the $d\mathbb{Z}$ -graded object $M(V)^{dk} = M^{kV}$. This construction is compatible with the restriction map along the inclusion of a subgroup $H \subset G$.

We are interested in the special case $G = C_2$ and with $V = \rho$ the real regular representation. In this case $M(\rho)^\bullet$ is an evenly graded object, with $M(\rho)^{2n} = M^{n\rho}$.

Example 3.12. If E is a C_2 -spectrum, then $\underline{E}(\rho)^\bullet$ is the evenly graded C_2 -space with

$$\underline{E}(\rho)^{2n} = \underline{E}^{n\rho} = \Omega^\infty S^{n\rho} \wedge E.$$

In this case

$$\text{res } \underline{E}(\rho)^{2n} = \underline{E}^{2n}.$$

Example 3.13. If E is a C_2 -spectrum then $E(\rho)^\bullet$ is the evenly graded abelian group with

$$E(\rho)^{2n} = E_{C_2}^{n\rho}(\text{pt}) = \pi_{-n\rho}^{C_2} E.$$

3.5. **Real oriented spectra and real bordism.**

3.5.1. Real orientations.

Definition 3.14. A *real oriented* spectrum is a C_2 -equivariant homotopy commutative ring spectrum E , equipped with an element $x \in \tilde{E}_{C_2}^\rho(\mathbf{CP}^\infty)$ whose restriction to $\tilde{E}_{C_2}^\rho(\mathbf{CP}^1)$ corresponds to the element 1 under the isomorphism

$$\tilde{E}_{C_2}^\rho(\mathbf{CP}^1) \approx \tilde{E}_{C_2}^\rho(S^\rho) \approx E^0(\text{pt}).$$

From the cofibration sequence (3.8) one can easily prove

Proposition 3.15 (Araki[4], Landweber[16]). *If E is a real oriented spectrum then the maps*

$$\begin{aligned} E_*^{C_2}[[x]] &\rightarrow E_{C_2}^*(\mathbf{CP}^\infty) \\ E_*^{C_2}[[x, y]] &\rightarrow E_{C_2}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \end{aligned}$$

are isomorphisms, where, in the bottom expression, x and y are the pullbacks of the real orientation x along the projections to the first and second factors.

The equivariant E -cohomology homomorphism induced by map

$$\mathbf{CP}^\infty \times \mathbf{CP}^\infty \rightarrow \mathbf{CP}^\infty$$

classifying the tensor product of the two universal “real” line bundles sends the real orientation x to

$$F(x, y) = \sum a_{ij} x^i y^j \in E_{C_2}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \approx E_*^{C_2}[[x, y]]$$

with

$$a_{ij} \in E_{(i+j-1)\rho}^{C_2} = E(\rho)^{-(i+j-1)\rho}.$$

As in the non-equivariant case, this defines a formal group law over $E_{C_2}^*$, and so, with the conventions of §3.4, a map of evenly graded rings

$$MU^\bullet \rightarrow E(\rho)^\bullet.$$

3.5.2. The real bordism spectrum. The universal example of a real oriented spectrum is the *real bordism* spectrum $MU_{\mathbb{R}}$ of Landweber [16] and Fujii [7], and later investigated by Araki [4], Hu-Kriz [12], and in [9]. The spectrum $MU_{\mathbb{R}}$ is the C_2 equivariant spectrum, constructed as Thom spectrum of the universal complex vector bundle over BU , with C_2 acting by complex conjugation. It can be realized as the C_2 -equivariant spectrum

$$(3.16) \quad \varinjlim S^{-n\rho} \wedge MU_{\mathbb{R}}(n)$$

associated to the maps

$$(3.17) \quad S^\rho \wedge MU_{\mathbb{R}}(n-1) \rightarrow MU_{\mathbb{R}}(n)$$

in which $MU_{\mathbb{R}}(m)$ denotes the Thom complex of the universal complex m -plane bundle over $BU(m)$ with C_2 acting by complex conjugation. The inclusion of the zero section

$$\mathbf{CP}^\infty \rightarrow MU_{\mathbb{R}}(1)$$

is an equivariant weak equivalence, and from (3.16) gives the universal real orientation

$$\Sigma^\infty \mathbf{CP}^\infty \rightarrow S^\rho \wedge MU_{\mathbb{R}}^\rho$$

of $MU_{\mathbb{R}}$.

The Schubert cell decomposition of Grassmannians equips the spectrum $MU_{\mathbb{R}}$ with a stable cell decomposition into cells of the form $D(n\rho)$. As observed by Araki [4] this implies

Proposition 3.18. *For every $n \in \mathbb{Z}$ the spectrum*

$$H\mathbb{Z} \wedge S^{n\rho} \wedge MU_{\mathbb{R}}$$

is pure. □

A much deeper result holds.

Proposition 3.19 (Araki [3, Theorem 4.6], Hu-Kriz [12, Theorem 2.28], [9, Theorem 6.1]). *The map*

$$MU^{\bullet} \rightarrow MU_{\mathbb{R}}(\rho)^{\bullet}$$

is an isomorphism, with inverse given by the restriction mapping

$$\text{res} : \pi_{k\rho}^{C_2} MU_{\mathbb{R}} \rightarrow \pi_{2k} MU.$$

As in [12, Lemma 2.17], one simple consequence of Proposition 3.19 is

Corollary 3.20. *If E is real oriented then E satisfies the condition of Lemma 3.2 and so the ring $E(\rho)^{\bullet}$ is an evenly graded commutative ring.* □

3.6. Hopf rings from real oriented spectra. Suppose that E is a real oriented spectrum. As in the previous section let $E(\rho)^{\bullet}$ be the evenly graded ring given by

$$E(\rho)^{2n} = E_{C_2}^{n\rho}(\text{pt}),$$

and

$$(3.21) \quad MU^{\bullet} \rightarrow E(\rho)^{\bullet}$$

the map classifying the formal group law.

By Corollary 3.20 and Example 3.12 the spaces

$$\underline{E}(\rho)^{\bullet}$$

form an evenly graded Hopf ring in \mathfrak{Spaces}^{C_2} over the evenly graded commutative ring $E(\rho)^{\bullet}$. Restricting scalars along (3.21) makes $\underline{E}(\rho)^{\bullet}$ into an evenly graded Hopf ring in \mathfrak{Spaces}^{C_2} over MU^{\bullet} . As in §1.2.1 this equips

$$\mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(\underline{E}(\rho)^{\bullet}) = H\mathbb{Z} \wedge \underline{E}(\rho)_{+}^{\bullet}$$

with an additive curve

$$x : \mathbf{M}(\mathbb{A}^1) = \mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(\mathbf{CP}^{\infty}) \rightarrow \mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(\underline{E}(\rho)^2)$$

which, by Theorem 1.29, is classified by a unique map

$$(3.22) \quad \mathbf{M}(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet}) \rightarrow \mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(\underline{E}(\rho)^{\bullet})$$

of evenly graded Hopf rings over MU^{\bullet} .

This applies in particular to the case $E = MU_{\mathbb{R}}$. For simplicity write

$$\mathcal{M}\mathcal{U}\mathcal{R}^{\bullet} = \mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(\underline{MU}_{\mathbb{R}}(\rho)^{\bullet}).$$

Theorem 2, our main result, is a consequence of the more refined

Theorem 3.23. *The induced map*

$$(3.24) \quad \mathbf{M}(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet}) \rightarrow \mathcal{M}\mathcal{U}\mathcal{R}^{\bullet}$$

is an isomorphism.

Note that Theorem 1.26 implies

Proposition 3.25. *The restriction*

$$\text{res } \mathbf{M}(\mathcal{M}\mathcal{U}_{RW}^\bullet) \rightarrow \text{res } \mathcal{M}\mathcal{U}\mathcal{R}^\bullet$$

of (3.24) is an isomorphism. \square

By Proposition 3.1 this means that to prove Theorem 3.23 it suffices to show that (3.24) induces an isomorphism of geometric fixed points.

3.7. Modified geometric fixed points. The discussion will take place in more ordinary terms if we slightly modify the geometric fixed point functor

$$(3.26) \quad \Phi^{C_2} : \mathbf{Mod}_{H\mathbb{Z}} \rightarrow \mathbf{Mod}_{\Phi^{C_2}H\mathbb{Z}}.$$

Since

$$\Phi^{C_2}(H\mathbb{Z}) = H\mathbb{F}_2[u] \quad |u| = 2,$$

(see for example [9, Proposition 7.5]), the functor (3.26) is a strongly additive symmetric monoidal functor

$$\Phi^{C_2} : \mathbf{Mod}_{H\mathbb{Z}} \rightarrow \mathbf{Mod}_{H\mathbb{F}_2[u]}.$$

This can be composed with

$$X \mapsto H\mathbb{F}_2 \underset{H\mathbb{F}_2[u]}{\wedge} X$$

to give a further strongly additive symmetric monoidal functor

$$\bar{\Phi}^{C_2} : \mathbf{Mod}_{H\mathbb{Z}} \rightarrow \mathbf{Mod}_{H\mathbb{F}_2}.$$

Example 3.27. If Y is a C_2 space then

$$\bar{\Phi}^{C_2}(H\mathbb{Z} \wedge Y_+) = H\mathbb{F}_2 \wedge Y_+^{C_2}.$$

Definition 3.28. A C_2 spectrum X is *bounded below* if the fixed point spectrum X^{C_2} and the underlying spectrum $\text{res } X$ have the properties

$$\begin{aligned} \pi_i X^{C_2} &= 0 & i &\ll 0 \\ \pi_i \text{res } X &= 0 & i &\ll 0 \end{aligned}$$

Lemma 3.29. *A map $X \rightarrow Y$ in $\mathbf{Mod}_{H\mathbb{F}_2[u]}$ of objects which are bounded below is an isomorphism if and only if*

$$H\mathbb{F}_2 \underset{H\mathbb{F}_2[u]}{\wedge} X \rightarrow H\mathbb{F}_2 \underset{H\mathbb{F}_2[u]}{\wedge} Y$$

is an isomorphism in $\mathbf{Mod}_{H\mathbb{F}_2}$.

Proof: By passing to the mapping cone we reduce to the statement that if X is bounded below and $H\mathbb{F}_2 \underset{H\mathbb{F}_2[u]}{\wedge} X$ is contractible then X is contractible. This follows easily from the cofibration sequence

$$S^2 \wedge X \xrightarrow{u} X \rightarrow H\mathbb{F}_2 \underset{H\mathbb{F}_2[u]}{\wedge} X.$$

\square

Since geometric fixed points preserves connectivity, Lemma 3.29 and Proposition 3.1 give

Corollary 3.30. *A map $X \rightarrow Y$ in $\mathbf{Mod}_{H\mathbb{Z}}$ of $H\mathbb{Z}$ modules which are bounded below is an isomorphism if and only if the maps*

$$\begin{aligned} \text{res } X &\rightarrow \text{res } Y \\ \bar{\Phi}^{C_2}(X) &\rightarrow \bar{\Phi}^{C_2}(Y) \end{aligned}$$

of underlying and modified geometric fixed points are isomorphisms. \square

Since the spectra $\mathbf{M}(\mathcal{M}\mathcal{U}_{\text{RW}}^{2m})$ and $\mathcal{M}\mathcal{U}\mathcal{R}_{2m}$ are bounded below, Proposition 3.25 and Corollary 3.30 reduce Theorem 3.23 to showing that the map

$$(3.31) \quad \bar{\Phi}^{C_2}\mathbf{M}(\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet) \rightarrow \bar{\Phi}^{C_2}(\mathcal{M}\mathcal{U}\mathcal{R}^\bullet)$$

is an isomorphism.

4. THE HOPF RING X^\bullet

The map (3.31) is a map in the category $\mathbf{Mod}_{H\mathbb{F}_2}$ which, by Proposition 2.19 is equivalent to \mathbf{Vect}_* via the functors

$$\mathbf{V} : \mathbf{Vect}_* \leftrightarrow \mathbf{Mod}_{H\mathbb{F}_2} : \pi_*$$

To locate everything in \mathbf{Vect}_* first note that the following diagram commutes

$$(4.1) \quad \begin{array}{ccc} \mathbf{Ab}_\bullet^{\text{fr}} & \xrightarrow{\mathbf{M}} & \mathbf{Mod}_{H\mathbb{Z}} \\ \Phi^1 \downarrow & & \downarrow \bar{\Phi}^{C_2} \\ \mathbf{Vect}_* & \xrightarrow{\mathbf{V}} & \mathbf{Mod}_{H\mathbb{F}_2}, \end{array}$$

where Φ^1 is the functor defined in §2. This means that we may identify the image of left hand side of (3.31) in \mathbf{Vect}_* with

$$\Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^\bullet).$$

The image of the right hand side is a Hopf ring over MU^\bullet equipped with the additive curve

$$u = \pi_* \bar{\Phi}^{C_2}(x) : \mathbb{A}^1(1) = \pi_* \bar{\Phi}^{C_2}\mathbf{M}(\mathbb{A}^1) \rightarrow \pi_* \bar{\Phi}^{C_2}(\mathcal{M}\mathcal{U}\mathcal{R}^\bullet),$$

and the image of the map (3.31) is the unique map of evenly graded Hopf rings over MU^\bullet classifying this additive curve.

The purpose of this section is to identify the right hand side of (3.31) with the mod 2 homology of an evenly graded Hopf ring X^\bullet over MU^\bullet in spaces, and from that, establish the decompositions (4.16) and (4.17), their compatibility, and the commutativity of the diagrams (4.20) and (4.21).

In all of the remaining discussion we will be concerned solely with homology with coefficients in \mathbb{F}_2 . To simplify the notation we will now use the symbol $H_*(-)$ to mean $H_*(-; \mathbb{F}_2)$.

4.1. The spaces X^{2k} . The relationship between geometric fixed points and suspension spectra (Example 3.27) gives

$$\begin{aligned} \bar{\Phi}^{C_2}\mathcal{M}\mathcal{U}\mathcal{R}^{2k} &= \bar{\Phi}^{C_2}\mathcal{F}_{H\mathbb{Z}}^{\text{mod}}(MU_{\mathbb{R}}(\rho)^{2k}) \\ &= \bar{\Phi}^{C_2}(H\mathbb{Z} \wedge MU_{\mathbb{R}}^{k\rho}_+) \\ &= H\mathbb{F}_2 \wedge (MU_{\mathbb{R}}^{k\rho}_+)^{C_2}. \end{aligned}$$

Let

$$X^{2k} = (\underline{MU}_{\mathbb{R}}^{k\rho})^{C_2}.$$

By functoriality, the sequence of spaces X^{2k} forms an evenly graded Hopf ring X^\bullet over MU^\bullet in $\text{ho } \mathfrak{S}\text{paces}$. The constant associated to $[a] \in MU^{2k}$ is obtained by passing to fixed points from the unique equivariant map

$$S^0 \rightarrow \underline{MU}_{\mathbb{R}}^{k\rho}$$

restricting to $a \in \pi_0 \underline{MU}^{2k} = MU^{2k}(\text{pt})$. By the above, the Hopf ring $\pi_* \bar{\Phi}^{C_2} \mathcal{MUR}^\bullet$ is $H_*(X^\bullet; \mathbb{F}_2)$, and the weight one additive curve u is the homology homomorphism induced by the map

$$(4.2) \quad \mathbf{RP}^\infty \rightarrow X^2$$

obtained by passing to fixed points from the real orientation

$$\mathbf{CP}^\infty \rightarrow \underline{MU}_{\mathbb{R}}^\rho.$$

The image of (3.31) in \mathbf{Vect}_* is therefore the unique map

$$(4.3) \quad \Phi^1(\mathcal{MURW}^\bullet) \rightarrow H_*(X^\bullet)$$

of Hopf rings over MU^\bullet , classifying the weight one additive curve coming from (4.2).

4.2. Decomposition. Our next aim is to decompose $H_*(X^{2k})$ in a manner compatible with the decomposition (2.34) of Proposition 2.33. This will be achieved by exploiting the fixed point inclusion

$$(4.4) \quad X^\bullet \rightarrow \underline{MU}^\bullet$$

which is map of evenly graded Hopf rings over MU^\bullet . Proposition 3.19 implies that (4.4) induces a bijection of path components. By Example 2.8 and Proposition 2.18 there is an algebra decomposition

$$(4.5) \quad H_*(X^{2k}) \approx H_*((X^{2k})') \otimes \mathbb{F}_2[MU^{2k}].$$

Since

$$H_*(\underline{MU}^\bullet) = \Phi^2(\mathcal{MURW}^\bullet)$$

composition of (4.3) with the fixed point inclusion gives a sequence

$$(4.6) \quad \Phi^1 \mathcal{MURW}^\bullet \rightarrow H_*(X^\bullet) \rightarrow \Phi^2 \mathcal{MURW}^\bullet$$

of Hopf rings over MU^\bullet .

Proposition 4.7. *The composition (4.6) is the Verschiebung.*

Proof: This follows from Proposition 2.21 and the fact that the fixed point inclusion

$$\mathbf{RP}^\infty \rightarrow \mathbf{CP}^\infty$$

induces the Verschiebung

$$\mathbb{A}^1(1) = H_*(\mathbf{RP}^\infty; \mathbb{F}_2) \rightarrow \mathbb{A}^1(2) = H_*(\mathbf{CP}^\infty; \mathbb{F}_2).$$

□

From Proposition 2.26 we conclude

Corollary 4.8. *The fixed point inclusion*

$$H_*(X^\bullet; \mathbb{F}_2) \rightarrow H_*(\underline{MU}^\bullet; \mathbb{F}_2) = \Phi^2(\mathcal{MU}_{RW}^\bullet)$$

is a surjective map of underlying \mathbb{F}_2 vector spaces. \square

To go further, smash the cofibration sequence

$$(C_2)_+ \rightarrow S^0 \rightarrow S^\sigma,$$

with $S^1 \wedge S^{(k-1)\rho} \wedge MU_{\mathbb{R}}$, take fixed points, and pass to zeroth spaces. This leads to the fibration sequence

$$(4.9) \quad \begin{array}{ccc} \underline{MU}^{2k-1} & \longrightarrow & BX^{2(k-1)} \\ & & \downarrow \\ & & X^{2k} \longrightarrow \underline{MU}^{2k}, \end{array}$$

in which we have written

$$BX^{2(k-1)} = (\Omega^\infty S^1 \wedge S^{(k-1)\rho} \wedge MU_{\mathbb{R}})^{C_2}.$$

This is justified by

Lemma 4.10. *The space $\Omega^\infty(S^1 \wedge S^{(k-1)\rho} \wedge MU_{\mathbb{R}})^{C_2}$ is connected.*

Proof: We have

$$\pi_0 \Omega^\infty S^1 \wedge S^{(k-1)\rho} \wedge MU_{\mathbb{R}}^{C_2} = \pi_{-(k-1)\rho-1}^{C_2} MU_{\mathbb{R}}$$

which vanishes by [9, Theorem 1.13 and Corollary 4.65] \square

Remark 4.11. The original proof of the vanishing result is in the unpublished manuscript [3, Theorem 4.6] of Araki. The first published version is due to Hu and Kriz [12, Theorem 4.11].

By Proposition 3.19, the sequence (4.9) restricts to a fibration sequence

$$(4.12) \quad \begin{array}{ccc} \underline{MU}^{2k-1} & \longrightarrow & BX^{2(k-1)} \\ & & \downarrow \\ & & (X^{2k})' \longrightarrow (\underline{MU}^{2k})'. \end{array}$$

Proposition 4.13. *The sequence*

$$(4.14) \quad \mathbb{F}_2 \rightarrow H_* BX^{2(k-1)} \rightarrow H_*(X^{2k})' \rightarrow H_*(\underline{MU}^{2k})' \rightarrow \mathbb{F}_2$$

is a short exact sequence of Hopf-algebras. .

Proof: By Corollary 4.8 the map

$$H^*((\underline{MU}^{2k})') \rightarrow H^*((X^{2k})')$$

is a monomorphism of connected graded Hopf algebras over \mathbb{F}_2 and hence a flat map of rings (Milnor-Moore [23, Theorem 4.4]). It follows from the Eilenberg-Moore spectral sequence that

$$H^*((X^{2k})') \otimes_{H^*((\underline{MU}^{2k})')} \mathbb{F}_2 \rightarrow H^*(BX^{2(k-1)})$$

is an isomorphism. The assertion to be proved is just the linear dual of this fact. \square

Propositions 4.13 and 4.7 imply that the restriction of the map

$$\Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k} \rightarrow H_*(X^{2k})$$

to the Verschiebung ideal factors through $H_*(BX^{2(k-1)})$, leading to the following diagram of short exact sequences of Hopf algebras

$$(4.15) \quad \begin{array}{ccccccc} \mathbb{F}_2 & \longrightarrow & K_{\text{RW}}^{2k} & \longrightarrow & (\Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' & \longrightarrow & H_*(\underline{MU}^{2k})' \longrightarrow \mathbb{F}_2 \\ & & \downarrow & & \downarrow & & \parallel \\ \mathbb{F}_2 & \longrightarrow & H_* BX^{2(k-1)} & \longrightarrow & H_*(X^{2k})' & \longrightarrow & H_*(\underline{MU}^{2k})' \longrightarrow \mathbb{F}_2 . \end{array}$$

As in Proposition 2.33 one may choose a commutative algebra section of the upper right map and, using it, construct isomorphisms

$$\begin{array}{l} K_{\text{RW}}^{2k} \otimes H_*(\underline{MU}^{2k})' \rightarrow (\Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \\ H_* BX^{2(k-1)} \otimes H_*(\underline{MU}^{2k})' \rightarrow H_*(X^{2k})' \end{array}$$

with respect to which the middle vertical map in (4.15) is the tensor product of the left and right vertical maps.

Putting this all together we have exhibited algebra isomorphisms

$$(4.16) \quad \Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k} \approx K_{\text{RW}}^{2k} \otimes \Phi^2(\mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \otimes \mathbb{F}_2[MU^{2k}]$$

$$(4.17) \quad H_* X^{2k} \approx H_* BX^{2k-2} \otimes \Phi^2(\mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \otimes \mathbb{F}_2[MU^{2k}]$$

with respect to which the map

$$(4.18) \quad \Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^{2k} \rightarrow H_* X^{2k}$$

is the tensor product of the identity map of $\Phi^2(\mathcal{M}\mathcal{U}_{\text{RW}}^{2k})' \otimes \mathbb{F}_2[MU^{2k}]$ with the map

$$K^{2k} \rightarrow H_* BX^{2(k-1)}.$$

Remark 4.19. Adjoint to the map $BX^{2(k-1)} \rightarrow X^{2k}$ is a map $X^{2(k-1)} \rightarrow \Omega X^{2k}$. The spectrum associated to these maps is the geometric fixed point spectrum $\Phi^{C_2} MU_{\mathbb{R}} = MO$.

4.3. Changing k . Recall the element

$$\beta_1(1) \in \mathbb{A}^1(1)$$

and its images (both denoted $b_1(1)$) in $(\Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^2)_1$ and $H_1 X^2$ under the universal additive curve, and the additive curve u of §4.1. Our proof that the map (4.18) is an isomorphism involves an induction on k . What relates the successive values of k is the operation $b_1(1) \circ (-)$, and the diagram

$$(4.20) \quad \begin{array}{ccc} \Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^{2(k-1)}) & \longrightarrow & H_* X^{2(k-1)} \\ b_1(1) \circ (-) \downarrow & & \downarrow b_1(1) \circ (-) \\ \Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^{2k}) & \longrightarrow & H_* X^{2k}, \end{array}$$

which commutes by virtue of the fact that

$$\Phi^1 \mathcal{M}\mathcal{U}_{\text{RW}}^\bullet \rightarrow H_* X^\bullet$$

is a map of Hopf rings. As explained in the discussion leading up to the statement of Proposition 2.36 the fact that $b_1(1)$ is primitive implies that the vertical maps in (4.20) induce maps of indecomposables from the top row. Since $b_1(1)$ maps to zero in $(\Phi^2 \mathcal{M}\mathcal{U}_{\text{RW}}^2)_1 = 0$, the images of the vertical maps are contained in the Hopf algebra kernels in (4.15). This leads to the commutative diagram

$$(4.21) \quad \begin{array}{ccc} Q(\Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^{2(k-1)})) & \longrightarrow & Q(H_* X^{2(k-1)}) \\ b_1(1) \circ (-) \downarrow & & \downarrow b_1(1) \circ (-) \\ K_{\text{RW}}^{2k} & \longrightarrow & H_* B X^{2(k-1)}. \end{array}$$

Finally, the fact that the real orientation $x \in MU_{\mathbb{R}C_2}^{\rho}(\mathbf{C}\mathbf{P}^{\infty})$ restricts to 1 under the isomorphism $M\tilde{U}_{\mathbb{R}}^2(S^{\rho}) \approx MU_{\mathbb{R}}^0(\text{pt})$ implies that the element $b_1(1) \in H_1 X^2$ corresponds to the element $[0] \in H_0 X^0$ under the map

$$\Sigma X^0 \rightarrow B X_0 \rightarrow X^2.$$

This implies that right column in (4.21) coincides with the homology suspension induced by

$$\Sigma X^{2(k-1)} \approx \Sigma \Omega B X^{2(k-1)} \rightarrow B X^{2(k-1)}$$

and so is the edge homomorphism of the Rothenberg-Steenrod spectral sequence

$$\text{tor}_{s,t}^{H_* X^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_{s+t} B X^{2(k-1)}.$$

5. PROOF OF THE MAIN THEOREM

Our main result is Theorem 3.23 which we restate for the reader's convenience

Theorem 5.1 (Theorem 3.23). *The induced map*

$$(5.2) \quad \mathbf{M}(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet}) \rightarrow \mathcal{M}\mathcal{U}\mathcal{R}^{\bullet}$$

is an isomorphism.

As explained in §3.7 the results of Ravenel-Wilson [24] reduce the above to showing that the induced map on modified geometric fixed points

$$(5.3) \quad \bar{\Phi}^{C_2} \mathbf{M}(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet}) \rightarrow \bar{\Phi}^{C_2}(\mathcal{M}\mathcal{U}\mathcal{R}^{\bullet})$$

is an isomorphism. This is a map in $\mathbf{Mod}_{H\mathbb{F}_2}$ which is equivalent to the category \mathbf{Vect}_* of graded vector spaces over \mathbb{F}_2 . As explained in §4.1, under this equivalence the map becomes

$$(5.4) \quad \Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet}) \rightarrow H_*(X^{\bullet}).$$

classifying the weight one additive curve u defined in §4.1. We will prove Theorem 5.1 by showing that (5.4) is an isomorphism.

The universal Hopf ring $\Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet})$ will come up frequently, in many diagrams, and with an additional index indicating the grading in \mathbf{Vect}_* . To keep things simple we will use the abbreviation

$$\mathcal{R}^{\bullet} = \Phi^1(\mathcal{M}\mathcal{U}_{\text{RW}}^{\bullet})$$

and write (5.4) as

$$(5.5) \quad \mathcal{R}^{\bullet} \rightarrow H_*(X^{\bullet}).$$

With this notation we isolate the statement that remains to be proved.

Proposition 5.6. *The map*

$$(5.7) \quad \mathcal{R}^\bullet \rightarrow H_*(X^\bullet).$$

described in §4.1 is an isomorphism.

Our proof of Proposition 5.6 follows the inductive argument of Chan [5]. As mentioned in the introduction, the result can also be quickly deduced from the more general result [15, Theorem 1.5] of Kitchloo-Wilson.

5.1. The stable range. In spectra there is a canonical weak equivalence

$$\varinjlim S^{-2k} \wedge \underline{MU}^{2k} \rightarrow MU$$

given by the unit of the adjunction between Σ^∞ and Ω^∞ , and the map

$$\varinjlim H_\bullet(S^{-2k} \wedge \underline{MU}^{2k}; \mathbb{Z}) \rightarrow H_\bullet(MU; \mathbb{Z})$$

is an isomorphism.

Lemma 5.8. *Let \mathcal{C} be an additive category with countable coproducts. If*

$$\mathbf{F} : \mathbf{Ab} \rightarrow \mathcal{C}$$

is a coproduct preserving functor, then the colimit

$$\varinjlim \mathbf{F}(H_* S^{-2k} \underline{MU}^{2k})$$

exists in \mathcal{C} and the map

$$\varinjlim \mathbf{F}(H_* S^{-2k} \underline{MU}^{2k}) \rightarrow \mathbf{F}(H_* MU)$$

is an isomorphism.

Proof: Since \mathcal{F} preserves coproducts, it suffices to show that the coequalizer diagram

$$(5.9) \quad \bigoplus H_* S^{-2k} \underline{MU}^{2k} \rightrightarrows \bigoplus H_* S^{-2k} \underline{MU}^{2k} \rightarrow H_* MU$$

becomes a coequalizer diagram after applying \mathcal{F} . However since $H_*(MU; \mathbb{Z})$ is a free abelian group, the diagram (5.9) extends to a split coequalizer. The lemma follows since split coequalizers are universal colimits. \square

Now consider the diagram

$$(5.10) \quad \begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{M}(H_\bullet S^{-2k} \wedge \underline{MU}^{2k}) & \longrightarrow & \mathbf{M}(H_\bullet S^{-2(k+1)} \wedge \underline{MU}^{2(k+1)}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}^{k\rho} & \longrightarrow & H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}^{(k+1)\rho} & \longrightarrow & \dots \end{array}$$

By Proposition 5.8 with $\mathbf{F} = \mathbf{M}$, the colimit of the top row exists in $\mathbf{Mod}_{H\mathbb{Z}}$ and is given by $\mathbf{M}(H_\bullet MU)$. The map from the bottom row to $H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}$ is a cone on the bottom row (in the sense of MacLane [20, §III.3, p. 67]), so there is a unique map

$$(5.11) \quad \mathbf{M}(H_\bullet MU) \rightarrow H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}$$

of cones on the rows of (5.10). Since the map from the top row to the bottom row of (5.10) becomes an isomorphism after applying res , the map (5.11) is an

isomorphism after applying res . Since $H\mathbb{Z} \wedge MU_{\mathbb{R}}$ is pure (Proposition 3.18) this implies that (5.11) is an isomorphism (Proposition 3.5).

Proposition 5.12. *The map $\mathcal{R}^{2k} \rightarrow H_*X^{2k}$ is an isomorphism in degrees less than $\max\{1, 2k\}$.*

Proof: In degree 0, the map $\mathcal{R}^{2k} \rightarrow H_*X^{2k}$ is the identity map of $\mathbb{F}_2[MU^{2k}]$, so it suffices to consider the case $k > 0$ (in which case $MU^{2k} = 0$), and work in reduced homology. We therefore wish to show that the map

$$\mathbf{M}(\tilde{H}_{\bullet}MU^{2k}) \rightarrow H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}^{k\rho}$$

becomes a $2k$ -equivalence¹ after applying modified geometric fixed points.

Consider the diagram

$$(5.13) \quad \begin{array}{ccc} \mathbf{M}(\tilde{H}_{\bullet}MU^{2k}) & \longrightarrow & \mathbf{M}(H_{\bullet}S^{2k} \wedge MU) \\ \downarrow & & \downarrow \sim \\ H\mathbb{Z} \wedge \underline{MU}_{\mathbb{R}}^{k\rho} & \longrightarrow & H\mathbb{Z} \wedge S^{k\rho} \wedge MU_{\mathbb{R}} \end{array}$$

extracted from the map of cones on (5.10) by tensoring the k^{th} -term with $H\mathbb{Z} \wedge S^{k\rho}$. We wish to show that the left vertical arrow becomes a $2k$ -equivalence after applying $\bar{\Phi}^{C_2}$. By the discussion above, the right vertical arrow is an isomorphism. The map

$$\tilde{H}_{\bullet}MU^{2k} \rightarrow H_{\bullet}S^{2k} \wedge MU$$

is a $4k$ -connected map of evenly graded free abelian groups. Since $\bar{\Phi}^{C_2} \circ \mathbf{M} = \Phi^1$ this means that the map

$$\bar{\Phi}^{C_2}\mathbf{M}(\tilde{H}_{\bullet}MU^{2k}) \rightarrow \bar{\Phi}^{C_2}\mathbf{M}(H_{\bullet}S^{2k} \wedge MU)$$

is a $2k$ -connected map of graded \mathbb{F}_2 vector spaces. For the bottom map, the fact that the spectrum $S^{k\rho} \wedge MU_{\mathbb{R}}$ is equivariantly $(k-1)$ -connected implies that the map

$$(5.14) \quad \Sigma^{\infty}\underline{MU}_{\mathbb{R}}^{k\rho} \rightarrow S^{k\rho} \wedge MU_{\mathbb{R}}$$

is equivariantly $2k$ -connected, hence so is the map on geometric fixed points, since geometric fixed points preserves connectivity. The identity

$$\bar{\Phi}^{C_2}H\mathbb{Z} \wedge W = H\mathbb{Z}/2 \wedge \Phi^{C_2}W$$

then implies that the bottom row of (5.13) becomes a $2k$ -equivalence after passing to modified geometric fixed points. It follows that after applying modified geometric fixed points to (5.13), the top, right, and bottom arrows become isomorphism in dimensions less than $2k$. The same is therefore true of the left vertical arrow. \square

¹Specifically, an epimorphism in degree $2k$ and an isomorphism in lower degrees.

5.2. **Facts about the algebras \mathcal{R}^{2k} .** We now collect the facts about the map (5.5) that will be used in the proof of Theorem 5.1.

For a graded free module M over a ring R write

$$\dim_\alpha M = \dim_\alpha^R M = \sum (\dim_R M_k) \alpha^k.$$

Let $p(k)$ be the number of partitions of k , defined by the expansion

$$\prod_{m \geq 1} (1 - \alpha^m)^{-1} = \sum_{k \geq 0} p(k) \alpha^k,$$

so that

$$\begin{aligned} \dim_\alpha \pi_* S^{2k} \wedge MU &= \sum p(m) \alpha^{2(k+m)} \\ \dim_\alpha \pi_* (\underline{MU}^{2k})' &= \sum_{k+m > 0} p(m) \alpha^{2(k+m)}. \end{aligned}$$

Since for any connected infinite loop space Z the map

$$\text{Sym } \pi_* X \otimes \mathbb{Q} \rightarrow H_*(X; \mathbb{Q})$$

is an isomorphism, and since $H_*(\underline{MU}^{2k})'$ is free abelian, we have

$$\dim_\alpha H_*(\underline{MU}^{2k})' = \prod_{k+m > 0} (1 - \alpha^{2(k+m)})^{-p(m)}.$$

It follows from 1.26² that

$$\dim_\alpha (\mathcal{R}^{2k})' = \prod_{k+m > 0} (1 - \alpha^{k+m})^{-p(m)},$$

and so

$$(5.15) \quad \dim_\alpha K_{\text{RW}}^{2k} = \frac{\dim_\alpha (\mathcal{R}^{2k})'}{\dim_\alpha H_*(\underline{MU}^{2k})'} = \prod_{k+m > 0} (1 + \alpha^{k+m})^{p(m)}.$$

We also know that

$$\pi_0 X^{2k} = \pi_0 \underline{MU}^{2k} = MU^{2k}$$

is a free abelian group of rank $p(-k)$.

Proposition 5.16. *The rings \mathcal{R}^{2k} , $(\mathcal{R}^{2k})'$ and the maps $(\mathcal{R}^{2k})' \rightarrow H_*(X^{2k})'$ have the following properties*

- i) *The Map $(\mathcal{R}^{2k})' \rightarrow H_*(X^{2k})'$ is an isomorphism in degrees less than $\max\{1, 2k\}$.*
- ii) *The map $b_1(1) \circ (-) : Q\mathcal{R}^{2k} \rightarrow K_{\text{RW}}^{2(k+1)} \subset (\mathcal{R}^{2(k+1)})'$ is a monomorphism and the following diagram commutes*

$$\begin{array}{ccc} Q\mathcal{R}^{2k} & \longrightarrow & QH_*X^{2k} \\ \downarrow b_1(1) \circ (-) & & \downarrow b_1(1) \circ (-) \\ \mathcal{R}^{2(k+1)} & \longrightarrow & H_*X^{2(k+1)}. \end{array}$$

- iii) *The algebra $\text{tor}_{s,t}^{\mathcal{R}^{2k}}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by elements in $\text{tor}_{1,t}$.*

²With a little more investment in setting things up, these purely algebraic facts can easily be given purely algebraic proofs. See, for example [11].

iv) *With respect to the grading on $\mathrm{tor}_{*,*}^{\mathcal{R}^{2k}}(\mathbb{F}_2, \mathbb{F}_2)$ in which the term $\mathrm{tor}_{s,t}$ has degree $s+t$, there is an equality of Poincaré series*

$$\dim_{\alpha}(\mathrm{tor}_{*,*}^{\mathcal{R}^{2k}}(\mathbb{F}_2, \mathbb{F}_2)) = \dim_{\alpha}(K_{RW}^{2(k+1)}).$$

Proof: Part i) is Proposition 5.12. The first assertion in part ii) is gotten by applying $\bar{\Phi}^{C_2}$, and the commutative diagram is (4.20). Part iii) is immediate from the fact that MU^{2k} is a free abelian group and $(\mathcal{R}^{2k})'$ is a polynomial algebra, so that the tor algebras in question are exterior algebras on the elements in degree $\mathrm{tor}_{1,t}$. For part iv), we use that fact that $(\mathcal{R}^{2k})'$ is a polynomial algebra and that

$$\dim_{\alpha} Q(\mathcal{R}^{2k})' = \sum_{k+m>0} p(m)\alpha^{k+m},$$

where Q denotes the indecomposables. This implies that

$$\mathrm{tor}_{*,*}^{(\mathcal{R}^{2k})'}(\mathbb{F}_2, \mathbb{F}_2)$$

is an exterior algebra on the suspension of the indecomposables, so that, with respect to total degree,

$$\dim_{\alpha} \mathrm{tor}_{*,*}^{(\mathcal{R}^{2k})'}(\mathbb{F}_2, \mathbb{F}_2) = \prod_{k+m>0} (1 + \alpha^{k+1+m})^{p(m)}.$$

Since MU^{2k} is a free abelian group of rank $p(-k)$ we can conclude

$$\begin{aligned} \dim_{\alpha} \mathrm{tor}_{*,*}^{\mathcal{R}^{2k}}(\mathbb{F}_2, \mathbb{F}_2) &= (1 + \alpha)^{p(-k)} \prod_{k+m>0} (1 + \alpha^{k+1+m})^{p(m)} \\ &= \prod_{k+m+1>0} (1 + \alpha^{k+1+m})^{p(m)}. \end{aligned}$$

The claim now follows from (5.15). \square

5.3. The induction. The assertion of Proposition 5.6 is that for all k , the map

$$(5.17) \quad \mathcal{R}^{2k} \rightarrow H_* X^{2k}$$

is an isomorphism. The argument will involve an induction on degree, so we are actually interested in the assertion that for some ℓ , the map

$$(5.18) \quad (\mathcal{R}^{2k})_{*\leq\ell} \rightarrow H_{*\leq\ell} X^{2k}$$

is an isomorphism. By the discussion in §4.2, the map (5.17) is the tensor product of the identity map of $\mathbb{F}_2[\pi_0 X^{2k}]$ with

$$(5.19) \quad (\mathcal{R}^{2k})' \rightarrow H_*(X^{2k})',$$

which in turn, is the tensor product of the identity map of $H_*(\underline{MU}^{2k})'$ with

$$(5.20) \quad K_{RW}^{2k} \rightarrow H_* BX^{2(k-1)}.$$

It follows that the condition that any one of these three maps be an isomorphism in degrees less than or equal to ℓ is equivalent to the condition that the other two be so.

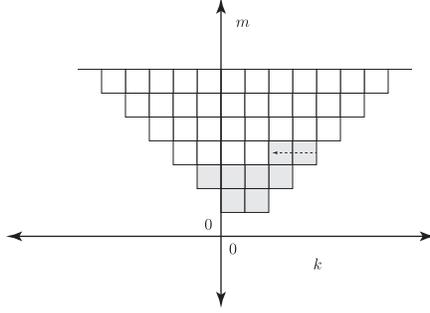


FIGURE 5.1. The Induction Region

Now each of these three maps is a map of Hopf algebras and is an isomorphism in degrees less than or equal to ℓ if and only if the corresponding map

$$(5.21) \quad \begin{aligned} Q\mathcal{R}^{2k} &\rightarrow QH_*X^{2k} \\ Q(\mathcal{R}^{2k})' &\rightarrow QH_*(X^{2k})' \\ QK_{RW}^{2k} &\rightarrow QH_*BX^{2(k-1)} \end{aligned}$$

of indecomposables is. This is clear for the second and third maps as both sides are connected graded Hopf algebras. It is so for the first since the map

$$Q\mathcal{R}^{2k} \rightarrow QH_*X^{2k}$$

is the sum of the map

$$Q(\mathcal{R}^{2k})' \rightarrow QH_*(X^{2k})'$$

and the identity map of

$$Q\mathbb{F}_2[\pi_0 X^{2k}] = \pi_0 X^{2k} \otimes \mathbb{F}_2.$$

Of this, what we will use is

Proposition 5.22. *For a given k and ℓ , if any of the following three maps is an isomorphism*

$$\begin{aligned} (\mathcal{R}^{2k})_{*\leq\ell} &\rightarrow H_{*\leq\ell}(X^{2k}) \\ (K_{RW}^{2k})_{*\leq\ell} &\rightarrow H_{*\leq\ell}(BX^{2(k-1)}) \\ (Q\mathcal{R}^{2k})_{*\leq\ell} &\rightarrow QH_{*\leq\ell}(X^{2k}) \end{aligned}$$

so are the other two. □

Following the inductive the argument of Chan [5] (and Ravenel-Wilson [25]) consider the assertion that

$$(5.23) \quad (\mathcal{R}^{2k})_{m+k} \rightarrow H_{m+k}(X^{2k})$$

is an isomorphism. By part i) of Proposition 5.16 we know this to be so for $k > m$ ($m+k < 2k$) and for $k < 1-m$ ($m+k < 1$). Among other things this means that it is an isomorphism for all $m \leq 0$. The region for which we do *not* know it to be an isomorphism is indicated by the boxes in Figure 5.1.

We will establish, by increasing induction on m , that for all k the map (5.23) is an isomorphism. The induction starts with $m \leq 0$.

For the inductive step fix m and assume that

$$(5.24) \quad (\mathcal{R}^{2k})_{m'+k} \rightarrow H_{m'+k}(X^{2k}).$$

is an isomorphism for all k and all $m' < m$. We will show by decreasing induction on k that it is an isomorphism for $m' = m$, that is, that the map (5.23) is an isomorphism.

The induction on k starts with $k = m+1$. Assume we have established that (5.23) is an isomorphism for some k . We wish to conclude that

$$(5.25) \quad (\mathcal{R}^{2(k-1)})_{m+k-1} \rightarrow H_{m+k-1}(X^{2(k-1)})$$

is an isomorphism. For this we will use the spectral sequence

$$\mathrm{tor}_{s,t}^{H_*(X^{2(k-1)})}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_{s+t}(BX^{2(k-1)}).$$

to show that the map

$$(Q\mathcal{R}^{2(k-1)})_{m+k-1} \rightarrow QH_{m+k-1}(X^{2(k-1)})$$

is a monomorphism, and then, by dimension count, an isomorphism. Combined with the induction hypothesis, this shows that

$$(Q\mathcal{R}^{2(k-1)})_{*\leq m+k-1} \rightarrow QH_{*\leq m+k-1}(X^{2(k-1)})$$

is an isomorphism, and so by Proposition 5.22 that

$$(\mathcal{R}^{2(k-1)})_{*\leq m+k-1} \rightarrow H_{*\leq m+k-1}(X^{2(k-1)})$$

is as well.

To simplify the notation a bit, write $A_* = H_*(X^{2(k-1)})$, and $I_* \subset A_*$ for the kernel of the counit (augmentation) $A_* \rightarrow \mathbb{F}_2$. Of course, for $t > 0$, $I_t = A_t$. The reduced bar construction for computing

$$\mathrm{tor}_{s,t} = \mathrm{tor}_{s,t}^{A_*}(\mathbb{F}_2, \mathbb{F}_2)$$

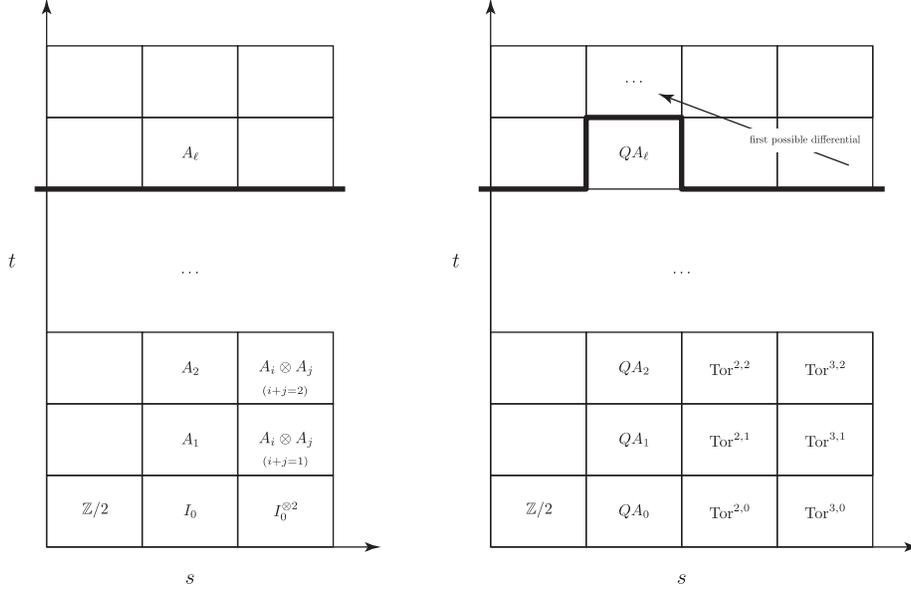
is depicted on the left in Figure 5.2. It serves as an $E_{s,t}^1$ -term for the spectral sequence, with d_1 the bar differential. The differential d_r has bidegree $(-r, r-1)$. The $E_{1,t}^2$ -term consists of the indecomposables $Q(A)_t = (I/I^2)_t$. All of the elements in the groups $\mathrm{tor}_{1,t}$ are permanent cycles. If it happens that for $t < \ell$, the algebra $\mathrm{tor}_{s,t}^{A_*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by $\mathrm{tor}_{1,t}$, then there can be no differentials into the terms $\mathrm{tor}_{s,t}$ for $t \leq \ell + 1$.

We apply these considerations with $\ell = m + k - 1$. Since the map $\mathcal{R}^{2(k-1)} \rightarrow H_*(X^{2(k-1)})$ is an isomorphism in degrees less than $\ell = m + k - 1$, we have for $t < \ell$, an isomorphism

$$\mathrm{tor}_{s,t}^{H_*(X^{2(k-1)})}(\mathbb{F}_2, \mathbb{F}_2) \approx \mathrm{tor}_{s,t}^{\mathcal{R}^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2).$$

Since $\mathrm{tor}_{0,t} = 0$ for $t > 0$ this holds in particular when $s + t \leq \ell$ and when $s + t = \ell + 1$ provided $s > 1$. By our induction hypothesis on m , part iii) and the discussion above, there are no differentials entering or leaving the $E_{s,t}^r$ term when $s + t = \ell + 1$. This means that the map

$$b_1(1) \circ (-) : QH_\ell(X^{2(k-1)}) \rightarrow H_{\ell+1}(BX^{2(k-1)})$$

FIGURE 5.2. The E_1 and E_2 -terms of the tor spectral sequence

is a monomorphism. Consider the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (Q\mathcal{R}^{2(k-1)})_\ell & \xrightarrow{b_1(1) \circ (-)} & (K_{\text{RW}}^{2k})_{\ell+1} & & \\
 & & \downarrow & & \downarrow \approx & & \\
 0 & \longrightarrow & QH_\ell X^{2(k-1)} & \xrightarrow{b_1(1) \circ (-)} & H_{\ell+1} BX^{2(k-1)} & \longrightarrow & M \longrightarrow 0
 \end{array}$$

in which the bottom row is exact. The top map is a monomorphism by part ii) of Theorem 5.16 and the right vertical map is an isomorphism by our induction hypothesis. This establishes the assertion that $(Q\mathcal{R}^{2(k-1)})_\ell \rightarrow QH_\ell(X^{2(k-1)})$ is a monomorphism.

Now for the dimension count. From the spectral sequence, the vector space M has a filtration whose associated graded vector space is

$$\bigoplus_{\substack{s+t=\ell+1 \\ s>1}} \text{tor}_{s,t}^{\mathcal{R}^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2),$$

and so

$$(5.26) \quad \dim M = \sum_{\substack{s+t=\ell+1 \\ s>1}} \dim \text{tor}_{s,t}^{\mathcal{R}^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2).$$

Now part iv) of Proposition 5.16 implies that

$$\begin{aligned}
 (5.27) \quad \dim H_{\ell+1}(BX^{2k}) &= \dim(K_{\text{RW}}^{2(k+1)})_{\ell+1} \\
 &= \sum_{s+t=\ell+1} \dim \text{tor}_{s,t}^{\mathcal{R}^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2).
 \end{aligned}$$

The difference between (5.27) and (5.26) is

$$\dim \operatorname{tor}_{1,\ell}^{\mathcal{R}^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2).$$

It follows that

$$\dim QH_\ell X^{2(k-1)} = \dim \operatorname{tor}_{1,\ell}^{\mathcal{R}^{2(k-1)}}(\mathbb{F}_2, \mathbb{F}_2) = \dim(Q\mathcal{R}^{2(k-1)})_\ell,$$

and so the map $(Q\mathcal{R}^{2(k-1)})_\ell \rightarrow QH_\ell(X^{2(k-1)})$ must be an isomorphism. This completes the proof.

APPENDIX A. NOTATION AND BASIC NOTIONS

Strongly additive categories. An *additive category* is a category \mathcal{C} satisfying the following conditions:

- i) The category \mathcal{C} posses finite products and coproducts. In particular \mathcal{C} has an initial object and a terminal object, corresponding to the empty coproduct and product.
- ii) \mathcal{C} is *pointed* in the sense that the map from the initial object to the terminal object is an isomorphism. This object is denoted $*$ and its existence makes all of the hom set $\mathcal{C}(X, Y)$ into *pointed sets* with base point the map $X \rightarrow * \rightarrow Y$.
- iii) For every finite collection of objects $\{X_i\}$ of \mathcal{C} , the map

$$\coprod X_i \rightarrow \prod X_i$$

whose i^{th} component is the identity map on the summand X_i and the base point $*$ on other summands is an isomorphism. This equips the objects $\mathcal{C}(X, Y)$ the structure of a commutative monoid.

- iv) For all X and Y , the commutative monoid $\mathcal{C}(X, Y)$ is an abelian group.

An additive category closed under arbitry coproducts is called *strongly additive*.

Symmetric monoidal categories. A *monoidal category* is a category \mathcal{C} equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ together with a natural isomorphism

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$$

satisfying the pentagon axiom, and for which there exists an object $\mathbf{1} \in \mathcal{C}$ which is a two-sided unit for \otimes , compatible with the $\alpha_{X,Y,Z}$ (see [20, VII, §1] [6, Chapter 2], and for further references [14]). In [6, Remark 2.2.9] Etingof et al. point out that the unit, if it exists, is unique up to unique isomorphism and so corresponds to a property and not a structure.

A *symmetric monoidal category* is a monoidal category \mathcal{C} equipped with a natural symmetry isomorphism

$$c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$$

for which

$$c_{Y,X} \circ c_{X,Y} = \operatorname{Id}_{X \otimes Y}$$

and the “hexagon” axiom described in [20, VII, §7] [6, §8.1] and [29, §1]. Etingof et al. [6, Exercise 8.1.6] point out that the identities [20, VII §7 (2)] are a consequence of the other axioms.

The functors between symmetric monoidal categories we consider are the *strong symmetric monoidal functors* in the sense of Thomason [29, §1]. (See also [6, §2.4] for an incisive and full discussion.)

We use (\mathcal{C}, \otimes) to indicate the data of a symmetric monoidal category, rather than $(\mathcal{C}, \otimes, \alpha, c)$, and denote the unit by $\mathbf{1}$.

In this paper a *strongly additive* symmetric monoidal category is a symmetric monoidal category (\mathcal{C}, \otimes) in which \mathcal{C} is additive, contains arbitrary coproducts, and the monoidal functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ commutes with coproducts in each variable. In particular the monoidal product is biadditive. A *strongly additive symmetric monoidal functor* between strongly additive symmetric monoidal categories is a (strong) symmetric monoidal functor which commutes with coproducts.

Grading. In this paper we are frequently concerned with evenly graded abelian groups, and it is useful to use the symbol \bullet as a wildcard matching even integers. So for example, for a space or spectrum X , the symbols $H_\bullet X$ and $\pi_\bullet X$ denote the even dimensional homology and homotopy groups of X regarded as evenly graded abelian groups. Similarly an evenly graded Hopf ring is denoted by something like \mathcal{H}^\bullet with \mathcal{R}^n indicating the component of degree n when n is even, and undefined if n is odd.

In equivariant homotopy theory we follow the convention, introduced by Hu and Kriz in [12], of using the symbol \star as a wildcard matching a real virtual representation, and using M_\star and M^\star to denote $RO(G)$ -graded objects.

Given an $RO(G)$ -graded object M^\star and a representation V of G of dimension d , there is a $d\mathbb{Z}$ -graded object $M(V)$ with $M(V)^{nd} = M^{nV}$ (§3.4).

For ordinary \mathbb{Z} -grading we use, as is customary, the symbol $*$ as in M_* and M^* .

Yoneda. We do not distinguish in notation between an object X in a category \mathcal{D} and the functor represented by X . Thus for $X, Y \in \mathcal{D}$ we write $X(Y) = \mathcal{D}(Y, X)$. This comes up most of the time when $\mathcal{D} = \mathbf{coalg} \mathcal{C}$ is the category of coalgebras in a symmetric monoidal category \mathcal{C} .

Spaces in a spectrum. If E is a spectrum and $n \in \mathbb{Z}$ we write

$$\underline{E}^n = \Omega^\infty \Sigma^n E$$

for the n^{th} space in the associated Ω -spectrum. This is a departure from the long established convention found throughout the literature [18, 19, 27, 30, 1, 2, 17, 31]. It is with some reluctance that we have chosen to make this convention, but it leads to clearer expressions in the situations of this paper. For example the \underline{E}^n represents the functor $E^n(X)$, so this is consistent with our convention for the Yoneda embedding in the case $\mathcal{C} = \mathbf{ho} \mathbf{Spaces}$ is the homotopy category of spaces

$$E^n(X) = \underline{E}^n(X) = \mathbf{ho} \mathbf{Spaces}(X, \underline{E}^n).$$

Further miscellaneous notation.

- $\mathbf{coalg} \mathcal{C}$ is the category of cocommutative counital coalgebras in a symmetric monoidal category \mathcal{C} (§1.1.1).
- $\mathbf{HopfRings}_{R^\bullet}(\mathcal{C})$ is the category of evenly graded Hopf ring in \mathcal{C} over the evenly graded ring R^\bullet .
- From §1 to §4 the symbol $H_* X$ refers the homology $H_*(X; \mathbb{Z})$ of a space or spectrum X . From §4 until the end of the paper $H_*(X)$ denotes homology with coefficients in \mathbb{F}_2 .
- For a commutative ring k , the category \mathbf{Mod}_k^* is the symmetric monoidal category of left k -modules with monoidal product the tensor product and symmetry constraint given by the Koszul sign rule (Example 1.6). The

- category \mathbf{Mod}_k^* is a strongly additive symmetric monoidal category. When $k = \mathbb{Z}$ we write \mathbf{Ab}_* instead of $\mathbf{Mod}_{\mathbb{Z}}^*$ (§1.2.2).
- For an (equivariant) E_∞ ring spectrum R , the category \mathbf{Mod}_R is the homotopy category of (equivariant) left R -modules (Example 1.7 and §3.1). The category \mathbf{Mod}_R is a strongly additive symmetric monoidal category under $X \wedge_R Y$.
 - For a (G -)space X , the free (equivariant) R -module generated by X is $\mathcal{F}_R^{\text{mod}}(X) = R \wedge X_+$ (Example 1.7 and §3.1).
 - \mathbb{A}^1 is the Hopf algebra $H_*(\mathbf{CP}^\infty)$ (§1.2.2). It has a graded basis $\{\beta_i \mid i = 0, 1, \dots\}$ in which β_i is the generator of $H_{2i}(\mathbf{CP}^\infty; \mathbb{Z})$ dual to x^i , where $x \in H^2(\mathbf{CP}^\infty; \mathbb{Z})$ is the first Chern class of the tautological line bundle.
 - $\mathcal{MU}_{\text{RW}}^\bullet$ is the Ravenel-Wilson Hopf ring (Definition 1.22).
 - \mathcal{D}^\bullet denotes the category of evenly graded objects of a category \mathcal{D} (§1.1.3).
 - $\mathbf{Ab}_\bullet^{\text{fr}}$ is the strongly additive symmetric monoidal category of evenly graded free abelian groups and tensor product (§1.2.3).
 - $\mathbf{C} : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Mod}_{H\mathbb{Z}}$ (1.30) is the unique strongly additive functor satisfying $\mathbf{C}(\mathbb{Z}[2n]) = H\mathbb{Z} \wedge S^{2n}$.
 - $\mathbf{Vect}_* = \mathbf{Mod}_{\mathbb{F}_2}^*$ is the (strongly additive) symmetric monoidal category of graded vector spaces over \mathbb{F}_2 and tensor products (§2.3)
 - $\mathbf{V} : \mathbf{Vect}_* \rightarrow \mathbf{Mod}_{H\mathbb{F}_2}$ (1.30) is the unique strongly additive functor satisfying $\mathbf{V}(\mathbb{F}_2[k]) = H\mathbb{F}_2 \wedge S^{2k}$ (§2.3).
 - if X is a pointed space then X' denotes the connected component of the base point.
 - If C is a (-1) -connected coalgebra in \mathbf{Mod}_k^* equipped with a coalgebra map $b : k \rightarrow A_0$, then C' denotes the connected component of C containing b (Definition 2.9).
 - For a coalgebra C the set $\pi_0 C = C(\mathbf{1})$ is the set of coalgebra maps $\mathbf{1} \rightarrow C$ (Definition 2.4).
 - $\Phi^k(-) : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Vect}_*$ is the strongly additive symmetric monoidal functor sending $\mathbb{Z}[2n]$ to $\mathbb{F}_2[kn]$ (§2.3).
 - $(-)^{\phi} : \mathbf{Vect}_* \rightarrow \mathbf{Vect}_*$ is the functor which doubles the grading: $V_{2n}^{\phi} = V_n$ (§2.4).
 - $\mathbf{v} : C \rightarrow C^{\phi}$ is the Verschiebung (§2.4).
 - $\mathbb{A}^1(k)$ is the Hopf algebra $\Phi^k \mathbb{A}^1 \in \mathbf{Vect}_*$, and has a graded basis $\{\beta_i(k) \mid i = 0, 1, \dots\}$ with $\beta_i(k) = \Phi^k \beta_i$ (Example 2.24).
 - $Q\mathcal{H}$ is the module of indecomposables in a Hopf algebra \mathcal{H} (2.35).
 - ρ is the real regular representation of C_2 .
 - $\mathbf{M} : \mathbf{Ab}_\bullet^{\text{fr}} \rightarrow \mathbf{Mod}_{H\mathbb{Z}}^T$ the unique strongly additive functor satisfying $\mathbf{M}(\mathbb{Z}[2n]) = S^{n\rho} \wedge H\mathbb{Z}$ (§3.3.2).
 - Given an $RO(G)$ -graded object M^* and a representation V of G of dimension d , $M(V)$ is the $d\mathbb{Z}$ -graded object with $M(V)^{nd} = M^{nV}$ (§3.4).
 - $\bar{\Phi}^{C_2}$ is the modified geometric fixed point functor (§3.7).
 - $\mathcal{R}^\bullet = \Phi^1 \mathcal{MU}_{\text{RW}}^\bullet$ (§5)

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