# HOPKINS: KERVAIRE INVARIANT OF IMMERSIONS

Note Title 2/11/2010

In 1930s Pontryagin introduced relationship

cobordism of framed manifolds $\longleftrightarrow$ homotopy groups of spheres

$M^k \subset \mathbb{R}^{n+k}$ $n \gg 0$ $S^{n+k} \longrightarrow S^n$

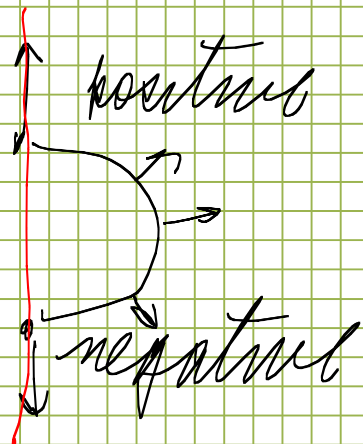
cobordism homotopy

$\Omega^{fr}_k$ $\pi_{k+n} S^n$ for $n \gg 0$

$\pi_k S^0$

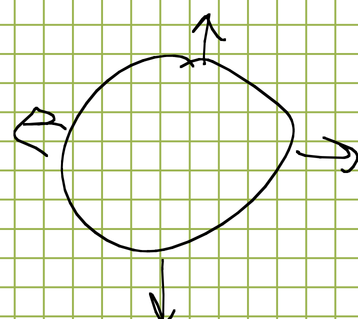
$k=0$

$M^0 \subset \mathbb{R}^1$

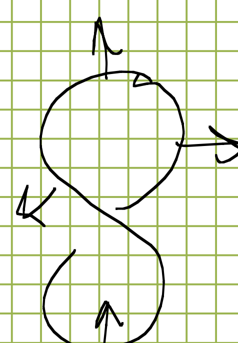



$\Omega_0^{fr} = \mathbb{Z}$ $\qquad$ $\pi_0 S^0 = \mathbb{Z}$

$k=1$ Will draw embedding in $\mathbb{R}^3$ as immersion in $\mathbb{R}^2$



bounds a disk



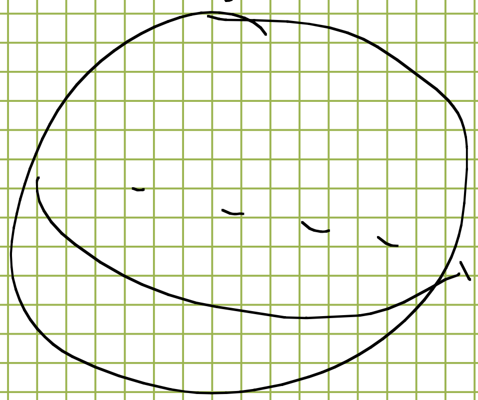
does not bound a disk

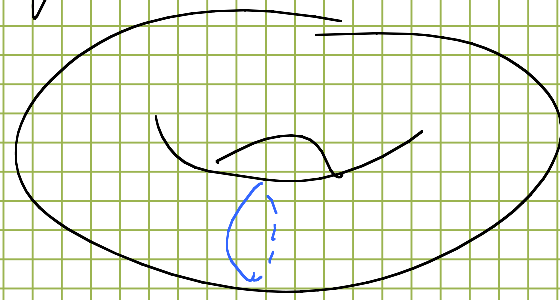
$\pi_1(S^0) = \mathbb{Z}/2$

$\Omega_2^{fr}$ turns out to $\mathbb{Z}/2$ but took a long time to understand. Framed 2-mfd is oriented and hence has a genus

$S^2$

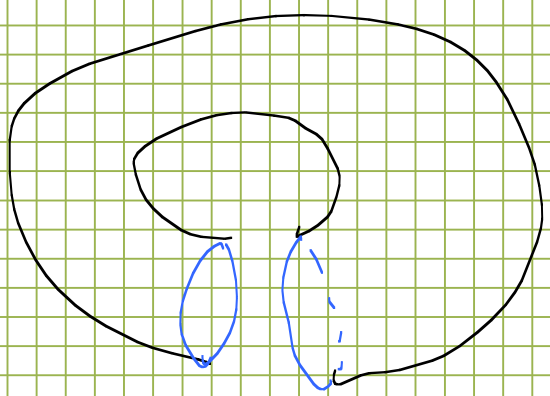
genus 0





has framing that extends to $D^3$

Any two framings differ by map $S^2 \to GL_n(\mathbb{R})$ and $\pi_2 GL_n(\mathbb{R}) = 0$



Framed surgery

The framing of the circle we cut must have trivial framing. The surgery obstruction is $\varphi : H_1(M; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$

$$\varphi(x+y) - \varphi(x) - \varphi(y) = \# \, x \cap y = \text{intersection number}$$

$$= B(x,y) = \text{bilinear nondegenerate form}$$

Two possible values of $\varphi$ in cohomology

$$\varphi(x,y) = \begin{cases} xy \\ x^2 + xy + x^2 \end{cases}$$

| 0 | $x$ | $y$ | $x+y$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 |

$$= \begin{cases} H(x,y) \\ Q(x,y) \end{cases}$$

For general $V$, $(V, \varphi) \cong H \oplus H \oplus \cdots \oplus H$

or $Q \oplus H \oplus \cdots \oplus H$

Witt group of such $(V, \varphi)$ is $\mathbb{Z}/2$

$$(V, \varphi) \xrightarrow{\text{Arf}} \mathbb{Z}/2$$

Apply this to $\varphi : H_1(M; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ to get framed cobordism invariant.

Note that $B(x, x) = \varphi(x + x) - \varphi(x) - \varphi(x)$

$= \varphi(2x) - 2\varphi(x) = 0$

since we are working mod 2

e.g. $H^1(\mathbb{RP}^2)$ does not have such a $\varphi$.

The map $q$ is not intrinsic to $M$ but depends on the framing

## E H Brown's variation

If we could define $q(e_1) = 1/2$ and $q(x) = x^2/2$

i.e. if we have a $\mathbb{Z}/4$-valued invariant

$$\overset{\shortparallel}{\tfrac{1}{2}\mathbb{Z}/2\mathbb{Z}} = \{0, 1/2, 1, 3/2\}$$

The resulting Witt gp is $\mathbb{Z}/8$ generated by $q(x) = x^2/2$ as above.

Given $(V, q)$ and $x \in V$ with $q(x) = 0$

$$V^\perp \longrightarrow V \xrightarrow{B(x,-)} \mathbb{Z}/2$$

$$(V^\perp/x, q) \sim (V, q)$$

Showing gen has order 8 is a fun exercise.

Geometric side: Let $\Sigma \looparrowright \mathbb{R}^3$ be an immersed surface. Define

$q : H_1(\Sigma; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/4$ defined by counting half twists of a ribbon in $\Sigma$ neighboring closed curve. (Belt trick)

Can only do a surgery on ker $q$.

The usual immersed Klein bottle $=K$ is

null cobordant.

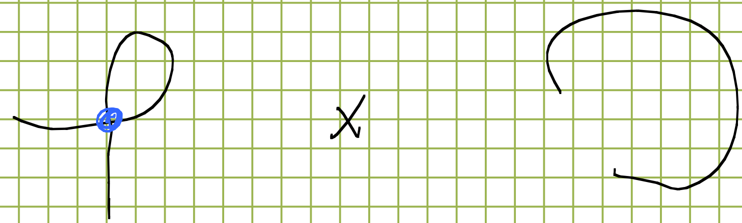
$\varphi(x+y) - \varphi(x) - \varphi(y) =$ intersection #.

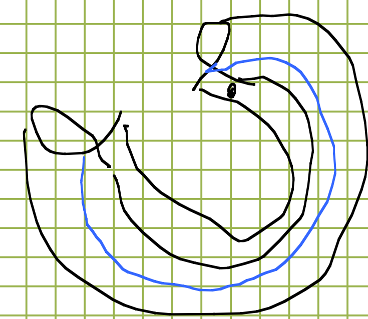
What does the gp $\mathbb{Z}/8$ look like.

Elt of order 2 is $\varphi(xe_1 + ye_2) = x^2 + xy + y^2$
immersed as in SLIDE 53, so both
gen's go in as figure eights.

There is an immersion of K via
nontrivial figure eight bundle over $S^1$.
representing elt of order 4. There are
2 ways to make a half twist before
gluing ends of cylinder $E \times I$ together

$\mathbb{RP}^2 \looparrowright \mathbb{R}^3$ Boy's surface





Take 3 of these and place them on $xy$, $xz$ and $yz$ planes. Get immersed surface bounded by 3 circles on the 3-planes (???). This and its mirror image represent $\pm 1 \in C_8$.