Note Title 2/11/2010

HOPKINS – THE KERVAIRE INVARIANT PROBLEM

Thm (HHR) If M is a smooth framed mfd with Kervaire invariant 1, then dim M is 2, 6, 14, 30, 62, or 126.

This solves a long standing problem.

1930s: Understood degree of $M^n \to S^n$, homology + cohomology.

Pontryagin studied maps $S^{n+k} \longrightarrow S^n$ inverse image of regular value is a framed M^k. Preimage of closed path is a cobordism between two such mfds.

Thus we get

cobordism classes of stably framed k-manifolds $\longleftrightarrow \pi_{n+k} S^n$ for $n \gg 0$

Pontryagin used this to study $\pi_{n+k} S^n$ for small k.

$\pi_n S^n = \mathbb{Z}$ (degree of a map)

$\pi_{n+1} S^n = \mathbb{Z}/2$ (two framings of S^1)

$\pi_{n+2} S^n \overset{?}{=} 0$ by mistake

Studied framed surfaces M and

framed surgery. There is an obstruction having to do with framings on closed curves leading to $\varphi : H^1(M) \longrightarrow \mathbb{Z}/2$

φ is <u>not</u> linear. $\varphi(x+y) - \varphi(x) - \varphi(y) = \int xy$

$\operatorname{Arf}(\varphi) \in \mathbb{Z}/2$ and $\pi_{n+2}(S^n) = \mathbb{Z}/2$

$M \longrightarrow \operatorname{Arf}(\varphi)$

<u>Q</u> When is every framed mfd cobordant to a "sphere?", i.e. a topological sphere.

<u>A</u> Always except in the dimensions of the theorem.

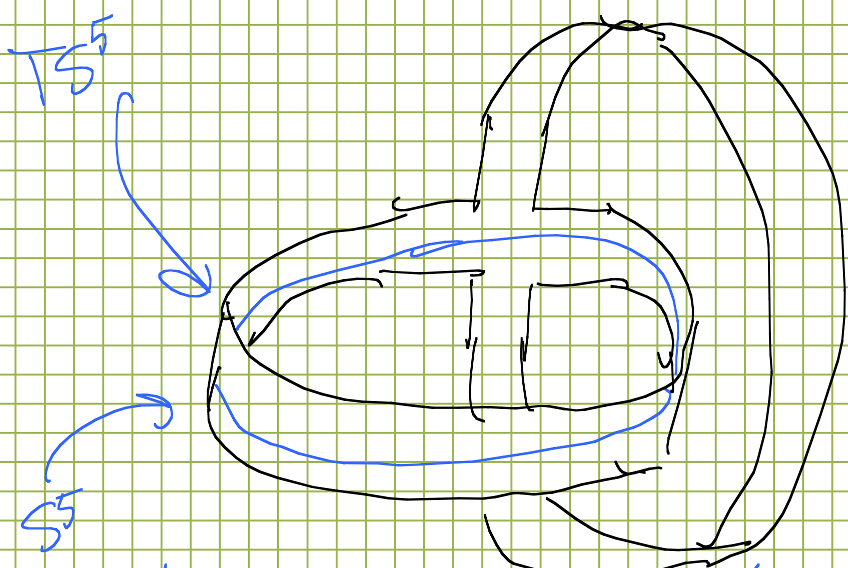
Late 1950s + early 60's work of Kervaire – Milnor.

Kervaire defined for an smooth framed M^{4k+2}

$$q : H^{2k+1}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$$

$$q(x+y) - q(x) - q(y) = \int xy$$

Showed that is vanishes for $k=2$



Kervaire's handcuffs

$\partial N^{10} = X_9$ homeo to S^9

$M^{10} = N^{10} \cup CX_9$

This M has no smooth structure

Could replace 5 by $2k+1$ giving N^{4k+2}

X_{4k+1} homeo to S^{4k+1}

M^{4k+2} PL manifold

Q is X_{4k+1} diffeo to S^{4k+1} ?

Q is M smoothable

Answer: Almost never.

KM could not answer these equivalent questions.

Q In which dims does there exist a smooth stably framed mfd of Kervaire invariant 1? This is answered by our theorem.

Browder 1969. M^k exists with $\Phi(M) = 1$

$\iff \exists\ \theta_j \in \pi_*^{st} S^0$ represented by h_j^2 in the Adams spectral sequence

$k = 2^{j+1} - 2$. dim M must have this form.

Barratt, Jones, Mahowald + Tangora showed $\exists\ \theta_j$ for $j \leq 5$ by 1984

θ_1, θ_2 and θ_3 were known before.

Many believed that all θ_j exist. Now that we know there are only 5 or 6 of them, we look for constructions related to exceptional Lie groups

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EHP sequence

$$\pi_k S^n \xrightarrow{E} \pi_{k+1} S^{n+1} \xrightarrow{H} \pi_{k+1} S^{2n+1} \xrightarrow{P} \pi_{k-1} S^n$$

This leads to an inductive process starting with $\pi_* S^1$.

Suppose $k = 2n$, so $\pi_{k+1} S^{2n+1} = \mathbb{Z}$. Generator maps to $[\iota_n, \iota_n] \in \pi_{2n-1} S^n$, the Whitehead square

Q1 For which j is $[\iota_n, \iota_n]$ in image of E^j ?

Q2 Is $[\iota_n, \iota_n]$ divisible by 2 ?

Q1 $[\iota_n, \iota_n]$ in image $E^j \iff S^n$ has j linearly independent vector fields [JAMES]
Solved by Adams 1962

Q2 For even n this is the Hopf invariant one problem, which was also solved by Adams. For n odd it is the Kervaire invariant question.

The proof We introduce a cohom theory Ω with

1. Detection Thm If $\exists\, \theta_j$, it has nontrivial image in $\pi_* \Omega$
2. Periodicity Thm $\Omega_{k+256} = \Omega_k$
3. Gap Theorem $\Omega_k = 0$ for $-4 < k < 0$.

2/12/10 Informal discussion

Want to prove reduction thm

Thm (Landweber) Let M^{2n} be a "real" mfd with $M^{C_2} = N^n$. If N^n is an unorientable boundary then M is C_2 cobordant to a free C_2-mfd.

For such an M^{2n} we have a double cover $M^{2n} \to \underbrace{M^{2n}/C_2}_{\bar{M}}$. Then $\int_{\bar{M}} w_1^{2n}$.

__Example__ $\mathbb{C}P^2$ fixed at $\mathbb{R}P^2$

$x^2+y^2=\lambda z^2$ quadric for $\lambda\in\mathbb{C}$

For $\lambda=1$ it is as above. For $\lambda=-1$, there are no real points so the C_2-action is free. The orbit space is $\mathbb{R}P^2$, where $w_1^2\neq 0$.

$$\pi_*\Phi^{C_2}MU_{\mathbb{R}}(\bar{x}_1,\ldots) = \text{``}\mathbb{Z}/2[a]\text{''} \quad |a|=|b|=2$$

(not obviously a ring)

$$\downarrow$$

$$\pi_*\Phi^{C_2}H\underline{\mathbb{Z}} = \mathbb{Z}/2[b]$$

The map above is related to characteristic numbers w_1^{2n} as in the example above for $n=1$.

The C_4 case

$$MU^{(2)} = MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \quad \text{has } C_4\text{-action}$$

$$\pi_* MU^{(2)} = \mathbb{Z}[x_1, \gamma x_1, x_2, \gamma x_2, \dots]$$

$$MU^{(2)} / (x_1, \gamma x_1, \dots) \overset{?}{\simeq} H\mathbb{Z}$$

As before this can be reduced to geometric fixed points

$$M \xrightarrow[\substack{\text{normal}\\ \text{bundle}\\ V\oplus W}]{\nu} BU \times BU \qquad C_4\text{-map}$$

$$(V, W) \longrightarrow (W, \bar{V})$$

$V \oplus \gamma^* V \simeq$ stable normal bundle

M has C_4 action restricting to a real structure over C_2. with involution γ

<u>Example</u> Let X be a real mfd. ^ It is a complex bundle V which is its stable normal bundle

$(X \times X, p_1^* V)$ is a C_4-mfd as above.

Thm Given a C_4-mfd M. M^{C_2} is unoriented. If $M^{C_4} = \partial N$ then M is cobordant to M' with $(M')^{C_4} = \phi$.

This is related to $\left(EC_{2+} \to S^0 \to \tilde{E}C_2\right) \wedge MU^{(2)}$.

Suppose $M^{C_4} = \phi$. M^{C_2} has a free action of $C_2' = C_4/C_2$. We can work as before and use

$$\int_{M^{C_2}/C_2'} w_1^{\dim}$$

__Example__ $\left(\mathbb{C}P^1 \times \mathbb{C}P^1\right)^{C_4} = \{ (a,b) : (a,b) = (b,\bar{a}) \}$

$= \mathbb{R}P^1$

Let $\widetilde{\mathbb{C}P}^1$ be defined by $x^2 + y^2 = -z^2$

(different complex structure)

???

A real cobordism is a map $N \to \mathbb{R}$ transverse at 0 and 1

A complex " $N \to \mathbb{C}$ "

equiv for C_2-actions on N and $\mathbb{C}$

e.g. $\{[x,y,z] \in \mathbb{C}P^2 : x^2 + y^2 - z^2 = 0\}$ is complex

cobordant to $\{[x,y,z] \in \mathbb{C}P^2 : x^2 + y^2 + z^2 = 0\}$