HOPKINS RT 2

Note Title 10/29/2010

This is the hardest part of the paper.

$$R(\infty) \longrightarrow H\underline{\mathbb{Z}}$$

We want this to be an equivalence

Nonequivariantly it is part of Milnor's computation of $\pi_* MU$.

Another approach: show it is an iso in $H\mathbb{Z}/p_*^G(-)$

This works in motivic htpy theory (H-Morel)

" " in $MU_{\mathbb{R}}$ (Hu-Kriz)

Need to know $H\mathbb{Z}/p_* H\underline{\mathbb{Z}}$. If we did (independently of our proof), it could lead to another proof. There could be an exotic Steenrod algebra.

Hill's talk reduced us to showing

$$\Phi^G R(\infty) \longrightarrow \Phi^G H\mathbb{Z}$$

$\pi_* = \begin{cases} \mathbb{Z}/2 & \text{even} \\ 0 & \text{odd} \end{cases}$ $\qquad \pi_* = \mathbb{Z}/2[b] \qquad |b| = 2$

We do not have a direct proof that $R(\infty)$ is a ring spectrum.

$$R(\infty) = MU^{((G))} \wedge_A S$$

where $A = S[G \cdot \bar{r}_1, \ldots]$

$$= MU^{((G))}/(G \cdot \bar{r}_1) \wedge_{MU^{((G))}} MU^{((G))}/(G \cdot \bar{r}_2) \cdots\cdots \quad (1)$$

The generator of π_{2^i} of $\Phi^{G} R(\infty)$ occurred when modding out by $G \cdot \bar{r}_{2^i-1}$

These are occurring in independent factors in (1), so it suffices to show b^{2^i} is in image for all i.

Special case: Show $b \in \pi_2 \Phi^{G} H\mathbb{Z}$ is in image. <u>The main diagram.</u>

$M_1 = MU^{((G))}/(G \cdot \bar{m}_1)$. Note $EP = EC_2$ where $C_2 = G/G'$

$G = C_{2^n}$

$G' = C_{2^{n-1}}$

$$\begin{tikzcd}
EP_+ \wedge MU^{((G))} \arrow[r] \arrow[d] & MU^{((G))} \arrow[r] \arrow[d] & \tilde{E}P \wedge MU^{((G))} \arrow[d] \\
EP_+ \wedge M_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & \tilde{E}P \wedge M_1 \arrow[d] \\
EP_+ \wedge H\mathbb{Z} \arrow[r] & H\mathbb{Z} \arrow[r] & \tilde{E}P \wedge H\mathbb{Z}
\end{tikzcd}$$

$y_1 \mapsto N\bar{m}_1 \in \pi_\rho \mapsto$ $\pi_\rho = \pi_1 \Phi^{G} MU^{((G))} = \pi_1 MO = 0$

0

Isotropy separation

b $\qquad b \in \pi_2$

odd π_* $\qquad \pi_*$ in dim 0 $\qquad$ even π_*

Consider $N\tilde{\gamma}_1 \in \pi_\rho MU^{((G))}$.

CLAIM: Image of $y_1 \in \pi_\rho EP_+ \wedge MU^{((G))}$ in $\pi_\rho(EP \wedge H\mathbb{Z}) = \mathbb{Z}/2$ is nonzero. The theorem follows by a diagram chase.

An alternate approach:

Consider the case $G = C_2$, so $EP = EG$.

$\pi_\rho EP \wedge MU_{\mathbb{R}}$ is bordism gp of real mfds and $M^{2d} \to EC_2$. This means C_2 acts freely on M.

This gives map $M^{2d}/C_2 \xrightarrow{\alpha} BC_2$

$\langle [M^{2d}/C_2], \alpha^{2d} \rangle$

$Y \longleftrightarrow$ Quadric $x^2+y^2-z^2$ S^2 with antipodal action

$S^2/C_2 = \mathbb{R}P^2$

$Y_{2k-1} \longleftrightarrow x_1^2 + \cdots x_?^2 = -z^2$ quadric = Grassmannian of 2-planes

How to do it for general G.

Induction on subgps means left column of diagram behaves nicely. Consider

$MU^{((C_8))}$

$\pi_0 \, MU^{((G))} = \mathbb{Z}$

π_2 " $= \mathbb{Z}\{G \cdot M_1\}$ $\quad C_{8+} \wedge_{C_2} S^{2\rho_2}$

π_4 $= \mathbb{Z}\{M_1, \gamma^2 M_1 \cdots\}$

$\vdots$

π_8 $\supset N\bar{m}_1$ $\quad S^{\rho_8}$

Suppose we can ignore induced cells so we have (in low dims)

Crudely $\quad MU^{((C_8))} \longrightarrow H\underline{\mathbb{Z}} \xrightarrow{\overline{Q}_1} \Sigma S^{\rho_8} \wedge H\underline{\mathbb{Z}}$ Milnor operation

So there is a fibration seq

$$\begin{array}{ccc} EP_+ \wedge S^{\rho_8} \wedge H\underline{\mathbb{Z}} & \longrightarrow & EP_+ \wedge MU^{((G))} \\ & & \downarrow \\ & & EP_+ \wedge H\underline{\mathbb{Z}} \end{array}$$

CLAIM The following seq is exact

$$\begin{tikzcd} \pi_\rho^G EP_+ \wedge S^{\rho_8} \wedge H\mathbb{Z} \arrow[r] & \pi_\rho^G EP_+ \wedge MU^{((G))} \arrow[d] & \ni y \\ & \pi_\rho^G EP_+ \wedge H\mathbb{Z} & \end{tikzcd}$$

We are ignoring induced slice cells in lower dims.

e.g. $\pi_\rho^G EP_+ \wedge C_{8+} \underset{C_2}{\wedge} S^{\rho_2} \wedge H\mathbb{Z} = \pi_{\rho_8} C_{8+} \wedge S^{\rho_2} \wedge H\mathbb{Z} = 0$

since $S^\rho \geq |G|$

The claim follows.

To show y has $\neq 0$ image, we need to show it a property that the image of the previous gp does not have.

$H \subset G$ index 2

$S^{\rho_8} \wedge H\mathbb{Z} \simeq 8$

$$\pi_\rho^G EC_{2+} \wedge S^{\rho_8} \wedge H\mathbb{Z}$$
$$\|$$
$$\pi_0^G EC_{2+} \wedge H\mathbb{Z}$$
$$\|$$
$$\pi_0^G C_{2+} \wedge H\mathbb{Z} \cong \pi_0^H H\mathbb{Z}$$

$EC_{2+} = C_{2+} \cup$ higher cells

Conclusion: $\pi_\rho^G EC_{2+} \wedge S^\rho \wedge H\mathbb{Z}$ is in image of transfer from π_*^H

Need to show y is <u>not</u> a transfer.

Suffices to show $N\bar{r}_1$ is not in image of transfer.

In $\pi_8^n MU^{((G))}$ we have N_{m_1}

The image of the transfer consists of elts of the form $x - \gamma x$ for a gen $\gamma \in G$.

In $\pi_8^n MU^{((G))}$ generates a summand on which γ acts via sign

It follows that y has nontrivial image