HU PERIODICITY

Note Title 10/28/2010

$$MU^{((C_8))} = N_{C_2}^{C_8} MU_{\mathbb{R}} \underset{u}{\equiv} MU^{(4)}$$

$$D \in \pi_\bigstar MU^{((C_8))}$$

$$\Xi = \tilde{\Omega} = D^{-1} MU^{((C_8))}$$

$$\Xi O = \Omega = \tilde{\Omega}^{C_8} = \tilde{\Omega}^{hC_8}$$

Periodicity $\pi_{256+*}\Omega = \pi_*\Omega$

Will look at slice SS for $MU^{((G))}$

We need $G = C_2, C_4, C_8$

Will need $RO(G)$-graded slice SS

Let V be a virtual rep of G

$$E_2^{s,t} = \pi_{V+t-s} P^{|V|+t}_{|V|+t} X \Longrightarrow \pi_{V+t-s} X$$

We need the case $V = -l\sigma$, σ = sign rep.
We will say twist $= -l$.

Special elements

1. For any rep V we have $S^0 \xrightarrow{a_V} S^V \in \pi^G_{-V} S^0$

$$a_{V \oplus W} = a_V a_W \qquad a = a_\sigma \in E_2^{1,1-\sigma}$$

(Hurewicz image)

For any G-spectrum X

$$\pi_* \underline{\Phi}^G X = a^{-1} \pi_* X \quad \text{where } \widetilde{EC_2} \wedge X \simeq S^{\infty\sigma} \wedge X$$

2. $M_1 \in \pi_{2,1}^{u} MU^{((G))}$

$\overline{M}_1 \in \pi_{1\rho_2}^{C_2} MU^{((G))}$ ρ_2 = regular rep of C_2

$N_{C_2}^{G} \overline{M}_1 \in \pi_{1\rho_G}^{G} MU^{((G))}$ ρ_G = reg rep of G

$f_1 = a_{\bar\rho_G} N(\overline{M}_1)$ $\bar\rho_G$ = reduced reg rep of G

$\in E_2^{(g-1)i, gi}$ $g = |G|$

3. For an oriented V we have

$$H_{|V|}^{G}(S^V, \underline{\mathbb{Z}}) \xrightarrow{\cong} H_{|V|}^{u}(S^V, \underline{\mathbb{Z}})$$

$$u_V \longmapsto \text{orientation}$$

$u_V \in \pi_{|V|-V}^{G} H\underline{\mathbb{Z}}$

For any V, $2V$ is oriented so u_{2V} exists

$$u_{V\oplus W} = u_V u_W$$

$$u = u_{2\sigma} \in E_2^{0,2-2\sigma}$$

Hu Kriz notation
$u = \sigma^{-2}$

Remark f_1 above can be constructed geometrically

C_8-mfd M with $V_M = V_1 \oplus V_2 \oplus V_3 \oplus V_4$

where V_i is complex and

$\gamma : V_i \xrightarrow{\cong} V_{i+1}$ and $\gamma^4 : V_i \circlearrowleft$ anti-linear involution

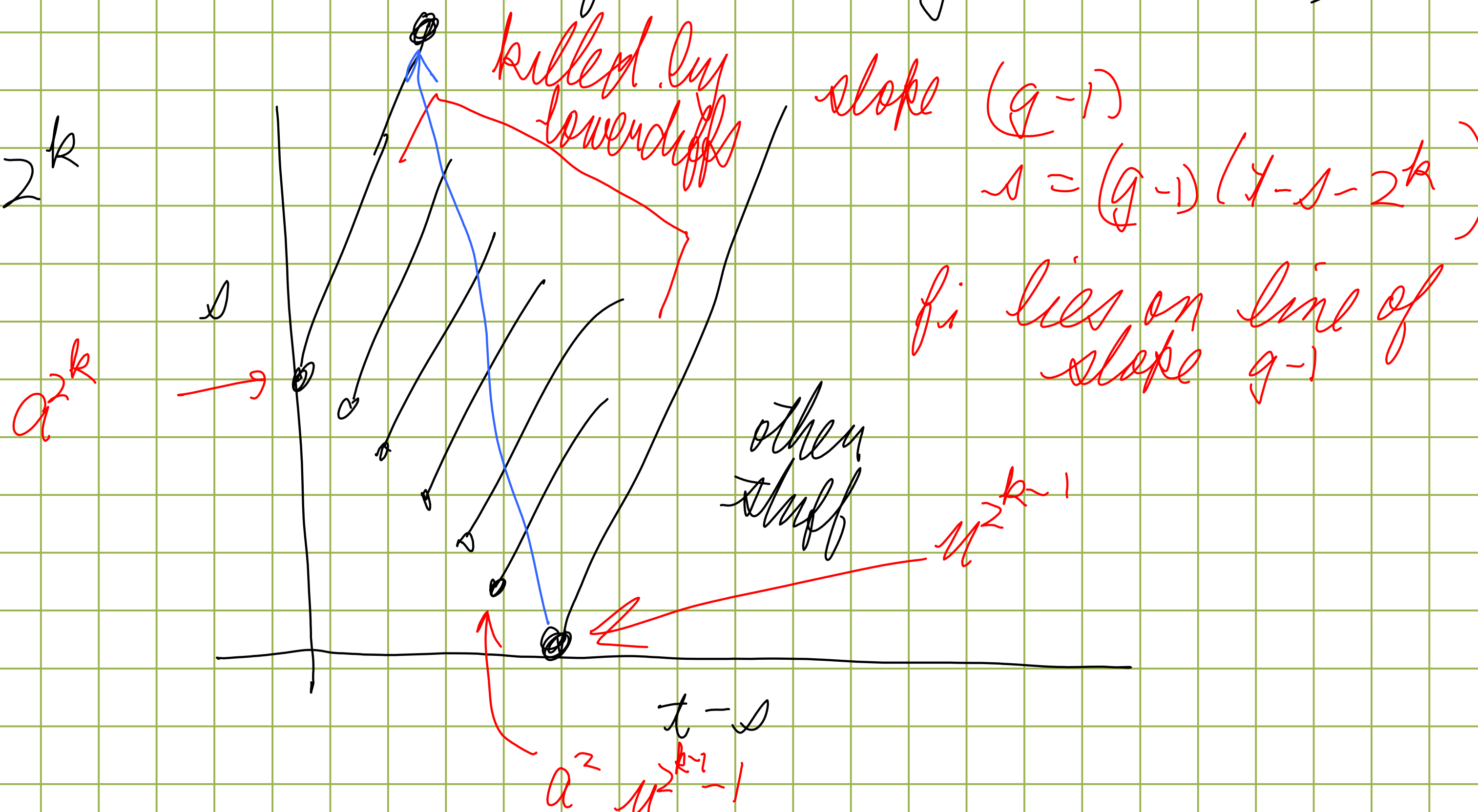
equiv spectrum on trivial universe $\longrightarrow MU^{((8))}$

In the slice SS for $V=-16$ twist $-\ell$:

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \longrightarrow E_2^{s,t-16}$$

<u>Thm</u> This is an iso for $s \geq (g-1)(t-s-\ell)$

For $\ell = 2^k$



Slice Differentials Thm. In the slice SS for $MU^{((G))}$ the diff on $u^{2^{k-1}}$ is

$$d_{1+(2^k-1)g}\left(u^{2^{k-1}}\right) = a^{2^k} f_{2^k-1}$$

twist as promised

Proof: $\Phi^G MU^{((G))} = MO$ and

$$\pi_* MO = \mathbb{Z}/2[h_i : i \neq 2^k-1]$$

$$f_i \longmapsto h_i$$

$$f_{2^k-1} \longmapsto 0$$

Argue by induction on k.

Assume true for $k' < k$, so lower powers of u are gone.

Inverting a gives SS converging to $MO_*[a^{\pm 1}]$.

4 diff cannot hit f_i for $i \neq 2^k - 1$.

The only possible diff on $u^{2^{k-1}}$ is the one indicated, i.

Also f_{2^k-1} must be killed by a power of a. This differential is the only thing that can kill it. QED

Inverting elements

$g = |G|$

Let $\overline{\Delta}_k^{(g)} = N_{C_2}^{G}(\overline{r}_{2^k-1}) \in \pi^{G}_{(2^k-1)\rho_G} MU^{((G))}$

Cor Inverting $\overline{\Delta}_k^{(g)}$ makes u^{2^k} a perm cycle.

Pf For $MU^{((G))}$, $d(u^{2^k}) = a^{2^{k+1}} f_{2^{k+1}-1}$. Will show this gets killed earlier

$$f_{2^{k+1}-1} \overline{\Delta}_k^{(g)} = \left(a_{\overline{\rho}}^{2^{k+1}} N(\overline{r}_{2^{k+1}-1})\right) N(\overline{r}_{2^k-1}))$$

$$= \left(a_{\overline{\rho}}^{2^k} N(\overline{r}_{2^{k+1}-1})\right)\left(a_{\overline{\rho}}^{2^k} N(\overline{r}_{2^k-1})\right)$$

$$= a_{\overline{\rho}}^{2^k} \overline{\Delta}_{k+1}^{(g)} f_{2^k-1}$$

$$= a_V^{2^k} \Delta_{k+1}^{(g)} a^{2^k} f_{2^k-1} \quad \text{for } V = \overline{\rho} - \sigma$$

$$= a_V^{2^k} \Delta_{k+1}^{(g)} d_{n'}(u^{2^{k-1}}) \quad \text{for } n' \text{ even}$$

etc. $\qquad$ QED

<u>Norm function</u> If $H \subseteq G$ and V is H-rep then $N_H^G S^V = S^{\mathrm{Ind}_H^G(V)}$

There is an identity

$$u_{\mathrm{Ind}_H^G V} = N_H^G(u_V)\, u_{\mathrm{Ind}_H^G(|V|)}$$

From this we find

$$\mathcal{U}_{2\rho_8} = \mathcal{U}_{8\sigma_8}\, N^{C_8}_{C_4}\left(\mathcal{U}_{4\sigma_4}\right)\, N^{C_8}_{C_8}\left(\mathcal{U}_{2\sigma_2}\right)$$

inverting $\bar{\Delta}^{(2)}_4$ makes $\mathcal{U}_{32\sigma_2} = \mathcal{U}^{16}_{2\sigma_2}$ a perm cycle

$\bar{\Delta}^{(4)}_2$ " $\mathcal{U}_{8\sigma_4} = \mathcal{U}^{2}_{4\sigma_4}$ "

$\bar{\Delta}^{(8)}_1$ " $\mathcal{U}_{4\sigma_8} = \mathcal{U}^{2}_{2\sigma_2}$ "

Then $D = \bar{\Delta}^{(8)}_1\, N^{C_8}_{C_4}\left(\Delta^{(4)}_2\right)\, N^{C_8}_{C_2}\left(\Delta^{(2)}_4\right) \in \pi^{C_8}_{19\rho_8}$

makes $\mathcal{U}_{32\rho_8}$ a perm cycle

Define $\bar{\Delta}_1^{(8)} = u_{2\rho_8} \left(\bar{\Delta}_1^{(8)}\right)^2$

$$\left(\Delta_1^{(8)}\right)^{32} = u_{32\rho_8} \left(\bar{\Delta}_1^{(8)}\right)^{32} \in \pi_{256} D^{-1} MU^{((C_8))}$$

If we forget to underlying spectrum,

$$u_{32\rho_8} : S^{256-32\rho_8} \longrightarrow D^{-1} MU^{((C_8))}$$

becomes a unit so

$\left(\Delta_1^{(8)}\right)^{16}$ is a perm cycle forgetting to a unit

$$\pi_*^u D^{-1} MU^{((C_8))} \xrightarrow[\cong]{\left(\Delta_1^{(8)}\right)^{16}} \pi_{*+256}^u D^{-1} MU^{((C_8))}$$

This means it also gives an equivalence

of finitely fixed pts

$\left(D^{-1} MU^{((C_8))}\right)^{hC_8}$ is 256-periodic.