Introduction to ∞-Categories

Lecture Notes

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Chapter 1

Introduction

These are lecture notes from an 8-week Ph.D. course on ∞-categories at the University of Copenhagen in Spring 2017.

Warning 1.0.1. These notes are very preliminary, and are probably full of mistakes, inaccuracies, and really terrible jokes. No warranty, use at own risk. The existence of current lecture notes does not imply the existence of future lecture notes. Comments and complaints (as well as large monetary donations) are very welcome!

Of course, nothing in these notes is original — in fact, for the first part of the course everything is basically due to Joyal, unless it's even older, and the presentation here is mostly stolen from Charles Rezk's lecture notes [Rez17].

∞-Categories live at the intersection of higher category theory and homotopy theory: On the one hand, they are an implementation of higher categories based on homotopy theory, and on the other hand, they are a higher-categorical language for talking about homotopy theories. This introductory chapter attempts to explain this statement, starting with some background on higher categories.

[TO DO: Add references!]

1.1 Higher Categories

The basic idea of higher categories is to add morphisms between morphisms, and then keep going: An (n,k)-category should be a structure with objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, ..., and n-morphisms between $(n-1)$-morphisms; these should all be invertible for $n > k$. We want to allow $n = \infty$, and it's usual to say n-category for (n,n)-category.

Warning 1.1.1. Following Lurie we'll use ∞-category as an abbreviation for $(\infty,1)$-category, *not* (∞,∞)-category. (The latter are sometimes known as ω-categories, at least in the strict case.)

Example 1.1.2. A 0-category is a set, a 1-category is a category, and a $(1,0)$-category is a groupoid.

At first glance, this doesn't seem so hard to make precise:

Definition 1.1.3. An n-category is a category enriched in $(n-1)$-categories. I.e. an n-category $\mathcal{C}$ has a collection of objects, for each pair of objects x, y an $(n-1)$-category $\mathcal{C}(x,y)$ of morphisms between them, for each object x a unit in $\mathcal{C}(x,x)$ and for each triple of objects x,y,z an associative composition $\mathcal{C}(x,y) \times \mathcal{C}(y,z) \to \mathcal{C}(x,z)$.

We can then define $(n, 0)$-categories or n-groupoids to be n-categories where all i-morphisms are invertible for $i = 1, \ldots, n$ and inductively take (n, k)-categories to be categories enriched in $(n-1, k-1)$-groupoids.

These structures are known as *strict* n-categories and n-groupoids. As the ominous word "strict" indicates, these are not the n-categories we're looking for. For one thing, presumably we wanted to define n-categories in the first place because we hoped there would be some interesting examples.

Let's consider a situation from topology we would hope to fit into this formalism: Every topological space X has a *fundamental groupoid* $\pi_{\leq 1}X$; this has points in X as objects and homotopy classes of paths as morphisms. Moreover, if X is a 1-type (meaning $\pi_i X = 0$ for $i > 1$) then we can recover X from $\pi_{\leq 1}X$. (More generally, any space can be truncated to form a 1-type by killing the higher homotopy groups, and we can recover precisely this 1-type from $\pi_{\leq 1}X$.) Similarly, we can define a fundamental 2-groupoid $\pi_{\leq 2}X$ of points, paths, and homotopy classes of homotopies between paths. It can be shown that 2-groupoids correspond to 2-types.[1] We would like to keep going: we would expect that every topological space X has a *fundamental n-groupoid* $\pi_{\leq n}X$ such that we can recover the underlying n-type of X from $\pi_{\leq n}X$. Unfortunately, Carlos Simpson showed that it is *impossible* to model the 3-type of S^2 by any strict 3-groupoid.

Why are we being punished like this? We did something morally reprehensible when we defined strict n-categories: Notice that the associativity criterion for 2-categories requires that for objects x, y, z, w the two composition functors

$$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \times \mathcal{C}(z, w) \to \mathcal{C}(x, w)$$

are *equal*. As morally upstanding citizens we know that it is very sinful to require functors to be equal — we are only permitted to ask for them to be *naturally isomorphic*. So we should improve the definition of 2-categories by replacing this equality by a natural isomorphism, the *associator*. This must then satisfy a coherence condition, the pentagon axiom, relating the different ways to apply associators for compositions of four morphisms. Similarly, the relations for the identity maps should be replaced be natural isomorphisms (the left and right *unitors*) satisfying coherence conditions (the triangle axioms). The resulting structure is called a (weak) 2-category or *bicategory*. It is possible to work with these[2], but it requires drawing lots of really big diagrams. Any bicategory, it turns out, is actually equivalent to a strict 2-category, which explains why we got away with strict 2-groupoids above.

People have also worked out how to define weak 3-categories (or tricategories) this way — it gets pretty complicated; as Simpson's example suggests, not every 3-category is equivalent to a strict one. It then turns out, as we hoped, that weak 3-groupoids correspond to 3-types.[3]

You can even find a definition of 4-categories in this style somewhere online, but explicitly writing out the coherence conditions gets increasingly annoying, not to say impossible — not to speak of actually working with these structures once you've defined them. So if you want to go on and define n-categories it

[1] This is essentially due to Whitehead, at least for connected 2-types.

[2] At least if you're Australian...

[3] This seems to be an unpublished result of Joyal and Tierney, and also in an unpublished thesis of O. Leroy in Montpellier; there is also a proof by Berger, which is actually published.

seems you have to find some clever way of packaging this coherence data; there were various approaches to this (by people like Batanin, Leinster, Baez–Dolan) proposed back in the 90s.

Alternatively, we can *cheat*. Grothendieck proposed that the relationship between n-types and n-groupoids we saw for low n above extends all the way to infinity:

Conjecture 1.1.4 (Grothendieck's Homotopy Hypothesis). *∞-groupoids are the same thing as homotopy types.*

If you're an upstanding category theorist, you would view this as a conjecture you want to prove about some model for ∞-groupoids.[4] Alternatively, if you're an unscrupulous homotopy theorist, you can turn this on its head and view it as a proposed definition of ∞-groupoids as *being* (some model for) homotopy types; then you can use this as a starting point to develop a theory of (∞, n)-categories for $n > 0$. As the remainder of this course will hopefully convince you, this turns out to work very well indeed.

How can we get a model for $(\infty, 1)$-categories in this way? If we think of homotopy types the traditional way, i.e. using topological spaces, and assume we can get away with a bit of strictness at the bottom, a natural first attempt is to consider *topological categories*:

Definition 1.1.5. A topological category is a category enriched in topological spaces. Thus a topological category $\mathbf{C}$ has a set of objects, for all pairs of objects x, y, a topological space $\mathbf{C}(x, y)$, and associative, unital and continuous composition maps

$$\mathbf{C}(x, y) \times \mathbf{C}(y, z) \to \mathbf{C}(x, z).$$

Alternatively, we can consider a more combinatorial model for homotopy types, such as simplicial sets (of which much more will be said later), and consider *simplicial categories*, meaning categories enriched in simplicial sets.

1.2 What Does This Have to Do with Homotopy Theory?

Ok, so we've asserted that simplicial categories give one possible model of $(\infty, 1)$-categories. But what does all this categorical nonsense have to do with homotopy theory? To answer this, let's consider the most basic structure in which we can talk about "doing homotopy theory", namely that of a *relative category*:

Definition 1.2.1. A *relative category* is a category $\mathbf{C}$ equipped with a class W of morphisms that we think of as *weak equivalences*; this should contain all the isomorphisms and be closed under composition. Usually we also want it to satisfy the *2-of-3 property*: if two out of the three morphisms (f, g, gf) are in W, then so is the third.

[4] In fact, Grothendieck originally proposed this in the context of a particular notion of ∞-groupoids whose definition he sketched; this has later been fleshed out by Ara and Maltsiniotis.

Examples 1.2.2. Topological spaces and weak homotopy equivalences; chain complexes of R-modules and quasi-isomorphisms.

If $(\mathbf{C}, W)$ is a relative category, we can define its *homotopy category*, which is the universal category where the morphisms in W are sent to isomorphisms. In the examples this gives the usual homotopy category of spaces, and the derived category of R. In practice passing to the homotopy category loses important information, and we can't "do homotopy theory" only by working there. Dwyer and Kan showed that we can do better: from any relative category we can extract a simplicial category where we "invert the morphisms in W up to homotopy". This turns out to capture all the "homotopy-invariant" information from the relative category we started with, so if we want we can think of this higher category as *being* the homotopy theory.

Unfortunately, working with simplicial categories turns out to be rather painful in practice, as it is difficult to make homotopy-invariant constructions. For example, suppose we have a functor $F\colon \mathbf{C} \to \mathbf{D}$ of simplicial categories. If we also have equivalences[5] $\phi_c\colon F(c) \xrightarrow{\sim} X_c$ for all objects c in $\mathbf{C}$, it seems reasonable to expect that we can find another functor F' taking c to X_c and a natural transformation from F to F'. This is false, however.

To avoid issues like this, Vogt introduced the notion of *homotopy-coherent diagrams* in simplicial categories.[6] The basic idea is that for a pair of composable morphisms $c \xrightarrow{f} c' \xrightarrow{g} c''$ in $\mathbf{C}$ we should not have an equality $F(gf) = F(g)F(f)$ but rather supply an edge between $F(gf)$ and $F(g)F(f)$ in the simplicial set $\mathbf{D}(c, c'')$. Similarly, for a triple of composable morphisms $c \xrightarrow{f} c' \xrightarrow{g} c'' \xrightarrow{h} c'''$ we should have a square (i.e. a compatible pair of 2-simplices)

$$\begin{tikzcd} F(h)F(g)F(f) \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & F(h)F(gf) \arrow[d, dash] \\ F(hg)F(f) \arrow[r, dash] & F(hgf) \end{tikzcd}$$

in $\mathbf{D}(c, c''')$. Cordier later noticed that you could define a *nerve* functor N from simplicial categories to simplicial sets, and that a homotopy coherent diagram of shape $\mathbf{C}$ in $\mathbf{D}$ we precisely the same thing as a map of simplicial sets $\mathrm{N}\mathbf{C} \to \mathrm{N}\mathbf{D}$. He also showed that these simplicial sets had a property previously introduced by Boardman and Vogt: they were *weak* or *restricted Kan complexes*.

Much later, Joyal realized that you could use this class of simplicial sets, which he rechristened *quasicategories*, to develop a model for $(\infty, 1)$-categories that is much easier to work with than simplicial categories — for example, many homotopy-invariant constructions can be implemented combinatorially using simplicial sets.

In the first part of this course, we'll work through the foundations of quasicategories, introducing analogues of many basic concepts from category theory. Later, we'll survey (in much less detail) some more advanced topics, which will get us into the work of Lurie.

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[5]Meaning that ϕ_c induces a weak homotopy equivalence $\mathbf{D}(d, F(c)) \to \mathbf{D}(d, X_c)$ for all d.

[6]This is not quite true — Vogt actually only defined homotopy-coherent diagrams in topological spaces of shape $\mathbf{C}$ for $\mathbf{C}$ an ordinary category. It's the same idea, however.

Chapter 2

"Review" of Simplicial Sets

Before we introduce quasicategories, in this chapter we discuss some rather formal aspects of the homotopy theory of simplicial sets, in particular Kan complexes. This is probably familiar to many, at least in part, but it'll be useful to go through it since quite a few things about quasicategories will run parallel to this story.

2.1 Simplicial Sets

Definition 2.1.1. The *simplicial indexing category* $\mathbb{\Delta}$ is the category with objects the ordered sets $[n] = \{0, \ldots, n\}$ and order-preserving maps between them; this is a skeleton of the category of non-empty finite ordered sets. A *simplicial set* is a presheaf of sets on $\mathbb{\Delta}$, i.e. a contravariant functor $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. We'll write X_n for $X([n])$ and Δ^n for the image of $[n]$ under the Yoneda embedding, i.e. the simplicial set $\mathrm{Hom}_{\mathbb{\Delta}}(-, [n])$.

[I'm assuming you've seen this before, and that you know about faces and degeneracies, and how to picture Δ^n geometrically.]

We write $\mathrm{Set}_\Delta := \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Set})$ for the category of simplicial sets.

2.2 Geometric Realization

Let $|\Delta^n|$ denote the *geometric n-simplex*, i.e. the subspace $\{(x_0, \ldots, x_n) : \sum x_i = 1, x_i \geq 0\}$ of $\mathbb{R}^{n+1}$. Taking geometric faces and degeneracies we get a functor $\mathbb{\Delta} \to \mathrm{Top}$ taking $[n]$ to $|\Delta^n|$. This has a unique extension to a colimit-preserving functor $|{-}|\colon \mathrm{Set}_\Delta \to \mathrm{Top}$. This is called *geometric realization*; it is the left adjoint to the *singular simplicial set* functor $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{Set}_\Delta$, given by $\mathrm{Sing}(T)_\bullet := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^\bullet|, T)$.

2.3 Lifting Properties

If $f\colon a \to b$ and $g\colon x \to y$ are morphisms in a category $\mathbf{C}$, we say that f has the *left lifting property* for g (and that g has the *right lifting property* for f) if for every commutative square

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & x \arrow[d, "g"] \\ b \arrow[r] & y \end{tikzcd}$$

there exists a lift $h\colon b \to x$, i.e. a morphism making the diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & x \arrow[d, "g"] \\ b \arrow[r] \arrow[ur, dashed, "h"] & y \end{tikzcd}$$

commute.

If S is some collection of maps in $\mathbf{C}$, we'll write $\mathrm{LLP}(S)$ for the class of maps that have the left lifting property for all the morphisms in S, and $\mathrm{RLP}(S)$ for all those that have the right lifting property. We'll also say an object x has a right lifting property if the map $x \to *$ to the terminal object does so.

2.4 Horns and Kan Complexes

Definition 2.4.1. The *horn* Λ^n_k is the subcomplex of Δ^n obtained by removing the interior and the face opposite the kth vertex. More precisely, it is given by

$$(\Lambda^n_k)_i = \{\sigma \in \Delta^n_i : \{0, \dots, k-1, k+1, \dots, n\} \nsubseteq \mathrm{im}\sigma\}.$$

Similarly, the *boundary* $\partial\Delta^n$ is given by removing the interior of Δ^n, so

$$(\partial\Delta^n)_i = \{\sigma \in \Delta^n_i : \{0, \dots, n\} \nsubseteq \mathrm{im}\sigma\}.$$

Homotopy types are particularly well described by *Kan complexes*:

Definition 2.4.2. A simplicial set X is a *Kan complex* if it has the right lifting property for the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for all n, k. More generally, a map that has the right lifting property for the horn inclusions is called a *Kan fibration*.

Example 2.4.3. If T is a topological space, then $\mathrm{Sing}(T)$ is always a Kan complex: Extending a map $\Lambda^n_k \to \mathrm{Sing}(T)$ to Δ^n is, by adjunction, the same thing as extending a map of topological spaces $|\Lambda^n_k| \to T$ to $|\Delta^n|$, and $|\Lambda^n_k| \to |\Delta^n|$ is a deformation retract.

2.5 Connected Components

Definition 2.5.1. If X is a simplicial set, the set $\pi_0 X$ of *connected components* of X is the quotient of X_0 by the equivalence relation $\sim$ generated by $p \sim q$ if there is an edge in X_1 from p to q.

We can equivalently define $\pi_0 X$ as the coequalizer of the two face maps $X_1 \rightrightarrows X_0$, which is (by a cofinality argument) the same as the colimit of X viewed as a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. Thus π_0 is a colimit-preserving functor, and is left adjoint to the constant simplicial set functor $\mathrm{Set} \to \mathrm{Set}_\Delta$.

The connected components are easier to describe for a Kan complex, as we have:

Lemma 2.5.2. *If X is a Kan complex, then $\sim$ is an equivalence relation on X_0.*

Sketch Proof. For transitivity we need to fill a horn of shape Λ^2_1, and for symmetry a horn of shape Λ^2_0 (with one edge degenerate). $\square$

2.6 Saturated Classes

We want to prove statements such as "every simplicial set can be replaced by a Kan complex" and "if X is a Kan complex then the internal Hom X^K is again a Kan complex for every K". To do this we need to introduce some rather formal machinery in the next few sections.

Definition 2.6.1. A class of morphisms S in Set_Δ is *saturated* if it contains all isomorphisms, and is closed under cobase change, composition, transfinite composition, coproducts, and retracts.

Remark 2.6.2. Let's expand that a bit:

- S is closed under cobase change if whenever we have a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

  with f in S, then f' is in S.

- S is closed under transfinite composition says, first of all, that if we have

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$$

  with all f_i in S, then the induced map $A_0 \to \operatorname{colim}_n A_n$ is in S. More generally, the same should hold for a colimit over *any* well-ordered set. (For a finite ordered set we get as a special case closure under composition in the obvious sense.)

- S is closed under coproducts if $f\colon A \to B$ and $f'\colon A' \to B'$ in S implies $f \amalg f'\colon A \amalg A' \to B \amalg B'$ is in S.

- S is closed under retracts if, given a commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"] \arrow[rr, bend left, equal] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B' \arrow[r] \arrow[rr, bend right, equal] & B \arrow[r] & B' \end{tikzcd}$$

  with f in S, then f' is in S.

Lemma 2.6.3. *For any class S, the class $\mathrm{LLP}(S)$ is saturated.*

Proof. Just check the class LLP(S) is closed under each of the required operations in turn. For example, if we have a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

with f in LLP(S), and a commutative square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "g"] \\ B' \arrow[r] & Y \end{tikzcd}$$

with g in S, then in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] & Y \end{tikzcd}$$

the dashed lift exists since $f \in$ LLP(S). But then the universal property of the pushout B' supplies a lift $B' \to X$ in the right-hand square. □

Definition 2.6.4. If S is a set of maps in Set_Δ, its *saturation* $\bar{S}$ is the smallest saturated class that contains S.

As an immediate consequence, we get:

Lemma 2.6.5. $\bar{S} \subseteq \mathrm{LLP}(\mathrm{RLP}(S))$.

Proof. Clearly $S \subseteq \mathrm{LLP}(\mathrm{RLP}(S))$. But $\mathrm{LLP}(\mathrm{RLP}(S))$ is saturated, so this implies $\bar{S} \subseteq \mathrm{LLP}(\mathrm{RLP}(S))$. □

Later we'll see, using the small object argument, that we actually have $\bar{S} = \mathrm{LLP}(\mathrm{RLP}(S))$.

Definition 2.6.6. A morphism of simplicial sets is *anodyne* if it lies in the saturated class generated by the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$.

Lemma 2.6.7. $\mathrm{RLP}(S) = \mathrm{RLP}(\bar{S})$.

Proof. Clearly $\mathrm{RLP}(\bar{S}) \subseteq \mathrm{RLP}(S)$. But if $f \in \mathrm{RLP}(S)$, then $S \subseteq \mathrm{LLP}(f)$. Since $\mathrm{LLP}(f)$ is a saturated class, this implies $\bar{S} \subseteq \mathrm{LLP}(f)$, hence $f \in \mathrm{RLP}(()\bar{S})$. □

Example 2.6.8. A map of simplicial sets is a Kan fibration if and only if it has the right lifting property for all anodyne maps.

Proposition 2.6.9. *The monomorphisms in* Set_Δ *are the saturated class generated by the boundary inclusions* $\partial\Delta^n \hookrightarrow \Delta^n$.

Sketch Proof. It's not hard to check that the monomorphisms are indeed a saturated class. For the other direction we need to show that any monomorphism $K \hookrightarrow X$ can be built by adding simplices along their boundary.

Every simplicial set X has a *skeletal filtration.* Let $\mathbb{\Delta}_{\leq n}$ denote the full subcategory of $\mathbb{\Delta}$ spanned by the object $[i]$ with $i \leq n$, and let $i_n \colon \mathbb{\Delta}_{\leq n} \to \mathbb{\Delta}$ be the inclusion. Then the *n-skeleton* of X is $\mathrm{sk}_n X := i_{n,!} i_n^* X$ — i.e. restrict X to $\mathbb{\Delta}_{\leq n}$ and take the left Kan extension back to $\mathbb{\Delta}$; this amounts to throwing out all i-simplices for $i > n$ and then adding back in the degenerate i-simplices on the n-simplices. An n-simplex of X is *non-degenerate* if it does not lie in $\mathrm{sk}_n X$, and for every n we have a pushout square

$$\begin{tikzcd}
\coprod_{\sigma \in X_n^{\mathrm{nd}}} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in X_n^{\mathrm{nd}}} \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1} X \arrow[r] & \mathrm{sk}_n X
\end{tikzcd}$$

where X_n^{nd} is the set of non-degenerate n-simplices. This shows that $\emptyset \to X$ lies in the saturated class generated by the boundary inclusions.

For a general monomorphism $K \hookrightarrow X$ we similarly observe that we have pushout squares

$$\begin{tikzcd}
\coprod_{\sigma \in X_n^{\mathrm{nd}} \setminus K_n} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in X_n^{\mathrm{nd}} \setminus K_n} \Delta^n \arrow[d] \\
K \amalg_{\mathrm{sk}_{n-1} K} \mathrm{sk}_{n-1} X \arrow[r] & K \amalg_{\mathrm{sk}_n K} \mathrm{sk}_n X.
\end{tikzcd}$$

Thus this also lies in the required saturated class. □

Definition 2.6.10. A morphism that has the right lifting property for all monomorphisms (or equivalently for the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$) is called a *trivial fibration.*

2.7 The Small Object Argument

Theorem 2.7.1 (Quillen's Small Object Argument)**.** *If S is any set of morphisms in* Set_Δ*, then any map $f \colon X \to Y$ of simplicial sets admits a factorization as* $X \xrightarrow{s} U \xrightarrow{t} Y$ *where* $s \in \bar{S}$ *and* $t \in \mathrm{RLP}(S)$.

Proof. We'll just prove the simplest case, which is all we'll use; this is where the maps $K \to L$ in S are all such that K has finitely many non-degenerate simplices.

Set $U_0 := X$; then we'll define a sequence of objects U_n and maps $s_n \colon U_n \to U_{n+1}$ with compatible maps $t_n \colon U_n \to Y$. Let A_n be the set of all commutative squares of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\sigma"'] & U_{n-1} \arrow[d, "t_{n-1}"] \\ L \arrow[r] & Y. \end{tikzcd}$$

Then we define U_{n+1} by the pushout

$$\begin{tikzcd} \coprod_{A_n} K \arrow[r] \arrow[d] & U_{n-1} \arrow[d, "s_{n-1}"] \\ \coprod_{A_n} L \arrow[r] & U_n \end{tikzcd}$$

The squares in A_n also determine a commutative square

$$\begin{tikzcd} \coprod_{A_n} K \arrow[r] \arrow[d] & U_{n-1} \arrow[d, "t_{n-1}"] \\ \coprod_{A_n} L \arrow[r] & Y, \end{tikzcd}$$

so the universal property of U_n gives a map $t_n \colon U_n \to Y$ such that $t_n \circ s_{n-1} = t_{n-1}$.

Now define $U := \operatorname{colim}_n U_n$ (with the colimit along the maps s_n) with $s \colon X \to U$ the natural map from U_0 to the colimit, and $t \colon U \to Y$ the map from the colimit induced by the compatible maps t_n. Each of the maps s_n is a cobase change of a map in S, so they all lie in $\bar{S}$, and s is a transfinite composite of these, so it is also in $\bar{S}$.

It remains to show that t is in $\mathrm{RLP}(S)$. Suppose then that we have a diagram

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "\sigma"'] & U \arrow[d, "t"] \\ L \arrow[r] & Y \end{tikzcd}$$

with $\sigma \in S$. The image of every simplex in K in U is the image of a simplex in some finite stage U_i in the colimit. Since K has finitely many non-degenerate simplices, we can choose n so that all of these simplices lie in the image of U_n, hence the map i factors as $K \to U_n \to U$. But then we have a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\sigma"] & U_n \arrow[r, "s_n"] & U_{n+1} \arrow[r] & U \arrow[d, "t"] \\ L \arrow[urr, dashed] \arrow[rrr] & & & Y, \end{tikzcd}$$

where the lift to U_{n+1} exists by definition of s_n. $\square$

Remark 2.7.2. The proof in the general case is essentially the same, but uses transfinite composition over a bigger ordinal Ω — we choose it so that all the sources of maps in S have fewer than Ω non-degenerate simplices.

Remark 2.7.3. Note that the construction of U in the proof is completely functorial, so we can strengthen the theorem to say that we can always choose such factorizations that are functorial.

Corollary 2.7.4. *Any morphism of simplicial sets factors as both an anodyne map followed by a Kan fibration, and as a monomorphism followed by a trivial fibration. In particular, for any simplicial set K there exists an anodyne map $K \to \tilde{K}$ such that $\tilde{K}$ is a Kan complex.*

Corollary 2.7.5. *If S is any set of maps, then $\bar{S} = \mathrm{LLP}(\mathrm{RLP}(S))$.*

Proof. We saw in Lemma 2.6.5 that $\bar{S} \subseteq \mathrm{LLP}(\mathrm{RLP}(S))$. So it remains to show that if $f\colon X \to Y$ is in $\mathrm{LLP}(\mathrm{RLP}(S))$ then $f \in \bar{S}$. By Theorem 2.7.1 we can factor f as $X \xrightarrow{s} U \xrightarrow{t} Y$ where $s \in \bar{S}$ and $t \in \mathrm{RLP}(S)$. Then f has the left lifting property for t, so we can choose a lift in the square

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & U \arrow[d, "t"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dashed, "z"] & Y. \end{tikzcd}$$

But this gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, "s"] & X \arrow[d, "f"] \\ Y \arrow[r, "z"] & U \arrow[r, "t"] & Y. \end{tikzcd}$$

Since $tz = \mathrm{id}_Y$ this makes f a retract of s. Since $\bar{S}$ is in particular closed under retracts, this means $f \in \bar{S}$. $\square$

2.8 Pushout Products

Definition 2.8.1. If $\mathbf{C}$ is a category with limits and colimits, then we can define the *pushout product* $i \square j$ of morphisms $i\colon X \to Y$ and $j\colon K \to L$ as the natural map

$$K \times Y \amalg_{K \times X} L \times X \to Y \times L.$$

There is also a dual version, using the internal Hom: If $f\colon A \to B$ is another morphism, we'll denote by $f^{\square i}$ the induced map

$$A^Y \to A^X \times_{B^X} B^Y.$$

Lemma 2.8.2. *Given $i\colon X \to Y$, $j\colon K \to L$, and $f\colon A \to B$, then $i \square j$ has the left lifting property for f if and only if i has the left lifting property for $f^{\square j}$.*

Proof. Expand out what it means to give a lift in terms of the universal property of the pushouts and pullbacks, and you'll see that under the product/internal Hom adjunction the two precisely correspond. [TO DO: Draw diagrams] □

Exercise 2.8.3. The functors $-\Box i$ and $(-)^{\Box i}$ are adjoint to each other as functors from the category of morhpisms in Set_Δ to itself.

Lemma 2.8.4. *For any classes of maps S and T, we have $\bar{S}\Box\bar{T} \subseteq \overline{S\Box T}$*

Proof. Let $U := \mathrm{RLP}(S\Box T)$. By Corollary 2.7.5, the class $\overline{S\Box T}$ can be identified with $\mathrm{LLP}(U)$. We first show that $\bar{S}\Box T \subseteq \mathrm{LLP}(U)$: Let Ξ denote the set of maps f such that $\{f\}\Box T \subseteq \mathrm{LLP}(U)$; then Ξ is equivalently the set of maps that have the left lifting property for $U^{\Box T}$. Hence Ξ is saturated; since $S \subseteq \Xi$ this implies $\bar{S} \subseteq \Xi$, i.e. $\bar{S}\Box T \subseteq \mathrm{LLP}(U)$. Next the same argument with $\bar{S}$ in place of T and T in place of S shows $\bar{S}\Box\bar{T} \subseteq \mathrm{LLP}(U)$. □

Proposition 2.8.5. $(\partial\Delta^n \to \Delta^n)\Box(\Lambda^m_k \to \Delta^m)$ *is anodyne.*

Proof. This is a calculation, which we won't do because it's annoying. We need to give a filtration of $\Delta^n \times \Delta^m$ starting with $\partial\Delta^n \times \Delta^m \amalg_{\partial\Delta^n\times\Lambda^m_k} \Delta^n \times \Lambda^m_k$ where each step is given by filling a single horn to a simplex. [TO DO: Reference for proof] □

Applying Lemma 2.8.4 we immediately get:

Corollary 2.8.6. *If f is anodyne and g is a monomorphism, then $f\Box g$ is anodyne.*

Now using Lemma 2.8.2 this gives:

Corollary 2.8.7.

(i) If p is a Kan fibration and i is anodyne, then $p^{\Box i}$ is a trivial fibration.

(ii) If p is a Kan fibration and i is a monomorphism, then $p^{\Box i}$ is a Kan fibration.

In particular, if X is a Kan complex and K is any simplicial set, then X^K is a Kan complex; we'll usually write $\mathrm{Map}(K, X)$ for this internal Hom X^K when X is Kan.

2.9 Homotopy Equivalences

Definition 2.9.1. A (simplicial) *homotopy* between two maps $f, g \colon X \to Y$ is a map $\phi \colon X \times \Delta^1 \to Y$ such that $\phi \circ (\mathrm{id} \times d_1) = f$, $\phi \circ (\mathrm{id} \times d_0) = g$. A morphism $f \colon X \to Y$ between Kan complexes is a *homotopy equivalence* if there exists $g \colon Y \to X$ and homotopies between fg and id_Y and between gf and id_X. In this case we say that g is a *homotopy inverse* to f.

Exercise 2.9.2. If Y is a Kan complex then for any simplicial set K homotopy is an equivalence relation on maps $K \to Y$.

Lemma 2.9.3. *A homotopy equivalence of Kan complexes induces an isomorphism on π_0.*

Proof. Suppose $f\colon X \to Y$ is a homotopy equivalence between Kan complexes, and choose a homotopy inverse g. The functor π_0 preserves products (since the products of simplicial sets and sets preserve colimits in each variable, and π_0 is a left adjoint, it's enough to check on simplices, where it's clear). Therefore a homotopy between two maps means they induce the same map on π_0; in particular $\pi_0 g$ is an inverse to $\pi_0 f$. □

Definition 2.9.4. Let $h\mathrm{Kan}$ denote the category with Kan complexes as objects, and the set of morphisms from X to Y given by $\pi_0\mathrm{Map}(X, Y)$. (Composition is well-defined since π_0 preserves products.)

Proposition 2.9.5. *The following are equivalent for a morphism $f\colon X \to Y$ of Kan complexes:*

(1) f is a homotopy equivalence.

(2) For all Kan complexes Z, the induced map $f^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is a homotopy equivalence.

(3) For all Kan complexes Z, the induced map $\pi_0 f^*\colon \pi_0\mathrm{Map}(Y, Z) \to \pi_0\mathrm{Map}(X, Z)$ is an isomorphism.

(4) f is an isomorphism in the category $h\mathrm{Kan}$.

Proof. If f is a homotopy equivalence with homotopy inverse g, then $g^*\colon \mathrm{Map}(X, Z) \to \mathrm{Map}(Y, Z)$ is a homotopy inverse to f^*: A homotopy $\phi\colon X \times \Delta^1 \to X$ from gf to id_X induces $\phi^*\colon \mathrm{Map}(X, Z) \to \mathrm{Map}(X \times \Delta^1, Z) \simeq \mathrm{Map}(\Delta^1, \mathrm{Map}(X, Z))$; this corresponds to a map $\mathrm{Map}(X, Z) \times \Delta^1 \to \mathrm{Map}(X, Z)$ which is a homotopy from $(gf)^* = f^*g^*$ to $\mathrm{id}_{\mathrm{Map}(X,Z)}$. Thus (1) implies (2).

Next (2) implies (3) by Lemma 2.9.3, and (3) is equivalent to (4) by definition of $h\mathrm{Kan}$. It remains to show that (4) implies (1). Write $[f] \in \pi_0\mathrm{Map}(X, Y)$ for the class represented by f and assume that it is an isomorphism. Then there exists $[g] \in \pi_0\mathrm{Map}(Y, X)$ represented by $g\colon Y \to X$, such that $[gf] = [g][f] = [\mathrm{id}_X]$ and $[fg] = [f][g] = [\mathrm{id}_Y]$. Since $\mathrm{Map}(X, X)$ and $\mathrm{Map}(Y, Y)$ are Kan complexes, this means that there exist $\Delta^1 \to \mathrm{Map}(X, X)$ and $\Delta^1 \to \mathrm{Map}(Y, Y)$ connecting gf to id_X and fg to id_Y. This says precisely that f is a homotopy equivalence. □

Corollary 2.9.6. *Homotopy equivalences between Kan complexes satisfy the 2-of-3 property.*

Proof. Immediate from condition (4) above. (It is also easy to check directly from the definition.) □

Lemma 2.9.7. *Suppose $f\colon X \to Y$ is a trivial fibration between Kan complexes. Then f is a homotopy equivalence.*

Proof. Since f is a trivial fibration, we can choose a section s of f by picking a lift in

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

Then $f \circ s = \mathrm{id}_Y$. Next, we pick a lift h in the square

$$\begin{tikzcd} X \amalg X \arrow[r, "{(sf, \mathrm{id}_X)}"] \arrow[d, "\mathrm{id}_X \times (\partial\Delta^1 \to \Delta^1)"'] & X \arrow[d, "f"] \\ X \times \Delta^1 \arrow[r, "f \circ \pi_X"'] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

Then h is precisely a homotopy between fs and id_X. □

2.10 Weak Homotopy Equivalences

Definition 2.10.1. We say a map $f\colon X \to Y$ of simplicial sets is a *weak homotopy equivalence* if for all Kan complexes K the map of Kan complexes $f^*\colon \mathrm{Map}(Y, K) \to \mathrm{Map}(X, K)$ is a homotopy equivalence.

Lemma 2.10.2. *Any anodyne map is a weak homotopy equivalence.*

Proof. If $f\colon X \to Y$ is anodyne and K is a Kan complex, then $f^*\colon \mathrm{Map}(Y, K) \to \mathrm{Map}(X, K)$ is a trivial fibration by Corollary 2.8.7. By Lemma 2.9.7 this implies f^* is a homotopy equivalence. □

Proposition 2.10.3. *The following are equivalent for a map of simplicial sets $f\colon X \to Y$:*

(1) f is a weak homotopy equivalence.

(2) For any Kan complex K, the induced map $\pi_0\mathrm{Map}(Y, K) \to \pi_0\mathrm{Map}(X, K)$ is an isomorphism.

Moreover, the weak homotopy equivalences satisfy the 2-of-3 property.

Proof. Using Theorem 2.7.1 we can choose a commutative square

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ Y \arrow[r, "\eta"'] & \tilde{Y} \end{tikzcd}$$

where ξ and η are anodyne and $\tilde{X}$ and $\tilde{Y}$ are Kan complexes. For any Kan complex K we then get a commutative square

$$\begin{tikzcd}
\mathrm{Map}(\tilde{Y}, K) \arrow[r, "\tilde{f}^*"] \arrow[d, "\eta^*"'] & \mathrm{Map}(\tilde{X}, K) \arrow[d, "\xi^*"] \\
\mathrm{Map}(Y, K) \arrow[r, "f^*"'] & \mathrm{Map}(X, K)
\end{tikzcd}$$

where the vertical maps are trivial fibrations, and hence homotopy equivalences by Lemma 2.9.7. Since 2-of-3 holds for (weak) homotopy equivalences between Kan complexes, we see that f is a weak homotopy equivalence if and only if $\tilde{f}$ is one. Moreover, since homotopy equivalences between Kan complexes are π_0-isomorphisms by Lemma 2.9.3 we see that condition (2) holds for f if and only if it holds for $\tilde{f}$. Thus (1) is equivalent to (2) for f since they are equivalent for $\tilde{f}$ by Proposition 2.9.5. □

If X is a Kan complex, for $p, q \in X_0$ we define $\mathrm{Map}_X(p, q)$ by the pullback square

$$\begin{tikzcd}
\mathrm{Map}_X(p, q) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1, X) \arrow[d] \\
\{(p, q)\} \arrow[r] & X \times X.
\end{tikzcd}$$

Since $\partial\Delta^1 \hookrightarrow \Delta^1$ is a monomorphism, the map $\mathrm{Map}(\Delta^1, X) \to X \times X$ is a Kan fibration, so $\mathrm{Map}_X(p, q)$ is again a Kan complex. If $p = q$ we write $\Omega_p X := \mathrm{Map}_X(p, p)$. We can then define $\pi_1(X, p) := \pi_0 \Omega_p X$. (This only gives the right answer if X is indeed a Kan complex; moreover, it is a group, as you would expect, but we won't prove it.) Proceeding inductively, we can define $\pi_n(X, x)$ too. We'll eventually prove the following inductive characterization of homotopy equivalences:

Theorem 2.10.4. *A map $f\colon X \to Y$ between Kan complexes is a (weak) homotopy equivalence if and only if*

- *$\pi_0 X \to \pi_0 Y$ is surjective,*
- *for all vertices $x, x' \in X_0$, $\mathrm{Map}_X(x, x') \to \mathrm{Map}_Y(fx, fx')$ is a (weak) homotopy equivalence.*

Exercise 2.10.5. Show that this pair of conditions implies that $\pi_0 X \to \pi_0 Y$ is an isomorphism.

Remark 2.10.6. This is a weak version of the Whitehead Theorem, which says that f is a homotopy equivalence if and only if $\pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism for all n and x. We probably won't prove that.

Remark 2.10.7. Weak homotopy equivalences of simplicial sets are often defined as the maps $X \to Y$ such that $|X| \to |Y|$ is a weak homotopy equivalence of topological spaces. This agrees with the definition above, but we definitely won't prove this.

Chapter 3

Introducing Quasicategories

3.1 Inner Horns and Nerves

To motivate the definition of quasicategories, we'll first note that we can think of categories as being certain simplicial sets.

Definition 3.1.1. We can view the partially ordered sets $[n]$ as categories; this determines a functor $\mathbb{\Delta} \to \mathrm{Cat}$. The *nerve* of a category $\mathbf{C}$ is the simplicial set $\mathrm{N}\mathbf{C} := \mathrm{Hom}_{\mathrm{Cat}}([\bullet], \mathbf{C})$.

Definition 3.1.2. The *inner horns* are the horns $\Lambda^n_k \hookrightarrow \Delta^n$ with $0 < k < n$.

Proposition 3.1.3. *A simplicial set X is isomorphic to the nerve of a category if and only if every inner horn $\Lambda^n_k \to X$ has a* unique *extension to $\Delta^n \to X$.*

To prove this, it's useful to first prove that nerves of categories have a property that makes them fairly simple: they are 2-*coskeletal* in the following sense:

Definition 3.1.4. For X a simplicial set, let $\mathrm{cosk}_n X := i_{n,*} i_n^* X$, where $i_{n,*}$ is right Kan extension along $i_n \colon \mathbb{\Delta}_{\leq n} \to \mathbb{\Delta}$. We say that X is n-*coskeletal* if the natural map $X \to \mathrm{cosk}_n X$ is an isomorphism.

Lemma 3.1.5. *The following are equivalent for a simplicial set X:*

(1) X is n-coskeletal.

(2) For every simplicial set K the map $\mathrm{Hom}(K, X) \to \mathrm{Hom}(\mathrm{sk}_n K, X)$ *is an isomorphism.*

(3) The map $\mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\mathrm{sk}_n \Delta^k, X)$ *is an isomorphism for all $k > n$.*

(4) For every $k > n$ the map $\mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\partial\Delta^k, X)$ *is an isomorphism.*

Proof. There are natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{cosk}_n X) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_{\leq n}, \mathrm{Set})}(i_n^* K, i_n^* X) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n K, X),$$

which shows that $\mathrm{Hom}(K, X) \to \mathrm{Hom}(\mathrm{sk}_n K, X)$ is an isomorphism if and only if $\mathrm{Hom}(K, X) \to \mathrm{Hom}(K, \mathrm{cosk}_n X)$, so (1) is equivalent to (2). Next (3) is a special case of (2), and also implies (2) since sk_n preserves colimits and every simplicial set is a colimit of simplices. Moreover, (4) follows immediately from (2), since $\partial\Delta^k$ contains all n-simplices of Δ^k if $k > n$. It remains to show that (4) implies (3); to do this we considre the skeletal filtration of Δ^k. We have $\mathrm{sk}_{k-1}\Delta^k = \partial\Delta^k$, and $\mathrm{sk}_{i-1}\Delta^k \to \mathrm{sk}_i\Delta^k$ is a cobase change of a coproduct of copies of $\partial\Delta^i \to \Delta^i$. By (4) the maps $\mathrm{Hom}(\mathrm{sk}_i\Delta^k, X) \to \mathrm{Hom}(\mathrm{sk}_{i-1}\Delta^k, X)$ are therefore all isomorphisms for $i = n+1, \ldots, k$, and so the composite $\mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\mathrm{sk}_n\Delta^k, X)$ is an isomorphism, which gives (3). □

Lemma 3.1.6. *If* $\mathbf{C}$ *is a category, then* $\mathrm{N}\mathbf{C}$ *is 2-coskeletal.*

Proof. By the previous lemma we must show that for $k > 2$ we have that $\mathrm{Hom}(\Delta^k, \mathrm{N}\mathbf{C}) \to \mathrm{Hom}(\mathrm{sk}_2\Delta^k, \mathrm{N}\mathbf{C})$ is an isomorphism. An element of the source is a sequence $(c_0 \xrightarrow{f_1} c_1 \to \cdots \to c_n)$ of composable morphisms in $\mathbf{C}$. This data is clearly uniquely determined by the images of the edges $i \to i+1$ in Δ^k. On the other hand, a map from the 2-skeleton of Δ^k to $\mathrm{N}\mathbf{C}$ is also uniquely determined by this data, as the 2-simplices precisely say that the remaining edges are sent to composites of the morphisms these edges are sent to. □

Proof of Proposition 3.1.3. First consider a category $\mathbf{C}$. For $n > 3$, $\mathrm{sk}_2\Lambda^n_k \cong \mathrm{sk}_2\Delta^n$, so since $\mathrm{N}\mathbf{C}$ is 2-coskeletal by Lemma 3.1.6 it suffices to consider 2- and 3-horns. A map from Λ^2_1 specifies two composable morphisms, which determines a unique 2-simplex, and fillers for 3-horns exist because composition is associative. [TO DO: Spell this out.]

Conversely, given a simplicial set X with unique inner horn fillers, we define a category $\mathbf{C}$. The objects of $\mathbf{C}$ are the 0-simplices in X, and $\mathrm{Hom}_{\mathbf{C}}(x, y)$ is the set of edges from x to y in X. Composition of these is defined by filling a Λ^2_1 and restriction to $\Delta^{\{0,2\}}$ — this is well-defined since the extension to Δ^2 is unique. Identities are given by degenerate edges — these behave like identitites since e.g. the composite of $f\colon x \to y$ with $s_0 y$ is determined by a unique 2-simplex, which must be the degenerate 2-simplex $s_1 f$.

We get a map $X \to \mathrm{N}\mathbf{C}$ by taking an n-simplex to its restrictions to $\Delta^{\{0,1\}}$, $\ldots$, $\Delta^{\{n-1,n\}}$. (In fact it suffices to define this on 2-simplices, since $\mathrm{N}\mathbf{C}$ is 2-coskeletal.) Now for any inner horn $\Lambda^n_k \to \Delta^n$ we have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n, X) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^n, \mathrm{N}\mathbf{C}) \arrow[d] \\
\mathrm{Hom}(\Lambda^n_k, X) \arrow[r] & \mathrm{Hom}(\Lambda^n_k, \mathrm{N}\mathbf{C})
\end{tikzcd}$$

where the vertical maps are isomorphisms. Since Λ^n_k is built from Δ^i's with $i < n$, if we know we have isomorphisms on $\mathrm{Hom}(\Delta^i, –)$ for $i < n$ we can conclude we have an isomorphism for $i = n$. Proceeding by induction, we get an isomorphism for every i provided we have an isomorphism on 0- and 1-simplices, which we have by definition of $\mathbf{C}$. □

Moreover, we have:

Lemma 3.1.7. *The nerve functor* $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{Set}_\Delta$ *is fully faithful.*

Proof. Since nerves are 2-coskeletal, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}\mathbf{C}, \mathrm{N}\mathbf{D}) \cong \mathrm{Hom}_{\mathrm{Fun}(\Delta^{\mathrm{op}}_{\leq 2}, \mathrm{Set})}(\mathrm{N}\mathbf{C}|_{\Delta^{\mathrm{op}}_{\leq 2}}, \mathrm{N}\mathbf{D}|_{\Delta^{\mathrm{op}}_{\leq 2}}).$$

Expanding out what such a natural transformation of functors on $\Delta^{\mathrm{op}}_{\leq 2}$ amounts to, we see that this is precisely the data of a functor. [TO DO: Spell this out.] □

This means that we can regard categories as being certain simplicial sets; we'll often take advantage of this and omit the nerve functor from our notation.

3.2 Quasicategories

We are finally ready to define the objects that constitute our models for ∞-categories:

Definition 3.2.1. A simplicial set is a *quasicategory* if it has the right lifting property for the inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, $0 < k < n$.

Remark 3.2.2. We see immediately that Kan complexes and nerves of categories are quasicategoires. Thus quasicategories include both categories and spaces, in the form of Kan complexes.

Remark 3.2.3. We think of the vertices (0-simplices) of a quasicategory $\mathcal{C}$ as its *objects* and its edges (1-simplices) as its *morphism*. The existence of fillers for Λ^2_1 then says that we can compose morphisms, but the lack of uniqueness tells us that such composites are not unique — the remaining requirements can be interpreted as saying that the space of possible composites is nevertheless *contractible*; we'll see a precise version of this statement later in Corollary 3.7.6.

3.3 The Homotopy Category

The nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{Set}_\Delta$ has a left adjoint; this is the unique colimit-preserving functor $h\colon \mathrm{Set}_\Delta \to \mathrm{Cat}$ that extends $[-]\colon \mathbb{\Delta} \to \mathrm{Cat}$. We call hK the *homotopy category* of the simplicial set K. In general this category is hard to describe — colimits of categories are usually not nice. For quasicategories we can get a much simpler description, however. To give this we first introduce some notation:

Definition 3.3.1. If K is a simplicial set and x, y are 0-simplices in K, let $K_1^{x,y}$ denote the set of 1-simplices σ of K such that $d_1\sigma = x$ and $d_0\sigma = y$. We define two relations $\sim_l$ and $\sim_r$ on the set $K_1^{x,y}$: We say $\sigma \sim_l \tau$ if there exists a 2-simplex μ such that $d_2\mu$ is degenerate, $d_0\mu = \sigma$ and $d_1\mu = \tau$:

$$
\begin{tikzcd}
 & x \arrow[dr, "\sigma"] & \\
x \arrow[ur, equal] \arrow[rr, "\tau"] & & y;
\end{tikzcd}
$$

similarly we say $\sigma \sim_r \tau$ if there exists a 2-simplex μ such that $d_0\mu$ is degenerate, $d_2\mu = \sigma$ and $d_1\mu = \tau$:

$$
\begin{tikzcd}
 & y \arrow[dr, equal] & \\
x \arrow[ur, "\sigma"] \arrow[rr, "\tau"] & & y
\end{tikzcd}
$$

Lemma 3.3.2. *Suppose $\mathcal{C}$ is a quasicategory and x, y are objects of $\mathcal{C}$. Then*

(i) Both $\sim_l$ and $\sim_r$ are equivalence relations on $\mathcal{C}_1^{x,y}$.

(ii) For $\sigma, \tau \in \mathcal{C}_1^{x,y}$ we have $\sigma \sim_l \tau$ if and only if $\sigma \sim_r \tau$.

Proof. Reflexivity for $\sim_r$ follows from the degenerate 2-simplices $s_1\sigma$:

$$\begin{tikzcd}
& y \arrow[dr, equal] & \\
x \arrow[ur, "\sigma"] \arrow[rr, "\sigma"'] & & y;
\end{tikzcd}$$

to show transitivity for $\sim_r$ suppose we have $\sigma \sim_r \tau$ and $\tau \sim_r \lambda$. Then we have a Λ^3_2 of the form

$$\begin{tikzcd}
& y \arrow[drr, equal] \arrow[ddr, equal] & & \\
& & & y \\
x \arrow[uur, "\sigma"] \arrow[urrr, "\lambda"] \arrow[rr, "\tau"'] & & y \arrow[ur, equal] &
\end{tikzcd}$$

where the 0-face is degenerate. Filling this, the 2-face witnesses $\sigma \sim_r \lambda$.

Next, we want to show the two relations are equal. If $\sigma \sim_l \tau$ we have a Λ^3_1 of the form

$$\begin{tikzcd}
& x \arrow[drr, "\sigma"] \arrow[ddr, "\sigma"] \arrow[ddl, equal] & & \\
& & & y \\
x \arrow[urrr, "\sigma"] \arrow[rr, "\tau"'] & & y \arrow[ur, equal] &
\end{tikzcd}$$

with the 0-face and 2-face degenerate. Filling this we get a 2-simplex witnessing $\sigma \sim_r \tau$. On the other hand, if $\sigma \sim_r \tau$ we have a Λ^3_2 of the form

$$\begin{tikzcd}
& x \arrow[drr, "\tau"] \arrow[ddr, "\tau"] \arrow[ddl, equal] & & \\
& & & y \\
x \arrow[urrr, "\sigma"] \arrow[rr, "\tau"'] & & y, \arrow[ur, equal] &
\end{tikzcd}$$

again with the 0-face and 2-face degenerate. Filling this we get a 2-simplex witnessing $\tau \sim_l \sigma$. Thus $\sim_l$ is symmetric, and using this we then see that $\sim_l = \sim_r$. ☐

Definition 3.3.3. Let $\mathcal{C}$ be a quasicategory. The *homotopy category* $h\mathcal{C}$ of $\mathcal{C}$ is the category whose objects are the 0-simplices of $\mathcal{C}$, with $h\mathcal{C}(x,y) := \mathcal{C}_1^{x,y}/\sim_l = \mathcal{C}_1^{x,y}/\sim_r$. If $[f] \in h\mathcal{C}(x,y)$ and $[g] \in h\mathcal{C}(y,z)$ are represented by 1-simplices f and g, then the composite $[g][f]$ is represented by $d_1\sigma$ for any 2-simplex σ such that $d_2\sigma = f$ and $d_0\sigma = g$; this is well-defined since given two such 2-simplices, with d_1 given by h and h', respectively, we can make a Λ^3_1 of the form

$$\begin{tikzcd}
& y \arrow[drr, "g"] \arrow[ddr, "g"] & & \\
& & & z \\
x \arrow[uur, "f"] \arrow[urrr, "h"] \arrow[rr, "h'"'] & & z; \arrow[ur, equal] &
\end{tikzcd}$$

filling this we get a 2-simplex witnessing $h \sim_r h'$. Identity maps are represented by degenerate 1-simplices.

Exercise 3.3.4. Check that this composition is associative and unital.

Definition 3.3.5. We define a map of simplicial sets $\mathcal{C} \to h\mathcal{C}$, determined by a map $\mathrm{sk}_2\mathcal{C} \to h\mathcal{C}$, which is the identity on 0-simplices, takes edges to their equivalence class in $h\mathcal{C}$, and 2-simplices to the composition relation they witness in $h\mathcal{C}$.

Proposition 3.3.6. *Suppose $\mathcal{C}$ is a quasicategory and $\mathbf{D}$ is an ordinary category. Then any map $F\colon \mathcal{C} \to \mathrm{N}\mathbf{D}$ factors uniquely as $\mathcal{C} \to \mathrm{N}h\mathcal{C} \to \mathrm{N}\mathbf{D}$.*

Proof. Since $\mathrm{N}\mathbf{D}$ is 2-coskeletal, the map F is determined by its restriction to $\mathrm{sk}_2\mathcal{C}$. The image of a 2-simplex σ exhibits an identity $F(d_1\sigma) = F(d_0\sigma) \circ F(\delta_2\sigma)$. If $\sigma \sim_l \tau$ then applying this to a 2-simplex witnessing the relation gives $F(\sigma) = F(\tau)$. There is therefore a (unique) way to factor the map on 1-simplices through $(\mathrm{N}h\mathcal{C})_1$. Moreover, the image of a 2-simplex only depends on its image in $h\mathcal{C}$, so we get the required unique factorization. □

Remark 3.3.7. This says precisely that $h\mathcal{C}$ as defined explicitly for a quasicategory $\mathcal{C}$ has the universal property $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathrm{N}\mathbf{D}) \cong \mathrm{Hom}_{\mathrm{Cat}}(h\mathcal{C}, \mathbf{D}$, and so gives the value of the left adjoint h of N at $\mathcal{C}$.

3.4 Subcategories

If $\mathcal{C}$ is a quasicategory and $\mathbf{D}$ is a subcategory of $h\mathcal{C}$, then the *subcategory* of $\mathcal{C}$ determined by $\mathbf{D}$ is given by the pullback square

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{N}\mathbf{D} \arrow[r] & h\mathcal{C}. \end{tikzcd}$$

Definition 3.4.1. A map of simplicial sets is an *inner fibration* if it has the right lifting property for the inner horn inclusions.

Lemma 3.4.2. *If $\mathbf{C}$ is an ordinary category and $\mathcal{E}$ is a quasicategory, then any map $F\colon \mathcal{E} \to \mathbf{C}$ is an inner fibration.*

Proof. Suppose we have a commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d] & \mathcal{E} \arrow[d, "F"] \\ \Delta^n \arrow[r, "\tau"'] & \mathbf{C}, \end{tikzcd}$$

with $0 < k < n$. Since $\mathcal{E}$ is a quasicategory, we can extend σ' to a map $\bar{\sigma}\colon \Delta^n \to \mathcal{E}$. Then $F\bar{\sigma}$ is a map $\Delta^n \to \mathbf{C}$ whose restriction to Λ^n_k is $\tau|_{\Lambda^n_k}$. But since inner horn fillers in $\mathbf{D}$ are unique by Proposition 3.1.3, it follows that $F\bar{\sigma} = \tau$, which means $\bar{\sigma}$ gives the required lift. □

Lemma 3.4.3. *The subcategory $\mathcal{D}$ of $\mathcal{C}$ determined by a subcategory $\mathbf{D}$ of $h\mathcal{C}$ is a quasicategory.*

Proof. In the pullback square

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathrm{N}\mathbf{D} \arrow[r] & h\mathcal{C}
\end{tikzcd}$$

the right vertical map is an inner fibration by Lemma 3.4.2. Since inner fibrations are closed under base change (being defined by a right lifting property), this means $\mathcal{D} \to \mathbf{D}$ is an inner fibration. Since inner fibrations are closed under composition, this implies $\mathcal{D}$ is a quasicategory (as this is equivalent to the map $\mathcal{D} \to *$ being an inner fibration). □

3.5 Equivalences in a Quasicategory

Definition 3.5.1. A morphism f in a quasicategory $\mathcal{C}$ is an *equivalence* if the morphism in $h\mathcal{C}$ represented by f is an isomorphism.

Definition 3.5.2. Let E^1 denote the (nerve of the) category with two objects and a unique morphism between any pair of objects.

A map $E^1 \to \mathcal{C}$ is a "coherent" notion of an equivalence in $\mathcal{C}$.

Lemma 3.5.3. *If a 1-simplex $\Delta^1 \to \mathcal{C}$ can be extended to a map $E^1 \to \mathcal{C}$ then it is an equivalence.*

Proof. The data included in the map $E^1 \to \mathcal{C}$ includes the appropriate 2-simplices that show the 1-simplices in E^1 are mapped to isomorphisms. [TO DO: Describe the nerve of E^1 explicitly and spell this out.] □

We will later see that this precisely characterizes the equivalences — so coherent equivalences are exactly the maps that give isomorphisms in the homotopy category.

3.6 Quasigroupoids

Definition 3.6.1. A quasicategory $\mathcal{G}$ is a *quasigroupoid* if all morphisms in $\mathcal{G}$ are equivalences, i.e. if all morphisms in $h\mathcal{G}$ are isomorphisms, or equivalently if $h\mathcal{G}$ is a groupoid.

Lemma 3.6.2. *A Kan complex is a quasigroupoid.*

Proof. If X is a Kan complex, then choosing fillers for outer horns we can make a post- and pre-inverse for every morphism in hX. □

Remark 3.6.3. Later we'll prove that the quasigroupoids are *precisely* the Kan complexes.

Definition 3.6.4. If $\mathbf{C}$ is a category, the *core* $\mathrm{Core}(\mathbf{C})$ is the subcategory of $\mathbf{C}$ containing only the isomorphisms.

Definition 3.6.5. Let $\mathcal{C}$ be a quasicategory. The *core* of $\mathcal{C}$ is the quasigroupoid $\mathrm{Core}(\mathcal{C})$ defined as the subcategory of $\mathcal{C}$ corresponding to the core of $h\mathcal{C}$.

3.7 Inner Anodyne Maps

Definition 3.7.1. A morphism of simplicial sets is *inner anodyne* if it lies in the saturated class generated by the inner horn inclusions.

Remark 3.7.2. By Lemma 2.6.7 the inner fibrations are precisely the maps that have the right lifting property for the inner anodyne maps.

Proposition 3.7.3. *For $i\colon \Lambda^n_k \hookrightarrow \Delta^n$ an inner horn and $j\colon \partial\Delta^m \hookrightarrow \Delta^m$ a boundary inclusion, the inclusion*

$$i\Box j\colon \Lambda^n_k \times \Delta^m \amalg_{\Lambda^n_k \times \partial\Delta^m} \Delta^n \times \partial\Delta^m \hookrightarrow \Delta^n \times \Delta^m$$

is inner anodyne.

Example 3.7.4. [TO DO: Draw and explain the case $n = 2$, $m = 1$.]

You can prove this by constructing a filtration of $\Delta^n \times \Delta^m$ starting with $\Lambda^n_k \times \Delta^m \amalg_{\Lambda^n_k \times \partial\Delta^m} \Delta^n \times \partial\Delta^m$ such that in each step you fill a unique inner horn. However, there is a stronger result that takes a bit less work:

Proposition 3.7.5. *The inner anodyne morphisms are precisely the saturation of $\{\Lambda^2_1 \hookrightarrow \Delta^2\}\Box\{\partial\Delta^m \hookrightarrow \Delta^m\}$.*

Proof. First we have to prove the case $n = 2$ of Proposition 3.7.3. At least for now I won't do this here — see [Lur09, Proposition 2.3.2.1]. [TO DO: Add this proof?] Next let S denote $\{\Lambda^2_1 \hookrightarrow \Delta^2\}\Box\{\partial\Delta^m \hookrightarrow \Delta^m\}$; since S is contained in the inner anodyne maps, so is its saturation $\bar{S}$. On the other hand, for $0 < j < n$ we can construct retract diagrams

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] \arrow[rr, bend left, equal] & \Lambda^2_1 \times \Delta^n \amalg_{\Lambda^2_1 \times \partial\Delta^n} \Delta^2 \times \partial\Delta^n \arrow[r] \arrow[d] & \Lambda^n_j \arrow[d] \\
\Delta^n \arrow[r, "s"] \arrow[rr, bend right, equal] & \Delta^2 \times \Delta^n \arrow[r, "r"] & \Delta^n,
\end{tikzcd}$$

where

$$s(y) = \begin{cases} (0, y), & y < j, \\ (1, y), & y = j, \\ (2, y), & y > j, \end{cases} \qquad r(x, y) = \begin{cases} y, & x = 0, y < j, \\ y, & x = 2, y > j, \\ j, & \text{otherwise.} \end{cases}$$

Since $\bar{S}$ is closed under retracts, this implies the inner horn inclusions lie in $\bar{S}$, and hence so do all the inner anodyne maps since $\bar{S}$ is saturated. $\square$

Combining Proposition 3.7.5 with Lemma 2.8.2, we get the following characterization of quasicategories:

Corollary 3.7.6.

(i) A simplicial set X is a quasicategory if and only if $X^{\Delta^2} \to X^{\Lambda^2_1} \cong X^{\Delta^1} \times_X X^{\Delta^1}$ is a trivial fibration.

(ii) A morphism $f\colon X \to Y$ is an inner fibration if and only if $X^{\Delta^2} \to Y^{\Delta^2} \times_{Y^{\Lambda^2_1}} X^{\Lambda^2_1}$ is a trivial fibration.

Intuitively, this says that a simplicial set is a quasicategory if and only if for any pair of composable edges, the space of composites is contractible.

Proof of Proposition 3.7.3. We have

$$\{\Lambda^n_j \hookrightarrow \Delta^n : 0 < j < n\} \square \{\partial\Delta^m \hookrightarrow \Delta^m\} \hookrightarrow \overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \{\partial\Delta^n \hookrightarrow \Delta^n\}} \square \{\partial\Delta^m \hookrightarrow \Delta^m\}$$

by Proposition 3.7.5. Then using Lemma 2.8.4 we get an inclusion

$$\overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \{\partial\Delta^n \hookrightarrow \Delta^n\}} \square \{\partial\Delta^m \hookrightarrow \Delta^m\} \hookrightarrow \overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \{\partial\Delta^n \hookrightarrow \Delta^n\} \square \{\partial\Delta^m \hookrightarrow \Delta^m\}}.$$

Now the set $\{\partial\Delta^n \hookrightarrow \Delta^n\} \square \{\partial\Delta^m \hookrightarrow \Delta^m\}$ consists of monomorphisms, and by Proposition 2.6.9 the monomorphisms are the saturated class generated by the boundary inclusions. This means we have an inclusion

$$\overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \{\partial\Delta^n \hookrightarrow \Delta^n\} \square \{\partial\Delta^m \hookrightarrow \Delta^m\}} \hookrightarrow \overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \overline{\{\partial\Delta^n \hookrightarrow \Delta^n\}}}.$$

Applying Lemma 2.8.4 again, and noting that the saturation of a saturated class is itself, we get

$$\overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \overline{\{\partial\Delta^n \hookrightarrow \Delta^n\}}} \hookrightarrow \overline{\{\Lambda^2_1 \hookrightarrow \Delta^2\} \square \{\partial\Delta^n \hookrightarrow \Delta^n\}},$$

which is the class of inner anodyne maps by Proposition 3.7.5. □

Applying Lemma 2.8.4 we immediately get:

Corollary 3.7.7. *If i is inner anodyne and j is a monomorphism, then $i \square j$ is inner anodyne.*

Now using Lemma 2.8.2 this gives:

Corollary 3.7.8. *(i) If j is a monomorphism and p is an inner fibration, then $p^{\square j}$ is an inner fibration.*

(ii) If j is inner anodyne and p is an inner fibration, then $p^{\square j}$ is a trivial fibration.

As a particularly important special case, we have:

Corollary 3.7.9. *Suppose $\mathcal{C}$ is a quasicategory. Then for any simplicial set K, the internal Hom $\mathcal{C}^K$ is a quasicategory.*

Definition 3.7.10. If $\mathcal{C}$ is a quasicategory and K is any simplicial set, then we'll write $\mathrm{Fun}(K, \mathcal{C})$ for the quasicategory $\mathcal{C}^K$, and we'll often refer to maps between quasicategories as *functors*. We then define $\mathrm{Map}(K, \mathcal{C})$ to be the quasigroupoid $\mathrm{Core}(\mathrm{Fun}(K, \mathcal{C}))$.

3.8 Categorical Equivalences of Quasicategories

Definition 3.8.1. A *natural transformation* between two maps $\mathcal{C} \to \mathcal{D}$ of quasicategories is a map $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ restricting appropriately. We say this is a *natural equivalence* if it is an equivalence in the quasicategory $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Remark 3.8.2. Since any functor preserves equivalences, it is immediate that if $h\colon \mathcal{C} \times \Delta^1 \to \mathcal{D}$ is a natural equivalence, then $h(c, -)\colon \Delta^1 \to \mathcal{D}$ is an equivalence for every $c \in \mathcal{C}$. We'll prove later that the converse is true: if a natural transformation is objectwise given by equivalences, then it is a natural equivalence. This is non-obvious since it requires us to pick inverses that fit together to a natural transformation.

Lemma 3.8.3. *If $h\colon \mathcal{C} \times \Delta^1 \to \mathcal{D}$ is a natural equivalence, then it induces a homotopy* $\mathrm{Core}(\mathcal{C}) \times \Delta^1 \to \mathrm{Core}(\mathcal{D})$.

Proof. Restricting h to $\mathrm{Core}(\mathcal{C})$ we get a map $\mathrm{Core}(\mathcal{C}) \times \Delta^1 \to \mathcal{D}$. But since h takes every object of c to an equivalence in $\mathcal{D}$, this restriction factors through $\mathrm{Core}(\mathcal{D})$. □

Lemma 3.8.4. *If $h\colon \mathcal{C} \times \Delta^1 \to \mathcal{D}$ is a natural equivalence, then the induced natural transformation $H\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \times \Delta^1 \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is a natural equivalence for every quasicategory $\mathcal{E}$.*

Proof. Choose an inverse $g\colon \Delta^1 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and 2-simplices exhibiting gf and fg as equivalent to degenerate edges. Then these induce induce the same data for H. □

Definition 3.8.5. A functor of quasicategories $F\colon \mathcal{C} \to \mathcal{D}$ is a *categorical equivalence* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural equivalences between GF and $\mathrm{id}_{\mathcal{C}}$, and between FG and $\mathrm{id}_{\mathcal{D}}$.

Remark 3.8.6. The functors Core, h, and π_0 all preserve products of quasicategories, and $\pi_0\mathrm{Core}(\mathcal{C}) \cong \pi_0\mathrm{Core}(h\mathcal{C})$.

Definition 3.8.7. Let $h\mathrm{QCat}$ denote the category with quasicategories as objects, and the set of morphisms from $\mathcal{C}$ to $\mathcal{D}$ given by $\pi_0\mathrm{Map}(\mathcal{C}, \mathcal{D})$. (Composition makes sense since $\pi_0\mathrm{Core}$ preserves products.)

Proposition 3.8.8. *The following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$ of quasicategories:*

(1) F is a categorical equivalence.

(2) For every quasicategory $\mathcal{E}$, the induced functor $F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is a categorical equivalence.

(3) For every quasicategory $\mathcal{E}$, the induced functor $F^*\colon h\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to h\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is an equivalence of categories.

(4) For every quasicategory $\mathcal{E}$, the induced map $F^*\colon \mathrm{Map}(\mathcal{D}, \mathcal{E}) \to \mathrm{Map}(\mathcal{C}, \mathcal{E})$ is a homotopy equivalence.

(5) For every quasicategory $\mathcal{E}$, the induced map $F^*\colon \pi_0\mathrm{Map}(\mathcal{D}, \mathcal{E}) \to \pi_0\mathrm{Map}(\mathcal{C}, \mathcal{E})$ is an isomorphism.

(6) F is an isomorphism in $h\mathrm{QCat}$.

Proof. To show that (1) implies (2), we observe that the data exhibiting F as a categorical equivalence induces the same data for F^*, as in the proof of Proposition 2.9.5, using Lemma 3.8.4. Now (2) implies (3) (since h preserves products) and (4) (using Lemma 3.8.3), and (4) implies (5) by Lemma 2.9.3. (5) is equivalent to (6) by the definition of $h\mathrm{QCat}$, and (6) implies (1) by expanding out what an inverse to F in $h\mathrm{QCat}$ is, as in the proof of Proposition 2.9.5. □

Corollary 3.8.9. *Categorical equivalences between quasicategories complexes satisfy the 2-of-3 property.*

Proof. Immediate from condition (6) above. (It is also easy to check directly from the definition.) □

Lemma 3.8.10. *Suppose $f\colon \mathcal{E} \to \mathcal{B}$ is a trivial fibration between quasicategories. Then it is a categorical equivalence.*

Proof. Since f is a trivial fibration, we can choose a section s of f by picking a lift in

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "f"] \\ \mathcal{B} \arrow[r, "\mathrm{id}_{\mathcal{B}}"'] \arrow[ur, dashed, "s"] & \mathcal{B} \end{tikzcd}$$

Then $f \circ s = \mathrm{id}_{\mathcal{B}}$. Next, we pick a lift h in the square

$$\begin{tikzcd} \mathcal{E} \amalg \mathcal{E} \arrow[r, "{(sf, \mathrm{id}_{\mathcal{E}})}"] \arrow[d] & \mathcal{E} \arrow[d, "f"] \\ \mathcal{E} \times E^1 \arrow[r, "f \circ \pi_{\mathcal{E}}"'] \arrow[ur, dashed, "h"] & \mathcal{B}. \end{tikzcd}$$

Then h is an E^1-homotopy between fs and $\mathrm{id}_{\mathcal{E}}$, which gives an equivalence in $\mathrm{Fun}(\mathcal{B}, \mathcal{B})$ as required. □

3.9 Weak Categorical Equivalences

Warning 3.9.1. We'll use (at least here) the term *weak categorical equivalence* by analogy with *weak homotopy equivalence*. However, these maps are usually just called categorical equivalences in the literature.

Definition 3.9.2. We say a map $f\colon X \to Y$ of simplicial sets is a *weak categorical equivalence* if for all quasicategories $\mathcal{C}$ the map of quasicategories $f^*\colon \mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(Y, \mathcal{C})$ is a categorical equivalence.

Lemma 3.9.3. *Any inner anodyne map is a categorical equivalence.*

Proof. If $i\colon I \to I'$ is inner anodyne, then for every quasicategory $\mathcal{C}$, the induced map $i^*\colon \mathrm{Fun}(I', \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ is a trivial fibration by Corollary 3.7.8. It is therefore a categorical equivalence by Lemma 3.8.10. □

Proposition 3.9.4. *For $f\colon X \to Y$ a map of simplicial sets, the following are equivalent:*

(1) f is a weak categorical equivalence.

(2) For all quasicategories $\mathcal{C}$, the map $f^*\colon \mathrm{hFun}(Y, \mathcal{C}) \to \mathrm{hFun}(X, \mathcal{C})$ is an equivalence of categories.

(3) For all quasicategories $\mathcal{C}$, the map $f^*\colon \mathrm{Map}(Y, \mathcal{C}) \to \mathrm{Map}(X, \mathcal{C})$ is a homotopy equivalence.

(4) For all quasicategories $\mathcal{C}$, the map $f^*\colon \pi_0\mathrm{Map}(Y, \mathcal{C}) \to \pi_0\mathrm{Map}(X, \mathcal{C})$ is an isomorphism.

Proof. By the small object argument we can construct a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\xi"] & Y \arrow[d, "\eta"] \\ \tilde{X} \arrow[r, "\tilde{f}"] & \tilde{Y} \end{tikzcd}$$

where ξ and η are inner anodyne. Thus we have for any quasicategory $\mathcal{C}$ a commutative square

$$\begin{tikzcd} \mathrm{Fun}(\tilde{X}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\tilde{Y}, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}(X, \mathcal{C}) \arrow[r] & \mathrm{Fun}(Y, \mathcal{C}) \end{tikzcd}$$

where the vertical maps are trivial fibrations. Thus the vertical maps are categorical equivalences by Lemma 3.8.10, and hence induce homotopy equivalences on cores by Lemma 3.8.3, equivalences of homotopy categories as h preserves products, and isomorphisms on π_0 of the cores by by Lemma 2.9.3. By the 2-of-3 property for the various types of equivalence, we conclude that all of (1)–(4) hold for f if and only if they hold for $\tilde{f}$. Since $\tilde{f}$ is a functor of quasicategories, (1)–(4) are equivalent for $\tilde{f}$ by Proposition 3.8.8, hence these conditions are also equivalent for f. □

We now turn to a less trivial criterion for categorical equivalences, which we'll (probably) prove later.

Definition 3.9.5. If $\mathcal{C}$ is a quasicategory and x, y are objects of $\mathcal{C}$, we define the *mapping space* $\mathrm{Map}_{\mathcal{C}}(x, y)$ by the pullback square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d] \\ \{(x, y)\} \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

We'll see later that this actually is a space, i.e. a Kan complex.

Remark 3.9.6. There's no natural way to choose a composition map $\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z)$. However, we have something slightly weaker: The inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ is inner anodyne, so the induced map $\mathrm{Fun}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \cong \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a trivial fibration. On fibres over x, y, z this together with $d_1^*\colon \mathrm{Fun}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ induces

$$\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \leftarrow \mathrm{Map}_{\mathcal{C}}(x, y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z),$$

where the first map is a trivial fibration. We can thus choose a section of it to get a composition map. This can also be upgraded to extract a *Segal category* from $\mathcal{C}$, which is another model for ∞-categories, but we won't discuss this here.

Definition 3.9.7. A map of quasicategory $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if the induced functor $hF\colon h\mathcal{C} \to h\mathcal{D}$ is essentially surjective, or equivalently if $\pi_0\mathrm{Core}(\mathcal{C}) \to \pi_0\mathrm{Core}(\mathcal{D})$ is surjective. F is *fully faithful* if for all $x, y \in \mathcal{C}$ the induced map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(Fx, Fy)$ is a homotopy equivalence.

Theorem 3.9.8. *A functor of quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

Rezk calls this the "fundamental theorem of quasicategories".

Chapter 4

Joins, Slices, and Colimits

4.1 Joins

For categories $\mathbf{C}$ and $\mathbf{D}$, their join $\mathbf{C} \star \mathbf{D}$ is the category with objects $\operatorname{ob}\mathbf{C} \amalg \operatorname{ob}\mathbf{D}$, and morphisms

$$\operatorname{Hom}(x, y) = \begin{cases} \operatorname{Hom}_{\mathbf{C}}(x, y), & x, y \in \mathbf{C}, \\ \operatorname{Hom}_{\mathbf{D}}(x, y), & x, y \in \mathbf{D}, \\ *, & x \in \mathbf{C}, y \in \mathbf{D}, \\ \emptyset, & x \in \mathbf{D}, y \in \mathbf{C}. \end{cases}$$

For example $\mathbf{C} \star [0]$ is $\mathbf{C}$ with a freely added terminal object, while $[0] \star \mathbf{C}$ is $\mathbf{C}$ with a freely added initial object. (In particular, $\star$ is definitely not symmetric!)

If S and T are ordered sets, viewed as categories, then $S \star T$ is the ordered set with underlying set $S \amalg T$ with ordering extending the given ones on S and T by requiring $s \leq t$ for every $s \in S$ and $t \in T$. For example, $[n] \star [m] = [n + m + 1]$. Here we can also allow n or m to be -1, setting $[-1] = \emptyset$.

Let $\mathbb{\Delta}_+$ denote $\mathbb{\Delta}$ with this added initial object $[-1]$ – this is a skeleton of the category of all ordered finite sets. Then $\star$ defines a monoidal structure on $\mathbb{\Delta}_+$.

We can use this to get an induced monoidal structure on $\operatorname{Fun}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathrm{Set})$, which we also denote $\star$. This is an example of a *Day convolution*; it is also characterized as the unique monoidal structure on $\operatorname{Fun}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathrm{Set})$ that preserves colimits in each variable and extends $\star$ on $\mathbb{\Delta}_+$ via the Yoneda embedding. Taking our fancy pants off, this amounts to the following explicit formula for $F \star G$:

$$(F \star G)([n]) = \coprod_{[a] \star [b] = [n]} F([a]) \times G([b]) = \coprod_{a + b = n - 1} F([a]) \times G([b]).$$

(Here a and b could be -1, of course.)

We can identify Set_Δ with the full subcategory of $\operatorname{Fun}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathrm{Set})$ spanned by the presheaves F such that $F([-1]) = *$. This subcategory is closed under $\star$, so $\star$ restricts to a monoidal structure on Set_Δ. Note that the unit in Set_Δ is $\emptyset$, since this corresponds to $[-1]$ under this identification.

Definition 4.1.1. We write $X^{\triangleleft}$ for $\Delta^0 \star X$ and $X^{\triangleright}$ for $X \star \Delta^0$.

Exercise 4.1.2. We have the following identifications:

- for categories $\mathbf{C}$ and $\mathbf{D}$, $\mathrm{N}\mathbf{C} \star \mathrm{N}\mathbf{D} \cong \mathrm{N}(\mathbf{C} \star \mathbf{D})$ (in particular $\Delta^n \star \Delta^m \cong \Delta^{n+m+1}$, but we already knew that)
- $(\partial\Delta^{n-1})^{\triangleleft} \cong \Lambda^n_0$,

- $(\partial\Delta^{n-1})^{\triangleright} \cong \Lambda^n_n$.

Lemma 4.1.3. *If $\mathcal{C}$ and $\mathcal{D}$ are quasicategories, then $\mathcal{C} \star \mathcal{D}$ is a quasicategory.*

Proof. Consider an inner horn $f \colon \Lambda^n_i \to \mathcal{C} \star \mathcal{D}$. If the image of f is contained in $\mathcal{C}$ or $\mathcal{D}$ then we can extend to a simplex since these are quasicategories. Otherwise, let i be the index such that the vertices $0, \dots, i$ land in $\mathcal{C}$ and $i+1, \dots, n$ land in $\mathcal{D}$. Then f restricts to maps $\Delta^i \to \mathcal{C}$ and $\Delta^{n-i-1} \to \mathcal{D}$ taking the join of these we get a map $\Delta^n \cong \Delta^i \star \Delta^{n-i-1} \to \mathcal{C} \star \mathcal{D}$ that restricts to f. $\square$

Remark 4.1.4. This proof is a bit unsatisfying, as it is not clear why this should work. The general reason is the there is an adjunction

$$F : \mathrm{Set}_{\Delta/\Delta^1} \rightleftarrows \mathrm{Set}_\Delta \times \mathrm{Set}_\Delta : G$$

where $F(X \to \Delta^1)$ extracts the fibres (X_0, X_1), while $G(X, Y) = X \star Y \to \Delta^0 \star \Delta^0 \cong \Delta^1$. For an inner horn Λ^n_i, the fibres of a map to Δ^1 are either the horn itselfand the empty set, if the map is constant, or simplices, if it is not. See [Joy08, Proposition 3.5] for a proof of this.

4.2 Slices

Lemma 4.2.1. *For $X \in \mathrm{Set}_\Delta$, the functors $X \star -$ and $- \star X$ from Set_Δ to $\mathrm{Set}_{\Delta,X/}$ preserve colimits.*

Remark 4.2.2. Viewed as functors from Set_Δ to Set_Δ these functors can't preserve colimits, since they do not preserve the initial object — $\emptyset$ is the unit for $\star$, so $X \star \emptyset \cong \emptyset \star X \cong X$.

Proof. It suffices to show that for every n the functor $(X \star -)_n$ from Set_Δ to $\mathrm{Set}_{X_n/}$ preserves colimits. Using the explicit formula for the join, this functor is

$$X_n \amalg X_{n-1} \times (-)_0 \amalg \cdots \amalg (-)_n$$

The functor $X_{n-1} \times (-)_0 \amalg \cdots \amalg (-)_n$ from Set_Δ preserves colimits (since colimits commute and the product of sets preserves colimits in each variable), and $X_n \amalg - \colon \mathrm{Set}_\Delta \to \mathrm{Set}_{X_n/}$ preserves colimits (since it is the left adjoint of the forgetful functor). $\square$

As a consequence, the functors $X \star -$ and $- \star X$ have right adjoints, both of which are functors $\mathrm{Set}_{\Delta,X/} \to \mathrm{Set}_\Delta$. We denote their images at $p \colon X \to Y$ by $Y_{p/}$ and $Y_{/p}$, respectively. More explicitly, if K is a simplicial set, then

$$\mathrm{Hom}(K, Y_{p/}) \cong \mathrm{Hom}_{X/}(X \star K, Y) \cong \left\{ \begin{tikzcd} X \arrow[d, hook] \arrow[dr, "p"] & \\ X \star K \arrow[r] & Y \end{tikzcd} \right\},$$

$$\mathrm{Hom}(K, Y_{/p}) \cong \mathrm{Hom}_{X/}(K \star X, Y) \cong \left\{ \begin{tikzcd} X \arrow[d, hook] \arrow[dr, "p"] & \\ K \star X \arrow[r] & Y \end{tikzcd} \right\},$$

As key special cases, for $y\colon \Delta^0 \to Y$ we have $Y_{y/}$ and $Y_{/y}$ described by

$$(Y_{y/})_n = \{\sigma\colon \Delta^{n+1} \to Y : \sigma(0) = y\},$$

$$(Y_{/y})_n = \{\sigma\colon \Delta^{n+1} \to Y : \sigma(n+1) = y\}.$$

Exercise 4.2.3. For $\mathbf{C}$ a category and $c \in \mathbf{C}$, we have $(\mathrm{N}\mathbf{C})_{/x} \cong \mathrm{N}(\mathbf{C}_{/x})$, $(\mathrm{N}\mathbf{C})_{x/} \cong \mathrm{N}(\mathbf{C}_{x/})$.

Slices have two kinds of functoriality: for $T \xrightarrow{j} \xrightarrow{p} X$, we have a map $X_{/p} \to X_{/pj}$ induced by $\mathrm{Hom}_{S/}(K \star S, X) \to \mathrm{Hom}_{T/}(K \star T, X)$, and for $S \xrightarrow{p} X \xrightarrow{f} Y$ we ahve $X_{/p} \to Y_{/fp}$ induced by $\mathrm{Hom}_{S/}(K \star S, X) \to \mathrm{Hom}_{S/}(K \star S, Y)$. These are compatible, in the sense that for $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$ we have a commutative square

$$\begin{tikzcd}
X_{/p} \arrow[r] \arrow[d] & X_{/pj} \arrow[d] \\
Y_{/fp} \arrow[r] & Y_{/fpj}.
\end{tikzcd}$$

(Of course, the same thing holds for slices under p.)

4.3 Pushout-Joins and Pullback-Slices

We now consider an analogue of the pushout-product using the join:

Definition 4.3.1. For $i\colon A \to B$ and $j\colon K \to L$, the *pushout-join* $i \boxtimes j$ is the natural map

$$A \star L \amalg_{A \star K} B \star K \to B \star L.$$

Note that as $\star$ is not symmetric, neither is $\boxtimes$.

Exercise 4.3.2. We have the following isomorphisms:

- $(\Lambda^n_j \hookrightarrow \Delta^n) \boxtimes (\partial\Delta^m \hookrightarrow \Delta^m) \cong (\Lambda^{n+m+1}_j \hookrightarrow \Delta^{n+m+1})$,
- $(\partial\Delta^m \hookrightarrow \Delta^m) \boxtimes (\Lambda^n_j \hookrightarrow \Delta^n) \cong (\Lambda^{m+n+1}_{m+j+1} \hookrightarrow \Delta^{n+m+1})$,
- $(\partial\Delta^m \hookrightarrow \Delta^m) \boxtimes (\partial\Delta^n \hookrightarrow \Delta^n) \cong (\partial\Delta^{m+n+1} \hookrightarrow \Delta^{m+n+1})$.

Next we want to introduce the dual construction, analogous to the pullback-Hom. Since $\star$ is non-symmetric and doesn't preserve $\emptyset$ this is a bit more complicated to describe: For $i\colon A \to B$, $i \boxtimes -$ and $- \boxtimes i$ are functors $\mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Fun}([1], \mathrm{Set}_{\Delta, B/})$ (where to identify the target we use that $i \boxtimes (\emptyset \to \emptyset) \cong B \xrightarrow{\mathrm{id}} B$), and they preserve colimits (since these are computed objectwise in $[1]$). Both therefore have right adjoints, these being functors $\mathrm{Fun}([1], \mathrm{Set}_{\Delta, B/}) \to \mathrm{Fun}([1], \mathrm{Set}_\Delta)$. We denote their value at $B \xrightarrow{p} X \xrightarrow{h} Y$ by $h^{i \boxtimes_p}$ and $h^{\boxtimes_p i}$, respectively. [This is Rezk's notation. Maybe use $h^{i\boxtimes}_{/p}$ and $h^{\boxtimes i}_{p/}$ instead?]

Explicitly, $h^{\boxtimes_p i}$ is the natural map

$$X_{/p} \to X_{/pi} \times_{Y_{/hpi}} Y_{/hp},$$

and $h^{i \boxast_p}$ is the natural map

$$X_{p/} \to X_{pi/} \times_{Y_{hpi/}} Y_{hp/}.$$

Proposition 4.3.3. *Given $i \colon A \to B$, $j \colon K \to L$, $h \colon X \to Y$, the following are equivalent:*

(i) $i \boxast j$ has the left lifting property for h.

(ii) i has the left lifting property for $h^{\boxast_q j}$ for all $q \colon L \to X$,

(iii) j has the left lifting property for $h^{i \boxast_p}$ for all $p \colon B \to X$.

Proof. Expand out the pushouts and pullbacks, as in the proof of Lemma 2.8.2. For instance, a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_{/q} \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & X_{/qj} \times_{Y_{/hqj}} Y_{/hq}
\end{tikzcd}$$

corresponds to

$$\begin{tikzcd}
\emptyset \star L \arrow[r] \arrow[rr, bend left, "q"] & A \star L \amalg_{A \star K} B \star K \arrow[r] \arrow[d, "i \boxast j"] & X \arrow[d, "h"] \\
& B \star L \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

$\square$

Exercise 4.3.4. Prove this using the adjunctions for $\boxast$ and the behaviour of $\boxast$ on identity morphisms, by reformulating the lifting problem as the existence of a factorization.

Lemma 4.3.5. *For S and T classes of maps in* Set_Δ*, we have $\bar{S} \boxast \bar{T} \subseteq \overline{S \boxast T}$.*

Proof. As the proof of Lemma 2.8.4 for $\square$. $\square$

4.4 Right and Left Fibrations

Definition 4.4.1. A map in Set_Δ is *left anodyne* if it is in the saturation of $\{\Lambda^n_i : 0 \leq i < n\}$ (the *left horns*) and *right anodyne* if it is in the saturation of $\{\Lambda^n_i : 0 < i \leq n\}$ (the *right horns*. A map that has the right lifting property for the left horns (or equivalently the left anodyne maps) is a *left fibration*, and we similarly define a *right fibration* to be a map that has the right lifting property for the right horns.

Combining Lemma 4.3.5 with Exercise 4.3.2, we get:

Proposition 4.4.2.

(i) *If i is right anodyne and j is a monomorphism, then $i \boxtimes j$ is inner anodyne.*

(ii) *If i is a monomorphism and j is left anodyne, then $i \boxtimes j$ is inner anodyne.*

(iii) *If i and j are monomorphisms then $i \boxtimes j$ is a monomorphism.*

Using Proposition 4.3.3 this immediately implies:

Corollary 4.4.3. *Suppose given an inner fibration $f\colon X \to Y$, and maps $p\colon S \to X$ and $j\colon T \to S$.*

(i) *If j is a monomorphism, then $f^{j \boxtimes p}\colon X_{p/} \to X_{pj/} \times_{Y_{fpj/}} Y_{fp/}$ is a left fibration. If moreover f is a trivial fibration, so is $f^{j \boxtimes p}$.*

(ii) *If j is a monomorphism, then $f^{\boxtimes_p j}\colon X_{/p} \to X_{/pj} \times_{Y_{/fpj}} Y_{/fp}$ is a right fibration. If moreover f is a trivial fibration, so is $f^{\boxtimes_p j}$.*

(iii) *If j is right anodyne, then $f^{j \boxtimes p}$ is a trivial fibration.*

(iv) *If j is left anodyne, then $f^{\boxtimes_p j}$ is a trivial fibration.*

As a key special case, we have:

Corollary 4.4.4. *Let $\mathcal{C}$ be a quasicategory. Given maps $p\colon S \to \mathcal{C}$ and $j\colon T \to S$, we have:*

(i) *If j is a monomorphism, then $\mathcal{C}_{p/} \to \mathcal{C}_{pj/}$ is a left fibration.*

(ii) *If j is a monomorphism, then $\mathcal{C}_{/p} \to \mathcal{C}_{/pj}$ is a right fibration.*

(iii) *If j is right anodyne, then $\mathcal{C}_{p/} \to \mathcal{C}_{pj/}$ is a trivial fibration.*

(iv) *If j is left anodyne, then $\mathcal{C}_{/p} \to \mathcal{C}_{/pj}$ is a trivial fibration.*

Corollary 4.4.5. *If $\mathcal{C}$ is a quasicategory, then for any map $p\colon S \to \mathcal{C}$ the map $\mathcal{C}_{p/} \to \mathcal{C}$ is a left fibration, and $\mathcal{C}_{/p} \to \mathcal{C}$ is a right fibration. In particular, both $\mathcal{C}_{p/}$ and $\mathcal{C}_{/p}$ are quasicategories.*

Remark 4.4.6. Consider a quasicategory $\mathcal{C}$ and a morphism $f\colon x \to y$ in $\mathcal{C}$. We have maps

$$\mathcal{C}_{/x} \xleftarrow{p} \mathcal{C}_{/f} \xrightarrow{q} \mathcal{C}_{/y}$$

induced by the inclusions $\{0\}, \{1\} \hookrightarrow \Delta^1$. Since $\{0\} \hookrightarrow \Delta^1$ is a left horn, the map p is a trivial fibration, so it admits a section $s\colon \mathcal{C}_{/x} \to \mathcal{C}_{/f}$. Composing with this gives a map $\mathcal{C}_{/x} \to \mathcal{C}_{/y}$ corresponding to composing $c \xrightarrow{g} x$ with f to get $c \xrightarrow{fg} y$.

Remark 4.4.7. If we are to think of $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$ as quasicategorical analogues of slice categories, then the fibres of $\mathcal{C}_{/x} \to \mathcal{C}$ and $\mathcal{C}_{x/} \to \mathcal{C}$ at y ought to be the mapping spaces $\mathrm{Map}_{\mathcal{C}}(y, x)$ and $\mathrm{Map}_{\mathcal{C}}(x, y)$. In fact, we ought to have, for instance, a pullback square

$$\begin{tikzcd}
\mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "\mathrm{ev}_1"] \\
\{x\} \arrow[r] & \mathcal{C}.
\end{tikzcd}$$

This is not literally true, but we will [?] see later that this is indeed the case up to categorical equivalence — so we do have a *homotopy* pullback square of this form.

4.5 Initial and Terminal Objects

Definition 4.5.1. Let $\mathcal{C}$ be a quasicategory. An object $x \in \mathcal{C}$ is *initial* if $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial fibration, and *terminal* if $\mathcal{C}_{/x} \to \mathcal{C}$ is a trivial fibration.

Lemma 4.5.2. *x is initial if and only if every map $\partial\Delta^n \to \mathcal{C}$ taking 0 to x extends to a map from Δ^n.*

Proof. Using the adjunction between joins and slices, we see that the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^n \arrow[r] & \mathcal{C} \end{tikzcd}$$

corresponds to

$$\begin{tikzcd} \Delta^0 \star \partial\Delta^n \amalg_{\partial\Delta^n} \Delta^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^{n+1} \arrow[r] & * \end{tikzcd}$$

where $\Delta^0 \star \partial\Delta^n \amalg_{\partial\Delta^n} \Delta^n \cong \partial\Delta^{n+1}$. □

Remark 4.5.3. To justify this definition of initial objects, suppose we know that the fibre of $\mathcal{C}_{x/} \to \mathcal{C}$ at y is $\mathrm{Map}_{\mathcal{C}}(x, y)$, and that a left fibration is trivial if and only if its fibres are contractible. Then x is initial if and only if $\mathrm{Map}_{\mathcal{C}}(x, y)$ is contractible for all $y \in \mathcal{C}$, just as you would expect.

Lemma 4.5.4. *Let $\mathcal{C}$ be a quasicategory, and let $\mathcal{C}_{\text{init}}$ denote the full subcategory of $\mathcal{C}$ spanned by the the initial objects. Then $\mathcal{C}_{\text{init}}$ is either empty or a contractible Kan complex.*

Proof. If $\mathcal{C}_{\text{init}}$ is not empty, then any map $f\colon \partial\Delta^n \to \mathcal{C}_{\text{init}}$ extends to a simplex $\Delta^n \to \mathcal{C}$ since $f(0)$ is an initial object. But then σ lands in $\mathcal{C}_{\text{init}}$ since all the vertices of σ lie in this full subcategory. Thus $\mathcal{C}_{\text{init}} \to \Delta^0$ is a trivial fibration. □

4.6 Limits and Colimits

Definition 4.6.1. Given a map $p\colon K \to \mathcal{C}$, where $\mathcal{C}$ is a quasicategory, a *colimit* of p is an initial object of $\mathcal{C}_{p/}$, and a *limit* of p is a terminal object of $\mathcal{C}_{/p}$.

Remark 4.6.2. Using Lemma 4.5.2, a colimit of $p\colon K \to \mathcal{C}$ is a map $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ extending p such that for every diagram

$$\begin{tikzcd} K \star \Delta^0 \arrow[r, hook] \arrow[rr, bend left, "\bar{p}"] & K \star \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & K \star \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

the dotted lift exists.

Remark 4.6.3. By Lemma 4.5.4, the subcategory of $\mathcal{C}_{p/}$ spanned by the colimits of p is either empty or contractible.

Proposition 4.6.4. *Given maps of simplicial sets $p\colon S \to X$ and $q\colon T \to X_{p/}$, corresponding to $\tilde{q}\colon S \star T \to X$, there is a natural isomorphism $X_{\tilde{q}/} \cong (X_{p/})_{q/}$.*

Proof. By definition of $X_{\tilde{q}/}$ we have a pullback square

$$\begin{tikzcd} \mathrm{Hom}(K, X_{\tilde{q}/}) \arrow[r] \arrow[d] & \mathrm{Hom}(S \star T \star K, X) \arrow[d] \\ \{q\} \arrow[r] & \mathrm{Hom}(S \star T, X). \end{tikzcd}$$

Similarly, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(K, (X_{p/})_{q/}) \arrow[r] \arrow[d] & \mathrm{Hom}(T \star K, X_{p/}) \arrow[r] \arrow[d] & \mathrm{Hom}(S \star T \star K, X) \arrow[d] \\ \{q\} \arrow[r] & \mathrm{Hom}(T, X_{p/}) \arrow[r] \arrow[d] & \mathrm{Hom}(S \star T, X) \arrow[d] \\ & \{p\} \arrow[r] & \mathrm{Hom}(S, X) \end{tikzcd}$$

where all the squares are Cartesian. The composite square in the top row is naturally identified with our first square, giving a natural isomorphism $\mathrm{Hom}(K, X_{\tilde{q}/}) \cong \mathrm{Hom}(K, (X_{p/})_{q/})$. $\square$

Corollary 4.6.5. *A cone $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$ is a colimit of $p := \bar{p}|_K$ if and only if $\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/}$ is a trivial fibration.*

Proof. By definition $\bar{p}$ is a colimit if and only if $(\mathcal{C}_{p/})_{\bar{p}/} \to \mathcal{C}_{p/}$ is a trivial fibration, and we have an isomorphism $(\mathcal{C}_{p/})_{\bar{p}/} \cong \mathcal{C}_{\bar{p}/}$. $\square$

Chapter 5

Joyal's Lifting Theorem and Applications

Our goal in this chapter is to prove Joyal's lifting theorem. This says that if $\mathcal{C}$ is a quasicategory and $\phi\colon \Lambda^n_0 \to \mathcal{C}$ is a left horn such that the image of the edge $0 \to 1$ is an equivalence in $\mathcal{C}$, then ϕ can be extended to a simplex. (Of course, the analogous statement for right horns Λ^n_n is also true, and we will actually prove a relative version for inner fibrations.) This turns out to imply many of the statements we claimed above, such as that quasigroupoids are Kan complexes.

5.1 Conservative Functors

Definition 5.1.1. A functor $F\colon \mathbf{C} \to \mathbf{D}$ of categories is *conservative* if F detects isomorphisms, i.e. if a morphism $f \in \mathbf{C}$ is an isomorphism if and only if $F(f)$ is an isomorphism.

Definition 5.1.2. We say a functor of quasicategories $p\colon \mathcal{E} \to \mathcal{B}$ is *conservative* if the induced functor $h\mathcal{E} \to h\mathcal{B}$ is conservative. Equivalently, p is conservative if a morphism in $\mathcal{E}$ is an equivalence if and only if its image in $\mathcal{B}$ is an equivalence.

Lemma 5.1.3. *Every right or left fibration of quasicategories is conservative.*

Proof. We consider the case of a left fibration $p\colon \mathcal{E} \to \mathcal{B}$; the proof for right fibrations is similar. Let $f\colon e \to e'$ be a morphism in $\mathcal{E}$ such that $p(f)$ is an equivalence. Let $\sigma\colon \Delta^2 \to \mathcal{B}$ be a 2-simplex that exhibits $\mathrm{id}_{p(e)}$ as the composite of $p(f)$ and its inverse. Then we have a diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \Delta^2 \arrow[r, "\sigma"'] \arrow[ur, dashed] & \mathcal{B}, \end{tikzcd}$$

where the map $\Lambda^2_0 \to \mathcal{E}$ is determined by f and id_e. We can choose a lift in this square since p is a left fibration, which implies that f has a post-inverse g. Now applying the same argument to g, which also maps to an equivalence in $\mathcal{B}$, we see that g also has a post-inverse. Thus g has both a pre- and a post-inverse, hence it is an equivalence, as is f. $\square$

Lemma 5.1.4. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a conservative functor of quasicategories. Then the commutative square*

$$\begin{tikzcd} \mathrm{Core}(\mathcal{E}) \arrow[r] \arrow[d, "\mathrm{Core}(p)"'] & \mathcal{E} \arrow[d, "p"] \\ \mathrm{Core}(\mathcal{B}) \arrow[r] & \mathcal{B} \end{tikzcd}$$

is Cartesian.

Proof. Since a morphism in $h\mathcal{E}$ is an isomorphism if and only if its image in $h\mathcal{B}$ is one, in the commutative diagram

$$\begin{tikzcd} \mathrm{Core}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Core}(h\mathcal{E}) \arrow[r] \arrow[d] & h\mathcal{E} \arrow[d] \\ \mathrm{Core}(h\mathcal{B}) \arrow[r] & h\mathcal{B}, \end{tikzcd}$$

the bottom square is a pullback. Since the top square is a pullback by definition of $\mathrm{Core}(\mathcal{E})$, this implies the outer square is a pullback. Now consider the commutative diagram

$$\begin{tikzcd} \mathrm{Core}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Core}(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathrm{Core}(h\mathcal{B}) \arrow[r] & h\mathcal{B}. \end{tikzcd}$$

Here the bottom square is a pullback by definition, and we just saw the outer square is a pullback. But then the top square is also a pullback. ☐

5.2 Isofibrations

Definition 5.2.1. A functor of categories $F\colon \mathbf{E} \to \mathbf{B}$ is an *isofibration* if for every $e \in \mathcal{E}$ and every isomorphism $f\colon F(e) \to b$ there exists an isomorphism $\bar{f}\colon e \to e'$ such that $F(\bar{f}) = f$.

Remark 5.2.2. F is an isofibration if and only if for every $e \in \mathcal{E}$ and every isomorphism $f\colon b \to F(e)$ there exists an isomorphism $\bar{f}\colon e' \to e$ such that $F(\bar{f}) = f$. To see this we just apply the dual lifting criterion to the *inverse* of f.

Remark 5.2.3. Isofibrations should be regarded as the appropriate notion of *fibrations* between categories. Indeed, they are the fibrations in the "canonical" model structure on categories, which is the unique model structure where the equivalences are the weak equivalences. This means that being an isofibration is not invariant under equivalence of categories — it is a notion that depends on the usual definition of the category of categories rather than the (more canonical) (2,1)-category of categories.

Definition 5.2.4. A functor of quasicategories $p\colon \mathcal{E} \to \mathcal{B}$ is an *isofibration* if p is an inner fibration and $hp\colon h\mathcal{E} \to h\mathcal{B}$ is an isofibration of ordinary categories.

Lemma 5.2.5. *The following are equivalent for an inner fibration $p\colon \mathcal{E} \to \mathcal{B}$ between quasicategories:*

(1) p is an isofibration.

(2) For every $e \in \mathcal{E}$ and every equivalence $f\colon p(e) \to b$ in $\mathcal{B}$ there exists an equivalence $\bar{f}\colon e \to e'$ such that $p(\bar{f}) = f$.

(3) For every $e \in \mathcal{E}$ and every equivalence $f\colon b \to p(e)$ in $\mathcal{B}$ there exists an equivalence $\bar{f}\colon e' \to e$ such that $p(\bar{f}) = f$.

Proof. It is clear that (2) and (3) imply (1). We will show that (1) implies (2); the proof that (1) implies (3) is the same using the alternative definition of isofibrations in Remark 5.2.2. Suppose $f\colon p(e) \to b$ is an equivalence in $\mathcal{B}$. Since $h(p)$ is an isofibration, there exists an equivalence $g\colon e \to e'$ in $\mathcal{E}$ such that $p(g)$ and f are the same morphism in $h\mathcal{B}$, i.e. $p(g) \sim_r f$. Let $\sigma\colon \Delta^2 \to \mathcal{B}$ be a 2-simplex that exhibits $p(g) \sim_r f$, then we have a commutative square

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "\tau"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \Delta^2 \arrow[r, "\sigma"'] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

where τ is given by $\tau|_{\Delta^{\{0,1\}}} = g$, $\tau|_{\Delta^{\{1,2\}}} = \mathrm{id}_{e'}$. As p is an inner fibration, we can choose a lift $\bar{\sigma}\colon \Delta^2 \to \mathcal{E}$ in this square; then $\bar{f} := \sigma|_{\Delta^{\{0,2\}}}$ lies over f, and is an equivalence since $\bar{f} \sim_r g$. □

Lemma 5.2.6. *All left and right fibrations between quasicategories are isofibrations.*

Proof. Consider a right fibration $p\colon \mathcal{E} \to \mathcal{B}$; the proof for left fibrations is essentially the same. Suppose $f\colon b \to p(e)$ is an equivalence in $\mathcal{B}$; then we have a commutative square

$$\begin{tikzcd} \Lambda^1_1 \arrow[r, "\{e\}"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

where we can choose a lift $\bar{f}\colon \Delta^1 \to \mathcal{E}$ since p is a right fibration. Then $\bar{f}$ is an equivalence since p is conservative by Lemma 5.1.3, and thus p is an isofibration by Lemma 5.2.5. □

Remark 5.2.7. As a consequence, we deduce that every right fibration between quasicategories satisfies condition (2) in Lemma 5.2.5, which is not an obvious consequence of the definition of right fibrations.

5.3 Joyal's Lifting Theorem

We are now ready to prove the first version of Joyal's lifting theorem for outer horns:

Theorem 5.3.1. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be an inner fibration between quasicategories, and let $f\colon x \to y$ be a morphism in $\mathcal{E}$ such that $p(f)$ is an equivalence in $\mathcal{B}$. Then the following are equivalent:*

(1) f is an equivalence.

(2) For $n \geq 2$, every diagram

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

admits a lift $\Delta^n \to \mathcal{E}$.

(3) For $n \geq 2$, every diagram

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

admits a lift $\Delta^n \to \mathcal{E}$.

Proof. We'll prove that (1) is equivalent to (2); the proof that (1) is equivalent to (3) is similar. Let's first prove that (2) holds if f is an equivalence: Suppose that we have a diagram

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "\alpha"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ & \Delta^n \arrow[r, "\beta"] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

By Exercise 4.3.2 we have that $\Lambda^n_0 \hookrightarrow \Delta^n$ is $(\{0\} \hookrightarrow \Delta^1) \boxtimes (\partial\Delta^{n-2} \hookrightarrow \Delta^{n-2})$ (where $n \geq 2$). Thus by Proposition 4.3.3 our lifting problem is equivalent to

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[rr] \arrow[d, hook] & & \mathcal{E}_{/\alpha|_{\Delta^{n-2}}} \arrow[d, "\phi"] \\ \Delta^1 \arrow[urr, dashed] \arrow[rr, "g"'] \arrow[ddrr, "f"] & & \mathcal{E}_{/\alpha|_{\partial\Delta^{n-2}}} \times_{\mathcal{B}_{/\beta|_{\partial\Delta^{n-2}}}} \mathcal{B}_{/\beta|_{\Delta^{n-2}}} \arrow[d, "\psi"] \\ & & \mathcal{E}_{/\alpha|_{\partial\Delta^{n-2}}} \arrow[d, "\eta"] \\ & & \mathcal{E}. \end{tikzcd}$$

Here ϕ and η are right fibrations by Corollary 4.4.3, and ψ is a right fibration since it is the pullback of the right fibration $\mathcal{B}_{/\beta|_{\Delta^{n-2}}} \to \mathcal{B}_{/\beta|_{\partial\Delta^{n-2}}}$. Thus $\eta\psi$ is conservative by Lemma 5.1.3, hence as f is an equivalence the edge g is also an equivalence. Then we can lift g to (an equivalence in) $\mathcal{E}_{/\alpha|_{\Delta^{n-2}}}$ since the right fibration ϕ is an isofibration by Lemma 5.2.6.

Now we assume that if (2) holds for f, then f is an equivalence. Choose a 2-simplex $\sigma\colon \Delta^2 \to \mathcal{B}$ with $\sigma|_{\Delta^{\{0,1\}}} = p(f)$, $\sigma|_{\Delta^{\{0,2\}}} = \mathrm{id}_x$, and $\sigma|_{\Delta^{1,2}}$ an inverse of $p(f)$. Then we have a commutative diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "\sigma'"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
\Delta^2 \arrow[r, "\sigma"'] & \mathcal{B},
\end{tikzcd}$$

where $\sigma'|_{\Delta^{\{0,1\}}} = f$ and $\sigma'|_{\Delta^{\{0,2\}}} = \mathrm{id}_x$. By (2) we can choose a 2-simplex $\bar{\sigma}\colon \Delta^2 \to \mathcal{E}$ extending σ' and lying over σ. Thus f has a post-inverse $g := \bar{\sigma}|_{\Delta^{\{1,2\}}}$. Next, we have map $\tau_0\colon \Lambda^3_0 \to \mathcal{B}$ of the form

$$\begin{tikzcd}
& py \arrow[rr, equal] \arrow[dr, "g"] & & py \\
px \arrow[ur, "pf"] \arrow[urrr, "pf"] \arrow[rr, equal] & & py, \arrow[ur, "f"'] &
\end{tikzcd}$$

so that $\tau_0|_{\Delta^{\{0,1,2}} = \sigma$, and $\tau_0|_{\Delta^{\{0,2,3\}}}$ and $\tau|_{\Delta^{\{0,1,3\}}}$ are degenerate. Since we already saw that (1) implies (2), using this for the inner fibration $\mathcal{B} \to \Delta^0$ we see that, as $p(f)$ is an equivalence, we can extend τ_0 to a 3-simplex $\tau\colon \Delta^3 \to \mathcal{B}$. There is a lift of $\tau|_{\Lambda^3_0}$ to $\bar{\tau}_0\colon \Lambda^3_0 \to \mathcal{E}$ with $\bar{\tau}_0|_{\Delta^{\{0,1,2}} = \bar{\sigma}$, and $\bar{\tau}_0|_{\Delta^{\{0,2,3\}}}$ and $\bar{\tau}_0|_{\Delta^{\{0,1,3\}}}$ degenerate. Using (2) we can now lift $\bar{\tau}_0$ to a 3-simplex $\bar{\tau}\colon \Delta^3 \to \mathcal{E}$, and the face $\bar{\tau}|_{\Delta^{\{1,2,3\}}}$ shows that g is also a pre-inverse of f, hence f is an equivalence. □

Corollary 5.3.2. *Suppose $\mathcal{C}$ is a quasicategory. Then the following are equivalent for a morphism f in $\mathcal{C}$:*

(1) f is an equivalence.

(2) Every map $\phi\colon \Lambda^n_0 \to \mathcal{C}$, $n \geq 2$, such that $\phi|_{\Delta^{\{0,1\}}} = f$, admits an extension to Δ^n.

(3) Every map $\phi\colon \Lambda^n_n \to \mathcal{C}$, $n \geq 2$, such that $\phi|_{\Delta^{\{n-1,n\}}} = f$, admits an extension to Δ^n.

5.4 Some Immediate Consequences

Applying Corollary 5.3.2 to a quasigroupoid, we get:

Corollary 5.4.1. *A quasigroupoid is a Kan complex.*

Proof. Since every morphism in a quasigroupoid is an equivalence, Corollary 5.3.2 implies that we can fill every outer horn. □

Another immediate corollary is:

Corollary 5.4.2. *If $p\colon X \to Y$ is a right or left fibration and Y is a Kan complex, then p is a Kan fibration.*

In particular:

Corollary 5.4.3. *If a simplicial set has the right lifting property for either the left or the right horns, then it is a Kan complex.*

Applying Proposition 4.3.3 on the other side from what we did in the proof of Theorem 5.3.1, we get the following reformulation:

Corollary 5.4.4. *The following are equivalent for a morphism $f\colon x \to y$ in a quasicategory $\mathcal{C}$:*

(1) f is an equivalence.

(2) $\mathcal{C}_{f/} \to \mathcal{C}_{x/}$ is a trivial fibration.

(3) $\mathcal{C}_{/f} \to \mathcal{C}_{/y}$ is a trivial fibration.

Corollary 5.4.5. *If $f\colon x \to y$ is an equivalence, then $\mathcal{C}_{x/}$ and $\mathcal{C}_{y/}$ are categorically equivalent, as are $\mathcal{C}_{/x}$ and $\mathcal{C}_{/y}$.*

Proof. We have maps $\mathcal{C}_{x/} \xleftarrow{p} \mathcal{C}_{f/} \xrightarrow{q} \mathcal{C}_{y/}$ induced by the inclusions $\{0\}, \{1\} \hookrightarrow \Delta^1$. Here q is a trivial fibration since $\{1\} \hookrightarrow \Delta^1$ is right anodyne, and p is a trivial fibration since f is an equivalence. Choosing a section of one of these we get the required categorical equivalence. □

Lemma 5.4.6. *If x is an object of a quasicategory $\mathcal{C}$, then id_x is an initial object of $\mathcal{C}_{x/}$.*

Proof. The lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[rr, bend left, "\mathrm{id}_x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{x/} \\
& \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

corresponds under the join-slice adjunction to

$$\begin{tikzcd}
\Delta^0 \star \{0\} \arrow[r] \arrow[rrr, bend left, "\mathrm{id}_x"] & \Delta^0 \star \partial\Delta^n \arrow[r, "\cong"] \arrow[d] & \Lambda^{n+1}_0 \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^0 \star \Delta^n \arrow[r, "\cong"] & \Delta^{n+1}, \arrow[ur, dashed] &
\end{tikzcd}$$

where a lift exists by Corollary 5.3.2 since id_x is an equivalence. □

5.5 Equivalences are Coherent Equivalences

Recall that E^1 denotes the category with two objects $0, 1$, and a unique morphism between any pair of objects — this is the "walking isomorphism". The n-simplices of $\mathrm{N}E^1$ can then be denoted by lists $(i_0, i_1, \ldots, i_n)$ where each i_k is 0 or 1. This simplex is non-degenerate precisely when $i_k \neq i_{k+1}$ for all k. This means we have two non-degenerate n-simplices for each n, namely $(0, 1, 0, 1, \ldots)$ and $(1, 0, 1, 0, \ldots)$.

Definition 5.5.1. Let $F_n E^1$ denote the subspace of $\mathrm{N}E^1$ containing all the k-simplices for $k \leq n$ and the non-degenerate n-simplex of the form $(0, 1, 0, \ldots)$.

The non-degenerate n-simplex σ of $F_n E^1$ intersects $F_{n-1}E^1$ on Λ^n_0: Every face of σ lies in $F_{n-1}E^1$ except $d_0\sigma$, which is $(1, 0, 1, \ldots)$. Thus we have a pushout square

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & F_{n-1}E^1 \arrow[d] \\ \Delta^n \arrow[r] & F_n E^1. \end{tikzcd}$$

Thus we have proved:

Lemma 5.5.2. *The map $F_n E^1 \hookrightarrow \mathrm{N}E^1$ is (left) anodyne for every n.*

Proposition 5.5.3. *For $\mathcal{C}$ a quasicategory, every equivalence $f\colon \Delta^1 \to \mathcal{C}$ extends to a map $E^1 \to \mathcal{C}$.*

Proof. If f is an equivalence, then it factors through $\mathrm{Core}(\mathcal{C})$, which is a Kan complex. Therefore we can extend it along the anodyne map $\Delta^1 = F_1 E^1 \hookrightarrow \mathrm{N}E^1$. $\square$

Using this, we get two useful characterizations of isofibrations:

Proposition 5.5.4. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is an inner fibration between quasicategories. Then the following are equivalent:*

(1) p is an isofibration.

(2) p has the right lifting property for $\{0\} \hookrightarrow E^1$.

(3) $\mathrm{Core}(p)\colon \mathrm{Core}(\mathcal{E}) \to \mathrm{Core}(\mathcal{B})$ is a Kan fibration.

Proof. Suppose (2) holds. Then it is clear that p is an isofibration as every equivalence in $\mathcal{B}$ extends to a map from E^1 by Proposition 5.5.3.

Conversely, if p is an isofibration, then to construct a lift in a square

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ E^1 \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

we inductively construct one in

$$\begin{tikzcd} F_nE^1 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ F_{n+1}E^1 \arrow[r] \arrow[ur, dashed] & \mathcal{B}. \end{tikzcd}$$

For $n = 0$ there exists a lift whose image is an equivalence since p is an isofibration, which lets us construct the lift for $n > 0$ using Theorem 5.3.1 since $F_{n+1}E^1$ is the pushout of F_nE^1 along a Λ^{n+1}_0 whose leading edge is mapped to an equivalence. Thus (1) implies (2).

Now we show that (3) holds if p is an isofibration. Consider a square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Core}(\mathcal{E}) \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Core}(\mathcal{B}). \end{tikzcd}$$

For inner horns, there exists a lift $\Delta^n \to \mathcal{E}$ since p is an inner fibration, but this factors through $\mathrm{Core}(\mathcal{E})$ since every edge is sent to an equivalence. For outer horns of dimension ≥ 2, Theorem 5.3.1 gives lifts to $\mathcal{E}$, which factor through $\mathrm{Core}(\mathcal{E})$ by the same argument. Finally, for Λ^1_0 and Λ^1_1, we have lifts since p is an isofibration.

If (3) holds, then p is an isofibration since lifts for $\mathrm{Core}(p)$ along the horns Λ^1_0 precisely give the required lifts for equivalences. □

Corollary 5.5.5. *If $p\colon \mathcal{E} \to \mathcal{B}$ is a conservative isofibration, then its fibres are Kan complexes.*

Proof. Since p is conservative, we have a pullback square

$$\begin{tikzcd} \mathrm{Core}(\mathcal{E}) \arrow[r] \arrow[d, "\mathrm{Core}(p)"'] & \mathrm{Core}(\mathcal{B}) \arrow[d, "p"] \\ \mathcal{E} \arrow[r] & \mathcal{B} \end{tikzcd}$$

by Lemma 6.4.1, so the fibres of p are the same as the fibres of $\mathrm{Core}(p)$. Moreover, since p is an isofibration the map $\mathrm{Core}(p)$ is a Kan fibration by Proposition 5.5.4 so these fibres are Kan complexes. □

5.6 Pushout-Product Version of Joyal's Lifting Theorem

Proposition 5.6.1. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of quasicategories. For $m, n \geq 1$ there exists a lift in any diagram*

$$\begin{tikzcd}
\Delta^{\{0,1\}} \times \{0\} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^m_0 \times \Delta^n \amalg_{\Lambda^m_0 \times \partial\Delta^n} \Delta^m \times \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d] \\
& \Delta^m \times \Delta^n \arrow[r] & \mathcal{D},
\end{tikzcd}$$

provided f is an equivalence.

Sketch Proof. You construct a filtration of $(\Lambda^m_0 \hookrightarrow \Delta^m)\square(\partial\Delta^n \hookrightarrow \Delta^n)$ such that in each step you fill either an inner horn or a 0th horn whose leading edge is mapped to f. (See [Joy08, Lemma 5.8].) $\square$

[TO DO: Draw a small example.]

Corollary 5.6.2. *If $p\colon \mathcal{E} \to \mathcal{B}$ is an isofibration of quasicategories, then for any monomorphism $j\colon K \hookrightarrow L$, the map*

$$p^{\square j}\colon \mathrm{Fun}(L, \mathcal{E}) \to \mathrm{Fun}(L, \mathcal{B}) \times_{\mathrm{Fun}(K,\mathcal{B})} \mathrm{Fun}(K, \mathcal{E})$$

is an isofibration.

Proof. Let S denote the set of morphisms j such that $p^{\square j}$ is an isofibration. Then S is saturated: by Proposition 5.5.4 j is in S precisely if $p^{\square j}$ has the right lifting property for $T = \{\Lambda^n_i \hookrightarrow \Delta^n : 0 < i < n\} \cup \{\{0\} \hookrightarrow E^1\}$. But this is equivalent to j having the left lifting property for $p^{\square T}$, so S is saturated by Lemma 2.6.3.

To see that S contains all monomorphisms it therefore suffices by Proposition 2.6.9 to show that the boundary inclusions $i\colon \partial\Delta^n \hookrightarrow \Delta^n$ lie in S. We know that $p^{\square i}$ is an inner fibration by Corollary 3.7.8, so it remains to show we can lift equivalences. If $n = 0$ then we have p, so this holds by assumption. For $n > 0$, the lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, \mathcal{E}) \arrow[d, "p^{\square i}"] \\
\Delta^1 \arrow[r, "f"'] \arrow[ur, dashed] & \mathrm{Fun}(\partial\Delta^n, \mathcal{E}) \times_{\mathrm{Fun}(\partial\Delta^n,\mathcal{B})} \mathrm{Fun}(\Delta^n, \mathcal{B}),
\end{tikzcd}$$

with f an equivalence, corresponds to

$$\begin{tikzcd}
\{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \arrow[r, "g"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
\Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B}.
\end{tikzcd}$$

Here $g|_{\Delta^1\times\{0\}}$ is an equivalence since f is one, which means a lift exists by Proposition 5.6.1. □

Remark 5.6.3. We only used that p was an isofibration to prove that $\emptyset \hookrightarrow \Delta^0$ was in S. Thus, the same proof implies that if $i\colon K \hookrightarrow L$ is a monomorphism such that i_0 is a bijection, then $p^{\square i}$ is an isofibration for any inner fibration p.

5.7 The Objectwise Criterion for Natural Equivalences

Proposition 5.7.1. *Let $j\colon K \hookrightarrow L$ be a monomorphism such that $j_0\colon K_0 \to L_0$ is a bijection. Then for every quasicategory $\mathcal{C}$,*

(i) for every diagram

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left] & \Lambda^2_0 \arrow[r] \arrow[d, hook] & \mathrm{Fun}(L,\mathcal{C}) \arrow[d, "j^*"] \\ & \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(K,\mathcal{C}) \end{tikzcd}$$

such that j^*f is an equivalence, there exists a lift.

(ii) the functor j^* is conservative.

Proof. We first prove that if (i) holds for a map j, then j^* is conservative. Consider $f\colon \Delta^1 \to \mathrm{Fun}(K,\mathcal{C})$ such that j^*f is an equivalence. Then we have a diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "a"] \arrow[d] & \mathrm{Fun}(L,\mathcal{C}) \arrow[d, "j^*"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dashed] & \mathrm{Fun}(K,\mathcal{C}), \end{tikzcd}$$

where b is chosen so that $b|_{\Delta^{\{0,1\}}} = j^*f$, $b|_{\Delta^{\{1,2\}}}$ is an inverse of j^*f, and $b|_{\Delta^{\{0,2\}}}$ is degenerate, and $a|_{\Delta^{\{0,1\}}} = f$, with $a|_{\Delta^{\{0,2\}}}$ degenerate. Then there exists a lift $b'\colon \Delta^2 \to \mathrm{Fun}(K,\mathcal{C})$, hence f has a post-inverse $g := b'|_{\Delta^{\{1,2\}}}$. Then the same argument applied to g shows this also has a post-inverse. Then g is an equivalence and hence so is f.

Now check that if S denotes the set of j such that j^* satisfies (i), then S is saturated. This is clear, though in the case of (transfinite) composites we need to use that for $j \in S$ the functor j^* is conservative, as we just saw. Using the skeletal filtration of a monomorphism, we now just need to show that (i) holds for $i\colon \partial\Delta^n \hookrightarrow \Delta^n$ where $n \geq 1$. In this case the lifting problem in (i) is adjoint to

$$\begin{tikzcd} \Delta^{\{0,1\}} \times \{0\} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^2_0 \times \Delta^n \amalg_{\Lambda^2_0 \times \partial\Delta^n} \Delta^2 \times \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^2 \times \Delta^n, \arrow[ur, dashed] & \end{tikzcd}$$

where f is an equivalence in $\mathcal{C}$. This lift exists by Proposition 5.6.1. □

Corollary 5.7.2. *A natural transformation $\eta\colon F \to G$, where F, G are functors of quasicategories $\mathcal{C} \to \mathcal{D}$, is a natural equivalence if and only if the maps $\eta_c\colon F(c) \to G(c)$ are equivalences in $\mathcal{D}$ for every $c \in \mathcal{C}$.*

Proof. The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ (where $\mathcal{C}_0 = \mathrm{sk}_0\mathcal{C}$ denotes the constant simplicial set on the 0-simplices of $\mathcal{C}$) satisfies the hypothesis of Proposition 5.7.1. Thus the induced map $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{D}) \cong \prod_{c \in \mathcal{C}_0} \mathcal{D}$ is conservative. □

Corollary 5.7.3. *For $\mathcal{C}$ a quasicategory and x, y objects of $\mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex.*

Proof. The projection $j^*\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is induced by the inclusion $j\colon \{0, 1\} \hookrightarrow \Delta^1$, which is bijective on 0-simplices. Therefore j^* is an isofibration by Corollary 5.6.2 and conservative by Proposition 5.7.1. Thus its fibres are Kan complexes by Corollary 5.5.5. □

Chapter 6

Fully Faithful and Essentially Surjective Functors

6.1 The Enriched Homotopy Category

Definition 6.1.1. If $\mathcal{C}$ is a quasicategory, its *enriched homotopy category* $\mathbf{h}\mathcal{C}$ is the hKan-enriched category defined as follows: The objects of $\mathbf{h}\mathcal{C}$ are the 0-simplices of $\mathcal{C}$. For objects x, y, the mapping object $\mathbf{h}\mathcal{C}(x, y)$ is the image in hKan of the Kan complex $\mathrm{Map}_{\mathcal{C}}(x, y)$. To define composition, recall from Remark 3.9.6 that for any three objects x, y, z in $\mathcal{C}$ we have maps

$$\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \leftarrow \mathrm{Map}_{\mathcal{C}}(x, y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z),$$

where the first map is a trivial fibration — these are the maps of fibres from $\mathcal{C}^{\Lambda^2_1} \leftarrow \mathcal{C}^{\Delta^2} \to \mathcal{C}^{\Delta^1}$. In hKan, the homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(x, y, z) \to \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z)$ becomes an isomorphism and so has a unique inverse. Using this we get the composition map $\mathbf{h}\mathcal{C}(x, y) \times \mathbf{h}\mathcal{C}(y, z) \to \mathbf{h}\mathcal{C}(x, z)$. To check the composition is associative we similarly consider the maps on fibres using $\mathcal{C}^{\Delta^3}$, and the identities in $\mathcal{C}$ clearly give identity maps in $\mathbf{h}\mathcal{C}$.

Remark 6.1.2. The homotopy category $h\mathcal{C}$ is the underlying category of $\mathbf{h}\mathcal{C}$, obtained by taking π_0 of the mapping spaces.

Warning 6.1.3. The enriched homotopy category $\mathbf{h}\mathcal{C}$ contains strictly less information than $\mathcal{C}$. Conversely, there are hKan-enriched categories that do not arise from any quasicategory; for instance, there are homotopy-associative H-spaces that do not lift to A_∞-spaces.

The following is immediate from the definitions:

Lemma 6.1.4. *A functor $f \colon \mathcal{C} \to \mathcal{D}$ of quasicategories is fully faithful and essentially surjective if and only if $\mathbf{h}f$ is a fully faithful and essentially surjective functor of hKan-enriched categories, or equivalently $\mathbf{h}f$ is an equivalence of hKan-enriched categories.* □

As an immediate consequence, we get:

Lemma 6.1.5. *Fully faithful and essentially surjective functors of quasicategories have the 2-of-3 property.*

Proof. Equivalences of hKan-enriched categories have the 2-of-3 property. □

Lemma 6.1.6. *If f is a categorical equivalence of quasicategories, then $\mathbf{h}f$ is an equivalence of hKan-enriched categories.*

Proof. It is enough to observe that $\mathbf{h}$ preserves products — then the data that exhibits f as a categorical equivalence induces under $\mathbf{h}$ data that exhibits $\mathbf{h}f$ as an equivalence. □

Combining these two lemmas, we get:

Lemma 6.1.7. *If f is a categorical equivalence of quasicategories, then f is fully faithful and essentially surjective.*

6.2 Trivial Fibrations of Kan Complexes

Definition 6.2.1. A map $p\colon X \to Y$ of simplicial sets is a *fibrewise deformation retraction* if there exist $s\colon Y \to X$ such that $ps = \mathrm{id}_Y$, and $k\colon X \times \Delta^1 \to X$ such that $k|_{X\times\{0\}} = \mathrm{id}_X$, $k|_{X\times\{1\}} = sp$, and $pk = p\pi_X$ for π_X the projection $X \times \Delta^1 \to X$. In other words, k is a fibrewise homotopy from id_X to sp.

Lemma 6.2.2. *Let $p\colon X \to Y$ be a Kan fibration. Then p is a trivial fibration if and only if p is a fibrewise deformation retract.*

Proof. If p is a trivial fibration then we can choose a section s by picking a lift in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dashed, "s"] & Y, \end{tikzcd}$$

and then we can choose fibrewise homotopy k as a lift in the diagram

$$\begin{tikzcd} X \times \partial\Delta^1 \arrow[r, "\mathrm{id}_X \amalg sp"] \arrow[d] & X \arrow[d, "p"] \\ X \times \Delta^1 \arrow[r, "p\pi_X"'] \arrow[ur, dashed, "k"] & Y. \end{tikzcd}$$

Conversely, if p is a fibrewise deformation retract, consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] & Y \end{tikzcd}$$

with i a monomorphism. Then using s and k we can construct a commutative square

$$\begin{tikzcd} A \times \Delta^1 \amalg_{A\times\{1\}} B \times \{1\} \arrow[r, "k(a\times\mathrm{id})\cup sb"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \times \Delta^1 \arrow[r, "b\pi_B"'] \arrow[ur, dashed, "\phi"] & Y \end{tikzcd}$$

where π_B is the projection $B \times \Delta^1 \to B$, and j is $i\square(\{1\} \hookrightarrow \Delta^1)$. Then j is anodyne by Corollary 2.8.6, and so a lift ϕ exists since p is a Kan fibration. Then $\phi|_{B\times\{0\}}$ gives the required lift in the original square. $\square$

Proposition 6.2.3. *A Kan fibration $p\colon X \to Y$ of Kan complexes is a trivial fibration if and only if it is a homotopy equivalence.*

Proof. We already saw in Lemma 2.9.7 that a trivial fibration of Kan complexes is a homotopy equivalence. Suppose that p is a homotopy equivalence, so there exist $f\colon Y \to X$ and homotopies $u\colon X \times \Delta^1 \to X$, $v\colon Y \times \Delta^1 \to Y$ from fp to id_X and pf to id_Y, respectively. We will deform this data to get a fibrewise deformation retract. First, since $Y \times \{0\} \hookrightarrow Y \times \Delta^1$ is anodyne, we can choose a lift α in

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ Y \times \Delta^1 \arrow[r, "v"'] \arrow[ur, dashed, "\alpha"] & Y. \end{tikzcd}$$

Let $s := \alpha|_{Y\times\{1\}}$, then $ps = v|_{Y\times\{1\}} = \mathrm{id}_Y$, so s is a section of p. Moreover, α is a homotopy from f to p, and combining $s\alpha$ with u gives a new homotopy $w\colon X \times \Delta^1 \to X$ from sp to id_X. Then consider

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "a"] \arrow[d] & X^X \arrow[d, "p^*"] \\ \Delta^2 \arrow[r, "b"'] & Y^X, \end{tikzcd}$$

t where $a|_{\Delta^{\{0,1\}}} = w$, $a|_{\Delta^{\{0,2\}}} = spw$, and b is the degenerate 2-simplex $s_1(pw)$. Since p^* is a Kan fibration Corollary 2.8.7, there exists a lift t. Then $k := t|_{\Delta^{\{1,2\}}}$ is a fibrewise homotopy from id_X to sp. Thus p is a fibrewise deformation retract, and so a trivial fibration by Lemma 6.2.2. □

Corollary 6.2.4. *If X is a Kan complex, then the map $X \to \Delta^0$ is a homotopy equivalence if and only if it is a trivial fibration.* □

Corollary 6.2.5. *If $j\colon A \hookrightarrow B$ is a monomorphism, then j is a weak homotopy equivalence if and only if $j^*\colon \mathrm{Map}(B,X) \to \mathrm{Map}(A,X)$ is a trivial fibration for all Kan complexes X.*

Proof. By definition j is a weak homotopy equivalence if and only if j^* is a homotopy equivalence for every Kan complex X. As j is a monomorphism, j^* is a Kan fibration by Corollary 2.8.7, and so it is a trivial fibration if and only if it is a homotopy equivalence by Proposition 6.2.3. □

Corollary 6.2.6. *If $j\colon A \hookrightarrow B$ is a monomorphism and a weak homotopy equivalence, and $p\colon X \to Y$ is a Kan fibration of Kan complexes, then $p^{\square j}$ is a trivial fibration.*

Proof. By Corollary 2.8.7, the map $p^{\square j}$ is a Kan fibration of Kan complexes, so by Proposition 6.2.3 it suffices to show that it is also a homotopy equivalence.

Consider the diagram

$$\begin{tikzcd}
\mathrm{Map}(B,X) \arrow[r, "p^{\square j}"] \arrow[rr, bend left, "j_X^*"] & \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y) \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}(A,X) \arrow[d] \\
& \mathrm{Map}(B,Y) \arrow[r, "j_Y^*"] & \mathrm{Map}(A,Y).
\end{tikzcd}$$

Here j_X^* and j_Y^* are Kan fibrations of Kan complexes and also homotopy equivalences, so they are trivial fibrations. Therefore the base change q is also a trivial fibration. By the 2-of-3 property, the map $p^{\square j}$ is then also a homotopy equivalence. $\square$

Corollary 6.2.7. *A map $p\colon X \to Y$ of Kan complexes is a Kan fibration if and only if it has the right lifting property for all monomorphisms that are weak homotopy equivalences.*

Proof. If p has the right lifting property for all monomorphisms that are weak homotopy equivalences, then in particular it has the right lifting property for the horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, so p is a Kan fibration. Conversely, if p is a Kan fibration then by Corollary 6.2.6 the map $p^{\square j}$ is a trivial fibration for any monomorphism j that is a weak homotopy equivalence. In particular, this means $p^{\square j}$ is surjective on 0-simplices, which translates into all lifting problems with j and p having solutions, i.e. p has the right lifting property for j. $\square$

Proposition 6.2.8. *A Kan fibration $p\colon X \to Y$ is a trivial fibration if and only if its fibres are contractible Kan complexes.*

Proof. If p is a trivial fibration, so is $X_y \to \{y\}$ for every 0-simplex y of Y, i.e. the fibres are contractible Kan complexes. Conversely, suppose p has contractible fibres and consider the lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "a"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "b"'] & Y.
\end{tikzcd}$$

Define $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ by

$$\gamma(i,j) = \begin{cases} i, & j = 0, \\ n, & j = 1. \end{cases}$$

Then $\gamma|_{\Delta^n \times \{0\}} = \mathrm{id}_{\Delta^n}$ and $\gamma|_{\Delta^n \times \{1\}}$ is constant. Since $\partial\Delta^n \times \{0\} \hookrightarrow \partial\Delta^n \times \Delta^1$ is anodyne, there exists a lift c in

$$\begin{tikzcd}
\partial\Delta^n \times \{0\} \arrow[rrr, "a"] \arrow[d, hook] & & & X \arrow[d] \\
\partial\Delta^n \times \Delta^1 \arrow[r, hook] \arrow[urrr, dashed, "c"] & \Delta^n \times \Delta^1 \arrow[r, "\gamma"] & \Delta^n \arrow[r, "b"] & Y.
\end{tikzcd}$$

Here $b\gamma|_{\Delta^n\times\{1\}}$ is constant at $b(n)$, so $c|_{\partial\Delta^n\times\{1\}}$ factors through the fibre $X_{b(n)}$. Since $X_{b(n)}$ is a contractible Kan complex, there exists a lift d in

$$\begin{tikzcd}
\partial\Delta^n \times \{1\} \arrow[r] \arrow[d, hook] & X_{b(n)} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \{1\} \arrow[r] \arrow[ur, dashed, "d"] & \{y\} \arrow[r] & Y.
\end{tikzcd}$$

We then get a commutative square

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \amalg_{\partial\Delta^n\times\{1\}} \Delta^n \times \{1\} \arrow[r, "c \cup d"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

Then $s|_{\Delta^n\times\{0\}}$ gives the required lift in the original square. □

Corollary 6.2.9. *Suppose*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a pullback square with p a Kan fibration and g surjective on 0-simplices. Then p' is a trivial fibration if and only if p is. If Y and Y' are Kan complexes, then p is a homotopy equivalence if and only if p' is one. □

6.3 Fully Faithful and Essentially Surjective Maps of Kan Complexes

Corollary 6.3.1. *Let $p\colon X \to Y$ be a Kan fibration of Kan complexes. Then p is fully faithful if and only if $p^{\square(\partial\Delta^1\hookrightarrow\Delta^1)}\colon X^{\Delta^1} \to Y^{\Delta^1} \times_{Y\times Y} X \times X$ is a trivial fibration.*

Proof. Consider the pullback square

$$\begin{tikzcd}
\coprod_{x,y\in X_0} \mathrm{Map}_X(x,y) \arrow[r] \arrow[d, "q'"'] & X^{\Delta^1} \arrow[d, "q"] \\
\coprod_{x,y\in X_0} \mathrm{Map}_Y(px,py) \arrow[r, "g"'] & Y^{\Delta^1} \times_{Y\times Y} X \times X
\end{tikzcd}$$

Here g is surjective on 0-simplices and q is a Kan fibration of Kan complexes. Therefore q is a homotopy equivalence if and only if q' is one, and clearly q' is a homotopy equivalence if and only if p is fully faithful. □

Theorem 6.3.2. *If $f\colon X \to Y$ is a map of Kan complexes, then f is fully faithful and essentially surjective if and only if f is a homotopy equivalence.*

Proof. We saw in Lemma 6.1.7 that if f is a homotopy equivalence then it is fully faithful and essentially surjective. Suppose therefore that f is fully faithful and essentially surjective. We can factor f as $X \xrightarrow{f'} X' \xrightarrow{f''} Y$ where f' is anodyne and f'' is a Kan fibration. Then f' is both fully faithful and essentially surjective and a homotopy equivalence, so since both classes have the 2-of-3 property it's enough to show that f'' is a homotopy equivalence. In other words, we may without loss of generality assume that f is a Kan fibration. Then we can consider the commutative diagram

$$
\begin{tikzcd}
 & Y \arrow[dr, "i"] \arrow[drr, "\mathrm{id}_Y"] & & \\
X \arrow[ur, "f"] \arrow[r, "j"] \arrow[dr, "\mathrm{id}_X"'] \arrow[rrr, bend left=60, "f"] & Y^{\Delta^1} \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\ulcorner", very near start] & Y^{\Delta^1} \arrow[r, "\mathrm{ev}_1"'] \arrow[d, "\mathrm{ev}_0"] & Y \\
 & X \arrow[r, "f"] & Y, &
\end{tikzcd}
$$

where $i\colon Y \to Y^{\Delta^1}$ is the "constant edge" map, adjoint to the projection $Y \times \Delta^1 \to Y$. Here, as $\{0\}, \{1\} \hookrightarrow \Delta^1$ are anodyne, the maps $\mathrm{ev}_0, \mathrm{ev}_1$ are trivial fibrations. Therefore the base change π_2 is also a trivial fibration, which means by the 2-of-3 property that j is a homotopy equivalence. Thus we see that f is a homotopy equivalence if and only if the map $r := \mathrm{ev}_1 \circ \pi$ is one.

Now consider the diagram

$$
\begin{tikzcd}
Y^{\Delta^1} \times_{Y^2} X^2 \arrow[r] \arrow[d] \arrow[rr, bend left, "r'"] & X^2 \arrow[r] & X \arrow[d, "f"] \\
Y^{\Delta^1} \times_{Y^2} X \times Y \arrow[r] \arrow[rr, bend right, "r"'] & X \times Y \arrow[r] & Y
\end{tikzcd}
$$

Since f is a Kan fibration and essentially surjective, it is surjective on 0-simplices, since we have lifts for $\{0\} \hookrightarrow \Delta^1$. Then as r is a Kan fibration of Kan complexes, we know from Corollary 6.2.9 that r is a homotopy equivalence if and only if r' is one. Finally, we consider the diagram

$$
\begin{tikzcd}
X^{\Delta^1} \arrow[r, "q"] \arrow[rr, bend left, "\mathrm{ev}_1"] & Y^{\Delta^1} \times_{Y^2} X^2 \arrow[r, "r'"] & X.
\end{tikzcd}
$$

Here again ev_1 is a trivial fibration since $\{1\} \hookrightarrow \Delta^1$ is anodyne. By Corollary 6.3.1 the map q is a trivial fibration since f is fully faithful. But then r' is a homotopy equivalence by the 2-of-3 property, so we conclude that f is also a homotopy equivalence, as required. ∎

6.4 Trivial Fibrations of Quasicategories

Lemma 6.4.1. *If $f\colon \mathcal{C} \to \mathcal{D}$ is a functor of quasicategories and $i\colon K \hookrightarrow L$ is a monomorphism, then*

$$\mathrm{Core}(\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L) \cong \mathrm{Core}(\mathcal{C}^K) \times_{\mathrm{Core}(\mathcal{D}^K)} \mathrm{Core}(\mathcal{D}^L).$$

[TO DO: Rewrite as $\mathrm{Core}(-)$ preserves pullbacks of quasicategories along isofibrations.]

Proof. Both sides are subobjects of $\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L$, which is a quasicategory since $i^*\colon \mathcal{D}^L \to \mathcal{D}^K$ is an inner fibration and $\mathcal{C}^K$ is a quasicategory by Corollary 3.7.8. The left-hand side is contained in the right-hand side, since all functors of quasicategories preserve equivalences. On the other hand, $i^*\colon \mathcal{D}^L \to \mathcal{D}^K$ is an isofibration by Corollary 5.6.2 and so $\mathrm{Core}(\mathcal{D}^L) \to \mathrm{Core}(\mathcal{D}^K)$ is a Kan fibration by Proposition 5.5.4. Thus, as $\mathrm{Core}(\mathcal{C}^K)$ is a Kan complex, the right-hand side is also a Kan complex, and so must be contained in the left-hand side as this is the maximal Kan complex contained in $\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L$. □

Proposition 6.4.2. *Suppose $p\colon \mathcal{C} \to \mathcal{D}$ is a functor of quasicategories. Then p is a trivial fibration if and only if p is an isofibration and a categorical equivalence.*

Proof. If p is a trivial fibration then it is an inner fibration and has the right lifting property for $\{0\} \hookrightarrow E^1$, so it is an isofibration by Proposition 5.5.4; it is also a categorical equivalence by Lemma 3.8.10.

Conversely, if p is a categorical fibration. It suffices to show that for any monomorphism $i\colon K \hookrightarrow L$, the map $\mathrm{Core}(p^{\square i})$ is surjective on 0-simplices, as this implies all lifting problems have solutions. In fact, $\mathrm{Core}(p^{\square i})$ is a trivial fibration: Since $p^{\square i}$ is an isofibration by Corollary 5.6.2, the map $\mathrm{Core}(p^{\square i})$ is a Kan fibration by Proposition 5.5.4. The maps $\mathrm{Core}(\mathcal{C}^L) \to \mathrm{Core}(\mathcal{D}^L)$ and $\mathrm{Core}(\mathcal{C}^K) \to \mathrm{Core}(\mathcal{D}^K)$ are similarly Kan fibrations of Kan complexes and categorical equivalences, so they are trivial fibrations by Proposition 6.2.3. Then in the diagram

$$\begin{tikzcd} \mathrm{Core}(\mathcal{C}^L) \arrow[r] \arrow[rr, bend left] & \mathrm{Core}(\mathcal{C}^K) \times_{\mathrm{Core}(\mathcal{D}^K)} \mathrm{Core}(\mathcal{D}^L) \arrow[r] & \mathrm{Core}(\mathcal{D}^L), \end{tikzcd}$$

the composite and the second map are both trivial fibrations, hence the first map is a homotopy equivalence by the 2-of-3 property. □

Corollary 6.4.3. *If $j\colon K \hookrightarrow L$ is a monomorphism, then j is a weak categorical equivalence if and only if $j^*\colon \mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ is a trivial fibration for all quasicategories $\mathcal{C}$.*

Proof. By definition, j is a weak categorical equivalence if and only if the map j^* is a categorical equivalence for every quasicategory $\mathcal{C}$. The map j^* is an isofibration by Corollary 5.6.2, so by Proposition 6.4.2 it is a trivial fibration if and only if it is a categorical equivalence. □

Proposition 6.4.4. *A functor of quasicategories $p\colon \mathcal{C} \to \mathcal{D}$ is an isofibration if and only if it has the right lifting property for every monomorphism that is a weak categorical equivalence.*

Proof. If p has this lifting property, then it is an inner fibration and has the right lifting property for $\{0\} \hookrightarrow E^1$, so it follows from Proposition 5.5.4 that p is an isofibration. Conversely, suppose p is an isofibration and $j: K \hookrightarrow L$ is a monomorphism that is a weak categorical equivalence. It suffices to prove that the map $p^{\square j}$ is surjective on 0-simplices. In fact, this map is a trivial fibration; it is an isofibration by Corollary 5.6.2, so by Proposition 6.4.2 it suffices to show it is a categorical equivalence. Consider the diagram

$$\begin{tikzcd} \mathcal{C}^L \arrow[r, "p^{\square j}"] \arrow[dr, "j^*_{\mathcal{C}}"'] & \mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L \arrow[r] \arrow[d, "q"] & \mathcal{D}^L \arrow[d, "j^*_{\mathcal{D}}"] \\ & \mathcal{C}^K \arrow[r] & \mathcal{D}^K. \end{tikzcd}$$

Here $j^*_{\mathcal{D}}$ and $j^*_{\mathcal{C}}$ are trivial fibrations by Corollary 6.4.3, hence so is the base change q. By the 2-of-3 property it follows that $p^{\square j}$ is a categorical equivalence, as required. □

6.5 Localization of Quasicategories

Definition 6.5.1. For $\mathcal{C}$ a quasicategory, X a simplicial set, and W a collection of edges in X, let $\mathrm{Fun}_{(W)}(X, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(X, \mathcal{C})$ spanned by the functors that take the edges in W to equivalences in $\mathcal{C}$.

We want to prove that there exists a quasicategory $X[W^{-1}]$ such that $\mathrm{Fun}_{(W)}(X, \mathcal{C})$ is categorically equivalent to $\mathrm{Fun}(X[W^{-1}], \mathcal{C})$ for every quasicategory $\mathcal{C}$. We will build up to this starting with the following observation:

Lemma 6.5.2. *For $i: \Delta^1 = F_1E^1 \hookrightarrow E^1$ and $\mathcal{C}$ a quasicategory, the restriction map $i^*: \mathrm{Fun}(E^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ factors through a trivial fibration $\mathrm{Fun}(E^1, \mathcal{C}) \to \mathrm{Fun}_{(\Delta^1)_1}(\Delta^1, \mathcal{C})$.*

Proof. i^* clearly factors through the full subcategory $\mathrm{Fun}_{(\Delta^1)_1}(\Delta^1, \mathcal{C})$, and we know $\mathrm{Fun}(E^1, \mathcal{C}) \to \mathrm{Fun}_{(\Delta^1)_1}(\Delta^1, \mathcal{C})$ is surjective on 0-simplices from Proposition 5.5.3. It therefore suffices to show that

$$\mathrm{Fun}_{(\Delta^1)_1}(F_nE^1, \mathcal{C}) \to \mathrm{Fun}_{(\Delta^1)_1}(F_{n-1}E^1, \mathcal{C})$$

is a trivial fibration for $n > 1$, where clearly $\mathrm{Fun}_{(\Delta^1)_1}(F_nE^1, \mathcal{C}) = \mathrm{Fun}(F_nE^1, \mathcal{C})$ for $n \geq 3$. The lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Fun}_{(\Delta^1)_1}(F_nE^1, \mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}_{(\Delta^1)_1}(F_{n-1}E^1, \mathcal{C}) \end{tikzcd}$$

corresponds to

$$\begin{tikzcd} \Delta^n \times \partial\Delta^m \amalg_{\Lambda^n_0 \times \partial\Delta^m} \Lambda^n_0 \times \Delta^m \arrow[r] \arrow[d] & F_nE^1 \times \partial\Delta^m \amalg_{F_{n-1}E^1 \times \partial\Delta^m} F_{n-1}E^1 \times \Delta^m \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \times \Delta^m \arrow[r] & F_nE^1 \times \Delta^m. \arrow[ur, dashed] \end{tikzcd}$$

Since the left-hand square is a pushout, it suffices to construct a lift to $\Delta^n \times \Delta^m$, which exists by Proposition 5.6.1. □

Next, we consider the special case where we replace X by a Kan complex, or in other words replace all the edges by equivalences:

Proposition 6.5.3. *Suppose X is a simplicial set and $\mathcal{C}$ is a quasicategory. Let $i\colon X \hookrightarrow \|X\|$ be an anodyne map to a Kan complex. Then the restriction map $i^*\colon \mathrm{Fun}(\|X\|, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ factors through a trivial fibration $p\colon \mathrm{Fun}(\|X\|, \mathcal{C}) \to \mathrm{Fun}_{(X_1)}(X, \mathcal{C})$.*

Proof. Since any functor $\|X\| \to \mathcal{C}$ must take every edge in the Kan complex $\|X\|$ to an equivalence in $\mathcal{C}$, i^* factors through $\mathrm{Fun}_{(X_1)}(X, \mathcal{C})$. Moreover, we know that i^* is an isofibration by Corollary 5.6.2, which implies that p is also an isofibration. It therefore suffices by Proposition 6.4.2 to show that p is a categorical equivalence. Next we want to observe that if p is a trivial fibration for one choice of i, then it is one for any other choice: If $i'\colon X \hookrightarrow X'$ and $i''\colon X \hookrightarrow X''$ are two anodyne maps with X', X'' Kan complexes, then there exists a homotopy equivalence $f\colon X' \to X''$ such that $fi' = i''$ — we use that $i'^*\colon \mathrm{Map}(X', X'') \to \mathrm{Map}(X, X'')$ is a trivial fibration, plus the 2-of-3 property. Then we have a commutative triangle

$$\begin{tikzcd} \mathrm{Fun}(X', \mathcal{C}) \arrow[rr, "f^*"] \arrow[dr, "p'"'] & & \mathrm{Fun}(X'', \mathcal{C}) \arrow[dl, "p''"] \\ & \mathrm{Fun}_{(X_1)}(X, \mathcal{C}) & \end{tikzcd}$$

where f^* is a categorical equivalence, hence p' is a trivial fibration if and only if p'' is one.

It therefore suffices to prove the result for one choice of i. Let $X' := X \amalg_{\coprod_{X_1} \Delta^1} \coprod_{X_1} E^1$ and let $\alpha\colon X \to X'$ denote the inclusion — this is anodyne by Lemma 5.5.2. Choose an *inner* anodyne map $\beta\colon X' \to X''$ with X'' a quasicategory, and let $i := \beta\alpha$. We claim that X'' is actually a Kan complex. To see this it suffices to show that hX'' is groupoid.

Any weak categorical equivalence $f\colon A \to B$ induces an equivalence of homotopy categories: Since h preserves products, for any category $\mathbf{C}$ the simplicial set $\mathrm{Fun}(A, \mathbf{C})$ is isomorphic to (the nerve of) the category $\mathrm{Fun}(hA, \mathbf{C})$, so hf induces an equivalence of categories $\mathrm{Fun}(hB, \mathbf{C}) \to \mathrm{Fun}(hA, \mathbf{C})$ for every category $\mathbf{C}$, so hf is an equivalence. It therefore suffices to show that hX' is a groupoid, since hX' is equivalent to hX''.

But by construction, every edge in X' factors through a map $E^1 \to X'$, and so any map $X' \to \mathcal{C}$ with $\mathcal{C}$ a quasicategory must factor through $\mathrm{Core}(\mathcal{C})$. For any category $\mathbf{C}$ we in particular have

$$\mathrm{Hom}_{\mathrm{Cat}}(hX', \mathbf{C}) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(X', \mathbf{C}) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(X', \mathrm{Core}(\mathbf{C})) \cong \mathrm{Hom}_{\mathrm{Cat}}(hX', \mathrm{Core}(\mathbf{C})),$$

so hX' is a groupoid.

The map p is the composite

$$\mathrm{Fun}(X'', \mathcal{C}) \xrightarrow{\beta^*} \mathrm{Fun}(X', \mathcal{C}) \xrightarrow{\alpha^*} \mathrm{Fun}_{(X_1)}(X, \mathcal{C}).$$

Here β^* is a trivial fibration by Corollary 3.7.8 since β is inner anodyne, and α^* is a trivial fibration since it is a pullback of products of the map $\mathrm{Fun}(E^1, \mathcal{C}) \to \mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{C})$, which is a trivial fibration by Lemma 6.5.2. Thus i^* is a trivial fibration, as required. □

Corollary 6.5.4. *Given $W \hookrightarrow X$, choose an anodyne map $W \hookrightarrow \|W\|$, with $\|W\|$ a Kan complex, and an inner anodyne map*

$$j \colon X \amalg_W \|W\| \hookrightarrow X[W^{-1}]$$

with $X[W^{-1}]$ a quasicategory. Then for any quasicategory $\mathcal{C}$, the restriction $\mathrm{Fun}(X[W^{-1}], \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ factors through a trivial fibration

$$\mathrm{Fun}(X[W^{-1}], \mathcal{C}) \to \mathrm{Fun}_{(W_1)}(X, \mathcal{C}).$$

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(X', \mathcal{C}) \arrow[r, "j^*"] & \mathrm{Fun}(X \amalg_W \|W\|, \mathcal{C}) \arrow[r, "\beta"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}_{(W_1)}(X, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(X, \mathcal{C}) \arrow[d] \\
& \mathrm{Fun}(\|W\|, \mathcal{C}) \arrow[r, "\gamma"] & \mathrm{Fun}_{(W_1)}(W, \mathcal{C}) \arrow[r] & \mathrm{Fun}(W, \mathcal{C}).
\end{tikzcd}$$

Here j^* is a trivial fibration since j is inner anodyne. The map labelled γ is a trivial fibration by Proposition 6.5.3, hence so is its base change β. The composite βj^* is then also a trivial fibration, as required. □

Example 6.5.5. Consider a relative category $(\mathbf{C}, W)$, i.e. a category $\mathbf{C}$ equipped with a collection of "weak equivalences" W. Then we have constructed a quasicategory $\mathbf{C}[W^{-1}]$ with the universal property that functors of quasicategories $\mathbf{C}[W^{-1}] \to \mathcal{D}$ correspond to functors $\mathbf{C} \to \mathcal{D}$ that take the morphisms in W to equivalences in $\mathcal{D}$. (Taking $\mathcal{D}$ to be an ordinary category, we see that the homotopy category $h\mathbf{C}[W^{-1}]$ is the (Gabriel-Zisman) localization of $\mathbf{C}$ at W.)

6.6 Fully Faithful and Essentially Surjective Maps of Quasicategories

We now want to prove that fully faithful and essentially surjective functors of quasicategories are categorical equivalences. We will deduce it from the following characterization of categorical equivalences:

Theorem 6.6.1. *The following are equivalent for a map of quasicategories $f \colon \mathcal{C} \to \mathcal{D}$:*

(1) f is a categorical equivalence.

(2) $f_* \colon \mathrm{Map}(\mathcal{A}, \mathcal{C}) \to \mathrm{Map}(\mathcal{A}, \mathcal{D})$ is a homotopy equivalence for all quasicategories $\mathcal{A}$.

(3) $f_* \colon \mathrm{Map}(K, \mathcal{C}) \to \mathrm{Map}(K, \mathcal{D})$ is a homotopy equivalence for all simplicial sets K.

(4) $f_* \colon \mathrm{Map}(\Delta^n, \mathcal{C}) \to \mathrm{Map}(\Delta^n, \mathcal{D})$ is a homotopy equivalence for all n.

(5) $f_* \colon \mathrm{Map}(\Delta^n, \mathcal{C}) \to \mathrm{Map}(\Delta^n, \mathcal{D})$ *is a homotopy equivalence for* $n = 0, 1$.

In order to reduce the proof to the case of isofibrations, we first show that we can factor any map of quasicategories as a categorical equivalence followed by an isofibration:

Definition 6.6.2. For $f \colon \mathcal{C} \to \mathcal{D}$ a functor of quasicategories, define a factorization $\mathcal{C} \xrightarrow{j} P(f) \xrightarrow{p} \mathcal{D}$ by the diagram

$$\begin{tikzcd}
& \mathcal{D} \arrow[dr, "g"] & & \\
\mathcal{C} \arrow[ur, "f"] \arrow[r] \arrow[dr, equal] & P(f) \arrow[r] \arrow[d, "s"] \arrow[rr, bend left, "p"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{D}) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"] & \mathcal{D} \\
& \mathcal{C} \arrow[r, "f"] & \mathcal{D}. &
\end{tikzcd}$$

Here $g \colon \mathcal{D} \to \mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{D})$ arises from the map $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ adjoint to the projection $\mathcal{D} \times \Delta^1 \to \mathcal{D}$, which factors through $\mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{D})$ (since identities are equivalences).

Lemma 6.6.3. *In the diagram above, the map* j *is a monomorphism and a categorical equivalence, and* p *is an isofibration.*

Proof. Consider

$$\mathrm{Fun}(E^1, \mathcal{D}) \xrightarrow{q} \mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{D}) \xrightarrow{\mathrm{ev}_i} \mathcal{D}.$$

Here the composite map is a trivial fibration (since $\{i\} \hookrightarrow E^1$ is a monomorphism and a categorical equivalence), as is q by Lemma 6.5.2. It follows that ev_i is also a categorical equivalence. The restrictions $\mathrm{ev}_i \colon \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ are isofibrations, so the restricted maps $\mathrm{ev}_i \colon \mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ are also isofibrations since we are restricting to a full subcategory. They are therefore trivial fibrations by Proposition 6.4.2. The base change s is therefore also a trivial fibration, hence j is a categorical equivalence by the 2-of-3 property; it is also a monomorphism since s is a post-inverse.

Now consider the diagram

$$\begin{tikzcd}
P(f) \arrow[r] \arrow[d] \arrow[dd, bend right, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}_{(\Delta^1_1)}(\Delta^1, \mathcal{D}) \arrow[d, "e"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "f \times \mathrm{id}"] \arrow[d] & \mathcal{D} \times \mathcal{D} \\
\mathcal{D} &
\end{tikzcd}$$

Here e is an isofibration, being a restriction of the isofibration $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\partial\Delta^1, \mathcal{D})$ to a full subcategory. Hence so is the base change s, which means that p is an isofibration, being a composite of two of them. $\square$

Next let's make a couple of simple observations:

Lemma 6.6.4. *If $p\colon \mathcal{E} \to \mathcal{B}$ is a trivial fibration of quasicategories, then* $\mathrm{Core}(\mathcal{E}) \to \mathrm{Core}(\mathcal{B})$ *is a trivial fibration.*

Proof. Consider a lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "a"] \arrow[d] & \mathrm{Core}(\mathcal{E}) \arrow[d, "\mathrm{Core}(p)"] \\
\Delta^n \arrow[r, "b"'] & \mathrm{Core}(\mathcal{B}.)
\end{tikzcd}$$

We can lift b to a map $c\colon \Delta^n \to \mathcal{E}$. If $n > 1$ then c takes all the edges of Δ^n to equivalences since they are contained in $\partial\Delta^n$, so c factors through $\mathrm{Core}(\mathcal{E})$. If $n = 1$ then we can choose c to be an equivalence in $\mathcal{E}$ since p is in particular an isofibration. Finally, if $n = 0$ then c factors through $\mathrm{Core}(\mathcal{E})$ since this contains all 0-simplices of $\mathcal{E}$. □

Lemma 6.6.5. *Given a monomorphism $i\colon K \hookrightarrow L$, we can construct a diagram*

$$\begin{tikzcd}
K \arrow[r, hook] \arrow[d, hook, "i"] & \|K\| \arrow[d, hook] \arrow[dr, hook, "j"] \arrow[dl, phantom, "\lrcorner"] & \\
L \arrow[r, hook] & L \amalg_K \|K\| \arrow[r, hook] & \|L\|,
\end{tikzcd}$$

with $\|K\|$ and $\|L\|$ Kan complexes and the horizontal maps all anodyne. Then if $q\colon \mathcal{E} \to \mathcal{B}$ is an isofibration of quasicategories, q has the right lifting property for $j\colon \|K\| \hookrightarrow \|L\|$ if and only if $\mathrm{Core}(q)$ *has the right lifting property for $i\colon K \hookrightarrow L$.*

Proof. Suppose $\mathrm{Core}(q)$ has the right lifting property for i. Since any map from the Kan complexes $\|K\|$ and $\|L\|$ to a quasicategory must factor through the core, it suffices to find a lift in the right-hand square in the diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "i"] & \|K\| \arrow[r, equal] \arrow[d] & \|K\| \arrow[r] \arrow[d] & \mathrm{Core}(\mathcal{E}) \arrow[d] \\
L \arrow[r] & L \amalg_K \|K\| \arrow[r] & \|L\| \arrow[r] & \mathrm{Core}(\mathcal{B}).
\end{tikzcd}$$

Here we can by assumption choose a lift $L \to \mathrm{Core}(\mathcal{E})$. This extends over the pushout $L \amalg_K \|K\|$ by its universal property. And since $L \amalg_K \|K\| \to \|L\|$ is anodyne we can choose a lift to $\|L\|$ as $\mathrm{Core}(q)$ is a Kan fibration by Proposition 5.5.4.

Conversely, suppose q has the right lifting property for j, and consider a lifting problem

$$\begin{tikzcd}
K \arrow[r, "a"] \arrow[d, "i"'] & \mathrm{Core}(\mathcal{E}) \arrow[d, "\mathrm{Core}(q)"] \\
L \arrow[r, "b"'] & \mathrm{Core}(\mathcal{B}).
\end{tikzcd}$$

Since $\mathrm{Core}(\mathcal{E})$ is a Kan complex, and $K \hookrightarrow \|K\|$ is anodyne, the map a factors through a map $a' \colon \|K\| \to \mathrm{Core}(\mathcal{E})$. Then we can extend b to a map b' from the pushout $L \amalg_K \|K\|$. Finally b' extends through the anodyne map $L \amalg_K \|K\| \to \|L\|$ to a map b''. We then have a commutative diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "i"] & \|K\| \arrow[r, equal] \arrow[d] & \|K\| \arrow[r, "a'"] \arrow[d] & \mathrm{Core}(\mathcal{E}) \arrow[d] \\
L \arrow[r] & L \amalg_K \|K\| \arrow[r] & \|L\| \arrow[r, "b''"] & \mathrm{Core}(\mathcal{B}).
\end{tikzcd}$$

Here there exists a lift in the rightmost square by assumption. □

Lemma 6.6.6. *There exists a set T such that for $q \colon \mathcal{E} \to \mathcal{B}$ an isofibration of quasicategories, q has the right lifting property for T if and only if* $\mathrm{Core}(q)$ *is a trivial fibration.*

Proof. By Lemma 6.6.5 the set $\{\|\partial\Delta^n\| \hookrightarrow \|\Delta^n\|\}$ has this property. □

Using this, we can prove the following key observation for the proof of Theorem 6.6.1:

Lemma 6.6.7. *Suppose $p \colon \mathcal{E} \to \mathcal{B}$ is an isofibration of quasicategories. Let S_p denote the set of monomorphisms i such that* $\mathrm{Core}(p^{\square i})$ *is a trivial fibration. Then:*

(i) S_p is saturated.

(ii) If $K \subseteq L$ and $\emptyset \hookrightarrow K$ and $\emptyset \hookrightarrow L$ are in S_p, then $K \hookrightarrow L$ is in S_p.

Proof. We first prove (i). Since p is an isofibration, for any monomorphism i the map $p^{\square i}$ is an isofibration by Corollary 5.6.2. Therefore, by Lemma $\mathrm{Core}(p^{\square i})$ is a trivial fibration if and only if $T \subseteq \mathrm{LLP}(p^{\square i})$. But that is equivalent to $i \in \mathrm{LLP}(p^{\square T})$, so S_p is the intersection of $\mathrm{LLP}(p^{\square T})$ and the set of monomorphisms; since both of these are saturated, so is S_p.

To prove (ii), consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(L,\mathcal{E}) \arrow[r, "\mathrm{Core}(p^{\square i})"'] \arrow[rr, bend left, "p_*^L"] & \mathrm{Map}(K,\mathcal{E}) \times_{\mathrm{Map}(K,\mathcal{B})} \mathrm{Map}(L,\mathcal{B}) \arrow[r, "q"'] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}(L,\mathcal{B}) \arrow[d] \\
& \mathrm{Map}(K,\mathcal{E}) \arrow[r, "p_*^K"] & \mathrm{Map}(K,\mathcal{B})
\end{tikzcd}$$

(where we have implicitly used Lemma 6.4.1). Here the maps p_*^K and p_*^L are both trivial fibrations by assumption, hence so is the base change q. By the 2-of-3 property, it follows that $\mathrm{Core}(p^{\square i})$ is a homotopy equivalence. Since p is an isofibration, $p^{\square i}$ is an isofibration by Corollary 5.6.2 and thus $\mathrm{Core}(p^{\square i})$ is a Kan fibration by Proposition 5.5.4. Therefore $\mathrm{Core}(p^{\square i})$ is a trivial fibration by Proposition 6.2.3. □

Proof of Theorem 6.6.1. We start by doing the easy implications: (1) implies (2) since the data of a categorical equivalence for p induces the data of a homotopy equivalence for p_*. On the other hand, (2) implies in particular that $\pi_0\mathrm{Map}(\mathcal{A}, \mathcal{C}) \to \pi_0\mathrm{Map}(\mathcal{A}, \mathcal{D})$ is an isomorphism for all $\mathcal{A}$, so f is an isomorphism in $h\mathrm{QCat}$, and so a categorical equivalence by Proposition 3.8.8.

(3) obviously implies both (2) and (4), and (4) implies (5). To see that (2) implies (3), observe that for any simplicial set K we can choose an inner anodyne map $j\colon K \hookrightarrow K'$ with K' a quasicategory. Then we have a commutative square

$$\begin{tikzcd} \mathrm{Map}(K', \mathcal{C}) \arrow[r, "f_*"] \arrow[d, "j^*"'] & \mathrm{Map}(K', \mathcal{D}) \arrow[d, "j^*"] \\ \mathrm{Map}(K, \mathcal{C}) \arrow[r, "f_*"'] & \mathrm{Map}(K, \mathcal{D}). \end{tikzcd}$$

Here the vertical maps are trivial fibrations by Lemma 6.6.4 and Corollary 3.7.8. By the 2-of-3 property, we see that if the top horizontal map is a homotopy equivalence, so is the bottom horizontal map.

Next, we prove that (4) implies (3). By factoring f as in Definition 6.6.2, we may (by the 2-of-3 property) without loss of generality assume that f is an isofibration. Consider the saturated class S_f from Lemma 6.6.7; (4) says that this contains $\emptyset \hookrightarrow \Delta^n$ for all n. Then we can show inductively on n (using the skeletal filtration) that it contains $\partial\Delta^n$, and so it contains $\partial\Delta^n \hookrightarrow \Delta^n$ by Lemma 6.6.7(ii). But that means it contains all monomorphisms by Proposition 2.6.9, which in particular gives (3).

Finally, we show that (5) implies (4), again assuming that f is an isofibration. If the saturated class S_f contains $\emptyset \hookrightarrow \Delta^j$ for all $j < n$ then using the skeletal filtration we see that it also contains $\emptyset \hookrightarrow K$ for all simplicial sets K with no non-degenerate simplices of dimension n or higher, in particular it contains $\emptyset \hookrightarrow \Lambda^n_i$ for $0 < i < n$. Moreover, S_f contains the inner anodyne maps by Corollary 3.7.8 and Lemma 6.6.4, so for every $n > 1$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}(\Delta^n, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^n, \mathcal{D}) \arrow[d] \\ \mathrm{Map}(\Lambda^n_i, \mathcal{C}) \arrow[r] & \mathrm{Map}(\Lambda^n_i, \mathcal{D}) \end{tikzcd}$$

where the vertical maps are trivial fibrations. We therefore see that if S_f contains $\emptyset \hookrightarrow \Delta^j$ for $j < n$ then it also contains $\emptyset \hookrightarrow \Delta^n$, provided $n > 1$. Inducting on n we then have that (5) implies (4). □

Remark 6.6.8. If we assume the existence of the Kan–Quillen model structure on Set_Δ together with standard facts about model categories, we can give a simpler proof of Theorem 6.6.1, without going through Lemmas 6.6.5–6.6.7. Specifically, to show that (4) implies (3) we can induct on the number of simplices in K: Suppose we have a pushout

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & K \arrow[d] \\
\Delta^n \arrow[r] & K',
\end{tikzcd}$$

then we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Map}(K',\mathcal{C}) \arrow[rr] \arrow[dr, two heads] \arrow[dd] & & \operatorname{Map}(K',\mathcal{D}) \arrow[dr, two heads] \arrow[dd] & \\
& \operatorname{Map}(K,\mathcal{C}) \arrow[rr, "\sim"] \arrow[dd] & & \operatorname{Map}(K,\mathcal{D}) \arrow[dd] \\
\operatorname{Map}(\Delta^n,\mathcal{C}) \arrow[rr, "\sim"] \arrow[dr, two heads] & & \operatorname{Map}(\Delta^n,\mathcal{D}) \arrow[dr, two heads] & \\
& \operatorname{Map}(\partial\Delta^n,\mathcal{C}) \arrow[rr, "\sim"] & & \operatorname{Map}(\partial\Delta^n,\mathcal{D}).
\end{tikzcd}$$

Here the left and right faces are pullback squares, and certain maps are (trivial) Kan fibrations, as indicated. In particular, the left and right faces are homotopy pullback squares, and therefore $\operatorname{Map}(K',\mathcal{C}) \to \operatorname{Map}(K',\mathcal{D})$ must also be a homotopy equivalence.

Corollary 6.6.9. *Suppose $f\colon \mathcal{C} \to \mathcal{D}$ is a functor of quasicategories. Then f is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

Proof. We saw in Lemma 6.1.7 that a categorical equivalence is fully faithful and essentially surjective. To prove the converse, using the factorization of Definition 6.6.2 we see, since both categorical equivalences and fully faithful and essentially surjective functors have the 2-of-3 property, that we may without loss of generality assume that f is an isofibration.

By Theorem 6.6.1 it suffices to show that $\operatorname{Core}(\mathcal{C}) \to \operatorname{Core}(\mathcal{D})$ and $\operatorname{Map}(\Delta^1,\mathcal{C}) \to \operatorname{Map}(\Delta^1,\mathcal{D})$ are homotopy equivalences. It is easy to see that if f is fully faithful and essentially surjective, then so is $\operatorname{Core}(f)$, hence this is a homotopy equivalence by Theorem 6.3.2. Next consider the diagram

$$\begin{tikzcd}
\coprod_{x,y\in\mathcal{C}_0} \operatorname{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d, "q'"] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Map}(\Delta^1,\mathcal{C}) \arrow[d, "q"] \arrow[dr, "f_*"] & \\
\coprod_{x,y\in\mathcal{C}_0} \operatorname{Map}_{\mathcal{D}}(fx,fy) \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[r, "r"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Map}(\Delta^1,\mathcal{D}) \arrow[d] \\
\mathcal{C}_0^{\times 2} \arrow[r] & \operatorname{Core}(\mathcal{C})^{\times 2} \arrow[r, "\operatorname{Core}(f)^{\times 2}"] & \operatorname{Core}(\mathcal{D})^{\times 2},
\end{tikzcd}$$

where the squares are all pullbacks. Since $\operatorname{Core}(f)^{\times 2}$ is a trivial fibration, so is its base change r, hence by the 2-of-3 property it suffices to show that the map

q is a homotopy equivalence. We know that $f^{\square(\partial\Delta^1 \hookrightarrow \Delta^1)}$ is an isofibration, hence $q := \mathrm{Core}(f^{\square(\partial\Delta^1 \hookrightarrow \Delta^1)})$ is a Kan fibration. Moreover, j is surjective on 0-simplices — indeed, it is an isomorphism on 0-simplices. By Corollary 6.2.9 we can then conclude that q is a homotopy equivalence if and only if q' is a homotopy equivalence. If f is fully faithful, then q' is certainly a homotopy equivalence, which completes the proof. $\square$

Chapter 7

The Joyal Model Structure

7.1 Model Categories

Definition 7.1.1. A *model category* is a category $\mathbf{C}$ with small limits and colimits, equipped with collections of maps W, C, F (called respectively the *weak equivalences*, *cofibrations*, and *fibrations*), such that:

- W satisfies the 2-of-3 property,
- Every morphism f in $\mathbf{C}$ factors as $p \circ i$ with $i \in C \cap W$, $p \in F$.
- $C \cap W = \mathrm{LLP}(F)$, $F = \mathrm{RLP}(C \cap W)$
- Every morphism f in $\mathbf{C}$ factors as $p \circ i$ with $i \in C$, $p \in F \cap W$.
- $C = \mathrm{LLP}(F \cap W)$, $F \cap W = \mathrm{RLP}(C)$.

The last four conditions can be summarized as: $(C \cap W, F)$ and $C, F \cap W)$ are *weak factorization systems*. We say an object X is *cofibrant* if $\emptyset \to X$ is a cofibration, and *fibrant* if $X \to *$ is a fibration.

Remark 7.1.2. From any model category $\mathbf{C}$ we can extract an ∞-category as the localization $\mathbf{C}[W^{-1}]$. (If $\mathbf{C}$ is a simplicial model category there is also a more explicit construction of this ∞-category using the coherent nerve.) Many interesting ∞-categories arise from model categories in this way. From the ∞-categorical point of view, a model structure is a useful tool for carrying out various computations and constructions in the associated ∞-category. For example, it can be shown that homotopy colimits in a model category describe colimits in the corresponding ∞-category, which can be a helpful way of computing certain specific examples of such colimits. A fairly reasonable analogy is that choosing a model structure on an ∞-category is like choosing coordinates on a manifold — if you're doing an abstract construction or definition, you don't want to use coordinates, but they can be unavoidable when doing computations.

Remark 7.1.3. The *homotopy category* of a model category $\mathbf{C}$ is $h\mathbf{C}[W^{-1}]$. A key result about model categories is that there is a much simpler description of this category: it is equivalent to the category $\mathrm{ho}\mathbf{C}$ with objects the fibrant-cofibrant objects of $\mathbf{C}$ and with $\mathrm{Hom}_{\mathrm{ho}\mathbf{C}}(x, y)$ the quotient of $\mathrm{Hom}_{\mathbf{C}}(x, y)$ by the equivalence relation of (left or right) homotopy between maps.

Remark 7.1.4. A more refined version of this construction (due to Dwyer–Kan[?]) gives an explicit description of the spaces of maps in the ∞-category $\mathbf{C}[W^{-1}]$: If $X^\bullet$ is a Reedy cofibrant cosimplicial object weakly equivalent to the constant cosimplicial object x, then the simplicial set $\mathrm{Hom}_{\mathbf{C}}(X^\bullet, y)$ is a Kan complex with the homotopy type of $\mathrm{Map}_{\mathbf{C}[W^{-1}]}(x,y)$. The set $\mathrm{Hom}_{\mathrm{ho}\mathbf{C}}(x,y)$ can be identified with $\pi_0\mathrm{Hom}_{\mathbf{C}}(X^\bullet, y)$.

7.2 The Joyal Model Structure

Theorem 7.2.1 (Joyal)**.** *There is a model structure on* Set_Δ *where the cofibrations are the monomorphisms and the fibrant objects are the quasicategories. Moreover, the weak equivalences are the weak categorical equivalences and the fibrations between fibrant objects are the isofibrations.*

Remark 7.2.2. A model structure is uniquely determined by its fibrant objects and cofibrations, so this model structure is unique.

Definition 7.2.3. A map of simplicial sets is a *categorical fibration* if it has the right lifting property for all monomorphisms that are weak categorical equivalences.

Let's write C for the set of monomorphisms, W for the set of weak categorical equivalences, and F for the set of categorical fibrations. To prove that the Joyal model structure exists, we must show that $(C \cap W, F)$ and $(C, W \cap F)$ are weak factorization systems. We start with the easy case:

Proposition 7.2.4. *A morphism of simplicial sets* $p\colon X \to Y$ *is a categorical fibration and a weak categorical equivalence if and only if it is a trivial fibration.*

Proof. Clearly a trivial fibration is necessarily a categorical fibration. Moreover, if p is a trivial fibration we can choose a section $s\colon Y \to X$ and a map $X \times E^1 \to X$ restricting to sp and id_X. These induce the data of a categorical equivalence for $p^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ for every quasicategory $\mathcal{C}$, so p is a categorical equivalence.

Now suppose p is a categorical fibration and a weak categorical equivalence. We can factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where j is a monomorphism and q is a trivial fibration. Then j is a weak categorical equivalence by the 2-of-3 property, so there exists a lift f in the square

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "j"'] & X \arrow[d, "p"] \\
Z \arrow[r, "q"'] \arrow[ur, dashed, "f"] & Y.
\end{tikzcd}$$

This gives a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "p"] & Z \arrow[r, "f"] \arrow[d, "q"] & X \arrow[d, "p"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y,
\end{tikzcd}$$

i.e. p is a retract of q. But then p is also a trivial fibration. $\square$

Corollary 7.2.5. *$(C, W \cap F)$ is a weak factorization system.*

Proof. By Proposition 7.2.4 this is just the weak factorization system of monomorphisms and trivial fibrations, which we already know exists (by the small object argument). ☐

To show that $(C \cap W, F)$ is a weak factorization system, we want to find a set S such that $C \cap W = \bar{S}$. Unfortunately there is no (known) way to find a nice description of such a set S, but we can formally show it exists.

Proposition 7.2.6. *There exists a functor $F\colon \mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Fun}([1], \mathrm{Set})$ such that:*

(i) $F(f)$ is a monomorphism for every morphism f.

(ii) $F(f)$ is a bijection if and only if f is a weak categorical equivalence.

(iii) F commutes with κ-filtered colimits for some regular cardinal κ.

(iv) F takes κ-small simplicial sets to κ-small sets.

Proof. Using the small object argument we can choose a functorial factorization of maps as an inner anodyne map followed by an inner fibration. This gives in particular a functor $Q\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ and a natural transformation $\eta\colon \mathrm{id} \to Q$ such that for every X, the map $\eta_X\colon X \to QX$ is inner anodyne and QX is a quasicategory. This induces a functor $Q\colon \mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Fun}([1], \mathrm{Set}_\Delta)$ such that for every f, the map Qf is a map of quasicategories that is a categorical equivalence if and only if f is a weak categorical equivalence.

Next, let $\Pi\colon \mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Fun}([1], \mathrm{Set}_\Delta)$ be the functor that assigns to $f\colon X \to Y$ the path fibration $P(f) \to Y$. Then $\Pi Q(f)$ is an isofibration of quasicategories for every f, and is a trivial fibration if and only if f is a weak categorical equivalence.

Now let $G\colon \mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Fun}([1], \mathrm{Set})$ be the functor

$$X \to Y \quad \mapsto \quad \coprod_n \mathrm{Hom}(\Delta^n, X) \to \coprod_n \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} \mathrm{Hom}(\Delta^n, Y).$$

Then $G\Pi Q(f)$ is surjective if and only if f is a weak categorical equivalence.

Finally, let $H\colon \mathrm{Fun}([1], \mathrm{Set}) \to \mathrm{Fun}([1], \mathrm{Set})$ be the functor taking $f\colon S \to T$ to $\mathrm{im}(f) \to T$. Then $H(f)$ is a monomorphism for every f, and f is surjective if and only if $H(f)$ is a bijection. We then get that $F := HG\Pi Q$ satisfies (i) and (ii).

(iii) and (iv) follow from the construction (using the innards of the small object argument for Q) and basic properties of κ-filtered colimits and κ-compact objects — we can take κ to be the smallest uncountable cardinal. ☐

Lemma 7.2.7. *Suppose $f\colon X \hookrightarrow Y$ is a monomorphism and a weak categorical equivalence. Then every κ-small subcomplex $A \subseteq Y$ is contained in a κ-small subcomplex B such that $B \cap X \hookrightarrow B$ is a weak categorical equivalence.*

Proof. For $A \subseteq Y$, let $f_A\colon A \cap X \hookrightarrow A$ denote the restriction of f. The simplicial set Y is a κ-filtered colimit of its κ-small subcomplexes, so

$$F(f) \cong \operatorname*{colim}_{\substack{A \subseteq Y \\ \kappa\text{-small}}} F(f_A).$$

If $F(f)$ is an isomorphism, then for any κ-small $A \subseteq Y$ there must exist a κ-small A' containing A such that a lift exists in the square

$$\begin{tikzcd} F_0(f_A) \arrow[r] \arrow[d] & F_0(f_{A'}) \arrow[d] \\ F_1(f_A) \arrow[r] \arrow[ur, dashed] & F_1(f_{A'}), \end{tikzcd}$$

— since $F_1(f_A)$ is κ-small, any lift $F_1(f_A) \to F_0(f)$ must factor through a finite stage in the colimit.

Iterating this, we obtain transfinite-inductively a sequence A_i indexed by $i < \kappa$ (taking colimits at limit ordinals). Take $B := \operatorname{colim}_{i<\kappa} A_i$; since κ is regular, this is a κ-small colimit, hence B is a κ-small simplicial set. Moreover, $F(f_B)$ is an isomorphism: the sections $F_1(f_{A_i}) \to F_0(f_{A_{i+1}})$ combine to an inverse in the colimit. Thus f_B is a categorical equivalence, as required. □

Proposition 7.2.8. *Let S be a set of representatives of isomorphism classes of monomorphisms $i\colon A \hookrightarrow B$ such that i is a weak categorical equivalence and B is κ-small. Then $\bar{S} = C \cap W$.*

Proof. We first observe that $W \cap C$ is indeed a saturated class. By Corollary 6.4.3 a monomorphism $j\colon K \hookrightarrow L$ is a weak categorical equivalence if and only if the map $j^*\colon \operatorname{Fun}(L, \mathcal{C}) \to \operatorname{Fun}(K, \mathcal{C})$ is a trivial fibration for all quasicategories $\mathcal{C}$. Thus j is a weak categorical equivalence if and only if $C \subseteq \operatorname{LLP}(\{\mathcal{C} \to \Delta^0\}^{\square i})$, or equivalently $i \in \operatorname{LLP}(\{\mathcal{C} \to \Delta^0\}^{\square C})$, where this is a saturated class. Thus $W \cap C$, which is the intersection of this class with the monomorphisms, is also saturated.

Since clearly $S \subseteq W \cap C$, it remains to show that $W \cap C \subseteq \bar{S}$. Consider a map $i\colon X \hookrightarrow Y$ in $W \cap C$. Let $\mathbf{P}$ be the partially ordered set of subsets K, $X \subseteq K \subseteq Y$, such that $X \to K$ is in $\bar{S}$. Since $\bar{S}$ is saturated, we have that every totally ordered subset of $\mathbf{P}$ has a maximal element, hence by Zorn's Lemma the partially ordered set $\mathbf{P}$ has a maximal element M; by the 2-of-3 property, the inclusion $X \hookrightarrow M$ is a weak categorical equivalence.

Suppose $M \neq Y$. Then there exists a κ-small subcomplex $A \subseteq Y$ not contained in M. By Lemma 7.2.7, there exists a κ-small subset B with $A \subseteq B \subseteq Y$, such that $B \cap X \hookrightarrow B$ is a weak categorical equivalence; then this lies in S by definition. Then $M \hookrightarrow M \cup B$ is cobase change of j, and so lies in $\bar{S}$, hence so does the composite $X \hookrightarrow M \cup B$. But then $M \cup B$ is contained in $\mathbf{P}$ and is strictly larger than M, which is a contradiction. Thus we must have $M = Y$, i.e. $i \in \bar{S}$. □

Now Corollary 2.7.5 immediately implies:

Corollary 7.2.9. *$(W \cap C, F)$ is a weak factorization system.*

Proof of Theorem 7.2.1. It remains only to observe that the isofibrations are the categorical fibrations between quasicategories by Proposition 6.4.4. □

Remark 7.2.10. By a very similar argument (which is originally due to Cisinski), you can show that there exists a model structure on Set_Δ such that

- the cofibraitons are the monomorphisms,
- the fibrant objects are the Kan complexes,
- the weak equivalences are the weak homotopy equivalences,
- the fibrations between fibrant objects are the Kan fibrations.

By uniqueness, this must be the standard Kan–Quillen model structure. However, it is less formal to show that the trivial fibrations are precisely the anodyne maps, or equivalently that the fibrations between arbitrary simplicial sets are still the Kan fibrations.

Remark 7.2.11. The homotopy category of the Joyal model structure can be identified with $h\mathrm{QCat}$. [TO DO: Prove this.]

Definition 7.2.12. A *Cartesian* model category is a model category $\mathbf{C}$ such that the category $\mathbf{C}$ is Cartesian closed (i.e. for every object $c \in \mathbf{C}$ the functor $c \times -$ has a right adjoint $\mathbf{Hom}(c, -)$) and if $i\colon A \to B$ and $j\colon K \to L$ are cofibrations, then

$$i \square j\colon A \times L \amalg_{A \times K} B \times B \to B \times L$$

is a cofibration, and is a weak equivalence if either i or j is one.

Remark 7.2.13. This condition on $i \square j$ says precisely that $- \times -$ is a *left Quillen bifunctor*. It is equivalent to: for every cofibration $i\colon A \to B$ and every fibration $p\colon X \to Y$, the map

$$p^{\square i}\colon \mathbf{Hom}(B, X) \to \mathbf{Hom}(B, Y) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A, X)$$

is a fibration, and a weak equivalence if either i or p is one.

Proposition 7.2.14. *The Joyal model structure is Cartesian.*

Proof. We consider monomorphisms i and j. Then $i \square j$ is obviously a monomorphism. If i is a weak categorical equivalence, then we want to show that $\mathrm{Fun}(i \square j, \mathcal{C})$ is a categorical equivalence for every quasicategory $\mathcal{C}$. We know that this map is an isofibration by Corollary 5.6.2, so it is a categorical equivalence if and only if it is a trivial fibration. Now if k is another monomorphism, we have $k \in \mathrm{LLP}(\mathrm{Fun}(i \square j, \mathcal{C}))$ if and only if $k \square j \in \mathrm{LLP}(\mathrm{Fun}(i, \mathcal{C}))$. The map $\mathrm{Fun}(i, \mathcal{C})$ is a trivial fibration, since i is a weak categorical equivalence, so we conclude that $\mathrm{Fun}(i \square j, \mathcal{C})$ has the right lifting property for all monomorphisms, i.e. is a trivial fibration. □

Remark 7.2.15. If $\mathbf{C}$ is a Cartesian model category, then $\mathbf{Hom}(X, -)$ gives a *derived* right adjoint to $X \times -$. In other words, (after appropriate (co)fibrant replacements) we get an adjunction between the induced functors on $h\mathbf{C}[W^{-1}]$. (Moreover, with quite a bit more work it can be shown that this extends to an adjunction between the induced functors on $\mathbf{C}[W^{-1}]$.) Thus, we see that $\mathrm{Fun}(K, \mathcal{C})$ for $\mathcal{C}$ a quasicategory really *does* represent the ∞-category of functors, in the sense that it's right adjoint to the Cartesian product in a derived sense. Note that this *fails* for simplicial categories — they do *not* form a Cartesian model category.

Chapter 8

Straightening and the Yoneda Lemma

8.1 Fibrations in Ordinary Category Theory

Given a functor $F\colon \mathbf{C} \to \mathrm{Cat}$, we can define a new category $\mathrm{Gr}(F)$ as follows:

- the objects of $\mathrm{Gr}(F)$ are pairs (c, x) with $c \in \mathbf{C}$ and $x \in F(c)$,
- a morphism $(c, x) \to (c', x')$ is a morphism $f\colon c \to c'$ in $\mathbf{C}$ and a morphism $\phi\colon F(f)(x) \to x'$ in $F(c')$.
- the composite of $(c \xrightarrow{f} c', F(f)(x) \xrightarrow{\phi} x')$ with $(c' \xrightarrow{f'} c'', F(f')(x') \xrightarrow{\phi'} x'')$ is $(c \xrightarrow{f'f} c'', F(f'f)(x) = F(f')(F(f)(x)) \xrightarrow{F(f')(\phi)} F(f')(x') \xrightarrow{\phi'} x'')$.

There is an obvious projection functor $\mathrm{Gr}(F) \to \mathbf{C}$ taking (c, x) to c.

Now we might ask the question: Can we go back? Or, more or less equivalently, can we characterize the functors $\mathbf{E} \to \mathbf{C}$ that arise in this way from functors $\mathbf{C} \to \mathrm{Cat}$? The answers to both questions lie in Grothendieck's theory of *fibrations* of categories.

Definition 8.1.1. Given a functor $p\colon \mathbf{E} \to \mathbf{B}$, we say that a morphism $f\colon x \to y$ in $\mathbf{E}$ is *p-coCartesian* if for all $e \in \mathbf{E}$ lying over $b \in \mathbf{B}$, the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{E}}(y, e) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{E}}(x, e) \arrow[d] \\
\mathrm{Hom}_{\mathbf{B}}(py, b) \arrow[r] & \mathrm{Hom}_{\mathbf{B}}(px, b)
\end{tikzcd}$$

is Cartesian. In other words, given $g\colon x \to e$ such that $p(g) = h \circ p(f)$ for some h, there exists a unique $\bar{h}\colon y \to e$ such that $g = \bar{h} \circ f$.

Definition 8.1.2. A functor $p\colon \mathbf{E} \to \mathbf{B}$ is a *Grothendieck opfibration* if for every $e \in \mathbf{E}$ and every morphism $f\colon p(e) \to b$, there exists a coCartesian morphism $\bar{f}\colon e \to e'$ with $p(\bar{f}) = f$.

Remark 8.1.3. There are also dual notions of *Cartesian morphisms* and *Grothendieck fibrations*; the quick definition is that p is a Grothendieck fibration if and only if p^{op} is a Grothendieck opfibration.

Warning 8.1.4. This definition of Grothendieck opfibrations is *not* invariant under equivalences of categories. (For example, with this definition a Grothendieck opfibration is necessarily an isofibration.) We can fix this by modifying the definition slightly, by instead requiring that for $e \in \mathbf{E}$ and

$f\colon p(e) \to b$ there exists a coCartesian morphism $\bar{f}\colon e \to e'$ and an isomorphism $\phi\colon b \xrightarrow{\sim} p(e')$ such that $p(\bar{f}) = \phi \circ f$. Functors satisfying this are known as *Street opfibrations.*

If $p\colon \mathbf{E} \to \mathbf{B}$ is a Grothendieck opfibration, and we choose[1] for every $e \in \mathbf{E}$ and $f\colon p(e) \to b$ a coCartesian morphism $e \to f_! e$, then we can almost define a functor from $\mathbf{B}$ to Cat:

- We send $b \in \mathbf{B}$ to the fibre $\mathbf{E}_b$ of p at b (since p is an isofibration we're allowed to take the strict fibre),
- For $f\colon b \to b'$ the functor $F(f)\colon \mathbf{E}_b \to \mathbf{E}_{b'}$ takes e to the target $f_! e$ of the chosen coCartesian morphism. A morphism $e \to e'$ is taken to the unique morphism $f_! e \to f_! e'$ factoring the composite $e \to e' \to f_! e'$ through $e \to f_! e$.

The problem is that we don't necessarily get $F(g) \circ F(f) = F(gf)$, since we don't know that the coCartesian morphisms $e \to f_! e \to g_!(f_! e)$ and $e \to (gf)_! e$ are *equal.* However, they are canonically isomorphism, so we get a natural isomorphism between $F(g) \circ F(f)$ and $F(gf)$. This, and further coherence data for triple composites, makes F a *pseudofunctor* $\mathbf{B} \to \mathrm{Cat}$.

A pseudofunctor is really the natural notion of functor to a 2-category, since it's morally dubious to ask for functors to be equal rather than naturally isomorphic, so we should be happy with this result.[2]

Theorem 8.1.5 (Grothendieck)**.** *The constructions we have described give an equivalence of 2-categories between Grothendieck opfibrations over* $\mathbf{B}$ *(and functors that preserve coCartesian morphisms) and pseudofunctors* $\mathbf{B} \to \mathrm{Cat}$*.*

Exercise 8.1.6. The fibres of a Grothendieck opfibration $p\colon \mathbf{E} \to \mathbf{B}$ are groupoids if and only if *all* morphisms in $\mathbf{E}$ are p-coCartesian.

8.2 From Left Fibrations to Functors

The equivalence of opfibrations and functors is a useful result in the case of 1-categories, as fibrations can be more convenient to work with the functors to Cat. For instance, one can often get a much cleaner definition of a fibration than of the associated functor, with the latter requiring making some arbitrary (albeit essentially unique) choices. Though this is perhaps a mere aesthetic advantage, the analogous result for ∞-categories is of paramount importance: It is essentially impossible to "write down" functors of ∞-categories $\mathcal{B} \to \mathrm{Cat}_\infty$, while it is often possible to define fibrations that we can then "straighten" to get the functor we want. Here we will consider this in the simplest case of left fibrations, which are the analogue of opfibrations whose fibres are groupoids.

To justify this assertion, observe that since $\Lambda^n_0 \hookrightarrow \Delta^n$ can be written as $\Delta^0 \star \partial\Delta^{n-1} \to \Delta^0 \star \Delta^{n-1}$, if $p\colon \mathcal{E} \to \mathcal{B}$ is a left fibration then $\mathcal{E}_{/e} \to \mathcal{B}_{/p(b)}$ is a trivial fibration for all $e \in \mathcal{E}$ by the join-slice adjunction of §4.2. Thus, for

[1] Such a choice is known as a *cleavage* of the fibration. I'm sure there's a bad joke to be made here.

[2] It is in fact possible to strictify a pseudofunctor to a strict 2-category to a strict functor. You really ought not to want to do it though...

$e \in \mathcal{E}$ and $f\colon p(e) \to b$, the space of morphisms $e \to e'$ over f is contractible. If you stare at this long enough, you'll realize it says that every morphism is coCartesian.

This suggests that we should be able to somehow extract a functor to spaces from a left fibration. To see the pieces of this functor, we need the following observation (which we won't prove — see [Lur09, Corollary 2.1.2.7]):

Proposition 8.2.1. *If i is left anodyne and j is a monomorphism, then $i \square j$ is left anodyne.*

By Lemma 2.8.2, this implies:

Corollary 8.2.2. *If p is a left fibration and i is a monomorphism, then $p^{\square i}$ is also a left fibration. If i is left anodyne, then $p^{\square i}$ is a trivial fibration.*

Definition 8.2.3. If $p\colon X \to Y$ is a left fibration and $f\colon K \to Y$ is an arbitrary map of simplicial sets, we write $\mathrm{Map}_Y(K, X)$ for the simplicial set defined by the pullback

$$\begin{tikzcd} \mathrm{Map}_Y(K,X) \arrow[r] \arrow[d] & X^K \arrow[d, "p^K"] \\ \{f\} \arrow[r] & Y^K; \end{tikzcd}$$

since p^K is again a left fibration by Corollary 8.2.2 the simplicial set $\mathrm{Map}_Y(K, X)$ is always a Kan complex, which justifies the choice of notation.

Corollary 8.2.4. *Given $p\colon X \to Y$ a left fibration and a diagram*

$$\begin{tikzcd} K \arrow[rr, hook, "i"] \arrow[dr, "\kappa"'] & & L \arrow[dl, "\lambda"] \\ & Y & \end{tikzcd}$$

with i left anodyne, then $i^*\colon \mathrm{Map}_Y(L, X) \to \mathrm{Map}_Y(K, X)$ is a trivial fibration. If i is just a monomorphism, then i^* is a Kan fibration.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_Y(L,X) \arrow[r] \arrow[d, "i^*"] \arrow[dr, phantom, "\ulcorner", very near start] & X^L \arrow[d, "p^{\square i}"] & \\ \mathrm{Map}_Y(K,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y^L \times_{Y^K} X^K \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X^K \arrow[d] \\ \{\lambda\} \arrow[r] & Y^L \arrow[r] & Y^K. \end{tikzcd}$$

Here the composite squares in the left row and the bottom column are Cartesian, as is the lower right square. It follows that the bottom left square is also Cartesian, hence so is the top left square. If i is left anodyne then $p^{\square i}$ is a trivial fibration by Corollary 8.2.2, which means that its base change $i^*\colon \mathrm{Map}_Y(L, X) \to \mathrm{Map}_Y(K, X)$ is also a trivial fibration. Similarly, if i is a monomorphism then $p^{\square i}$ is a left fibration, hence is i^* a left fibration $\square$

In particular, given an edge $\Delta^1 \to Y$ from y to y', the inclusions $\{0\}, \{1\} \to \Delta^1$ give a diagram

$$X_y \cong \mathrm{Map}_Y(\Delta^{\{0\}}, X) \xleftarrow{\sim} \mathrm{Map}_Y(\Delta^1, X) \to \mathrm{Map}_Y(\Delta^{\{1\}}, X) \cong X_{y'},$$

where the left-hand morphism is a trivial fibration and the right-hand morphism is a left fibration. Up to choosing a section of this trivial fibration we thus get a map of Kan complexes $X_y \to X_{y'}$.

Given a 2-simplex $\sigma\colon \Delta^2 \to Y$ we similarly get a diagram

$$\begin{tikzcd}
 & & \mathrm{Map}_Y(\Delta^{\{1\}}, X) & & \\
 & \mathrm{Map}_Y(\Delta^{\{0,1\}}, X) \arrow[ur] \arrow[dl, "\sim"'] & \mathrm{Map}_Y(\Delta^2, X) \arrow[u] \arrow[l, "\sim"] \arrow[r] \arrow[d, "\sim"] \arrow[dll, "\sim"] \arrow[drr, "*"] & \mathrm{Map}_Y(\Delta^{\{1,2\}}, X) \arrow[ul, "\sim"'] \arrow[dr] & \\
\mathrm{Map}_Y(\Delta^{\{0\}}, X) & & \mathrm{Map}_Y(\Delta^{\{0,2\}}, X) \arrow[ll, "\sim"] \arrow[rr] & & \mathrm{Map}_Y(\Delta^{\{2\}}, X)
\end{tikzcd}$$

Up to inverting trivial fibrations, this gives commutative a diagram

$$\begin{tikzcd}
 & X_{\sigma(1)} \arrow[dr] & \\
X_{\sigma(0)} \arrow[ur] \arrow[rr] & & X_{\sigma(2)};
\end{tikzcd}$$

in a sense, this is precisely the data contained in the 2-simplex labelled $*$ in the diagram above.

We'd like to systematize this construction to get a functor from Y to the ∞-category $\mathcal{S}$ of spaces. For this, we use the *category of simplices* of Y.

Definition 8.2.5. Given $X \in \mathrm{Set}_\Delta$, we define a category $\mathbb{\Delta}_{/X}$ as follows:

- the objects are pairs $([n], \sigma\colon \Delta^n \to X)$,
- a morphism $([n], \sigma) \to ([m], \tau)$ is a morphism $f\colon [n] \to [m]$ in $\mathbb{\Delta}$ such that the diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "f"] \arrow[dr, "\sigma"'] & & \Delta^m \arrow[dl, "\tau"] \\
 & X &
\end{tikzcd}$$

  commutes.

Remark 8.2.6. The obvious projection $\mathbb{\Delta}_{/X} \to \mathbb{\Delta}$ is the Grothendieck fibration for X, viewed as a functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$.

Definition 8.2.7. The obvious "forgetful functor" $\mathbb{\Delta}_{/X} \to \mathrm{Set}_{\Delta/X}$ taking $([n], \Delta^n \xrightarrow{\sigma} X)$ to $\Delta^n \xrightarrow{\sigma} X$ extends to a unique colimit preserving functor $L_X\colon \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_{/X}, \mathrm{Set}) \to \mathrm{Set}_{\Delta/X}$ with right adjoint R_X given by

$$(Y \to X) \mapsto (([n], \Delta^n \xrightarrow{\sigma} X) \mapsto \mathrm{Hom}_X(\Delta^n, Y)).$$

Exercise 8.2.8. The adjunction $L_X \dashv R_X$ is an adjoint equivalence of categories.

We can "upgrade" this by adding another simplicial direction: define $\mathbb{\Delta}_{/X} \times \mathbb{\Delta} \to \mathrm{Set}_{\Delta/X}$ by

$$(([n], \sigma), [m]) \mapsto \Delta^n \times \Delta^m \to \Delta^n \xrightarrow{\sigma} X.$$

This extends to a unique colimit-preserving functor $\mathbf{L}_X : \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_{/X}, \mathrm{Set}_\Delta) \to \mathrm{Set}_{\Delta/X}$. This has a right adjoint $\mathbf{R}_X : \mathrm{Set}_{\Delta/X} \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}_{/X}, \mathrm{Set}_\Delta)$; the functor $\mathbf{R}_X(Y \to X)$ takes $([n], \sigma : \Delta^n \to X)$ to $\mathrm{Map}_X(\Delta^n, Y)$.

Warning 8.2.9. The pictures above in the case Δ^n (with $n = 1, 2$) show only a part of $\mathbf{R}_{\Delta^n}(Y \to \Delta^n)$, namely the (interesting) part corresponding to non-degenerate simplices in Δ^n. There are also (infinitely many) additional objects corresponding to degenerate simplices.

Let W_X denote the set of morphisms in $\mathbb{\Delta}_{/X}$ whose images $f : [n] \to [m]$ in $\mathbb{\Delta}$ satisfy $f(0) = 0$. If $p : Y \to X$ is a left fibration, then it follows from Corollary 8.2.4 that the functor $\mathbf{R}_X(p)$ takes the morphisms in W_X to trivial fibrations in Set_Δ.

Thus, viewed $\mathcal{S}$ as the quasicategory obtained by inverting the weak homotopy equivalences in Set_Δ, for a left fibration p the functor $\mathbf{R}_X(p)$ induces a functor of quasicategories

$$\mathbb{\Delta}^{\mathrm{op}}_{/X}[W_X^{-1}] \to \mathcal{S}.$$

We can construct a map of simplicial sets $i : \mathrm{N}\mathbb{\Delta}^{\mathrm{op}}_{/X} \to X$, called the "initial vertex map", as follows:

- On 0-simplices, i takes $\sigma : \Delta^n \to X$ to $\sigma(0) \in X_0$.
- A 1-simplex of $\mathrm{N}\mathbb{\Delta}^{\mathrm{op}}_{/X}$ corresponds to $f : [n] \to [m]$ and $\sigma : \Delta^m \to X$ (this being f viewed as a morphism $\sigma \circ f \to \sigma$); i takes this to the 1-simplex $\sigma|_{\Delta^{0,f(0)}}$ in X.
- A k-simplex of $\mathrm{N}\mathbb{\Delta}^{\mathrm{op}}_{/X}$ corresponds to a k-simplex $[n_0] \to [n_1] \to \cdots \to [n_k]$ in $\mathrm{N}\mathbb{\Delta}$ together with $\sigma : \Delta^{n_k} \to X$; if we write f_i for the composite map $[n_i] \to [n_k$ then i takes this to the k-simplex $\sigma|_{\Delta^{0,f_{k-1}(0),\ldots,f_1(0),f_0(0)}}$ in X.

This map takes edges in W_X to degenerate edges in X; if X is a quasicategory it therefore induces a morphism $\mathbb{\Delta}^{\mathrm{op}}_{/X}[W_X^{-1} \to X$.

Theorem 8.2.10 (Joyal?, Dwyer–Kan?, Stevenson [Ste15])**.** *The map*

$$\mathbb{\Delta}^{\mathrm{op}}_{/X}[W_X^{-1}] \to X$$

is a categorical equivalence.

Vague Idea of Proof. We won't prove this. The idea is to first show you can work simplex by simplex and thus reduce to checking it for $X = \Delta^n$. In this case i is a functor $\mathbb{\Delta}^{\mathrm{op}}_{/[n]} \to [n]$ and you can explicitly write down an functor $l : [n] \to \mathbb{\Delta}^{\mathrm{op}}_{/\Delta^n}$ (the unique n-simplex that doesn't have any morphisms in W_X, which is $n \to (n-1)n \to (n-2)(n-1)n \to \cdots \to 1\cdots n \to 01\cdots n$) such that $il = \mathrm{id}$, and a natural transformation $li \to \mathrm{id}$ given by a morphism in W_X for every object. After inverting W_X this because a natural equivalence, which means l is a pseudo-inverse to i. $\square$

From a left fibration $p\colon \mathcal{E} \to \mathcal{B}$ with $\mathcal{B}$ a quasicategory, we thus have

$$\mathcal{B} \xleftarrow{\sim} \Delta^{\mathrm{op}}_{/\mathcal{B}}[W_{\mathcal{B}}^{-1}] \to \mathcal{S},$$

from which we can extract the desired functor $\mathcal{B} \to \mathcal{S}$ — the "straightening" of p.

8.3 The Straightening Theorem

We would like to enhance this construction of functors from left fibrations to an equivalence of ∞-categories

$$\mathrm{Fun}(\mathcal{B}, \mathcal{S}) \simeq \mathrm{Cat}^{L}_{\infty/\mathcal{B}},$$

where $\mathrm{Cat}^{L}_{\infty/\mathcal{B}}$ is the full subcategory of $\mathrm{Cat}_{\infty/\mathcal{B}}$ spanned by the left fibrations. This equivalence is the "straightening theorem" for left fibrations, which was first proved by Lurie [Lur09]. By now there are several nicer proofs, due to Heuts–Moerdijk [HM16] and Stevenson [Ste15]. They all construct the desired equivalence on the model category level, but using different functors (which are equivalent on the ∞-category level). Here we'll just give a brief sketch of Stevenson's approach.

First, we recall some results about model categories:

Theorem 8.3.1 (Joyal)**.** *For $X \in \mathrm{Set}_{\Delta}$, there exists a (unique) model structure on $\mathrm{Set}_{\Delta/X}$ such that*

- *the cofibrations are the monomorphisms,*
- *the fibrant objects are the left fibrations over X,*
- *the fibrations between fibrant objects are the left fibrations,*
- *the weak equivalences between fibrant objects are the categorical equivalences (equivalently, these are the fibrewise homotopy equivalences).*

This is the covariant model structure.

Theorem 8.3.2. *If $\mathbf{C}$ is a category (or more generally a simplicial category), then there exists a (unique) model structure on $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_{\Delta})$ such that*

- *the fibrations are the natural transformations $\eta\colon F \to G$ such that $\eta_c\colon F(c) \to G(c)$ is a Kan fibration for all $c \in \mathbf{C}$,*
- *the weak equivalences are the natural transformations $\eta\colon F \to G$ such that $\eta_c\colon F(c) \to G(c)$ is a weak homotopy equivalence for all $c \in \mathbf{C}$.*

This is the projective model structure *(for the Kan–Quillen model structure on Set_{Δ}).*

Theorem 8.3.3. *If* $\mathbf{M}$ *is a decent*[3] *model category and* S *is a set of cofibrations in* $\mathbf{M}$*, then there exists a (unique) model structure on* $\mathbf{M}$ *where*

- *the new cofibrations are the old cofibrations,*
- *the new fibrant objects are the* S*-local old fibrant objects, meaning those that have the right lifting property for* $(\partial\Delta^n \hookrightarrow \Delta^n)\square s$ *for all* n *and* $s \in S$*,*
- *the new weak equivalences between the new fibrant objects are the old weak equivalences.*

This is the Bousfield localization *of* $\mathbf{M}$ *at* S*; we'll write* $L_S\mathbf{M}$ *for this model category.*

Example 8.3.4. The covariant model structure on $\mathrm{Set}_{\Delta/X}$ is the Bousfield localization of the slice model structure induced by the Joyal model structure with respect to the maps

$$\begin{tikzcd} \Lambda^n_0 \arrow[rr, hook] \arrow[dr] & & \Delta^n \arrow[dl] \\ & X. & \end{tikzcd}$$

Theorem 8.3.5 (Stevenson)**.**

(i) The adjunction

$$\mathbf{L}_X : \mathrm{Fun}(\triangle^{\mathrm{op}}_{/X}, \mathrm{Set}_\Delta) \rightleftarrows \mathrm{Set}_{\Delta/X} : \mathbf{R}_X$$

is a Quillen adjunction, where $\mathrm{Fun}(\triangle^{\mathrm{op}}_{/X}, \mathrm{Set}_\Delta)$ *is equipped with the projective model structure and* $\mathrm{Set}_{\Delta/X}$ *with the covariant model structure.*

(ii) Since $\mathbf{L}_X$ *takes the morphisms in (the Yoneda image of)* W_X *to left anodyne maps, there is an induced Quillen adjunction*

$$\mathbf{L}_X : L_{W_X}\mathrm{Fun}(\triangle^{\mathrm{op}}_{/X}, \mathrm{Set}_\Delta) \rightleftarrows \mathrm{Set}_{\Delta/X} : \mathbf{R}_X.$$

This is a Quillen equivalence.

To extract our desired functor of ∞-categories from this, we use the following results:

Theorem 8.3.6 (Dwyer–Kan?)**.** *If* $\mathbf{C}$ *is a small simplicial category,* W *is a set of morphisms in* $\mathbf{C}$*, and* $\mathbf{C} \to L_W\mathbf{C}$ *is a simplicial category model for the localization of* $\mathbf{C}$ *at* W*, then the induced Quillen adjunction*

$$L_W\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta) \rightleftarrows \mathrm{Fun}(L_W\mathbf{C}, \mathrm{Set}_\Delta)$$

is a Quillen equivalence.

Theorem 8.3.7. *Let* $\mathbf{C}$ *be a small (fibrant) simplicial category. The* ∞*-category associated to the projective model structure on* $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta)$ *is the functor* ∞*-category* $\mathrm{Fun}(\mathrm{N}\mathbf{C}, \mathcal{S})$ *(where* $\mathrm{N}\mathbf{C}$ *is the coherent nerve of* $\mathbf{C}$*).*

[3]Simplicial, combinatorial, and left proper is certainly enough.

Combining all these results and using the facts that

- Quillen equivalences of model categories induces equivalences of ∞-categories,
- slice model structures model slice ∞-categories,
- for X a quasicategory, the covariant model structure on $\mathrm{Set}_{\Delta/X}$ models the full subcategory $\mathrm{Cat}^{L}_{\infty/X}$ of $\mathrm{Cat}_{\infty/X}$ spanned by the left fibration[4]

we get:

Theorem 8.3.8 (Straightening for left fibrations)**.** *For $\mathcal{C}$ an ∞-category, there is an equivalence of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Cat}^{L}_{\infty/\mathcal{C}}.$$

Example 8.3.9. If X is a Kan complex, then a left fibration over X is just a Kan fibration, and the covariant model structure on $\mathrm{Set}_{\Delta/X}$ is just the slice model structure for the Kan–Quillen model structure on Set_{Δ}. So in this case the straightening theorem reduces to

$$\mathrm{Fun}(X, \mathcal{S}) \simeq \mathcal{S}_{/X}.$$

In particular, we get an equivalence

$$\mathrm{Core}(\mathcal{S}_{/X}) \simeq \mathrm{Map}(X, \mathcal{S}) \simeq \mathrm{Map}(X, \mathrm{Core}(\mathcal{S})).$$

Choose an object F in $\mathcal{S}$. Under this equivalence the maps to X whose fibres are equivalent to F correspond to the maps from X to the full subcategory of $\mathrm{Core}(\mathcal{S})$ spanned by F, which is $B\mathrm{Aut}(F)$. In other words $\mathrm{Map}(X, B\mathrm{Aut}(F))$ is equivalent to the space of maps to X with fibre F — that is, $B\mathrm{Aut}(F)$ is the classifying space for fibrations with fibre F.

8.4 Twisted Arrow ∞-Categories

Recall that the classical Yoneda Lemma says that for a category $\mathbf{C}$, the functor $y\colon \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathrm{Set})$ induced by $\mathrm{Hom}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathrm{Set}$ gives a natural isomorphism $\mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathrm{Set})}(y(c), F) \cong F(c)$ for $F \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathrm{Set})$. This is, of course, a key result in category theory, so we would like to have an analogue for ∞-categories. The first issue we have to confront is that we need some way to construct a mapping space functor $\mathrm{Map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ for an ∞-category $\mathcal{C}$. In some sense this information is contained in the projection $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$, whose fibres are the mapping spaces of $\mathcal{C}$. We can instead work with a "twisted" version of this, which gives a right fibration we can straighten:

Definition 8.4.1. If $\mathbf{C}$ is a category, then the *twisted arrow category* $\mathrm{Tw}(\mathbf{C})$ of $\mathbf{C}$ has as objects the morphisms of $\mathbf{C}$, and a morphism from $f\colon x \to y$ to

[4] This is easy to see, given the previous point, as we have a fully faithful inclusion between the simplicial categories of fibrant objects.

$f'\colon x' \to y'$ is a commutative diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \\ x' \arrow[r, "f'"] & y' . \arrow[u] \end{tikzcd}$$

There is then a projection $\mathrm{Tw}(\mathbf{C}) \to \mathbf{C} \times \mathbf{C}^{\mathrm{op}}$ that takes a morphism to its source and target.

Exercise 8.4.2. The projection $\mathrm{Tw}(\mathbf{C}) \to \mathbf{C} \times \mathbf{C}^{\mathrm{op}}$ is the Grothendieck fibration for $\mathrm{Hom}_{\mathbf{C}}$.

Definition 8.4.3. Let $\epsilon\colon \mathbb{\Delta} \to \mathbb{\Delta}$ be the functor $[n] \mapsto [n] \star [n]^{\mathrm{op}} \cong [2n+1]$. Composition with ϵ induces a functor $\epsilon^*\colon \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$. If $\mathcal{C}$ is a quasicategory we define $\mathrm{Tw}(\mathcal{C}) := \epsilon^*\mathcal{C}$. The natural inclusions $[n], [n]^{\mathrm{op}} \to \epsilon([n])$ induce a natural morphism $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Exercise 8.4.4. If $\mathbf{C}$ is a category, then $\mathrm{Tw}(\mathrm{N}\mathbf{C}) \cong \mathrm{N}\mathrm{Tw}(\mathbf{C})$.

Theorem 8.4.5 (Lurie [Lur17, Proposition 5.2.1.3])**.** *If $\mathcal{C}$ is a quasicategory, then $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a right fibration; in particular, $\mathrm{Tw}(\mathcal{C})$ is a quasicategory.*

This is proved by explicitly checking the lifting property.

For $c \in \mathcal{C}$, let $\mathrm{Tw}(\mathcal{C})_c \to \mathcal{C}$ denote the pullback of $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ along $\mathcal{C} \times \{c\} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Theorem 8.4.6 (Lurie [Lur17, Proposition 5.2.1.10])**.** *For $c \in \mathcal{C}$, there is a natural morphism $\mathcal{C}_{/c} \to \mathrm{Tw}(\mathcal{C})_c$ over $\mathcal{C}$, and this is a categorical equivalence.*

In particular, the fibre of $\mathrm{Tw}(\mathcal{C})$ at c, c' can be identified with $\mathrm{Map}_{\mathcal{C}}(c, c')$ (except we haven't prove that the fibres of slice ∞-categories are mapping spaces).

Straightening the right fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ this gives a functor $\mathrm{Map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. The values of this functor on objects are mapping spaces; more can be said, see [Lur17, Proposition 5.2.1.10].

8.5 The Yoneda Lemma

The functor $\mathrm{Map}_{\mathcal{C}}$ induces a functor $y\colon \mathcal{C} \to \mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Moreover, under the straightening equivalence $\mathcal{P}(\mathcal{C}) \simeq \mathrm{Cat}^{R}_{\infty/\mathcal{C}}$ this corresponds to a functor taking $c \in \mathcal{C}$ to the right fibration $\mathcal{C}_{/c} \to \mathcal{C}$.

We want to prove that $\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(y(c), F)$ is equivalent to $F(c)$. Equivalently, we want to prove that for a right fibration $p\colon \mathcal{E} \to \mathcal{C}$, we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}(\mathcal{C}_{/c}, \mathcal{E}) \simeq \mathcal{E}_c.$$

Lemma 8.5.1. *If i is a monomorphism, then $\Delta^0 \star i$ is left anodyne and $i \star \Delta^0$ is right anodyne.*

Proof. It is easy to see that $\Delta^0 \star i$ lies in the saturated class generated by $\Delta^0 \star \{\partial\Delta^n \hookrightarrow \Delta^n\}$ But $\Delta^0 \star (\partial\Delta^n \hookrightarrow \Delta^n)$ is isomorphic to $\Lambda^{n+1}_0 \hookrightarrow \Delta^{n+1}$ so this is in turn contained in the class of left anodyne maps. $\square$

Proposition 8.5.2. *If $\mathcal{C}$ is a quasicategory then an object $x \in \mathcal{C}$ is initial if and only if $\{x\} \hookrightarrow \mathcal{C}$ is left anodyne, and terminal if and only if this inclusion is right anodyne.*

Proof. We consider the case of terminal objects. If x is terminal, let j denote the inclusion $\{x\} \colon \mathcal{C}$. We know $j^{\triangleright} = j \star \Delta^0$ is right anodyne, by Lemma 8.5.1. We will show that j is a retract of $j^{\triangleright}$, which implies it is right anodyne since this is by definition a saturated class and so is closed under retracts. We then want to construct a lift in the diagram

$$\begin{tikzcd}
\{x\} \arrow[r] \arrow[d] \arrow[rr, bend left, equal] & \{x\}^{\triangleright} \arrow[r] \arrow[d] & \{x\} \arrow[d] \\
\mathcal{C} \arrow[r] \arrow[rr, bend right, equal] & \mathcal{C}^{\triangleright} \arrow[r, dashed] & \mathcal{C}.
\end{tikzcd}$$

This is equivalent to finding a lift in

$$\begin{tikzcd}
\mathcal{C} \amalg_{\{x\}} \{x\}^{\triangleright} \arrow[r] \arrow[d] & \mathcal{C} \\
\mathcal{C}^{\triangleright}, \arrow[ur, dashed] &
\end{tikzcd}$$

which in turn is equivalent to finding a lift in

$$\begin{tikzcd}
\{x\} \arrow[r] \arrow[d, "j"'] & \mathcal{C}_{/x} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed] & \mathcal{C}.
\end{tikzcd}$$

Since x is terminal, $\mathcal{C}_{/x} \to \mathcal{C}$ is a trivial fibration, so this lift exists.

Now suppose $j \colon \{x\} \hookrightarrow \mathcal{C}$ is right anodyne. Then as $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration, we can choose a lift in

$$\begin{tikzcd}
\mathcal{C} \amalg_{\{x\}} \{x\}^{\triangleright} \arrow[r] \arrow[d] & \mathcal{C} \\
\mathcal{C}^{\triangleright}. \arrow[ur, dashed, "s"] &
\end{tikzcd}$$

Using this lift s we can show that x is terminal: given $f \colon \partial\Delta^n \to \mathcal{C}$ with $f(n) = x$, we get $sf \colon \partial\Delta^n \to \mathcal{C}_{/x}$ taking n to id_x; this corresponds to a map $fs \colon \Lambda^{n+1}_{n+1} \cong \partial\Delta^n \star \Delta^0 \to \mathcal{C}$ that restricts to f on $\partial\Delta^n$ and takes $n \to (n+1)$ to id_x. Since id_x is an equivalence, by Corollary 5.3.2 there exists an extension of fs to Δ^{n+1}. Restricting to $\Delta^{\{0,\ldots,n\}}$ we get the desired extension of f. $\square$

This implies the Yoneda Lemma:

Corollary 8.5.3. *If $\mathcal{E} \to \mathcal{C}$ is a right fibration between quasicategories, then for all $x \in \mathcal{C}$ the map $\mathrm{Map}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}) \to \mathrm{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}) \cong \mathcal{E}_x$ is a trivial fibration.*

Proof. The inclusion $\{x\} \hookrightarrow \mathcal{C}_{/x}$ is right anodyne by Proposition 8.5.2, and so induces a trivial fibration on $\mathrm{Map}_{\mathcal{C}}(-, \mathcal{E})$ by (a dual version of) Corollary 8.2.4. $\square$

In particular, we get:

Corollary 8.5.4. *The Yoneda embedding $y\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is fully faithful.*

Proof. We have

$$\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(y(c), y(c')) \simeq \mathrm{Map}_{\mathcal{C}}(\mathcal{C}_{/c}, \mathcal{C}_{/c'}) \simeq (\mathcal{C}_{/c'})_c \simeq \mathrm{Map}_{\mathcal{C}}(c, c'). \qquad \square$$

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