Hill on Kervaire Invariant MFO 9-20-10

Note Title 9/20/2010

Properties of Ω

1) If Kervaire classes exist, their image in $\pi_* \Omega$ is nonzero

2) $\pi_{-2} \Omega = 0$

3) $\pi_{k+256} \Omega = \pi_k \Omega$

$$\Omega = \left(\vec{\Delta}^{-1} MU^{(4)}\right)^{C_8}$$

Could also consider $\left(\tilde{\Delta}^{-1} MU^{(4)}\right)^{hC_8}$, which is easier to understand. We prove the two are equivalent.

III. For a large family of $D \in \pi_* MU^{(n)}$,

$$\pi_{-2}\left(D^{-1} MU^{(n)}\right)^{C_{2^{n+1}}} = 0$$

IV. For a slightly smaller family

$$\pi_*\left(D^{-1} MU^{(n)}\right)^{hC_{2^{n+1}}} \text{ is periodic.}$$

For the same (smaller) family

$$(\ )^{hG} = (\ )^{G}.$$

Slice Spectral Sequence

Existence and $E_2 \Longrightarrow$ Gap theorem

Some basic differentials and geometry

(III) $\Longrightarrow$ periodicity + fixed point theorems

Equivariant homotopy has 2 big kinds of htpy gps. ① For a G-space X and $H \subset G$ we have $\pi_k^H X := \pi_k(X^H)$

This leads to a Mackey functor

② For a virtual rep $V = V_0 - V_1$ we have

$\pi_V(X) := [S^V, X]_G = [S^{V_0}, \Sigma^{V_1} X]_G$. This is $RO(G)$ graded homotopy

A *Mackey functor* is a diagram of abelian gps on the subgps of G.

$\pi_k^H(X) = [S^k, X]_H = \pi_k(X^H)$

$= [G_+ \wedge_H S^k, X]_G = [(G/H)_+ \wedge S^k, X]_G$

(since $\mathrm{Hom}_{\mathbb{Z}(H)}(M, N) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G \otimes_{\mathbb{Z}(H)} M, N)$)

For $H \leq K$, the map $G/H \to G/K$ induces

$$\begin{tikzcd}
{[G/K_+ \wedge S^k, X]_G} \arrow[d] \arrow[r, phantom, "="] & \pi_k X^K \arrow[d, "\text{restriction}"] \\
{[G/H_+ \wedge S^k, X]_G} \arrow[r, phantom, "="] & \pi_k X^H \arrow[u, dashed, bend left, "\text{transfer}"]
\end{tikzcd}$$

If $\sigma \in \mathrm{Aut}(G/H)$ then σ acts on $\pi_k^H(X)$

If $\underline{M}$ is a Mackey functor, there is a Eilenberg–Mac Lane spectrum $H\underline{M}$ with

$$\pi_k H\underline{M} = \begin{cases} \underline{M} & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

The constant Mackey functor $\underline{\mathbb{Z}}$

value on each subgp is $\mathbb{Z}$

restriction maps are identity

transfer is mult by index.

Warning $\pi_0 S^0 \neq \underline{\mathbb{Z}}$ tom Dieck

There is an eqvr filtration of any eqvr spectrum X that refines the ordinary Postnikov tower and is useful. It is not the eqvr Postnikov filtration.

Slice Theorem The slice associated graded for $MU^{(2^n)}$ is a wedge of

$$\bigvee_{p\in I} C_{2^{n+1}+} \wedge_H S^{\frac{|p|}{|H_p|}\rho_{H_p}} \wedge H\underline{\mathbb{Z}}$$

$$\pi_*^{\{e\}} MU^{(2^n)} = \mathbb{Z}[r_1, \gamma r_1, \gamma^2 r_1, \cdots \gamma^{2^n-1} r_1, r_2 \cdots]$$

γ = generator of $C_{2^{n+1}} = G$

G acts on monomials (up to sign) and has orbit set I. For $p\in I$, $H_p = \mathrm{Stab}(p)$.

Observe $\quad S^{k\rho_G} \wedge (G_+ \wedge_H S^{\ell\rho_H})$

$$= G_+ \wedge_H \left(S^{(\ell + k|G/H|)\rho_H} \right)$$

<u>Corollary</u> There is a filtration of $\Sigma^{k\rho_G} MU^{(2^n)}$ with associated graded has the same form as before.

$$\pi_*^G \left(G_+ \wedge_H S^{k\rho_H} \wedge H\underline{\mathbb{Z}} \right) = \pi_*^H \left(S^{k\rho_H} \wedge H\underline{\mathbb{Z}} \right) = H_*^H \left(S^{k\rho_H}, \underline{\mathbb{Z}} \right)$$

Example $\quad G = C_2 \ : \ S^{\rho_{C_2}} = S^{1+\sigma} = \mathbb{C} \qquad \sigma = \text{sign rep}$

Chain complex $\qquad\qquad\qquad\qquad S^1 \vee (C_{2+} \wedge J)^2)$

$2 \qquad \mathbb{Z}^2 = \mathbb{Z}[C_2]$

$\qquad\qquad \downarrow$ fold map

$1 \qquad \mathbb{Z}$

sign rep $\qquad$ rotation by $\pi/2$

$G = C_4 \qquad \rho_{C_4} = 1 \oplus \overline{\rho}_{C_4} = 1 + \sigma + \lambda$

By analyzing fixed pts we find

$$\underbrace{\overset{1}{\mathbb{Z}} \xleftarrow{\varepsilon} \overset{2}{\mathbb{Z}[C_2]}}_{\text{same as in } C_2\text{-case.}} \xleftarrow{1-\gamma} \overset{3}{\mathbb{Z}[C_4]} \xleftarrow{1+\gamma} \overset{4}{\mathbb{Z}[C_4]}$$

Thm For any k and $G = C_{2^n}$,

$$\underline{\pi}_{-2}^{G} S^{k\rho_G} \wedge H\underline{\mathbb{Z}} = 0$$

Pf Obvious for $k > 0$ for connectivity. For $k < 0$ we get the dual to the complex for $-k\rho_G$.

For $k = -1$ or -2 the relevant portion of the chain cx, though nontrivial, has trivial homology.

-1 or -2   -2 or -3

$$\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^2$$

$Hom(\mathbb{Z}, -)$ $\mathbb{Z} \xrightarrow{1} \cancel{\mathbb{Z}}$