THE $\mathbb{C}$-MOTIVIC ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR TOPOLOGICAL MODULAR FORMS

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ABSTRACT. We analyze the $\mathbb{C}$-motivic (and classical) Adams-Novikov spectral sequence for the $\mathbb{C}$-motivic modular forms spectrum *mmf* (and for the classical topological modular forms spectrum *tmf*). We primarily use purely algebraic techniques, with a few exceptions. Along the way, we settle a previously unresolved detail about the multiplicative structure of the homotopy groups of *tmf*.

1. INTRODUCTION

The topological modular forms spectrum *tmf* plays an essential role in the study of the stable homotopy groups of spheres [Bau08] [Beh20] [DFHH14] [Goe10] [Hop95] [Hop02] [HM14] [Rez02]. The unit map $S \to tmf$ from the sphere spectrum to *tmf* detects much of the structure of the stable homotopy groups of S, including the elements η (1-stem), ν (3-stem), ϵ (8-stem), κ (14-stem), $\overline{\kappa}$ (20-stem), and many additional elements. The unit map is far from injective (for example, σ (7-stem) maps to zero in *tmf*), so it does not detect all of the stable homotopy groups of spheres. Moreover, it is also not surjective. The computation of the *tmf*-Hurewicz image is a difficult problem that leads to the identification of infinite v_2-periodic families in the stable homotopy groups of spheres [BMQ20].

The spectrum *tmf* serves as an approximation to the sphere spectrum. This approximation is highly suitable for testing theories and for developing computational techniques. The structure of *tmf* is complicated enough to exhibit the complex phenomena related to the computation of stable homotopy groups, but it is also simple enough to be computed exhaustively. We have found that the study of *tmf* is an indispensable step along the way to understanding the sphere spectrum.

By comparison, the spectrum *ko* is arguably too simple to serve as a test case for computational theories. For example, its Adams spectral sequence collapses, so its homotopy reduces to an entirely algebraic problem. Neither the Adams nor the Adams-Novikov spectral sequence collapses for *tmf*. However, the analysis of *tmf* does not involve crossing differentials or crossing extensions in the sense of [IWX20, Section 2.1]. This means that the homotopy of *tmf* does not share the most delicate parts of the homotopy groups of spheres.

Bruner and Rognes [BR21] have recently exhaustively studied the Adams spectral sequence for *tmf*. They completely determine the additive and (primary) multiplicative structure of the stable homotopy groups of *tmf*, with one exception.

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The goal of this manuscript is to carry out the Adams-Novikov spectral sequence for *tmf*. In fact, we will work in the more general $\mathbb{C}$-motivic context and compute the motivic Adams-Novikov spectral sequence for the $\mathbb{C}$-motivic modular forms spectrum *mmf*. The classical computation is easily recovered from the motivic computation by an algebraic localization.

More specifically, there is a certain motivic element τ. Inverting τ has the effect of collapsing $\mathbb{C}$-motivic computations to classical computations. In particular, τ-torsion phenomena disappear in the classical context. Henceforth, we will work in the $\mathbb{C}$-motivic context. The interested reader can easily recover classical computations from our work by inverting τ.

From another perspective, we also compute the $\mathbb{C}$-motivic effective slice spectral sequence for *mmf*, since it agrees with the Adams-Novikov spectral sequence over $\mathbb{C}$. This identification of spectral sequences does not appear to be cleanly stated in the literature, but it is a computational consequence of the weight 0 result of [Lev15, Theorem 1].

Our goal is not merely to record the details of the Adams-Novikov spectral sequence, which have previously appeared in [Bau08]. More specifically, we have attempted to give proofs that are as algebraic as possible. Such algebraic proofs are less likely to contain subtle mistakes, and they are more easily verifiable by machine. The motivic context provides us with additional algebraic tools that are not accessible in the strictly classical context. We also correct a few oversights and minor mistakes in the analysis of [Bau08].

1.1. **Algebraic philosophy.** We do not use any information from the sphere spectrum as input for our computations. We do, however, assume full knowledge of the algebraic structure of the motivic Adams and motivic Adams-Novikov E_2-pages for *mmf*. This is consistent with our goal of using algebraic techniques whenever possible. It is also consistent with our philosophy that the role of *tmf* is to inform us about the sphere spectrum. By comparison, in [BR21] it is necessary to import the relation $\eta^2\kappa = 0$ to *tmf* from previous knowledge of the sphere spectrum. Fortunately for us, we have the relation $h_1^2 d = 0$ in the Adams-Novikov E_2-page for *mmf*. Because there are no elements in higher filtration, the relation $\eta^2\kappa = 0$ therefore has an entirely algebraic proof.

A computation involving the Adams or Adams-Novikov spectral sequence breaks into two main stages. The first stage is entirely algebraic and involves the computation of the E_2-page. In the modern era, this first stage is typically conducted by machine. The computation of the E_2-pages for *tmf* is not elementary, but it can be done manually with enough patience [Bae] [Bau08, Section 7] [BR21] [Rez02, Section 18].

The second stage of the process involves the computation of differentials and hidden extensions. This stage typically requires input from topology, so it cannot be fully automated because it is not entirely algebraic.

Our contribution is to recognize that much of this topological second stage actually can be carried out using only algebraic information. The key idea is to use the additional structure of the motivic context in order to pass back and forth between the Adams and Adams-Novikov spectral sequences. Each E_2-page tells us some things about the homotopy groups of *tmf*. The information contained in these E_2-pages does overlap, but not perfectly. The union of the information in both E_2-pages is strictly greater than the information in either one of the E_2-pages.

We give several concrete examples of information available in only one of the two E_2-pages.

(1) In the classical Adams E_2-page for *tmf*, we have the relation $h_1^4 = 0$. This implies the relation $\eta^4 = 0$ in homotopy. However, in the classical Adams-Novikov E_2-page, the element h_1^4 is non-zero and is hit by an Adams-Novikov d_3 differential.

Thus, the relation $\eta^4 = 0$ has an entirely algebraic proof, but only in the Adams spectral sequence.

(2) In fact, the relation $h_1^4 = 0$ is a consequence of the Massey product $h_1^2 = \langle h_0, h_1, h_0 \rangle$ in the Adams E_2-page. In the classical Adams-Novikov E_2-page, the corresponding Massey product $\langle 2, h_1, 2 \rangle$ is zero. Consequently, the Toda bracket $\eta^2 = \langle 2, \eta, 2 \rangle$ has an entirely algebraic proof, but only in the Adams spectral sequence.

(3) In the classical Adams-Novikov E_2-page for *tmf*, we have the relation $h_2^3 = h_1 c$. This implies the relation $\nu^3 = \eta\epsilon$. However, in the classical Adams E_2-page, we have $h_2^3 = 0$. In fact, there is a hidden ν extension from h_2^2 to $h_1 c$ in the Adams spectral sequence. Thus, the relation $\nu^3 = \eta\epsilon$ has an entirely algebraic proof, but only in the Adams-Novikov spectral sequence.

(4) In fact, the relation $h_2^3 = h_1 c$ is a consequence of the Massey product $c = \langle h_2, h_1, h_2 \rangle$ in the Adams-Novikov E_2-page. In the classical Adams E_2-page, the corresponding Massey product is zero. Consequently, the Toda bracket $\epsilon = \langle \nu, \eta, \nu \rangle$ has an entirely algebraic proof, but only in the Adams-Novikov spectral sequence. See Lemma 2.20 for more detail on this example.

In order to obtain one key Adams-Novikov differential, we use Bruner's theorem on the interaction between algebraic Steenrod operations [May70] and Adams differentials in the context of the Adams spectral sequence. We refer to [Bru84, Theorem 2.2] for a precise readable statement; see also [BMMS86] and [Mäk73]. The practical implementation of Bruner's theorem requires only algebraic information in the form of algebraic Steenrod operations on Ext groups. These operations can be computed by machine, although not as effectively as the additive and multiplicative structure of the Ext groups. The algebraic Steenrod operations are additional structure on top of what topologists usually think of as "standard homological algebra".

In the context of the Adams-Novikov spectral sequence, we also rely on the Leibniz rule in the form $d_r(x^k) = kx^{k-1}d_r(x)$. Philosophically, this formula is connected to Bruner's theorem, although we do not know how to make a precise connection. As in the case of Bruner's theorem, it feels like slightly more information than is usually considered in standard homological algebra.

We also draw attention to Proposition 4.5, in which we establish a hidden 2 extension in the 110-stem. Here we use some information about the homotopy groups of mmf/τ^2. One might argue that this information is not entirely of an algebraic nature. By comparison, the corresponding 2 extension in the Adams spectral sequence is hidden, but not particularly difficult [BR21, Theorem 9.8(110)].

1.2. **Techniques.** Section 2.10 describes a particularly powerful method for studying the $\mathbb{C}$-motivic Adams-Novikov spectral sequence in a way that has no classical analogue. There is a map $q : mmf/\tau \to \Sigma^{1,-1} mmf$ that can be viewed as projection to the top cell of the 2-cell *mmf*-module mmf/τ. The homotopy of mmf/τ is entirely understood in an algebraic sense since it is isomorphic to the classical Adams-Novikov E_2-page for *tmf*. Moreover, the map q maps onto the homotopy of *mmf* that is annihilated by τ. Thus q can be used to detect structure in *mmf* that is related to classes that are annihilated by τ.

In practice, many specific questions about hidden extensions do not directly involve elements that are annihilated by τ. Frequently, if we multiply these elements by a power of τ and a power of g, then we end up with elements that are annihilated by τ. We can use q to understand these latter elements, and finally deduce information about the original elements. Table 5 lists numerous specific examples of this process. The majority of hidden extensions can be handled very easily in this way, although a few extensions require more complicated arguments.

We avoid the use of Toda brackets whenever possible, but occasionally they are inevitable. In those cases where we must compute a Toda bracket, we once again rely exclusively on algebraic techniques. Namely, our Toda brackets arise from corresponding Massey products in either the Adams or Adams-Novikov E_2-page. The Moss Convergence Theorem [Mos70] says that such algebraic Massey products detect Toda brackets in "well-behaved" situations. In practice, all of the situations that we study are well-behaved.

1.3. **The differentials on Δ^k.** Having carried out the entire analysis of the motivic Adams-Novikov spectral sequence for *mmf*, we can see in hindsight that there are a few key steps from which all of the other miscellaneous computations follow. Our experience shows that the key steps involve the differentials on elements of the form $2^j\Delta^k$. This is not particularly surprising; we expect the element Δ to play a dominant role since it represents v_2-periodicity.

First, we establish $d_5(\Delta) = \tau^2 h_2 g$ in Proposition 3.8. This follows immediately by comparison to the Adams spectral sequence, in which $\tau^2 h_2 g$ is already zero in the E_2-page. Thus, we have an algebraic proof for $d_5(\Delta)$. Then the Leibniz rule implies that $d_5(\Delta^2) = 2\tau^2\Delta h_2 g$.

The Leibniz rule also implies that $d_5(\Delta^4) = 4\tau^2\Delta^3 h_2 g$. However, $4\tau^2\Delta^3 h_2 g$ is zero in the Adams-Novikov E_2-page. Because of the hidden 2 extension from $2\tau^2 h_2$ to $\tau^3 h_1^3$, the element $\tau^3\Delta^3 h_1^3 g$ ought to play the role of $4\tau^2\Delta^3 h_2 g$. This strongly suggests that there should be a differential $d_7(\Delta^4) = \tau^3\Delta^3 h_1^3 g$. In fact, this formula is correct (see Proposition 3.14), but it requires some work to give a precise proof.

Our solution, once again, is to play the Adams and Adams-Novikov spectral sequences against each other. We used the Adams E_2-page to obtain the Adams-Novikov differential $d_5(\Delta)$. Then we used the Leibniz rule in the Adams-Novikov spectral sequence to obtain $d_5(\Delta^2)$. In turn, this last Adams-Novikov differential implies an Adams differential $d_2(\Delta^2)$, or $d_2(w_2)$ in the notation of [BR21]. Next, we obtain an Adams differential $d_3(\Delta^4)$, or $d_3(w_2^2)$ in the notation of [BR21], by applying Bruner's theorem on the interaction between squaring operations and Adams differentials [BMMS86] [Bru84]. Finally, the Adams differential $d_3(\Delta^4)$ implies that there is an Adams-Novikov differential $d_7(\Delta^4)$. For more details, see Sections 3.3 and 3.4. Curiously, precise statements about the Adams-Novikov differential $d_7(\Delta^4)$ are missing from [Bau08] [HM14] [Rez02].

1.4. **Main results.** Our main results are expressed in the charts in Section 7. For completeness, we express this in the form of a main theorem.

Theorem 1.1. *The charts in Section 7 represent the $\mathbb{C}$-motivic Adams-Novikov spectral sequence for the motivic modular forms spectrum mmf, including complete descriptions of*

- *the E_2-page.*
- *all differentials.*
- *the E_∞-page.*
- *all hidden extensions by 2, η, and ν.*

The proof of Theorem 1.1 consists of the sum of a long list of miscellaneous computations, which are carried out throughout the manuscript. See especially the tables in Section 6. These tables summarize the main computational facts, and they give cross-references to more detailed proofs of each fact.

Our work is not as complete as [BR21] because we have not completely analyzed the multiplicative structure. In principle, this could be done using the same techniques. We do study one family of multiplicative relations in more detail. Bruner and Rognes identify a family ν_k of elements in the homotopy of *tmf*. They mostly determine the products among these elements, but they leave one case unresolved. Our techniques settle this last detail about the 2-primary multiplicative structure of the homotopy of *tmf*.

Theorem 1.2. *In the context of* [BR21], $\nu_4\nu_6 = \nu\nu_2 M$.

Theorem 1.2 is proved later as Corollary 5.12. In fact, it is a consequence of the more general Theorem 5.10, which offers a graceful simultaneous analysis of products $\nu_j\nu_k$. Bruner and Rognes empirically observed the formula

$$\nu_i\nu_j = (i+1)\nu\nu_{i+j}.$$

Our proof shows that the coefficients $(i+1)$ arise naturally from the Leibniz rule

$$d_5(\Delta^{i+1}) = (i+1)\Delta^i d_5(\Delta).$$

1.5. **Future directions.** Our work raises some questions that deserve further study.

Problem 1.1. Compute the $\overline{\kappa}$-periodic $\mathbb{C}$-motivic spectrum $mmf[\overline{\kappa}^{-1}]$.

Frequently, we detect elements and relations by first computing their products with various powers of g or $\overline{\kappa}$. In other words, much of the structure of *mmf* is reflected in the $\overline{\kappa}$-periodic spectrum $mmf[\overline{\kappa}^{-1}]$. This motivic spectrum is non-trivial, but its homotopy is entirely annihilated by τ^{11}. Consequently, its Betti realization is trivial, and it represents purely "exotic" motivic phenomena. We mention that [BBC23] also studies g-periodic phenomena in *tmf*, although not in a way that is particularly close to our perspective.

Problem 1.2. Develop better technology to deduce the differential $d_7(\Delta^4) = \tau^3\Delta^3 h_1^3 g$ directly from the differential $d_5(\Delta) = \tau^2 h_2 g$.

It is conceivable that $d_7(\Delta^4)$ could be deduced directly from $d_5(\Delta)$ using a variant of Bruner's theorem that would apply in the Adams-Novikov spectral sequence, but we have not even formulated a precise statement of such a variant. There is a connection between Bruner's theorem and the Leibniz rule $d_r(x^2) = 2xd_r(x)$, but the precise relationship is not clear to us.

Another possible approach to Problem 1.2 might involve an enriched E_2-page in which the 2 extension from $2\tau^2 h_2$ to $\tau^3 h_1^3$ is not hidden.

Problem 1.3. Construct a spectral sequence whose E_2-page reflects the algebraic structure of both the Adams and Adams-Novikov E_2-pages.

We frequently pass back and forth between the Adams and Adams-Novikov spectral sequences. In order to facilitate these transitions, Section 2.5 introduces a notion of correspondence between elements of the Adams spectral sequence and elements of the Adams-Novikov spectral sequence.

This setup feels like a preliminary attempt to describe a richer connection between the two spectral sequences. It would be much more convenient and effective to compute in just a single spectral sequence that reflects the algebraic structure of both the Adams and Adams-Novikov spectral sequences. There are some preliminary indications that "bimotivic homotopy theory" (also known as $H\mathbb{F}_2$-synthetic BP-synthetic homotopy theory) provides a context for this.

1.6. **Outline.** We begin in Section 2 with a discussion of tools that we will use to carry out our explicit computations. We describe both the motivic Adams and motivic Adams-Novikov spectral sequences for *mmf*, and we establish notation for elements in these spectral sequences. We also establish notation for certain homotopy elements that we will use later. We draw particular attention to Sections 2.9 and 2.10, which establish a powerful tool for detecting hidden extensions. The basic idea is to use the motivic spectrum mmf/τ, whose homotopy is entirely algebraic.

Our explicit computations begin in Section 3, where we establish all of the Adams-Novikov differentials. The propositions in this section are mostly in order of increasing length of differentials. However, we make some exceptions to this general rule to preserve the logical order, so each result only depends on previously proved results.

Once the Adams-Novikov differentials are computed, we proceed to compute all hidden extensions by 2, η, and ν in Section 4. Most of these extensions follow immediately by comparison to the homotopy of mmf/τ, but there are several cases with more difficult proofs.

Finally, in Section 5, we consider an explicit family of products that are particularly interesting. Our results on these products fill a gap in the product structure of $\pi_* tmf$, as described in [BR21].

1.7. **Conventions.** We work exclusively at the prime 2. There are interesting aspects to the computation of *tmf* at the prime 3 ([Bau08, Chapter 5], [DFHH14], [BR21, Chapter 13]), but we do not address that topic. We use the motivic Adams-Novikov spectral sequence to compute the homotopy groups of the 2-localization of *mmf*. We also use the E_2-page of the motivic Adams spectral sequence, which actually converges to the homotopy groups of the 2-completion of *mmf*. The distinction between localization and completion is not essential since only finitely generated abelian groups appear in our work. For expository simplicity, these localizations or completions do not appear in our notation. For example, the symbol $\mathbb{Z}$ refers to the integers localized at 2, or to the 2-adic integers. Similarly, $\pi_{*,*} mmf$ refers to the motivic stable homotopy groups of the 2-localization (or 2-completion) of *mmf*.

The adjective "motivic" always refers exclusively to the $\mathbb{C}$-motivic context. We consider no base fields other than $\mathbb{C}$.

Many of our explicit results are labelled with the degrees in which they occur. These degrees may help the reader navigate the overall computation, especially in finding the relevant elements on Adams-Novikov charts.

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2. Background

In this section, we discuss the techniques that we will use later to carry out our computations.

2.1. **The $\mathbb{C}$-motivic modular forms spectrum *mmf*.** There is a $\mathbb{C}$-motivic E_∞-ring spectrum *mmf* that can be viewed as the analogue of the classical topological modular forms spectrum *tmf* [GIKR22]. The Betti realization of *mmf* is the classical spectrum *tmf*. Moreover, the cohomology of *mmf* is $A /\!\!/ A(2)$, where A denotes the $\mathbb{C}$-motivic Steenrod algebra and $A(2)$ is the subalgebra generated by Sq^1, Sq^2, and Sq^4.

2.2. **The $\mathbb{C}$-motivic Adams spectral sequence for *mmf*.** We abbreviate the motivic Adams spectral sequence for *mmf* by mAss. The cohomology of $\mathbb{C}$-motivic $A(2)$ is the E_2-page of the mAss. The manuscript [Isa09] computes the cohomology of $\mathbb{C}$-motivic $A(2)$ using the motivic May spectral sequence, and it gives a complete description of its ring structure. The mAss E_2-page consists entirely of algebraic information, which we take as given. We grade the mAss E_2-page in the form (s, f, w), where s is the topological stem, f is the Adams filtration, and w is the motivic weight.

The motivic Adams differentials are recorded in [Isa18]. However, this manuscript does not depend on previous knowledge of any Adams differentials, neither classical nor motivic. For completeness, we provide self-contained proofs for two Adams differentials in Proposition 3.19.

We adopt the notation of [Isa09] and [Isa18] for the mAss. For the reader's convenience, Table 1 provides a concordance between our notation and the notation of [BR21]. Beware that the motivic generators u and Δu have no classical counterparts because they are annhilated by τ.

Table 1: Generators of the motivic Adams E_2-page for *mmf*

(s,f,w)	[Isa09]	[BR21]
$(0,1,0)$	h_0	h_0
$(1,1,1)$	h_1	h_1
$(3,1,2)$	h_2	h_2
$(8,3,5)$	c	c_0
$(8,4,4)$	P	w_1
$(11,3,7)$	u	
$(12,3,6)$	a or α	α
$(14,4,8)$	d	d_0
$(15,3,8)$	n or ν	β
$(17,4,10)$	e	e_0
$(20,4,12)$	g	g
$(25,5,13)$	Δh_1	γ
$(32,7,17)$	Δc	δ
$(35,7,19)$	Δu	
$(48,8,24)$	Δ^2	w_2

2.3. **The $\mathbb{C}$-motivic Adams-Novikov spectral sequence for *mmf*.** The E_2-page of the classical Adams-Novikov spectral sequence for *tmf* is given by $\mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*tmf)$, where BP denotes the Brown-Peterson spectrum. Analogously to the classical Adams-Novikov spectral sequence, one can construct a motivic Adams-Novikov spectral sequence by resolving with respect to the motivic Brown-Peterson spectrum. We abbreviate the motivic Adams-Novikov spectral sequence by mANss. We grade the mANss E_2-page in the form (s,f,w), where s is the topological stem, f is the Adams-Novikov filtration, and w is the motivic weight.

The mANss is easy to describe in classical terms. The motivic E_2-page can be obtained from its classical analogue by first assigning a third degree, called the weight, to be half of the total degree for each class, then adjoining a polynomial generator τ of degree $(0,0,-1)$ (see, e.g. [HKO11][Isa19]). More explicitly, a classical element x in degree (s,f) corresponds to a family of elements $\{\tau^n x | n \geq 0\}$ in the mANss, where the motivic element x has degree $\left(s,f,\frac{s+f}{2}\right)$.

The E_2-page of the mANss consists entirely of algebraic information, which we take as given. For our purposes, the best way to compute this E_2-page is by the algebraic Novikov spectral sequence, which is worked out in detail in [Bae].

Remark 2.1. The E_2-page of the classical Adams-Novikov spectral sequence for *tmf* is the cohomology of a version of the elliptic curve Hopf algebroid ([Rez02][Bau08]). By the change-of-rings theorem [Rez02, Theorem 15.3], this is the same as the cohomology of the

Hopf algebroid $(BP_*tmf, BP_*BP \otimes_{BP_*} BP_*tmf)$. See [Rez02, Proposition 15.7 and Section 20] for more details. We do not rely on this perspective.

2.4. **Notation for the motivic Adams-Novikov spectral sequence.** Table 2 lists the multiplicative generators for the mANss E_2-page for *mmf*. These generators are the starting point of our computation.

Table 2: Generators of the motivic Adams-Novikov E_2-page for *mmf*

(s, f, w)	generator
$(0, 0, -1)$	τ
$(1, 1, 1)$	h_1
$(3, 1, 2)$	h_2
$(5, 1, 3)$	$h_1 v_1^2$
$(8, 0, 4)$	P
$(8, 2, 5)$	c
$(12, 0, 6)$	$4a$
$(14, 2, 8)$	d
$(20, 4, 12)$	g
$(24, 0, 12)$	Δ

One must be slightly careful with the definitions of some of these elements because they belong to cyclic groups of order greater than 2. In these cases, there is more than one possible generator. Specifically, this issue arises for the elements h_2, P, $4a$, g, and Δ. For P, $4a$, and g, we simply choose arbitrary generators.

Remark 2.2. $(3, 1, 2)$ The choice of h_2 makes little practical difference to us, as long as it is a generator of the mANss E_2-page in degree $(3, 1, 2)$. For definiteness, we take h_2 to represent the homotopy element ν, assuming an a priori definition of ν (for example, by appealing to the homotopy of the sphere spectrum or by appealing to a geometric construction of ν involving quaternionic multiplication).

The choice of Δ also makes little practical difference. We choose Δ in such a way to make our formulas easier to write. See Remark 3.9 and Remark 5.8 for more details. Note that the choice of Δ depends on a previous choice of h_2.

Remark 2.3. $(12, 0, 6)$ The notation $4a$ does not appear to be natural and deserves some explanation. There are two closely related reasons why we find this notation to be convenient. First, the element $4a$ is detected in the algebraic Novikov spectral sequence [Bae] by an element $h_0^2 a$. Second, the element $2 \cdot 4a$ turns out to be a permanent cycle that detects an element in $\pi_{12,6}mmf$. This same homotopy element is detected by $h_0^3 a$ in the Adams spectral sequence for *mmf*.

The element g is a permanent cycle and therefore represents a homotopy class $\overline{\kappa}$. Multiplication by g provides regular structure to the mANss for *mmf*. We typically sort elements into families that are related by g multiplication. In other words, when we consider a particular element x, we also typically consider the elements xg^k for all $k \geq 0$ at the same time.

Taken together, Figures 1 and 3 depict the E_2-page of the mANss for *mmf* graphically. The careful reader should superimpose these figures in order to obtain a full picture of the mANss. Figure 1 depicts a regular v_1-periodic pattern in the E_2-page, to be discussed in detail in Section 2.7. Figure 3 depicts the remaining classes.

2.5. **Comparison between the mANss and the mAss.**

Definition 2.4. Let a be a permanent cycle in the mANss for mmf, and let b be a permanent cycle in the mAss for mmf. The elements a and b **correspond** if there exists a non-zero element in $\pi_{*,*}mmf$ that is detected by a in the mANss for mmf and is detected by b in the mAss for mmf.

Remark 2.5. Beware that a permanent cycle may detect more than one element in $\pi_{*,*}mmf$, depending on the presence of permanent cycles in higher filtration. We ask only that the cosets detected by a and b intersect; they need not coincide. We give an explicit example.

The element P of the mANss E_∞-page detects two elements of $\pi_{8,4}mmf$ because of the presence of τc in higher filtration. On the other hand, the element P of the mAss E_∞-page detects infinitely many elements (which differ only by a 2-adic unit factor) because of the presence of Ph_0^k in higher filtration for $k \geq 1$. This is an example of a corresponding pair of elements that do not detect precisely the same coset of homotopy elements.

Remark 2.6. It is possible that a single element of the mANss corresponds to two different elements of the mAss. For example, the element P of the mANss detects two elements of $\pi_{8,4}mmf$ because of the presence of τc in higher filtration. These two homotopy elements are detected by τc and by P in the mAss. Consequently, the mANss element P corresponds to the mAss element P, and it also corresponds to the mAss element τc. Fortunately, this kind of complication never arises for us in practice. For example, none of the correspondences listed in Table 4 exhibit this type of behavior.

Remark 2.7. The element 2 of the mANss E_∞-page detects a single element in homotopy since there are no elements in higher filtration. On the other hand, the element h_0 of the mAss E_∞-page detects infinitely many elements in homotopy, all of which differ by a 2-adic unit factor, because of the presence of h_0^k in higher filtration. Consequently, while 2 and h_0 are a corresponding pair, they do not detect the same sets of homotopy elements. Rather, the homotopy elements detected by 2 form a subset of the homotopy elements detected by h_0.

Among the corresponding pairs listed in Table 4, the same phenomenon occurs for h_2, g, Δh_1, and $4\Delta^2$. In all of these cases, the homotopy elements detected by the mANss E_∞-page element form a subset of the homotopy elements detected by the mAss E_∞-page element.

Multiplicative structure respects corresponding pairs. The following proposition establishes this principle precisely.

Proposition 2.8. *Let a and a' be elements of the mANss E_∞-page, and let b and b' be elements of the mAss E_∞-page. If a corresponds to a', b corresponds to b', and ab and $a'b'$ are non-zero; then ab corresponds to $a'b'$.*

Proof. Let a and a' detect a homotopy element α, and let b and b' detect a homotopy element β. Then ab and $a'b'$ detect the product $\alpha\beta$. ∎

Remark 2.9. The motivic Thom reduction map $BP \to H\mathbb{F}_2$ induces a map from the mANss for mmf to the mAss for mmf. This map detects some corresponding pairs but not all of them. Namely, it detects the pairs involving h_1, h_2, and g. These are the elements for which there is no filtration shift between the mANss and the mAss.

2.6. **Homotopy elements.** Table 3 lists some notation that we use for elements in the homotopy of mmf. We use the same symbols as in [BR21] for our motivic versions. Beware that some of our homotopy elements may not be exactly compatible under Betti realization with the ones in [BR21]. We discuss the details of these ambiguities in the following paragraphs.

We define elements in homotopy by specifying the elements in the mANss E_∞-page that detect them. In some cases, it is already easy to see that these detecting elements survive to the E_∞-page. For example, there are no possible targets for differentials on h_1 and h_2; nor can they be hit by differentials. Beware that we do not yet know that some of these detecting elements actually survive to the E_∞-page. This will only become apparent after our analysis of Adams-Novikov differentials.

In some cases, there are E_∞-page elements in higher filtration. When this occurs, the specified element in the E_∞-page detects more than one element in homotopy. For example, the element τh_1^3 lies in filtration higher than the filtration of h_2. Therefore, h_2 detects two distinct elements in homotopy. In Table 3, this ambiguity occurs only for ν, κ_4, and the elements of the form ν_k.

The choice of ν is of little practical signficance to us. For definiteness, we may use an a priori definition of ν, as discussed in Remark 2.2. The choices of ν_k will be discussed later in Definition 5.4. The choice of κ_4 is immaterial for our purposes, so it can be an arbitrary generator of $\pi_{110,56}$.

Table 3: Some elememts of $\pi_{*,*}mmf$

(s,w)	name	detected by
$(1,1)$	η	h_1
$(3,2)$	ν	h_2
$(8,5)$	ϵ	c
$(14,8)$	κ	d
$(20,12)$	$\bar{\kappa}$	g
$(25,13)$	η_1	Δh_1
$(27,14)$	ν_1	$2\Delta h_2$
$(51,26)$	ν_2	$\Delta^2 h_2$
$(96,48)$	D_4	$2\Delta^4$
$(99,50)$	ν_4	$\Delta^4 h_2$
$(110,56)$	κ_4	$\Delta^4 d$
$(123,62)$	ν_5	$2\Delta^5 h_2$
$(147,74)$	ν_6	$\Delta^6 h_2$
$(192,96)$	M	Δ^8

Remark 2.10. $(20,4,12)$ Bruner and Rognes choose $\overline{\kappa}$ by reference to the unit map $S \to tmf$, together with a prior choice of $\overline{\kappa}$ in $\pi_{20}S$. For our purposes, we only need that $\overline{\kappa}$ is detected by g in the mANss E_∞-page, so we may choose $\overline{\kappa}$ to be compatible with the one in [BR21].

There is a slight complication with $\overline{\kappa}$. In [Isa19] and [IWX20], the symbol $\overline{\kappa}$ is used for an element of $\pi_{20,11}S^{0,0}$ that is detected by τg in the motivic Adams spectral sequence. The point is that g does not survive the May spectral sequence, so it does not exist in the motivic Adams spectral sequence.

Here, we use $\overline{\kappa}$ for an element of $\pi_{20,12}mmf$. This element is detected by g in the Adams spectral sequence for mmf. The unit map $S^{0,0} \to mmf$ takes $\overline{\kappa}$ to $\tau\overline{\kappa}$.

Remark 2.11. Bruner and Rognes refer to the "edge homomorphism" in order to specify certain elements in $\pi_* tmf$. From the perspective of the Adams-Novikov spectral sequence, this edge homomorphism takes a particularly convenient form that can be easily described as a surjection followed by an injection. The surjection takes $\pi_* tmf$ onto its quotient by elements that are detected in strictly positive Adams-Novikov filtration. In other words, the surjection maps $\pi_* tmf$ onto the Adams-Novikov E_∞-page in filtration 0. Then the injection is the inclusion of the Adams-Novikov E_∞-page into the Adams-Novikov E_2-page in filtration 0. In other words, the edge homomorphism detects the homotopy elements that are detected in Adams-Novikov filtration 0. This description of the edge homomorphism applies equally well in the setting of $\pi_{*,*} mmf$ and the motivic Adams-Novikov spectral sequence.

The edge homomorphism depends on the choice of Δ (see Remark 3.9). Beware that our choice of Δ does not guarantee that our edge homomorphism is identical to the one discussed in [BR21]. Consequently, our definitions of the homotopy elements D_4 and M in Table 3 may not be the same as [BR21, Definition 9.22]. All possible choices of Δ differ by multiples of 2, so Δ^k is well-defined up to multiples of 2^k. Therefore, our choices of D_4 and M agree with the Bruner-Rognes definitions up to multiples of 16 and 256 respectively.

2.7. **v_1-periodicity.** Part of the mANss for *mmf* reflects v_1-periodic homotopy. The pattern of differentials in this part is similar to the Adams-Novikov differentials for *ko* (see [Bau08, page 31]). We consider this part separately and omit them from computations of higher differentials. Beware that we are not employing an intrinsic definition of v_1-periodic homotopy. Rather, we are simply observing some specific structure in the mANss for *mmf*.

In the mANss E_2-page, consider elements of the form $\tau^a h_1^b P^m (4a)^\epsilon \Delta^n$, where ϵ equals 0 or 1 and $m + \epsilon > 0$. We refer to these elements as the v_1-periodic classes.

Note that 1 and Δ^n (as well as their τ multiples and h_1 multiples) are excluded from this family of elements. The knowledgeable reader may observe that these powers of Δ satisfy an intrinsic definition of v_1-periodicity. Our family is constructed for its practical convenience, not for its intrinsic properties. The v_1-periodic elements, as we have defined them, only interact with each other through the Adams-Novikov differentials. However, the powers of Δ support Adams-Novikov differentials that take values outside of the v_1-periodic family. Consequently, we consider them in conjunction with the non-v_1-periodic elements.

Figures 1 and 2 display the v_1-periodic portions of the mANss E_2-pages and E_∞-pages respectively. Our other charts exclude the v_1-periodic family.

2.8. **The spectrum mmf/τ.** Consider the cofiber sequence

$$\Sigma^{0,-1} mmf \xrightarrow{\tau} mmf \xrightarrow{i} mmf/\tau \xrightarrow{q} \Sigma^{1,-1} mmf \tag{2.12}$$

of *mmf*-modules. The spectrum mmf/τ is a 2-cell *mmf*-module, in the sense that it is built from two copies of *mmf*. We refer to i as inclusion of the bottom cell, and we refer to q as projection to the top cell.

The mANss for mmf/τ has a particularly simple algebraic form. The E_2-page is isomorphic to the E_2-page of the classical Adams-Novikov spectral sequence for *tmf*, except that it has a third degree. However, this additional degree carries no extra information since it equals half of the total degree, i.e., the sum of the stem and the Adams-Novikov filtration.

Moreover, the mANss for mmf/τ collapses. There are no differentials, so the E_∞-page equals the E_2-page. Even better, there are no possible hidden extensions for degree reasons. Consequently, the homotopy of mmf/τ is isomorphic to the classical Adams-Novikov E_2-page for *tmf*. Therefore, we take the homotopy of mmf/τ as given since it is entirely algebraic information. The results discussed in this paragraph are *tmf* versions of the results in [Isa19, Section 6.2], which are stated for the sphere spectrum.

We use the notation of Table 2 in order to describe homotopy elements in $\pi_{*,*} mmf/\tau$. On the other hand, we need to be more careful about notation for elements in $\pi_{*,*} mmf$. We can specify elements in $\pi_{*,*} mmf$ by giving detecting elements in the mANss E_∞-page, but this only specifies homotopy elements up to higher filtration. See Section 2.6 for more discussion of choices of elements in $\pi_{*,*}mmf$.

The mAss for mmf/τ is isomorphic to the algebraic Novikov spectral sequence, for which we have complete information [Bae]. This is a *tmf* version of the results in [GWX21], which are stated for the sphere spectrum.

2.9. **Inclusion and projection.** We discuss the inclusion i and the projection q from Equation (2.12) in more detail. Many of these ideas first appeared in [Isa19, Chapter 5] in more primitive forms.

We already observed that both i and q are *mmf*-module maps. Note that the inclusion i is a ring map, but the projection q is not. They induce maps of motivic Adams-Novikov spectral sequences. These spectral sequence maps are in fact module maps over the mANss for *mmf*. Similarly, the induced maps of homotopy groups are $\pi_{*,*}mmf$-module maps.

We describe the inclusion $i : mmf \to mmf/\tau$ of the bottom cell in computational terms. If α is a homotopy element that is not a multiple of τ, then $i(\alpha)$ is an element of the mANss E_2-page that detects α. On the other hand, if α is a multiple of τ, then $i(\alpha)$ is zero. This fact is closely related to the observation that the motivic Adams-Novikov spectral sequence is the same as the τ-Bockstein spectral sequence.

Table 3 gives a number of values of i. For example, we have $i(\eta) = h_1$. In fact, we have defined the elements in the middle column of the table to have the appropriate values under i.

For later use, we describe the computational implication that $q : mmf/\tau \to \Sigma^{1,-1}mmf$ is an *mmf*-module map. Let α be an element of $\pi_{*,*}mmf$, and let x be an element of $\pi_{*,*}mmf/\tau$. The object mmf/τ is a right *mmf*-module, and

$$x \cdot \alpha = x \cdot i(\alpha),$$

where the dot on the left side represents the module action and the dot on the right side represents the multiplication of the ring spectrum mmf/τ. Then we have that

$$q(x) \cdot \alpha = q(x \cdot \alpha) = q(x \cdot i(\alpha)), \tag{2.13}$$

where the dot on the left represents multiplication in *mmf*; the dot in the center represents the *mmf*-module action on mmf/τ; and the dot on the right represents multiplication in mmf/τ.

We need a precise statement about the values of q. Our desired statement has essentially the same content as [BHS19, Theorem 9.19(1c)], which we reformulate into a form that is more convenient for us.

Proposition 2.14. *Let x be an element of the mANss E_2-page that is not divisible by τ, and suppose that there is a non-zero motivic Adams-Novikov differential $d_{2r+1}(x) = \tau^r y$. If we consider x as an element of $\pi_{*,*}mmf/\tau$, then the element $q(x)$ of $\pi_{*,*}mmf$ is detected by $-\tau^{r-1}y$ in the mANss E_∞-page.*

Proof. The proof is a chase of the right side of the diagram

$$\begin{tikzcd}
mmf/\tau \arrow[r, "\tau^r"] & mmf/\tau^{r+1} \arrow[r] & mmf/\tau^r \arrow[r, "\beta"] & mmf/\tau \\
mmf \arrow[u, "i"] \arrow[r, "\tau^r"] \arrow[d, "\tau^{r-1}"'] & mmf \arrow[u] \arrow[r] \arrow[d, "="'] & mmf/\tau^r \arrow[u, "="'] \arrow[r] \arrow[d] & mmf \arrow[u, "i"'] \arrow[d, "\tau^{r-1}"] \\
mmf \arrow[r, "\tau"'] & mmf \arrow[r, "i"'] & mmf/\tau \arrow[r, "q"'] & mmf,
\end{tikzcd}$$

in which the rows are cofiber sequences. We start with the element x in $\pi_{*,*}mmf/\tau$ in the bottom row. This element lifts to mmf/τ^r in the middle row by [BHS19, Theorem 9.19] because x survives to the E_{2r+1}-page. The map β is the "Bockstein" mentioned in [BHS19, Theorem 9.19], so we have that $\beta(x)$ equals $-y$ in the upper right corner of the diagram. Then $-y$ lifts to an element of $\pi_{*,*}mmf$ in the middle row that is detected by $-y$. Finally, multiply by τ^{r-1} to obtain $q(x)$. ∎

Remark 2.15. Proposition 2.14 requires that x supports a non-zero Adams-Novikov differential. On the other hand, suppose that x is a permanent cycle. Then x is in the image of i, and $q(x) = 0$ since the composition qi is zero.

2.10. **Hidden extensions.** We briefly review the notion of hidden extensions in spectral sequences. We adopt the following definition of hidden extensions.

Definition 2.16. [Isa19, Definition 4.1.2] Let α be an element in the target of a multiplicative spectral sequence, and suppose that α is detected by an element a in the E_∞-page of the spectral sequence. A hidden extension by α is a pair of elements b and c of the E_∞-page such that:

(1) the product $a \cdot b$ equals zero in the E_∞-page.
(2) the element b detects an element β in the target such that c detects the product $\alpha \cdot \beta$.
(3) if there exists an element β' of the target that is detected by b' such that $\alpha \cdot \beta'$ is detected by c, then the filtration of b' is less than or equal to the filtration of b.

We will use projection q to simplify our analysis of hidden extensions. We shall show that two different products in $\pi_{*,*}mmf$ are the image of the same element in $\pi_{*,*}mmf/\tau$. Therefore, they are equal.

Method 2.17. Suppose that α is not divisible by τ, so $i(\alpha) = a$, where a is an element of the mANss that detects α. Consider a possible hidden α extension from b to c in the mANss for mmf. If b and c detect classes β and γ that are annihilated by τ, then β and γ are in the image of projection q to the top cell. Let $\overline{b}$ and $\overline{c}$ be their pre-images in $\pi_{*,*}(mmf/\tau)$. Since this latter object is algebraic and completely known, we can determine whether $\overline{b}$ and $\overline{c}$ are related by an extension by mere inspection.

Equation (2.13) shows that

$$q(\overline{b} \cdot a) = q(\overline{b} \cdot i(\alpha)) = q(\overline{b}) \cdot \alpha = \beta \cdot \alpha,$$

where the first two dots represent multiplication in mmf/τ, while the last two dots represent multiplication in mmf. If $\overline{b} \cdot a$ equals $\overline{c}$, then $\beta \cdot \alpha$ equals $q(\overline{c}) = \gamma$, and there is a hidden α extension from b to c.

On the other hand, if $\overline{b} \cdot a$ equals zero, then $\beta \cdot \alpha$ equals zero, and there is not a hidden α extension from b to c.

In practice, Method 2.17 is very effective for determining hidden extensions. The main restriction is that it only applies to extensions between classes that are annihilated by τ.

Example 2.18. $(54, 2, 28)$ We illustrate Method 2.17 with a concrete example of the hidden 2 extension from $\Delta^2 h_2^2$ to $\tau^4 dg^2$ in the 54-stem. In this example, we assume some knowledge of the relevant Adams-Novikov differentials (see Section 3). Consequently, one should view this example as a deduction of a hidden extension from previously determined differentials.

First, multiply by τg. If we establish a hidden 2 extension from $\tau\Delta^2 h_2^2 g$ to $\tau^5 dg^3$ in the 74-stem, then we can immediately conclude the desired extension in the 54-stem. This step already requires motivic technology, since both $\Delta^2 h_2^2 g$ and dg^3 are hit by classical Adams-Novikov differentials.

The key point is that the two elements under consideration in the 74-stem are non-zero but annihilated by τ. They are annihilated by τ because of the differentials $d_5(\Delta^3 h_2) = \tau^2\Delta^2 h_2^2 g$ and $d_{13}(2\Delta^3 h_2) = \tau^6 dg^3$, to be proved later in Propositions 3.8 and 3.16.

The elements $\tau\Delta^2 h_2^2 g$ and $\tau^5 dg^3$ represent classes in $\pi_{74,39}mmf$ that are annihilated by τ. Therefore, these elements lie in the image of $q : \pi_{75,38}mmf/\tau \to \pi_{74,39}mmf$.

By Proposition 2.14, the preimages in $\pi_{75,38}mmf/\tau$ are $\Delta^3 h_2$ and $2\Delta^3 h_2$ respectively. These two elements are connected by a 2 extension. Therefore, their images under q are also connected by a 2 extension.

2.11. **Toda brackets.** For background on Massey products and Toda brackets, including statements of the May convergence theorem and the Moss convergence theorem, we refer readers to [Tod62], [May69], [Mos70] and also [Isa19], [BK21].

Massey products in the E_2-page of an Adams or Adams-Novikov spectral sequence are algebraic information since they are part of the structure of Ext groups. Some Toda brackets in homotopy can be deduced directly from these Massey products using the Moss convergence theorem. In order to apply this theorem, one must establish the absence of crossing differentials. Whenever we apply the Moss convergence theorem, there will be no possible crossing differentials. In other words, the crossing differentials condition is satisfied for algebraic reasons. Thus, the Toda brackets that we use are algebraic in the sense that they can be deduced directly from the algebraic structure of Ext.

Remark 2.19. In general, Massey products and Toda brackets are defined as sets, not elements. An equality of the form $\langle \alpha, \beta, \gamma \rangle = \delta$ means that

(1) δ is contained in the bracket;
(2) the bracket has zero indeterminacy.

The following lemma gives an explicit example of an algebraic deduction of a Toda bracket. See Table 3 for an explanation of the notation.

Lemma 2.20. $(8, 3, 5)$ *The Toda bracket $\langle \nu, \eta, \nu \rangle$ in $\pi_{8,5}mmf$ is detected by c and has no indeterminacy.*

Proof. The proof follows several steps:

(1) Establish the Massey product $c = \langle h_2, h_1, h_2 \rangle$ in the E_2-page of the mANss.
(2) Check that there are no crossing differentials.
(3) Check that the Toda bracket $\langle \nu, \eta, \nu \rangle$ is well-defined and that it has no indeterminacy.
(4) Apply the Moss convergence theorem to the Massey product and deduce the desired Toda bracket.

For step (1), we check the following statements:

(a) The Massey product is well-defined because of the relation $h_1 h_2 = 0$ in the E_2-page of the mANss for *mmf* (see Figure 3).
(b) The element c is contained in the Massey product [Bau08, Equation (7.3)] [Bae].
(c) The indeterminacy is trivial by inspection. In more detail, the indeterminacy equals $h_2 \cdot E_2^{5,1,3}$. The only non-zero element of $E_2^{5,1,3}$ is $h_1 v_1^2$, and $h_2 \cdot h_1 v_1^2 = 0$. This last relation holds already in the E_2-page of the motivic algebraic Novikov spectral sequence [Bae].

For step (2), we need to check for crossing differentials for the relation $h_1 h_2$ in degree $(4, 2, 3)$. We are looking for non-zero Adams-Novikov differentials in degrees $(5, f, 3)$, where $f < 1$. There are no possible sources for such differentials (see Figure 3).

For step (3), we check that the Toda bracket is well-defined because $\eta\nu$ is zero in $\pi_{4,3}mmf$ for degree reasons. The indeterminacy equals $\nu \cdot \pi_{5,3}mmf$, which is zero for degree reasons.

For step (4), we apply the Moss convergence theorem. The theorem implies that there exists an element in $\langle h_2, h_1, h_2 \rangle$ that is a permanent cycle and that detects an element in $\langle \nu, \eta, \nu \rangle$. Since there are no indeterminacies for both the Massey product and the Toda bracket, the permanent cycle must be c. ∎

3. Differentials

In this section, we compute all differentials in the mANss for *mmf*, proving hidden extensions and Toda brackets only as needed along the way. Our results are presented in logical order, so each proof only depends on earlier results. We return to a more exhaustive study of hidden extensions later in Section 4.

Theorem 3.1. *Table 6 lists all of the non-zero differentials on all of the indecomposable elements of each mANss E_r-page.*

Proof. The differentials are proved in the various propositions later in this section. The last column of Table 6 indicates the specific proposition that proves each differential.

Some indecomposables do not support differentials. In most cases, this follows for degree reasons, i.e., because there are no possible targets. Proposition 3.30 handles two slightly more difficult cases. ∎

All differentials follow from straightforward applications of the Leibniz rule to the ones listed in Table 6.

3.1. d_3 differentials.

Proposition 3.2. $(5,1,3)$ $d_3(h_1 v_1^2) = \tau h_1^4$.

Proof. In the mAss E_2-page, h_1^4 is a non-zero element that is annihilated by τ. By inspection, h_1^4 corresponds to the element of the same name in the mANss. Therefore, τh_1^4 must be hit by an Adams-Novikov differential, and there is only one possibility. ∎

Proposition 3.3. $(12,0,6)$ $d_3(4a) = \tau P h_1^3$.

Proof. For degree reasons, $d_3(P) = 0$. Thus Proposition 3.2 implies that $d_3(P \cdot h_1 v_1^2) = \tau P h_1^4$. We have the relation $P \cdot h_1 v_1^2 = h_1 \cdot 4a$ in the Adams-Novikov E_2-page. Note that this relation arises from a hidden h_1 extension from $h_0^2 a$ to $\overline{Ph_1^4}$ in the algebraic Novikov spectral sequence [Bae]. Therefore, $4a$ must also support a d_3 differential, and there is only one possibility. ∎

The Leibniz rule, combined with Proposition 3.2 and Proposition 3.3, implies some additional d_3 differentials. By inspection, the other multiplicative generators do not support d_3 differentials.

Remark 3.4. All of the d_3 differentials are h_1-periodic, in the sense that they can be computed in the localization of the mANss E_2-page in which h_1 is inverted. This localized spectral sequence computes the homotopy of the η-periodic spectrum $mmf[\eta^{-1}]$. See [GI15, Section 6.1] for a related discussion.

3.2. Corresponding pairs.

Earlier in Section 2.5, we discussed the notion of elements from the mANss and from the mAss that correspond. Having computed the d_3 differentials, we are now in a position to establish a number of corresponding pairs that will be used in later arguments.

Theorem 3.5. *Table 4 lists some pairs of elements that correspond.*

Table 4: Some corresponding elements in the motivic Adams and motivic Adams-Novikov spectral sequences

mANss degree	mANss element	mAss element	mAss degree
$(0,0,0)$	2	h_0	$(0,1,0)$
$(1,1,1)$	h_1	h_1	$(1,1,1)$
$(3,1,2)$	h_2	h_2	$(3,1,2)$
$(14,2,8)$	d	d	$(14,4,8)$
$(20,4,12)$	g	g	$(20,4,12)$
$(25,1,13)$	Δh_1	Δh_1	$(25,5,13)$
$(27,1,14)$	$2\Delta h_2$	an	$(27,6,14)$
$(48,0,24)$	$4\Delta^2$	$\Delta^2 h_0^2$	$(48,10,24)$
$(110,2,56)$	$\Delta^4 d$	$\Delta^4 d$	$(110,20,56)$

Proof. We discuss the correspondence between $2\Delta h_2$ and an in detail. Most of the other corresponding pairs are established with essentially the same argument. Some slightly more difficult cases are established later in Lemmas 3.10 and 3.34.

For degree reasons, the element $2\Delta h_2$ of the mANss for mmf cannot support an Adams-Novikov differential, nor can it be hit by an Adams-Novikov differential. (Beware that Δh_2 does support a differential.) Therefore, $2\Delta h_2$ detects some element α in $\pi_{27,14}mmf$.

The inclusion $i : mmf \to mmf/\tau$ induces a map

(3.6)
$$\begin{tikzcd} E_2(mmf) \arrow[r] \arrow[d, Rightarrow] & E_2(mmf/\tau) \arrow[d, Rightarrow] \\ \pi_{*,*}mmf \arrow[r] & \pi_{*,*}mmf/\tau \end{tikzcd}$$

of motivic Adams spectral sequences. The spectral sequence on the right is identified with the algebraic Novikov spectral sequence that converges to the classical Adams-Novikov E_2-page for *tmf* [GWX21].

The element α in the lower left corner maps to $2\Delta h_2$ in the lower right corner. This latter element is detected by an in filtration 6 in the upper right corner [Bae]. Therefore, α is detected in the upper left corner in filtration at most 6. The only possible value is an. ∎

Remark 3.7. Previous knowledge of the d_3 differentials is required in order to conclude that $2\Delta h_2$ (and other elements as well) does not support an Adams-Novikov differential. For example, it is conceivable that $d_{25}(2\Delta h_2) = \tau^{12}h_1^{26}$. However, we already know that $\tau^{12}h_1^{26}$ is hit by the differential $d_3(\tau^{11}h_1^{22} \cdot h_1 v_1^2)$.

3.3. d_5 **differentials.** Having determined all d_3 differentials, one can mechanically compute the E_4-page. Through the 22-stem, no additional differentials are possible for degree reasons, so the E_4-page equals the E_∞-page in that range.

Proposition 3.8. $(24,0,12)$ *There exists a generator Δ of the mANss E_2-page in degree* $(24,0,12)$ *such that* $d_5(\Delta) = \tau^2 h_2 g$.

Proof. The mAss element $h_2 g$ is annihilated by τ^2 in the E_2-page. Moreover, $\tau h_2 g$ does not support a hidden τ extension in the mAss because of the presence of $\overline{\tau h_2 g}$ in the homotopy of mmf/τ. More precisely, projection to the top cell takes $\overline{\tau h_2 g}$ to $\tau h_2 g$, so $\tau h_2 g$ must detect homotopy elements that are annihilated by τ.

The mANss element $h_2 g$ corresponds to the mAss element $h_2 g$ because of Table 4 and Proposition 2.8. Therefore, $\tau^2 h_2 g$ must be hit by an Adams-Novikov differential. The only possibility is a d_5 differential whose source is in degree $(24, 0, 12)$. Since $\tau^2 h_2 g$ is not a multiple of 2, the source of the differential must be a generator. ■

Remark 3.9. $(24, 0, 12)$ Proposition 3.8 does not uniquely specify Δ. Since $4\tau^2 h_2 g$ is zero in the mANss E_2-page, Δ is only well-defined up to multiples of 4. Later in Remark 5.8 we will make a further refinement in the definition of Δ. Also note that the choice of Δ depends on a previous choice of h_2, as in Remark 2.2.

The Leibniz rule, together with Proposition 3.8, implies additional d_5 differentials. The other multiplicative generators of the E_5-page do not support differentials.

Of particular note is the differential

$$d_5(\Delta^2) = 2\Delta d_5(\Delta) = 2\tau^2 \Delta h_2 g.$$

This easy computation is an Adams-Novikov version of Bruner's theorem on the interaction between Adams differentials and algebraic squaring operations [BMMS86] [Bru84]. However, its corresponding Adams differential $d_2(\Delta^2) = \tau^2 ang$ is not as easy to obtain by direct analysis of the Adams spectral sequence [BR21]. The difficulty is that Δ^2 is not the value of an algebraic squaring operation since Δ is not present in the Adams E_2-page. By "postponing" the differential that hits $\tau^2 h_2 g$ from algebra to topology, we obtain an easier argument for the differential on Δ^2.

Lemma 3.10. $(48, 0, 24)$ *The element $4\Delta^2$ of the mANss for mmf corresponds to $\Delta^2 h_0^2$ in the mAss for mmf.*

Proof. Having established that $d_5(\Delta^2) = 2\tau^2 \Delta h_2 g$ as a consequence of the Leibniz rule and Proposition 3.8, we conclude that $4\Delta^2$ does not support an Adams-Novikov differential for degree reasons. (Beware that $2\Delta^2$ does support a differential, but we do not need to know that already.) Note that $4\Delta^2$ is detected in the algebraic Novikov spectral sequence by $\Delta^2 h_0^2$, which has filtration 10. Using the argument in the proof of Theorem 3.5, we conclude that $4\Delta^2$ corresponds to an element in the mAss with filtration at most 10. However, there are three possibilities: Δ^2, $\Delta^2 h_0$, and $\Delta^2 h_0^2$.

The top horizontal map of Diagram (3.6) takes Δ^2 and $\Delta^2 h_0$ to elements of the same name. These elements detect Δ^2 and $2\Delta^2$ in the Adams-Novikov E_2-page. This means that $4\Delta^2$ cannot correspond to Δ^2 or $\Delta^2 h_0$. ■

3.4. d_7 **differentials.** The main goal of this section is to establish some d_7 differentials in Proposition 3.14 and Proposition 3.21. In order to obtain these differentials, we will need some hidden extensions and some later differentials. We establish these other results first, in order to preserve strict logical order.

Lemma 3.11. $(3, 1, 2)$ *There is a hidden* 2 *extension from* $2h_2$ *to* τh_1^3.

Proof. According to Table 4 and Proposition 2.8, the mANss element $2h_2$ corresponds to the mAss element $h_0 h_2$. The element $h_0 h_2$ supports an h_0 extension in the mAss E_2-page, so $2h_2$ must support a 2 extension in the mANss. There is only one possible target for this extension. ■

Remark 3.12. The hidden extension of Lemma 3.11 is the first in an infinite family of similar hidden extensions from the elements $2h_2 g^k$ to the elements $\tau h_1^3 g^k$. For $k \geq 1$, these extensions are "exotic" in the sense that they do not occur classically, since both $2h_2 g^k$ and $h_1^3 g^k$ are the targets of classical Adams-Novikov differentials.

Lemma 3.13. $(27, 1, 14)$ *There is a hidden* 2 *extension from* $2\Delta h_2$ *to* $\tau \Delta h_1^3$.

Proof. We already observed in Table 4 that $2\Delta h_2$ and $\Delta h_1 \cdot h_1^2$ correspond to an and Δh_1^3 in the mAss. In the mAss E_2-page, we have the relation $h_0 \cdot an = \tau \Delta h_1^3$. Therefore, there must be a hidden 2 extension between the corresponding Adams-Novikov elements. ■

Proposition 3.14.

(1) $(24, 0, 12)$ $d_7(4\Delta) = \tau^3 h_1^3 g$.
(2) $(48, 0, 24)$ $d_7(2\Delta^2) = \tau^3 \Delta h_1^3 g$.

Proof. Proposition 3.8 says that $\tau^2 h_2 g$ is hit by an Adams-Novikov differential, so $2\tau^2 h_2 g$ is also hit by an Adams-Novikov differential. Remark 3.12 says that there is a hidden 2 extension from $2h_2 g$ to $\tau h_1^3 g$. Therefore, $\tau^3 h_1^3 g$ is hit by a differential, and there is just one possible source for this differential.

The proof for the second differential is essentially the same. We need a hidden 2 extension from $2\Delta h_2 g$ to $\tau \Delta h_1^3 g$, which follows from Lemma 3.13 and multiplication by g. ■

Remark 3.15. Proposition 3.8 and Proposition 3.14 show that both $2\tau h_2 g^k$ and $\tau^2 h_1^3 g^k$ are annihilated by τ. In hindsight, we can see that the hidden 2 extensions connecting them are examples of Method 2.17. Their pre-images in mmf/τ are $2\Delta g^{k-1}$ and $4\Delta g^{k-1}$, which are related by 2 extensions.

However, beware that we needed the hidden 2 extension from $2h_2$ to τh_1^3 in order to establish the differential $d_7(4\Delta)$. An independent proof of Lemma 3.11 is necessary in order to avoid a circular argument.

Before finishing the analysis of the d_7 differential in Proposition 3.21, we deduce some higher differentials.

Proposition 3.16. $(75, 1, 38)$ $d_{13}(2\Delta^3 h_2) = \tau^6 d g^3$.

Proof. We have the relation $ang \cdot an = \tau^4 d g^3$ in the mAss E_2-page because of the relations $a^2 n = \tau d \cdot \Delta h_1$ and $\Delta h_1 \cdot n = \tau^3 g^2$ [Isa09, Theorem 4.13]. According to Table 4 and Proposition 2.8, the mANss elements $2\Delta h_2 g$, $2\Delta h_2$, d, and g correspond to the mAss elements ang, an, d, and g. This means that there is a hidden $2\Delta h_2$ extension from $2\Delta h_2 g$ to $\tau^4 d g^3$ in the mANss.

Using the Leibniz rule and Proposition 3.8, we already know that $2\tau^2 \Delta h_2 g$ is hit by the differential $d_5(\Delta^2)$. Therefore, $\tau^6 d g^3$ must also be hit by a differential. There are two possibilities for this differential: $d_{11}(\tau \Delta^3 h_1^3)$ and $d_{13}(\Delta^3 h_2)$. However, $\tau \Delta^3 h_1^3$ is a product $\tau(\Delta h_1)^3$ of permanent cycles, so it cannot support a differential. ■

Remark 3.17. The proof of Proposition 3.16 contains an example of Method 2.17. There is a hidden $2\Delta h_2$ extension from $2\tau \Delta h_2 g$ to $\tau^5 d g^3$. Both of these elements are annihilated by τ. Their pre-images under projection to the top cell of mmf/τ are Δ^2 and $2\Delta^3 h_2$ respectively, which are related by a $2\Delta h_2$ extension.

Proposition 3.18. $(56, 2, 29)$ $d_9(\Delta^2 c) = \tau^4 h_1 d g^2$.

Proof. Recall from Example 2.18 that there is a hidden 2 extension from $\Delta^2 h_2^2$ to $\tau^4 d g^2$. The argument for this hidden extension uses Proposition 3.8 and Proposition 3.16. Therefore, $\tau^4 h_1 d g^2$ must be hit by a differential because $2h_1 = 0$. There is only one possible differential. ■

Proposition 3.19. *In the mAss for mmf, we have the Adams differentials:*

(1) $(48, 8, 24)$ $d_2(\Delta^2) = \tau^2 ang$.
(2) $(96, 16, 48)$ $d_3(\Delta^4) = \tau^8 n g^4$.

Proof. We start with the Adams-Novikov differential $d_5(\Delta^2) = 2\tau^2\Delta h_2 g$. We know from Table 4 and Proposition 2.8 that $2\Delta h_2 g$ corresponds to the element ang in the mAss. Therefore, $\tau^2 ang$ must be hit by some Adams differential, and the only possibility is that $d_2(\Delta^2)$ equals $\tau^2 ang$.

Next, we apply Bruner's theorem on the interaction between Adams differentials and algebraic squaring operations. We refer to [BR21, Theorem 5.6] for a precise readable statement, although [Bru84], [BMMS86] and [Mäk73] are preceding references. We apply Bruner's theorem with $x = \Delta^2$, $r = 2$, and $i = 8$; so $s = 8$, $t = 56$, $v = v(48) = 1$, and $\overline{a} = h_0$. We obtain that

$$d_* \operatorname{Sq}^8(\Delta^2) = \operatorname{Sq}^9 d_2(\Delta^2) \dotplus h_0 \cdot \Delta^2 \cdot d_2(\Delta^2) = \operatorname{Sq}^9(\tau^2 ang) + h_0 \cdot \Delta^2 \cdot \tau^2 ang = \operatorname{Sq}^9(\tau^2 ang).$$

Next, we compute that $\operatorname{Sq}^9(\tau^2 ang) = \tau^4 \cdot \tau\Delta h_1 \cdot n^2 \cdot g^2$, using the Cartan formula for algebraic squaring operations, as well as the formulas $\operatorname{Sq}^2(a) = \tau\Delta h_1$, $\operatorname{Sq}^3(n) = n^2$, and $\operatorname{Sq}^4(g) = g^2$ [BR21, Theorem 1.20]. Finally, apply the relation $\Delta h_1 \cdot n = \tau^3 g^2$ to obtain the Adams differential $d_3(\Delta^4) = \tau^8 ng^4$. ∎

Remark 3.20. The careful reader may object to our use of a motivic version of Bruner's theorem in the proof of Proposition 3.19, while only the classical version of the theorem has a published proof. In fact, this concern is irrelevant here. One can use the classical Bruner's theorem to establish the classical Adams d_3 differential and then deduce the motivic version of the differential.

Proposition 3.21. $(96, 0, 48)$ $d_7(\Delta^4) = \tau^3\Delta^3 h_1^3 g$.

Proof. Table 4 shows that the mANss element $4\Delta^2$ corresponds to the mAss element $\Delta^2 h_0^2$. Therefore, Proposition 2.8 shows that the mANss element $16\Delta^4$ corresponds to the mAss element $\Delta^4 h_0^4$.

Proposition 3.19 shows that Δ^4 does not survive the mAss. Therefore, $\Delta^4 h_0^4$ does not detect homotopy elements that are divisible by 16. Consequently, the corresponding element $16\Delta^4$ in the mANss does not detect homotopy elements that are divisible by 16. This means that Δ^4 must support an Adams-Novikov differential.

There are two possible values for this differential: $\tau^3\Delta^3 h_1^3 g$ and $\tau^9 h_1 dg^4$. However, Proposition 3.18 shows that the latter element is already hit by the differential $d_9(\tau^5\Delta^2 cg^2) = \tau^9 h_1 dg^4$. ∎

3.5. d_9 **differentials.** At this point, we have determined all differentials d_r for $r \leq 7$. It remains to study higher differentials, although some higher differentials have already been determined in earlier propositions. We continue to proceed roughly in order of increasing values of r, although we occasionally need some Toda brackets, hidden extensions, and later differentials to preserve strict logical order.

Proposition 3.22. $(171, 1, 86)$ $d_{13}(2\Delta^7 h_2) = \tau^6\Delta^4 dg^3$.

Proof. The argument is nearly identical to the proof of Proposition 3.16. The mAss E_2-page relation $\Delta^4 ang \cdot an = \tau^4\Delta^4 dg^3$ implies that there is a hidden $2\Delta h_2$ extension from $2\Delta^5 h_2 g$ to $\tau^4\Delta^4 dg^3$ in the mANss. We already know that $2\tau^2\Delta^5 h_2 g$ is hit by the differential $d_5(\Delta^6)$. Therefore, $\tau^6\Delta^4 dg^3$ must also be hit by a differential.

There are two possibilities for this differential: $d_{11}(\tau\Delta^7 h_1^3)$ and $d_{13}(2\Delta^7 h_2)$. The former possibility is ruled out by the decomposition $\tau\Delta^6 h_1^2 \cdot \Delta h_1$ and the observation that both $\Delta^6 h_1^2$ and Δh_1 survive past the E_{11}-page for degree reasons. ∎

Lemma 3.23. $(150, 2, 76)$ *There is a hidden* 2 *extension from* $\Delta^6 h_2^2$ *to* $\tau^4\Delta^4 dg^2$.

Proof. The proof is similar to the argument in Example 2.18. We already know the differentials $d_5(\Delta^7 h_2) = \tau^2\Delta^6 h_2^2 g$ and $d_{13}(2\Delta^7 h_2) = \tau^6\Delta^4 dg^3$ from Propositions 3.8 and 3.22. Therefore, projection to the top cell detects a hidden 2 extension from $\tau\Delta^6 h_2^2 g$ to $\tau^5\Delta^4 dg^3$. Finally, use τg multiplication to deduce the hidden 2 extension on $\Delta^6 h_2^2$. ■

Proposition 3.24.

(1) $(80, 2, 41)$ $d_9(\Delta^3 c) = \tau^4\Delta h_1 dg^2$.
(2) $(176, 2, 89)$ $d_9(\Delta^7 c) = \tau^4\Delta^5 h_1 dg^2$.

Proof. We saw in Example 2.18 that $\tau^4 dg^2$ detects a multiple of 2. Therefore, $\Delta h_1 \cdot \tau^4 dg^2$ must detect zero since Δh_1 does not support a 2 extension for degree reasons. Therefore, $\tau^4\Delta h_1 dg^2$ must be hit by a differential, and there is only one possibility.

The argument for the second differential is nearly identical. Lemma 3.23 shows that the element $\tau^4\Delta^4 dg^2$ detects a multiple of 2. Therefore, $\Delta h_1 \cdot \tau^4\Delta^4 dg^2$ must detect zero, and there is only one differential that can hit it. ■

Proposition 3.25. $(152, 2, 77)$ $d_9(\Delta^6 c) = \tau^4\Delta^4 h_1 dg^2$.

Proof. The argument is similar to the proof of Proposition 3.18. Lemma 3.23 shows that $\tau^4\Delta^4 dg^2$ detects a multiple of 2. Therefore, $\tau^4\Delta^4 h_1 dg^2$ must be hit by a differential because $2h_1 = 0$. There is only one possible differential. ■

Lemma 3.26. $(25, 1, 13)$ *The Toda bracket $\langle \eta, \nu, \tau^2\overline{\kappa}\rangle$ is detected by Δh_1 and has indeterminacy detected by $P^3 h_1$.*

Proof. By inspection, the Toda bracket is well-defined and has indeterminacy detected by $P^3 h_1$ (which is a v_1-periodic element).

We use the Moss convergence theorem in the mAss for *mmf*. By [Isa09, Definition 4.4(1)], we have the Massey product $\Delta h_1 = \langle h_1, h_2, \tau^2 g\rangle$ in the E_2-page of the mAss for *mmf*. There are no possible crossing differentials in the mAss for *mmf*.

Finally, Table 4 implies that the mAss elements h_1, h_2, and $\tau^2 g$ detect η, ν, and $\tau^2\overline{\kappa}$ respectively (see also Table 3). ■

Lemma 3.27. $(25, 1, 13)$ *There is a hidden ν extension from Δh_1 to $\tau^2 cg$.*

Proof. Lemmas 2.20 and 3.26 show that the Toda brackets $\langle \nu, \eta, \nu\rangle$ and $\langle \eta, \nu, \tau^2\overline{\kappa}\rangle$ are detected by c and Δh_1 respectively.

The hidden ν extension follows from the shuffling relation

$$\nu\langle \eta, \nu, \tau^2\overline{\kappa}\rangle = \langle \nu, \eta, \nu\rangle\tau^2\overline{\kappa}.$$

■

Lemma 3.28. $(25, 1, 13)$ *There is a hidden η extension from $2\Delta h_2$ to $\tau^2 cg$.*

Proof. As in the proof of Lemma 3.27, the element $\tau^2 cg$ detects $\langle \eta, \nu, \tau^2\overline{\kappa}\rangle\nu$, which equals $\eta\langle \nu, \tau^2\overline{\kappa}, \nu\rangle$. Therefore, $\tau^2 cg$ is the target of a hidden η extension. There are two possible sources for such an extension: $\tau\Delta h_1^3$ and $2\Delta h_2$. The former possibility is ruled out by Lemma 3.13, which shows that $\tau\Delta h_1^3$ is the target of a hidden 2 extension. ■

Proposition 3.29.

(1) $(49, 1, 25)$ $d_9(\Delta^2 h_1) = \tau^4 cg^2$.
(2) $(73, 1, 37)$ $d_9(\Delta^3 h_1) = \tau^4\Delta cg^2$.
(3) $(145, 1, 73)$ $d_9(\Delta^6 h_1) = \tau^4\Delta^4 cg^2$.
(4) $(169, 1, 85)$ $d_9(\Delta^7 h_1) = \tau^4\Delta^5 cg^2$.

Proof. It follows from Lemma 3.28 that there is a hidden η extension from $2\Delta h_2 g$ to $\tau^2 cg^2$. Proposition 3.8 and the Leibniz rule imply that $d_5(\Delta^2) = 2\tau^2\Delta h_2 g$. Therefore, $\tau^4 cg^2$ must be hit by some differential, and there is only one possibility.

Having established the first differential, we can compute that

$$d_9(\Delta^3 h_1^2) = \Delta h_1 \cdot d_9(\Delta^2 h_1) = \tau^4 \Delta h_1 cg^2.$$

Since $\Delta^3 h_1^2 = \Delta^3 h_1 \cdot h_1$, it follows that $d_9(\Delta^3 h_1)$ equals $\tau^4 \Delta cg^2$.

Similarly,

$$d_9(\Delta^7 h_1^2) = \Delta^5 h_1 \cdot d_9(\Delta^2 h_1) = \tau^4 \Delta^5 h_1 cg^2,$$

from which it follows that $d_9(\Delta^7 h_1)$ equals $\tau^4\Delta^5 cg^2$. However, we need to observe that $d_9(\Delta^5 h_1)$ is zero. The only possible non-zero value for $d_9(\Delta^5 h_1)$ is $\tau^4\Delta^3 cg^2$, but this is ruled out by the observation that $\tau^4\Delta^3 cg^2$ supports a d_9 differential by Proposition 3.24.

Finally, note that $d_9(\Delta^7 h_1^2) = \Delta h_1 \cdot d_9(\Delta^6 h_1)$. The value of $d_9(\Delta^7 h_1^2)$ was computed in the previous paragraph. It follows that $d_9(\Delta^6 h_1)$ equals $\tau^4\Delta^4 cg^2$. ■

Proposition 3.30.

(1) $d_9(\Delta^4 c) = 0$.
(2) $d_9(\Delta^5 c) = 0$.

Proof. It follows from Proposition 3.29 that $\tau^4\Delta^4 cg^2$ and $\tau^4\Delta^5 cg^2$ are targets of d_9 differentials, so they cannot support d_9 differentials. This implies that $\Delta^4 c$ and $\Delta^5 c$ cannot support d_9 differentials. ■

The Leibniz rule, together with the differentials given in Propositions 3.24, 3.25, 3.29, and 3.30, determines all d_9 differentials.

3.6. d_{11} **differentials.**

Lemma 3.31. $(14, 2, 8)$ *There is a hidden ϵ extension from d to $\tau h_1^2 g$.*

Proof. We will show that there is a hidden ϵ extension from $h_1 d$ to $\tau h_1^3 g$. The desired extension follows immediately.

The relation $h_1 c = h_2^3$ in the mANss E_2-page implies that $\eta\epsilon$ equals ν^3. Also, the relation $h_2^2 d = 4g$ implies that $\nu^2\kappa = 4\overline{\kappa}$. Then

$$\eta\epsilon\kappa = \nu^3\kappa = 4\nu\overline{\kappa} = \tau\eta^3\overline{\kappa}.$$

The last equality uses the hidden 2 extension from $2h_2$ to τh_1^3, as shown in Lemma 3.11. ■

Lemma 3.32. $(39, 3, 21)$ *There is a hidden ν extension from $\Delta h_1 d$ to $\tau^3 h_1^2 g^2$.*

Proof. The element $\Delta h_1 d$ detects the product $\eta_1 \cdot \kappa$. Lemma 3.27 implies that $\nu \cdot \eta_1 \cdot \kappa$ equals $\tau^2\epsilon\kappa\overline{\kappa}$. Lemma 3.31 implies that this last product equals $\tau^3\eta^2\overline{\kappa}^2$, which is detected by $\tau^3 h_1^2 g^2$. ■

Proposition 3.33.

(1) $(62, 2, 32)$ $d_{11}(\Delta^2 d) = \tau^5 h_1 g^3$.
(2) $(158, 2, 80)$ $d_{11}(\Delta^6 d) = \tau^5\Delta^4 h_1 g^3$.

Proof. The element $\tau^5 h_1^2 g^3$ detects $\tau^5\eta^2\overline{\kappa}^3$. Lemma 3.32 implies that $\tau^5\eta^2\overline{\kappa}^3$ equals $\tau^2\nu\overline{\kappa} \cdot \eta_1 \cdot \kappa$. Because of Proposition 3.8, we know that $\tau^2\nu\overline{\kappa}$ is zero. Therefore, $\tau^5 h_1^2 g^3$ is hit by some differential. The only possibility is that $d_{11}(\Delta^2 h_1 d) = \tau^5 h_1^2 g^3$. It follows that $d_{11}(\Delta^2 d) = \tau^5 h_1 g^3$.

For the second formula, multiply by the permanent cycle $\Delta^4 h_1$ to see that $d_{11}(\Delta^6 h_1 d)$ equals $\tau^5\Delta^4 h_1^2 g^3$. It follows that $d_{11}(\Delta^6 d)$ equals $\tau^5\Delta^4 h_1 g^3$. ■

3.7. d_{13} **differentials.** We have already established some d_{13} differentials in Propositions 3.16 and 3.22 because we needed those results in order to compute shorter differentials. We now finish the computation of the d_{13} differentials.

Lemma 3.34. $(110, 2, 56)$ *The element $\Delta^4 d$ of the mANss for mmf corresponds to the element of the same name in the mAss for mmf.*

Proof. We have already analyzed all possible Adams-Novikov differentials of length 11 or less, and there are no other possible values for a differential on $\Delta^4 d$. Therefore, $\Delta^4 d$ is a permanent cycle in the mANss for *mmf*.

Now the argument given in the proof of Theorem 3.5 applies. The mANss element $\Delta^4 d$ is detected in filtration 20 in the Adams E_2-page for mmf/τ. Therefore, $\Delta^4 d$ corresponds to an element of the mAss with Adams filtration at most 20. There is only one possible element in the mAss with sufficiently low filtration. ∎

Lemma 3.35.

(1) $(39, 3, 21)$ *There is a hidden η extension from $\Delta h_1 d$ to $2\tau^2 g^2$.*
(2) $(135, 3, 69)$ *There is a hidden η extension from $\Delta^5 h_1 d$ to $2\tau^2 \Delta^4 g^2$.*

Proof. Table 4 shows that the elements Δh_1 and d in the mANss for *mmf* correspond to elements of the same name in the mAss for *mmf*. The product $\Delta h_1 \cdot h_1 d$ is non-zero in the mAss E_2-page and also in the mAss E_∞-page because there are no possible differentials that could hit it. (Note that this product is non-zero in the motivic context, but the corresponding classical product is zero in the E_2-page of the Adams spectral sequence for *tmf*.)

Therefore, $\Delta h_1 d$ must support a hidden η extension in the mANss for *mmf*. There are three possible targets for this extension: $\tau^2 g^2$, $2\tau^2 g^2$, and $3\tau^2 g^2$. The first and last possibilities are ruled out by the relation $2\eta = 0$.

The argument for the second extension is nearly identical. Table 4 and Proposition 2.8 imply that the mANss element $\Delta^5 h_1 d$ corresponds to the mAss element $\Delta^4 \cdot \Delta h_1 \cdot d$. The product $\Delta^4 \cdot \Delta h_1 \cdot h_1 d$ is non-zero in the mAss E_∞-page, so $\Delta^5 h_1 d$ must support a hidden η extension in the mANss. The only possible target for this extension is $2\tau^2 \Delta^4 g^2$. ∎

Proposition 3.36.

(1) $(81, 3, 42)$ $d_{13}(\Delta^3 h_1 c) = 2\tau^6 g^4$.
(2) $(177, 3, 90)$ $d_{13}(\Delta^7 h_1 c) = 2\tau^6 \Delta^4 g^4$.

Proof. Lemma 3.35 implies that there is a hidden η extension from $\Delta h_1 d g^2$ to $2\tau^2 g^4$. Proposition 3.24 shows that $\tau^4 \Delta h_1 d g^2$ is hit by a differential. Therefore, $2\tau^6 g^4$ must also be hit by a differential. There is only one possible source for this differential.

The proof for the second formula is similar. There is a hidden η extension from $\Delta^5 h_1 d g^2$ to $2\tau^2 \Delta^4 g^4$. Since $\tau^4 \Delta^5 h_1 d g^2$ is hit by a differential, $2\tau^6 \Delta^4 g^4$ must also be hit by a differential. ∎

3.8. d_{23} **differentials.**

Lemma 3.37. $(75, 3, 38)$ *There is a hidden η_1 extension from $\tau \Delta^3 h_1^3$ to $\tau^9 g^5$.*

Proof. According to Table 4, the mANss elements Δh_1 and g correspond to elements of the same name in the mAss. In the mAss E_2-page, the relations given in [Isa09, Theorem 4.13] imply that $\tau(\Delta h_1)^4 = \tau^9 g^5$. Therefore, in the mANss, $\tau^9 g^5$ detects the product $\tau \eta_1^4$. On the other hand, $\tau \Delta^3 h_1^3$ detects the product $\tau \eta_1^3$ in the mANss. ∎

Remark 3.38. $(75, 3, 39)$ Beware that $\Delta^3 h_1^3$ does not support a hidden η_1 extension. Rather, it supports a non-hidden extension since $\Delta^4 h_1^4$ is non-zero. However, $\Delta^4 h_1^4$ is annihilated by τ, which allows for the hidden extension on $\tau\Delta^3 h_1^3$.

Proposition 3.39. $(121, 1, 61)$ $d_{23}(\Delta^5 h_1) = \tau^{11} g^6$.

Proof. The hidden extension of Lemma 3.37 implies that there is a hidden η_1 extension from $\tau\Delta^3 h_1^3 g$ to $\tau^9 g^6$. We already know that $\tau^3\Delta^3 h_1^3 g$ is zero because of the differential $d_7(\Delta^4)$ from Proposition 3.21. Therefore, $\tau^{11} g^6$ must be the value of some differential, and there is only one possibility. ■

4. Hidden extensions

In Section 3, we established several hidden extensions in the mANss for *mmf* as steps towards computing differentials. In this section, we finish the analysis of all hidden extensions by 2, η, and ν. Our work does not completely determine the ring structure of $\pi_{*,*}mmf$ because there exist hidden extensions by other elements. Up to one minor uncertainty, the entire ring structure of $\pi_* tmf$ is determined in [BR21].

Theorem 4.1. *Up to multiples of g and Δ^8, Tables 7, 8 and 9 list all hidden extensions by 2, η, and ν in the mANss for mmf.*

Proof. Some of the non-zero hidden extensions are established in the previous results because we needed them to compute Adams-Novikov differentials. The remaining non-zero hidden extensions are proved in the following results. The last columns of the tables indicate the specific proofs for each extension.

There are some possible hidden extensions that turn out not to occur. Most of these possibilities can be ruled out using Method 2.17. For example, consider the possible hidden η extension from $\tau\Delta h_1^3$ to $\tau^2 cg$. Because of multiplication by τg, we may instead consider the possible hidden η extension from $\tau^2\Delta h_1^3 g$ to $\tau^3 cg^2$. These last two elements are annihilated by τ, so they are in the image of projection to the top cell. By inspection, there is no η extension in the homotopy of mmf/τ in the appropriate degree.

A few miscellaneous cases remain, but their proofs are straightforward. For example,

- $(65, 3, 34)$ there is no hidden 2 extension from $\Delta^2 h_2 d$ to $\tau^3\Delta h_1 g^2$ because the latter element supports an h_1 extension.
- $(24, 0, 12)$ there is no hidden ν extension from 8Δ to $\tau\Delta h_1^3$ because the first element is annihilated by g while the second element is not.

■

Proposition 4.2. *Table 5 lists some hidden extensions in the mANss for mmf.*

Table 5: Some hidden extensions deduced from Method 2.17

(s, f, w)	source	type	target	reason	
$(51, 1, 26)$	$2\Delta^2 h_2$	2	$\tau\Delta^2 h_1^3$	$d_5(2\Delta^3) = 2\tau^2\Delta^2 h_2 g$	$d_7(4\Delta^3) = \tau^3\Delta^2 h_1^3 g$
$(54, 2, 28)$	$\Delta^2 h_2^2$	2	$\tau^4 dg^2$	$d_5(\Delta^3 h_2) = \tau^2\Delta^2 h_2^2 g$	$d_{13}(2\Delta^3 h_2) = \tau^6 dg^3$
$(99, 1, 50)$	$2\Delta^4 h_2$	2	$\tau\Delta^4 h_1^3$	$d_5(2\Delta^5) = 2\tau^2\Delta^4 h_2 g$	$d_7(4\Delta^5) = \tau^3\Delta^4 h_1^3 g$
$(123, 1, 62)$	$2\Delta^5 h_2$	2	$\tau\Delta^5 h_1^3$	$d_5(\Delta^6) = 2\tau^2\Delta^5 h_2 g$	$d_7(2\Delta^6) = \tau^3\Delta^5 h_1^3 g$
$(147, 1, 74)$	$2\Delta^6 h_2$	2	$\tau\Delta^6 h_1^3$	$d_5(2\Delta^7) = 2\tau^2\Delta^6 h_2 g$	$d_7(4\Delta^7) = \tau^3\Delta^6 h_1^3 g$
$(51, 1, 26)$	$\Delta^2 h_2$	η	$\tau^2\Delta cg$	$d_5(\Delta^3) = \tau^2\Delta^2 h_2 g$	$d_9(\Delta^3 h_1) = \tau^4\Delta cg^2$
$(99, 1, 50)$	$\Delta^4 h_2$	η	$\tau^9 g^5$	$d_5(\Delta^5) = \tau^2\Delta^4 h_2 g$	$d_{23}(\Delta^5 h_1) = \tau^{11} g^6$
$(123, 1, 62)$	$2\Delta^5 h_2$	η	$\tau^2\Delta^4 cg$	$d_5(\Delta^6) = 2\tau^2\Delta^5 h_2 g$	$d_9(\Delta^6 h_1) = \tau^4\Delta^4 cg^2$

Table 5: Some hidden extensions deduced from Method 2.17

(s,f,w)	source	type	target	reason	
$(124,6,63)$	$\tau^2\Delta^4 cg$	η	$\tau^9\Delta h_1 g^5$	$d_9(\Delta^6 h_1) = \tau^4\Delta^4 cg^2$	$d_{23}(\Delta^6 h_1^2) = \tau^{11}\Delta h_1 g^6$
$(129,3,66)$	$\Delta^5 h_1 c$	η	$\tau^7\Delta^2 h_1^2 g^4$	$d_9(\Delta^7 h_1^2) = \tau^4\Delta^5 h_1 cg^2$	$d_{23}(\Delta^7 h_1^3) = \tau^{11}\Delta^2 h_1^2 g^6$
$(147,1,74)$	$\Delta^6 h_2$	η	$\tau^2\Delta^5 cg$	$d_5(\Delta^7) = \tau^2\Delta^6 h_2 g$	$d_9(\Delta^7 h_1) = \tau^4\Delta^5 cg^2$
$(161,3,82)$	$\Delta^6 h_2 d$	η	$\tau^3\Delta^5 h_1^2 g^2$	$d_5(\Delta^7 d) = \tau^2\Delta^6 h_2 dg$	$d_{11}(\Delta^7 h_1 d) = \tau^5\Delta^5 h_1^2 g^3$
$(0,0,0)$	4	ν	τh_1^3	$d_5(\Delta h_2 d) = 4\tau^2 g^2$	$d_7(4\Delta g) = \tau^3 h_1^3 g^2$
$(48,0,24)$	$4\Delta^2$	ν	$\tau\Delta^2 h_1^3$	$d_5(\Delta^3 h_2 d) = 4\tau^2\Delta^2 g^2$	$d_7(4\Delta^3 g) = \tau^3\Delta^2 h_1^3 g^2$
$(51,1,26)$	$2\Delta^2 h_2$	ν	$\tau^4 dg^2$	$d_5(2\Delta^3) = 2\tau^2\Delta^2 h_2 g$	$d_{13}(2\Delta^3 h_2) = \tau^6 dg^3$
$(57,3,30)$	$\Delta^2 h_2^3$	ν	$2\tau^4 g^3$	$d_5(\Delta^3 h_2^2) = \tau^2\Delta^2 h_2^3 g$	$d_{13}(\Delta^3 h_2^3) = 2\tau^6 g^4$
$(96,0,48)$	$4\Delta^4$	ν	$\tau\Delta^4 h_1^3$	$d_5(\Delta^5 h_2 d) = 4\tau^2\Delta^4 g^2$	$d_7(4\Delta^5 g) = \tau^3\Delta^4 h_1^3 g^2$
$(144,0,72)$	$4\Delta^6$	ν	$\tau\Delta^6 h_1^3$	$d_5(\Delta^7 h_2 d) = 4\tau^2\Delta^6 g^2$	$d_7(4\Delta^7 g) = \tau^3\Delta^6 h_1^3 g^2$
$(147,1,74)$	$2\Delta^6 h_2$	ν	$\tau^4\Delta^4 dg^2$	$d_5(2\Delta^7) = 2\tau^2\Delta^6 h_2 g$	$d_{13}(2\Delta^7 h_2) = \tau^6\Delta^4 dg^3$
$(153,3,78)$	$\Delta^6 h_2^3$	ν	$2\tau^4\Delta^4 g^3$	$d_5(\Delta^7 h_2^2) = \tau^2\Delta^6 h_2^3 g$	$d_{13}(\Delta^7 h_2^3) = 2\tau^6\Delta^4 g^4$

Proof. All of these extensions follow from Method 2.17, using the differentials in the last two columns of Table 5. To illustrate, we discuss the first extension in the table. In order to obtain the extension from $2\Delta^2 h_2$ to $\tau\Delta^2 h_1^3$, we can establish a hidden 2 extension from $2\tau\Delta^2 h_2 g$ to $\tau^2\Delta^2 h_1^3 g$. Then the desired extension follows immediately.

The elements $2\tau\Delta^2 h_2 g$ and $\tau^2\Delta^2 h_1^3 g$ are annihilated by τ in the E_∞-page of the mANss for *mmf*. Therefore, they detect elements in $\pi_{71,37}mmf$ that are in the image of $\pi_{72,36}mmf/\tau$ under projection to the top cell. By inspection, these preimages are $2\Delta^3$ and $4\Delta^3$. These latter elements are connected by a 2 extension, so their images are also connected by a 2 extension.

The other extensions have essentially the same proof. First multiply by an appropriate power of g. Then pull back to $\pi_{*,*}mmf/\tau$, where the extension is visible by inspection. ∎

Remark 4.3. $(124,6,63)$ The hidden η extension from $\tau^2\Delta^4 cg$ to $\tau^9\Delta h_1 g^5$ in Table 5 deserves further discussion. Note that $\Delta^4 cg$ and $\tau\Delta^4 cg$ support η extensions that are not hidden. However, $\tau^2\Delta^4 h_1 cg$ is zero, so $\tau^2\Delta^4 cg$ can support a hidden η extension. This explains why the E_∞-page chart in Figure 5 shows both an h_1 extension and a hidden η extension on the element $\Delta^4 cg$ in the 124-stem.

The subtleties of this situation are illuminated by consideration of homotopy elements. Let α be an element of $\pi_{124,65}mmf$ that is detected by $\Delta^4 cg$. The element $\tau^2\alpha$ is detected by $\tau^2\Delta^4 cg$. The hidden η extension implies that $\tau^2\eta\alpha$ is detected by $\tau^9\Delta h_1 g^5$.

Now let β be an element in $\pi_{122,64}$ that is detected by $\Delta^4 h_2^2 g$. Note that $\tau^2\beta$ must be zero because $\tau^2\Delta^2 h_2^2 g$ is zero and because there are no E_∞-page elements in higher filtration. Then $\nu\beta$ is detected by $h_2 \cdot \Delta^4 h_2^2 g$, which equals $\Delta^4 h_1 cg$.

Both $\eta\alpha$ and $\nu\beta$ are detected by the same element of the E_∞-page, but they are not equal. The first product is not annihilated by τ^2, while the latter product is annihilated by τ^2. In fact, the difference between $\eta\alpha$ and $\nu\beta$ is detected by $\tau^7\Delta h_1 g^5$. This phenomenon corresponds to the classical relation $\nu^2\nu_4 = \eta\epsilon_4 + \eta_1\overline{\kappa}^4$ [BR21, Proposition 9.17].

Remark 4.4. $(65,3,34)$ The chart in [Bau08] shows a hidden η extension from $\Delta^2 h_2 d$ to $\Delta h_1^2 g^2$ in the 66-stem. According to Definition 2.16, this is not a hidden extension because of the presence of $\Delta h_1 g^2$ in higher filtration.

Nevertheless, there is a relevant point here about multiplicative structure. Because of the presence of $\tau^3 \Delta h_1 g^2$ in higher filtration, the element $\Delta^2 h_2 d$ detects two homotopy elements. One of these elements is annihilated by η, and one is not. The product $\nu_2 \kappa$ is one of the two homotopy elements that are detected by $\Delta^2 h_2 d$. In fact, $\nu_2 \kappa$ is the homotopy element that is not annihilated by η. This follows from the hidden η extension from $\Delta^2 h_2$ to $\tau^2 \Delta c g$ and the hidden κ extension from $\Delta c g$ to $\tau \Delta h_1^2 g^2$.

Proposition 4.5. $(110, 2, 56)$ *There is a hidden* 2 *extension from* $\Delta^4 d$ *to* $\tau^6 \Delta^2 h_1^2 g^3$.

Proof. The proof is a variation on Method 2.17, in which we use the long exact sequence

$$\pi_{*,*} mmf \longrightarrow \pi_{*,*} mmf/\tau^2 \longrightarrow \pi_{*-1,*+2} mmf \xrightarrow{\tau^2} \pi_{*-1,*} mmf$$

induced by the cofiber sequence

$$mmf \longrightarrow mmf/\tau^2 \longrightarrow \Sigma^{1,-2} mmf \xrightarrow{\tau^2} \Sigma^{1,0} mmf.$$

We will show that there is a hidden 2 extension from $\tau^4 \Delta^4 d g^3$ to $\tau^{10} \Delta^2 h_1^2 g^6$. The desired 2 extension follows immediately by multiplication by $\tau^4 g^3$.

Recall from Proposition 3.22 that there is a differential $d_{13}(2\Delta^7 h_2) = \tau^6 \Delta^4 d g^3$. Also, it follows from Proposition 3.39 that there is a differential $d_{23}(\Delta^7 h_1^3) = \tau^{11} \Delta^2 h_1^2 g^6$.

Therefore, $\tau^4 \Delta^4 d g^3$ and $\tau^{10} \Delta^2 h_1^2 g^6$ detect elements in $\pi_{170,88} mmf$ that are annihilated by τ^2. Hence they have preimages in $\pi_{171,86} mmf/\tau^2$ under projection to the top cell. By inspection, these preimages are $2\Delta^7 h_2$ and $\tau \Delta^7 h_1^3$.

In the mANss for mmf, there is a differential $d_5(\Delta^7) = \tau^2 \Delta^6 h_2 g$. However, in the mANss for mmf/τ^2, the element $\tau^2 \Delta^6 h_2 g$ is already zero in the E_2-page. Therefore, Δ^7 is a permanent cycle in the mANss for mmf/τ^2.

Recall the hidden 2 extension from $2h_2$ to τh_1^3 established in Lemma 3.11. Multiplication by Δ^7 gives a hidden 2 extension in the mANss E_∞-page for mmf/τ^2 from $2\Delta^7 h_2$ to $\tau \Delta^7 h_1^3$.

Finally, apply projection to the top cell to obtain the hidden 2 extension from $\tau^4 \Delta^4 d g^3$ to $\tau^{10} \Delta^2 h_1^2 g^6$. ∎

Proposition 4.6. $(50, 2, 26)$ *There is a hidden* ν *extension from* $\Delta^2 h_1^2$ *to* $\tau^2 \Delta h_1 c g$.

Proof. This follows from Δh_1 multiplication on the hidden extension from Δh_1 to $\tau^2 c g$ established in Lemma 3.27. ∎

The next several lemmas establish some Toda brackets that we will use to deduce further hidden extensions. All of these Toda brackets are deduced from algebraic information, i.e., from Massey products in the mANss E_2-page.

Lemma 4.7. $(32, 2, 17)$ *The Toda bracket* $\langle \nu^2, 2, \eta_1 \rangle$ *is detected by* Δc *and has no indeterminacy.*

Proof. We have the Massey product $c = \langle h_2^2, h_0, h_1 \rangle$ in the motivic algebraic Novikov E_2-page [Bae]. The May convergence theorem [May69] [BK21, Theorem 4.16] implies that $c = \langle h_2^2, 2, h_1 \rangle$ in the mANss E_2-page. Multiply by Δ to obtain

$$\Delta c = \langle h_2^2, 2, h_1 \rangle \Delta = \langle h_2^2, 2, \Delta h_1 \rangle.$$

The second equality holds because there is no indeterminacy by inspection.

There are no crossing differentials, so the Moss convergence theorem [Mos70, Theorem 1.2] [BK21, Theorem 4.16] implies that Δc detects the Toda bracket. By inspection, the bracket has no indeterminacy. ∎

Lemma 4.8. $(128,2,65)$ *The Toda bracket* $\langle \nu_2^2, 2, \eta_1 \rangle$ *is detected by* $\Delta^5 c$ *and has no indeterminacy.*

Proof. As in the proof of Lemma 4.8, we have the Massey product $c = \langle h_2^2, 2, h_1 \rangle$ in the mANss E_2-page. Multiply by Δ^5 to obtain

$$\Delta^5 c = \Delta^4 \langle h_2^2, 2, h_1 \rangle \Delta = \langle \Delta^4 h_2^2, 2, \Delta h_1 \rangle.$$

The second equality holds because there is no indeterminacy by inspection.

There are no crossing differentials, so the Moss convergence theorem [Mos70, Theorem 1.2] [BK21, Theorem 4.16] implies that $\Delta^5 c$ detects the Toda bracket. By inspection, the bracket has no indeterminacy. ■

Lemma 4.9. $(35,7,21)$ *The Toda bracket* $\langle \nu^2, 2, \epsilon\bar{\kappa} \rangle$ *is detected by* $h_1 dg$ *and has no indeterminacy.*

Proof. We have the Massey product $h_1 dg = \langle h_2^2, h_0, cg \rangle$ in the motivic algebraic Novikov E_2-page [Bae]. The May convergence theorem [May69] [BK21, Theorem 4.16] implies that $h_1 dg = \langle h_2^2, 2, cg \rangle$ in the mANss E_2-page.

There are no crossing differentials, so the Moss convergence theorem [Mos70, Theorem 1.2] [BK21, Theorem 4.16] implies that $h_1 dg$ detects the Toda bracket. By inspection, the bracket has no indeterminacy. ■

Lemma 4.10. $(131,7,69)$ *The Toda bracket* $\langle \nu_2^2, 2, \epsilon\bar{\kappa} \rangle$ *is detected by* $\Delta^4 h_1 dg$ *and has no indeterminacy.*

Proof. As in the proof of Lemma 4.9, we have the Massey product $h_1 dg = \langle h_2^2, 2, cg \rangle$ in the mANss E_2-page. Multiply by Δ^4 to obtain

$$\Delta^4 h_1 dg = \Delta^4 \langle h_2^2, h_0, cg \rangle = \langle \Delta^4 h_2^2, h_0, cg \rangle.$$

The second equality holds because there is no indeterminacy by inspection.

There are no crossing differentials, so the Moss convergence theorem [Mos70, Theorem 1.2] [BK21, Theorem 4.16] implies that $\Delta^4 h_1 dg$ detects the Toda bracket. By inspection, the bracket has no indeterminacy. ■

Proposition 4.11. *There are hidden* ν *extensions:*

(1) $(32,2,17)$ *from* Δc *to* $\tau^2 h_1 dg$.
(2) $(128,2,65)$ *from* $\Delta^5 c$ *to* $\tau^2 \Delta^4 h_1 dg$.

Proof. Recall from Lemma 4.7 that the Toda bracket $\langle \nu^2, 2, \eta_1 \rangle$ is detected by Δc. We have

$$\langle \nu^2, 2, \eta_1 \rangle \nu = \langle \nu^2, 2, \nu \cdot \eta_1 \rangle = \langle \nu^2, 2, \tau^2 \epsilon\bar{\kappa} \rangle.$$

The first equality holds because there is no indeterminacy by inspection. The second equality follows from the hidden ν extension of Lemma 3.27. Lemma 4.9 implies that $\tau^2 h_1 dg$ detects the last Toda bracket.

The proof for the second hidden extension is nearly identical. Consider the equalities

$$\langle \nu_2^2, 2, \eta_1 \rangle \nu = \langle \nu_2^2, 2, \nu \cdot \eta_1 \rangle = \langle \nu_2^2, 2, \tau^2 \epsilon\bar{\kappa} \rangle,$$

and use Lemma 4.8 and Lemma 4.10. ■

Proposition 4.12. *There are hidden* ν *extensions:*

(1) $(97,1,49)$ *from* $\Delta^4 h_1$ *to* $\tau^9 g^5$.
(2) $(122,2,62)$ *from* $\Delta^5 h_1^2$ *to* $\tau^9 \Delta h_1 g^5$.
(3) $(147,3,75)$ *from* $\Delta^6 h_1^3$ *to* $\tau^9 \Delta^2 h_1^2 g^5$.

Proof. We prove the third hidden extension. Then the first two hidden extensions follow from multiplication by Δh_1.

Proposition 4.5 and Lemma 3.23 imply that there is a hidden 4ν extension from $\Delta^6 h_2$ to $\tau^{10}\Delta^2 h_1^2 g^5$. We also have a hidden 2 extension from $2\Delta^6 h_2$ to $\tau\Delta^6 h_1^3$, as shown in Proposition 4.2. It follows that there must be a hidden ν extension from $\Delta^6 h_1^3$ to $\tau^9\Delta^2 h_1^2 g^5$. ■

Proposition 4.13. $(110, 2, 56)$ *There is a hidden ϵ extension from $\Delta^4 d$ to $\tau\Delta^4 h_1^2 g$.*

Proof. We showed in Lemma 3.31 that there is a hidden ϵ extension from d to $\tau h_1^2 g$. Multiply by $\Delta^4 h_1$ to obtain a hidden ϵ extension from $\Delta^4 h_1 d$ to $\tau\Delta^4 h_1^2 g$. Finally, use h_1 multiplication to obtain the hidden extension on $\Delta^4 d$. ■

Proposition 4.14. $(135, 3, 69)$ *There is a hidden ν extension from $\Delta^5 h_1 d$ to $\tau^3\Delta^4 h_1^2 g^2$.*

Proof. By Lemma 3.26, the element Δh_1 detects the Toda bracket $\langle \eta, \nu, \tau^2\overline{\kappa}\rangle$. Recall from Table 3 that κ_4 is an element of $\pi_{110,56}mmf$ that is detected by the permanent cycle $\Delta^4 d$. Then the element $\Delta^5 h_1 d$ detects $\langle \eta, \nu, \tau^2\overline{\kappa}\rangle\kappa_4$. Now shuffle to obtain

$$\nu\langle \eta, \nu, \tau^2\overline{\kappa}\rangle\kappa_4 = \langle \nu, \eta, \nu\rangle\tau^2\overline{\kappa}\cdot\kappa_4.$$

Recall from Lemma 2.20 that $\epsilon = \langle \nu, \eta, \nu\rangle$. Also recall from Proposition 4.13 that there is a hidden ϵ extension from $\Delta^4 d$ to $\tau\Delta^4 h_1^2 g$. We conclude that $\epsilon\cdot\tau^2\overline{\kappa}\cdot\kappa_4$ is detected by $\tau^3\Delta^4 h_1^2 g^2$. ■

5. The elements ν_k

The multiplicative structure of classical $\pi_* tmf$ at the prime 2 has been completely computed, with one exception [BR21, p. 19]. We will use the mANss for *mmf* in order to resolve this last piece of 2-primary multiplicative structure.

As discussed in Remark 2.11, our choices of homotopy elements are not necessarily strictly compatible with the choices in [BR21]. However, our choices do agree up to multiples of certain powers of 2. Our computations below in Proposition 5.9, Theorem 5.10, Corollary 5.12, Proposition 5.13, and Proposition 5.15 lie in groups of order at most 8, so the possible discrepancies are irrelevant.

We will frequently multiply by the element $\tau\overline{\kappa}$ in $\pi_{20,11}mmf$ in order to detect elements and relations. Beware that multiplication by $\tau\overline{\kappa}$ is not injective in general. However, in all degrees that we study, multiplication by $\tau\overline{\kappa}$ is in fact an isomorphism.

Recall the projection $q : mmf/\tau \to mmf$ to the top cell that was discussed in detail in Section 2.9. We will rely heavily on this map in order to transfer the algebraic information in $\pi_{*,*}mmf/\tau$ into homotopical information about $\pi_{*,*}mmf$.

Lemma 5.1. *The element $q(\Delta^{k+1})$ of $\pi_{*,*}mmf$ is detected by $-(k+1)\tau\Delta^k h_2 g$ in Adams-Novikov filtration 5.*

Proof. If $k+1$ is not a multiple of 4, then we have the non-zero differential $d_5(\Delta^{k+1}) = (k+1)\tau^2\Delta^k h_2 g$. Proposition 2.14 implies that $q(\Delta^{k+1})$ is detected by $-(k+1)\tau\Delta^k h_2 g$.

If $k+1$ is congruent to 4 modulo 8, then we have the non-zero differential $d_7(\Delta^{k+1}) = \tau^3\Delta^k h_1^3 g$. Proposition 2.14 implies that $q(\Delta^{k+1})$ is detected by $\tau^2\Delta^k h_1^3 g$ in filtration 7. This implies that $q(\Delta^{k+1})$ is detected by zero in filtration 5.

If $k+1$ is a multiple of 8, then Δ^k is a permanent cycle, so $q(\Delta^{k+1})$ equals zero. This implies that $q(\Delta^{k+1})$ is detected by zero in filtration 5. ■

Remark 5.2. For uniformity, we have stated Lemma 5.1 for all values of k. As shown in the proof of the lemma, there are in fact three cases, depending on the value of k. If $k+1$ is not a multiple of 4, then $-(k+1)\tau\Delta^k h_2 g$ is a non-zero element in the mANss E_∞-page.

On the other hand, if $k+1$ is a multiple of 4, then $-(k+1)\tau\Delta^k h_2 g$ is zero in the E_∞-page since $\tau\Delta^k h_2 g$ is an element of order 4. In these cases, the lemma says that $q(\Delta^{k+1})$ is detected by zero in filtration 5. In other words, $q(\Delta^{k+1})$ is detected in filtration strictly greater than 5, if it is non-zero. In fact, $q(\Delta^{k+1})$ is detected by $\tau^2\Delta^k h_1^3 g$ in filtration 7 when $k+1$ is congruent to 4 modulo 8. Also, $q(\Delta^{k+1})$ is zero when $k+1$ is a multiple of 8 because Δ^{k+1} is a permanent cycle.

Lemma 5.3. *The element $q(\Delta^{k+1})$ is a multiple of $\tau\overline{\kappa}$.*

Proof. Lemma 5.1 shows that $q(\Delta^{k+1})$ is detected by $-(k+1)\tau\Delta^k h_2 g$. By inspection, all possible values of $q(\Delta^{k+1})$ are multiples of $\tau\overline{\kappa}$. ∎

Definition 5.4. Let ν_k be the element of $\pi_{24k+3,12k+2}mmf$ such that $q(\Delta^{k+1})$ equals $-\tau\overline{\kappa}\cdot\nu_k$.

Note that ν_k exists because of Lemma 5.3. Multiplication by $\tau\overline{\kappa}$ is an isomorphism in the relevant degrees, so ν_k is specified uniquely. We choose a minus sign in the defining formula of Definition 5.4 for later convenience.

Remark 5.5. Bruner and Rognes consider ν_3 and ν_7 to be "honorary" members of the family of elements ν_k. They are not multiplicative generators; ν_3 is non-zero but decomposable, and ν_7 equals zero. Definition 5.4 also implies that ν_7 is zero. This follows from the observation that $q(\Delta^8)$ equals zero since Δ^8 is a permanent cycle.

The careful reader will note that the elements ν_k were already partially defined in Table 3 in Section 2.6. The following lemma shows that the two approaches to ν_k are compatible. Table 3 leaves some ambiguity in the definition of ν_k, and Definition 5.4 resolves that ambiguity.

Lemma 5.6. *The element ν_k is detected by $(k+1)\Delta^k h_2$ in Adams-Novikov filtration 1.*

Proof. Lemma 5.1 determines the mANss E_∞-page elements that detect $q(\Delta^{k+1})$. Then Definition 5.4 means that $-\tau\overline{\kappa}\cdot\nu_k$ is detected by those same elements. Multiplication by τg is an isomorphism in the relevant degrees, so the detecting elements for ν_k are then determined. ∎

Remark 5.7. Similarly to Remark 5.2, Lemma 5.6 includes three cases. If $k+1$ is not a multiple of 4, then $(k+1)\Delta^k h_2$ is a non-zero element of the mANss E_∞-page. If $k+1$ is a multiple of 4, then $(k+1)\Delta^k h_2$ is zero since $\Delta^k h_2$ is an element of order 4. This means that ν_k is detected in filtration strictly greater than 1, if it is non-zero. In fact, ν_k is detected by $\tau\Delta^k h_1^3$ in filtration 3 if $k+1$ is congruent to 4 modulo 8, and ν_k is zero if $k+1$ is a multiple of 8.

Remark 5.8. Earlier in Remark 2.2, we chose h_2 so that it detects the element ν. Lemma 5.6 shows that ν_0 is also detected by h_2, but that does not guarantee that it equals ν because of the presence of τh_1^3 in higher filtration. We can only conclude that ν and ν_0 are equal up to multiples of 4.

If ν equals $5\nu_0$, then we compute that

$$q(5\Delta) = -5\tau\overline{\kappa}\cdot\nu_0 = -\tau\overline{\kappa}\cdot\nu.$$

So we may replace Δ by 5Δ, if necessary, and assume without loss of generality that ν_0 equals ν. This replacement is compatible with our previous choice of Δ in Remark 3.9, which specified Δ only up to multiples of 4.

Proposition 5.9. $\nu_{k+8} = \nu_k \cdot M$.

Proof. Using Equation (2.13), we have

$$q(\Delta^{k+9}) = q(\Delta^{k+1}\cdot\Delta^8) = q(\Delta^{k+1}\cdot i(M)) = q(\Delta^{k+1})\cdot M = -\tau\overline{\kappa}\cdot\nu_k\cdot M.$$

Here we are using that $i(M) = \Delta^8$, which is equivalent to the definition that M is detected by Δ^8 (see Table 3).

On the other hand, $q(\Delta^{k+9})$ equals $-\tau\overline{\kappa}\cdot\nu_{k+8}$ by Definition 5.4. Finally, multiplication by $-\tau\overline{\kappa}$ is an isomorphism in the relevant degrees. ∎

Proposition 5.9 means that for practical purposes, we only need to consider the elements ν_k for $0 \leq k \leq 7$.

Theorem 5.10.

$$\nu_j\nu_k = (k+1)\nu_{j+k}\nu_0.$$

Proof. The proof splits into two cases, depending on whether $k+1$ is a multiple of 4. First, we handle the (more interesting) situation when $k+1$ is not a multiple of 4. We address the case when $k+1$ is a multiple of 4 below in a separate Proposition 5.13. The proof techniques for the two cases are similar, but the details are somewhat different.

Multiplication by $\tau\overline{\kappa}$ is an isomorphism in the relevant degrees, so it suffices to establish our relation after multiplication by $\tau\overline{\kappa}$.

Using Equation (2.13), we have

$$\begin{aligned} q((k+1)\Delta^{j+k+1}h_2) &= q(\Delta^{j+k+1}\cdot(k+1)h_2) = q(\Delta^{j+k+1}\cdot i((k+1)\nu_0)) = \\ &= q(\Delta^{j+k+1})\cdot(k+1)\nu_0 = -\tau\overline{\kappa}\cdot\nu_{j+k}\cdot(k+1)\nu_0. \end{aligned}$$

Here we are using that $i((k+1)\nu_0) = (k+1)h_2$; in other words, $(k+1)\nu_0$ is detected by $(k+1)h_2$. This requires that $k+1$ is not a multiple of 4. Otherwise, $(k+1)\nu_0$ is a multiple of τ, and $i((k+1)\nu_0)$ is zero.

We will now compute $q((k+1)\Delta^{j+k+1}h_2)$ another way. We have $i(\nu_k) = (k+1)\Delta^k h_2$; in other words, ν_k is detected by the non-zero element $(k+1)\Delta^k h_2$, as shown in Lemma 5.6. This requires that $k+1$ is not a multiple of 4. Otherwise, ν_k is a multiple of τ, and $i(\nu_k)$ is zero.

Then we have

$$q((k+1)\Delta^{j+k+1}h_2) = q(\Delta^{j+1}\cdot(k+1)\Delta^k h_2) = q(\Delta^{j+1}\cdot i(\nu_k)) = q(\Delta^{j+1})\cdot\nu_k = -\tau\overline{\kappa}\cdot\nu_j\cdot\nu_k.$$

∎

Remark 5.11. The exact form of the equation in Theorem 5.10 is guided by the structure of our proof. One could also write

$$\nu_i\nu_j = (i+1)\nu\nu_{i+j},$$

which more closely aligns with the notation in [BR21]. All of the elements ν_k are in odd stems, so they pairwise anti-commute.

Corollary 5.12. $(246, 2, 124)$ *$\nu_4\nu_6 = \nu\nu_2 M$.*

Proof. Theorem 5.10 implies that $\nu_4\nu_6$ equals $7\nu_{10}\nu_0$, which equals $-7\nu_0\nu_{10}$ by graded commutativity. By Remark 5.8 and Proposition 5.9, the latter expression equals $-7\nu\nu_2 M$. Finally, $\nu\nu_2 M$ belongs to a group of order 4, so $-7\nu\nu_2 M$ equals $\nu\nu_2 M$. ∎

We now return to the case of Theorem 5.10 in which $k+1$ is a multiple of 4.

Proposition 5.13. *If $k+1$ is a multiple of 4, then $\nu_j\cdot\nu_k = (k+1)\nu_{j+k}\nu_0$.*

Proof. First, let $k+1$ be a multiple of 8, so ν_k is zero. The element $\nu_{j+k}\nu_0$ belongs to a group whose order divides 8, so $(k+1)\nu_{j+k}\nu_0$ is zero. In other words, the equality holds because both sides are zero.

Next, let $k+1$ be congruent to 4 modulo 8. Let α be an element of $\pi_{*,*}mmf$ that is detected by $\Delta^k h_1^3$. The element ν_k is detected by $\tau\Delta^k h_1^3$, according to Remark 5.7. Since there are no elements in higher filtration, we can conclude that ν_k equals $\tau\alpha$. We have

$$q(\Delta^{j+k+1}h_1^3) = q(\Delta^{j+1}\cdot\Delta^k h_1^3) = q(\Delta^{j+1}\cdot i(\alpha)) = q(\Delta^{j+1})\cdot\alpha = -\tau\overline{\kappa}\cdot\nu_j\cdot\alpha = -\overline{\kappa}\cdot\nu_j\cdot\nu_k.$$

Now we add the assumption that $j+1$ is not congruent to 4 modulo 8. Given the assumption that $k+1$ is congruent to 4 modulo 8, we get that $j+k+1$ is not congruent to 7 modulo 8. Then $\Delta^{j+k+1}h_1^3$ is a permanent cycle, so $q(\Delta^{j+k+1}h_1^3)$ is zero. Together with the computation in the previous paragraph, this implies that $\nu_j\cdot\nu_k$ is zero since multiplication by $\overline{\kappa}$ is an isomorphism in the relevant degrees. Note also that $(k+1)\nu_{j+k}\nu_0$ is zero because it belongs to a group whose order divides 4.

Finally, we must consider the case when $j+1$ is congruent to 4 modulo 8, i.e., that $j+k+1$ is congruent to 7 modulo 8. Then $q(\Delta^{j+k+1}h_1^3)$ is detected by $\tau^{10}\Delta^{j+k-4}h_1^2g^6$ because of Proposition 2.14 and the differential $d_{23}(\Delta^{j+k+1}h_1^3) = \tau^{11}\Delta^{j+k-4}h_1^2g^6$. This means that $-\overline{\kappa}\cdot\nu_j\cdot\nu_k$ is detected by $\tau^{10}\Delta^{j+k-4}h_1^2g^6$. It follows that $\nu_j\cdot\nu_k$ is detected by $\tau^{10}\Delta^{j+k-4}h_1^2g^5$. Finally, this latter element also detects $(k+1)\nu_{j+k}\nu_0$ because of the hidden 2 extensions in the 150-stem and their multiples under Δ^8 multiplication (see Table 7). ■

Remark 5.14. As shown in the proof, most cases of Proposition 5.13 hold because both sides of the equation are zero. Both sides of the equation are non-zero precisely when $j+1$ and $k+1$ are congruent to 4 modulo 8.

Bruner and Rognes establish some relations that reduce the ambiguity in their definitions of ν_k. Finally, we will show that our elements defined in Definition 5.4 satisfy those same relations. We have already discussed the choice of ν_0 in Remark 5.8. The only additional requirements are the relations

$$\begin{aligned}\nu_0 D_4 &= 2\nu_4\\ \nu_1\nu_5 &= 2\nu_0\nu_6\\ \nu_2\nu_4 &= 3\nu_0\nu_6.\end{aligned}$$

The first formula is proved in Proposition 5.15, while the last two are specific instances of Theorem 5.10.

Proposition 5.15. $(99,1,50)$ $\nu_0 D_4 = 2\nu_4$.

Proof. Because of Lemma 5.6, both products are detected by $2\Delta^4 h_2$. However, they are not necessarily equal because of the presence of $\tau\Delta^4 h_1^3$ in higher filtration. We will show that $\tau\overline{\kappa}\cdot\nu D_4$ equals $\tau\overline{\kappa}\cdot 2\nu_4$. Our desired relation follows immediately because multiplication by $\tau\overline{\kappa}$ is an isomorphism in the relevant degree.

Using Equation (2.13), we have

$$q(2\Delta^5) = q(\Delta\cdot 2\Delta^4) = q(\Delta\cdot i(D_4)) = q(\Delta)\cdot D_4 = -\tau\overline{\kappa}\cdot\nu\cdot D_4.$$

Here we are using that $i(D_4) = 2\Delta^4$, which is equivalent to the definition that D_4 is detected by $2\Delta^4$ (see Table 3). On the other hand, we also have

$$q(2\Delta^5) = q(\Delta^5\cdot 2) = q(\Delta^5\cdot i(2)) = q(\Delta^5)\cdot 2 = -\tau\overline{\kappa}\cdot\nu_4\cdot 2.$$

■

6. Tables

Table 6: Adams-Novikov differentials

(s,f,w)	x	r	$d_r(x)$	proof
$(5,1,3)$	$h_1v_1^2$	3	τh_1^4	Proposition 3.2
$(12,0,6)$	$4a$	3	$\tau P h_1^3$	Proposition 3.3
$(24,0,12)$	Δ	5	$\tau^2 h_2 g$	Proposition 3.8
$(24,0,12)$	4Δ	7	$\tau^3 h_1^3 g$	Proposition 3.14
$(48,0,24)$	$2\Delta^2$	7	$\tau^3 \Delta h_1^3 g$	Proposition 3.14
$(96,0,48)$	Δ^4	7	$\tau^3 \Delta^3 h_1^3 g$	Proposition 3.21
$(49,1,25)$	$\Delta^2 h_1$	9	$\tau^4 c g^2$	Proposition 3.29
$(56,2,29)$	$\Delta^2 c$	9	$\tau^4 h_1 d g^2$	Proposition 3.18
$(73,1,37)$	$\Delta^3 h_1$	9	$\tau^4 \Delta c g^2$	Proposition 3.29
$(80,2,41)$	$\Delta^3 c$	9	$\tau^4 \Delta h_1 d g^2$	Proposition 3.24
$(145,1,73)$	$\Delta^6 h_1$	9	$\tau^4 \Delta^4 c g^2$	Proposition 3.29
$(169,1,85)$	$\Delta^7 h_1$	9	$\tau^4 \Delta^5 c g^2$	Proposition 3.29
$(152,2,77)$	$\Delta^6 c$	9	$\tau^4 \Delta^4 h_1 d g^2$	Proposition 3.25
$(176,2,89)$	$\Delta^7 c$	9	$\tau^4 \Delta^5 h_1 d g^2$	Proposition 3.24
$(62,2,32)$	$\Delta^2 d$	11	$\tau^5 h_1 g^3$	Proposition 3.33
$(158,2,80)$	$\Delta^6 d$	11	$\tau^5 \Delta^4 h_1 g^3$	Proposition 3.33
$(75,1,38)$	$2\Delta^3 h_2$	13	$\tau^6 d g^3$	Proposition 3.16
$(81,3,42)$	$\Delta^3 h_1 c$	13	$2\tau^6 g^4$	Proposition 3.36
$(171,1,86)$	$2\Delta^7 h_2$	13	$\tau^6 \Delta^4 d g^3$	Proposition 3.22
$(177,3,90)$	$\Delta^7 h_1 c$	13	$2\tau^6 \Delta^4 g^4$	Proposition 3.36
$(121,1,61)$	$\Delta^5 h_1$	23	$\tau^{11} g^6$	Proposition 3.39

Table 7: Hidden 2 extensions

(s,f,w)	source	target	proof
$(3,1,2)$	$2h_2$	τh_1^3	Lemma 3.11
$(27,1,14)$	$2\Delta h_2$	$\tau \Delta h_1^3$	Lemma 3.13
$(51,1,26)$	$2\Delta^2 h_2$	$\tau \Delta^2 h_1^3$	Proposition 4.2
$(54,2,28)$	$\Delta^2 h_2^2$	$\tau^4 d g^2$	Example 2.18
$(99,1,50)$	$2\Delta^4 h_2$	$\tau \Delta^4 h_1^3$	Proposition 4.2
$(110,2,56)$	$\Delta^4 d$	$\tau^6 \Delta^2 h_1^2 g^3$	Proposition 4.5
$(123,1,62)$	$2\Delta^5 h_2$	$\tau \Delta^5 h_1^3$	Proposition 4.2
$(147,1,74)$	$2\Delta^6 h_2$	$\tau \Delta^6 h_1^3$	Proposition 4.2
$(150,2,76)$	$\Delta^6 h_2^2$	$\tau^4 \Delta^4 d g^2$	Proposition 4.2

Table 8: Hidden η extensions

(s,f,w)	source	target	proof
$(27,1,14)$	$2\Delta h_2$	$\tau^2 cg$	Lemma 3.28
$(39,3,21)$	$\Delta h_1 d$	$2\tau^2 g^2$	Lemma 3.35
$(51,1,26)$	$\Delta^2 h_2$	$\tau^2 \Delta cg$	Proposition 4.2
$(99,1,50)$	$\Delta^4 h_2$	$\tau^9 g^5$	Proposition 4.2
$(123,1,62)$	$2\Delta^5 h_2$	$\tau^2 \Delta^4 cg$	Proposition 4.2
$(124,6,63)$	$\tau^2 \Delta^4 cg$	$\tau^9 \Delta h_1 g^5$	Proposition 4.2
$(129,3,66)$	$\Delta^5 h_1 c$	$\tau^7 \Delta^2 h_1^2 g^4$	Proposition 4.2
$(135,3,69)$	$\Delta^5 h_1 d$	$2\tau^2 \Delta^4 g^2$	Proposition 4.2
$(147,1,74)$	$\Delta^6 h_2$	$\tau^2 \Delta^5 cg$	Proposition 4.2
$(161,3,82)$	$\Delta^6 h_2 d$	$\tau^3 \Delta^5 h_1^2 g^2$	Proposition 4.2

Table 9: Hidden ν extensions

(s,f,w)	source	target	proof
$(0,0,0)$	4	τh_1^3	Proposition 4.2
$(25,1,13)$	Δh_1	$\tau^2 cg$	Lemma 3.27
$(32,2,17)$	Δc	$\tau^2 h_1 dg$	Proposition 4.11
$(39,3,21)$	$\Delta h_1 d$	$\tau^3 h_1^2 g^2$	Lemma 3.32
$(48,0,24)$	$4\Delta^2$	$\tau \Delta^2 h_1^3$	Proposition 4.2
$(50,2,26)$	$\Delta^2 h_1^2$	$\tau^2 \Delta h_1 cg$	Proposition 4.6
$(51,1,26)$	$2\Delta^2 h_2$	$\tau^4 dg^2$	Proposition 4.2
$(57,3,30)$	$\Delta^2 h_2^3$	$2\tau^4 g^3$	Proposition 4.2
$(96,0,48)$	$4\Delta^4$	$\tau \Delta^4 h_1^3$	Proposition 4.2
$(97,1,49)$	$\Delta^4 h_1$	$\tau^9 g^5$	Proposition 4.12
$(122,2,62)$	$\Delta^5 h_1^2$	$\tau^9 \Delta h_1 g^5$	Proposition 4.12
$(128,2,65)$	$\Delta^5 c$	$\tau^2 \Delta^4 h_1 dg$	Proposition 4.11
$(135,3,69)$	$\Delta^5 h_1 d$	$\tau^3 \Delta^4 h_1^2 g^2$	Proposition 4.14
$(144,0,72)$	$4\Delta^6$	$\tau \Delta^6 h_1^3$	Proposition 4.2
$(147,1,74)$	$2\Delta^6 h_2$	$\tau^4 \Delta^4 dg^2$	Proposition 4.2
$(147,3,75)$	$\Delta^6 h_1^3$	$\tau^9 \Delta^2 h_1^2 g^5$	Proposition 4.12
$(153,3,78)$	$\Delta^6 h_2^3$	$2\tau^4 \Delta^4 g^3$	Proposition 4.2

Table 10: Some Toda brackets

(s,f,w)	Toda bracket	detected by	indet	proof	used in
$(8,2,5)$	$\langle \nu, \eta, \nu \rangle$	c	0	Lemma 2.20	3.27, 4.14
$(25,1,13)$	$\langle \eta, \nu, \tau^2 \overline{\kappa} \rangle$	Δh_1	$P^3 h_1$	Lemma 3.26	3.27, 3.28, 4.14
$(32,2,17)$	$\langle \nu^2, 2, \eta_1 \rangle$	Δc	0	Lemma 4.7	4.11
$(128,2,65)$	$\langle \nu_2^2, 2, \eta_1 \rangle$	$\Delta^5 c$	0	Lemma 4.8	4.11
$(35,7,21)$	$\langle \nu^2, 2, \epsilon \overline{\kappa} \rangle$	$h_1 dg$	0	Lemma 4.9	4.11
$(131,7,69)$	$\langle \nu_2^2, 2, \epsilon \overline{\kappa} \rangle$	$\Delta^4 h_1 dg$	0	Lemma 4.10	4.11

7. Charts

The following charts display the E_2-page, E_9-page, and E_∞-page of the mANss for *mmf*. Each of these pages is free as a module over $\mathbb{Z}[\Delta^8]$, where Δ^8 is a class in the 192-stem. For legibility, we display the v_1-periodic elements on separate charts. See Section 2.7 for discussion of v_1-periodicity. To obtain the full E_2-page, one must superimpose Figures 1 and 3. To obtain the full E_∞-page, one must superimpose Figures 2 and 5.

We describe each chart in slightly more detail.

- Figure 1 shows the v_1-periodic portion of the mANss E_2-page, together with all differentials that are supported by the displayed elements.
- Figure 2 shows the v_1-periodic portion of the mANss E_∞-page.
- Figure 3 shows the non-v_1-periodic portion of the mANss E_2-page, together with all d_3, d_5, and d_7 differentials that are supported by the displayed elements.
- Figure 4 shows the non-v_1-periodic portion of the mANss E_9-page, together with all differentials that are supported by the displayed elements.
- Figure 5 shows the non-v_1-periodic portion of the mANss E_∞-page, together with all hidden extensions by 2, η, and ν.

7.1. **Elements.** For each fixed stem and filtration, the mANss consists of a $\mathbb{Z}[\tau]$-module. We use a graphical notation to describe these modules. Our notation represents the associated graded object of a filtration that is related to the powers of 2.

- An open box □ indicates a copy of $\mathbb{Z}[\tau]$ in the associated graded object.
- A solid gray dot • indicates a copy of $\mathbb{F}_2[\tau]$ in the associated graded object.
- A solid colored dot indicates a copy of $\mathbb{F}_2[\tau]/\tau^r$ in the associated graded object. The value of r is encoded in the color of the dot, as shown in Table 11.
- Short vertical lines indicate extensions by 2.

Our graphical notation has the advantages of flexibility, compactness, and convenience. We illustrate with two examples.

Example 7.1. In Figure 3 at degree $(48, 0)$, one sees . This notation indicates a copy of $\mathbb{Z}[\tau]$. More precisely, it represents the filtration $4\mathbb{Z}[\tau] \subseteq 2\mathbb{Z}[\tau] \subseteq \mathbb{Z}[\tau]$ whose filtration quotients are $\mathbb{Z}[\tau]$, $\mathbb{F}_2[\tau]$, and $\mathbb{F}_2[\tau]$. This particular filtration is relevant for our mANss computation because $2\mathbb{Z}[\tau]$ is the subgroup of d_5 cycles, and $4\mathbb{Z}[\tau]$ is the subgroup of d_7 cycles.

Example 7.2. In Figure 5 at degree $(120, 24)$, one sees . This notation indicates the $\mathbb{Z}[\tau]$-module

$$\frac{\mathbb{Z}[\tau]}{8, 4\tau^2, 2\tau^6, \tau^{11}},$$

which is somewhat cumbersome to describe in traditional notation. More precisely, it represents the filtration

$$\frac{4\mathbb{Z}[\tau]}{8, 4\tau^2} \subseteq \frac{2\mathbb{Z}[\tau]}{8, 4\tau^2, 2\tau^6} \subseteq \frac{\mathbb{Z}[\tau]}{8, 4\tau^2, 2\tau^6, \tau^{11}}.$$

whose filtration quotients are $\mathbb{F}_2[\tau]/\tau^2$, $\mathbb{F}_2[\tau]/\tau^6$, and $\mathbb{F}_2[\tau]/\tau^{11}$. The blue, magenta, and orange dots correspond to these filtration quotients, as shown in Table 11.

Table 11: Color interpretations for elements

n	color
$\mathbb{F}_2[\tau]$	• gray
$\mathbb{F}_2[\tau]/\tau$	• red
$\mathbb{F}_2[\tau]/\tau^2$	• blue
$\mathbb{F}_2[\tau]/\tau^3$	• green
$\mathbb{F}_2[\tau]/\tau^4$	• cyan
$\mathbb{F}_2[\tau]/\tau^5$	• brown
$\mathbb{F}_2[\tau]/\tau^6$	• magenta
$\mathbb{F}_2[\tau]/\tau^{11}$	• orange

7.2. **Differentials.** Lines of negative slope indicate Adams-Novikov differentials. The differentials are colored according to their lengths, as described in Table 12. These color choices are compatible with our choice of colors for τ torsion in Section 7.1, in the following sense. An Adams-Novikov d_{2r+1} differential always takes the form $d_{2r+1}(x) = \tau^r y$, and it creates τ^r torsion in the following page. We use matching colors for d_{2r+1} and for τ^r torsion.

Table 12: Color interpretations for Adams-Novikov differentials

color	slope	d_r
red	-3	d_3
blue	-5	d_5
green	-7	d_7
cyan	-9	d_9
brown	-11	d_{11}
magenta	-13	d_{13}
orange	-23	d_{23}

7.3. **Extensions.**

- Solid lines of slope 1 indicate h_1 multiplications. The colors of these lines are determined by the τ torsion of the targets.
- Arrows of slope 1 indicate infinite families of elements that are connected by h_1 multiplications. The colors of the arrows reflect the τ torsion of the elements.
- Solid lines of slope 1/3 indicate h_2 multiplications. The colors of these lines are determined by the τ torsion of the targets.
- Dashed lines indicate hidden extensions by 2, η, and ν. Some of these lines are curved solely for the purpose of legibility.
- The colors of dashed lines indicate the τ torsion of the targets of the extensions. For example, the vertical dashed line in the 23-stem of Figure 5 is blue because its value $\tau h_1^3 g$ is annihilated by τ^2.

Figure 5 shows an h_1 extension and also a hidden η extension on the element $\Delta^4 cg$ in degree $(124, 6, 65)$. See Remark 4.3 for an explanation.

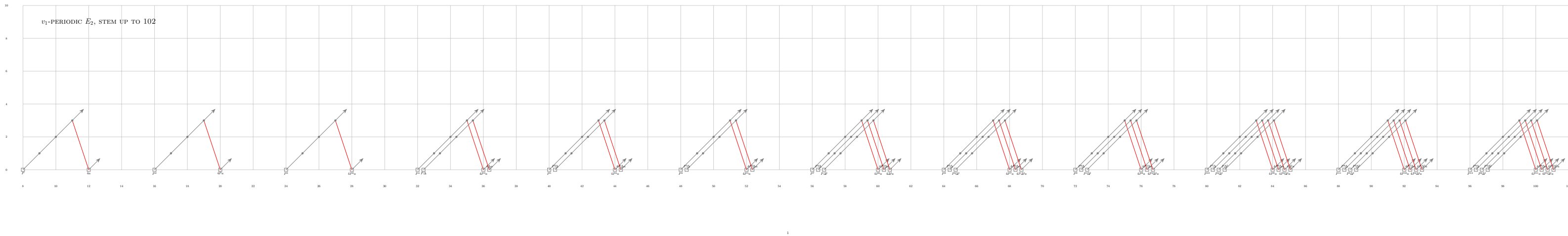

FIGURE 1. The v_1-periodic portion of the $\mathbb{C}$-motivic Adams-Novikov E_2-page for *mmf*

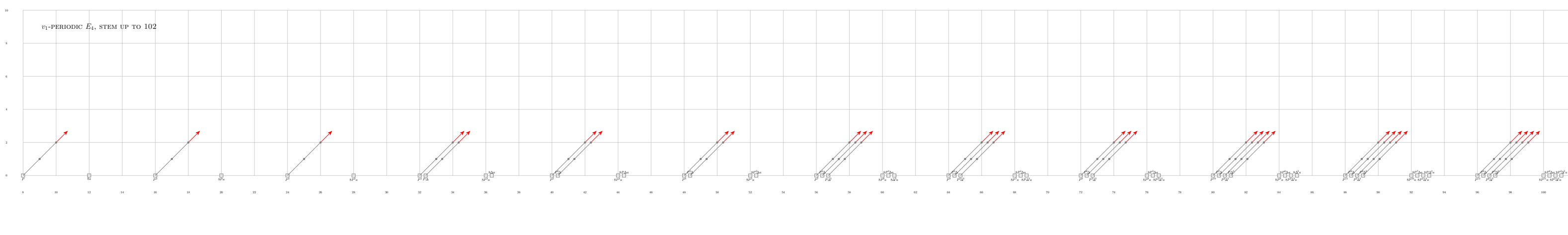

FIGURE 2. The v_1-periodic portion of the $\mathbb{C}$-motivic Adams-Novikov E_∞-page for *mmf*

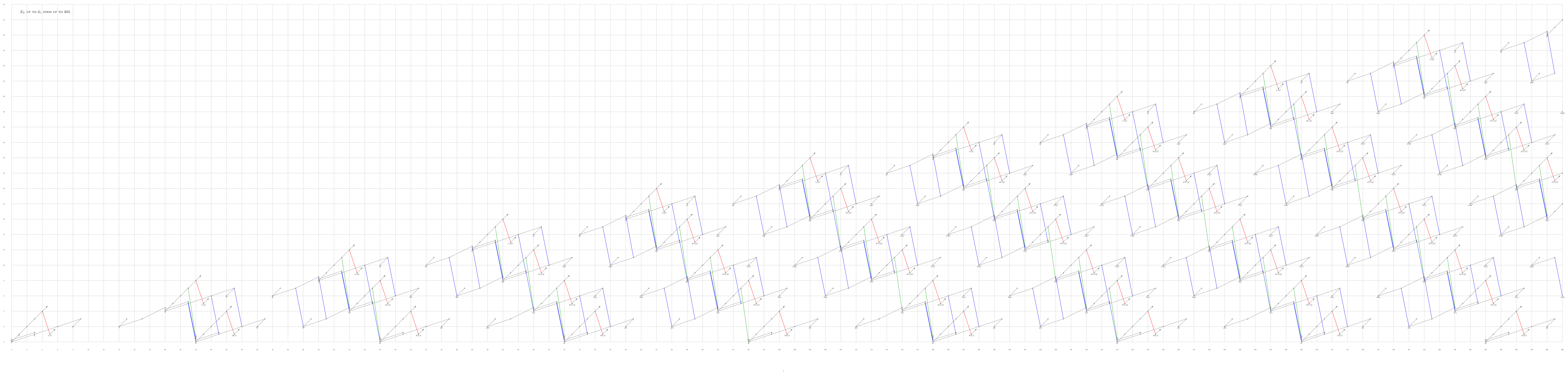

FIGURE 3. The $\mathbb{C}$-motivic Adams-Novikov E_2-page for mmf with differentials of length at most 7

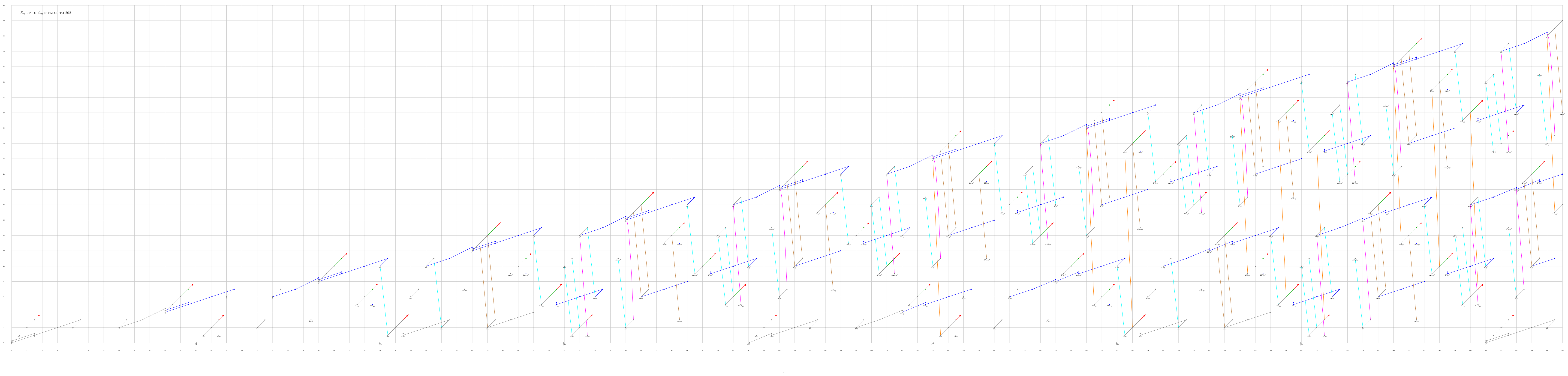

FIGURE 4. The $\mathbb{C}$-motivic Adams-Novikov E_9-page for mmf with differentials of length at least 9

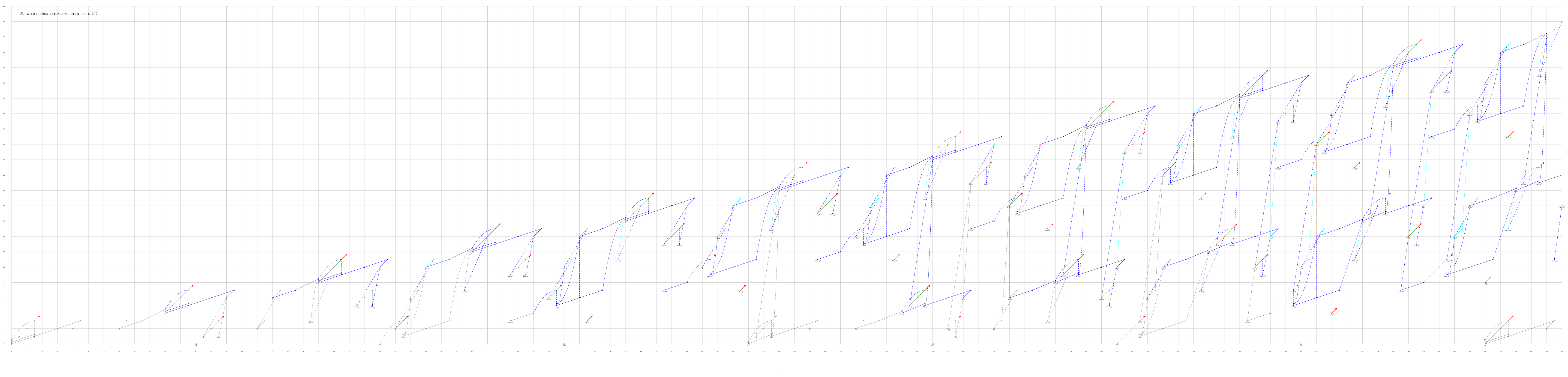

FIGURE 5. The $\mathbb{C}$-motivic Adams-Novikov E_∞-page for mmf with hidden extensions by 2, η, and ν

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