Stable homotopy groups of spheres

Daniel C. Isaksen[a,1,2], Guozhen Wang[b,1], and Zhouli Xu[c,d,1]

[a]Department of Mathematics, Wayne State University, Detroit, MI 48202; [b]Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China; [c]Department of Mathematics, University of California San Diego, La Jolla, CA 92093; and [d]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

We discuss the current state of knowledge of stable homotopy groups of spheres. We describe a computational method using motivic homotopy theory, viewed as a deformation of classical homotopy theory. This yields a streamlined computation of the first 61 stable homotopy groups and gives information about the stable homotopy groups in dimensions 62 through 90. As an application, we determine the groups of homotopy spheres that classify smooth structures on spheres through dimension 90, except for dimension 4. The method relies more heavily on machine computations than previous methods and is therefore less prone to error. The main mathematical tool is the Adams spectral sequence.

stable homotopy groups of spheres | smooth structures on spheres | Adams spectral sequence | motivic homotopy theory

One of the most fundamental problems in topology is to determine the set of homotopy classes $[S^{n+k}, S^n]$ of continuous based maps $f : S^{n+k} \to S^n$ between spheres. For $n + k \geq 1$, these sets have a natural group structure, and they are abelian when $n + k \geq 2$. Despite their essential topological importance, they are notoriously difficult to compute. Detailed computations exist only when k is at most ~ 30 (1–6).

Theorem 0.1 summarizes some basic structural results about $[S^{n+k}, S^n]$.

Theorem 0.1.

1) $[S^{n+k}, S^n] = 0$ when $k < 0$.
2) $[S^n, S^n] = \mathbb{Z}$.
3) $[S^{n+k}, S^n]$ is a finite group, except when $k = 0$, or n is even and $k = n - 1$ [Serre finiteness theorem (7)].

Unfortunately, **Theorem 0.1** gives little information about the groups $[S^{n+k}, S^n]$ when $k > 0$.

The Freudenthal suspension theorem (8) provides a relationship between the groups $[S^{n+k}, S^n]$ for fixed k and varying n. The suspension map induces a sequence

$$\cdots \to [S^{n-1+k}, S^{n-1}] \to [S^{n+k}, S^n] \to [S^{n+1+k}, S^{n+1}] \to \cdots$$

of group homomorphisms, and these homomorphisms are in fact isomorphisms when $n > k + 1$. The stable value $[S^{n+k}, S^n]$ for n sufficiently large is known as the kth stable homotopy group π_k.

The stable homotopy groups π_k enjoy additional structure that make them more amenable to computation than the unstable groups. The rest of this article is entirely concerned with the stable homotopy groups. While the study of stable homotopy groups gives much information about the structure of the unstable groups, it does not give complete information. Stable homotopy group information does not tell us much about the unstable groups when $n \leq k + 1$.

Theorem 0.1 implies that π_k is zero if $k < 0$, that π_0 is isomorphic to $\mathbb{Z}$, and that π_k is a finite group for all $k > 0$. Because each group π_k is finite, it makes sense to study the groups one prime at a time. More specifically, we can compute the p-primary component of the group π_k for all primes p and then reassemble these components into a uniquely determined finite abelian group.

History has demonstrated the effectiveness of this p-primary approach. For the remainder of this article, we will focus on the 2-primary components of π_k, except that we record odd pri-

mary information in Tables 1 and 2 as a convenience for the reader. We write $\pi_k^\wedge$ for the 2-primary component of π_k. There are plenty of interesting phenomena to study at odd primes, but we leave that discussion for other work. See Ravenel's comprehensive book (5) for an extensive source of information on computations at odd primes.

We will not discuss the long history of stable homotopy group computations thoroughly. Some of the key results include the original definitions of homotopy groups by Čech and Hurewicz, the discovery of the Hopf maps (9), the connection to framed cobordism due to Pontryagin (10, 11), Serre's method using fiber sequences and Eilenberg–Mac Lane spaces (12), the Adams spectral sequence (13), the May spectral sequence (14), Toda's work involving higher compositions and the EHP sequence (6), deeper analysis of the Adams spectral sequence by Mahowald (with Barratt, Bruner, and Tangora) (15–17), the Adams–Novikov spectral sequence (18), and Kochman's approach involving the Atiyah–Hirzebruch spectral sequence for the Brown–Peterson spectrum BP (19, 20).

The stable homotopy groups have important applications in the study of high-dimensional manifolds. See 2. *Groups of Homotopy Spheres* for more discussion of one such application.

1. Stable Homotopy Group Computations

We use the $\mathbb{C}$-motivic homotopy theory of Morel and Voevodsky (21), which has a richer structure than classical homotopy theory, to deduce information about stable homotopy groups. In practice, our procedure works remarkably well. Already we have obtained nearly complete information to dimension 90, extending well beyond the previously known range that ended at dimension 61.

Significance

The geometric objects of interest in algebraic topology can be constructed by fitting together spheres of varying dimensions. The homotopy groups of spheres describe the ways in which spheres can be attached to each other. From the viewpoint of algebraic topology, detailed knowledge of these groups would lead to a classification of geometric objects. We summarize the state of previous knowledge of the stable homotopy groups of spheres through dimension 61 and provide information in dimensions 62 through 90. It is a highly counterintuitive fact that some high-dimensional spheres can be viewed in more than one way from the perspective of differentiability. We enumerate these differentiable structures through dimension 90, except for dimension 4.

Table 1. Stable homotopy groups in dimensions 1 to 61

k	v_1-torsion at the prime 2	v_1-torsion at odd primes	v_1-periodic	Group of smooth structures
1	·	·	2	·
2	·	·	2	·
3	·	·	8·3	·
4	·	·	·	?
5	·	·	·	·
6	2	·	·	·
7	·	·	16·3·5	b_2
8	2	·	2	2
9	2	·	2^2	2·2^2
10	·	3	2	2·3
11	·	·	8·9·7	b_3
12	·	·	·	·
13	·	3	·	3
14	2·2	·	·	2
15	2	·	32·3·5	b_4·2
16	2	·	2	2
17	2^2	·	2^2	2·2^3
18	8	·	2	2·8
19	2	·	8·3·11	b_5·2
20	8	3	·	8·3
21	2^2	·	·	2·2^2
22	2^2	·	·	2^2
23	2·8	3	16·9·5·7·13	b_6·2·8·3
24	2	·	2	2
25	·	·	2^2	2·2
26	2	3	2	2^2·3
27	·	·	8·3	b_7
28	2	·	·	2
29	·	3	·	3
30	2	3	·	3
31	2^2	·	64·3·5·17	b_8·2^2
32	2^3	·	2	2^3
33	2^3	·	2^2	2·2^4
34	2^2·4	·	2	2^3·4
35	2^2	·	8·27·7·19	b_9·2^2
36	2	3	·	2·3
37	2^2	3	·	2·2^2·3
38	2·4	3·5	·	2·4·3·5
39	2^5	3	16·3·25·11	b_{10}·2^5·3
40	2^4·4	3	2	2^4·4·3
41	2^3	·	2^2	2·2^4
42	2·8	3	2	2^2·8·3
43	·	·	8·3·23	b_{11}
44	8	·	·	8
45	2^3·16	9·5	·	2·2^3·16·9·5
46	2^4	3	·	2^4·3
47	2^3·4	3	32·9·5·7·13	b_{12}·2^3·4·3
48	2^3·4	·	2	2^3·4
49	·	3	2^2	2·2·3
50	2^2	3	2	2^3·3
51	2·8	·	8·3	b_{13}·2·8
52	2^3	3	·	2^3·3
53	2^4	·	·	2·2^4
54	2·4	·	·	2·4
55	·	3	16·3·5·29	b_{14}·3
56	·	·	2	·
57	2	·	2^2	2·2^2
58	2	·	2	2^2
59	2^2	·	8·9·7·11·31	b_{15}·2^2
60	4	·	·	4
61	·	·	·	·

n stands for $\mathbb{Z}/n$, $n{\cdot}m$ stands for $\mathbb{Z}/n \oplus \mathbb{Z}/m$, and n^j stands for $(\mathbb{Z}/n)^j$. Underlined symbols indicate contributions from Θ_n^{bp}, and b_k stands for $2^{2k-2}(2^{2k-1}-1)$ times the numerator of $8\zeta(1-2k)$, where ζ is the Riemann zeta function.

Isaksen et al.

Table 2. Stable homotopy groups in dimensions 62 to 90

k	v_1-torsion at the prime 2	v_1-torsion at odd primes	v_1-periodic	Group of smooth structures
62	2^4	3	·	$2^3 \cdot 3$
63	$2^2 \cdot 4$	·	$128 \cdot 3 \cdot 5 \cdot 17$	$\underline{b_{16}} \cdot 2^2 \cdot 4$
64	$2^5 \cdot 4$	·	2	$2^5 \cdot 4$
65	$2^7 \cdot 4$	3	2^2	$2 \cdot 2^8 \cdot 4 \cdot 3$
66	$2^5 \cdot 8$	·	2	$2^6 \cdot 8$
67	$2^3 \cdot 4$	·	$8 \cdot 3$	$\underline{b_{17}} \cdot 2^3 \cdot 4$
68	2^3	3	·	$2^3 \cdot 3$
69	2^4	·	·	$2 \cdot 2^4$
70	$2^5 \cdot 4^2$	·	·	$2^5 \cdot 4^2$
71	$2^6 \cdot 4 \cdot 8$	·	$16 \cdot 27 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37$	$\underline{b_{18}} \cdot 2^6 \cdot 4 \cdot 8$
72	2^7	3	2	$2^7 \cdot 3$
73	2^5	·	2^2	$2 \cdot 2^6$
74	4^3	3	2	$2 \cdot 4^3 \cdot 3$
75	2	9	$8 \cdot 3$	$\underline{b_{19}} \cdot 2 \cdot 9$
76	$2^2 \cdot 4$	5	·	$2^2 \cdot 4 \cdot 5$
77	$2^5 \cdot 4$	·	·	$2 \cdot 2^5 \cdot 4$
78	$2^3 \cdot 4^2$	3	·	$2^3 \cdot 4^2 \cdot 3$
79	$2^6 \cdot 4$	·	$32 \cdot 3 \cdot 25 \cdot 11 \cdot 41$	$\underline{b_{20}} \cdot 2^6 \cdot 4$
80	2^8	·	2	2^8
81	$2^3 \cdot 4 \cdot 8$	3^2	2^2	$2 \cdot 2^4 \cdot 4 \cdot 8 \cdot 3^2$
82	$2^5 \cdot 8$ or $2^4 \cdot 8$ or $2^3 \cdot 4 \cdot 8$	$3 \cdot 7$	2	$2^6 \cdot 8 \cdot 3 \cdot 7$ or $2^5 \cdot 8 \cdot 3 \cdot 7$ or $2^4 \cdot 4 \cdot 8 \cdot 3 \cdot 7$
83	$2^3 \cdot 8$ or $2^3 \cdot 4$	5	$8 \cdot 9 \cdot 49 \cdot 43$	$\underline{b_{21}} \cdot 2^3 \cdot 8 \cdot 5$ or $\underline{b_{21}} \cdot 2^3 \cdot 4 \cdot 5$
84	2^6 or 2^5 or	3^2	·	$2^6 \cdot 3^2$ or $2^5 \cdot 3^2$ or
85	$2^6 \cdot 4^2$ or $2^5 \cdot 4^2$ or $2^4 \cdot 4^3$	3^2	·	$2^6 \cdot 4^2 \cdot 3^2$ or $2^5 \cdot 4^2 \cdot 3^2$ or $2^4 \cdot 4^3 \cdot 3^2$
86	$2^5 \cdot 8^2$ or $2^4 \cdot 8^2$ or $2^3 \cdot 4 \cdot 8^2$ or $2^2 \cdot 4 \cdot 8^2$	$3 \cdot 5$	·	$2^5 \cdot 8^2 \cdot 3 \cdot 5$ or $2^4 \cdot 8^2 \cdot 3 \cdot 5$ or $2^3 \cdot 4 \cdot 8^2 \cdot 3 \cdot 5$ or $2^2 \cdot 4 \cdot 8^2 \cdot 3 \cdot 5$
87	2^8 or 2^7 or $2^6 \cdot 4$ or $2^5 \cdot 4$	·	$16 \cdot 3 \cdot 5 \cdot 23$	$\underline{b_{22}} \cdot 2^8$ or $\underline{b_{22}} \cdot 2^7$ or $\underline{b_{22}} \cdot 2^6 \cdot 4$ or $\underline{b_{22}} \cdot 2^5 \cdot 4$
88	$2^4 \cdot 4$	·	2	$2^4 \cdot 4$
89	2^3	·	2^2	$2 \cdot 2^4$
90	$2^3 \cdot 8$ or $2^2 \cdot 8$	3	2	$2^4 \cdot 8 \cdot 3$ or $2^3 \cdot 8 \cdot 3$

n stands for $\mathbb{Z}/n$, $n \cdot m$ stands for $\mathbb{Z}/n \oplus \mathbb{Z}/m$, and n^j stands for $(\mathbb{Z}/n)^j$. Underlined symbols indicate contributions from Θ_n^{bp}, and b_k stands for $2^{2k-2}(2^{2k-1}-1)$ times the numerator of $8\zeta(1-2k)$, where ζ is the Riemann zeta function.

We briefly describe the main breakthrough here, and we elaborate further in *6. Algebraicity of the Cofiber of* τ. Motivic homotopy theory is equipped with a parameter τ. If we set τ equal to zero, then we obtain a category that has a purely algebraic description and is thus amenable to machine computation. We can pull back information from this algebraic category to the $\mathbb{C}$-motivic category, thus transporting machine data to $\mathbb{C}$-motivic homotopical information. Finally, we can transform $\mathbb{C}$-motivic homotopical information to classical information by inverting τ.

Theorem 1.1. *Tables* 1 *and* 2 *describe the stable homotopy groups* π_k *for all values of* k *up to 90.*

We adopt the following notation in Tables 1 and 2. An integer n stands for the cyclic abelian group $\mathbb{Z}/n$, the expression $n \cdot m$ stands for the direct sum $\mathbb{Z}/n \oplus \mathbb{Z}/m$, the symbol $\cdot$ by itself stands for the trivial group, and n^j stands for the direct sum of j copies of $\mathbb{Z}/n$. Table 1 gives previously known information. Table 2 is derived from ref. 22.

Tables 1 and 2 describe each group π_k as the direct sum of three subgroups. The first and third columns describe the 2-primary component of each group, while the second and third columns describe the odd primary components of each group. See *3. v_1-Periodic Stable Homotopy Groups* for a brief explanation of the meaning of the v_1-torsion and v_1-periodic subgroups.

The last columns of Tables 1 and 2 describe the groups of homotopy spheres that classify smooth structures on spheres in dimensions at least 5. See *2. Groups of Homotopy Spheres* and **Theorem 2.1** for more details.

Starting in dimension 82, there remain some uncertainties in the v_1-torsion subgroups at the prime 2. In most cases, these uncertainties mean that the order of π_k is known only up to a factor of 2. Unknown Adams differentials are the source of these problems, to be discussed in *4. The Adams Spectral Sequence*. In a few cases, the additive group structure is also not completely determined due to possible hidden 2 extensions.
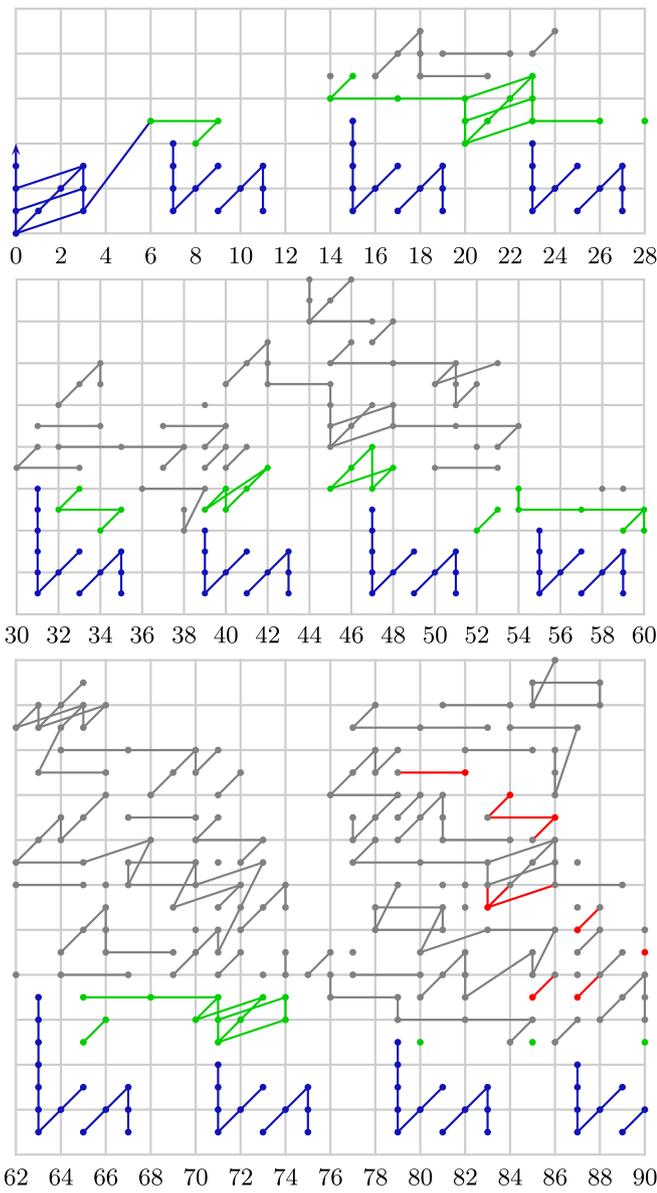
MATHEMATICS

Fig. 1. The 2-primary stable homotopy groups.

Fig. 1 displays the 2-primary stable homotopy groups in a graphical format. Vertical chains of n dots in column k indicate a copy of $\mathbb{Z}/2^n$ in $\pi_k^\wedge$. The nonvertical lines indicate additional multiplicative structure that we will not discuss here. The blue dots represent the v_1-periodic subgroups. The green dots are associated to the topological modular forms spectrum *tmf*; the precise relationship is too complicated to describe here. Finally, the red dots indicate uncertainties. This type of graphical description of stable homotopy groups is a modification by Allen Hatcher of Adams spectral sequence charts (23).

The orders of individual 2-primary stable homotopy groups do not follow a clear pattern, with large increases and decreases seemingly at random. However, an empirically observed pattern emerges if we consider the cumulative size of the groups, i.e., the product of the orders of all 2-primary stable homotopy groups in dimensions 1 through k.

Our data strongly suggest that asymptotically, there is a linear relationship between k^2 and the logarithm of this product of orders. In other words, the number of dots in Fig. 1 in columns 1 through k is asymptotically linearly proportional to k^2. Thus, in

extending from dimension 61 to dimension 90, the overall size of the computation more than doubles.

Conjecture 1.2. *Let $f(k)$ be the product of the orders of the 2-primary stable homotopy groups in dimensions 1 through k. Then $\log f(k) = O(k^2)$.*

One interpretation of this conjecture is that the expected value of the logarithm of the order of the 2-primary component of π_k grows linearly in k.

2. Groups of Homotopy Spheres

An important application of stable homotopy group computations is to the work of Kervaire and Milnor (24) on the classification of smooth structures on spheres in dimensions at least 5. Let Θ_n be the group of h-cobordism classes of homotopy n-spheres. This group classifies the differential structures on S^n for $n \geq 5$. It has a subgroup Θ_n^{bp}, which consists of homotopy spheres that bound parallelizable manifolds. See also ref. 25 for a survey on this subject.

Kervaire and Milnor (24) showed that Θ_n^{bp} is cyclic, and they determined the order of this group in terms of Bernoulli numbers. They also showed how Θ_n can be determined via exact sequences from Θ_n^{bp}, the cokernel π_n/J of the J-homomorphism, and the Kervaire invariant. See *3. v_1-Periodic Stable Homotopy Groups* for further discussion of the cokernel of J. Despite the recent breakthrough of Hill, Hopkins, and Ravenel (26), there is still one remaining unknown value of the Kervaire invariant in dimension 126. See the introduction of ref. 27 for a more detailed discussion of these ideas.

Theorem 2.1. *The last columns of Tables 1 and 2 describe the groups Θ_n for $n \leq 90$, with the exception of $n = 4$. The underlined symbols denote the contributions from Θ_n^{bp}, and b_k is $2^{2k-2}(2^{2k-1} - 1)$ times the numerator of $8\zeta(1 - 2k)$, where ζ is the Riemann zeta function.*

The first few values, and then estimates, of the numbers b_k (for $k \geq 2$) are given by the sequence

$$28, \ 992, \ 8128, \ 261632, \ 1.45 \times 10^9,$$
$$6.71 \times 10^7, \ 1.94 \times 10^{12}, \ 7.54 \times 10^{14}, \ldots.$$

We restate the following conjecture from ref. 27, which is based on the current knowledge of stable homotopy groups and a problem proposed by Milnor (25).

Conjecture 2.2. *In dimensions greater than 4, the only spheres with unique smooth structures are S^5, S^6, S^{12}, S^{56}, and S^{61}.*

Uniqueness in dimensions 5, 6, and 12 was known to Kervaire and Milnor (24). Uniqueness in dimension 56 is established in ref. 28 and uniqueness in dimension 61 is established in ref. 27.

Conjecture 2.2 is equivalent to the claim that the group Θ_n is not of order 1 for dimensions greater than 61. This conjecture has been confirmed in all odd dimensions by the second and the third authors (27) based on the work of Hill, Hopkins, and Ravenel (26) and in even dimensions up to 140 by Behrens, Hill, Hopkins, and Mahowald (29).

3. v_1-Periodic Stable Homotopy Groups

Adams provided the first infinite families of elements in the stable homotopy groups (30). Within the stable homotopy groups, there is a regular repeating pattern of subgroups. These subgroups are known as the "v_1-periodic stable homotopy groups," and they are closely related to the "image of J."

Theorem 3.1. *Table 3 gives the v_1-periodic stable homotopy groups inside of $\pi_k^\wedge$ for all $k \geq 2$. Here $\nu(k+1)$ is the 2-primary factor of the integer $k + 1$ (30).*

The odd-primary v_1-periodic stable homotopy groups can be similarly described using odd-primary factors of integers.

The v_1-periodic subgroups are direct summands of the stable homotopy groups. Their complementary summands are known

Table 3.   The 2-primary v_1-periodic stable homotopy groups

k	v_1-periodic subgroup of $\pi_k^\wedge$
0 mod 8	2
1 mod 8	2^2
2 mod 8	2
3 mod 8	8
7 mod 8	$2\nu(k+1)$

as the "v_1-torsion subgroups." The language involving periodicity and torsion derives from the theory of nilpotence and periodicity due to Devinatz, Hopkins, Ravenel, Smith, and others (31–33), which goes beyond the scope of this article.

The cokernel of the J-homomorphism appeared in *2. Groups of Homotopy Spheres* as an ingredient in classifying smooth structures on spheres. This cokernel is slightly different from the v_1-torsion part of π_n at the prime 2. In dimensions $8m+1$ and $8m+2$, there are classes detected by $P^m h_1$ and $P^m h_1^2$ in the Adams spectral sequence. These classes are v_1-periodic, in the sense that they are detected by the $K(1)$-local sphere. However, they are also in the cokernel of the J-homomorphism.

4. The Adams Spectral Sequence

The most powerful tool for computing $\pi_k^\wedge$ is the Adams spectral sequence (13). Information about ordinary cohomology, together with its higher structure in the form of cohomology operations, yields information about the stable homotopy groups. See ref. 34 for a graphical representation of the Adams spectral sequence.

The Steenrod algebra A is the ring of stable operations on cohomology with $\mathbb{F}_2$ coefficients (35). It is the associative algebra generated by the Steenrod squaring operations Sq^i modulo the Adem relations, which are easy to write down explicitly but hard to handle conceptually. The Steenrod algebra also has a coproduct that gives it the structure of a Hopf algebra.

Following ideas of Milnor (36), it turns out to be much easier to work with the $\mathbb{F}_2$-dual. In other words, we consider $A_* = \mathrm{Hom}_{\mathbb{F}_2}(A, \mathbb{F}_2)$. Then the product and coproduct on A become a coproduct and a product on A_*, respectively, and A_* is again a Hopf algebra.

The Hopf algebra A_* is much easier to describe. It is isomorphic to the polynomial ring $\mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$, where the coproduct is given by the formula

$$\zeta_i \mapsto \sum_{k=0}^{i} \zeta_{i-k}^{2^k} \otimes \zeta_k.$$

By convention, we let ζ_0 equal 1.

The next step in the Adams spectral sequence program is to consider the derived groups $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ of $\mathrm{Hom}_A(\mathbb{F}_2, \mathbb{F}_2)$, in the sense of homological algebra. Roughly speaking, these Ext groups capture higher information about the structure of A, including generators, relations, relations among relations, etc.

The groups $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ serve as the input to the Adams spectral sequence. Up to dimension 13, these algebraic Ext groups give complete information about $\pi_k^\wedge$. However, in higher dimensions, further complications occur. Specifically, one must compute Adams differentials. These differentials measure the deviation between algebra and homotopy. In practice, the computation of these differentials is the limiting factor in our knowledge of stable homotopy groups.

In even higher dimensions, the algebraic groups $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ themselves become difficult to compute directly. The May spectral sequence is the best way to compute these groups by hand (14).

In the modern era, the most efficient way to compute Ext groups is by machine. Bruner (37–39), Nassau (40), and G.W.

(41) have constructed various efficient algorithms that provide a wealth of algebraic data, far surpassing our ability to interpret. The most extensive computations extend beyond dimension 200. For practical purposes, we can take this Ext data as given. Computer-assisted techniques are likely to continue to grow in importance in the computation of stable homotopy groups.

Beyond dimension 30, the analysis of Adams differentials becomes more difficult (15–17, 28). The stable homotopy groups possess higher structure in the form of Massey products and Toda brackets, and this higher structure leads to additional information about differentials. However, these arguments are notoriously tricky, and the published literature contains several incorrect computations.

The practical limit of this style of argument occurs at dimension 61. See ref. 28 for a thorough accounting of the Adams spectral sequence through dimension 59. The article (27) employs strenuous efforts to obtain just two more stable homotopy groups in dimensions 60 and 61. Beyond dimension 61, these methods are simply no longer practical.

5. The Motivic Adams Spectral Sequence

Morel and Voevodsky (21, 42) developed motivic homotopy theory in the mid-1990s as a means of importing homotopical techniques into algebraic geometry. This program found great success in Voevodsky's resolutions of the Milnor conjecture (43) and the Bloch–Kato conjecture (44).

Motivic homotopy theory is bigraded, so all invariants, including cohomology and stable homotopy groups, are bigraded. There is a bigraded family of spheres $S^{p,q}$ that serve as the basic building blocks of motivic homotopy theory. While motivic homotopy theory can be studied over any base field, we will focus only on the case when the base field is $\mathbb{C}$. We may further simplify the theory somewhat by considering only cellular objects and by taking appropriate 2-completions.

The additional structure contained in $\mathbb{C}$-motivic homotopy theory provides a new tool for computing classical stable homotopy groups (22). There are now two entirely topological models for 2-complete cellular $\mathbb{C}$-motivic homotopy theory (45, 46). In these models, all computations can be derived from first principles, using only well-known standard classical computations. Therefore, even though our work is very much inspired by motivic homotopy theory, it is logically independent of the deep and difficult foundational and computational results in motivic homotopy theory, including refs. 21, 42, 43, 47, and 48.

In fact, very recent work of Robert Burklund, D.C.I., and Z.X. demonstrates that alternative versions of these topological models allow us to construct new exotic homotopy theories with additional computational consequences. More details will appear in forthcoming work.

The $\mathbb{C}$-motivic cohomology of a point takes the form $\mathbb{F}_2[\tau]$, where τ has degree $(0, 1)$. The dual $\mathbb{C}$-motivic Steenrod algebra $A_*^{\mathbb{C}}$ takes the form

$$\frac{\mathbb{F}_2[\tau][\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 = \tau \xi_{i+1}},$$

where the coproduct is given by the formulas

$$\tau_i \mapsto \tau_i \otimes 1 + \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \tau_k, \qquad \xi_i \mapsto \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k.$$

By convention, we let ξ_0 equal 1.

Comparison to the classical dual Steenrod algebra illuminates the subtleties of the $\mathbb{C}$-motivic dual Steenrod algebra. After inverting τ, the element ξ_{i+1} becomes decomposable, so $A_*^{\mathbb{C}}[\tau^{-1}]$ is a polynomial algebra over $\mathbb{F}_2[\tau^{\pm 1}]$ on generators τ_i.

This recovers the classical dual Steenrod algebra, where τ_i and ξ_{i+1} correspond to ζ_i and ζ_i^2, respectively.

On the other hand, after setting τ equal to zero, the result is an exterior algebra on generators τ_i tensored with a polynomial algebra on generators ξ_{i+1}. This structure is analogous to Milnor's description of the classical dual Steenrod algebra at odd primes (36).

As in the classical case, $\mathrm{Ext}_{A^{\mathbb{C}}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])$ can be computed by machine in a large range. Then $\mathbb{C}$-motivic Adams differentials can be determined by the standard methods. As in the classical case, dimension 61 seems to be the practical limit of this approach.

6. Algebraicity of the Cofiber of τ

There is a map $S^{0,-1} \to S^{0,0}$ in the $\mathbb{C}$-motivic stable homotopy category that induces multiplication by τ in $\mathbb{C}$-motivic cohomology. Therefore, we use the same notation τ for this map between spheres. Let S/τ be the mapping cone (or cofiber) of τ. Surprisingly, this stable two-cell complex turns out to have a remarkably algebraic structure.

The Adams–Novikov spectral sequence is a remarkably effective tool for computing stable homotopy groups (5). The input for this spectral sequence is the purely algebraic object $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, in which BP is the Brown–Peterson spectrum (49) and BP_*BP is the dual of the ring of its stable operations. The algebraic input itself is complicated and can be computed in a range by machine or by hand using the algebraic Novikov spectral sequence (18, 50).

Theorem 6.1. *The $\mathbb{C}$-motivic Adams spectral sequence that computes the motivic stable homotopy groups of S/τ is isomorphic to the algebraic Novikov spectral sequence* (51).

In fact, **Theorem 6.1** is a computational corollary of other more structural results. In particular, Gheorghe demonstrated that the $\mathbb{C}$-motivic spectrum S/τ is an E_∞-ring object in an essentially unique way (52), and the homotopy category of cellular S/τ-modules is equivalent to a derived category of BP_*BP-comodules (51).

Deformation theory provides a unifying perspective on this circle of ideas. The key insight is that $\mathbb{C}$-motivic cellular stable homotopy theory is a deformation of classical stable homotopy

theory (51), after completing at a prime p. From this perspective, the generic fiber of $\mathbb{C}$-motivic stable homotopy theory is classical stable homotopy theory, and the special fiber has an entirely algebraic description. The special fiber is the category of BP_*BP-comodules, or equivalently, the category of quasicoherent sheaves on the moduli stack of one-dimensional formal groups.

Theorem 6.1 is particularly valuable for computation. It means that the Adams spectral sequence for S/τ can be computed in an entirely algebraic manner, i.e., can be computed by machine in a large range. This observation leads to the following innovative program for computing classical stable homotopy groups: 1) Compute the $\mathbb{C}$-motivic Adams E_2-page by machine in a large range. 2) Compute the algebraic Novikov spectral sequence by machine in a large range, including all differentials and multiplicative structure. 3) Use **Theorem 6.1** to deduce the structure of the motivic Adams spectral sequence for S/τ. 4) Use the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \to S/\tau \to S^{1,-1}$$

and naturality of Adams spectral sequences to pull back and push forward Adams differentials for S/τ to Adams differentials for the motivic sphere. 5) Apply a variety of ad hoc arguments to deduce additional Adams differentials for the motivic sphere. 6) Invert τ to obtain the classical Adams spectral sequence and the classical stable homotopy groups.

The machine-generated data that we use in steps 1 and 2 are available at ref. 41.

As the dimension increases, the ad hoc arguments of step 5 become more and more complicated. Eventually, this approach will break down when the ad hoc arguments become too complicated to resolve. It is not yet clear when that will occur.

1. M. Mimura, On the generalized Hopf homomorphism and the higher composition. I. *J. Math. Kyoto Univ.* **4**, 171–190 (1964).
2. M. Mimura, M. Mori, N. Oda, Determination of 2-components of the 23- and 24-stems in homotopy groups of spheres. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **29**, 1–42 (1975).
3. M. Mimura, H. Toda, The $(n+20)$-th homotopy groups of n-spheres. *J. Math. Kyoto Univ.* **3**, 37–58 (1963).
4. N. Oda, On the 2-components of the unstable homotopy groups of spheres. I. *Proc. Japan Acad. Ser. A Math. Sci.* **53**, 202–205 (1977).
5. D. C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres* (Pure and Applied Mathematics, Academic Press, Inc., Orlando, FL, 1986), vol. 121.
6. H. Toda, *Composition Methods in Homotopy Groups of Spheres* (Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1962), vol. 49.
7. J.-P. Serre, Groupes d'homotopie et classes de groupes abéliens. *Ann. Math.* **58**, 258–294 (1953).
8. H. Freudenthal, Über die Klassen der Sphärenabbildungen I. Große Dimensionen. *Compositio Math.* **5**, 299–314 (1938).
9. H. Hopf, Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Math. Ann.* **104**, 637–665 (1931).
10. L. S. Pontryagin, Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one. *Dokl. Akad. Nauk SSSR* **70**, 957–959 (1950).
11. V. A. Rohlin, On a mapping of the $(n+3)$-dimensional sphere into the n-dimensional sphere. *Dokl. Akad. Nauk SSSR* **80**, 541–544 (1951).
12. J.-P. Serre, Homologie singulière des espaces fibrés. Applications. *Ann. Math.* **54**, 425–505 (1951).
13. J. F. Adams, On the structure and applications of the Steenrod algebra. *Comment Math. Helv.* **32**, 180–214 (1958).
14. J. P. May, "The cohomology of restricted Lie algebras and of Hopf algebras: Application to the Steenrod algebra," PhD thesis, Princeton University, Princeton, NJ (1964).
15. M. G. Barratt, M. E. Mahowald, M. C. Tangora, Some differentials in the Adams spectral sequence. II. *Topology* **9**, 309–316 (1970).

16. R. Bruner, A new differential in the Adams spectral sequence. *Topology* **23**, 271–276 (1984).
17. M. Mahowald, T. Martin, Some differentials in the Adams spectral sequence. *Topology* **6**, 349–369 (1967).
18. S. P. Novikov, Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.* **31**, 855–951 (1967).
19. S. O. Kochman, *Stable Homotopy Groups of Spheres: A Computer-Assisted Approach* (Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1990), vol. 1423.
20. S. O. Kochman, M. E. Mahowald, "On the computation of stable stems" in *The Čech Centennial (Boston, MA, 1993)* (Contemporary Math, American Mathematical Society, Providence, RI, 1995), Vol. 181, pp. 299–316.
21. F. Morel, V. Voevodsky, $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.* **90**, 45–143 (1999).
22. D. C. Isaksen, G. Wang, Z. Xu, More stable stems. arXiv:2001.04511v2 (17 June 2020).
23. A. Hatcher, Pictures of stable homotopy groups of spheres. pi.math.cornell.edu/~hatcher/stemfigs/stems.pdf. Accessed 4 September 2020.
24. M. A. Kervaire, J. W. Milnor, Groups of homotopy spheres. I. *Ann. Math.* **77**, 504–537 (1963).
25. J. Milnor, Differential topology forty-six years later. *Notices Am. Math. Soc.* **58**, 804–809 (2011).
26. M. A. Hill, M. J. Hopkins, D. C. Ravenel, On the nonexistence of elements of Kervaire invariant one. *Ann. Math.* **184**, 1–262 (2016).
27. G. Wang, Z. Xu, The triviality of the 61-stem in the stable homotopy groups of spheres. *Ann. Math.* **186**, 501–580 (2017).
28. D. C. Isaksen, Stable stems. *Mem. Am. Math. Soc.* **262**, 1269 (2019).
29. M. Behrens, M. Hill, M. J. Hopkins, M. Mahowald, Detecting exotic spheres in low dimensions using coker J. *J. Lond. Math. Soc.* **101**, 1173–1218 (2020).
30. J. F. Adams, On the groups $J(X)$. IV. *Topology* **5**, 21–71 (1966).
31. E. S. Devinatz, M. J. Hopkins, J. H. Smith, Nilpotence and stable homotopy theory. I. *Ann. Math.* **128**, 207–241 (1988).
32. M. J. Hopkins, J. H. Smith, Nilpotence and stable homotopy theory. II. *Ann. Math.* **148**, 1–49 (1998).

33. D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory* (Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1992), Vol. 128.

34. D. C. Isaksen, G. Wang, Z. Xu, Classical and ℂ-motivic Adams charts (2020). s.wayne.edu/isaksen/adams-charts. Accessed 4 September 2020.

35. N. E. Steenrod, *Cohomology Operations* (Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1962), Vol. 50.

36. J. Milnor, The Steenrod algebra and its dual. *Ann. Math.* **67**, 150–171 (1958).

37. R. R. Bruner, "Calculation of large Ext modules" in *Computers in Geometry and Topology* (Lecture Notes in Pure and Applied Mathematics, Dekker, NY, 1989), Vol. 114, pp. 79–104.

38. R. R. Bruner, "Ext in the Nineties"in *Algebraic Topology: Oaxtepec, 1991*, M. C. Tangora, Ed. (Contemporary Mathematics, American Mathematical Society, Providence, RI, 1993), Vol. 146, pp. 71–90.

39. R. R. Bruner, "The cohomology of the mod 2 Steenrod algebra: A computer calculation," Wayne State University Research Report, **37** (1997).

40. C. Nassau, Nullhomotopie. www.nullhomotopie.de. Accessed 4 September 2020.

41. G. Wang, Data files for the algebraic Novikov, Adams, and Adams-Novikov spectral sequences. github.com/pouiyter/morestablestems. Accessed 4 September 2020.

42. F. Morel, Théorie homotopique des schémas. *Asterisque* **256**, vi+119, (1999).

43. V. Voevodsky, Motivic cohomology with **Z**/2-coefficients. *Publ. Math. Inst. Hautes Études Sci.* **98**, 59–104 (2003).

44. V. Voevodsky, On motivic cohomology with **Z**/*l*-coefficients. *Ann. Math.* **174**, 401–438 (2011).

45. B. Gheorghe, D. C. Isaksen, A. Krause, N. Ricka, ℂ-motivic modular forms. arXiv:1810.11050 (25 October 2018).

46. P. Pstragowski, Synthetic spectra and the cellular motivic category. arXiv:1803.01804 (5 March 2018).

47. V. Voevodsky, Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.* **98**, 1–57 (2003).

48. V. Voevodsky, Motivic Eilenberg-Maclane spaces. *Publ. Math. Inst. Hautes Études Sci.* **112**, 1–99 (2010).

49. E. H. Brown, Jr., F. P. Peterson, A spectrum whose Z_p cohomology is the algebra of reduced p^{th} powers. *Topology* **5**, 149–154 (1966).

50. H. R. Miller, "Some algebraic aspects of the Adams-Novikov spectral sequence," PhD thesis, Princeton University, Princeton, NJ (1975).

51. B. Gheorghe, G. Wang, Z. Xu, The special fiber of the motivic deformation of the stable homotopy category is algebraic. arXiv:1809.09290 (14 May 2020).

52. B. Gheorghe, The motivic cofiber of τ. *Doc. Math.* **23**, 1077–1127 (2018).

MATHEMATICS