H Miller

The Theory of Green Functors

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Introduction

With field coefficients being an essential tool for calculations in non-equivariant cohomology, one could expect some kind of field coefficients to play an even more significant role in equivariant cohomology where the homological barrier to calculations is far more formidable. In a search for the Green functor analog of a field, the "Green-fields" of greatest interest should be the analogs of $\mathbb{Q}$ and $\mathbb{Z}/p$. Since the Burnside ring is the equivariant analog of the integers, these fields should be obtained from the "prime ideals" of the Burnside ring considered as a Green functor rather than a ring. Finding a field of fractions for an integral domain which is not a field complicated the location of these "Green fields" and led to a general investigation of Green functors as the Mackey functor analogs of rings. Thus, this project became, for the analogs of rings, a rough draft equivalent of an undergraduate text on the basics of ring theory.

We introduce Mackey functor analogs of almost every basic concept in ring theory--from prime ideals to nilpotent elements so it is difficult to keep track of when a word is used in its ordinary sense and when it is used in its Mackey functor sense. We chose to underline the Mackey functor terminology. Thus, a Green functor is a <u>ring</u>, and we hope to locate <u>fields</u> by studying the prime <u>ideals</u> of the Burnside <u>ring</u>. Does anyone have a better notation?

Section 1 of these notes is a basic introduction to Mackey functors. We have a new definition for them--as additive functors from a small additive category $\mathcal{B}$--which is much cleaner than previous definitions. In Section 1 we also show that the category of Mackey functors has a "tensor product" which we denote □. Using it, the multiplication for a ring R can be described as a map

$$\phi : R \square R \longrightarrow R$$

of Mackey functors. This description is much easier to work with than the older "pairings" description. Section 2 summarizes the formal aspects of ring theory, showing that the category of modules over a ring R is a perfectly respectable abelian category. We define such concepts as submodules, ideals, and chain conditions and introduce our definition of a field--a commutative ring with no nontrivial ideals.

Section 3 is devoted to relation between a ring R (or a module M) and its values R(b) (or M (b)) at the various objects in $\mathcal{B}$. We define concepts like integral domain and prime ideal which can only be defined in terms of elements. We also describe the basic connections between rings and rings. In this section, we encounter our first big surprise: Even in a field, a non-zero element may have more than one multiplicative inverse.

Section 4 summarizes the basic results of induction theory and introduces two new ideas. First, we show that most of classical induction theory is just a seach for units in rings of endomorphisms. Second, we take advantage of our definition of Mackey functors as additive functors from a small additive category by showing that another major aspect of induction theory is just very simple sheaf theory. The techniques of sheaf theory promise to yield some nice results here.

In Section 5, we begin a rather technical study of an especially well-behaved class of Mackey functors which includes fields, integral domains, division rings and simple modules over any ring. Here too, we show that any Galois extension $[F_1, F_2]$ can be regarded as a single field. We also show that representation theory sits inside ring theory as the study of modules over certain fields and and integral domains. The fields give the well-behaved half of representation theory and the domains give modular representation theory. Note that, for us, representation theory is commutative--not non-commutative--ring theory.

Section 6 is devoted to the study of Mackey functors modulo p^n. Here, we compute the prime and primary ideals of the Burnside ring. Another surprise appears. The Burnside ring is a commutative Noetherian ring, but primary decomposition does not work. An ideal of the Burnside ring can be decomposed--quite formally--into irreducible ideals, but the irreducible ideals need not be primary. Only very incomplete results are available on the irreducible ideals of the Burnside ring and these are not included in these notes. The basic message seems to be that the Burnside ring expresses the misbehavior of the integer primes that divide the order of the group in question by the difference between the irreducible and primary ideals. Thus, one expects to have to work a bit to understand the irreducible ideals.

Section 7 deals with integral domains and fields. We characterize these rings in terms of ordinary ring and field theory. Further, we show that the category of modules over a field has homological dimension zero.

The ring $A \otimes \mathbb{Z}[1/|G|]$ unfairly discriminates against some perfectly respectable maximal ideals in the Burnside ring and should be avoided. Section 8 introduces the correct replacement for this fashionable ring.

I would like to thank Andreas Blass, Zig Fiedorowicz, Mel Hochster, and Craig Huneke for innumerable helpful conversations. Moreover, Craig Huneke and Mel Hochster provided parts (b) and (f) of Proposition 7.2 respectively. Also, I would like to thank Jim McClure for sharing his notes on computational techniques in equivariant ordinary cohomology. Reading McClure's notes brought me to a full understanding of the central importance of the Weyl groups in the study of the Mackey functors.

1. An introduction to Mackey functors

This section contains a brief overview of the approach to Mackey functors developed in my earlier notes. In this approach, a Mackey functor is a contravariant additive functor from a small additive category $\mathcal{B}$ --which I call the Burnside category--to the category Ab of abelian groups. The section begins with a description of $\mathcal{B}$. The really new aspect of my approach to Mackey functors is the introduction of the box product $M \square N$ of two Mackey functors M and N; this can be characterized as a universal object for pairings of Mackey functors in the sense of Dress (( ), p 195). The main purpose of this section is to introduce this box product construction and to explore its basic properties.

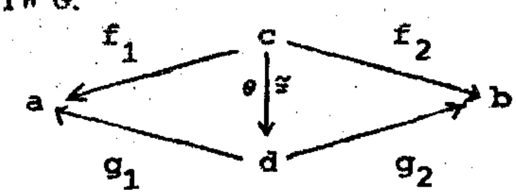
In order to define the category $\mathcal{B}$, we need to establish some basic notation. Throughout these notes, we work with Mackey functors for a fixed finite group G. The category $\mathcal{B}$ is constructed from the category $\hat{G}$ of finite G-sets and G-maps. The set of G-maps between finite G-sets a and b is denoted $\langle a,b\rangle$. For finite G-sets a and b, we write $a \prec b$ to indicate that there is a map of G-sets from a to b. If H and K are subgroups of G with H contained in K (denoted $H \leq K$), then the normalizer of H in K is denoted $N_K H$ and the Weyl group $N_K H/H$ is called $W_K H$. The class of subgroups of K conjugate in K to H is denoted $[H]_K$. If H and L are subgroups of K, then we write $[H]_K \leq [L]_K$ to indicate that H is conjugate (in K) to a subgroup (not necessarily proper) of L. If K is G, then we drop the subscripts in the above notation. Note that, for H, $L \subset K$, there is a K map from K/H to K/L if and only if $[H]_K \leq [L]_K$. Note also that the set of K maps of K/H into itself is isomorphic to $W_K H$. We denote the K-set

K/K by 1_K and the G-set G/G by 1. The number of elements in a set X is denoted $|X|$.

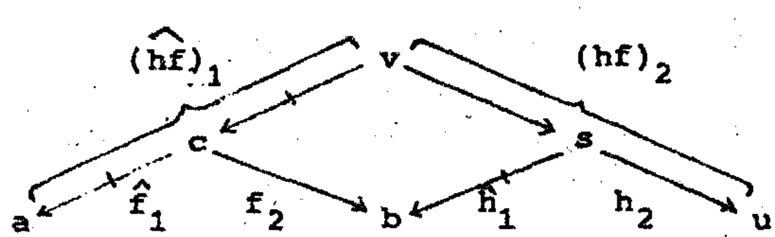
In order to describe $\mathcal{B}$, we first introduce a category $\mathcal{B}^+$. The objects of $\mathcal{B}$ and $\mathcal{B}^+$ are the finite G-sets (usually denoted by the small letters a, b, c, d, s, u, v, w). The maps from a to b in $\mathcal{B}^+$ have the form

$$f : a \overset{\hat{f}_1}{\longleftarrow\!\!\!\!+\!\!\!-} c \xrightarrow{f_2} b$$

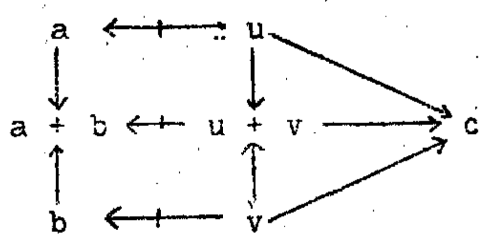
where $f_1 : c \longrightarrow a$ and $f_2 : c \longrightarrow b$ are maps in $\hat{G}$. The bar on the arrow ($\longleftarrow\!\!\!\!+\!\!\!-$) and the hat on $\hat{f}_1$ indicate that, in $\mathcal{B}^+$, $\hat{f}_1$ is considered as a map from a to c rather than a map from c to a. Maps of the form $f_2 : c \longrightarrow b$ (with $\hat{f}_1$ the identity) induce the restriction maps in familiar Mackey functors like the representation ring and so are called restrictions. They will often be generically designated by r. Maps of the form $a \longleftarrow\!\!\!\!+\!\!\!- c$ in $\mathcal{B}^+$ correspond to induction or transfer maps in the representation ring and are called transfers. They will often be designated by t. Two arrows, f and g, determine the same map in $\mathcal{B}^+$ if there is an isomorphism $\theta : c \longrightarrow d$ in $\hat{G}$, making the diagram below commute in $\hat{G}$.

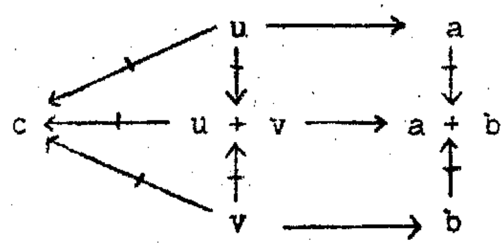


Composition in $\mathcal{B}^+$ is defined using pullbacks as in the diagram below for hf.



It is easy to check that the empty G-set 0 is both an initial and a terminal object for $\mathcal{B}^+$. If we denote the disjoint union of G-sets a and b by a + b, then it is easy to see that the diagram on the left below defines a one-to-one correspondence between maps out of a + b in $\mathcal{B}^+$ and pairs of maps out of a and b. Thus, a + b is the coproduct in $\mathcal{B}^+$. Similarly, the diagram om the right defines a correspondence which gives that a + b is the product of a and b in $\mathcal{B}^+$.





Since $\mathcal{B}^+$ has a zero object and biproducts, it follows formally ((cw), p 194) that the hom sets of $\mathcal{B}^+$ are abelian monoids and that composition is bilinear. It is easy to check that, in fact, the hom sets are free abelian monoids.

We obtain our category $\mathcal{B}$ from $\mathcal{B}^+$ by applying the usual construction to turn abelian monoids into abelian groups. Thus, the objects of $\mathcal{B}$ are the finite G-sets and the hom sets of $\mathcal{B}$ are free abelian groups whose elements are formal differences of maps in $\mathcal{B}^+$. Clearly 0 remains the zero object and a + b remains the biproduct of a and b in $\mathcal{B}$. We denote the set of maps in $\mathcal{B}$ from a to b by [a, b].

There is an obvious functor from $\mathcal{B}$ to its opposite category $\mathcal{B}^{op}$ which is the identity on objects and sends a map

$$f : a \xleftarrow{f_1} c \xrightarrow{f_2} b$$

to the map

$$Df : b \xleftarrow{f_2} c \xrightarrow{f_1} a$$

Occasionally, we will have a pair of functors F and G, one covariant and one contravariant, from $\mathcal{B}$ into some other category and a family of maps

$$\eta : Fa \longrightarrow Ga$$

which we will assert to be a natural transformation. In any such statement, an application of D to either functor to correct the variance is implicit.

If a and b are finite G-sets, then we denote their Cartesian product by $a \times b$. This cannot be the categorical product of a and b in $\mathcal{B}$ since that product is a + b; however, taking Cartesian products provides a natural pairing of $\mathcal{B}$ into itself which should be thought of as a tensor product. For any a, b, and c in $\mathcal{B}$, there is a natural isomorphism.

(1) $$[a \times b,\ c] \cong [a,\ Db \times c]$$

(note the use of D to correct the variance) which implies that $Db \times ?$ is right adjoint to $? \times b$ so that $\mathcal{B}$ is a symmetric monoidal closed category. Thinking of $\times$ as a tensor product and recalling the vector space isomorphism Hom $(V,W) \cong V^* \otimes W$ should make the adjunction above seem more natural.

Now that $\mathcal{B}$ is defined, we define a Mackey functor M to be a contravariant additive functor from $\mathcal{B}$ to the category Ab of abelian groups. We denote the category of Mackey functors by $\mathcal{M}$; it is clearly an abelian category satisfying the axiom AB5 needed for homological algebra. Using Dress's description of Mackey functors (see ( )), it is fairly easy to see that this definition of Mackey functors agrees with the older definitions (see my earlier notes or ( ) for details).

There is one obvious family of examples of Mackey functors--namely the representable functors $[?,\ b]$ for $b \in \mathcal{B}$.

<u>Definition 1.1</u> For any $b \in \mathcal{B}$, the representable functor $[?, b]$ is denoted $A_b \in \mathcal{M}$. The functor $A_1 = A$ is called the Burnside <u>ring</u>. The motivation for calling A the Burnside <u>ring</u> is that the value A(G/H) of A at the orbit G/H is the Burnside ring of H.

By the work of Day (( )), the symmetric monoidal structure on $\mathcal{B}$ induces a symmetric monoidal closed structure on the functor category $\mathcal{M}$. That is, for any two Mackey functors M and N, we have a tensor product-like construction $M \square N$. This construction is commutative and associative (up to natural isomorphisms) and it has unit A. The functor $? \square N$ has a right adjoint $\langle N, ? \rangle$ which we define below.

<u>Definition 1.2</u> For any Mackey functor M, and any $b \in \mathcal{B}$, let M_b be the Mackey functor defined on objects by $M_b(a) = M(b \times a)$ (for $a \in \mathcal{B}$) and on maps in the obvious fashion. Note that, by the adjunction isomorphism (1), the two possible interpretations of A_b are equivalent.

<u>Definition 1.3</u> For any Mackey functor M and N, the Mackey functor $\langle M, N \rangle$ is given on objects by

$$\langle M, N \rangle (b) = \text{Nat trans} (M, N_b) \qquad \text{for } b \in \mathcal{B} .$$

That is, the value of $\langle M, N \rangle$ at b is the maps in $\mathcal{M}$ from M to N_b.

Anyone interested in a precise definition of $M \square N$ should consult my earlier notes or Day's article (( )). For our purposes, it suffices that $\square$ and $\langle\ ,\ \rangle$ are adjoint and that $M \square N$ is completely characterized by the following result.

<u>Proposition 1.4</u> If L, M, and N are Mackey functors, then there is a one-to-one correspondence between maps

$$\theta : M \square N \longrightarrow L$$

and pairings $\theta : (M,N) \longrightarrow L$ in the sense of Dress (( ), p 195). A pairing is a collection of maps

$$\theta_b : Mb \otimes Nb \longrightarrow Lb \qquad \text{for } b \in \mathcal{B}$$

such that if $r : a \longrightarrow b$ is a restriction and $t : b \longleftarrow a$ is the associated transfer, then

$$r(\theta_b(x, y)) = \theta_a(r(x), r(y))$$
$$t(\theta_a(r(x), y')) = \theta_b(x, t(y'))$$
$$t(\theta_a(x', r(y))) = \theta_b(t(x'), y)$$

for $x \in M(b)$, $x' \in M(a)$, $y \in N(b)$, $y' \in N(a)$.

Readers unfamiliar with the relations above may acquire some feel for them by thinking about the relation between restriction, induction, and multiplication in the representation ring of G (This is the classic example of a pairing of Mackey functors) or by considering the relation between cup products in ordinary cohomology and the transfer map associated to a bundle or covering space.

The above characterization of $M \square N$ is generally the right one to use in constructing a map $M \square N \longrightarrow L$ and it occasionally suffices of analyzing the behavior of such a map. However, the following more sophisticated characterization is usually easier to use for analyzing maps of the form $M \square N \longrightarrow L$.

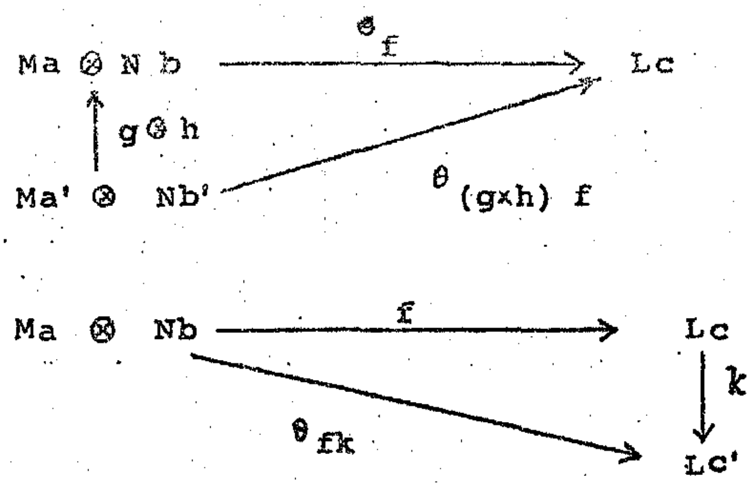
<u>Proposition 1.5</u> If M, N and L are Mackey functors, then a map

$$\theta : M \square N \longrightarrow L$$

determines and is determined by a family of maps

$$\theta_f : Ma \otimes Nb \longrightarrow Lc$$

indexed on the maps $f : c \longrightarrow a \times b$ in $\mathcal{B}$, such that, for maps $g : a \longrightarrow a'$, $h : b \longrightarrow b'$ and $k : c' \longrightarrow c$, the following diagrams commute



When working with a fixed pairing θ, we denote the maps θ_f by one of the following

$$Ma \otimes Nb \xrightarrow{\ f\ } Lc$$

$$Ma \otimes Nb \xrightarrow{c \to a\times b} Lc$$

The relation between the families θ of the two propositions is that the map

$$\theta : Mb \otimes Nb \longrightarrow Lb$$

of 1.4 is the map θ_Δ (from $\Delta : b \longrightarrow b \times b$) of 1.5. The map θ_f (from $f : c \longrightarrow a \times b$) of 1.5 is obtained from the maps of 1.4 as the following composite

$$Ma \otimes Nb \xrightarrow{\pi_1 \otimes \pi_2} M(a\times b) \otimes N(a \times b) \xrightarrow{\ \theta\ } L(a\times b) \xrightarrow{\ f\ } Lc$$

where $\pi_1 : a\times b \longrightarrow a$ and $\pi_2 : a \times b \longrightarrow b$ are the projections.

Any reader who is put off by the strangeness of the diagrams of Proposition 1.5 may rest assured--or be fairly warned--that if he continues to read this diligently these diagrams will become old familiar friends.

We need one more basic result on $\Box$ and $\langle\ ,\ \rangle$ --namely, the relations among these two functors, the representable functors A_b and the functors M_b of Definition 1.2.

Lemma 1.6 For any Mackey functor M and any a and b in $\mathcal{B}$, there are natural isomorphisms

$$A_a \square M \cong M \square A_a \cong M_a \cong \langle A_a, M\rangle$$

$$A_a \square A_b \cong A_{a\times b}$$

$$\langle A_a, A_b\rangle \cong A_{a\times b}.$$

Note that D must be used repeatedly to make sense of the naturality of these isomorphisms. We will generally think of M_b as $A_b \square M$ and therefore adopt the convention that it is covariant in b.

One more formality is needed to complete our introduction to Mackey functors. By the Yoneda lemma, for any Mackey functor M and any $b \in \mathcal{B}$, there is a one-to-one correspondence between maps

$$x : A_b \longrightarrow M$$

and elements $x \in M(b)$. As the category theorists have taught us, we make absolutely no distinction between the map x and the element x. Any reader who forgets this triviality will frequently find himself lost.

2. An introduction to *rings* and *modules*

In this section, we dispose of the purely formal aspects of *ring* theory. *Rings* (hitherto known as Green functors) and *modules* over *rings* are, of course, defined diagrammatically in the usual fashion. The elementary examples of *rings*--including the representable *rings*, polynomial *rings*, and endomorphism *rings*--are introduced. We show that the category of *modules* over any *ring* R is an abelian category enjoying all the pleasant properties of the category of ordinary modules over an ordinary ring. This section concludes with a discussion of *ideals* and of those concepts in *ring* theory--like chain conditions--that can be defined purely in terms of *ideals* without any reference to elements.

Anyone unaccustomed to the diagrammatic definitions of rings and modules may find ((CW), p 166-171) helpful in the definitions below.

Definitions 2.1 (a) A *ring* R consists of a Mackey functor R together with maps

$$1_R : A \longrightarrow R$$
$$\varphi : R \square R \longrightarrow R$$

such that the diagrams below commute

$$\begin{array}{ccc} R \square R \square R & \xrightarrow{\varphi \square 1} & R \square R \\ \big\downarrow{\scriptstyle 1 \square \varphi} & & \big\downarrow{\scriptstyle \varphi} \\ R \square R & \xrightarrow{\varphi} & R \end{array} \qquad \begin{array}{ccccc} A \square R & \xrightarrow{1_R \square 1} & R \square R & \xleftarrow{1 \square 1_R} & R \square A \\ & {\scriptstyle \cong}\searrow & \big\downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle \cong} & \\ & & R & & \end{array}$$

where the unlabeled isomorphisms are those expressing the fact that A is the unit for $\square$. The *ring* R is said to be commutative if the diagram

$$\begin{array}{ccc} R \square R & \xrightarrow[\cong]{\tau} & R \square R \\ & \searrow^{\varphi} \quad {}^{\varphi}\swarrow & \\ & R & \end{array}$$

commutes where τ is the commutativity isomorphism for $\square$.

(b) A left R-module for a ring R consists of a Mackey functor M together with an action map

$$\xi : R \square M \longrightarrow M$$

such that the diagrams below commute.

$$\begin{array}{ccc} R \square R \square M & \xrightarrow{\varphi \square 1} & R \square M \\ \downarrow{\scriptstyle 1 \square \xi} & & \downarrow{\scriptstyle \xi} \\ R \square M & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} A \square M & \xrightarrow{1_R \square 1} & R \square M \\ & \underset{\cong}{\searrow} & \downarrow{\scriptstyle \xi} \\ & & M \end{array}$$

A right R-module is defined analogously. If R is commutative, then the two notions coincide. A submodule N of M is just a subfunctor of M closed under the action of R. Note that R is both a left and right module over itself.

(c) A left ideal of R is just a submodule of R considered as a left module over itself. Right ideals are defined similarly. An (two-sided) ideal of R is just a subfunctor of R which is both a left and a right ideal.

(d) Homomorphisms of rings and modules are just maps of Mackey functors making the obvious diagrams commute.

Examples 2.2 (a) The unit isomorphisms $A \square A \cong A$ and $A \square M \cong M$ make A into a commutative ring and any Mackey functor M into an A-module. The unit map $1_R : A \longrightarrow R$ for any ring R is a ring homomorphism so that R may be thought of as an A-algebra.

(b) For any $b \in \mathcal{B}$, the maps

$$t : A \longrightarrow A_b$$
$$A_b \square A_b \cong A_{b\times b} \xrightarrow{\hat{\Delta}} A_b$$

derived from the isomorphisms of Lemma 1.6 and the maps $t : 1 \leftarrow\!\!+\!\!- b$ and $\hat{\Delta} : b \times b \leftarrow\!\!+\!\!- b$ make A_b into a commutative ring. Analogous maps make M_b an A_b-module for any Mackey functor M. Any transfer map $t : a \leftarrow\!\!+\!\!- b$ induces a ring homomorphism $A_a \longrightarrow A_b$ making A_b an algebra over A_a.

(c) If R and S are rings, then so is $R \square S$ under the maps

$$A \cong A \square A \xrightarrow{1_R \square 1_S} R \square S$$
$$R \square S \square R \square S \cong R \square R \square S \square S \xrightarrow{\varphi_R \square \varphi_S} R \square S$$

In particular, for any $b \in \mathcal{B}$ and any ring R, R_b is a ring and an algebra over both A_b and R via the maps

$$A_b \cong A_b \square A \xrightarrow{1 \square 1_R} A_b \square R \cong R_b$$
$$R \cong A \square R \xrightarrow{t \square 1} A_b \square R \cong R_b$$

Also, if M is an R-module, then M_b is an R_b module.

(d) If R is any ring, then R^{op} is the ring consisting of the same Mackey functor R, the same identity element and the multiplication

$$R^{op} \square R^{op} \cong R \square R \xrightarrow[\cong]{\tau} R \square R \xrightarrow{\varphi} R = R^{op}$$

--that is, the multiplication of R in the reverse order. Of course, if R is commutative, then the two rings are the same. Note that R is a left $R \square R^{op}$-module in the usual fashion and a two-sided ideal I of R is just an $R \square R^{op}$ submodule of R.

(e) If R is a commutative ring and $b \in \mathcal{B}$, then define the polynomial ring in one variable x_b of "rank" b to be the Mackey functor

$$R[x_b] = \bigoplus_{n=0}^{\infty} \left(R_{b^n}\right)/\Sigma_n$$

where Σ_n acts on R_{b^n} by permuting the copies of b in b^n. The identity element is the composite $A \xrightarrow{1_R} R = R_{b^0} \longrightarrow R[x_b]$ and the multiplication is derived from the maps

$$R_{b^m} \square R_{b^n} \cong (R \square R)_{b^{n+m}} \xrightarrow{\varphi} R_{b^{n+m}}$$

If S is an R-algebra, then it is easy to see that there is a one-to-one correspondence between elements of S(b) and <u>ring</u> homomorphism $R[x_b] \longrightarrow S$. A few minutes of playing with the images of maps from $A[x_b]$ into A should suffice to convince anyone that polynomial <u>rings</u> are strange and beautiful beasts full of mystery.

(f) If R is a <u>ring</u> and M is an R-<u>module</u>, then $\langle M, M\rangle_R$ is the subfunctor of $\langle M, M\rangle$ consisting of R-<u>module</u> homomorphisms. The Mackey functors $\langle M, M\rangle_R$ and $\langle M, M\rangle$ are <u>rings</u> under composition with identity elements the identity map $1: M \longrightarrow M$. Note that if R is commutative, then $\langle M, M\rangle_R$ can be given an R-<u>module</u> structure in the usual fashion. These endomorphism <u>rings</u> play a central role in our presentation of induction theory.

(g) If C is an abelian group and M is a Mackey functor, then we define the Mackey functor $C \otimes M$ by $(C \otimes M)(b) = C \otimes Mb$ for $b \in \mathcal{B}$. Clearly, if D is an ordinary ring and R is a <u>ring</u>, then $D \otimes R$ is a <u>ring</u> and if C is a D-module and M is an R-<u>module</u>, then $C \otimes M$ is a $D \otimes R$-<u>module</u>. Thus, one obvious source of <u>rings</u> is to take an ordinary ring D and form $D \otimes A$. To describe the result, we introduce the category $D \otimes \mathcal{B}$ with objects the finite G-sets. The set of maps from a to b in $D \otimes \mathcal{B}$ is just $D \otimes [a, b]$. With this notation, we have

<u>Lemma 2.3</u> If D is any ring, then the following categories are isomorphic

a) The category of modules over the ring $D\otimes A$

b) The category of contravariant additive functors from $\mathcal{B}$ to the category of D-modules.

c) The category of contravariant additive functors from $D\otimes\mathcal{B}$ to abelian groups.

The proof of this result is an elementary exercise in manipulating abelian functor categories. All three views of $D\otimes A$ modules have their applications.

For any ring R, we call the category of left R-modules R-mod and the category of right R-modules mod-R. One of the main purposes of this section is to show that these two categories enjoy all of the nice properties one usually associates with the category of modules over an ordinary ring. We begin with tensor products. If M and N are right and left R-modules respectively, then we can define the box product $M \square_R N$ over R as the coequalizer

$$M \square R \square N \rightrightarrows M \square N \longrightarrow M \square_R N$$

of the two action maps. If R, S, and T are three rings and M and N are an S-R and an R-T bimodule respectively, then $M \square_R N$ is an S-T bimodule and the usual associativity results hold for these tensor products. If M and N are both left (or right) R-modules, then we can define the Mackey functor $\langle M, N\rangle_R$ of R-module maps from M to N as a subfunctor of $\langle M, N\rangle$. Again, the usual bimodule remarks apply to $\langle M, N\rangle_R$. We record the basic properties of these constructions below.

Proposition 2.4 (a) If M an N are right and left R-modules respectively, then there are natural isomorphisms

(i) $M_c \square_R R_b \cong M_{c\times b}$

(ii) $R_b \square_R N_c \cong N_{b\times c}$

(iii) $\langle R_b, M_c\rangle_R \cong M_{b\times c}$ (v) $M_c \square_R N_b \cong (M \square_R N)_{c\times b}$

(iv) $\langle R_b, N_c\rangle_R \cong N_{b\times c}$ for all $b, c \in \mathcal{B}$

(b) If R, S, and T are rings and B, C, and D are S-R, R-T, and S-T bimodules respectively, then there is a natural isomorphism

$$\hom_{S-T}(B \square_R C, D) \cong \hom_{S-R}(B, \langle C, D\rangle_T)$$

(c) If R is commutative and M and N are R-modules, then $M \square_R N$ and $\langle M, N\rangle_R$ have natural R-module structures. Further, for R-modules M, N, and L, there is a natural isomorphism

$$\hom_R(M \square_R N, L) \cong \hom_R(M, \langle N, L\rangle_R).$$

Thus, R-mod is a symmetric monoidal closed category.

Except for the use of the flatness of A_b (see below) in the proof of (a), the proofs of these results are indistinguishable from the proofs for ordinary rings and modules.

We turn now to the behavior of limits and colimits in module categories. We have already observed that the category $\mathcal{M}$ of Mackey functors is an abelian category satisfying Grothendieck's condition AB5. As in the case of an ordinary ring, the functors $R \square ?$ (or $? \square R$) and $\langle R, ?\rangle$ provide a left and right adjoint respectively to the forgetful functor from R-mod (or mod-R) to $\mathcal{M}$. Thus, limits and colimits in R-mod (and mod-R) are obtained by taking the analogous limits and colimits in $\mathcal{M}$ and applying the natural R-module structures. It follows that R-mod and mod-R are abelian categories satisfying condition AB5. Note that limits in $\mathcal{M}$, R-mod and mod-R are taken point-wise. For example

$$\Big(\prod_{i \in I} M_i\Big)(a) = \prod_{i \in I} M_i a$$

$$(\bigoplus_{i\in I} M_i)(a) = \bigoplus_{i\in I} M_i a$$

for any indexed family $\{M_i\}_{i\in I}$ and $a\in\mathcal{B}$. Also a sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only if the sequences

$$0 \longrightarrow M'a \longrightarrow Ma \longrightarrow M''a \longrightarrow 0$$

are exact for all $a\in\mathcal{B}$. From this observation, Lemma 1.6 and Proposition 2.4(a), it follows that the functors $?\square A_a$, $?\ \square_R R_a$ and $R_a \square_R ?$ are exact for all $a\in\mathcal{B}$. Thus, the representable functors A_a are flat in $\mathcal{M}$ and the functors R_a are flat in R-mod and mod-R. As is always the case in a functor category, the representable functors A_a are projective and, as a family, they generate M. Further, if

$$c = \sum_{H\leq G} G/H \quad ,$$

then A_c is a projective generator for $\mathcal{M}$. Any projective in $\mathcal{M}$ is a direct summand of a direct sum of copies of A_c and so is flat by the usual argument. Since $R_a \cong R\square A_a$ is the free R-<u>module</u> (left or right) generated by A_a, it follows that R_a is projective in R-mod (or mod R) for any $a\in\mathcal{B}$. Also, R_c is a projective generator for R-mod (or mod-R). Again, it follows formally that any projective in R-mod (or mod-R) is flat. Being AB5-categories with a projective generator, $\mathcal{M}$, R-mod and mod-R all have enough injectives. Thus, they are perfectly respectable categories in which to do homological algebra. In particular, Tor and Ext derived functors exist for $\square$, $\square_R$, $\langle , \rangle$, and $\langle , \rangle_R$. The only hitch in all of this is that $\mathcal{M}$ is known to have infinite homological dimension. We will discuss the homological dimension of R-mod in a few special cases in later sections.

Remark 2.5 The good behavior of tensor products noted above suggests the possibility of translating into our context the Morita description of equivalences of module categories in terms of tensor products. However, since tensor products always commute with the functors $? \square A_b$, any direct translation of Morita theory would be applicable only to equivalences with the same commutativity property. Any work on Morita theory is further complicated by the fact that R-mod is not generated by R, but by R_c where c is $\sum_{H \leq G} G/H$. In spite of the generator problem, the usual proofs of the Morita characterization of equivalences appear to go through for those equivalences commuting with the functors $? \square A_b$. Some of our results in later sections involve equivalences between module categories over rings in two different categories of Mackey functors (that is, the ambient group G changes). It might be profitable to search for some generalization of Morita theory--along the lines of recent work on Morita theory for functor categories--which would describe these equivalences .

Some concepts in ring theory--like chain conditions--can be expressed purely in terms of the behavior of the submodules of a given module; such concepts translate formally to ring theory. In particular, we have the following obvious definitions.

Definition 2.6 (a) A left or right module over a ring R is Noetherian (Artinian) if every non-empty collection of submodules has a maximal (minimal) element.

(b) A ring R is left or right Noetherian (Artinian) if it is Noetherian (Artinian) as a left or right module over itself.

(c) A module M is simple if it has no non-trivial submodules and is semisimple if it is a direct sum of simple modules.

(d) A ring R is simple if it has no nontrivial (two-sided) ideals and is semisimple if it is a direct sum of simple rings.

(e) A division ring is a non-zero ring with no non-trivial left or right ideals.

(f) A field is a non-zero commutative ring with no non-trivial ideals.

(g) A maximal submodule N of a module M is a submodule strictly contained in M and not strictly contained in any other submodule.

(h) A maximal left (right or two-sided) ideal is a left (right or two-sided) ideal which is not the whole ring and which is not strictly contained in any other left (right or two-sided) ideal that is not the whole ring.

(i) A left (right or two-sided) ideal I is irreducible if, whenever $I = P \cap Q$ for P and Q left (right or two-sided) ideals, we have $I = P$ or $I = Q$.

Many basic results carry over without change in their statements or proofs. For example, we have

Lemma 2.7(a) If N is an R-submodule of an R-module M, then there is a one-to-one correspondence between R-submodules of M/N and R-submodules of M which contain N.

(b) If I is an ideal of R, then R/I is simple if and only if I is maximal (as a two-sided ideal).

(c) If I is an ideal of R, then R/I is a division ring if and only if I is maximal both as a left and a right ideal.

(d) If I is an <u>ideal</u> of a commutative <u>ring</u> R, then R/I is a <u>field</u> if and only if I is maximal.

(e) Any left (right or two-sided) <u>ideal</u> of a <u>ring</u> R other than the whole <u>ring</u> is contained in a maximal left (right or two-sided) <u>ideal</u>.

(f) If R is a left (or right) Noetherian <u>ring</u>, then any left (or right) <u>ideal</u> of R is a finite intersection of irreducible left (or right) <u>ideals</u>.

Note that the <u>rings</u> A_b for $b \in \mathcal{B}$ are Noetherian because any <u>ideal</u> is determined by its values at the orbits G/H for $H \leq G$ and each $A_b(G/H)$ is a finitely generated free abelian group. Also, if F is a field, then $F \otimes A_b$ is Artinian for the same reason. Lemma 2.7(f) suggests that it should be possible to classify all the <u>ideals</u> of A_b. As we will show in Section 4, such a classification would be quite useful in induction theory.

Some results do not carry over. For example, if M is a simple left R-<u>module</u>, then M need not be the quotient of R by a maximal <u>ideal</u>. The problem is that M(1) need not be nonzero. What is true is that if $M(b) \neq 0$ for $b \in \mathcal{B}$, then M is the quotient of the left R-<u>module</u> R_b by a maximal R-<u>submodule</u>. It is not necessary for this <u>submodule</u> to be a <u>module</u> over R_b.

Since simple <u>modules</u> are unexpectedly complicated, it is not clear how the Jacobson <u>radical</u> of a <u>ring</u> should be defined. The annihilator of an R-<u>module</u> M is just the kernel of the action map

$$R \longrightarrow \langle M, M \rangle.$$

It is clearly a two-sided <u>ideal</u> of R. The left Jacobson <u>radical</u> of R could be defined as the intersection of the annihilators of the simple left R-module or as the intersection of the maximal left ideals. It's not clear that these two possible definitions agree.

Note that the usual operations on ideals--like $I \cap J$, $I + J$ and IJ for ideals I and J are well-defined. In particular, IJ is the image of the map

$$I \square J \longrightarrow R \square R \xrightarrow{\varphi} R$$

and $I + J$ is the image of the map

$$R \square (I \oplus J) \longrightarrow R \square (R \oplus R) \longrightarrow R$$

if I and J are left ideals or

$$R \square (I \oplus J) \square R \longrightarrow R \square (R \oplus R) \square R \longrightarrow R$$

if I and J are two-sided ideals.

Remark 2.8 The observant reader may have already noted that the isomorphism $A \cong A \square A$ provides A with a Hopf algebra structure and the maps

$$A_b \xrightarrow{\Delta} A_{b \times b} \cong A_b \square A_b$$
$$A_b \longrightarrow A \qquad \text{(from } b \longrightarrow 1)$$

provide A_b with a coalgebra structure for any $b \in \mathcal{B}$. If $b \neq 1$, then A_b is not a Hopf algebra because the unit map does not behave properly with respect to either the counit or the comultiplication (and dually for the counit, unit and multiplication). These structures have apparently never been investigated--perhaps because there is no analog of Propositions 1.4 and 1.5 applicable to copairings. Nevertheless, it seems reasonable that an understanding of Hopf algebras would contribute to the understanding of equivariant Hopf spaces.

3. Rings, rings and elements.

Having described the known formal properties of rings and module categories, we now begin to investigate the basic structure of individual rings and modules. First, we introduce elements into our discussion. Untilizing elements, we define such basic concepts as principal ideals, units, zero divisors , integral domains, and prime (and primary) ideals. Some of the usual basic properties--and some surprises--follow easily from these definitions. To tie rings and modules to a more familiar world, we investigate the relations among a ring R, an R-module M and their values R(b) and M(b) at the elements b of $\mathcal{B}$. These relations yield the basic properties of simple modules, division rings, fields, and integral domains which we will exploit in later sections.

If R is a ring and M is an R-module, then by the Yoneda lemma, we can think of elements r of R (a) and m of M (b) (for a, b$\in\mathcal{B}$) as maps

$$A_a \xrightarrow{r} R \qquad\qquad A_b \xrightarrow{m} M$$

The composite

$$A_{a\times b} \cong A_a \square A_b \xrightarrow{r \square m} R \square M \xrightarrow{\xi} M$$

tells us that the product rm of r and m is an element of M(a×b). In particular, for elements r $\in$ R(a) and s $\in$ R(b), the product rs is in R(a×b). This is exactly the result one might expect by analogy with graded rings. We call these products "external" to distinguish them from the internal products which are defined later in this section. Experience suggests that one should always work with, and think in terms of, external (rather than internal) products whenever possible because they carry more information and are a closer analog to products in ordinary rings than are internal products.

Here we collect a host of element-dependent definitions.

Definition 3.1 (a) The principal left and two-sided ideals associated to an element $r \in R(b)$ are the images of the maps

$$R \square A_b \xrightarrow{1 \square r} R \square R \xrightarrow{\varphi} R$$

and

$$R \square A_b \square R \xrightarrow{1 \square r \square 1} R \square R \square R \xrightarrow{\varphi} R$$

respectively.

(b) An element $r \in R(b)$ is a right (left) unit if its associated left (right) principal ideal is all of R. An element $r \in R(b)$ is a unit if it is both a left and a right unit.

(c) An element $r \in R(a)$ is a zero divisor if there is an object $b \in \mathcal{B}$ and a non-zero element $s \in R(b)$ such that rs or sr is zero in $R(a \times b)$. It is sometimes useful to call this a b-zero divisor; the set of b in $\mathcal{B}$ for which r is a b-zero divisor tells how badly r misbehaves. Note that we can define annihilators of elements in a module in an analogous fashion.

(d) An element $r \in R(a)$ is (externally) nilpotent if there is an $n > 0$ such that r^n is zero in $R(a^n)$.

(e) A non-zero commutative ring D is an (integral) domain if it has no non-zero zero divisors.

(f) An ideal P of a commutative ring R is prime if it is not all of R and if, when rs is in $P(a \times b)$ for $r \in R(a)$ and $s \in R(b)$, either $r \in P(a)$ or $s \in P(b)$.

(g) An ideal I of a commutative ring R is primary if it is not all of R and if, whenever $rs \in I(a*b)$ for $r \in R(a)$ and $s \in R(b)$, either $r \in I(a)$ or $s^n \in I(b^n)$ for some $n > 0$.

Except for a few strange twists like 3.2(a) below, the expected basic results hold for the standard reasons. More results on units appear in Corollary 3.13.

Proposition 3.2(a) An element $x \in R(b)$ is a right unit if and only if the identity element $I_R : A \longrightarrow R$ can be written as the composite

$$A \xrightarrow{u} R \,\square\, A_b \xrightarrow{1 \,\square\, x} R \,\square\, R \xrightarrow{\varphi} R$$

for some map $u: A \longrightarrow R \,\square\, A_b$. Such a map corresponds to an element u of R(b) which may be thought of as a left inverse for x. However, u need not be unique even if x is a unit or even when R is commutative. Left units have an analogous description.

(b) The external product of two left (right or two-sided) units is a left (right or two-sided) unit.

(c) If $x \in R(a)$ maps to a left (right or two-sided) unit $y \in R(b)$ by any map $f : b \longrightarrow a$ in $\mathcal{B}$, then x is a left (right or two-sided) unit.

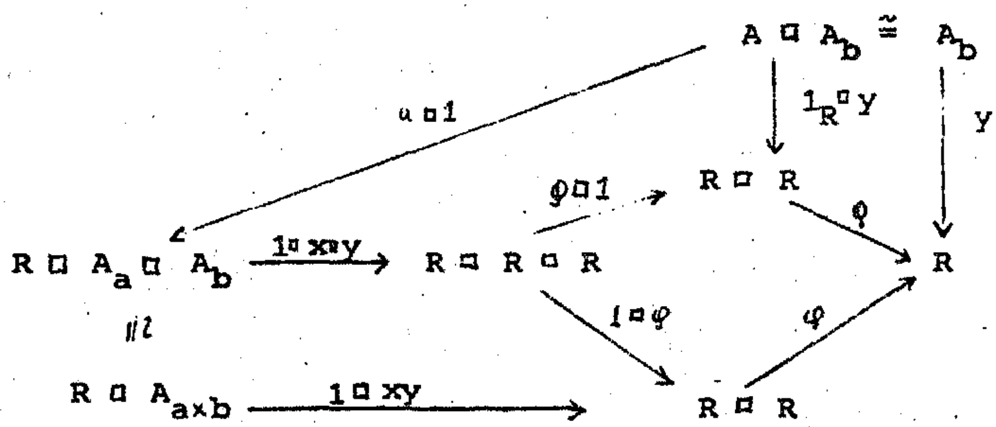
(d) A unit is not a zero divisor.

(e) Every non-zero element of a division ring is a unit. Thus, a division ring has no non-zero zero divisors and a field is an integral domain.

(f) If P is an ideal in a commutative ring R, then P is prime if and only if R/P is an integral domain. Further, P is primary if and only if every zero divisor in R/P is nilpotent.

Proof Part (a) is just the Yoneda lemma. Part (e) follows trivially from (d) and the definitions. Part (f) follows trivially from the definitions. Part (c) follows because any one or two-sided ideal containing x must contain y. The following proofs of parts (b) and (d) are a good illustration of an application of (a) and of the proof techniques peculiar to rings.

Let $y \in R(b)$ and let $x \in R(a)$ be a right unit with left inverse $u \in R(a)$. The commuting diagram below indicates that y is in the principal left ideal generated by $x y \in R(a \times b)$



If y is also a right unit, then any left ideal containing y must be all of R and we have (b). If xy is zero, then so is y since it is in the trivial ideal and we have (d).

The motivation for the diagram is that we want to say uxy = y, but this can't be said directly in terms of products because $uxy \in R(a \times a \times b)$ and $y \in R(b)$.

Remark 3.3 A word of caution about principal ideals is necessary. We say that an R-module M is finitely generated if there exist elements $x_i \in Ma_i$ for $1 \leq i \leq n$ such that the map

$$R \,\square\, \left(\bigoplus_{i=1}^{n} A_{a_i}\right) \xrightarrow{1 \,\square\, (\oplus x_i)} R \,\square\, \left(\bigoplus_{i=1}^{n} M\right) \longrightarrow M$$

is surjective. Via the isomorphism $\left(\bigoplus_{i=1}^{n} Ma_i\right) \cong M\left(\sum_{i=1}^{n} a_i\right)$, we see that

M is actually generated by a single element in $M(\sum_{i=1}^{n} a_i)$. Thus, any finitely generated module is, in fact, monogenic and any finitely generated ideal is principal. Anyone familiar with the ideal generated by x and y in the ordinary polynomial ring $\mathbb{Z}[x,y]$ will regard this behavior of rings as a bit strange. If we say that an ideal is strictly principal if it is generated by an element in $R(G/H)$ for some $H \leq G$, then we obtain a class of principal ideals which behave in a more intuitive fashion. Since the generator for the category R-mod is R_c where $c = \sum_{H\leq G} G/H$, this class of strictly principal ideals may be too small for some purposes; a better class might be those principal ideals generated by a single element of $R(c)$.

The key to understanding integral domains and simple modules is the following definition.

Definition 3.4 A subgroup H of G is a characteristic subgroup of a Mackey functor M if the map $M \longrightarrow M_{G/H}$ (from $1 \leftarrow\!\!\!+\!\!- G/H$) is injective and $M(G/K) = 0$ unless $[H] \leq [K]$. A Mackey functor M is said to have a characteristic subgroup if some $H \leq G$ is a characteristic subgroup of M. The basic properties of characteristic subgroups are

Lemma 3.5 (a) If M is H-characteristic, then $M = 0$ if and only if $M(G/H) = 0$. Thus, if a non-zero Mackey functor M has a characteristic subgroup, then that subgroup is unique up to conjugacy.

(b) A Mackey functor M has a characteristic subgroup if and only if for every $b \in \mathcal{B}$ with $Mb \neq 0$, the map $M \longrightarrow M_b$ determined by $1 \leftarrow\!\!\!+\!\!- b$ is injective.

Proof (a) For any $b \in \mathcal{B}$, $G/H \times b$ breaks up as a sum $\sum G/H_i$ of orbits with $[H_i] \leq [H]$ and H does not appear among the H_i unless there is a map $G/H \longrightarrow b$. Thus, if M is H-characteristic, then $M_{G/H}(b)$ is either zero or a direct sum of copies of M(G/H). Since $M \longrightarrow M_{G/H}$ is injective, it follows that M is zero if M(G/H) is. If H and K are both characteristic subgroups for $M \neq 0$, then M(G/H) and M(G/K) are both non-zero and we must have $[H] = [K]$.

(b) If M is H-characteristic, then $M_b \neq 0$ only if there is a map $G/H \longrightarrow b$. For such b, the map $M \longrightarrow M_{G/H}$ factors through the map $M \longrightarrow M_b$ and so this second map must be injective. On the other hand, assume the maps $M \longrightarrow M_b$ are injective when $M(b) \neq 0$. Let H be a smallest (in terms of number of elements) subgroup with $M(G/H) \neq 0$ (We can assume $M \neq 0$ since 0 is H-characteristic for every subgroup H). Suppose $M(G/K) \neq 0$. Then $M_{G/H}(G/K) = M(G/H \times G/K) \neq 0$. But then M(G/L) $\neq 0$ for some orbit G/L in $G/H \times G/K$. If $[H] \nleq K$, then we must have $[L] < [H]$ which is impossible by the minimal nature of $[H]$.

<u>Corollary 3.6</u> (a) If M is a simple <u>module</u> over a <u>ring</u> R, then M has a characteristic subgroup.

(b) Simple <u>rings</u> (which include <u>fields</u> and division <u>rings</u>) have characteristic subgroups.

<u>Proof</u> (a) Let M be a simple <u>module.</u> For any $b \in \mathcal{B}$, the map $M \longrightarrow M_b$ is either zero or injective. If M(b) is not zero, then the map $M(b) \longrightarrow M_b(b) = M(b \times b)$ is a split injection (by the map $M(b \times b) \longrightarrow M(b)$ from $\Delta : b \longrightarrow b \times b$) and so is not zero.

(b) A simple <u>ring</u> R is a simple <u>module</u> over $R \square R^{op}$.

The basic map $M \longrightarrow M_b$ has an alternate description from which it follows that integral domains have characteristic subgroups.

Lemma 3.7 Let M be a module over a ring R and let $1_b \in R(b)$ be the restriction of $1_R \in R(1)$. Then the map

$$M \longrightarrow M_b$$

is just (external) multiplication by 1_b.
The proof is just a diagram chase using Proposition 1.5.

Corollary 3.8 (a) If R is a ring such that $1_b \in R(b)$ is not a zero divisor whenever it is non-zero, then R has a characteristic subgroup.

(b) Rings whose only zero divisors are nilpotent have characteristic subgroups. In particular, integral domains have characteristic subgroups. Also, if P is a primary ideal in a commutative ring R, then R/P has a characteristic subgroup.

The proof of part (a) of this corollary follows from Proposition 3.9(b) below which gives that R(b) = 0 if and only if $1_b = 0$. For (b), note that if 1_b is not zero, then it is not nilpotant because R(b) is a direct summand of $R(b^n)$.

Corollaries 3.6 and 3.8 should suffice to convince those interested in Mackey functors that Mackey functors with characteristic subgroups are important. Section 5 is devoted to a detailed study of their very pleasant properties.

We have just about exhausted what can be said (to date) about ring theory without appealing to ring theory. The following proposition surveys the basic connection between rings and rings. The proofs are all easy exercises in chasing diagrams of the form introduced in Proposition 1.5.

Proposition 3.9 Let R be a ring and M be a left module over R.

(a) R(1) is a ring, and for any $b \in \mathcal{B}$, M(b) is an R(1) module. The unit of R(1) is the element $1_R : A \longrightarrow R$. The multiplication on R(1) and the action of R(1) on M(b) are given by

$$R(1) \otimes R(1) \xrightarrow{1 \cong 1\times 1} R(1) \qquad R(1) \otimes M(b) \xrightarrow{b \cong 1\times b} M(b)$$

Any map $f : b \longrightarrow a$ in $\mathcal{B}$ induces an R(1) module map $f : M(a) \longrightarrow M(b)$.

(b) For any $b \in \mathcal{B}$, R(b) is a ring and M(b) is a module over R(b). The unit of R(b) is the restriction 1_b of $1_R \in R(1)$. The (internal) multiplication on R(b) and the action of R(b) on M(b) are given by

$$R(b) \otimes R(b) \xrightarrow{\Delta:\, b \to b\times b} R(b) \qquad R(b) \otimes M(b) \xrightarrow{\Delta:\, b \to b\times b} M(b).$$

(c) Any restriction map (or conjugation) $r: a \longrightarrow b$ in $\mathcal{B}$ induces a ring homomorphism

$$r : R(b) \longrightarrow R(a)$$

In particular, the restriction $R(1) \longrightarrow R(b)$ is a ring homomorphism and R(1) acts on M(b) through this map. Note that transfers need not induce ring honomorphisms!

(d) If R is a division ring, then R(1) is a division ring and for $x \in R(1)$, any inverse

$$t: A \longrightarrow R$$

(as in Proposition 3.2a) is the actual inverse of x and so is uniquely determined. Note that for $b \neq 1$, R(b) need not be a division ring; it usually has zero divisors (in the ordinary ring sense).

(e) If R is commutative, then so are the R(b) for $b \in \mathcal{B}$.

(f) If R is an integral domain, then R(1) is an integral domain and if R is a field, then so is R(1). Again, R(b), for $b \neq 1$, can have zero divisors.

(g) If $P \subset R$ is a prime (or primary) ideal, then $P(1) \subset R(1)$ is a prime (or primary) ideal.

(h) If R is commutative, then An $x \in R(b)$ is (externally) nilpotent if, and only if, it is nilpotent when considered as an element of the ordinary ring R(b).

(i) If $x \in R(b)$ is a b-zero divisor in R, then x is a zero divisor when considered as an element in the ordinary ring R(b).

Results like (d), (f), and (i) above begin to illustrate the notational problem of keeping rings and rings separate. Certainly, matters become confusing when an element $r \in R(b)$ is a unit in the ring R and a zero divisor in the ring R(b).

Remark 3.10 The action of R(1) on M(b) for all $b \in \mathcal{B}$ can be given a Mackey functor description. For any Mackey functor M and any $b \in \mathcal{B}$, there is a natural map

$$M(b) \otimes A_b \longrightarrow M$$

which takes $m \otimes f$ to $f(m) \in M(a)$ for $m \in M(b)$ and $f \in A_b(a) = [a, b]$. For a ring R, this gives a map $R(1) \otimes A \longrightarrow R$ which can easily be seen to be a ring homomorphism. The action of R(1) on M(b) (for any R-module M) is via this ring homomorphism.

The maps $R(1) \otimes R(b) \xrightarrow{b \cong 1 \times b} R(b)$ of Proposition 1.5 induce a map

$$\theta : R(1) \otimes R \longrightarrow R$$

for any ring R. This map is a ring homomorphism if R is commutative (or more generally if R(1) is in the center of R). Also, the map $R \cong \mathbb{Z} \otimes R \longrightarrow R(1) \otimes R$ determined by $1_R \in R(1)$, is a right inverse

...

for θ. It seems likely that this pair of maps will be useful in relating the structure of R(1) and R.

Since R(1) is a ring when R is a ring, the following definition makes sense.

Definition 3.11 If R is a ring, then the integral characteristic of R is the characteristic of R(1). The charactistic of R is the kernel of the unit map $A \longrightarrow R$ (which is an ideal in A).

Note that if R is a division ring, field, or integral domain, then its integral characteristic is a prime. Also, if P is a primary ideal in a commutative ring R, then R/P has a prime power integral characteristic. We will see in Section 6 that for division rings, fields, and domains the characteristic ideal is determined by the integral characteristic p and the characteristic subgroup H. This ideal in A is denoted q(H,p).

The key to understanding units in a ring is the following corollary of Proposition 1.5 which can be used to compute principal ideals.

Proposition 3.12 If $L \square M \longrightarrow N$ is a map of Mackey functors and $m \in M(b)$ for $b \in \mathcal{B}$, then the map

$$L_b \cong L \square A_b \xrightarrow{1 \square m} L \square M \longrightarrow N$$

is given by

$$L(c\times b) \xrightarrow{1\otimes m} L(c\times b) \otimes M(b) \xrightarrow{c\times b \xrightarrow{1\times\Delta} c\times b\times b} N(c\times b) \xrightarrow{t} N(c)$$

for any $c \in \mathcal{B}$. Here the first map takes $\ell \in L(c\ b)$ to $\ell\otimes m$ and the last map is the transfer $c \longleftarrow\!\!\!+ c\times b$ determined by the projection $c\times b \longrightarrow c$.

Corollary 3.13 (a) An element $x \in R(b)$ is a right unit if and only if there is a $u \in R(b)$ such that $t(u \circ x) = 1_R \in R(1)$ where $u \circ x$ is the internal product in $R(b)$ and $t : 1 \leftarrow\!\!\!+\, b$ is the transfer. The element u is a left inverse of x in the sense of Proposition 3.2(a). A dual result applies to left units so $u \in R(b)$ is a left inverse of x if and only if $x \in R(b)$ is a right inverse of u.

(c) For any $b \in \mathcal{B}$, $R(b)$ contains a one-sided unit if and only if the transfer map $R(b) \longrightarrow R(1)$ is surjective.

(d) The image of the map $R_b \longrightarrow R$ induced by $b \longrightarrow 1$ is the principal left (or right) ideal of R generated by $1_b \in R(b)$. Thus, $R(b)$ contains a one-sided unit if and only if 1_b is a unit.

(e) $R(b)$ contains a unit if and only if there is an $x \in R(b)$ whose associated principal two-sided ideal is all of R.

(f) If $x, y \in R(b)$ and the internal product $x \circ y \in R(b)$ is a right unit, then y is a right unit. Also, if $x \circ y$ is a left unit, then x is a left unit.

(g) If $x \in R(a)$ and $y \in R(b)$ and the external product $xy \in R(a \times b)$ is a right unit, then so is y and if xy is a left unit, so is x.

Proof (a) An element $x \in R(b)$ is a right unit if and only if 1_R is in the left ideal generated by x. The condition for 1_R to be in this ideal can be seen immediately from Proposition 3.12.

(c) This is a trivial corollary of (a) since $R(b) \longrightarrow R(1)$ is a map of $R(1)$ modules.

(d) This follows from Proposition 3.12 by inspection.

(e) If $R(b)$ contains a <u>unit</u> then the associated left <u>ideal</u> of $1_b \in R(b)$ is all of R so the two-sided principal <u>ideal</u> must be all of R also. On the other hand, the value at 1 of the two-sided <u>ideal</u> generated by $x \in R(b)$ is just the image of the map

$$R \sqcup R^{op}(b) \longrightarrow R \sqcup R^{op}(b) \otimes R(b) \xrightarrow{\Delta} R(b) \xrightarrow{t} R(1)$$

obtained from the action of $R \sqcup R^{op}$ on R. If this principal ideal is all of R, then $t : R(b) \longrightarrow R(1)$ must be surjective and the remainder of (e) follows from (c).

(f) It suffices to show that $x \cdot y$ is in the left <u>ideal</u> generated by y and the right <u>ideal</u> generated by x. The image of $x \otimes y$ under the map

$$R(b) \otimes R(b) \xrightarrow{\hat{\Delta} \otimes 1} R(b \times b) \otimes R(b) \xrightarrow{b \times b \xrightarrow{1 \times \Delta} b \times b \times b} R(b \times b) \xrightarrow{\hat{\pi}_1} R(b)$$

can be computed to be $x \circ y$ so that $x \circ y$ is in the left ideal generated by y. The other result follows similarly.

(g) The external product $xy \in R(a \times b)$ is the internal product of $\pi_1 x$ and $\pi_2 y$ where $\pi_1 : a \times b \longrightarrow a$ and $\pi_2 : a \times b \longrightarrow b$ are projections. The result now follows from (f) and Proposition 3.2(c).

4. Remarks on induction theory

Our basic tools for analyzing Mackey functors in subsequent sections are induction theorems. Roughly speaking, an induction theorem for a Mackey functor M says that there is a b in $\mathcal{B}$ such that all the values of M are determined by the values Ma for a $\prec$ b in $\mathcal{B}$. The classical induction theorems are those which assert that, for some ring R, the R-representation ring of any finite group G is determined by the R-representation rings of some class of small subgroups of G. The induction theorems of interest to us here are those applicable to division rings, simple modules and integral domains.

This section provides a summary of the induction-theoretic results we need later. It divides naturally into two parts. In the first, we introduce the three basic types of induction theorems we employ and describe the relations among them. This material is drawn from Dress's basic article on induction theory ( ). The only new result in the first part is the observation that if one thinks in terms of units in endomorphism rings, then one acquires a new intuition for the basic results. The second part of this section is devoted to apparently new results on the type of induction we employ most often. The key to these results is a new understanding of the relation between Amitsur cohomology and induction theory in terms of sheaf theory for abelian functor categories.

The simpliest sort of induction theorem is like the classical theorem which asserts that every representation of a finite group G can be obtained by induction from representations of the elementary subgroups of G. In our notation, such a result says that, for some b in $\mathcal{B}$, the transfer map

$$M(b) \longrightarrow M(1)$$

is surjective. Such a result puts an upper bound on the size of M(1); however, to completely determine M(1), it is necessary to specify the kernel of the induction map. More sophisticated versions of this type of theorem specify the kernel, but we do not need them here.

For our purposes, it is more useful to have M(1) as a subgroup of some group than as a quotient group. Thus, the form of induction theorem we employ most often is the following:

<u>Definition 4.1</u> For $b \in \mathcal{B}$, a Mackey functor M satisfies b-injective induction if the diagram

$$M \longrightarrow M_b \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} M_{b \times b}$$

obtained from the diagram

$$b \times b \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} b \longrightarrow 1$$

in $\hat{G}$, is an equalizer diagram.

Note that this form of induction describes the whole of M and not just the value M(1).

By examining the decomposition of $c \times b$ and $c \times b \times b$ (for $c \in \mathcal{B}$) into orbits, one can easily see that, if M satisfies b-injective induction, then the value of M at any c in $\mathcal{B}$ is determined by the values Ma for $a \prec b$. We have already noted that for certain Mackey functors--such as division <u>rings</u>, integral <u>domains</u>, and simple <u>modules</u>--there is an $H \leq G$ such that M(G/K) is zero unless $[H] \leq [K]$. If such a Mackey functor satisfied G/H-injective induction, then clearly it would be almost trivial to compute all of its values. We will see that this is exactly what happens for division rings, fields, and nice integral domains.

Unfortunately, b-injective induction--for our purposes, the most useful form of induction--seems almost impossible to prove directly. For this reason, we are forced to consider two much stronger forms of induction.

Definition 4.2 (see ( )) For $b \in \mathcal{B}$, a Mackey functor M is b-projective if the transfer map

$$M_b \longrightarrow M$$

is a split surjection and is b-injective if the restriction map

$$M \longrightarrow M_b.$$

is a split injection.

Our first objective in this section is to establish Dress's basic results relating the types of induction defined above. Note that, for any ring R, the surjectivity of $R(b) \longrightarrow R(1)$ is equivalent to the existence of a unit in R(b). This observation is the key to our approach to induction.

In order to relate the various types of induction, we must first study the ring $\langle M, M \rangle$ of endomorphisms of a Mackey functor M. By Definition 1.3, an element f in $\langle M, M \rangle(b)$ for $b \in \mathcal{B}$, is just a map

$$f: M \longrightarrow M_b = \langle A_b, M \rangle.$$

By the adjunction between $? \square A_b$ and $\langle A_b, ? \rangle$, such an f may be regarded as a map

$$\tilde{f} : M \square A_b \cong M_b \longrightarrow M.$$

For b and c in $\mathcal{B}$, an element of $\langle M, M\rangle(b\times c)$ may be viewed in any of the following forms:

$$h : M \longrightarrow M_{b\times c}$$

$$\tilde{h} : M_{b\times c} \longrightarrow M$$

$$h_{b,c} : M_b \longrightarrow M_c$$

$$h_{c,b} : M_c \longrightarrow M_b$$

The following basic lemma relates these forms, characterizes the transfer for $\langle M, M\rangle$ and describes the composition of maps which gives $\langle M, M\rangle$ is ring structure.

Lemma 4.3 (a) For any map $f : M \longrightarrow M_b$, the map $\tilde{f} : M_b \longrightarrow M$ is the composite

$$M_b \xrightarrow{f_b} M_{b\times b} \longrightarrow M$$

where the second map comes from the map $b\times b \xleftarrow{\hat{\Delta}} b \longrightarrow 1$ in $\mathcal{B}$.

(b) For any map $\tilde{f}: M_b \longrightarrow M$, the map $f : M \longrightarrow M_b$ is the composite

$$M \longrightarrow M_{b\times b} \xrightarrow{\tilde{f}_b} M_b$$

where the first map comes from the map $1 \longleftarrow b \xrightarrow{\Delta} b\times b$ in $\mathcal{B}$.

(c) The image of $f\in\langle M, M\rangle(b)$ under the transfer map $\langle M, M\rangle(b) \longrightarrow \langle M, M\rangle(1)$ is given by either of the following composities:

$$M \xrightarrow{f} M_b \longrightarrow M$$

$$M \longrightarrow M_b \xrightarrow{f} M$$

where the unlabeled maps both come from the projection $b \longrightarrow 1$ in $\hat{G}$.

(d) If $f\in\langle M, M\rangle(b)$ and $g\in\langle M, M\rangle(c)$, then the external product $fg\in\langle M, M\rangle(b\times c)$ is given by either of the following composites:

$$fg : M \xrightarrow{g} M_c \xrightarrow{f_b} M_{b\times c}$$

$$\widetilde{(fg)} : M_{b\times c} \xrightarrow{\tilde{g}_b} M_b \xrightarrow{\tilde{f}} M.$$

The equivalence between b-injectivity and b-projectivity now follows easily.

<u>Proposition 4.4</u> (see ( )) For any Mackey functor M and any $b \in \mathcal{B}$, the following are equivalent:

(a) M is b-projective.

(b) M is b-injective.

(c) $\langle M, M\rangle$ (b) contains a <u>unit</u> for the <u>ring</u> $\langle M, M\rangle$.

(d) M is a direct summand of M_b

<u>Proof</u> By Lemma 4.3(c), statements (a) and (b) are just the two ways of saying that the identity map $1_M : M \longrightarrow M$ is in the image of the transfer $\langle M, M\rangle (b) \longrightarrow \langle M, M\rangle (1)$. By Corollary 3.13(c), this is equivalent to (c). Clearly, either (a) or (b) implies (d). To see that (d) implies the others, let f, $g \in \langle M, M\rangle$ (b) be maps representing M as a direct summand of M_b via the diagram

$$1_M : M \xrightarrow{g} M_b \xrightarrow{\tilde{f}} M.$$

By Lemma 4.3(a), (c), and (d) above, this composite is just the image of the internal product $f\circ g : M \longrightarrow M_b$ under the transfer map $\langle M, M\rangle (b) \longrightarrow \langle M, M\rangle (1)$. Thus, the internal product $f\circ g$ is a <u>unit</u> in $\langle M, M\rangle$ (b) and we have (c). Note that ~~by Corollary 3.13(f) both~~ f and g are one-sided <u>units</u> in $\langle M, M\rangle$ (b).

Dress's basic result on induction theory is now the result of a trivial observation about <u>units</u>.

Corollary 4.5 (see ( )) For any ring R and any $b \in \mathcal{B}$, the following are equivalent:

(a) Every (left or right) R-module M is b-projective.

(b) R is b-projective.

(c) The transfer map $R(b) \longrightarrow R(1)$ is surjective.

Proof Clearly, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). For (c) $\Rightarrow$ (a), note that (c) asserts that there is a unit in R(b). For any R-module M, the image of this unit under the action map

$$R \longrightarrow \langle M, M \rangle$$

is also a unit and so M is b-projective by Proposition 4.4.

Note that Proposition 3.2(e) now gives that if F is a field (or division ring) and $b \in \mathcal{B}$ with $F(b) \neq 0$, then any F-module V is b-projective. This is the key to our characterization of fields and their modules in Section 7.

To complete our survey of the basic results of induction theory, it suffices to show that b-projectivity implies b-injective induction.

Proposition 4.6 (see ( )) If the Mackey functor M is b-projective for some b in $\mathcal{B}$, then it satisfies b-injective induction.

Proof Let $\theta: M_b \longrightarrow M$ be any map representing the restriction map $M \longrightarrow M_b$ as a split injection. It is easy to see that $\theta: M_b \longrightarrow M$ and $\theta_b : M_{b\times b} \longrightarrow M_b$ represent

$$M \longrightarrow M_b \rightrightarrows M_{b\times b}$$

as a split equalizer ((CW), p 145).

Remark 4.7 Since the purpose of an induction theorem is to reduce the problem of computing the values of a Mackey functor M to that of computing its values on certain small subgroups of G, it is clearly desirable to identify the smallest collection of subgroups for which M satisfies induction. If by induction we mean b-projectivity for some b, then locating this collection of smallest subgroups translates into finding the least b in $\mathcal{B}$ (least with respect to $\prec$) for which $1_b : M \longrightarrow M_b$ is a unit in $\langle M, M \rangle$ (b). By Proposition 3.2(c) it suffices to consider those b in $\mathcal{B}$ of the form $\sum_{i=1}^{n} G/H_i$ with the conjugacy classes H_i all distinct. Certainly there is at least one minimal (with respect to $\prec$) such b for which 1_b is a unit. If b and b' are two such, then $1_{b\times b'} : M \longrightarrow M_{b\times b'}$ is a unit--being the exterior product of 1_b and $1_{b'}$. It follows that $b \cong b'$ --otherwise we would have $b\times b' \prec b$ and $b \not\prec b\times b'$ which yields a violation of the minimal nature of b. If b is the unique minimal element of $\mathcal{B}$ which is a sum of distinct orbits and for which 1_b is a unit in $\langle M, M \rangle$ (b), then by Proposition 3.2(c), M is a-projective for a in $\mathcal{B}$ if and only if $b \prec a$. This minimal element b is sometimes called the defect set or vertex of M.

The real difficulty which arises in working with b-injective induction instead of b-projectivity is that there is no general analog of Corollary 4.5 for b-injective induction. In fact, even for $G = \mathbb{Z}/2$, there is an integral <u>domain</u> satisfying b-injective induction with <u>modules</u> which do not satisfy b-injective induction. As a result, the only way of obtaining modules satisfying b-injective induction (which are not also b-projective) seems to be to construct them. Fortunately, this is easy.

<u>Definition 4.8</u> For any $b \in \mathcal{B}$ and any Mackey functor M, the (zero dimensional) b-Amitsur cohomology, H_bM, is the equalizer

$$H_bM \longrightarrow M_b \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} M_{b\times b}$$

Since $M \longrightarrow M_b$ equalizes the pair $M_b \rightrightarrows M_{b\times b}$ it factors uniquely as

$$\begin{array}{ccc} M & \longrightarrow & M_b \\ & \eta \searrow \quad \nearrow & \\ & H_bM & \end{array}$$

Note that the assignment of H_bM to M is a functor and η is a natural transformation. There are higher dimensional Amitsur cohomology groups (see ( )) which we will not discuss; hence we write H_bM instead of the usual $\overset{0}{H}_bM$.

<u>Proposition 4.9</u> (a) For any $b \in \mathcal{B}$ and any Mackey functor M, the Mackey functor H_bM satisfies b-injective induction. Thus, H_b is a functor from the category of Mackey functors to the category $\mathcal{M}_b$ of Mackey functors satisfying b-injective induction.

(b) The map $\eta: M \longrightarrow H_bM$ is universal among maps from M into Mackey functors satisfying b-injective induction. Thus η is an isomorphism if and only if M satisfies b-injective induction.

(c) The functor $H_b : \mathcal{M} \longrightarrow \mathcal{M}_b$ is left adjoint to the inclusion functor $\mathcal{M}_b \longrightarrow \mathcal{M}$.

<u>Proof</u> By its definition, η is an isomorphism if and only if M satisfies b-injective induction. Thus, to prove (a), it suffices to show that

$$\eta : H_bM \longrightarrow H_b(H_bM)$$

is an isomorphism. This follows from the diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\eta} & H_bM & \hookrightarrow & M_b & \rightrightarrows & M_{b\times b} \\
\eta\downarrow & & \downarrow\eta & & \cong\downarrow\eta & & \cong\downarrow\eta \\
H_b & \longrightarrow & H_b(H_bM) & \longrightarrow & H_b(M_b) & \rightrightarrows & H_b(M_{b\times b}) \\
\downarrow & & \downarrow & & \downarrow & & \\
M_b & \longrightarrow & (H_bM)_b & \hookrightarrow & (M_b)_b & &
\end{array}$$

in which the second and third rows are obtained by applying H_b and ? □ A_b respectively to the first row. The functor M_b is b-projective since $\Delta : b \longrightarrow b\times b$ induces a splitting of the restriction map $M_b \longrightarrow M_{b\times b}$. The isomorphisms and injections indicated above follow from this and the fact that H_bM is a subobject of M_b.

Since

$$H_bM \longrightarrow M_b \xrightarrow{\eta} H_b(M_b)$$

is an injection,

$$\eta : H_bM \longrightarrow H_b(H_bM)$$

must be an injection. Using the fact that $H_b(H_bM) \longrightarrow H_b(M_b)$ equalizes the pair $H_b(M_b) \rightrightarrows H_b(M_{b\times b})$ and the fact that H_bM is the equalizer of the pair $M_b \rightrightarrows M_{b\times b}$, it is easy to check that

$$\eta : H_bM \longrightarrow H_b(H_bM)$$

is surjective and therefore an isomorphism. The rest of the proof is formal nonsense.

The crux of the proposition is that we can canonically convert any Mackey functor into one satisfying b-injective induction. Moreover, this process of producing Mackey functors satisfying b-injective induction has a host of nice properties. For example, we have

Proposition 4.10 (a) The functor H_b (regarded as a functor from $\mathcal{M}$ to $\mathcal{M}_b$ or from $\mathcal{M}$ to $\mathcal{M}$) is left exact. In fact, H_b preserves all limits.

(b) If $M \in \mathcal{M}_b$ and $N \in \mathcal{M}$, then $\langle N,M\rangle \in \mathcal{M}_b$ and the map

$$\langle H_bN, M\rangle \xrightarrow{\eta^*} \langle N,M\rangle$$

is an isomorphism.

(c) If we define $M \square_b N$ to be $H_b(M \square N)$, then there is a natural isomorphism

$$\mathcal{M}_b(M \square_b N, L) = \mathcal{M}_b(M, \langle N,L\rangle)$$

for M, N and $L \in \mathcal{M}_b$. Thus, $\mathcal{M}_b$ is a symmetric monoidal closed category. The unit for $\square_b$ is H_bA.

(d) There is a natural map

$$\theta : H_bM \square H_bN \longrightarrow H_b(M \square N) = M \square_b N$$

for any Mackey functors M and N. Thus, if R is a ring and M is an R-module, then H_bR is a ring and H_bM is an H_bR-module. The identity element and multiplication of H_bR are

$$A \xrightarrow{\eta} H_bA \xrightarrow{H_b 1_R} H_bR$$

$$H_bR \square H_bR \xrightarrow{\theta} H_b(R \square R) \xrightarrow{H_b\varphi} H_bR$$

The action of H_bR on H_bM is given by

$$H_bR \square H_bM \xrightarrow{\theta} H_b(R \square M) \xrightarrow{H_b\xi} H_bM.$$

(e) Any Mackey functor which satisfies b-injective induction is a module over H_bA.

Proof Part (a) follows from the fact that limits commute with limits and the functor $? \square A_b$ preserves all limits.

For (b), if $M \in \mathcal{M}_b$, then $\langle N,M\rangle \in \mathcal{M}_b$ for any $N \in \mathcal{M}$ because $\langle N,?\rangle$ preserves limits and commutes with $? \square A_b$. That the map η^* is an isomorphism follows from the chain of isomorphisms

$$\begin{aligned}
\mathcal{M}_b(L, \langle N,M\rangle) &\cong \mathcal{M}(L, \langle N,M\rangle) \\
&\cong \mathcal{M}(N, \langle L,M\rangle) \\
&\cong \mathcal{M}_b(H_bN, \langle L,M\rangle) \\
&\cong \mathcal{M}(H_bN, \langle L,M\rangle) \\
&\cong \mathcal{M}(L, \langle H_bN,M\rangle) \\
&\cong \mathcal{M}_b(L, \langle H_bN,M\rangle)
\end{aligned}$$

for any $L \in \mathcal{M}_b$.

For (c), we have

$$\begin{aligned}
\mathcal{M}_b(M \square_b N,L) &\cong \mathcal{M}(M \square N,L) \\
&= \mathcal{M}(M, \langle N,L\rangle) \\
&= \mathcal{M}_b(M, \langle N,L\rangle)
\end{aligned}$$

for any M,N,L in $\mathcal{M}_b$. That H_bA is a unit for $\square_b$ follows from (b) since $\langle H_bA,N\rangle \cong \langle A,N\rangle \cong N$ for $N \in \mathcal{M}_b$.

For (d), the map Θ comes from $\eta : M \square N \longrightarrow H_b(M \square N)$ via the chain of adjunctions

$$\begin{aligned}
\mathcal{M}(M \square N, H_b(M \square N)) &\cong \mathcal{M}(M, \langle N,H_b(M \square N)\rangle) \\
&\cong \mathcal{M}(M, \langle H_bN,H_b(M \square N)\rangle) \\
&\cong \mathcal{M}_b(H_bM, \langle H_bN,H_b(M \square N)\rangle) \\
&\cong \mathcal{M}(H_bM, \langle H_bN,H_b(M \square N)\rangle) \\
&\cong \mathcal{M}(H_bM \square H_bN, H_b(M \square N))
\end{aligned}$$

The rest of (d) follows by inspection. Part (e) is a special case of (d) since any Mackey functor is an A-module and $H_bM \cong M$ if M satisfies b-injective induction. It should be noted that (c) through (e) follow from (b) by standard results in the theory of closed categories.

Note that the converse of (e) is false even for the group $G = Z/2$.

If R is a ring satisfying b-injective induction, then Proposition 4.10(d) suggests an approach to studying an R-module M which fails to satisfy b-injective induction. First, the R-module $H_b M$ must be understood and the the map $\eta: M \longrightarrow H_b M$ must be analyzed. If the map $M \longrightarrow M_b$ is injective, then so is η and this procedure has proved to be enlightening.

The right way to understand Propositions 4.9 and 4.10 is to recall that the category M of Mackey functors is a functor category and to note that the condition that

$$M \longrightarrow M_b \rightrightarrows M_{b \times b}$$

be an equalizer is the sheaf condition for a rather simple additive topology on $\mathcal{B}$. The functor H_b is just the sheafification functor. The best sources for additive sheaf theory seem to be Popescu ( ), Schubert ( ), and Stenström ( ). From them, we obtain

Proposition 4.11 The category $\mathcal{M}_b$ is an abelian category satisfying condition AB5. The functor $H_b(A_c)$, where $c = \sum_{H \leq G} G/H$, is a projective generator and $\mathcal{M}_b$ has enough projectives and injectives.

It seems quite likely that much of the work in Stenström ( ) on topologies for ordinary rings could be extended to apply to rings. Such an extension should offer considerable insight into $\mathcal{M}_b$ and b-injective induction.

Remark 4.12 Regarded as a functor from $\mathcal{M}$ to $\mathcal{M}$, H_b is left exact but not usually right exact. As a result, it has derived functors. These are easily seen to be the higher dimensional Amitsur cohomology groups H_b^n of Dress (   ). Many of the properties he asserts for them follow trivially from this observation.

5. H-characteristic and H-determined Mackey functors

From Section 3, recall that H is the characteristic subgroup of Mackey functor M if the restriction map $M \longrightarrow M_{G/H}$ is injective and if M(G/K) is zero unless $[H] \leq [K]$. Not every Mackey functor has a characteristic subgroup; but since fields, division rings, simple modules over any ring, and integral domains all have characteristic subgroups, the class of Mackey functors with characteristic subgroups is quite important. In this section, we introduce the machinery needed to investigate the structure of these Mackey functors. We also examine two other closely related classes of Mackey functors.

Definitions 5.1 Let $H \leq G$

(a) A Mackey functor M is H-bounded if M(G/K) = 0 for $[K] < [H]$ and M(G/H) ≠ 0 if M is non-zero.

(b) A Mackey functor M is H-determined if it is H-bounded and satisfies G/H-injective induction.

Note that H-characteristic Mackey functors are H-bounded, and H-determined Mackey functors are H-characteristic. Note also that if M is H-bounded, then $H_{G/H}M$ is H-determined. Clearly, the zero Mackey functor is H-bounded and H-determined for any $H \leq G$. A non-zero Mackey functor has at least one bounding subgroup (since there are only finitely many subgroups) and may have more than one. A non-zero H-characteristic Mackey functor has a unique (up to conjugacy) bound--namely, H. A non-zero Mackey functor need not be determined by a subgroup, but if it is determined, then a determining subgroup is also a characteristic subgroup and is therefore unique up to conjugacy.

From Corollaries 3.6 and 4.5, we obtain that division rings and fields have determining subgroups. Our basic result in this section is a classification of Mackey functors with determining subgroups.

For any subgroup H of G, the set of maps $\langle G/H, G/H\rangle$ is isomorphic to the Weyl group WH. Thus, for any Mackey functor M, M(G/H) has a WH-action, and evaluation at G/H gives a forgetful functor from the category of Mackey functors to the category of modules over the group ring $\mathbb{Z}[WH]$. Our characterization of H-determined Mackey functors is that this forgetful functor becomes an equivalence of categories when it is restricted to the full subcategory of $\mathcal{M}$ consisting of H-determined Mackey functors. Note that if R is a ring, then, by Proposition 3.9(c), WH acts on R(G/H) by ring automorphisms.

For any Mackey functor M, the image of the restriction map $M(1) \longrightarrow M(G/H)$ is contained in the set $M(G/H)^{WH}$ of WH-invariant elements of M(G/H). If M is H-characteristic, then the map $M(1) \longrightarrow M(G/H)$ is injective and we identify M(1) with its image in $M(G/H)^{WH}$. In particular, if M is H-determined, then M(1) is exactly $M(G/H)^{WH}$.

It should be obvious by now that this section is going to be littered with WH-actions. Unfortunately, the natural choices for these actions are a confused jumble of left and right actions. To bring some order into this chaos, we adopt the convention that all WH-actions are from the left (by acting through inverses when the natural action is on the right).

Our basic tool for working with H-bounded Mackey functors is the following elementary observation about finite G-sets.

Lemma 5.2 If $H \leq G$ and b is a finite G-set, then there is a one-to-one correspondence between orbits in $G/H \times b$ isomorphic to G/H and the set of maps $\langle G/H, b\rangle$. The correspondence is given by taking $f \in \langle G/H, b\rangle$ to the image of the map

$$G/H \xrightarrow{(1,f)} G/H \times b$$

From this, we obtain

Corollary 5.3 If M is an H-bounded Mackey functor and $b \in \mathcal{B}$, then the map

$$M(G/H \times b) \xrightarrow[\cong]{\oplus(1,f)} \bigoplus_{f \in \langle G/H, b\rangle} M(G/H)$$

is an isomorphism. Further, the effect of the projection maps $G/H \times b \xrightarrow{\pi_1} G/H$ and $G/H \times b \xrightarrow{\pi_2} b$ on M is described by the diagrams

$$\begin{array}{ccc} M(G/H \times b) & \xrightarrow{\oplus(1,f)} & \bigoplus_{f \in \langle G/H, b\rangle} M(G/H) \\ & {\scriptstyle \pi_1^*}\nwarrow \quad \nearrow{\scriptstyle \Delta} & \\ & M(G/H) & \end{array}$$

$$\begin{array}{ccc} M(G/H \times b) & \xrightarrow{\oplus(1,f)} & \bigoplus_{f \in \langle G/H, b\rangle} M(G/H) \\ & {\scriptstyle \pi_2^*}\nwarrow \quad \nearrow{\scriptstyle \oplus f} & \\ & M(b) & \end{array}$$

The transfer maps associated to the projections are described by analogous diagrams.

The basic implications of this corollary for H-bounded and H-characteristic Mackey functors are summarized by the following:

Proposition 5.4 (a) An H-bounded Mackey functor M is H-characteristic if and only if for every $b \in \mathcal{B}$ and every non-zero x in $M(b)$, there is a restriction map $f : G/H \longrightarrow b$ with $f(x)$ non-zero in $M(G/H)$.

(b) An H-characteristic, commutative ring R is an integral domain if and only if for every non-zero pair x, y in $R(G/H)$, there is a $g \in WH$ such that the internal product $x \circ (gy)$ is non-zero in $R(G/H)$.

(c) If R is an integral domain with characteristic subgroup H, then $R(GH)^{WH}$ is an integral domain and the non-zero elements of $R(G/H)^{WH}$ are not zero-divisors in R.

(d) An H-characteristic ring R is a division ring if and only if for every non-zero $x \in R(G/H)$ there exist $y, z \in R(G/H)$ such that $t(x \circ y) = t(z \circ x) = 1_R \in R(1)$ where $t : R(G/H) \longrightarrow R(1)$ is the transfer and the two products are internal.

(e) If M is H-bounded, then for any b in $\mathcal{B}$, the composite

$$M(b) \xrightarrow{t} M(1) \xrightarrow{r} M(G/H)$$

(where t is the transfer and r is the restriction) is just

$$(1) \qquad rt(x) = \sum_{\ell \in \langle G/H, b \rangle} \ell(x) \qquad \text{for } x \in M(b).$$

In particular, the composite $rt : M(G/H) \longrightarrow M(G/H)$ is just the trace of the WH-action. Note that when M is H-characteristic, formula (1) actually describes the transfer $t : M(b) \longrightarrow M(1)$.

Proof (a) If M is H-bounded, then M is H-characteristic if and only if the map

$$M(b) \xrightarrow{\pi_2} M(G/H \times b)$$

is injective for every b in $\mathcal{B}$. Part (a) follows immediately from the description of this map in Corollary 5.3.

(b) Assume first that R is an integral domain. Then for any non-zero x,y in R(G/H), the external product $xy \in R(G/H \times G/H)$ must be non-zero. It is easy to check that xy goes to the tuple $(x \cdot gy)_{g \in WH}$ under the isomorphism

$$R(G/H \times G/H) \cong \bigoplus_{g \in WH} R(G/H)$$

of Corollary 5.3. Now assume that the indicated condition on R(G/H) holds. For any non-zero $x \in R(a)$ and $y \in R(b)$ we must show that the product $xy \in R(a \times b)$ is non-zero. By (a), there exists maps $f: G/H \longrightarrow a$ and $h: G/H \longrightarrow b$ such that $\hat{f}(x)$ and $\hat{h}(y)$ are non-zero in R(G/H). The diagram

$$\begin{array}{ccc} R(a) \otimes R(b) & \xrightarrow{1} & R(a \times b) \\ \downarrow f \otimes h & & \downarrow f \times h \\ R(G/H) \otimes R(G/H) & \xrightarrow{1} & R(G/H \times G/H) \end{array}$$

commutes by Proposition 1.5 and it suffices to show that the external product of f(x) and h(y) is non-zero. This follows by reversing the first half of our argument.

(c) Part (c) is immediate from (b).

(d) If R is a division ring, then the required condition on R(G/H) holds because it is just the assertion that every non-zero element of R(G/H) is a unit. Assume the indicated condition on R(G/H) holds, then for any $b \in \mathcal{B}$ and any non-zero x in R(b), there is a map $f : G/H \longrightarrow b$ with $\hat{f}(x)$ non-zero in R(G/H). But then f(x) is a unit so x must be by Proposition 3.2(c).

(e) The map $rt : M(b) \longrightarrow M(G/H)$ comes from the composite

$$G/H \longrightarrow 1 \longleftarrow\!\!\!+\!\!- b$$

in $\mathcal{B}$. This composite is the same as the map

$$G/H \xleftarrow{\hat{\pi}_1} G/H \times b \xrightarrow{\pi_2} b.$$

Part (e) now follows from the characterizations of $\hat{\pi}_1$ and π_2 in Corollary 5.3.

Proposition 5.4 completes our basic remarks about H-bounded Mackey functors, and we turn now to the problem of constructing a functor from $\mathbb{Z}[WH]$-modules to Mackey functors which, in some sense, undoes the effect of evaluating at G/H.

<u>Definition 5.5</u> If V is a $\mathbb{Z}[WH]$-module, then the Mackey functor $J_{G/H}V$ is defined on b in $\mathcal{B}$ by

$$(J_{G/H}V)(b) = \Big(\bigoplus_{f\in\langle G/H,b\rangle} V\Big)^{WH}$$

Here, WH acts both on each of the summands V and on the indexing set $\langle G/H, b\rangle$ by precomposition (using the fact that $WH = \langle G/H, G/H\rangle$). If h is a map in $\mathcal{B}$ given by

$$h : a \xleftarrow{\hat{h}_1} y \xrightarrow{h_2} b,$$

then $\bar{h} : (J_{G/H}V)(b) \longrightarrow (J_{G/H}V)(a)$ is defined by the diagram

$$\begin{array}{ccc}
(J_{G/H}V)(b) = \Big(\bigoplus_{f\in\langle G/H,b\rangle} V\Big)^{WH} & \xrightarrow{\quad \bar{h} \quad} & \Big(\bigoplus_{f'\in\langle G/H,a\rangle} V\Big)^{WH} = (J_{G/H}V)(a) \\
\Big\downarrow \oplus\pi_{h_2\ell} & & \Big\downarrow \pi_{f'} \\
\bigoplus_{\substack{\ell\in\langle G/H,\, y\rangle \\ f' = h_1\ell}} V & \xrightarrow{\quad \nabla \quad} & V
\end{array}$$

where $\pi_{f'}$ is the projection onto the summand indexed by $f' : G/H \longrightarrow a$. It is easy to check that $J_{G/H}V$ is a Mackey functor. Any map $j : U \longrightarrow V$ of $\mathbb{Z}[WH]$-modules induces a map

$$J_{G/H}\,j : J_{G/H}U \longrightarrow J_{G/H}V$$

so $J_{G/H}$ is a functor from the category of $\mathbb{Z}[WH]$-modules to the category of Mackey functors. Note that the map

$$\oplus g : V \longrightarrow \left(\bigoplus_{g\in WH} V\right)^{WH} = (J_{G/H}V)(G/H)$$

induces an isomorphism between V and $(J_{G/H} V)(G/H)$.

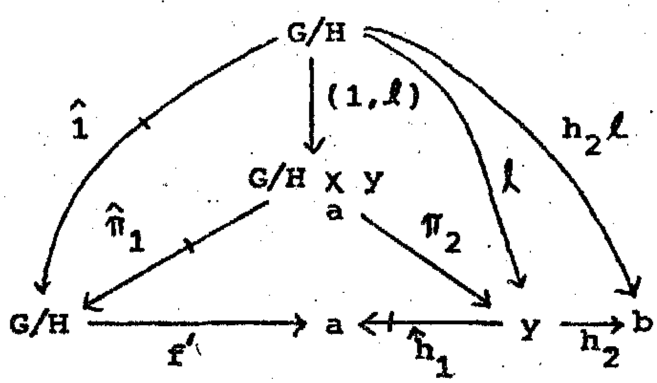
It is easy to check the following lemma:

<u>Lemma 5.6</u> For any $\mathbb{Z}[WH]$-module V, $J_{G/H}V$ is H-determined.

In fact, the definition of $J_{G/H}V$ on objects may be recovered from the assumptions that $J_{G/H}V$ is H-bounded, $(J_{G/H}V)(G/H) = V$, and the diagram

$$\mathcal{J}_{G/H}V \longrightarrow (J_{G/H}V)_{G/H} \rightrightarrows (J_{G/H}V)_{G/H\times G/H}$$

is an equalizer. The motivation for the definition of $J_{G/H}V$ on maps comes from the diagram below in $\mathcal{B}$.



<u>Remark 5.7</u> The description we have given for $J_{G/H}$ is the easiest one to use for proving that $J_{G/H}$ is a functor, but it obscures the real simplicity of $(J_{G/H}V)(b)$ for $b\in\mathcal{B}$. There are two alternate descriptions of $J_{G/H}$ which give a better feel for its value at any b in $\mathcal{B}$. For any $b\in\mathcal{B}$ and any $f : G/H \longrightarrow b$ in $\hat{G}$, let W_f be the subgroup of WH which fixes f as an element of $\langle G/H, b\rangle$: that is, $g\in W_f$ if and only if the diagram

$$\begin{array}{ccc} G/H & \xrightarrow{g} & G/H \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f} & \\ & b & \end{array}$$

commutes. For any $Z[WH]$ -module V and any subgroup W of WH, let V^W be the W-invariant elements of V. Our first description of $J_{G/H}(b)$ is

(2) $$J_{G/H}V(b) \cong \bigoplus_{f \in \langle G/H, b\rangle / WH} V^{W_f} \qquad \text{for } b \in \mathcal{B}$$

where the sum runs over the orbits of $\langle G/H, b\rangle$ under the action of WH. This description is not entirely natural because the subgroup W_f depends on the choice of f within its orbit; a different choice would yield a conjugate subgroup and an isomorphic fixed point module. The lack of naturality in the choices of the f's makes the description of the effect of a map $h : a \longrightarrow b$ on $J_{G/H}V$ hard to describe in terms of isomorphism (2). One notational trick seems very useful here. For $b \in \mathcal{B}$, $f : G/H \longrightarrow b$ and $v \in V^{W_f}$, let $v_f \in J_{G/H}V(b)$ be the element which is v in the place corresponding to f and zero else where. Then v_f is a canonical choice for an inverse image of $v \in J_{G/H}V(G/H) = V$ under the map

$$f^*: J_{G/H}V(b) \longrightarrow J_{G/H}V(G/H).$$

Our second alternate description of $J_{G/H}V$ applies directly only to $J_{G/H}V(G/K)$ for $[H] \leq [K]$. For any $g \in G$ with $g^{-1}Hg \subset K$, let W^g be the subgroup

$$(NH \cap gKg^{-1})/\, H$$

of WH. Then there is an isomorphism

$$(J_{G/H}V)(G/K) \cong \bigoplus_{[g^{-1}Hg]_K} V^{W^g}$$

where the sum runs over a set of $g \in G$ such that the subgroups

$g^{-1}Hg \leq K$ form a set of representatives of the K-conjugacy classes of K-subgroups which are G-conjugate to H. The subgroups W^g depend on the choices of the $g \in G$ so this isomorphism is not entirely natural and it is hard to describe the effect of a map $h : a \longrightarrow b$ in on $J_{G/H}V$. The connection between the two definitions is that the maps $G/H \longrightarrow G/K$ in G are in one-to-one correspondence with the K-conjugacy classes of K-subgroups which are G-conjugate to H.

The proofs of both of these alternate descriptions are easy manipulations of the original definition.

For any Mackey functor M, M(G/H) is a WH-module and it is natural to attempt to compare M and $J_{G/H}(M(G/H))$. For any $b \in \mathcal{B}$, we have a map $\lambda_b : M(b) \longrightarrow J_{G/H}(M(G/H))(b)$ defined by

$$\lambda_b : M(b) \xrightarrow{\oplus f} \Big(\bigoplus_{f \in \langle G/H, b\rangle} M(G/H) \Big)^{WH} = J_{G/H}(M(G/H))(b).$$

However, for an arbitrary M, the maps λ_b need not fit together to form a map

$$\lambda : M \longrightarrow J_{G/H}(M(G/H))$$

of Mackey functors. Conditions for the existence of λ as a map of Mackey functors and the basic properties of λ are as follows:

<u>Lemma 5.8</u> (a) For any M, a necessary and sufficient condition for the existence of $\lambda : M \longrightarrow J_{G/H}(M(G/H))$ is that the transfer maps

$$\hat{f} : M(G/K) \longrightarrow M(G/H)$$

are zero for every $f : G/K \longrightarrow G/H$ in $\hat{G}$. In particular, if M is G/H bounded, then λ exists.

(b) If λ exists, then M is H-determined if and only if λ is an isomorphism.

(c) If λ exists, then it induces a map $\bar{\lambda}: H_{G/H} M \longrightarrow J_{G/H}(M(G/H))$ making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \lambda\ } & J_{G/H}(M(G/H)) \\ & {\scriptstyle \eta}\searrow & \nearrow {\scriptstyle \bar{\lambda}} \\ & H_{G/H}M & \end{array}$$

commute. The map $\bar{\lambda}$ is an isomorphism if and only if M is H-bounded.

The proofs of (a) and (b) are easy diagram chases. The proof of (c) follows from the observation that for any M and any $b \in \Omega$, $(H_b M)(b) = M(b)$.

Let ${}_{G/H}\mathcal{M}$ be the full subcategory of $\mathcal{M}$ consisting of the Mackey functors for which λ is defined. Our basic technical tool for analyzing H-characteristic Mackey functors and our classification of H-determined Mackey functors are given in the following proposition.

<u>Proposition 5.9</u> (a) The map

$$\lambda: M \longrightarrow J_{G/H}(M(G/H))$$

and the isomorphism

$$J_{G/H}V(G/H) \xrightarrow{\ \cong\ } V$$

of Definition 5.5 are the unit and counit respectively of an adjunction between the "evaluation at G/H" functor

$${}_{G/H}\mathcal{M} \longrightarrow \mathbb{Z}[WH] - \text{modules}$$

and

$$J_{G/H} : \mathbb{Z}[WH]\text{-modules} \longrightarrow {}_{G/H}\mathcal{M}$$

(b) The two functors above restrict to a natural equivalence between the category of $\mathbb{Z}[WH]$-modules and the category of H-determined Mackey functors.

We want to understand rings and modules bounded by H, so we need to relate pairings of Mackey functors to some sort of pairings of $\mathbb{Z}[WH]$-modules.

Proposition 5.10 If U, V, and X are $\mathbb{Z}[WH]$-modules, then there is a one-to-one correspondence between Mackey functor pairings

$$J_{G/H}U \boxtimes J_{G/H}V \longrightarrow J_{G/H}X$$

and WH maps

$$U \otimes V \longrightarrow X$$

where WH acts diagonally on $U \otimes V$.

Proof Given a WH-map $\theta : U \otimes V \longrightarrow X$, we define a family of maps

$$\theta_b : J_{G/H}U(b) \otimes J_{G/H}V(b) \longrightarrow J_{G/H}X(b)$$

satisfying the conditions of Proposition 1.4 by the diagram

$$\begin{array}{ccc}
\left(\bigoplus_{f \in \langle G/H, b\rangle} U\right)^{WH} \otimes \left(\bigoplus_{f' \in \langle G/H, b\rangle} V\right)^{WH} & \xrightarrow{\theta_b} & \left(\bigoplus_{f'' \in \langle G/H, b\rangle} X\right)^{WH} \\
\Big\downarrow \pi_{f''} \otimes \pi_{f''} & & \Big\downarrow \pi_{f''} \\
U \otimes V & \longrightarrow & X
\end{array}$$

where $\pi_{f''}$ is projection onto the summand corresponding to f''. The existence of the required pairing of Mackey functors follows from Proposition 1.4.

Given a map $\bar{\theta} : J_{G/H}U \boxtimes J_{G/H}V \longrightarrow J_{G/H}X$, we recover the map $\theta : U \otimes V \longrightarrow X$ by taking $b = G/H$ in the description of Mackey functor pairings in Proposition 1.4.

Remark 5.11 The proposition above can be fancied up considerably (or totally obscured--depending on one's point of view) with a little closed category theory. The real key to its proof is the fact that

$(J_{G/H}U \square J_{G/H}V)(G/H)$ is $U \otimes V$ with the diagonal WH-action. In fact, a fancier version of Proposition 5.10 would assert that if X is a $\mathbb{Z}[WH]$-module and M and N are Mackey functors in ${}_{G/H}\mathcal{U}$, one of which is H-bounded, then there is a one-to-one correspondence between pairings

$$M \square N \longrightarrow J_{G/H}X$$

and $\mathbb{Z}[WH]$ -maps

$$M(G/H) \otimes N(G/H) \longrightarrow X.$$

The key to the proof of this is the observation that $M \square N$ is then H-bounded (and so in ${}_{G/H}\mathcal{M}$) and $M \square N(G/H)$ is just $M(G/H) \otimes N(G/H)$ with the diagonal WH-action.

From Proposition 5.10, we obtain a description of H-determined rings.

Corollary 5.12 $J_{G/H}$ induces a one-to-one correspondence (up to isomorphisms) between H-determined rings and pairs $(S, \theta: WH \longrightarrow Aut(S))$ where S is an ordinary ring and θ is a representation of WH in the group of ring automorphisms of S. The correspondence pairs commutative rings and commutative rings.

We denote the ring corresponding to (S, θ) by S_θ and the ring corresponding to the trivial map $WH \longrightarrow Aut(S)$ by S_H.

Of course, for any pair $(S, \theta: WH \longrightarrow Aut(S))$, the H-determined modules over the ring S_θ correspond exactly to pairs (V, ξ) where V is a $\mathbb{Z}[WH]$-module and

$$\xi: S \otimes V \longrightarrow V$$

is a map of $\mathbb{Z}[WH]$-modules, but this description is rather awkward. To obtain a better one, we define the ring $S[\theta]$ to be the free S-module generated by the set WH with multiplication given on generators sg,

$s'g'$ (for $s, s' \in S$; $g, g' \in WH$) by

$$(sg)(s'g') = (s\,\theta(g)(s'))(gg').$$

The ring $S[\theta]$ has the same S-module structure as the group ring $S[WH]$, but the multiplication of $S[\theta]$ incorporates the action of WH on S (whereas the multiplication of $S[WH]$ does not). It is easy to see that $S[\theta]$ -modules correspond exactly to the pairs (V, ρ) above, so we have

<u>Proposition 5.13</u> For any pair $(S, \theta: WH \longrightarrow Aut(S))$, the functor $J_{G/H}$ restricts to a natural equivalence between the category of $S[\theta]$-modules and the category of H-determined S_θ-<u>modules</u>.

<u>Examples 5.14</u> (a) Let F' be a field extension of F with Galois group G (hereafter indicated by $[F', F; G]$). The Weyl group of the trivial subgroup $\{e\}$ is G, so the pair $(F', 1: G \longrightarrow G)$ determines a commutative <u>ring</u> F'_1. The transfer

$$F'_1(G/\{e\}) = F' \longrightarrow (F')^G = F'_1(1)$$

is just the trace of the extension $[F', (F')^G; G]$. It follows immediately from Proposition 5.4(d) that F'_1 is a <u>field</u>.

(b) Generalizing (a), if $[F', F; \mathcal{H}]$ is a field extension and $\theta: WH \longrightarrow \mathcal{H}$ is a homomorphism, then F'_θ is a commutative <u>ring</u>. It follows immediately from Proposition 5.4(b) that F'_θ is an integral <u>domain</u>. The transfer

$$F'_\theta(G/H) = F' \longrightarrow (F')^{\mathcal{H}} = F'_\theta(1)$$

is just the trace of the extension $[F', (F')^{\mathcal{H}}; \mathcal{H}]$ multiplied by the order of the kernel of θ. Thus, if $|\ker\theta|$ is prime to the characteristic of F', then F'_θ is a <u>field</u>. In Section 7, we show that this is one of two basic sources of <u>fields</u>. We discuss <u>modules</u> over F'_θ in Section 7.

(c) For any ring S, there is a <u>ring</u> $S_{\{e\}}$ obtained by taking the trivial representation of $W\{e\} = G$ in the automorphism group of S. By Proposition 5.13, there is an equivalence of categories between the category of S[G]-modules and the category of $\{e\}$-determined $S_{\{e\}}$-modules. This rather quaint view of S-valued representation theory might have applications because if S is commutative, then so is $S_{\{e\}}$ (unlike S[G]), and if S is an integral domain, then $S_{\{e\}}$ is an integral domain. This suggests the possibility of applying the techniques of commutative algebra--in so far as they extend to commutative rings--to representation theory. Note that the transfer

$$S_{\{e\}}(G/\{e\}) \longrightarrow S_{\{e\}}(1)$$

is just multiplication by the order of G. Thus, if the characteristic of S and the order of G are relatively prime, then both $S_{\{e\}}$ and all its modules are $\{e\}$-projective. Further, if F is a field, then $F_{\{e\}}$ is a field if and only if the characteristic of F does not divide the order of G. Thus, the well-behaved part of field-valued representation theory corresponds to the study of modules over certain fields and modular representation theory corresponds to the study of modules over certain integral domains.

Remark 5.15 The functors $J_{G/H}$ can be used to construct a curious natural filtration on Mackey functors. Partition the set of subgroups of G into sets $S_0, S_1, \ldots, S_n$ defined inductively by letting S_0 be the set consisting only of the trivial subgroup $\{e\}$ and S_i (for $i \geq 1$) be the set consisting of those subgroups which are not in S_{i-1} and whose proper subgroups are all in $\bigcup_{j=0}^{i-1} S_j$. Thus, S_1 is the set of cyclic subgroups of prime order, and S_2 is the set of subgroups which are not cyclic of prime order, but which have no subgroups other than cyclic groups of prime order. Define a decreasing filtration on any Mackey functor M inductively by $M_0 = M$ and

$$M_{i+1} = \ker\left[\bigoplus_{H\in S_i} \lambda_{G/H} : M_i \longrightarrow \bigoplus_{H\in S_i} J_{G/H}(M_i(G/H))\right]$$

It is easy to check that M_i is in ${}_{G/H}\mathcal{M}$ for $H\in S_i$ so the required maps $\lambda_{G/H}$ are defined. In fact, if we define $\mathcal{M}_i$ to be the full subcategory of $\mathcal{M}$ whose objects are the Mackey functors N with

$$N(G/K) = 0 \qquad \text{for } K\in \bigcup_{j=1}^{i-1} S_j$$

then $M_i\in \mathcal{M}_i$ and our procedure defines a sequence of functors.

$$\mathcal{M} = \mathcal{M}_0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \quad \cdots \quad \longrightarrow \mathcal{M}_n = Ab$$

where n is the integer with $S_n = \{G\}$. These functors are right adjoints to the inclusions.

Of course, applying this filtration to any chain complex or cocomplex in $\mathcal{M}$ produces a spectral sequence. The spectral sequences obtained in this way from the cellular chains and cochains of a G-space (or spectrum) X and those obtained from a projective or injective resolution of any Mackey functor are currently under investigation.

6. Prime and primary ideals revisited

If S is any ring and P is an irreducible two-sided ideal of S--or if S is a commutative ring and P is a primary ideal of S--then S/P is a ring with integral characteristic p^n for some integer $n \geq 1$ and some prime p (p = 0 is possible). For this reason, rings with characteristic p^n merit special investigation. In this section, we begin such an investigation by considering rings R with integral characteristic p^n and a characteristic subgroup H. This class of rings includes, of course, rings of the form S/P where S is commutative and P is a primary ideal of S. From this study, we obtain a description of the prime and primary ideals of the Burnside ring. The techniques employed should be applicable to the study of the prime and primary ideals of other rings.

The key to understanding the mod p behavior (for $p \neq 0$) of any ring--or any Mackey functor--seems to be an understanding of certain chains of subgroups--which we call p-towers--in our ambient group G.

Definition 6.1 For any $H \leq G$ and any prime p, H_p is the minimal normal subgroup of H with H/H_p a p-group. The group H^p is a subgroup of G corresponding to a p-Sylow subgroup of NH_p/H_p which contains H/H_p. Thus, we have $H \leq H^p$, $H_p \lhd H^p$ and H^p/H_p is a p-group (by $K \lhd J$, we mean that K is a normal subgroup of J). Note that H^p is defined only up to conjugacy in G. For p =0, we take $H_p = H^p = H$ for convenience in stating results. The p-tower associated to H in G is the collection of subgroups K with $[H_p] \leq [K] \leq [H^p]$.

For convenience, we transcribe here (from (   )) the properties of H_p, H^p and p-towers which we need.

Lemma 6.2 (see ( )) (a) H_p is a characteristic subgroup of H.

(b) If $[H] \le [K]$, then $[H_p] \le [K_p]$.

(c) If $[H_p] \le [K] \le [H^p]$, then $[H_p] = [K_p]$

(d) If $H \triangleleft K$ and K/H is a p-group, then $[H_p] = [K_p]$.

(e) The prime p does not divide the order of $W(H^p)$, but if $H < L \le H^p$, then the order of $W_L H$ is p^m for some $m \ge 1$.

(f) If H, K are subgroups of $L \le G$ and $[H_p] = [K_p]$, then for any L-set X,

$$|\langle L/H,\ X\rangle_L| \equiv |\langle L/K,\ X\rangle_L| \mod p$$

where $\langle X, Y\rangle_L$ is the set of L-maps from the L-set X to the L-set Y.

The well-behaved rings with integral characteristic p^n seem to be those which have a bound H and are G/H^p-projective. The simplest examples of such rings are given by

Proposition 6.3 If R is a ring with integral characteristic p^n ($p \ne 0$) and characteristic subgroup H, then R is G/H^p-projective.

Proof By Lemma 6.2(e) and (f), p does not divide $|\langle G/H,\ G/H^p\rangle|$ so there is an integer m with $m\,|\langle G/H,\ G/H^p\rangle| \equiv 1 \mod p^n$. To compute the transfer $t(m\cdot 1_{G/H^p}) \in R(1)$, we think of R(1) as a submodule of R(G/H) and apply Proposition 5.4 (e). This gives

$$\begin{aligned} t(m\cdot 1_{G/H^p}) &= \sum_{f \in \langle G/H,\, G/H^p\rangle} f(m\cdot 1_{G/H^p}) \\ &= \sum m\cdot f(1_{G/H^p}) \\ &= \sum m\cdot 1_{G/H} \\ &= m\,|\langle G/H,\ G/H^p\rangle| \equiv 1 \mod p^n \end{aligned}$$

Thus, $m\cdot 1_{G/H^p}$ is a unit and R is G/H^p-projective.

If R has characteristic zero, then we have $t(1_{G/H}) = |WH|\cdot 1_R \in R(1)$ so the conclusion above need not hold unless $|WH| = 1$ or $R(1)$ is a rational vector space.

If R is an H-bounded, G/H^p-projective ring with integral characteristic p^n ($p \neq 0$), then the proper way to study R seems to be to compute the transfer maps out of $R(G/K)$ for the subgroups K in the p-tower determined by H. These maps may be hard to compute in R, but they are easy to compute in $H_{G/H}R$ if we think of $H_{G/H}R$ as $J_{G/H}(R(G/H))$ and apply Remark 5.7. If R is H-characteristic, then the map $\eta: R \longrightarrow H_{G/H}R$ is injective and the computations in $H_{G/H}R$ are especially useful.

Proposition 6.4 (a) Let R be an H-determined ring (which we think of as $J_{G/H}(R(G/H))$. Let $f : G/H \longrightarrow G/K$ be a map in $\hat{G}$, $v \in R(G/H)^{W_f}$, and $v_f \in R(G/K)$ (as discussed in Remark 5.7). If $\hat{h} : R(G/K) \longrightarrow R(G/L)$ is the transfer map induced by $h : G/K \longrightarrow G/L$ in $\hat{G}$, then

$$\hat{h}\,(v_f) = \sum_{g \in W_{hf}/W_f} (gv)_{hf}$$

where the sum runs over a set of coset representatives for W_{hf}/W_f considered as a subgroup of WH/W_f.

(b) If $H \leq K < L \leq N_{H^p}H$, $v \in R(G/H)^{WH}$ and $\pi: G/H \longrightarrow G/K$ and $\pi': G/K \longrightarrow G/L$ are the projections, then

$$\hat{\pi}'(\, v_\pi \,) \in pR(G/L)$$

(c) Let R be a ring with integral characteristic p and characteristic subgroup H such that WH acts trivially on $R(G/H)$ and let $H < K$ with $|K/H| = p$. Then every transfer $R(G/H) \longrightarrow R(G/K)$ is zero.

The proposition above (and other results) suggests that, for a ring R with integral characteristic p^n and bound H, the behavior of the transfers out of $R(G/K)$ (for $K < H^p$ in the p-tower determined by H) is closely related to the trace of the WH-action on $R({}^G/H)$. If there are elements in $R(G/H)$ whose trace is a unit in $R(G/H)$, then R should be G/K-projective for some $K < H^p$. Otherwise, elements in the image of these transfers tend to be nilpotent.

Our objective for the remainder of this section is to show how the results above can be applied to determine the prime and primary <u>ideals</u> of a <u>ring</u>. If P and Q are primary <u>ideals</u> of a <u>ring</u> R, then we would like to know when P is contained in Q. This is, of course, equivalent to knowing whether or not there is a <u>ring</u> surjection $R/P \longrightarrow R/Q$ commuting with the projections from R. The existence of such a map imposes fairly stringent conditions on the characteristic subgroups and integral characteristics of R/P and R/Q.

<u>Proposition 6.5</u> Let R and S be (non-zero) <u>rings</u> with characteristic subgroups H and K and integral characteristics p^m and q^n (p and q prime) respectively. The existence of a map $R \longrightarrow S$ imposes the following conditions on H, K, p^m, and q^n:

(1) Either $p = q$ and $n \geq m$ or $p = 0$

(2) If $p \neq 0$, then $[H] \leq [K] \leq [H^p]$

(3) If $p = q = 0$, then $[H] = [K]$

(4) If $p = 0$, $q \neq 0$, then either $[H] = [K]$ or $q^n \mid |WH|$ and $[H] \leq [K] \leq [H^q]$

<u>Proof</u> Note that if $R(b) \neq 0$, then it has characteristic p^m because the map $R \longrightarrow R_b$ is injective. We denote the identity elements in $R(b)$ and $S(b)$ as $1_{R,b}$ and $1_{S,b}$ respectively.

Condition (1) follows from the existence of a ring map $R(1) \longrightarrow S(1)$.

Since $1_{R,G/K} \in R(G/K)$ maps to $1_{S,G/K} \in S(G/K)$ and $1_{S,G/K} \neq 0$, we must have $[H] \leq [K]$ for any choice of p and q. If $p \neq 0$, then $1_{R,G/H^p}$ is a unit by Proposition 6.3 and so must go to a unit in S. This forces $[K] \leq [H^p]$. If $p = 0$, then $1_{R,G/H}$ transfers to $|WH| \cdot 1_R$ in $R(1)$, so either $|WH| \cdot 1_S$ is zero in $S(1)$ or $1_{S,G/H}$ is non-zero. For $p = q = 0$, this forces $[H] = [K]$. If $p = 0$, $q \neq 0$, and $[H] \neq [K]$, then we must have $q^n \mid |WH|$ so that $|WH| \cdot 1_S$ is zero. Since $1_{R,G/H^q}$ transfers to $|\langle G/H, G/H^q\rangle| \cdot 1_R$ and q does not divide $|\langle G/H, G/H^q\rangle|$, the image $1_{S,G/H^q}$ of $1_{R,G/H^q}$ is a unit in $S(G/H^q)$ and we have $[K] \leq [H^q]$.

The behavior of the primary ideals of the Burnside ring shows that, for $m = 1$ in the proposition above, the indicated restraints are the only general ones imposed by the existence of ring map $R \longrightarrow S$. If $m \neq 1$, then the existence of ring maps $R \longrightarrow S$ seems to be a rather messy problem.

To obtain a description of the prime and primary ideals of the Burnside ring, we consider the rings $(\mathbb{Z}/p^n)_H$ obtained from Corollary 5.12. The only zero divisors in $(\mathbb{Z}/p^n)_H$ are nilpotent (by Proposition 5.4) so the kernel of the identity element map $A \longrightarrow (\mathbb{Z}/p^n)_H$ is a primary ideal of A, which we call $q(H,p^n)$. The ideal $q(H,p)$ (for any $H \leq G$) is prime since $(\mathbb{Z}/p)_H$ is an integral domain. These definitions and Propositions 6.3, 6.4 and 6.5 suffice to describe the prime and primary ideals of A.

Theorem 6.6 The ideals $q(H,p^n)$ include all of the primary ideals of the Burnside ring A. Further,

(a) $q(H,p^n) \neq q(K,q^m)$ unless $[H] = [K]$ and $p^n = q^m$

(b) The only prime ideals of A are the $q(H,p)$ ($p = 0$ is allowed). The only maximal ideals are the prime ideals of the form $q(H^p,p)$ for $p \neq 0$.

(c) The minimal prime ideals are the $q(H,0)$.

(d) The prime ideal $q(H,p)$ is contained in the prime ideal $q(K,q)$ if and only if $[H] \leq [K] \leq [H^q]$ and either $p = 0$ or $p = q$.

(e) $q(H,p^m) \subset q(H,p^n)$ for $m \geq n$

(f) The ring $A/q(H,p^n)$ is the image of A in $(\mathbb{Z}/p^n)_H$. If $|WH|$ is a unit in $\mathbb{Z}/p^n$, then $A/q(H,p^n)$ is isomorphic to $(\mathbb{Z}/p^n)_H$ and is G/H-projective.

(g) If p does not divide either $|H|$ or $|WH|$, then the localization of A at the prime ideal $q(H,p)$ is $(A/q(H,0)) \otimes \mathbb{Z}_{(p)}$ where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p. In particular, $(A/q(H,0)) \otimes Q$ is the field of fractions of $A/q(H,0)$.

Note that no comment is made on the relation between $q(H,p^n)$ and $q(K,p^m)$ for $m > 1$ and $[H] \leq [K] \leq [H^p]$. The relation between these two ideals seems to be a fairly hard problem. Also note that the localization of A at $q(H,p)$ is not described if $q(H,p)$ does not meet the conditions in (g); it is not clear that the localization exists for such $q(H,p)$. Of course, by the localization of R at a prime ideal P, we mean a ring map $\theta : R \longrightarrow S$ with $\theta(R-P)$ contained in the units of S which is universal among ring maps with this property. The basic source of the problem of obtaining localizations is that inverses to units need not be unique.

Proof of 6.6 Let P be a primary ideal of A and let p^n and H be the integral characteristic and characteristic subgroup of A/P. The map $A(G/H) \longrightarrow A/P(G/H)$ is surjective so $A/P(G/H)$ is generated by the images of the elements

$$G/H \xleftarrow{\hat{f}} G/J \longrightarrow 1$$

in $A(G/H) = [G/H, 1]$. We can assume $J \leq H$. The elements for which $J \neq H$ vanish in $A/P(G/H)$ because they factor through $A/P(G/J)$ which is zero. Thus, $A/P(G/H)$ has a single generator and must be $\mathbb{Z}/p^n$. Since $A/P(1)$ sits inside the elements of $A/P(G/H)$ invariant under the WH-action, $A/P(1)$ must be isomorphic to $A/P(G/H)$ (via the restriction map) and $A/P(G/H)$ must be fixed by WH. Thus, $H_{G/H}A/P$ is $(\mathbb{Z}/p^n)_H$ by Lemma 5.8(e). Since the map $A/P \longrightarrow H_{G/H}A/P$ is injective, P must be the kernel $q(H,p^n)$ of the inclusion of the identity element $A \longrightarrow H_{G/H}A/P = (\mathbb{Z}/p^n)_H$.

To establish (a), apply Proposition 6.5 to $A/q(H,p^n)$ and $A/q(H,q^m)$.

For part (b), note that $A/q(H,p^n)(1)$ contains zero divisors unless $n=1$ so the only prime ideals are the $q(H,p)$. No $q(H,0)$ can be maximal since $A/q(H,0)(1) = \mathbb{Z}$ which is not a field. If $p \neq 0$ and $H \neq H^p$, then $A/q(H,p)$ is not a field because, by Proposition 6.4(c), $1_{G/H} \in A/q(H,p)(G/H)$ is not a unit. For $p \neq 0$, $A/q(H^p,p)$ is G/H^p-projective (by Proposition 6.3) and is therefore isomorphic to $(\mathbb{Z}/p)_{H^p}$ which is a field by Proposition 5.4(d). Thus, the $q(H^p,p)$ (for $p \neq 0$) are maximal and are the only maximal ideals.

Of course, the ideal $q(H,p^n)$ is contained in $q(K,q^m)$ if and only if there is a map

$$A/q(H,p^n) \longrightarrow A/q(k,q^m).$$

Part (c) and the "only if" half of part (d) follow from this observation and Proposition 6.5. To prove the "if" part of (d), it suffices to show that if $p \neq 0$ and $H < K \leq H^p$, then $q(H,p) \subset q(K,p)$. By the solvability of p-groups, there is a group J with $H \triangleleft J \leq K$ such that J/H has order p. By Lemma 5.8 and Proposition 6.4, there is a map

$$\lambda : A/q(H,p) \longrightarrow J_{G/J}(A/q(H,p)(G/J))$$

and therefore a map

$$\lambda' : A/q(H,p) \longrightarrow J_{G/J}((\mathbb{Z}/p)_H (G/J)).$$

Let V be a direct summand of

$$(\mathbb{Z}/p)_H (G/J) = \bigoplus_{g \in \langle G/H, G/J\rangle/WH} \mathbb{Z}/p$$

corresponding to an orbit of the action of WJ on $\langle G/H, G/J\rangle/WH$. Then $J_{G/J}(V)$ is an integral domain by Proposition 5.4(b) and the kernel of the identity element map

$$A \longrightarrow J_{G/J}(V)$$

must be $q(J,p)$. But this map factors as

$$A \longrightarrow A/q(H,p) \xrightarrow{\lambda'} J_{G/J}((\mathbb{Z}/p)_H(G/J)) \longrightarrow J_{G/J}(V).$$

Thus, $q(H,p) \subset q(J,P)$ and an inductive application of this process gives $q(H,p) \subset q(K,p)$ for $H \leq K \leq H^p$.

Part (e) follows from the obvious existence of a <u>ring</u> map $(\mathbb{Z}/p^n)_H \longrightarrow (\mathbb{Z}/p^m)_H$ for $m \leq n$.

For (f), the fact that $A/q(H,p^n)$ is the image of A in $(\mathbb{Z}/p^n)_H$ follows from the definition. The rest of (f) follows from the observation that $t(1_{G/H}) = |WH| \cdot 1_{(\mathbb{Z}/p^n)_H}$ in $(\mathbb{Z}/p^n)_H$.

For (g), let R be the ring $A/q(H,0) \otimes Z_{(p)}$. Since p does not divide $|WH|$, $1_{G/H}$ is a unit in R and R is G/H-projective. Since p does not divide $|H|$, there is an x in $A(G/H) - q(H,p)(G/H)$ such that the exterior product $q(H,0)x$ is zero. Thus, $q(H,0)$ must be in the kernel of any ring map $\varphi : A \longrightarrow S$ which takes $A - q(H,p)$ (and hence x) into the units of S. Further, any such map must factor as

$$A \xrightarrow{\theta} R \longrightarrow S$$

since $q \cdot 1_A$ is not in $q(H,p)(1)$ if q is an integer prime other than p. Thus, it suffices to show that the image of $A - q(H,p)$ under $\theta : A \longrightarrow R$ consists of units. Since R is G/H-projective, it suffices to see that the image of $A(G/H) - q(H,p)(G/H)$ in $R(G/H)$ consists of units. This image is easily seen to consist of elements of the form $(n/m)\theta(x)$ where n amd m are suitably chosen integers prime to p.

Remark 6.7 Our description of the prime ideals of A is somewhat different from the usual description of the prime ideals of the Burnside ring A(1). To compare the two descriptions, we let

$$\varphi^K_H : A(G/K) \longrightarrow Z$$

be the usual map of the Burnside ring of K into integers which is associated to the subgroup H of K (see ( ), p 203). Let $\tilde{q}_K(H,p^n)$ be the primary ideal of the Burnside ring A(G/K) of K determined by the condition

$$\varphi^K_H \equiv 0 \bmod p^n.$$

It is easy to see that the connection between $q(H,p^n)$ and the $\tilde{q}_K(H,p^n)$ is given by

$$q(H,p^n)\,(G/K) = \bigcap_{\substack{J \leq K \\ [J] = [H]}} \tilde{q}_K(J,p^n)$$

Here, if there are no such J, then the intersection is, by convention, all of A(G/K).

Remark 6.8 Commutative algebraists will no doubt be disturbed by the existence of a commutative Noetherian ring in which there is a finte, non-zero number of prime ideals between two prime ideals (like $q(H_p,p)$ and $q(H^p,p)$; this situation cannot occur in ordinary ring theory. The resolution of this difficulty is that if $P_1 \subset P_2 \subset P_3$ are prime ideals of A, then for any G/K, either $P_2(G/K) = P_1(G/K)$ or $P_2(G/K) = P_3(G/K)$. Thus, locally--with respect to the G/K--A behaves like an ordinary commutative Noetherian ring should, but globally, its behavior is more complex.

In ( ), Dress describes the relationship between the prime ideals of the Burnside ring and the ideals $\mathrm{im}(A(a) \longrightarrow A(1))$ and $\ker(A(a) \longrightarrow A(a))$. These results have important applications to induction theory (like Corollary 2, p 207 of ( )) and, from them, it should be possible to extract descriptions of the prime ideals of the rings A_a and H_aA for $a \in \mathcal{B}$. For this reason, we record there their generalization to results on the Burnside ring.

Proposition 6.9 (a) If K^b is the kernel of the map $A \longrightarrow A_b$ (for $b \in \mathcal{B}$), then $K^b < q(H,p)$ if and only if $G/H_p \prec b$. Moreover

$$K^b = \bigcap_{G/H \prec b} q(H,0)$$

(b) For any $b \in \mathcal{B}$, the ideal $(1_b) \subset A$ (which is the image of $A_b \longrightarrow A$) is contained in $q(H,p)$ if and only if $G/H \not\prec b$.

(c) For any pair a, b in $\mathcal{B}$,

$$A = K^{b} + (1_{a})$$

if and only if $G/H^{p} \prec a$ for every H with $G/H \prec b$.

7. Integral domains and fields

In this section, we analyze the structure of (integral) domains and fields. Our first main result is a complete description of the H-determined domains D for which D(1) is a field. Any domain is a subring of such a domain so the classification problem is reduced to determining the subdomains of a domain. Our classification result is applicable to any field and we employ it to study modules over fields. In particular, we show that any module over a field F is projective in the category F-mod of F-modules. We also consider the question of fields containing a given domain. Since fields of fractions need not exist, this is an important and curious topic.

Throughout this section, by ring (and ring) we mean a commutative ring (or ring). Certainly, the analogous problems of non-commutative rings without zero divisors and of division rings should be investigated.

A number of trace-like functions are needed for our analysis of domains, so we begin by introducing a notion of trace which includes all of them.

Definition 7.1 Let W' be a subgroup of a finite group W and let N be a $\mathbb{Z}[W]$-module.

For any $x \in N^{W'}$, we define $tr_{W/W'}x$ by

$$tr_{W/W'}x = \sum_{gW' \in W/W'} gx$$

where the sum is indexed on the cosets of W' in W. We write tr_W for $tr_{W/\{e\}}$. Note that $tr_{W/W'}(x)$ is in N^W and that it does not depend on the choice of the coset representatives g since W' fixes x.

By Corollary 3.8, any domain has a characteristic subgroup H, and in this section, we restrict attention to those domains D with a fixed characteristic subgroup H. If D is such a domain, then the group WH acts on the ring D(G/H)* by ring automorphisms. Proposition 5.4(b) gives the rather curious property of this action which is equivalent to D being a domain. Our first objective is to describe exactly what such an action implies about the ring D(G/H)*.

Keeping in mind two non-trivial examples of rings with such actions may make reading what follows easier. Consider the quotient ring $\mathbb{Z}[x,y]/(xy)$ of the polynomial ring $\mathbb{Z}[x,y]$ by the ideal generated by the product xy and let $\mathbb{Z}/2$ act on this quotient by permuting x and y. Consider also the ring obtained from this quotient by inverting all the non-zero invariant elements; this fraction ring is isomorphic to the product of the rings $\mathbb{Z}(x)$ and $\mathbb{Z}(y)$ of rational functions.

For the moment, we forget about rings and introduce a little ring theory to illuminate the structure of D(G/H).

Proposition 7.2 Let S be a commutative ring (with unit) and W be a finite group which acts on S (not necessarily effectively) by ring automorphisms in such a way that, for any pair of non-zero elements x and y in S, there is a g in W with $x(gy) \neq 0$. Then

(a) The non-zero invariant elements of S are non-zero divisors. In particular, S^W is an integral domain.

(b) S contains no non-zero nilpotent elements.

(c) S can be written as a finite product $\prod_{i=1}^{n} S_i$ of rings S_i such that the S_i are all isomorphic and no S_i contains a non-trivial idempotent.

(d) If W_1 is the subgroup of W taking S_1 to itself, then the action of W_1 on S_1 satisfies the hypothesis of this proposition and the ring $S_1^{W_1}$ of W_1-invariant elements of S_1 is isomorphic to S^W.

(e) Every element in S_1 satisfies a monic polynomial with coefficients in $S_1^{W_1}$ (the same applies to S and S^W).

(f) If $S_1^{W_1} = S^W$ is a field and K is the kernel of the action of W_1 on S_1, then S_1 is a normal separable field extension of $S_1^{W_1}$ with Galois group W_1/K.

<u>Proof</u> Part (a) is obvious.

For (b), assume that $x \in S$ is a non-zero nilpotent element and let k_0 be the largest integer with $x^{k_0} \neq 0$. There is a $g_1 \in W$ with $x^{k_0}(g_1x) \neq 0$ and thus $g_1 \neq e$. Let k_1 be the largest integer with $x^{k_0}(g_1x)^{k_1} \neq 0$; such a k_1 exists since g_1x is also nilpotent. There exists a $g_2 \in W$ with $x^{k_0}(g_1x)^{k_1} g_2 x \neq 0$. Again, $g_2 \neq e, g_1$ and there is a largest integer k_2 with $x^{k_0}(g_1x)^{k_1}(g_2x)^{k_2} \neq 0$. Clearly, this process can be continued until we run out of elements in W and thereby obtain a contradiction.

For (c), assume that S contains a non-trivial idempotent e. Such an idempotent cannot be fixed by W since $(1 - e)e = 0$. Any product of the form

(1) $\quad (g_1e)(g_2e)\ldots(g_ke) \qquad$ for $k \geq 1$, $g_i \in W$ for $1 \leq i \leq k$

is also idempotent. Let e' be a product of maximal length among the non-zero products of the form (1) (By length, we mean the number of distinct factors multiplied together). Let W' be the subgroup of W fixing e'; since e' is a nontrivial idempotent, $W' \neq W$. The trace $tr_{W/W'}\, e'$ is an idempotent because, clearly, either $e' = g\,e'$ or $e'(ge') = 0$ for any g in W. Being a W-invariant idempotent, $tr_{W/W'}\, e'$ must be either 0 or 1 and it is not 0 because $e'(tr_{W/W}\, e') = e'$. Thus, we have a product decomposition of S by

$$S = \prod_{gW' \in W/W'} S(ge')$$

The group W' acts on Se'. For any non-zero pair xe', ye' in Se', there is a $g \in W$ with $xe'g(ye') = 0$. This g must be in W' since $e'(ge') = 0$ otherwise. We have shown that the action of W' on Se' satisfies the hypothesis of this proposition, so if Se' contains a nontrivial idempotent, we can iterate the decomposition process. Since W' is strictly smaller than W, only finitely many iterations are possible and the last possible iteration produces the required decomposition. Note that the factors of the decomposition above, and thus of our final decomposition, are all isomorphic because W acts transitively on the orthogonal idempotents inducing the decomposition.

For (d), it suffices to show that, in the notation of the proof of (c), $(Se')^{W'} = S^W$. The map $a \longrightarrow ae'$ induces an injection of S^W into $(Se')^{W'}$. Suppose $x \in (Se')^{W'}$. Then $y = tr_{W/W'}(x)$ is in S^W and $ye' = x$. Thus $(Se')^{W'} = S^W$.

For (e), let $s \in S_1$ and define $p(x)$ by

$$p(x) = \prod_{g \in W_1} (x - gs)$$

Clearly, the coefficients of $p(x)$ are in $S_1^{W_1}$ and $p(s) = 0$. Note that we can replace S_1 by S and W_1 by W to obtain a monic polynomial with coefficients in S^W for any $s \in S$.

For (f), it suffices to show that S_1 is a field. Then it must be a normal, separable extension of $S_1^{W_1}$ with Galois group W_1/K. Let s be a non-zero element of S_1. From (e) and the fact that $S_1^{W_1}$ is a field, we obtain an equation of the form

$$s^n\,(sq(s) - 1) = 0$$

where $q(x)$ is a polynomial with coefficients in $S_1^{W_1}$. If $n = 0$, then $q(s)$ is an inverse for s. If $n \neq 0$, then it must be one. Otherwise, the element $s\,(sq(s) - 1)$ would be a non-zero nilpotent in S_1 and, by (b), there are none. Then we have

$$s^2(q(s))^2 = sq(s)$$

so that $sq(s)$ is an idempotent. The only idempotents in S_1 are 0 and 1, and if $sq(s) = 0$, then $s = 0$ by our equation. Thus, $sq(s) =$ 1 and $q(s)$ is the required inverse.

Remark 7.3 The correct way to understand Proposition 7.2 seems to be to think of S_1 as a representation of W_1 over $S_1^{W_1} = S^W$. The induced representation $\bar{S}$ of W over S^W has the form

$$\bar{S} = \bigoplus_{gW_1 \in W/W_1} S_1$$

The S^W module $\bar{S}$ can be made into a ring by giving it the product ring structure and it can be shown that W acts on $\bar{S}$ by ring automorphisms. Further, $\bar{S}$ is isomorphic to S by an isomorphism which preserves the W-actions.

Let R be another ring with a W-action satisfying the conditions of Proposition 7.2 and let $\theta: R \longrightarrow S$ be a ring homomorphism which commutes with the W-actions. We wish to compare the decompositions of

R and S given by the proposition. We have

$$R = \prod_{j=1}^{m} R_j \qquad\qquad S = \prod_{i=1}^{n} S_i .$$

Let U_1 fix R_1 and W_1 fix S_1 and let $\{d_j\}_{1 \le j \le m}$ and $\{e_i\}_{1 \le i \le n}$ be the indecomposible idempotents inducing the decompositions. Clearly, $\theta(d_j)$ is an idempotent in S and therefore a sum of some of the e_i. We may as well assume

$$\theta(d_1) = e_1 + e_2 + \ldots + e_k.$$

Note that n = mk. If $g \in W$ fixes e_1, then we must have $gd_1 = d_1$ by the orthogonality of the idempotents. Thus, $W_1 \subset U_1$. Let K and L be the kernels of the actions of U_1 on R_1 and W_1 on S_1 respectively and let $\mathcal{G} = U_1/K$ and $\mathcal{H} = W_1/L$. We think of $\mathcal{G}$ and $\mathcal{H}$ as "Galois" groups of R_1 over $R_1^{U_1}$ and S_1 over $S_1^{W_1}$.

There is a map $\theta_1 : R_1 \longrightarrow S_1$ given by

$$\theta_1(x) = \theta(x)\, e_1 .$$

This map is W_1 equivariant. Using the induced representations view of Remark 7.3, it is easy to see that θ_1, completely determines R, S, and θ.

Remark 7.4 If R is any ring with a W action satisfying the conditions of Proposition 7.2, then R^W is an integral domain with a field of fractions $(R^W)^{-1} R^W$. We can invert the non-zero elements of R^W in R to obtain $S = (R^W)^{-1}R$. The action of W on R extends to an action of W on S which also satisfies the conditions of Proposition 7.2. Note that $S^W = (R^W)^{-1} R^W$. Because the non-zero elements of R^W are not zero divisors in R, the natural map

$$\theta : R \longrightarrow (R^W)^{-1}R = S$$

is injective. This is an important example of the sort of extension discussed above.

The implications of the results above for a domain D should be fairly obvious. We use the notation $D_1(G/H)$, W_1H, K (or K_D), $\mathcal{Y}$ (or $\mathcal{Y}_D$) and e_1 (or e_1^D) to designate the structural data for $D(G/H)$ given by Proposition 7.2. Note that if D is H-determined, then D is completely determined by the W_1-module $D_1(G/H)$ and if, further, $D(1) = D(G/H)^{WH}$ is a field, then computing D is just an extended exercise in ordinary Galois theory. Any domain D with characteristic subgroup H inbeds in $H_{G/H}D$ which is H-determined. Further, if F is the field of fractions of $H_{G/H}D(1)$, then the domain $F \otimes_{H_{G/H}D(1)} H_{G/H}D$ is H-determined and field valued at $1 \in \mathcal{B}$. Thus, it can be completely analyzed using Galois theory, and then we can try to recover the structure of D via the inclusion

$$D \longrightarrow H_{G/H}D \longrightarrow F \otimes_{H_{G/H}D(1)} H_{G/H}D.$$

If D is an H-determined integral domain with D(1) a field, then it is natural to ask if D is a field. From Proposition 5.4(d), we see that the answer to this question depends only on the transfer map $t: D(G/H) \longrightarrow D(1)$; If we think of $D(1)$ as the WH-invariant elements in $D(G/H)$, then t is just the trace tr_W. It is fairly easy to see that D is a <u>field</u> if and only if there is an element x in $D_1(G/H)$ with $tr_W(x) = 1$. This trace is given by the formula

$$tr_W(x) = |K| \cdot \sum_{gW_1 \in W/W_1} g\, tr_{\mathcal{W}}(x) \qquad \text{for } x \in D_1(G/H)$$

where the sum runs over the cosets of W_1 in W/W_1. Since $D_1(G/H)$ is a normal, separable extension of $D_1(G/H)^{W_1H}$, there is an x in $D_1(G/H)$ with $tr_{\mathcal{W}}(x) = e_1$ so that $tr_W(x) = |K| \cdot 1_D$. Thus, we have

<u>Proposition 7.5</u> If D is an H-determined integral <u>domain</u> such that $D(1)$ is a field, then D is a <u>field</u> if and only if the characteristic of $D(1)$ does not divide the order of the kernel K of the action of W_1H on $D_1(G/H)$.

If we find ourselves stuck with an H-determined integral <u>domain</u> D such that $D(1)$ is a field, but D is not a <u>field</u>, then it is reasonable to consider the ways we might imbed it in a <u>field</u>. There are two distinct operations which may be performed on D--either independently or in concert--to obtain a <u>field</u> into which D imbeds. Both of these are best visualized by thinking of $D(G/H)$ as the induced WH representation obtained from the W_1H representation $D_1(G/H)$. The first process, which can always be used to produce a field, is to think of $D_1(G/H)$, not as a W_1H representation, but as a representation of some proper subgroup W of W_1H. If V is the WH-representation induced from the W-representation $D_1(G/H)$, then V can be given a product ring structure (as a D(1)-module, it is just a sum of copies of $D_1(G/H)$) in such a way that WH acts on V by ring automorphisms. It is easy to check that the ring $J_{G/H}(V)$ is an integral domain into which D imbeds. Further, $J_{G/H}(V)$ is a field if and only if the order of $K \cap W$ is not divisible by the characteristic p of D(1). Clearly, taking W to be the trivial subgroup always produces a field. Note that $J_{G/H}(V)(1)$ is $D_1(G/H)^W$ which could be strictly larger than $D_1(G/H)^{W_1H} = D(1)$.

The second approach to converting the domain D into a field is not always applicable. For this approach, we try to obtain an extension field F of the field $D_1(G/H)$ to which the action of W_1H on $D_1(G/H)$ can be extended. If such an extension F exists, then the kernel K' of the action of W_1H on F will be smaller (unless the extension is purely inseparable--in which case, it is of no interest). If U is the WH-representation obtained from the W_1H representation F, then $J_{G/H}(U)$ is an integral domain into which D imbeds; it is a field if and only if p does not divide the order of K'.

Of course, these two processes can be combined to obtain other integral domains into which D imbeds and some of these may be fields.

Example 7.6 Let $G = \mathbb{Z}/2$ and consider the domain $D = A/q(\{e\}, 2)$ where A is the Burnside ring of $\mathbb{Z}/2$. We write the $\mathbb{Z}/2$ set $\mathbb{Z}/2/\{e\}$ as $\mathbb{Z}/2$. It is easy to see that $D(\mathbb{Z}/2)$ is $\mathbb{Z}/2$ with trivial $\mathbb{Z}/2$ action. Our two extension processes produce fields F_1 and F_2 The field F_1 produced by the first method has $F_1(\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with the permutation $\mathbb{Z}/2$ action. The field F_2 has, as $F_2(\mathbb{Z}/2)$, the field with four elements with $\mathbb{Z}/2$ acting as the Galois group. Clearly, $F_1 \neq F_2$. It is easy to see that there are no rings in either field strictly between D and the field. Thus, the domain D does not have a field of fractions in any obvious sense.

Remark 7.7 The nonexistence of fields of fractions in certain cases (and, more generally, of localizations) is a rather disappointing aspect of the theory of rings. However, it is not clear that this defect is as serious, or even as real, as it seems. There are at least two possible resolutions to this problem which deserve consideration.

The first possible resolution is that our notation of a unit may be too simplistic. Consider the fields F_1 and F_2 of Example 7.6. If $x \in F_2(\mathbb{Z}/2)$ is a generator of the field with four elements, then it is both a unit in F_2 and a unit in the ring $F_2(\mathbb{Z}/2)$. On the other hand, the element (1,0) in $F_1(\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is a unit in F_1 but not a unit in $F_1(\mathbb{Z}/2)$. It may be that the right way to specify the localization

$$\theta : R \longrightarrow S^{-1} R$$

of a ring R at a multiplicative subset S is to specify

how it is to be a unit.

The second possible approach is derived from the observation that, in the polynomial ring $D[x_b]$ generated by one variable x_b at $b \in \Omega$, there is a polynomial $p_s(x_b) \in D[x_b](b)$ associated to each $s \in D(b)$ whose "solutions" are the inverses of s in D(b). If $\theta : D \longrightarrow R$ is a ring map and $r \in R(b)$ is an inverse for $\theta(s)$, then there is a unique map

$$D[x_b] / p_s(x) \longrightarrow R$$

which takes x_b to r. In particular, the fields F_1 and F_2 of Example 7.6 are both quotient rings of the ring $D[x_{\mathbb{Z}/2}] / p(x_{\mathbb{Z}/2})$ where $p(x_{\mathbb{Z}/2})$ is the polynomial in $D[x_{\mathbb{Z}/2}](\mathbb{Z}/2)$ whose "solutions" would be inverses to $1_{\mathbb{Z}/2}$ in $D(\mathbb{Z}/2)$. Thus, the correct way to study localizations may be to investigate polynomial rings. It seems likely that the first approach to localization--by saying how something is to be a unit--can be described in terms of the second by using suitable polynomials.

Let us assume now that by some means--fair or foul--we have obtained a field F with characteristic subgroup H. Then $F(1)$ is certainly a field and, being G/H-projective, F is H-determined. Thus, Proposition 7.2 applies to describe F completely in terms of data we designate by $F_1(G/H)$, W_1H, K, $\mathcal{H}$ and e_1. Our objective is to understand the modules over F. Clearly, if V is an F module, then e_1 splits off an $F(1)$- subspace $V_1(G/H)$ of the $F(1)$ vector space $V(G/H)$. Further, $V_1(G/H)$ is a vector space over $F_1(G/H)$ and W_1H acts on $V_1(G/H)$ in such a way that the map

$$F_1(G/H) \otimes V_1(G/H) \longrightarrow V_1(G/H)$$

is W_1H equivariant when $F_1(G/H) \otimes V_1(G/H)$ is given the diagonal W_1H action. We can define a twisted group ring $F_1(G/H)[\theta]$ (where θ: $W_1H \longrightarrow \mathrm{Aut}(F_1(G/H))$ gives the action of W_1H on $F_1(G/H)$) as in Proposition 5.13 and thereby obtain a complete description of F-modules. Our principal objective, for the moment, is to show that every F-module is projective in the category F-mod of F-modules. For this problem, the twisted group ring view of F-mod is unnecessarily complicated.

Proposition 7.8 If F is a field and V is a module over F, then V is projective in the category of F-modules.

Proof Let $\varphi : U \longrightarrow U'$ be a surjection between F-modules and let $\theta : V \longrightarrow U'$ be a map of F-modules. We must construct a lifting $\tilde{\theta} : V \longrightarrow U$ of θ so that $\varphi\tilde{\theta} = \theta$. If suffices to construct a map

$$\hat{\theta} : V_1(G/H) \longrightarrow U_1(G/H)$$

of $F_1(G/H)$-vector spaces which commutes with the W_1H actions and makes the diagram

(3)
$$\begin{array}{ccc} & & V_1(G/H) \\ & \overset{\hat{\theta}}{\swarrow} & \downarrow \theta_1 \\ U_1(G/H) & \xrightarrow{\varphi_1} & U_1'(G/H) \end{array}$$

commute. Clearly, there is a map $f : V_1(G/H) \longrightarrow U_1(G/H)$ of $F_1(G/H)$ vector spaces which makes the diagram commute, and our only problem is to make f equivariant. Let $u \in F_1(G/H)$ with $tr_{\mathcal{H}}\, u = 1$; such a u exists because $F_1(G/H)$ is a separable normal extension of $F_1(G/H)^{W_1H}$. Then define $\hat{\theta} : V_1(G/H) \longrightarrow U_1(G/H)$ by

$$\hat{\theta}(x) = \frac{1}{|K|} \cdot \sum_{g \in W_1 H} g(uf(g^{-1}x)) \qquad \text{for } x \in V_1(G/H).$$

It is easy to check that $\hat{\theta}$ is a map of $F_1(G/H)$ vector spaces commuting with the W_1H actions and making diagram (3) commute. Note that $1/|K|$ has to make sense because F is a field.

8. Rings of Interest

Here, as always, A is the Brunside ring. The rings of interest are

$$B = \bigoplus_{[H] \leq G} H_{G/H}(A/q(H,O))$$

$$C = \bigoplus_{[H] \leq G} (A/q(H,O)) \otimes \mathbb{Z}[1/|WH|]$$

The best way to think of these two is as subrings of $A \otimes Q$ so that

$$A \subset B \subset C \subset A \otimes Q$$

The ring B is obtained from A by adding to A(G/H) the idempotents which split $A(G/H) \otimes Q$ for every $H \leq G$. The ring C is obtained from A by adding the elements $e_H/|WH|$ to A where e_H is the idempotent in $A(1) \otimes Q$ which corresponds to the subgroup H. The ring C contains B because the $1/|WH|$ factor generates all the idempotents in the $A(G/H) \otimes Q$ by various transfers and multiplications.

The point of B is that it is--in some sense--the integral closure of the ring A in the ring $A \otimes Q$. A prime ideal lifting theorem which does for A exactly what the standard theorem does for finding the prime ideals of A(1) is a distinct possibility that is beyond the scope of this paper.

C is more important. It should be used in place of $A \otimes Z[1/|G|]$ All the nice theorems about $Z[1/|G|]$-valued Mackey functors can be extended to results about modules over C. Note that

$$C \subset A \otimes \mathbb{Z}[1/|G|] \subset A \otimes Q.$$

The advantage of C is that it preserves the maximal prime ideals

$q(H^P, p)$ for the primes p which divide $|G|$ whereas $A \otimes Z[1/|G|]$ obliterates them. The maximal primes $q(H^P, p)$ are perfectly respectable in A considered as a ring and there is no reason to throw them away. Another advantage of C is that it should make good sense--and be perfectly well behaved--for a compact Lie group where $A \otimes Z[1/|G|]$ is available only if one uses tom Dieck's--apparently not well understood--substitite for $|G|$. Certainly, for compact Lie groups, C should preserve vastly more information and become correspondingly more important.

Note that the summands $H_{G/H}(A/q(H,0))$ of B are G/H-determined and so well behaved and computable. The summand $(A/q(H,0)) \otimes Z[1/|WH|]$ of C is G/H-projective and, along with its modules, is utterly well behaved. Also, the ring C has homological dimension one. In fact, the category of modules over C is isomorphic to the sum

$$\bigoplus_{[H] \leq G} Z[1/|WH|]\,[WH] \text{ -modules}$$

where $Z[1/|WH|]\,[WH]$ is the group ring of WH with coefficients in $Z[1/|WH|]$. The fudge factor $1/|WH|$ is exactly what is needed in modules over a group ring to get homological dimension one.

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