Topological Cyclic Homology of Local Fields

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Abstract

We introduce a new approach to compute topological cyclic homology using the descent spectral sequence. We carry out the computation for a p-adic local field with $\mathbb{F}_p$-coefficient.

Contents

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1 Introduction

Fix a prime number p. Let K be a p-adic local field, i.e. a finite extension of $\mathbb{Q}_p$. In this paper, we introduce the *descent spectral sequence* to compute topological cyclic homology $\mathrm{TC}_*(\mathcal{O}_K)$ and its variants $\mathrm{TC}^-_*(\mathcal{O}_K)$ and $\mathrm{TP}_*(\mathcal{O}_K)$. In fact, we carry out the computation in the modulo p case, and obtain the structure of $\mathrm{TC}_*(\mathcal{O}_K;\mathbb{F}_p)$, which in turn determines the mod p algebraic K-theory of $\mathcal{O}_K$ by the cyclotomic trace map. Moreover, our approach for computing topological cyclic homology may apply to more general cases. In a forthcoming paper [8], we will treat the case of locally complete intersection schemes over $\mathbb{Z}_p$.

The descent spectral sequence is constructed using relative topological Hochschild homology through an Adams type resolution of the base ring. More precisely, let $\mathbf{k}$ be the residue field of $\mathcal{O}_K$, and let $W(\mathbf{k})$ be the ring of Witt vectors over $\mathbf{k}$. Let $\mathbb{S}_{W(\mathbf{k})}$ be the spherical Witt vectors (cf. [9, §5.2]). Let $\mathbb{N}$ be the additive monoid of natural numbers, and let $\mathbb{S}_{W(\mathbf{k})}[z]$ be the E_∞-ring spectrum

$$\mathbb{S}_{W(\mathbf{k})} \otimes_{\mathbb{S}} \Sigma^\infty \mathbb{N}_+.$$

We have a map of E_∞-ring spectra $\mathbb{S}_{W(\mathbf{k})}[z] \to \mathcal{O}_K$ sending z to ϖ_K. Using this map, we may define the relative topological Hochschild homology $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$, which has the structure of an E_∞-cyclotomic spectrum by [2]. Therefore we may further define relative periodic topological cyclic homology $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ and relative negative topological cyclic homology $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$.

By taking an Adams type resolution

$$\mathbb{S}_{W(\mathbf{k})} \to \mathbb{S}_{W(\mathbf{k})}[z] \to \mathbb{S}_{W(\mathbf{k})}[z_1, z_2] \to \dots \qquad (\star)$$

of $\mathbb{S}_{W(\mathbf{k})}$, we obtain a cosimplicial E_∞-cyclotomic spectrum $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})$. This gives rise to the descent spectral sequences

$$E_1^{i,j}(\mathrm{THH}(\mathcal{O}_K)) = \mathrm{THH}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i}) \Rightarrow \mathrm{THH}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}),$$

$$E_1^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)) = \mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i}) \Rightarrow \mathrm{TC}^-_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}),$$

and

$$E_1^{i,j}(\mathrm{TP}^-(\mathcal{O}_K)) = \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i}) \Rightarrow \mathrm{TP}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$$

(see §5 for more details); they are analogues of the Adams spectral sequence in the category of cyclotomic spectra. Combining the last two spectral sequences we obtain the third spectral sequence

$$E_2^{i,j}(\mathrm{TC}(\mathcal{O}_K)) \Rightarrow \mathrm{TC}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}),$$

where $E_2^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)), E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K))$ and $E_2^{i,j}(\mathrm{TC}(\mathcal{O}_K))$ are related by a long exact sequence induced from the fiber sequence

$$\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \to \mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \xrightarrow{\mathrm{can}-\varphi} \mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}).$$

By similar procedures as in the construction of the Adams spectral sequence, we show that the E_2-term of the decent spectral sequence for $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ (resp.

$\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$) is isomorphic to the cobar complex for $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ (resp. $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$) with respect to the Hopf algebroid

$$(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]))$$

(resp. $(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])))$. To understand the structure of these Hopf algebroids, we make use of the theory of δ-rings. Recall that a δ-ring structure on a p-torsionfree commutative ring A is equivalent to the datum of a ring map $\varphi : A \to A$ lifting the Frobenius on A/p; the corresponding δ-structure is given by

$$\delta(x) = \frac{\varphi(x) - x^p}{p}.$$

Note that there is a Frobenius map φ on $W(\mathbf{k})[z]$ which is the Frobenius on $W(\mathbf{k})$ and sends z to z^p. This makes $W(\mathbf{k})[z]$ into a δ-ring. In fact, the Frobenius map φ makes all $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})$ into δ-rings.

Let ϖ_K be a uniformizer of $\mathcal{O}_K$. Using results of [2], we know that $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is isomorphic to the completion of $W(\mathbf{k})[z]$ with respect to the filtration defined by powers of $E_K(z)$, which is a minimal polynomial for ϖ_K over $W(\mathbf{k})$. Determining the structures of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ and $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ is one of the key steps of the paper. It turns out that the former is isomorphic to the completed δ-ring obtained by adjoining

$$h = \frac{\varphi(z_1 - z_2)}{\varphi(E_K(z_1))}$$

to $W(\mathbf{k})[z_1,z_2]$. Consequently, we obtain an explicit description of the Hopf algebroid $(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]))$, which is isomorphic to the associated graded Hopf algebroid of $(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]))$ (see §3, §4 for more details).

The explicit description of $(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]))$ allows us to compute E_2-terms of the descent spectral sequences using standard relative injective resolutions. In §5, we first do this for $\mathrm{THH}(\mathcal{O}_K)$. Then we use the Nygaard filtration on the cobar complex for $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ to construct the algebraic Tate spectral sequence

$$E_2(\mathrm{THH}(\mathcal{O}_K))[\sigma^{\pm}] \Rightarrow E_2(\mathrm{TP}(\mathcal{O}_K)).$$

We also construct the algebraic homotopy fixed points spectral sequence

$$E_2(\mathrm{THH}(\mathcal{O}_K))[v] \Rightarrow E_2(\mathrm{TC}^-(\mathcal{O}_K))$$

in a similar way.

It turns out that extra complication occurs when apply the above approach to compute E_2-terms of the mod p descent spectral sequences. To remedy, we introduce a refinement of the Nygaard filtration on $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p)$ and compute all the differentials of the refined algebraic Tate spectral sequence in §6. Using refined algebraic Tate differentials, we compute E_2-terms of the descent spectral sequences for $\mathrm{TC}^-_*(\mathcal{O}_K; \mathbb{F}_p)$ and $\mathrm{TP}_*(\mathcal{O}_K; \mathbb{F}_p)$ in §7. In §8, we compute the E_2-term of the spectral sequence for $\mathrm{TC}_*(\mathcal{O}_K; \mathbb{F}_p)$. In §9, using the Bott elements in $\mathbb{K}_2(\mathbb{Q}_p(\zeta_{p^n}))$, we deduce that the constant term of $E_K(z)$ has to be equal to p. Putting these together, we finally conclude our main result:

Theorem 1.1. *Let $d = [K(\zeta_p) : K]$. Then we explicitly construct*

$$\beta, \lambda, \gamma, \alpha_1^{(1)}, \dots, \alpha_{e_K}^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{e_K}^{(2)}, \dots, \alpha_1^{(d)}, \dots, \alpha_{e_K}^{(d)} \in \mathrm{TC}_*(\mathcal{O}_K; \mathbb{F}_p)$$

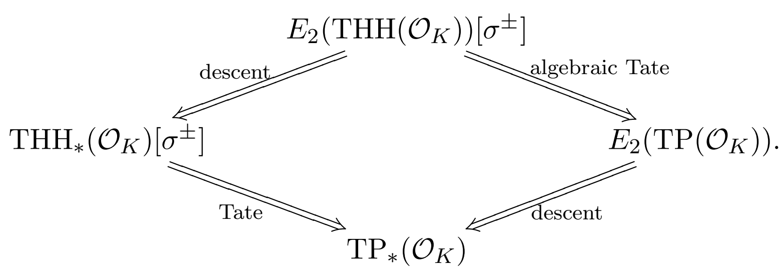
with $|\beta| = 2d$, $|\lambda| = -1$, $|\gamma| = 2d+1$, $|\alpha_i^{(j)}| = 2j-1$, such that

$$\mathrm{TC}_*(\mathcal{O}_K; \mathbb{F}_p) \cong \mathbb{F}_p[\beta]\{1, \lambda, \gamma, \lambda\gamma\} \oplus \mathbf{k}[\beta]\{\alpha_i^{(j)} | 1 \le i \le e_K, 1 \le j \le d\}$$

as $\mathbb{F}_p[\beta]$-modules. In particular, $\mathrm{TC}_*(\mathcal{O}_K; \mathbb{F}_p)$ is a free $\mathbb{F}_p[\beta]$-module.

Topological cyclic homology is an important tool for understanding algebraic K-theory. The case of p-adic local fields has been extensively studied by many people. For example, the case p odd and $e_K = 1$ is computed in [4] and [13]. The case p odd and e_K arbitrary is computed in [5]. The case $p = 2$ and $e_K = 1$ is computed in [12].

These prior works adopt a similar strategy, which is different form ours; the difference may be summarized by the following diagram



In the works mentioned above, one starts with the descent spectral sequence for $\mathrm{THH}(\mathcal{O}_K)$, which collapses at E_2-term in consideration of degrees. Then one applies the Tate spectral sequence to compute $\mathrm{TP}_*(\mathcal{O}_K)$. The hard part is to compute the Tate differentials, and the main technique for doing this is to inductively determine the structures of the Tate spectral sequences for all finite subgroups of the circle group $\mathbb{T}$.

Our approach proceeds in another direction. We first run the (mod p) algebraic Tate spectral sequence to compute the E_2-term of the descent spectral sequence for $\mathrm{TP}(\mathcal{O}_K)$. Since the cobar complex can be described explicitly thanks to the determination of the Hopf algebroid $(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]))$, the computation of algebraic Tate differentials is purely algebraic. It follows that the descent spectral sequence for $\mathrm{TP}_*(\mathcal{O}_K)$ collapses at the E_2-term in consideration of degrees. Indeed, it turns out that the structure of the algebraic Tate spectral sequence is similar to the structure of the Tate spectral sequence (see Remark 6.42). That is to say, by the resolution ($\star$) and the Nygaard filtration, we transform the problem of computing the Tate differentials, which is topological in nature, to a purely algebraic problem which in turn can be solved explicitly.

As indicated in the diagram, our approach consists of two steps. The first one is to determine the algebraic Tate differentials, which is purely algebraic. The other one is the computation of the descent spectral sequence for TP. As mentioned at the beginning, we will apply this approach to study the topological cyclic homology of locally complete intersection schemes over $\mathbb{Z}_p$ in a forthcoming paper. It turns out that for those schemes, the descent spectral sequence for TP is no longer degenerate. Their size are bounded by the number of generators of the sheaf of regular functions.

Finally, in §10, we observe that the decent spectral sequence computing $\mathrm{TC}_*(\mathcal{O}_K;\mathbb{F}_p)$ is reminiscent of the motivic spectral sequence computing $\mathbb{K}_*(K,\mathbb{F}_p)$. We expect that the decent spectral sequence will provide some incarnation of the motivic spectral sequence in the p-adic setting.

Relation with other works

The present work started with an attempt to compute $\mathrm{TC}(\mathcal{O}_K)$ using the spectral sequence introduced by Bhatt-Morrow-Scholze relating the prisms and topological cyclic homology [2]. In fact, one may resolve $\mathcal{O}_K$ by perfectoids in the quasi-syntomic site, and obtain a complex similar to $(\star)$ but having p-fractional powers of z_i's. Moreover, the E_1-term of the resulting spectral sequence has the descent spectral sequence as a subcomplex consisting of terms with integer exponents. We conjecture that the E_1-terms of these two spectral sequences are quasi-isomorphic, i.e. the subcomplex consisting of terms with non-integer exponents is acyclic.

In [6], Krause-Nikolaus also introduce a descent style spectral sequence to compute the topological Hochschild homology of quotients of DVRs. Their work also recover the main result of Lindenstrauss-Madsen [7] as ours (Corollary 5.19).

Notation and conventions

Fix a prime p. Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbf{k}$. Denote by $K_0 = W(\mathbf{k})[1/p]$ the maximal unramified subextension of K over $\mathbb{Q}_p$. Let e_K and f_K be the ramification index and inertia degree of K over $\mathbb{Q}_p$ respectively. Let ϖ_K be a uniformizer of $\mathcal{O}_K$, and let $E_K(z)$ be a minimal polynomial of ϖ_K over K_0. Let μ denote the leading coefficient of $E_K(z)$.

Warning: Throughout this paper, all Nygaard filtrations involved only jump at even numbers. For our purpose, we rescale the index of Nygaard filtrations by 2 after Convention 6.7.

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2 Cyclotomic structures on relative THH

Recall that the relative topological Hochschild homology is defined by the cyclic bar construction over the base.

Definition 2.1. Let E be an E_∞-ring spectrum, and let A be an E_∞-algebra over E. The relative topological Hochschild homology of A over E is defined as

$$\mathrm{THH}(A/E) = A^{\otimes_E \mathbb{T}}$$

in the ∞-category of E_∞-ring spectra, which is universal among the objects of $\mathbb{T}$-equivariant E_∞-E-algebras with a (non-equivariant) map from A.

The universal property of relative THH implies the following multiplicative property

$$\mathrm{THH}(A_1/E_1) \otimes_{\mathrm{THH}(A_2/E_2)} \mathrm{THH}(A_3/E_3) \cong \mathrm{THH}(A_1 \otimes_{A_2} A_3/E_1 \otimes_{E_2} E_3). \tag{2.2}$$

In general, relative topological Hochschild homology may not have cyclotomic structures. For example, the Hochschild homology $\mathrm{HH}(-) = \mathrm{THH}(-/\mathbb{Z})$ is not cyclotomic ([11, III.1.10]). However, we may put more conditions on the base to obtain a natural cyclotomic structure on the resulting relative THH.

Lemma 2.3. *The following are true.*

(1) Let E be an E_∞-cyclotomic spectrum such that the underlying $\mathbb{T}$-action is trivial. If the augmentation map

$$\mathrm{THH}(E) \to E$$

is a map of E_∞-cyclotomic spectra, then the functor $\mathrm{THH}(-/E)$ can be lifted to a functor from E_∞-E-algebras to E_∞-cyclotomic spectra.

(2) Moreover, suppose we have a commutative diagram of E_∞-cyclotomic spectra

$$\begin{array}{ccc} \mathrm{THH}(E_1) & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ \mathrm{THH}(E_2) & \longrightarrow & E_2 \end{array}$$

with trivial $\mathbb{T}$-actions on E_1 and E_2 such that it extends to a commutative diagram of E_∞-ring spectra

$$\begin{array}{ccccc} \mathrm{THH}(E_1) & \longrightarrow & E_1 & \longrightarrow & A_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(E_2) & \longrightarrow & E_2 & \longrightarrow & A_2. \end{array}$$

Then the natural map $\mathrm{THH}(A_1/E_1) \to \mathrm{THH}(A_2/E_2)$ is a map of E_∞-cyclotomic spectra.

Proof. Part (1) is essentially [2, Construction 11.5]. In fact, by (2.2), we get

$$\mathrm{THH}(X/E) \cong \mathrm{THH}(X) \otimes_{\mathrm{THH}(E)} E$$

in the ∞-category of E_∞-ring spectra. Since the forgetful functor from E_∞-cyclotomic spectra to E_∞-ring spectra is symmetric monoidal and preserves small colimits, we may lift $\mathrm{THH}(X/E)$ as the pushout of $\mathrm{THH}(X) \leftarrow \mathrm{THH}(E) \to E$ in the ∞-category of E_∞-cyclotomic spectra. Part (2) follows immediately. $\square$

Definition 2.4. When the condition of Lemma 2.3(1) holds, we set the relative negative cyclic homology

$$\mathrm{TC}^-(-/E) = \mathrm{THH}(-/E)^{h\mathbb{T}}$$

and the relative periodic cyclic homology

$$\mathrm{TP}(-/E) = (\mathrm{THH}(-/E)^{t\mathbb{T}}.$$

As in the absolute case, for any prime l, the cyclotomic structure on $\mathrm{THH}(-/E)$ induces the Frobenius

$$\varphi_l : \mathrm{TC}^-(-/E) = \mathrm{THH}(-/E)^{h\mathbb{T}} \to (\mathrm{THH}(-/E)^{tC_l})^{h\mathbb{T}}. \tag{2.5}$$

Moreover, there is the canonical map

$$\mathrm{can} : \mathrm{TC}^-(-/E) \cong (\mathrm{THH}(-/E)^{hC_l})^{h(\mathbb{T}/C_l)} = (\mathrm{THH}(-/E)^{hC_l})^{h\mathbb{T}} \to (\mathrm{THH}(-/E)^{tC_l})^{h\mathbb{T}}. \tag{2.6}$$

The relative topological cyclic homology is defined by the fiber sequence

$$\mathrm{TC}(-/E) \to \mathrm{TC}^-(-/E) \xrightarrow{\prod_l(\varphi_l^{h\mathbb{T}}-\mathrm{can})} \mathrm{TP}(-/E). \tag{2.7}$$

Using the argument of [11, Lemma II 4.2], we have

$$\mathrm{TP}(-/E;\mathbb{Z}_p) \cong (\mathrm{THH}(-/E)^{tC_p})^{h\mathbb{T}}. \tag{2.8}$$

Taking p-completion on (2.5), (2.6), we get

$$\varphi : \mathrm{TC}^-(-/E;\mathbb{Z}_p) \to \mathrm{TP}(-/E;\mathbb{Z}_p), \quad \mathrm{can} : \mathrm{TC}^-(-/E;\mathbb{Z}_p) \to \mathrm{TP}(-/E;\mathbb{Z}_p)$$

and the fiber sequence

$$\mathrm{TC}(-/E;\mathbb{Z}_p) \to \mathrm{TC}^-(-/E;\mathbb{Z}_p) \xrightarrow{\varphi-\mathrm{can}} \mathrm{TP}(-/E;\mathbb{Z}_p).$$

As in the absolute case, there are the homotopy fixed point spectral sequence

$$E_2^{i,j} = \mathrm{THH}_*(-/E)[v] \Rightarrow \mathrm{TC}^-_{i-j}(-/E) \tag{2.9}$$

and the Tate spectral sequence

$$E_2^{i,j} = \mathrm{THH}_*(-/E)[\sigma^{\pm 1}] \Rightarrow \mathrm{TP}_{i-j}(-/E), \tag{2.10}$$

where $|v| = -2, |\sigma| = 2$, and $\mathrm{can}(v) = \sigma^{-1}$. The *Nygaard filtration* $\mathcal{N}^{\geq \bullet}$ is defined to be the filtration on the abutment of the Tate spectral sequence; it is multiplicative as the Tate spectral sequence is multiplicative. When the Tate spectral sequence collapses at the E_2-term, we denote by p_j the natural projection

$$\mathcal{N}^{\geq j}(\mathrm{TP}_0(-/E)) \to \mathrm{THH}_j(-/E).$$

Recall that by [2, Proposition 11.3], $\mathbb{S}[z]$ admits an E_∞-cyclotomic structure over $\mathrm{THH}(\mathbb{S}[z])$ in which the $\mathbb{T}$-action is trivial and the Frobenius sends z to z^p.

Proposition 2.11. *The following are true.*

1. *There is a functorial E_∞-cyclotomic structure on* $\mathrm{THH}(-/\mathbb{S}_{W(\mathbf{k})})$.
2. *There is a functorial E_∞-cyclotomic structure on* $\mathrm{THH}(-/\mathbb{S}_{W(\mathbf{k})}[z])$.

Proof. We set the Frobenius on $\mathbb{S}_{W(\mathbf{k})}$ to be the unique E_∞-automorphism inducing the Frobenius on π_0. It follows that the resulting cyclotomic structure on $\mathbb{S}_{W(\mathbf{k})}$ agrees with the p-completion of the cyclotomic structure on $\mathrm{THH}(\mathbb{S}_{W(\mathbf{k})})$ via the augmentation map [11, IV.1.2]. This yields (1) by Lemma 2.3.

For (2), note that

$$\mathbb{S}_{W(\mathbf{k})}[z] \cong \mathbb{S}_{W(\mathbf{k})} \otimes_{\mathbb{S}} \mathbb{S}[z]$$

in the ∞-category of E_∞-ring spectra. We then define the cyclotomic structure on $\mathbb{S}_{W(\mathbf{k})}[z]$ using the cyclotomic structures on $\mathbb{S}_{W(\mathbf{k})}$ and $\mathbb{S}[z]$, and the monoidal structure on the ∞-category of E_∞-cyclotomic spectra. We conclude (2) by (1) and Lemma 2.3. □

Remark 2.12. Since $\mathbb{S}_{W(\mathbf{k})}$ is the p-completion of $\mathrm{THH}(\mathbb{S}_{W(\mathbf{k})})$, it follows that

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \cong \mathrm{THH}(\mathcal{O}_K) \otimes_{\mathrm{THH}(\mathbb{S}_{W(\mathbf{k})})} \mathbb{S}_{W(\mathbf{k})}$$

is isomorphic to the p-completion of $\mathrm{THH}(\mathcal{O}_K)$. Similarly, $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is isomorphic to the p-completion of $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}[z])$.

Remark 2.13. By the previous remark, we see that $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$, $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$, $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$, $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$, $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ and $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ are isomorphic to p-completions of $\mathrm{TC}(\mathcal{O}_K)$, $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}[z])$, $\mathrm{TC}^-(\mathcal{O}_K)$, $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}[z])$, $\mathrm{TP}(\mathcal{O}_K)$ and $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}[z])$ respectively.

Note that the composite

$$\mathbb{S}[z] \to \mathrm{THH}(-/\mathbb{S}[z]),$$

which is a map of E_∞-cyclotomic spectra, induces

$$i_C : \mathbb{S}[z]^{h\mathbb{T}} \to \mathrm{TC}_0^-(-/\mathbb{S}[z]; \mathbb{Z}_p), \quad i_P : (\mathbb{S}[z]^{tC_p})^{h\mathbb{T}} \to \mathrm{TP}_0(-/\mathbb{S}[z]; \mathbb{Z}_p).$$

Recall that $\mathbb{S}[z]$ is equipped with the trivial $\mathbb{T}$-action. In the following, when the context is clear, we abusively use z to denote the its images under i_C and $\mathrm{can} \circ i_C$.

Proposition 2.14. *We have $\varphi(z) = z^p$.*

Proof. Recall that the Frobenius φ_p on $\mathbb{S}[z]$ is the composite

$$\mathbb{S}[z] \xrightarrow{z \mapsto z^p} \mathbb{S}[z] \to \mathbb{S}[z]^{hC_p} \xrightarrow{\mathrm{can}} \mathbb{S}[z]^{tC_p}.$$

It follows that the composite

$$\varphi_p^{h\mathbb{T}} : \mathbb{S}[z]^{h\mathbb{T}} \xrightarrow{z \mapsto z^p} \mathbb{S}[z]^{h\mathbb{T}} \to \mathbb{S}[z]^{h\mathbb{T}} \xrightarrow{\mathrm{can}} (\mathbb{S}[z]^{tC_p})^{h\mathbb{T}}$$

satisfies $\varphi_p^{h\mathbb{T}}(z) = \mathrm{can}(z)^p$. On the other hand, it is straightforward to see

$$\varphi \circ i_C = i_P \circ \varphi_p^{h\mathbb{T}} \quad \text{and} \quad \mathrm{can} \circ i_C = i_P \circ \mathrm{can}.$$

The desired result follows. □

Now we specialize to the case of $\mathcal{O}_K$.

Theorem 2.15. *We have the following results on homotopy groups.*

(1) *We have*
$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathcal{O}_K[u],$$
where $u \in \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is any lift of the Bökstedt element in $\mathrm{THH}_2(\mathbf{k})$.

(2) *The Tate spectral sequence for $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ degenerates at the E_2-term. Consequently,*
$$\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])[\sigma^{\pm 1}]$$
with $|\sigma| = 2$.

(3) *We have*
$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong W(\mathbf{k})[[z]],$$
and the natural map $p_0 : \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{THH}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ corresponds to
$$W(\mathbf{k})[[z]] \xrightarrow{z \mapsto \varpi_K} \mathcal{O}_K.$$

(4) *The homotopy fixed point spectral sequence for $\mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ degenerates at the E_2-term. Consequently, the canonical map induces*
$$\mathrm{TC}_j^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]).$$
for $j \leq 0$.

(5) *Under the isomorphisms in (3) and (4), the Frobenius*
$$\varphi : \mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$$
corresponds to the map $W(\mathbf{k})[[z]] \to W(\mathbf{k})[[z]]$ which is the Frobenius on $W(\mathbf{k})$ and sends z to z^p.

(6) *We have*
$$\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])[u,v]/(uv - E_K(z))$$
where u is a lift of the u given in (1) under p_0 and $|v| = -2$ satisfying $\varphi(u) = \sigma$ and $\mathrm{can}(v) = \sigma^{-1}$. As a consequence, under the isomorphisms in (3), the Nygaard filtration on $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is given by
$$\mathcal{N}^{\geq 2j}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) = \mathcal{N}^{\geq 2j+1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) = (E_K(z))^j, \quad j \geq 0. \tag{2.16}$$
Moreover, we can make the constant term of $E_K(z)$ the same for all K.

Proof. By Remark 2.13, we see that all the statements except the last assertion of (6) follow immediately from [2, Proposition 11.10]. In fact, the argument given in loc. cit. is enough to show the following statement:

(6)' For any $u' \in \mathrm{TC}_2^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ lifting the u given in (1), there exist $v' \in \mathrm{TC}_{-2}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ and $\sigma' \in \mathrm{TP}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ such that

$$\mathrm{can}(v') = \sigma'^{-1}, \quad \varphi(u') = \sigma', \quad \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])[\sigma'^{\pm 1}]$$

and

$$\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])[u', v']/(u'v' - E_K(z)),$$

where $E_K(z)$ is a minimal polynomial of ϖ_K over K_0.

In the following, we give a proof of (6) based on (6)'. Firstly, by [11], there exists some nonzero $u_{\mathbb{F}_p} \in \mathrm{TC}_2^-(\mathbb{F}_p)$ such that

$$\mathrm{can}(u_{\mathbb{F}_p}) = p\tau\varphi(u_{\mathbb{F}_p})$$

for some $\tau \in \mathbb{Z}_p{}^\times$. Let $u_{\mathbf{k}}$ be the image of $u_{\mathbb{F}_p}$ along $\mathbb{F}_p \to \mathbf{k}$. By Lemma 2.3, the commutative diagram

$$\begin{array}{ccc} \mathbb{S}_{W(\mathbf{k})}[z] & \longrightarrow & \mathbb{S}_{W(\mathbf{k})} \\ {\scriptstyle z\mapsto\varpi_K}\downarrow & & \downarrow \\ \mathcal{O}_K & \longrightarrow & \mathbf{k} \end{array}$$

induces a map of E_∞-cyclotomic spectra $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{THH}(\mathbf{k})$. By (3) and (4), the induced map

$$\mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{TC}_0^-(\mathbf{k})$$

corresponds to the map $W(\mathbf{k})[[z]] \xrightarrow{z\mapsto 0} W(\mathbf{k})$, which is surjective. Moreover, by (6), $\mathrm{TC}_2^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is free of rank 1 over $\mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$. Hence

$$\mathrm{TC}_2^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{TC}_2^-(\mathbf{k})$$

is surjective as well.

Now take a lift u' of $u_{\mathbf{k}}$ in $\mathrm{TC}_2^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$. Using (6)', we have v', σ' such that

$$\mathrm{can}(v') = \sigma'^{-1}, \quad \varphi(u') = \sigma', \quad \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])[\sigma'^{\pm 1}]$$

and

$$\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \cong \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])[u', v']/(u'v' - E_K(z)).$$

Now by the construction of u', we deduce that the constant term of $E_K(z)$ is equal to $p\tau$, yielding the desired result. □

3 Structure of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$

This section is devoted to determining the structure of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$. Here we regard $\mathcal{O}_K$ as an $\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]$-algebra via the map $\mathbb{S}_{W(\mathbf{k})}[z_1, z_2] \xrightarrow{z_1, z_2 \mapsto \varpi_K} \mathcal{O}_K$.

Proposition 3.1. $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ *has a natural* E_∞*-cyclotomic structure.*

Proof. By the multiplicative property of relative THH (2.2), we have

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \cong \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1]) \otimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_2]).$$

The cyclotomic structures of $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_i]), i = 1, 2$, and the symmetric monoidal structure on the ∞-category of E_∞-cyclotomic spectra give rise to the E_∞-cyclotomic structure on $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. □

For $\heartsuit \in \{\mathrm{THH}, \mathrm{TC}, \mathrm{TC}^-, \mathrm{TP}\}$, the left unit η_L and right unit η_R are the maps

$$\heartsuit(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \heartsuit(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$$

induced by $z \mapsto z_1$ and $z \mapsto z_2$ respectively. For $? \in \{z, u, v, \sigma\}$, we denote by $?_1$ and $?_2$ the images of ? under the left and right units respectively. In the following, we regard $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ as an $\mathcal{O}_K[u_1]$-module via η_L.

Let I be the kernel of $W(\mathbf{k})[z_1,z_2] \to \mathcal{O}_K$ sending z_1, z_2 to ϖ, and let $t_{z_1-z_2}$ denote the image of $z_1 - z_2$ in $\mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ under the composite of isomorphisms

$$I/I^2 \cong \mathrm{HH}_2(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2]) \cong \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]).$$

Lemma 3.2. *The graded algebra associated to the filtration on* $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *defined by powers of* u_1 *is isomorphic to* $\mathcal{O}_K[u_1] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t_{z_1-z_2}\rangle$*, where* $\mathcal{O}_K\langle t\rangle$ *denotes the one variable divided power polynomial algebra over* $\mathcal{O}_K$*.*

Proof. By Theorem 2.15(1), we have

$$\begin{aligned} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])/(u_1) &\cong \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \otimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{\mathbb{W}(\mathbf{k})}[z_1])} \mathrm{THH}(\mathcal{O}_K/\mathcal{O}_K) \\ &\cong \mathrm{THH}(\mathcal{O}_K/\mathcal{O}_K[z_2]). \end{aligned} \tag{3.3}$$

Since $\mathrm{THH}_*(\mathcal{O}_K/\mathcal{O}_K[z]) \cong \mathrm{HH}_*(\mathcal{O}_K/\mathcal{O}_K[z]) \cong \mathcal{O}_K\langle t\rangle$ for any generator $t \in \mathrm{HH}_2(\mathcal{O}_K/\mathcal{O}_K[z])$, we deduce that the u_1-Bockstein spectral sequence collapses since everything is concentrated in even degrees. Hence the associated graded algebra is isomorphic to $\mathcal{O}_K[u_1] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t\rangle$. Note that under the isomorphism (3.3), $t_{z_1-z_2}$ maps to a generator of $\mathrm{THH}_2(\mathcal{O}_K/\mathcal{O}_K[z])$. This yields the desired result. □

The following result follows immediately.

Corollary 3.4. *We have that* $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *is a* p*-torsionfree integeral domain.*

Corollary 3.5. *Both the Tate spectral sequence for* $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *and the homotopy fixed point spectral sequence for* $\mathrm{TC}^-_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *degenerate at the* E_2*-term. Moreover,* $\mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *and* $\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *are concentrated in even degrees. The canonical morphism induces*

$$\mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \cong \mathcal{N}^{\geq j}\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]).$$

In particular,

$$\mathrm{can}: \mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \to \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$$

is an isomorphism for $j \leq 0$*.*

Proof. By Lemma 3.2, $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ is concentrated in even degrees. It follows that both the Tate spectral sequence and the homotopy fixed point spectral sequence degenerate at the E_2-term; $\mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ and $\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ are concentrated in even degrees. The rest of the corollary follows immediately. □

Remark 3.6. In general, for $n \geq 0$, we may regard $\mathcal{O}_K$ as an $\mathbb{S}_{W(\mathbf{k})}[z_1,\dots,z_n]$-module by sending all z_i to ϖ_K. Using the argument of Lemma 3.2 inductively, one easily shows that $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,\dots,z_n])$ is concentrated in even degrees. Consequently, Corollary 3.5 generalizes to this case.

Lemma 3.7. *The graded algebra associated to the Nygaard filtration of* $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *is isomorphic to* $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$.

Proof. This follows from Corollary 3.5. □

The following two results follow immediately.

Corollary 3.8. *For* $a \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$*, it has Nygaard filtration* j *if and only if* pa *has Nygaard filtration* j.

Corollary 3.9. *We have that* $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *is a* p*-torsionfree integral domain.*

Henceforth, by Corollary 3.5, we may identify $\mathrm{TC}^-_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ with $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ via the canonical map, and regard the Frobenius as an endomorphism on $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. By Proposition 2.14, we have

$$\varphi(z_1) = z_1^p, \qquad \varphi(z_2) = z_2^p. \tag{3.10}$$

Lemma 3.11. *If* $a \in \mathcal{N}^{\geq 2j}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$*, then* $\varphi(a)$ *is divisible by* $\varphi(E_K(z_1))^j$.

Proof. By Corollary 3.5, the Tate spectral sequence for $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ collapses at the E_2-term. We may write $a = \sigma_1^{-j}a_0$ for some

$$a_0 \in \mathcal{N}^{\geq 2j}\mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) = \mathrm{TC}^-_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]).$$

Hence by Theorem 2.15,

$$\varphi(a) = \varphi(\sigma_1^{-1})^j\varphi(a_0) = \varphi(v_1)^j\varphi(u_1)^j\sigma_1^{-j}\varphi(a_0) = \varphi(E_K(z_1))^j\sigma_1^{-j}\varphi(a_0),$$

yielding the desired result. □

Remark 3.12. By Theorem 2.15(3), $E_K(z)$ has Nygaard filtration 2 in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$. Hence $E_K(z_i)$ has Nygaard filtration 2 in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. By Lemma 3.11, $\varphi(E_K(z_i))$ is divisible by $\varphi(E_K(z_{3-i}))$ for $i=1,2$. Thus $\varphi(E_K(z_1))\varphi(E_K(z_2))^{-1}$ is a unit in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$.

Definition 3.13. For a ring R equipped with a multiplicative decreasing filtration $\mathcal{N}^{\geq\bullet}$, we call the topology on R defined by the filtration $\mathcal{N}^{\geq\bullet}$ the $\mathcal{N}$-topology. We define the $(p,\mathcal{N})$*-topology* on R to be the topology in which $\{(p^j)+\mathcal{N}^{\geq j}\}_{j\geq 0}$ forms a basis of open neighborhoods of 0.

Clearly R becomes a topological ring under either the $\mathcal{N}$ or the $(p, \mathcal{N})$-topology.

Remark 3.14. By Theorem 2.15, it is straightforward to see that $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is complete and separated under either the $\mathcal{N}$ or the $(p, \mathcal{N})$-topology. Moreover, the Frobenius is continuous with respect to the $(p, \mathcal{N})$-topology, but not the $\mathcal{N}$-topology.

Lemma 3.15. *Both the $\mathcal{N}$ and $(p, \mathcal{N})$-topology on $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ are complete and separated.*

Proof. The assertion for the $\mathcal{N}$-topology follows from the isomorphism

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \cong \mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$$

given by Corollary 3.5 and the fact that $\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ are all complete with respect to the $\mathcal{N}$-topology. For the $(p, \mathcal{N})$-topology, we first note that by Lemma 3.2, $THH_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ are all p-complete. By degeneration of the Tate spectral sequence, this implies that for each $j \geq 0$,

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])/\mathcal{N}^{\geq j}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$$

is p-complete and separated. Hence the $(p, \mathcal{N})$-completeness (resp. separateness) follows from the $\mathcal{N}$-completeness (resp, separateness). □

Lemma 3.16. *The Frobenius on $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ is continuous with respect to the $(p, \mathcal{N})$-topology.*

Proof. By Lemma 3.11, we have $\varphi((p^{2j}) + \mathcal{N}^{\geq 2j}) \subset (p^{2j}) + \mathcal{N}^{\geq 2j}$. The desired result follows. □

In the rest of this section, we give an explicit description of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$. To this end, we make use of the theory of δ-rings. Recall that a δ-ring structure on a p-torsionfree commutative ring A is equivalent to the datum of a ring map $\varphi : A \to A$ lifting the Frobenius on A/p; the corresponding δ-structure is given by

$$\delta(x) = \frac{\varphi(x) - x^p}{p}.$$

Using Theorem 2.15(3), we deduce that

$$p_0 : \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \to \mathrm{THH}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \cong \mathcal{O}_K$$

sends z_i to ϖ_K. It follows that $z_1 - z_2$ lies in

$$\ker(p_0) = \mathcal{N}^{\geq 2}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]).$$

By Lemma 3.11, there exists $h \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ such that

$$h\varphi(E_K(z_1)) = \varphi(z_1 - z_2) = z_1^p - z_2^p.$$

For $k \geq 0$, we inductively define $f^{(k)} \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])[1/p]$ by setting $f^{(0)} = z_1 - z_2$,

$$f^{(k+1)} = \frac{-(f^{(k)})^p + \delta^k(h)E_K(z_1)^{p^{k+1}}}{p}. \tag{3.17}$$

Proposition 3.18. *For $k \geq 0$, $f^{(k)} \in \mathcal{N}^{\geq 2p^k}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ and*

$$\delta^k(h)\varphi(E_K(z_1))^{p^k} = \varphi(f^{(k)}) \tag{3.19}$$

Proof. We will proceed by induction to show that

$$f^{(k)} \in W(\mathbf{k})[z_1, z_2][h, \dots, \delta^{k-1}(h)] \cap \mathcal{N}^{\geq 2p^k}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$$

and

$$\delta^k(h)\varphi(E_K(z_1))^{p^k} = \varphi(f^{(k)}).$$

The initial case is obvious. Now suppose for some $l \geq 0$, the claim holds for all $0 \leq k \leq l$. By Lemma 3.11, we first deduce that $\delta^k(h) \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ for all $k \leq l$. Using (3.19) for $k = l$, we get

$$\begin{aligned} f^{(l+1)} &= \frac{-(f^{(l)})^p + \delta^l(h)E_K(z_1)^{p^{l+1}}}{p} \\ &= \frac{(\varphi(f^{(l)}) - (f^{(l)})^p) + (\delta^l(h)E_K(z_1)^{p^{l+1}} - \delta^l(h)\varphi(E_K(z_1))^{p^l})}{p} \\ &= \delta(f^{(l)}) - \delta^l(h)\delta(E_K(z_1)^{p^l}). \end{aligned} \tag{3.20}$$

By Theorem 2.15(5), we deduce that $\delta(E_K(z_1)^{p^l}) \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$. By inductive hypothesis, we conclude $f^{(l+1)} \in W(\mathbf{k})[z_1, z_2][h, \dots, \delta^l(h)] \subset \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$. By inductive hypothesis and Remark 3.12, $pf^{(l+1)}$ has Nygaard filtration $\geq p^{l+1}$. Hence $f^{(l+1)}$ has Nygaard filtration $\geq p^{l+1}$ by Corollary 3.8.

It remains to show (3.19) for $k = l+1$. To this end, applying φ on (3.17) for $k = l$ and using inductive hypothesis, we get

$$\begin{aligned} \varphi(f^{(l+1)}) &= \frac{-\varphi(f^{(l)})^p + \varphi(\delta^l(h))\varphi(E_K(z_1))^{p^{l+1}}}{p} \\ &= \frac{-\delta^l(h)^p\varphi(E_K(z_1))^{p^{l+1}} + \varphi(\delta^l(h))\varphi(E_K(z_1))^{p^{l+1}}}{p} \\ &= \delta^{l+1}(h))\varphi(E_K(z_1))^{p^{l+1}}. \end{aligned}$$

□

Lemma 3.21. *The set of elements $\{f^{(k)} | k \geq 0\}$ generates, over $W(\mathbf{k})[z_1, z_2]$, a dense subring R of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ with respect to the $\mathcal{N}$-topology.*

Proof. It reduces to show that for all $j \geq 0$,

$$p_{2j} : R \cap \mathcal{N}^{\geq 2j}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \to \mathrm{THH}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$$

is surjective.

Firstly, by Theorem 2.15, we see that $p_2(E_K(z)) = u$, yielding

$$p_2(E_K(z_1)^j) = u_1^j$$

by functoriality of the Tate spectral sequence. To conclude, by Lemma 3.2, it suffices to show that $p_{2j}(R)$ contains $t_{z_1-z_2}^{[j]}$ for all $j \geq 0$.

By the commutative diagram

$$\begin{array}{ccc} \mathcal{N}^{\geq 2}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) & \longrightarrow & \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \\ \downarrow & & \downarrow \cong \\ \mathcal{N}^{\geq 2}\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2]) & \longrightarrow & \mathrm{HH}_2(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2]) \end{array}$$

one immediately checks that $f^{(0)}$ and $t_{z_1-z_2}$ have the same image in $\mathrm{HH}_2(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$. Hence $p_2(f^{(0)}) = (t_{z_1-z_2})$. For $k \geq 0$, we have

$$-(f^{(k)})^p \equiv pf^{(k+1)} \mod E_K(z_1)^{p^{k+1}}$$

by (3.17). By induction, we deduce that for all $k \geq 0$, $t_{z_1-z_2}^{[p^k]}$ lies in the image of $R \cap \mathcal{N}^{\geq 2p^k}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. Note that in the divided power polynomial algebra $\mathcal{O}_K\langle t\rangle$, for $j = j_0+pj_1+\cdots+j_kp^k$ with $0 \leq j_i \leq p-1$, $t^{[j]}$ is equal to $t^{j_0}(t^{[p]})^{j_1}\cdots(t^{[p^k]})^{j_k}$ up to a unit of $\mathbb{Z}_p$. It follows that the image of $R\cap\mathcal{N}^{\geq 2j}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ contains $t_{z_1-z_2}^{[j]}$ for all $j \geq 0$. □

Remark 3.22. In Corollary 4.17, we will prove that $p_{2p^k}(f^{(k)})$ is equal to $t_{z_1-z_2}^{[p^k]}$ up to a unit of $\mathbb{Z}_p$.

Proposition 3.23. *We have that* $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ *is a* δ*-ring. Moreover,* δ *is continuous with respect to the* $(p,\mathcal{N})$*-topology.*

Proof. We equip $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ with the $(p,\mathcal{N})$-topology. Firstly, by Corollary 3.8, it is straightforward to see that the map

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \xrightarrow{a\mapsto pa} \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$$

is strict. By Lemma 3.15, we get that the ideal (p) is closed in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. Moreover, it follows that to prove the lemma, it reduces to show that for all $a \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$, $\phi(a) = \varphi(a) - a^p$ is divisible by p in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. By Lemma 3.16, ϕ is continuous on $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. Now by the proof of Proposition 3.18, we see that R is a δ-ring. That is, $\phi(R) \subset (p)$. Since R is dense by Lemma 3.21, we conclude that $\phi^{-1}((p))$, which is a closed subset, is forced to be $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$, yielding the desired result. □

To describe the structure of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$, we compare it with the relative periodic cyclic homology $\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$. In the following, for a commutative ring R and an ideal $I \subset R$, denote by $D_R(I)$ the divided power envelop of I in R. We equip it with the *Nygaard filtration* $\mathcal{N}^{\geq\bullet}$ where $\mathcal{N}^{\geq j}D_R(I)$ is the R-submodule generated by $I^{[l]}$ for all $l \geq j$. For an R-module M, denote by $R\langle M\rangle$ the divided power envelop $D_{R[M]}((M))$.

Recall the following derived version of the Hochschild-Kostant-Rosenberg theorem (cf. [1, Theorem 3.27]):

Theorem 3.24. *Let R be a commutative ring, and let I be a complete intersection ideal of R. Let $A = R/I$. Then*

$$\mathrm{HH}_*(A/R) \cong A\langle I/I^2\rangle$$

under the canonical isomorphism $I/I^2 \cong \mathrm{HH}_2(A/R)$. Moreover, $\mathrm{HP}_0(A/R)$ is isomorphic to the completion of the divided power envelope $D_R(I)$ with respect to the Nygaard filtration.

Now let I be the kernel of $W(\mathbf{k}) \xrightarrow{z\mapsto\varpi_K} \mathcal{O}_K$ (resp. $W(\mathbf{k})[z_1,z_2] \xrightarrow{z_1,z_2\mapsto} \mathcal{O}_K$), and let $t_{E_K(z)}$ (resp. $t_{E_K(z_i)}$) denote the image of $E_K(z)$ (resp. $E_K(z_i)$) in $\mathrm{HH}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ under the isomorphism $I/I^2 \cong \mathrm{HH}_2(\mathcal{O}_K/W(\mathbf{k})[z])$ (resp. $I/I^2 \cong \mathrm{HH}_2(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$).

The following result follows from Theorem 3.24 immediately.

Corollary 3.25. *The following are true.*

(1) W have

$$\mathrm{HH}_*(\mathcal{O}_K/W(\mathbf{k})[z]) \cong \mathcal{O}_K\langle t_{E_K(z)}\rangle.$$

(2) We have

$$\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z]) \cong D_{W(\mathbf{k})[z]}((E_K(z)))^{\wedge}_{\mathcal{N}}.$$

(3) We have

$$\mathrm{HH}_*(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2]) \cong \mathcal{O}_K\langle t_{E_K(z_1)}, t_{z_1-z_2}\rangle.$$

(4) We have

$$\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2]) \cong D_{W(\mathbf{k})[z_1,z_2]}((E_K(z_1), z_1-z_2))^{\wedge}_{\mathcal{N}}.$$

By Definition 3.13, we may consider the $\mathcal{N}$ and $(p,\mathcal{N})$-topologies for $HP_0(\mathcal{O}_K/W(\mathbf{k})[z])$ and $HP_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$ as well.

Lemma 3.26. *Both $\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z])$ and $\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$ are complete and separated with respect to the $\mathcal{N}$ and $(p,\mathcal{N})$-topologies.*

Proof. For the $\mathcal{N}$-topology, it follows directly from Corollary 3.25(2), (4) respectively. On the other hand, for the $(p,\mathcal{N})$-topology, as in the proof of Lemma 3.15, it suffices to show that graded pieces of the Nygaard filtrations, which are isomorphic to $\mathrm{HH}_*(\mathcal{O}_K/W(\mathbf{k})[z])$ and $\mathrm{HH}_*(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$ repsectively, are all p-complete and separated. This in turn follows directly from Corollary 3.25(1), (3). □

Lemma 3.27. *Both the natural maps*

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z])$$

and

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \to \mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$$

are injective and strict with respect to the Nygaard filtration. Moreover, both maps are strict with respect to the $(p,\mathcal{N})$-topology.

Proof. Since both maps are compatible with the Nygaard filtration, for the first assertion, it reduces to show that the induced maps on graded pieces

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \to \mathrm{HH}_*(\mathcal{O}_K/W(\mathbf{k})[z]) \tag{3.28}$$

and

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \to \mathrm{HH}_*(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2]) \tag{3.29}$$

are all injective. For the second assertion, it is sufficient to show that both (3.28) and (3.29) are strict under the p-adic topology.

Firstly, note that under the isomorphism

$$\mathrm{HH}_2(\mathcal{O}_K/W(\mathbf{k})[z]) \cong \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),$$

$t_{E_K(z)}$ maps to u up to a unit of $\mathcal{O}_K$. We therefore conclude that (3.28) is injective and strict with respect to the p-adic topology by Theorem 2.15(1) and Corollary 3.25(1).

By Lemma 3.2, we deduce that $\mathrm{THH}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ is a successive extension of $\mathcal{O}_K u_1^{2l} t_{z_1-z_2}^{[2j-2l]}$ for $l = 0,1,\dots,j$. On the other hand, by Corollary 3.25(3), we see that $\mathrm{HH}_{2j}(W(\mathbf{k})/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ is a successive extension of $\mathcal{O}_K t_{E_K(z_1)}^{[2l]} t_{z_1-z_2}^{[2j-2l]}$ for $j = 0,1,\dots,k$. Since for each $0 \le l \le j$,

$$\mathcal{O}_K u_1^{2l} t_{z_1-z_2}^{[2j-2l]} \to \mathcal{O}_K t_{E_K(z_1)}^{[2l]} t_{z_1-z_2}^{[2j-2l]}$$

is injective and strict with respect to the p-adic topology, we conclude that (3.29) is injective and strict with respect to the p-adic topology. □

Lemma 3.30. *The δ-ring structure on $W(\mathbf{k})[z_1,z_2]$ extends to a δ-ring structure, which is continuous with respect to the $(p,\mathcal{N})$-topology, on $\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1,z_2])$.*

Proof. By [3, Corollary 2.38] and [3, Lemma 2.17], the δ-ring structure on $W(\mathbf{k})[z_1,z_2]$ extends to a δ-ring structure on $D_{W(\mathbf{k})[z_1,z_2]}((E_K(z_1), z_1-z_2))$, which is continuous with respect to the $(p,\mathcal{N})$-topology. It follows that the δ-ring structure naturally extends to the completion of $D_{W(\mathbf{k})[z_1,z_2]}((E_K(z_1), z_1-z_2))$ with respect to the $(p,\mathcal{N})$-topology. On the other hand, by the proof of Lemma 3.26, we see that the $(p,\mathcal{N})$-completion of $D_{W(\mathbf{k})[z_1,z_2]}((E_K(z_1), z_1-z_2))$ is naturally isomorphic to $D_{W(\mathbf{k})[z_1,z_2]}((E_K(z_1), z_1-z_2))^{\wedge}_{\mathcal{N}}$. The lemma follows. □

In the following, we equip $\mathrm{HP}_0(\mathcal{O}_K/W(k)[z_1,z_2])$ with the δ-ring structure given by Lemma 3.30.

Lemma 3.31. *The natural map*

$$\iota : \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \to \mathrm{HP}_0(\mathcal{O}_K/W(k)[z_1,z_2])$$

is a map of δ-rings.

Proof. Since both sides are p-torsionfree, it reduces to show that ι commutes with Frobenius. By induction, for every $i \geq 0$, we may find $g_i, g_i' \in W(\mathbf{k})[z_1, z_2]$ such that

$$g_i \delta^i(h) = g_i'.$$

It follows that $\iota(\varphi(g_i))\iota(\varphi(\delta^i(h))) = \iota(\varphi(g_i'))$. Note that

$$\iota(\varphi(g_i)) = \varphi(\iota(g_i)), \qquad \iota(\varphi(g_i')) = \varphi(\iota(g_i')).$$

Since $\mathrm{HP}_0(\mathcal{O}_K/W(\mathbf{k})[z_1, z_2])$ is an integral domain, this implies that $\iota(\varphi(\delta^i(h))) = \varphi(\iota(\delta^i(h)))$. We conclude the lemma by the facts that under the $(p, \mathcal{N})$-topology, $W(\mathbf{k})[z_1, z_2][h, \delta(h), \dots]$ is dense in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ (Lemma 3.21), and that both ι and φ are continuous (Lemma 3.27, Proposition 3.23, Lemma 3.30). $\square$

Corollary 3.32. *We have that* $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ *is isomorphic to the closure of the sub-δ-ring of* $D_{W(\mathbf{k})[z_1,z_2]}((E_K(z_1), z_1 - z_2))^{\wedge}_{\mathcal{N}}$ *generated by* $W(\mathbf{k})[z_1, z_2]$ *and* $\iota(h)$ *under either the $\mathcal{N}$-topology or the $(p, \mathcal{N})$-topology.*

Proof. This follows from the combination of Lemma 3.15, Lemma 3.21, Lemma 3.26, Lemma 3.27 and Lemma 3.31. $\square$

4 Hopf algebroid

In this section, we will show the pairs $(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]))$ and $(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]))$ form Hopf algebroids in appropriate categories. We first recall some basis on complete filtered modules.

Let R be a ring equipped with a complete decreasing filtration. We consider the category of complete filtered R-modules. For two complete filtered R-modules M, N, we define their tensor product in the category of complete filtered R-modules, i.e. the completed tensor product $M \hat{\otimes}_R N$, to be the completion of the filtered R-module $M \otimes_R N$.

Definition 4.1. Let M be a complete filtered R-module equipped with a filtration $\mathcal{N}^{\geq \bullet}$. We say M is free and locally finite over R if there exists $\{m_i\}_{i \in I} \subset M$ such that the following conditions hold.

1. For any j, there are only finitely many $i \in I$ such that $m_i \notin \mathcal{N}^{\geq j} M$.
2. The induced morphism $\oplus_{i \in I} R x_i \xrightarrow{x_i \to m_i} M$ of filtered R-modules is an isomorphism after taking completion.

Definition 4.2. Let S be a graded ring, and let M be a graded S-module with the grading $\mathrm{Gr}^\bullet$. We say M is free and locally finite over S if there exists $\{m_i\}_{i \in I} \subset M$ such that the following conditions hold.

1. For any j, there are only finitely many $i \in I$ such that m_i has non-zero component in $\mathrm{Gr}^k M$ for some $k \leq j$.
2. The induced morphism $\oplus_{i \in I} S x_i \xrightarrow{x_i \to m_i} M$ of graded S-modules is an isomorphism.

For a complete filtered R-module M, one easily checks that M is free and locally finite over R if and only if the associated graded module $\mathrm{Gr}^\bullet(M)$ is free and locally finite over $\mathrm{Gr}^\bullet(R)$. Suppose M and N are free and locally finite over R. Then $M\hat{\otimes}_R N$ is also free and locally finite over R. Moreover, we have

$$\mathrm{Gr}^\bullet(M\hat{\otimes}_R N) \cong \mathrm{Gr}^\bullet M \otimes_{\mathrm{Gr}^\bullet R} \mathrm{Gr}^\bullet N.$$

Proposition 4.3. *(1) Both η_L and η_R exhibit $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ as a free and locally finite filtered $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$-module.*

(2) Both η_L and η_R exhibit $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ as a free and locally finite graded $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$-module.

Proof. Since $(\mathrm{Gr}^\bullet(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])), \mathrm{Gr}^\bullet(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]))$ is isomorphic to $(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]), \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]))$, it reduces to show (2). We only need to treat the case of η_L. By Lemma 3.2, we see that $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])/(u_1)$ is a free $\mathcal{O}_K$-module with a basis of degrees $0, 2, 4, \dots$ respectively. Using Corollary 3.4, we may further deduce that such a basis lifts to a basis of $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ over $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ with the same degrees. Hence $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ is free and locally finite over $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ via η_L. □

Corollary 4.4. *We have that $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ and $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ are flat over $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ and $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ respectively.*

For $1 \le i \le n$, consider the natural maps

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, \dots, z_i]) \otimes_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}[z_i])} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_i, \dots, z_n]) \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, \dots, z_n]) \tag{4.5}$$

and

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}[z_1, \dots, z_i]) \otimes_{\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}[z_i])} \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_i, \dots, z_n]) \to \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_i, \dots, z_n]).$$

By Remark 3.6, the Tate spectral sequence for $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}[z_1, \dots, z_n])$ degenerates at the E_2-term. It follows that the Nygaard filtration on $\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}[z_1, \dots, z_n])$ is complete by the same argument as in the proof of Lemma 3.15. Hence the second map induces

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}[z_1, \dots, z_i]) \hat{\otimes}_{\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}[z_i])} \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_i, \dots, z_n]) \to \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_i, \dots, z_n]). \tag{4.6}$$

Lemma 4.7. *Both (4.5) and (4.6) are isomorphisms.*

Proof. The first assertion follows from the multiplicative property of relative THH. This in turn implies that (4.6) becomes an isomorphism after taking associated graded algebras on both sides. Thus (4.6) itself is an isomorphism. □

In the following, we regard $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ (resp. $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$) as a bimodule over $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ (resp. $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$) via the left and right units. Consider the following commutative diagram of E_∞-spectra over $\mathcal{O}_K$:

$$\begin{array}{ccc} \mathbb{S}_{W(\mathbf{k})}[z] & \xrightarrow{\eta_R} & \mathbb{S}_{W(\mathbf{k})}[z_1, z_2] \\ \downarrow{\scriptstyle \eta_L} & & \downarrow{\scriptstyle z_i \mapsto z_i} \\ \mathbb{S}_{W(\mathbf{k})}[z_1, z_2] & \xrightarrow{z_i \mapsto z_{i+1}} & \mathbb{S}_{W(\mathbf{k})}[z_1, z_2, z_3], \end{array} \tag{4.8}$$

and regard $\mathcal{O}_K$ as an $\mathbb{S}_{W(\mathbf{k})}[z_1, z_2, z_3]$-module by sending z_i to ϖ_K.

Corollary 4.9. *The diagram (4.8) induces*

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \otimes_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \cong \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2, z_3]) \tag{4.10}$$

and

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \hat{\otimes}_{\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])} \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \cong \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2, z_3]). \tag{4.11}$$

We define coproduct Δ on $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ over $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$(resp. $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ over $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$) as the composite of

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \xrightarrow{z_1 \mapsto z_1, z_2 \mapsto z_3} \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2, z_3]) \tag{4.12}$$

and (4.11) (resp. the composite of

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \xrightarrow{z_1 \mapsto z_1, z_2 \mapsto z_3} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2, z_3]))$$

and (4.10)). The counit and conjugation are defined as

$$\varepsilon : \mathrm{TP}_0(\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \xrightarrow{z_i \mapsto z} \mathrm{TP}_0(\mathbb{S}_{W(\mathbf{k})}[z]) \tag{4.13}$$

and

$$c : \mathrm{TP}_0(\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \xrightarrow{z_i \mapsto z_{3-i}} \mathrm{TP}_0(\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \tag{4.14}$$

(resp. $\varepsilon : \mathrm{THH}_*(\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \xrightarrow{z_i \mapsto z} \mathrm{THH}_*(\mathbb{S}_{W(\mathbf{k})}[z])$ and

$$c : \mathrm{THH}_*(\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \xrightarrow{z_i \mapsto z_{3-i}} \mathrm{THH}_*(\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]))$$

respectively.

By standard arguments as in the construction of Adams spectral sequences, we have:

Proposition 4.15. *(1) The pair* $\big(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])\big)$ *forms a Hopf algebroid in the category of complete filtered rings with the coproduct, counit and conjugation given above.*

(2) The pair $\big(THH_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), THH_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])\big)$ *forms a Hopf algebroid in the category of graded rings with the coproduct, counit and conjugation given above.*

In the following, we give an explicit description of the Hopf algebroid

$$(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]), \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])).$$

Lemma 4.16. *For any $i \geq 0$, $\delta^i(h) \in \mathcal{N}^{\geq 2}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$.*

Proof. Firstly, it is clear that $\varepsilon(\varphi(z_1 - z_2)) = 0$ and $\varepsilon(\varphi(E_K(z_1))) = \varphi(E_K(z))$. It follows that $\varepsilon(h) = 0$. Hence for all $i \geq 0$,

$$\varepsilon(\delta^i(h)) = \delta^i(\varepsilon(h)) = 0.$$

On the other hand, note that ε induces an isomorphism

$$\mathrm{Gr}^0(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])) \cong \mathrm{Gr}^0(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])).$$

This implies that

$$\ker(\varepsilon) \subset \mathcal{N}^{\geq 2}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]).$$

The lemma follows. □

Corollary 4.17. *We have*

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \cong \mathcal{O}_K[u_1] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t_{z_1 - z_2}\rangle. \tag{4.18}$$

Proof. By Lemma 4.16 and (3.17), we get that $-(f^{(k)})^p$ and $pf^{(k+1)}$ have the same image in $\mathrm{THH}_{2p^{k+1}}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$. Using the argument of Lemma 3.21, we conclude that the images of $\{f^{(k)}\}_{k \geq 0}$ in $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$ generates $t^{[j]}_{z_1 - z_2}$ for all $j \geq 0$ over $\mathbb{Z}_p$. This allows us to define the $\mathcal{O}_K[u_1]$-linear map

$$\mathcal{O}_K[u_1] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t\rangle \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]), \quad t^{[j]} \mapsto t^{[j]}_{z_1 - z_2}.$$

By Lemma 3.2, this map induces isomorphisms on graded pieces under the u_1-filtrations. Hence it is an isomorphism. □

Proposition 4.19. *Under the isomorphism (4.18), we have*

$$u_2 = u_1 - E'_K(\varpi_K)t_{z_1 - z_2}, \tag{4.20}$$

and

$$\Delta(t^{[i]}_{z_1 - z_2}) = \sum_{0 \leq j \leq i} t^{[j]}_{z_1 - z_2} \otimes t^{[i-j]}_{z_1 - z_2}, \quad \varepsilon(t_{z_1 - z_2}) = 0. \tag{4.21}$$

Proof. Using Theorem 2.15, we have that $u_2 = p_2(E_K(z_2))$. We conclude (4.20) by writing

$$E_K(z_2) = E_K(z_1) - E'_K(z_1)(z_1 - z_2) + (z_1 - z_2)^2 F(z_1)$$

for some F. For (4.21), since $z_1 - z_2$ maps to $z_1 - z_3$ under (4.12), we conclude by the binomial expansion

$$(z_1 - z_3)^i = \sum_{0 \leq j \leq i} \frac{i!}{j!(i-j)!}(z_1 - z_2)^j (z_2 - z_3)^{i-j}.$$

□

5 The decent spectral sequence

Consider the $\mathbb{S}_{W(\mathbf{k})}[z]$-Adams resolution for $\mathbb{S}_{W(\mathbf{k})}$:

$$\mathbb{S}_{W(\mathbf{k})} \to \mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}, \tag{5.1}$$

where $\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}$ denotes the n-fold tensor product of $\mathbb{S}_{W(\mathbf{k})}[z]$ over $\mathbb{S}_{W(\mathbf{k})}$. It induces the augmented cosimplicial cyclotomic E_∞-spectra

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \to \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}), \tag{5.2}$$

$$\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \to \mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}) \tag{5.3}$$

and

$$\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \to \mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}). \tag{5.4}$$

By the multiplicative property of relative THH, $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n})$ is equivalent to the n-fold tensor product of $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ over $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$. Hence (5.2) is an Adams resolution for $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ in the category of $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$-modules.

Proposition 5.5. *The Adams resolution (5.2) induces*

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \cong \mathrm{Tot}(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})). \tag{5.6}$$

Proof. By [10, Proposition 2.14], the fiber of

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \to \mathrm{Tot}_n(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})) \tag{5.7}$$

is homotopy equivalent to the n-fold smash product of the fiber of

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \to \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \tag{5.8}$$

with itself. It follows that the fiber of (5.7) is $n-1$-connected as the fiber of (5.8) is 0-connected. The proposition follows. ◻

Corollary 5.9. *The cosimplicial spectra (5.3), (5.4) induce*

$$\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \cong \mathrm{Tot}(\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})) \tag{5.10}$$

and

$$\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}) \cong \mathrm{Tot}(\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})). \tag{5.11}$$

Proof. The claim for TC^- is clear since

$$\mathrm{Tot}(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))^{h\mathbb{T}} \cong \mathrm{Tot}(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})^{h\mathbb{T}}).$$

For the case of TP, first note that

$$\mathrm{Tot}_n(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))_{h\mathbb{T}} \cong \mathrm{Tot}_n(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})_{h\mathbb{T}}).$$

Since the fiber of (5.7) is $(n-1)$-connected by the proof of Proposition 5.5, the fiber of

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})_{h\mathbb{T}} \to \mathrm{Tot}_n(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet}))_{h\mathbb{T}}$$

is $(n-1)$-connected as well. Hence the fiber of

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})_{h\mathbb{T}} \to \mathrm{Tot}_n(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})_{h\mathbb{T}})$$

is $(n-1)$-connected. We thus conclude

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})_{h\mathbb{T}} \cong \mathrm{Tot}(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})_{h\mathbb{T}}),$$

yielding the claim for TP. □

Using Proposition 5.5 and Corollary 5.9, the coskeleton filtrations of $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})$, $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})$ and $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})$ give rise to spectral sequences computing $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$, $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ and $\mathrm{TC}^-_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ respectively.

- The *descent spectral sequence* for $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$:

$$E_1^{i,j}(\mathrm{THH}(\mathcal{O}_K)) = \mathrm{THH}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i}) \Rightarrow \mathrm{THH}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}).$$

  By Lemma 4.7 and Corollary 4.4, the E_1-term may be identified with the cobar complex for $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ with respect to the Hopf algebroid

$$(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]), \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])).$$

  It follows that

$$E_2^{i,j}(\mathrm{THH}(\mathcal{O}_K)) \cong \mathrm{Ext}^{i,j}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])).$$

- The *descent spectral sequence* for $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$:

$$E_1^{i,j}(\mathrm{TP}(\mathcal{O}_K)) = \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i}) \Rightarrow \mathrm{TP}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}).$$

  By Lemma 4.7 and Corollary 4.4, the j-th row of the E_1-term may be identified with the cobar complex for $\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ with respect to the Hopf algebroid $(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]))$. It follows that

$$E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K)) \cong \mathrm{Ext}^{i}_{\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])).$$

- The *descent spectral sequence* for $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$:

$$E_1^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)) = \mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i}) \Rightarrow \mathrm{TC}^-_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}).$$

Remark 5.12. Indeed, the E_2-term of the descent spectral sequence for $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ may also be identified as an Ext-group in the category of complete filtered comodules over filtered Hopf algebroids. The details will be given in [8].

Using (5.10) and (5.11), we may also construct a spectral sequence computing $\mathrm{TC}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$. Firstly, the maps can, φ from relative TC^- to relative TP induce the maps of cosimplical E_∞-spectra

$$\mathrm{can}, \varphi : \mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}) \to \mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}).$$

Define $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})_{(n)}$ to be the fiber of

$$\mathrm{can} - \varphi : \mathrm{Tot}_n(\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})) \to \mathrm{Tot}_{n-1}(\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet})).$$

By construction, we get

$$\frac{\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})_{(n-1)}}{\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})_{(n)}} \cong \frac{\mathrm{Tot}_{n-1}(\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))}{\mathrm{Tot}_n(\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))} \oplus \Sigma^{-1} \frac{\mathrm{Tot}_{n-2}(\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))}{\mathrm{Tot}_{n-1}(\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))}.$$

The tower $\{\mathrm{TC}(\mathcal{O}_K)_{(n)}\}_{n \geq 0}$ gives rise to the *descent spectral sequence* for $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$:

$$E_1^{i,j}(\mathrm{TC}(\mathcal{O}_K)) \Rightarrow \mathrm{TC}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}),$$

where $E_1(\mathrm{TC}(\mathcal{O}_K))$ may be identified with

$$E_1(\mathrm{TC}^-(\mathcal{O}_K)) \xrightarrow{\mathrm{can}-\varphi} E_1(\mathrm{TP}(\mathcal{O}_K)).$$

Consequently, there is a multiplicative spectral sequence

$$\tilde{E}_2^{i,k,j}(\mathrm{TC}(\mathcal{O}_K)) \Rightarrow E_2^{i+k,j}(\mathrm{TC}(\mathcal{O}_K)),$$

where

$$\tilde{E}_2^{i,0,j}(\mathrm{TC}(\mathcal{O}_K)) \cong \ker(\mathrm{can} - \varphi : E_2^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)) \to E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K))),$$

$$\tilde{E}_2^{i,1,j}(\mathrm{TC}(\mathcal{O}_K)) \cong \mathrm{coker}(\mathrm{can} - \varphi : E_2^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)) \to E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K)))$$

and

$$\tilde{E}_2^{i,k,j}(\mathrm{TC}(\mathcal{O}_K)) = 0$$

for $k \neq 0, 1$.

In the rest of this section, we will compute $E_2^{i,j}(\mathrm{THH}(\mathcal{O}_K))$ explicitly. To this end, first note that it follows from Corollary 4.17 and (4.21) that the left $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$-linear map

$$D : \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]) \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]),$$

which sends $t_{z_1-z_2}^{[i]}$ to $t_{z_1-z_2}^{[i-1]}$, is a map of left $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$-modules. It follows that the complex

$$0 \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \xrightarrow{\eta_L} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2] \xrightarrow{a \mapsto D(a)dz} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])dz \to 0, \quad (5.13)$$

where dz has degree 2, is a relative injective resolution for $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ as left $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$-modules.

Proposition 5.14. *We have that* $\mathrm{Ext}^{i,j}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ *is computed by the complex*

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]) \xrightarrow{(D_0\circ\eta_R)dz} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])dz, \tag{5.15}$$

where the left $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$*-linear map*

$$D_0 : \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])\rangle \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$$

is given by $D_0(t_{z_1-z_2}) = 1$ *and* $D_0(t^{[i]}_{z_1-z_2}) = 0$ *for* $i \neq 1$.

Proof. Using (5.13), we first get that $\mathrm{Ext}^{i,j}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ is computed by the complex

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]))\\ &\xrightarrow{f\mapsto(D\circ f)dz} \mathrm{Hom}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]))dz.\end{aligned} \tag{5.16}$$

Recall that for a (commutative) Hopf algebroid (A,Γ), a left Γ-module M and an A-module N, there is a canonical isomorphism

$$\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_\Gamma(M,\Gamma\otimes_A N), \quad f\mapsto \tilde{f} = (\mathrm{id}\otimes f)\circ\Delta. \tag{5.17}$$

It is straightforward to check that D_0 corresponds to D under this isomorphism. It follows that (5.16) may be identified with the $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$-linear complex

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]))\\ &\xrightarrow{f\mapsto(D\circ\tilde{f})dz} \mathrm{Hom}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]))dz.\end{aligned} \tag{5.18}$$

Note that under the isomorphism (5.17), the identity map on $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$ corresponds to η_R . We thus conclude the proposition by the isomorphism

$$\mathrm{Hom}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])) \cong \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]),$$

which sends f to $f(1)$. □

The following results follow immediately.

Corollary 5.19. *We have*

$$\mathrm{Ext}^{0,0}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])) \cong \mathcal{O}_K$$

and

$$\mathrm{Ext}^{1,2n}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])) \cong \mathcal{O}_K/(nE'_K(\varpi_K)), n\geq 0.$$

The other Ext*-groups vanish. As a consequence, the descent spectral sequence for* $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ *collapses at the* E_2*-term.*

Remark 5.20. Corollary 5.19 recovers the main result of [7].

In the remainder of this section, we introduce the *algebraic Tate spectral sequence* and the *algebraic homotopy fixed points spectral sequence*. Note that the E_1-terms of the descent spectral sequences for TC^- and TP are equipped with the Nygaard filtration. This gives rise to the algebraic homotopy fixed points spectral sequence

$$E_1^{i,j,k}(\mathrm{TC}^-(\mathcal{O}_K)) = H^i(\mathrm{Gr}^{2k}(\mathrm{TC}_j^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))) \Rightarrow E_2^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)), \tag{5.21}$$

and the algebraic Tate spectral sequence

$$E_1^{i,j,k}(\mathrm{TP}(\mathcal{O}_K)) = H^i(\mathrm{Gr}^{2k}(\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}))) \Rightarrow E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K)). \tag{5.22}$$

They are multiplicative spectral sequences. Moreover, by Remark 3.6, we see that the graded pieces of the Nygaard filtrations of $E_1(\mathrm{TP}(\mathcal{O}_K))$ together with the induced d_1-differentials may be identified with part of $E_1(\mathrm{THH}(\mathcal{O}_K))[\sigma^{\pm 1}]$ in the sense that

$$E_1^{i,j,k}(\mathrm{TP}(\mathcal{O}_K)) \cong E_2^{i,2k}(\mathrm{THH}(\mathcal{O}_K))\sigma^j.$$

Since the algebraic homotopy fixed points spectral sequence is a truncation of the algebraic Tate spectral sequence, using Corollary 5.19, the following result follows immediately.

Proposition 5.23. *Both $E_2(\mathrm{TC}^-(\mathcal{O}_K))$ and $E_2(\mathrm{TP}(\mathcal{O}_K))$ are concentrated in $E_2^{0,*}$ and $E_2^{1,*}$. In particular, both the decent spectral sequences for $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ and $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})})$ collapse at the E_2-term.*

6 Refined algebraic Tate differentials

In this section, we consider mod p version of decent spectral sequences. To compute the E_2-terms of mod p descent sequences for $\mathrm{TP}(\mathcal{O}_K)$ and $\mathrm{TC}^-(\mathcal{O}_K)$, we introduce refined version of algebraic Tate and algebraic homotopy fixed point spectral sequences, and completely determine the refined algebraic Tate differentials.

By Lemma 4.7 and induction on n, we get that $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n})$ are p-torsionfree for all $n \geq 1$. Hence $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n})$ and $\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n})$ are all p-torsionfree as well by degeneracy of the Tate and homotopy fixed point spectral sequences respectively. It follows that for $n \geq 1$,

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}; \mathbb{F}_p) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{F}_p,$$
$$\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}; \mathbb{F}_p) = \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

and

$$\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}; \mathbb{F}_p) = \mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

This in turn implies the degeneracy of of the Tate and homotopy fixed point spectral sequences for $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}; \mathbb{F}_p)$ and $\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes n}; \mathbb{F}_p)$ respectively.

Moreover, analogues of Proposition 5.5 and Corollary 5.9 hold as well. Thus the coskeleton filtrations of the cosimplicial spectra

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet};\mathbb{F}_p),\quad \mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet};\mathbb{F}_p),\quad \mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet};\mathbb{F}_p)$$

give rise to *mod p decent spectral sequences* computing $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$, $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$ and $\mathrm{TC}^-_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$ as follows.

- The descent spectral sequence for $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$:

  $$E_1^{i,j}(\mathrm{THH}(\mathcal{O}_K);\mathbb{F}_p) = \mathrm{THH}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i};\mathbb{F}_p) \Rightarrow \mathrm{THH}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p).$$

  The E_1-term may be identified with the cobar complex for $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)$ with respect to the Hopf algebroid

  $$(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p), \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)).$$

  Hence

  $$E_2^{i,j}(\mathrm{THH}(\mathcal{O}_K);\mathbb{F}_p) \cong \mathrm{Ext}^{i,j}_{\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)}(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)).$$

- The descent spectral sequence for $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$:

  $$E_1^{i,j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p) = \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i};\mathbb{F}_p) \Rightarrow \mathrm{TP}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p).$$

  The j-th row of the E_1-term may be identified with the cobar complex for $\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)$ with respect to the Hopf algebroid

  $$(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p), \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)).$$

  It follows that

  $$E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p) \cong \mathrm{Ext}^{i}_{\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)}(\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)).$$

- The descent spectral sequence for $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$:

  $$E_1^{i,j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p) = \mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes i};\mathbb{F}_p) \Rightarrow \mathrm{TC}^-_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p).$$

- The descent spectral sequence for $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$:

  $$E_1^{i,j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p) \Rightarrow \mathrm{TC}_{j-i}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p).$$

  Similarly, there is a spectral sequence

  $$\tilde{E}_2^{i,k,j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p) \Rightarrow E_2^{i+k,j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p),$$

  where

  $$\tilde{E}_2^{i,0,j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p) \cong \ker(\mathrm{can}-\varphi : E_2^{i,j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p) \to E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p),$$
  $$\tilde{E}_2^{i,1,j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p) \cong \mathrm{coker}(\mathrm{can}-\varphi : E_2^{i,j}(\mathrm{TC}^-(\mathcal{O}_K)) \to E_2^{i,j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$$

  and

  $$\tilde{E}_2^{i,k,j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p) = 0$$

  for $k \neq 0, 1$.

In the following, we will first compute the E_2-term of the descent spectral sequence for $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$. To simplify the notations, from now on for

$$? \in \{z, z_i, \sigma, \sigma_i, u, u_i, v, v_i, t_{z_1-z_2}\},$$

we denote its image in the mod p reduction by the same symbol. Moreover, we abusively use z, z_i to denote their images in $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)$ and $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)$ respectively under p_0 . Under these notations, we have

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p) \cong W(\mathbf{k})[[z]] \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbf{k}[[z]]$$

and

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p) \cong \mathcal{O}_K[u] \otimes_{\mathbb{Z}} \mathbb{F}_p \cong (\mathcal{O}_K/(p))[u] = \mathbf{k}[z]/(z^{e_K})[u],$$

where z corresponds to $\overline{\varpi_K}$ under the last identification. Moreover, we have

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p) \cong (\mathcal{O}_K\langle t_{z_1-z_2}\rangle \otimes_{\mathcal{O}_K} \mathcal{O}_K[u_1]) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong (\mathbf{k}[z]/(z_1^{e_K})[u_1]\langle t_{z_1-z_2}\rangle.$$

Recall that the leading coefficient of $E_K(z)$ is denoted by μ.

Proposition 6.1. *The following are true.*

(1) $E_2^{0,*}(\mathrm{THH}(\mathcal{O}_K);\mathbb{F}_p)$ *is the* $\mathbf{k}$*-vector space freely generated by*

$$\begin{cases} z^l u^n, \quad 1 \le l \le e_K - 1 \text{ or } p \mid e_K n, & \text{if } e_K > 1 \\ u^n, \quad p \mid n, & \text{if } e_K = 1. \end{cases}$$

(2) $E_2^{1,*}(\mathrm{THH}(\mathcal{O}_K);\mathbb{F}_p)$ *is the* $\mathbf{k}$*-vector space freely generated by the set of cocycles*

$$\begin{cases} z_1^l(u_1^{n-1}t_{z_1-z_2} - (n-1)E_K'(z_1)u_1^{n-2}t_{z_1-z_2}^{[2]}), \quad 0 \le l \le e_K - 2 \text{ or } p \mid e_K n, & \text{if } e_K > 1 \\ \sum_{j=1}^{l} \frac{(n-1)!}{(n-j)!}(-\bar{\mu})^j u_1^{n-j} t_{z_1-z_2}^{[j]}, \quad p \mid n, & \text{if } e_K = 1. \end{cases}$$

(3) For $i \neq 0, 1$, $E_2^{i,*}(\mathrm{THH}(\mathcal{O}_K);\mathbb{F}_p) = 0$.

Proof. By similar argument as in the proof of Proposition 5.14, we get that $E_2(\mathrm{THH}(\mathcal{O}_K);\mathbb{F}_p)$ is computed by the complex

$$0 \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p) \xrightarrow{(D_0 \circ \eta_R)dz} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)dz \to 0. \tag{6.2}$$

This implies (3) immediately. Using (4.20) and $E_K'(z) \equiv e_K \mu z^{e_K-1} \mod p$, (6.2) may be identified with

$$0 \to (\mathbf{k}[z]/(z^{e_K}))[u] \xrightarrow{f(u) \mapsto -e_K\bar{\mu}z^{e_K-1}f'(u)dz} (\mathbf{k}[z]/(z^{e_K}))[u]dz \to 0. \tag{6.3}$$

Then a short computation shows that H^0 is the $\mathbf{k}$-vector space freely generated by

$$\begin{cases} z^l u^n, \quad 1 \le l \le e_K - 1 \text{ or } p \mid e_K n, & \text{if } e_K > 1 \\ u^n, \quad p \mid n, & \text{if } e_K = 1. \end{cases}$$

and H^1 is the $\mathbf{k}$-vector space freely generated by the set of cocycles

$$\begin{cases} z^l u^{n-1} dz, \quad 0 \le l \le e_K - 2 \text{ or } p \mid e_K n, & \text{if } e_K > 1 \\ u^{n-1} dz, \quad p \mid n, & \text{if } e_K = 1. \end{cases}$$

To compare (6.2) with the cobar complex, for $n \ge 1$, set

$$u^{(n)} = \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!} (-E_K'(z))^{j-1} u_1^{n-j} t_{z_1 - z_2}^{[j]} = \frac{u_1^n - u_2^n}{n E_K'(z)} \in \mathrm{THH}_*(\mathcal{O}_K / \mathbb{S}_{W(\mathbf{k})}[z_1, z_2]). \tag{6.4}$$

It is straightforward to see that $u^{(n)}$ is a cocycle in the cobar complex for $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$. Now consider the diagram

$$\begin{array}{ccc} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p) & \xrightarrow{(D_0 \circ \eta_R) dz} & \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p) dz \\ \downarrow \mathrm{id} & & \downarrow \beta \\ \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p) & \xrightarrow{\eta_L - \eta_R} & \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p), \end{array} \tag{6.5}$$

where β is the $\mathbf{k}[z]$-linear map sending $u^n dz$ to $\overline{u^{(n+1)}}$. By (6.4), it is straightforward to check that (6.12) is commutative. Thus it gives rise to a morphism from (6.2) to the cobar complex of $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p)$. Note that the right vertical map of (6.12) is injective. Since both (6.2) and the cobar complex compute the E_2-term of the descent spectral sequence, we deduce that (6.12) induces a quasi-isomorphism. Finally, note that if $e_K > 1$, then $E_K'(z)^2 = 0$ in $\mathbf{k}[z]/(z^{e_K})$. Now the proposition follows. □

Remark 6.6. The extra complication of $E_2(\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}; \mathbb{F}_p))$ originates from the "accidental" filtration clash of the differentials

$$z^{m e_K} \mapsto m e_K \bar{\mu} z_1^{m e_K - 1} dz$$

in degree $2m$. To remedy, we introduce the refined Nygaard filtration as follows.

Convention 6.7. *From now on, we rescale the index of Nygaard filtrations by* 2*. That is,* $\mathcal{N}^{\ge j}$ *takes place of* $\mathcal{N}^{\ge 2j}$*.*

Definition 6.8. Let M be a filtered $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p)$-module with the filtration $\mathcal{N}^{\ge \bullet}$. Define a refinement of $\mathcal{N}^{\ge \bullet}$ on M by setting

$$\mathcal{N}^{\ge j + \frac{m}{e_K}} M = z^m \mathcal{N}^{\ge j} M + \mathcal{N}^{\ge j+1} M$$

for $j \in \mathbb{Z}, 0 \le m < e_K$, and call it the refined filtration of $\mathcal{N}^{\ge \bullet}$. Note that under the refined filtrations, M is still a filtered $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p)$-module.

In the following, regard both $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p)$ and $\mathrm{TC}_*^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p)$ as $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{F}_p)$-modules via $z \mapsto z_1$. We call the refined filtration of Nygaard filtration the *refined Nygaard filtration*. Note that we may refine the Nygaard filtration of $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$ via both η_L and η_R. However, since

$$z_1^m - z_2^m \in \mathcal{N}^{\ge 1} \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$$

for $m \ge 1$, we get that both ways end up with the same filtration. Combining Corollary 4.17 and Proposition 4.19, we reach the following result.

Lemma 6.9. *Under the refined Nygaard filtration, the associated graded Hopf algebroid of*

$$\big(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p),\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)\big)$$

is

$$(\mathbf{k}[z],\mathbf{k}[z_1]\otimes_{\mathbf{k}}\mathbf{k}\langle t_{z_1-z_2}\rangle),$$

in which the following holds.

(1) If $e_K=1$, then $z_2=z_1+t_{z_1-z_2}$. If $e_K>1$, then $z_2=z_1$; in this case the Hopf algebroid becomes the Hopf algebra

$$(\mathbf{k}[z],\mathbf{k}[z]\otimes_{\mathbf{k}}\mathbf{k}\langle t_{z_1-z_2}\rangle).$$

(2) The coproduct Δ and counit ε satisfy

$$\Delta(t^{[i]}_{z_1-z_2})=\sum_{0\le j\le i}t^{[j]}_{z_1-z_2}\otimes t^{[i-j]}_{z_1-z_2},\quad \varepsilon(t^{[i]}_{z_1-z_2})=0$$

for all $i\geq 0$.

The refined Nygaard filtration on E_1-terms of the descent spectral sequences for $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$ and $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$ give rise to the *refined algebraic Tate spectral sequence*

$$\tilde{E}^{i,j,k}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)=H^i(\mathrm{Gr}^k(\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})))\Rightarrow E^{i,j}_2(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$$

and the *refined algebraic homotopy fixed points spectral sequence*

$$\tilde{E}^{i,j,k}_{\frac{1}{e_K}}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)=H^i(\mathrm{Gr}^k(\mathrm{TC}^-_j(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes\bullet})))\Rightarrow E^{i,j}_2(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p).$$

They are multiplicative spectral sequences with $\tilde{E}_r$-terms for all $r\in\frac{1}{e_K}\mathbb{Z}_{\geq 0}$. Moreover, by Remark 3.6, Lemma 6.9 and the functoriality of Tate spectral sequence, we see that $\tilde{E}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ may be identified with the cobar complex for $\mathbf{k}[z][\sigma^{\pm1}]$ with respect to the Hopf algebroid $(\mathbf{k}[z],\mathbf{k}[z_1]\otimes_{\mathbf{k}}\mathbf{k}\langle t\rangle)$, and $\tilde{E}_{\frac{1}{e_K}}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ is a truncation of $\tilde{E}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$.

Lemma 6.10. *The following are true.*

(1) If $e_K>1$, then

$$\tilde{E}^{0,j,*}_{1-\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)\cong\mathbf{k}[z]\sigma^j,\quad \tilde{E}^{1,0,*}_{1-\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)\cong\mathbf{k}[z_1]t_{z_1-z_2}\sigma^j.$$

Moreover, $d_{1-\frac{1}{e_K}}(z\sigma^j)=t_{z_1-z_2}\sigma^j$.

(2) If $e_K=1$, then

$$\tilde{E}^{0,j,*}_1(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)\cong\mathbf{k}[z^p]\sigma^j,\quad \tilde{E}^{1,j,*}_1(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)\cong\oplus_{p|n}\mathbf{k}\sum_{j=1}^{n}\frac{(n-1)!}{(n-j)!}(-1)^j z_1^{n-j}t^{[j]}_{z_1-z_2}\sigma^j.$$

(3) For $i \neq 0, 1$, $\tilde{E}^{i,j,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p) = 0$.

Proof. By functoriality of the Tate spectral sequence, we have

$$\sigma_1\sigma_2^{-1} - 1 \in \mathcal{N}^{\geq 1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]).$$

It follows that $\sigma_1 = \sigma_2$ in graded pieces of the cobar complex for $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z])$. Therefore we reduce to the case $j = 0$.

By a similar argument as for Proposition 5.14, we first see that $\mathrm{Ext}_{\mathbf{k}[z_1]\otimes_{\mathbf{k}}\mathbf{k}\langle t\rangle}(\mathbf{k}[z], \mathbf{k}[z])$ is computed by the complex

$$0 \to \mathbf{k}[z] \xrightarrow{f(z)\mapsto -f'(z)dz} \mathbf{k}[z]dz \to 0. \tag{6.11}$$

Then we proceed as in the poof of Proposition 6.1. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{k}[z] & \xrightarrow{f(z)\mapsto -f'(z)dz} & \mathbf{k}[z]dz \\ \downarrow \mathrm{id} & & \downarrow \beta \\ \mathbf{k}[z] & \xrightarrow{\eta_L - \eta_R} & \mathbf{k}[z_1] \otimes_{\mathbf{k}} \mathbf{k}\langle t\rangle, \end{array} \tag{6.12}$$

where β is the $\mathbf{k}[z]$-linear (under η_L) map sending $z^n dz$ to $\sum_{j=0}^{n} \frac{n!}{(n-j)!}(-1)^j z_1^{n-j} t^{[j+1]}_{z_1 - z_2}$. By a similar argument as in the proof of Proposition 6.1, we deduce that it gives rise to an quasi-isomorphism between (6.11) and the cobar complex. This yields the desired result on cohomology of the cobar complex. Finally, when $e_K > 1$, the differential of the cobar complex sends

$$z^n \in \mathcal{N}^{\geq \frac{n}{e_K}} \setminus \mathcal{N}^{\geq \frac{n+1}{e_K}}$$

to

$$z_2^n - z_1^n = \sum_{1\leq j\leq n} \binom{n}{j} (z_2 - z_1)^j z_1^{n-j},$$

which belongs to $\mathcal{N}^{\geq \frac{n}{e_K}+1-\frac{1}{e_K}}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$. It follows that

$$\tilde{E}^{*,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p) = \tilde{E}^{*,0,*}_{\frac{2}{e_K}}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p) = \cdots = \tilde{E}^{*,0,*}_{1-\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p)$$

and $d_{1-\frac{1}{e_K}}(z) = t_{z_1 - z_2}$. □

Corollary 6.13. *Both $E_2(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p)$ and $E_2(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p)$ are concentrated in $E_2^{0,*}$ and $E_2^{1,*}$. In particular, both the decent spectral sequences for $\mathrm{TC}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}; \mathbb{F}_p)$ and $\mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}; \mathbb{F}_p)$ collapse at the E_2-term.*

Convention 6.14. *Motivated by the results of Lemma 6.10, in what follows, denote $t_{z_1-z_2}$ by dz. When $e_K = 1$, denote*

$$\sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!}(-1)^j z_1^{n-j} t^{[j]}_{z_1-z_2},$$

which is formally equal to $\frac{z_1^n - z_2^n}{n}$, by $z_1^{n-1}dz$.

Under Convention 6.14, we may reformulate Lemma 6.10(1), (2) as follows.

Corollary 6.15. *For $e_K > 1$, we have*

$$\tilde{E}^{*,j,*}_{1-\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p) \cong \mathbf{k}[z]\sigma^j \oplus \mathbf{k}[z_1]dz\sigma^j,$$

and $d_{1-\frac{1}{e_K}}(z\sigma^j) = dz\sigma^j$. For $e_K = 1$, we have

$$\tilde{E}^{*,j,*}_{1}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p) \cong \mathbf{k}[z^p]\sigma^j \oplus z_1^{p-1}\mathbf{k}[z_1^p]dz\sigma^j.$$

In the rest of this section, we will compute the higher refined algebraic Tate differentials. We first treat the case of 0-stems. In the following, when the context is clear, for $j \in \mathbb{Z}_{\geq 0}$, we will simply denote $\mathcal{N}^{\geq j}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ by $\mathcal{N}^{\geq j}$. For $r \in \frac{1}{e_K}\mathbb{Z}_{\geq 0}$, we denote $\mathcal{N}^{\geq r}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)$ by $\mathcal{N}^{\geq r}$, and denote by $(p,\mathcal{N}^{\geq r})$ the preimage of $\mathcal{N}^{\geq r}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)$ under the natural projection

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2]) \to \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p).$$

In the following, for $a \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$ (resp. $a \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)$), we denote by $\nu(a)$ the smallest $j \in \mathbb{Z}_{\geq 0}$ (resp. $r \in \frac{1}{e_K}\mathbb{Z}_{\geq 0}$) such that $a \in \mathcal{N}^{\geq j}$ (resp. $a \in \mathcal{N}^{\geq r}$). Since the associated graded algebra are integral in both cases, we have $\nu(ab) = \nu(a) + \nu(b)$.

Lemma 6.16. *We have $\xi_0 = -\delta(f^{(0)})/f^{(0)} \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. Moreover,*

$$\xi_0 \equiv z_1^{p-1} \mod (p,\mathcal{N}^{\geq \frac{p-2}{e_K}+1}).$$

In particular, $\xi_0 \in (p,\mathcal{N}^{\geq \frac{p-1}{e_K}})$.

Proof. For the first claim, we have

$$\begin{aligned}
\delta(f^{(0)}) &= \frac{\varphi(f^{(0)}) - (f^{(0)})^p}{p} \\
&= \frac{z_1^p - (z_1 - f^{(0)})^p - (f^{(0)})^p}{p} \\
&= -f^{(0)}(z_1^{p-1} - \frac{p-1}{2}z_1^{p-2}f^{(0)} + \cdots + ((-1)^p+1)\frac{(f^{(0)})^{p-1}}{p}).
\end{aligned}$$

Note that $\frac{(-1)^p+1}{p} \in \mathbb{Z}$. Hence

$$\xi_0 = z_1^{p-1} - \frac{p-1}{2}z_1^{p-2}f^{(0)} + \cdots + ((-1)^p+1)\frac{(f^{(0)})^{p-1}}{p}$$

belongs to $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$. For $1 \leq i \leq p-1$, $\frac{p-1-i}{e_K} + i \geq \frac{p-2}{e_K} + 1$. Thus for such i, $z_1^{p-1-i}(f^{(0)})^i \in (p,\mathcal{N}^{\geq \frac{p-2}{e_K}+1})$. This implies that $\xi_0 - z_1^p \in (p,\mathcal{N}^{\geq \frac{p-2}{e_K}+1})$, yielding the second claim. ∎

Put $\tilde{\mu} = -\frac{\mu^p}{\delta(E_K(z_1))}$.

Lemma 6.17. *We have*

$$\varphi(f^{(0)}) \equiv \tilde{\mu} z_1^{pe_K+p-1} f^{(0)} \mod (p, \mathcal{N}^{\geq 2p}).$$

Proof. Recall that $h\varphi(E_K(z_1)) = \varphi(f^{(0)})$. Note that

$$\varphi(E_K(z_1)) \equiv \mu^p z_1^{pe_K} \mod p.$$

Thus

$$\varphi(f^{(0)}) \equiv \mu^p z_1^{pe_K} h \mod p. \tag{6.18}$$

On the other hand, using (3.20) for $l = 0$, we have

$$f^{(1)} = \delta(f^{(0)}) - h\delta(E_K(z_1)). \tag{6.19}$$

Since $f^{(1)} \in \mathcal{N}^{\geq p}$, we get

$$h \equiv \delta(f^{(0)})/\delta(E_K(z_1)) \mod \mathcal{N}^{\geq p}.$$

Combining this with Lemma 6.16 and the fact that $pe_K + \frac{p-2}{e_K} + 2 \geq 2p$, we deduce that

$$\tilde{\mu} z_1^{pe_K+p-1} f^{(0)} \equiv \tilde{\mu}\xi_0 z_1^{pe_K} f^{(0)} = \mu^p z_1^{pe_K} \delta(f^{(0)})/\delta(E_K(z_1)) \equiv \mu^p z_1^{pe_K} h \equiv \varphi(f^{(0)}) \mod (p, \mathcal{N}^{\geq 2p}),$$

concluding the lemma. ∎

Lemma 6.20. *Suppose $p > 2$ and $e_K > 1$. Then for $l \geq 1$,*

$$\varphi^l(f^{(0)}) \equiv \tilde{\mu}^{\frac{p^l-1}{p-1}} z_1^{(pe_K+p-1)\frac{p^l-1}{p-1}} f^{(0)} \mod (p, \mathcal{N}^{\geq p^l(1+\frac{1}{p-1}+\frac{1}{e_K})}) \tag{6.21}$$

Proof. We will establish the lemma by induction on l. The case $l = 1$ follows from Lemma 6.17 and the inequality $\frac{1}{e_K} + \frac{p}{p-1} \leq \frac{1}{2} + \frac{3}{2} = 2$.

Now suppose the claim holds for some $l \geq 1$. Raising both sides of (6.21) to the p-th power, we get

$$\varphi^{l+1}(f^{(0)}) \equiv \tilde{\mu}^{\frac{p^{l+1}-p}{p-1}} z_1^{(pe_K+p-1)\frac{p^{l+1}-p}{p-1}} \varphi(f^{(0)}) \mod (p, \mathcal{N}^{\geq p^{l+1}(1+\frac{1}{p-1}+\frac{1}{e_K})}). \tag{6.22}$$

Using Lemma 6.17 again, we have

$$\tilde{\mu}^{\frac{p^{l+1}-p}{p-1}} z_1^{(pe_K+p-1)\frac{p^{l+1}-p}{p-1}} \varphi(f^{(0)}) \equiv \tilde{\mu}^{\frac{p^{l+1}-1}{p-1}} z_1^{(pe_K+p-1)\frac{p^{l+1}-1}{p-1}} f^{(0)} \mod (p, \mathcal{N}^{\geq \frac{p^{l+2}-p^2}{p-1}+\frac{p^{l+1}-p}{e_K}+2p}). \tag{6.23}$$

On the other hand, it is straightforward to see that

$$\frac{p^{l+2}-p^2}{p-1} + \frac{p^{l+1}-p}{e_K} + 2p \geq p^{l+1}(1 + \frac{1}{p-1} + \frac{1}{e_K}). \tag{6.24}$$

Putting (6.22), (6.23) and (6.24) together, we prove the induction step. ∎

Lemma 6.25. *For $p = 2$ and $l \geq 1$, we have*

$$(f^{(1)})^{2^l} \in (2, \mathcal{N}^{\geq 2^{l+1}(1+\frac{1}{4})}).$$

Proof. Recall that by construction, we have

$$2f^{(2)} = -(f^{(1)})^2 + \delta^2(h)E_K(z_1)^4.$$

By Lemma 4.16, $\delta^2(h) \in \mathcal{N}^{\geq 1}$. It follows that

$$(f^{(1)})^2 \in (2, \mathcal{N}^{\geq 5}).$$

We thus conclude by raising to the 2^{l-1}-th power. □

Lemma 6.26. *Suppose $p = 2$ and $e_K > 3$. Then for $l \geq 1$,*

$$\varphi^l(f^{(0)}) \equiv \tilde{\mu}^{2^l-1} z_1^{(2^l-1)(2e_K+1)} f^{(0)} \mod (2, \mathcal{N}^{\geq 2^l(2+\frac{1}{e_K})-\frac{2}{e_K}}).$$

Proof. We proceed by induction on l. The case $l = 1$ follows from Lemma 6.17. Now suppose the claim holds for some $l \geq 1$. Using (6.18), (6.19), we first have

$$\varphi(f^{(0)}) \equiv \mu^2 z_1^{2e_K} h \equiv \tilde{\mu} z_1^{2e_K}(\xi_0 f^{(0)} + f^{(1)}) \mod 2$$

Raising to the power of 2^l, we get

$$\varphi^{l+1}(f^{(0)}) \equiv \tilde{\mu}^{2^l} z_1^{2^{l+1}e_K}(\xi_0^{2^l}\varphi^l(f^{(0)}) + (f^{(1)})^{2^l}) \mod 2.$$

By induction hypothesis, we have

$$\varphi^l(f^{(0)}) \equiv \tilde{\mu}^{2^l-1} z_1^{(2^l-1)(2e_K+1)} f^{(0)} \mod (2, \mathcal{N}^{\geq 2^l(2+\frac{1}{e_K})-\frac{2}{e_K}}).$$

It follows that

$$\varphi^l(f^{(0)}) \in (2, \mathcal{N}^{\geq (2^l-1)(2+\frac{1}{e_K})+1}).$$

On the other hand, using Lemma 6.16, we get

$$\xi_0^{2^l} \equiv z_1^{2^l} \mod (2, \mathcal{N}^{\geq 2^l}).$$

Putting these together, we deduce that

$$\tilde{\mu}^{2^l} z_1^{2^{l+1}e_K} \xi_0^{2^l} \varphi^l(f^{(0)}) \equiv \tilde{\mu}^{2^{l+1}-1} z_1^{2^{l+1}e_K+2^l} \varphi^l(f^{(0)}) \mod (2, \mathcal{N}^{\geq (2^l-1)(2+\frac{1}{e_K})+2^{l+1}+2^l+1}).$$

and

$$\tilde{\mu}^{2^{l+1}-1} z_1^{2^{l+1}e_K+2^l} \varphi^l(f^{(0)}) \equiv \tilde{\mu}^{2^{l+1}-1} z_1^{(2^{l+1}-1)(2e_K+1)} f^{(0)} \mod (2, \mathcal{N}^{\geq 2^{l+1}(2+\frac{1}{e_K})-\frac{2}{e_K}}).$$

Clearly $(2^l - 1)(2 + \frac{1}{e_K}) + 2^{l+1} + 2^l + 1 > 2^{l+1}(2 + \frac{1}{e_K}) - \frac{2}{e_K}$. Hence we get

$$\tilde{\mu}^{2^l} z_1^{2^{l+1}e_K} \xi_0^{2^l} \varphi^l(f^{(0)}) \equiv \tilde{\mu}^{2^{l+1}-1} z_1^{(2^{l+1}-1)(2e_K+1)} f^{(0)} \mod (2, \mathcal{N}^{\geq 2^{l+1}(2+\frac{1}{e_K})-\frac{2}{e_K}}). \tag{6.27}$$

Finally, by previous lemma, we have

$$\tilde{\mu}^{2^l} z_1^{2^{l+1}e_K}(f^{(1)})^{2^l} \in (2, \mathcal{N}^{\geq 2^{l+1}+2^{l+1}(1+\frac{1}{4})}) \subset (2, \mathcal{N}^{\geq 2^{l+1}(2+\frac{1}{e_K})-\frac{2}{e_K}}). \tag{6.28}$$

Combining (6.27) and (6.28), we conclude the induction step. □

Proposition 6.29. *Suppose $p>2, e_K>1$ or $p=2, e_K>3$. Then for $n\geq 0, l=v_p(n), n'=\frac{n}{p^l}$, the refined algebraic Tate differential satisfies*

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n)=n'\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+n-1}dz, \tag{6.30}$$

which is non-zero in $\tilde{E}^{1,0,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}_{\frac{p^{l+1}-1}{p-1}}$. Moreover, the targets of (6.30) are all different.

Proof. First note that

$$\nu(z_1^{pe_K\frac{p^l-1}{p-1}+n-1}(z_1-z_2))=\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}.$$

On the other hand, since $z_1^{p^l}-z_2^{p^l}\equiv\varphi^l(f^{(0)}) \mod p$ in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2])$, by Lemma 6.20 and Lemma 6.26, we get

$$\nu(z_1^{p^l}-z_2^{p^l}-\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+p^l-1}(z_1-z_2))>\frac{p^{l+1}-1}{p-1}+\frac{p^l-1}{e_K}.$$

Hence

$$\nu(z_1^{p^l}-z_2^{p^l})=\nu(\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+p^l-1}(z_1-z_2))=\frac{p^{l+1}-1}{p-1}+\frac{p^l-1}{e_K}.$$

Write

$$z_1^n-z_2^n=z_1^{n'p^l}-z_2^{n'p^l}=-\sum_{0\leq i\leq n'-1}(-1)^{n'-i}\binom{n'}{i}z_1^{ip^l}(z_1^{p^l}-z_2^{p^l})^{n-i}.$$

It is straightforward to see

$$\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}=\nu(z_1^{(n'-1)p^l}(z_1^{p^l}-z_2^{p^l}))<\nu(z_1^{ip^l}(z_1^{p^l}-z_2^{p^l})^{n-i})$$

for $i\leq n-2$. Note that $\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}=(\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K})+\frac{n}{e_K}$. We thus deduce that

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n)=n'z_1^{(n'-1)p^l}(z_1^{p^l}-z_2^{p^l})=n'\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+n-1}dz.$$

It remains to show that the targets of (6.30) are all different; note that this will automatically imply that the right hand side of (6.30) is non-zero. Put $\tilde{n}=pe_K\frac{p^l-1}{p-1}+n$. Since $v_p(n)=l$, we get $l=v_p(\tilde{n}+\frac{pe_K}{p-1})$. Consequently, n is uniquely determined by $\tilde{n}$. This yields the desired result. □

Now we treat the remaining cases. The strategy is to compare them with the known cases.

Proposition 6.31. *The result of Proposition 6.29 holds for all p and e_K.*

Proof. Choose an integer $m > 3$ coprime to p, and let $K' = K(\varpi_K^{\frac{1}{m}})$; the ramification index of K' is $e_{K'} = me_K$, and the corresponding Eisenstein polynomial for $\varpi_K^{\frac{1}{m}}$ is $E_{K'}(z) = E_K(z^m)$. Now the commutative diagram

$$\begin{array}{ccc} \mathbb{S}_{W(\mathbf{k})}[z] & \xrightarrow{z \mapsto z^m} & \mathbb{S}_{W(\mathbf{k})}[z] \\ \big\downarrow{\scriptstyle z \mapsto \varpi_K} & & \big\downarrow{\scriptstyle z \mapsto \varpi_K^{\frac{1}{m}}} \\ \mathcal{O}_K & \longrightarrow & \mathcal{O}_{K'} \end{array}$$

induces a map of cosimplicial cyclotomic spectra

$$T_m : \mathrm{TP}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p) \to \mathrm{TP}(\mathcal{O}_{K'}/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p).$$

Define the "less refined" Nygaard filtration on $\mathrm{TP}_*(\mathcal{O}_{K'}/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p)$ to be the filtration $\mathcal{N}^{\geq r}\mathrm{TP}_*(\mathcal{O}_{K'}/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}; \mathbb{F}_p)$ for $r \in \frac{1}{e_K}\mathbb{Z}_{\geq 0}$, which in turn induces the "less refined" algebraic Tate spectral sequence $\tilde{E}'(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$. Clearly T_m is compatible with filtrations. Thus it induces a morphism of spectral sequences

$$T_m : \tilde{E}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p) \to \tilde{E}'(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p).$$

By similar argument as for Proposition 6.1 and Lemma 6.10, we first obtain that if $e_K > 1$, then $\tilde{E}'^{*,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$ is isomorphic to $\mathbf{k}[z] \oplus \mathbf{k}[z_1]dz$, where dz denotes $t_{z_1 - z_2}$. If $e_K = 1$, then $\tilde{E}'^{0,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$ is the $\mathbf{k}$-vector space freely generated by $\{z^n | m \nmid n \text{ or } p \mid n\}$, and $\tilde{E}'^{1,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$ is the $\mathbf{k}$-vector space freely generated by the set of cocycles $\{z_1^n dz | m \nmid n+1 \text{ or } p \mid n+1\}$, where $z_1^n dz$ denotes

$$z_1^s((z_1^m)^{k-1} t_{z_1 - z_2} - (k-1) m z_1^{m-1} (z_1^m)^{k-2} t^{[2]}_{z_1 - z_2}), \quad 0 \leq s \leq m-1 \text{ and } s + (k-1)m = n,$$

which is formally equal to $\frac{z_1^{n+m} - z_2^{n+m}}{kmz_1^{m-1}}$; for $j \neq 0, 1$, $\tilde{E}'^{j,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p) = 0$. Under our convention of notations, it is straightforward to verify

$$T_m(z^n) = z^{mn}, \qquad T_m(z_1^n dz) = m z_1^{mn+m-1} dz; \tag{6.32}$$

note that right hand side of the second equality is just formally equal to $z_1^{mn} dz^m$. Combining with Lemma 6.10 and Corollary 6.15, we see that

$$T_m : \tilde{E}^{*,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p) \to \tilde{E}'^{*,0,*}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$$

is injective. To proceed, we need the following result.

Lemma 6.33. *For $n \geq 0, l = v_p(n)$, where $l \geq 1$ if $e_K = 1$, $n' = \frac{n}{p^l}$, the natural projection*

$$\phi : \tilde{E}'^{1,0,\frac{p^{l+1}-1}{p-1} + \frac{n-1}{e_K}}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p) \to \frac{\tilde{E}'^{1,0,\frac{p^{l+1}-1}{p-1} + \frac{n-1}{e_K}}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)}{\bigoplus_{i \neq pme_K \frac{p^l-1}{p-1} + mn-1} \mathbf{k} z_1^i dz} \cong \mathbf{k} z_1^{pme_K \frac{p^l-1}{p-1} + mn-1} dz$$

factors through $\tilde{E}'^{1,0,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}_{\frac{p^{l+1}-1}{p-1}}$. *Moreover,*

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^{mn}) \in \tilde{E}'^{1,0,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}_{\frac{p^{l+1}-1}{p-1}}$$

maps to $n'\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}} z_1^{pme_K\frac{p^l-1}{p-1}+mn-1}dz$ *via this projection. In particular,* $d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^{mn})$ *is non-zero.*

Proof. By first half of Proposition 6.29, if $z^t \in \tilde{E}'^{0,0,\frac{t}{e_K}}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$ has non-trivial contribution to $\tilde{E}'^{1,0,\frac{k-1}{e_K}}_{\frac{k-t}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$, then

$$\frac{p^{l'+1}-1}{p-1}+\frac{t-1}{me_K}=\frac{k-1}{e_K}+\frac{s}{me_K} \text{ for some } 0\le s\le m-1, \tag{6.34}$$

where $l'=v_p(t)$. By the second half of Proposition 6.29, t is uniquely determined by (k,s). In particular, if

$$k=e_K\frac{p^{l+1}-1}{p-1}+n, \quad s=m-1,$$

then t has to be equal to mn. Moreover, when (6.34) holds, we see from the argument of Proposition 6.1 and Lemma 6.10 that the image of z^t in $\tilde{E}'^{1,0,\frac{k-1}{e_K}}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$ is contained in the subspace generated by the cocycles $z_1^{m(k-1)+s'}dz, 0\le s'\le s$. Putting these together, we deduce that

$$\ker(\tilde{E}'^{1,0,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)\to \tilde{E}'^{1,0,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}_{\frac{p^{l+1}-1}{p-1}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$$

is contained in the subspace generated by $z_1^{pme_K\frac{p^l-1}{p-1}+mn-s}dz, 2\le s\le m$, yelding the first half of the lemma. Using (6.30), we conclude the second half of the lemma. □

Now we prove the proposition. We first show that $z^n\in\tilde{E}^{0,0,\frac{n}{e_K}}_{\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ survives to the $\tilde{E}_{\frac{p^{l+1}-1}{p-1}}$-term. We do this by induction. Suppose z^n survives to some $\tilde{E}_r$-term with $\frac{1}{e_K}\le r<\frac{p^{l+1}-1}{p-1}$. That is,

$$d(z^n)\in\mathcal{N}^{\ge r+\frac{n-1}{e_K}}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p).$$

Since $T_m(z^n)=z^{mn}$, which survives to the $\tilde{E}'_{\frac{p^{l+1}-1}{p-1}}$-term by Lemma 6.33, we have

$$T_m(d(z^n))=d(T_m(z^n))\in\mathcal{N}^{\ge r+\frac{n}{e_K}}\mathrm{TP}_0(\mathcal{O}_{K'}/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p).$$

Then the injectivity of $\tilde{E}_{\frac{1}{e_K}}^{1,0,r+\frac{n-1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p) \to \tilde{E}_{\frac{1}{e_K}}^{\prime 1,0,r+\frac{n-1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$ implies that $d(z^n) = d(\alpha)$ for some $\alpha \in \mathcal{N}^{r+\frac{n-1}{e_K}}\mathrm{TP}_0(\mathcal{O}_K;\mathbb{F}_p)$. Now

$$d(T_m(\alpha)) = T_m(d(\alpha)) = T_m(d(z^n)) = 0 \in \mathcal{N}^{r+\frac{n-1}{e_K}}\mathrm{TP}_0(\mathcal{O}_{K'}/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p),$$

we get $T_m(\alpha) \in \tilde{E}_{\frac{1}{e_K}}^{\prime 0,0,r+\frac{n-1}{e_K}}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$. By the explicit description of $\tilde{E}_{\frac{1}{e_K}}^{\prime 0,0,*}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ and $\tilde{E}_{\frac{1}{e_K}}^{\prime 0,0,*}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$, we conclude $\alpha \in \tilde{E}_{\frac{1}{e_K}}^{\prime 0,0,r+\frac{n-1}{e_K}}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$. Thus

$$d(z^n) = d(\alpha) = 0 \in \mathcal{N}^{r+\frac{n-1}{e_K}}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p),$$

yielding

$$d(z^n) \in \mathcal{N}^{\geq r+\frac{n}{e_K}}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p).$$

Once we know z^n survives to the $\tilde{E}_{\frac{p^{l+1}-1}{p-1}}$-term, since $\tilde{E}_{\frac{1}{e_K}}^{1,0,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}(\mathrm{TP}(\mathcal{O}_K;\mathbb{F}_p))$ is generated by $z_1^{pe_K\frac{p^l-1}{p-1}+n-1}dz$, we may suppose

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n) = \lambda z_1^{pe_K\frac{p^l-1}{p-1}+n-1}dz.$$

Applying the second half of Lemma 6.33, we get

$$\lambda = n'\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}.$$

The rest is the same as in the proof of Proposition 6.29. □

Remark 6.35. In fact, employing the result of Proposition 6.31 in the argument of Lemma 6.33 will prove the following fact: for $r \in \frac{1}{e_K}\mathbb{Z}_{\geq 1}\cup\{\infty\}$, if $\tilde{E}_r^{\prime 1,0,\frac{k-1}{e_K}+1}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is non-zero, that is $z_1^{k-1}dz$ is not in the image of $d_{r-\frac{1}{e_K}}$, then the natural projection

$$\tilde{E}_{\frac{1}{e_K}}^{\prime 1,0,\frac{k-1}{e_K}+1}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p) \to \frac{\tilde{E}_{\frac{1}{e_K}}^{\prime 1,0,\frac{k-1}{e_K}+1}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)}{\oplus_{i\neq mk-1}\mathbf{k}z_1^i dz} \cong \mathbf{k}z_1^{mk-1}dz$$

factors through $\tilde{E}_r^{\prime 1,0,\frac{k-1}{e_K}+1}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$. In particular, $T_m(z_1^{k-1}dz)$ is non-zero in $\tilde{E}_r^{\prime 1,0,\frac{k-1}{e_K}+1}(\mathrm{TP}(\mathcal{O}_{K'});\mathbb{F}_p)$.

Next we investigate the differentials on non-zero stems. To this end, put

$$\epsilon = \sigma_1\sigma_2^{-1}, \quad \epsilon_0 = \frac{\varphi(E_K(z_1))}{\varphi(E_K(z_2))};$$

by Remark 3.12, the latter is well-defined. By the functoriality of Tate spectral sequence, we have

$$\epsilon \in 1 + \mathcal{N}^{\geq 1}.$$

Using Theorem 2.15(6), we get

$$\frac{\epsilon}{\varphi(\epsilon)} = \frac{\varphi(\sigma_1^{-1})\sigma_1}{\varphi(\sigma_2^{-1})\sigma_2} = \frac{\varphi(v_1)\varphi(u_1)}{\varphi(v_2)\varphi(u_2)} = \epsilon_0. \tag{6.36}$$

Let $\bar{\epsilon}, \bar{\epsilon}_0$ be the images of ϵ, ϵ_0 in $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$ respectively. It follows that

$$\bar{\epsilon}_0 = \bar{\epsilon}^{1-p} \equiv 1 \mod \mathcal{N}^{\geq 1}.$$

Then it is straightforward to see that for $i \geq 0$,

$$\bar{\epsilon}_0^{p^i} \equiv 1 \mod \mathcal{N}^{\geq p^i}, \tag{6.37}$$

and

$$\prod_{i=0}^{\infty} \bar{\epsilon}_0^{p^i} = \bar{\epsilon}, \tag{6.38}$$

where the LHS takes limit under the $\mathcal{N}$-topology.

Lemma 6.39. *Fix an integer j. Then for $r \in \frac{1}{e_K}\mathbb{N}$, $m, k \in \mathbb{N}$ such that*

$$p^k > j, \quad \min\{p^m, p^k\} > r,$$

we have

$$z^{(p^k-j)e_K \frac{p^{m+1}-p}{p-1}} \sigma^j \in \tilde{E}_{1-\frac{1}{e_K}}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p)$$

survives to the $\tilde{E}_r$-term.

Proof. Consider

$$\alpha = (\prod_{i=1}^{m} \varphi^i(E_K(z)/\mu))^{p^k-j} \sigma^j \in \mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]).$$

Clearly α is a lift of $z^{(p^k-j)e_K \frac{p^{m+1}-p}{p-1}} \sigma^j$. We have

$$\eta_L(\alpha) = (\prod_{i=1}^{m} \varphi^i(E_K(z_1)/\mu))^{p^k-j} \sigma_1^j$$

and

$$\eta_R(\alpha) = (\prod_{i=1}^{m} \varphi^i(E_K(z_2)/\mu))^{p^k-j} \sigma_2^j = \eta_L(\alpha)\epsilon^{-j} \prod_{i=0}^{m-1} \varphi^i(\epsilon_0)^{j-p^k} = \eta_L(\alpha)(\epsilon^{-1} \prod_{i=0}^{m-1} \varphi^i(\epsilon_0))^j \prod_{i=0}^{m-1} \varphi^i(\epsilon_0)^{-p^k}.$$

By (6.37) and (6.38), we deduce that

$$\bar{\epsilon}^{-1}\prod_{i=0}^{m-1}\varphi^i(\bar{\epsilon}_0)\equiv 1 \mod \mathcal{N}^{\geq p^m}$$

and

$$\prod_{i=1}^{m}\varphi^i(\bar{\epsilon}_0)^{-p^k}\equiv 1 \mod \mathcal{N}^{\geq p^k}.$$

It follows that $\eta_L(z^{(p^k-j)e_K\frac{p^{m+1}-p}{p-1}}\sigma^j)-\eta_R(z^{(p^k-j)e_K\frac{p^{m+1}-p}{p-1}}\sigma^j)\in\mathcal{N}^{\geq(p^k-j)\frac{p^{m+1}-p}{p-1}+\min\{p^m,p^k\}}$, concluding the lemma. □

Proposition 6.40. *For $n\geq 0, j\in\mathbb{Z}, l=v_p(n-\frac{pe_Kj}{p-1})$, and $n'\equiv p^{-l}(n-\frac{pe_Kj}{p-1}) \mod p$, we have*

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n\sigma^j)=n'\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+n-1}\sigma^j dz, \tag{6.41}$$

which is non-zero in $\tilde{E}^{1,j,\frac{p^{l+1}-1}{p-1}+\frac{n-1}{e_K}}_{\frac{p^{l+1}-1}{p-1}}$. Moreover, the targets of (6.41) are all different.

Proof. Choose $k,m\in\mathbb{N}$ such that

$$p^k>j,\quad \min\{m,k\}>l.$$

Thus

$$(p^k-j)e_K\frac{p^{m+1}-p}{p-1}-n\equiv\frac{pe_Kj}{p-1}-n \mod p^{l+1}.$$

It follows that $(p^k-j)e_K\frac{p^{m+1}-p}{p-1}=n+sp^l$ with $s\equiv -n' \mod p$. By Lemma 6.39, $z^{(p^k-j)e_K\frac{p^{m+1}-p}{p-1}}\sigma^j$ survives to the $\tilde{E}_{\frac{p^{l+1}-1}{p-1}}$-term. Hence

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^{(p^k-j)e_K\frac{p^{m+1}-p}{p-1}}\sigma^j)=d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^{n+sp^l}\sigma^j)=0.$$

By Leibniz rule and Proposition 6.29, we deduce that

$$z_2^{sp^l}d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n\sigma^j)=-z_1^n\sigma^j d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^{sp^l})=-s\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+sp^l-1+n}\sigma^j dz.$$

Recall that both η_L and η_R define the refined Nygaard filtrations. It follows that

$$d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n\sigma^j)=n'\bar{\tilde{\mu}}^{\frac{p^l-1}{p-1}}z_1^{pe_K\frac{p^l-1}{p-1}+n-1}\sigma^j dz.$$

The rest is similar to the proof of Proposition 6.29: put $\tilde{n}=pe_K\frac{p^l-1}{p-1}+n$, then

$$l=v_p(\tilde{n}-\frac{pe_K(j-1)}{p-1}).$$

That is, n is uniquely determined by $\tilde{n}$. □

Remark 6.42. We see similarity between refined algebraic Tate differentials and Tate differentials in prior works. More precisely, z, z^{e_K}, σ and dz correspond to ϖ_K, $\tau_K\alpha_K$, τ_K^{-1} and $\tau_K\varpi_K d\log\varpi_K$ in [5, Theorem 5.5.1] respectively; for $p=2$ and $e_K=1$, σ, $z^2\sigma$ and $z\sigma^2dz$ correspond to t^{-1}, te_4 and e_3 in [12, Theorem 8.14] respectively; for p odd and $e_K=1$, σ, $z^p\sigma^{p-1}$ and $z^{p-1}\sigma^p dz$ correspond to t^{-1}, tf and e in [13, Theorem 7.4] respectively.

7 E_2-term of mod p descent spectral sequence I

In this section, we compute E_2-terms of the mod p descent spectral sequences for $\mathrm{TC}^-(\mathcal{O}_K)$ and $\mathrm{TP}(\mathcal{O}_K)$.

Proposition 7.1. *For $j\in\mathbb{Z}$, $E_2^{0,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is non-zero if and only if $j\geq 0$ and $p-1|e_Kj$. If this condition holds, then $E_2^{0,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is the 1-dimensional $\mathbf{k}$-vector space generated by a cocycle with leading term $z^{\frac{pe_Kj}{p-1}}\sigma^j$. Moreover, the canonical map induces*

$$E_2^{0,*}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)\cong E_2^{0,*}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p).$$

Proof. By Proposition 6.40, we deduce that $d_{\frac{p^{l+1}-1}{p-1}-\frac{1}{e_K}}(z^n\sigma^j)=0$ is equivalent to

$$l<v_p(n-\frac{pe_Kj}{p-1}).$$

Thus $z^n\sigma^j$ has non-trivial contribution to $\tilde{E}_\infty(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ if and only if

$$n=\frac{pe_Kj}{p-1}.$$

This concludes the first two assertions. For the last one, since

$$\mathrm{can}:\mathrm{TC}^-_*(\mathcal{O}_K;\mathbb{F}_p)\to\mathrm{TP}_*(\mathcal{O}_K;\mathbb{F}_p)$$

is injective, we have

$$\mathrm{can}:E_2^{0,*}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)\to E_2^{0,*}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$$

is injective as well. On the other hand, when $E_2^{0,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is non-zero, by Theorem 2.15, we have $z^{\frac{pe_Kj}{p-1}}\sigma^j=\bar{\mu}^{-j}z^{\frac{e_Kj}{p-1}}u^j\in E_2^{0,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$. Thus

$$\mathrm{can}:E_2^{0,*}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)\to E_2^{0,*}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$$

is also surjective. □

Proposition 7.2. *The $\mathbf{k}$-vector space $E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is freely generated by a set of cocycles whose leading terms are*

- $z_1^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz$ *with* $l \geq 1, b \in \mathbb{Z}$ *satisfying*

$$-\frac{e_K(j-1)}{p^{l-1}} < b < pe_K - \frac{e_K j}{p^{l-1}}, \quad p \nmid b, \quad b \equiv -e_K(j-1) \mod p-1,$$

and

- $z_1^{\frac{pe_K(j-1)}{p-1}-1}\sigma^j dz$*, if* $j > 1$ *and* $p-1 \mid e_K(j-1)$.

Proof. We first treat the case of $e_K > 1$. In this case, by Corollary 6.15, we see that $E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is generated over $\mathbf{k}$ by cocycles which are detected by $\{z_1^{n-1}\sigma^j dz\}_{n\geq 1}$. By Proposition 6.40, $z_1^{n-1}\sigma^j dz$ is hit by $z^m\sigma^j$ if and only if

$$pe_K\frac{p^l-1}{p-1} + m = n \tag{7.3}$$

with $l = v_p(m - \frac{pe_K j}{p-1}) < \infty$. In this case, it follows that

$$n \equiv m - \frac{pe_K}{p-1} \mod p^{l+1},$$

yielding $l = v_p(m - \frac{pe_K j}{p-1}) = v_p(n - \frac{pe_K(j-1)}{p-1})$. Hence m is uniquely determined by n, j.

Now put $l = v_p(n - \frac{pe_K(j-1)}{p-1})$. If $l = \infty$, then by previous argument $z_1^{n-1}\sigma^j dz$ is not hit by any $z^m\sigma^j$; in this case it follows that $j > 1$, $p-1 \mid e_K(j-1)$ and $n = \frac{pe_K(j-1)}{p-1}$.

If $l < \infty$, then we may write

$$n = \frac{pe_K(j-1)+bp^l}{p-1}$$

for some $b \in \mathbb{Z}$ satisfying

$$p \nmid b, \quad b \equiv -(j-1)e_K \mod p-1, \quad \frac{p(j-1)e_K + bp^l}{p-1} \geq 1;$$

the last one is equivalent to

$$bp^l + pe_K j \geq p - 1 + pe_K. \tag{7.4}$$

On the other hand, by (7.3), $z_1^{n-1}\sigma^j dz$ is not hit by any refined algebraic Tate differential if and only if

$$n - \frac{pe_K(p^l-1)}{p-1} < 0. \tag{7.5}$$

Note that (7.5) implies that $l \geq 1$. Conversely, if $l \geq 1$, then (7.4) plus (7.5) is equivalent to

$$-\frac{e_K(j-1)}{p^{l-1}} < b < pe_K - \frac{e_K j}{p^{l-1}},$$

concluding the desired result. Finally, note that all the resulting leading terms $z_1^{n-1}\sigma^j$ satisfy $p|n$. Thus by Corollary 6.15, the above argument applies equally to the case of $e_K = 1$. ∎

Proposition 7.6. *For $j \geq 1$, $E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ is freely generated over $\mathbf{k}$ by a set of cocycles whose leading terms are*

- $$z_1^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz$$

  with $l \geq 0$, $b \in \mathbb{Z}$ satisfying

  $$-\frac{e_K(j-1)}{p^l} < b < pe_K - \frac{e_K j}{p^l}, \quad p \nmid b, \quad b \equiv -e_K(j-1) \mod p-1,$$

  and
- $z_1^{\frac{pe_K(j-1)}{p-1}-1}\sigma^j dz$ *with $j > 1$ and $p-1 \mid e_K(j-1)$.*

Proof. Recall that the refined algebraic homotopy fixed points spectral sequence is a truncation of the refined algebraic Tate spectral sequence. More precisely, for

$$z^n\sigma^j \in \tilde{E}_{\frac{1}{e_K}}(TP(\mathcal{O}_K);\mathbb{F}_p) \quad (\text{resp. } z_1^{n-1}\sigma^j dz \in \tilde{E}_{\frac{1}{e_K}}(TP(\mathcal{O}_K);\mathbb{F}_p)),$$

it belongs to $\tilde{E}_{\frac{1}{e_K}}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ is equivalent to $je_K \leq n$ (resp. $(j-1)e_K \leq n-1$).

Therefore, using the argument of Proposition 7.2, we deduce that for

$$z^{n-1}\sigma^j dz \in \tilde{E}_{\frac{1}{e_K}}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p),$$

it is not hit by any refined algebraic homotopy fixed points differential if and only if

$$n = \frac{pe_K(j-1)}{p-1}$$

or

$$n - pe_K\frac{p^l-1}{p-1} < je_K \tag{7.7}$$

for $l = v_p(n - \frac{pe_K(j-1)}{p-1})$.

In the first case, we have $j > 1$ and $p-1 \mid e_K(j-1)$. Conversely, under this condition, it is straightforward to verify that $z^{\frac{pe_K j}{p-1}}\sigma^j dz$ belongs to $\tilde{E}_{\frac{1}{e_K}}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$.

In the second case, we may write $n = \frac{pe_K(j-1)+bp^l}{p-1}$ with

$$p \nmid b, \quad b \equiv -e_K(j-1) \mod p-1.$$

Moreover, the conditions $n \geq 1$ plus (7.7) is equivalent to

$$-\frac{e_K(j-1)}{p^l} < b < pe_K - \frac{e_K j}{p^l}.$$

Finally, if b satisfies all these conditions, then it is straightforward to check that $z_1^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz$ belongs to $\tilde{E}_{\frac{1}{e_K}}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$. $\square$

Lemma 7.8. *For $j \geq 1$, the kernel of the canonical map*

$$\mathrm{can}: E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p) \to E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$$

is an $e_K j$-dimensional $\mathbf{k}$*-vector space freely generated by a set of cocycles whose leading terms are*

$$z_1^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz$$

with $l \geq 0, b \in \mathbb{Z}$ satisfying

$$p \nmid b, \quad b \equiv -e_K(j-1) \mod p-1, \quad p^{l-1} \leq \frac{e_K j}{pe_K - b} < p^l. \tag{7.9}$$

Proof. By Propositions 7.2, 7.6, we obtain that the kernel of can is freely generated by a set of cocycles which are detected by $z_1^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz$ with $l \geq 0, b \in \mathbb{Z}$ satisfying

$$-\frac{e_K(j-1)}{p^l} < b < pe_K - \frac{e_K j}{p^l},\ p \nmid b,\ b \equiv -e_K(j-1) \mod p-1,\ b \notin (-\frac{e_K(j-1)}{p^{l-1}}, pe_K - \frac{e_K j}{p^{l-1}}).$$

It is straightforward to see that the first condition plus the last conditions is equivalent to

$$pe_K - \frac{e_K j}{p^{l-1}} \leq b < pe_K - \frac{e_K j}{p^l}, \tag{7.10}$$

which in turn is equivalent to the last condition of (7.9).

It remains to count the number of cocycles. To this end, first note that (7.10) implies that

$$pe_K(1-j) \leq b < pe_K.$$

Conversely, for any $e_K(1-j) \leq m < e_K$, there is exactly one $b \in [pm, pm+p-1]$ satisfying the first two conditions of (7.9). Moreover, for any $b \in [pe_K(1-j), pe_K)$, there is exactly one l satisfying (7.10). We thus conclude that the number of such cocycles is $e_K - e_K(1-j) = e_K j$. □

8 E_2-term of mod p descent spectral sequence II

In this section, we compute the E_2-term of the mod p descent spectral sequence for $\mathrm{TC}(\mathcal{O}_K)$. Firstly, we study the action of Frobenius on $E_2(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$.

Lemma 8.1. *For $n \geq e_K j$, we have*

$$\varphi(z^n\sigma^j) = \bar{\mu}^{-pj} z^{p(n-e_K j)}\sigma^j.$$

Proof. Using Theorem 2.15, we have

$$\varphi(z^n\sigma^j) = \varphi(z^{n-e_K j})\bar{\mu}^{-pj}\varphi(E_K(z)\sigma)^j = \bar{\mu}^{-pj} z^{p(n-e_K j)}\varphi(u)^j = \bar{\mu}^{-pj} z^{p(n-e_K j)}\sigma^j.$$

□

Lemma 8.2. *If $e_K > 1$, then*

$$\varphi(\sigma_1(z_1 - z_2)) \equiv -z_1^{p-1}\sigma_1(z_1 - z_2) \mod \mathcal{N}^{\geq \frac{p}{e_K}+1}.$$

Proof. In $\mathrm{TC}_2^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$, we have

$$\varphi(\sigma_1(z_1 - z_2)) = \varphi(\sigma_1 E_K(z_1))\frac{\varphi(z_1 - z_2)}{\varphi(E_K(z_1))} = h\varphi(u_1) = h\sigma_1.$$

Using (3.20) for $l = 0$ and the fact that $f^{(1)} \in \mathcal{N}^{\geq p}$, we get

$$h \equiv \delta(z_1 - z_2)/\delta(E_K(z_1)) \mod \mathcal{N}^{\geq p}.$$

By Lemma 6.16, we have

$$\delta(z_1 - z_2) \equiv -z_1^{p-1}(z_1 - z_2) \mod (p, \mathcal{N}^{\geq \frac{p-2}{e_K}+2}).$$

On the other hand, a short computation shows that

$$\delta(E_K(z_1)) \equiv 1 \mod (p, \mathcal{N}^{\frac{p}{e_K}}).$$

Putting these together, we conclude

$$h \equiv -z_1^{p-1}(z_1 - z_2) \mod (p, \mathcal{N}^{\geq r_0}),$$

where

$$r_0 = \min(p, \frac{2p-1}{e_K} + 1, \frac{p-2}{e_K} + 2) \geq \frac{p}{e_K} + 1$$

as $e_K > 1$. This yields the desired result by modulo p. □

Lemma 8.3. *If $\alpha \in \mathcal{N}^{\geq m}\mathrm{TC}_{2j}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$, then*

$$\varphi(\alpha) \in \mathcal{N}^{\geq p(m-j)}\mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p).$$

Proof. Write $m = m_0 + \frac{m_1}{e_K}$ with $m_0 \geq j$, $0 \leq m_1 < e_K$. Then there exist

$$x \in \mathcal{N}^{\geq m_0}\mathrm{TC}_{2j}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p), \quad y \in \mathcal{N}^{\geq m_0+1}\mathrm{TC}_{2j}^-(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$$

such that $\alpha = z_1^{m_1}x + y$. By a variant of the proof of Lemma 3.11, we get $\varphi(x)$ divisible by $\varphi(E_K(z_1))^{m_0-j}$, yielding

$$\varphi(x) \in \mathcal{N}^{\geq p(m_0-j)}\mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p).$$

Similarly, we get $\varphi(y) \in \mathcal{N}^{\geq p(m_0+1-j)}\mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p)$. It follows that

$$\varphi(\alpha) = z_1^{pm_1}\varphi(x) + \varphi(y) \in \mathcal{N}^{\geq p(m-j)}\mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{F}_p).$$

□

Proposition 8.4. *For $j \geq 1$, if $\alpha \in E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ is detected by $z_1^{n-1}\sigma^j dz$, then $\varphi(\alpha) \in E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is detected by*

$$-\bar{\mu}^{-p(j-1)} z_1^{p(n-e_K(j-1))-1}\sigma^j dz.$$

Before proving Proposition 8.4, note that the map

$$z_1^{n-1}\sigma^j dz \mapsto z_1^{p(n-e_K(j-1))-1}\sigma^j dz$$

gives rise to a bijection between leading terms of the cocycles given in Propositions 7.2 and Proposition 7.6 respectively. Therefore, granting Proposition 8.4, we obtain the following results.

Corollary 8.5. *For $j \geq 1$, $\varphi : E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p) \to E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$ is an isomorphism.*

Corollary 8.6. *Suppose $\alpha \in E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ has refined Nygaard filtration m.*

(1) For $j \geq 1$, the filtration of $\varphi(\alpha)$ is higher than (resp. lower than, equal to) the filtration of α if and only if

$$m > \frac{pj-1}{p-1} - \frac{1}{e_K} \quad (\textit{resp. } m < \frac{pj-1}{p-1} - \frac{1}{e_K},\ m = \frac{pj-1}{p-1} - \frac{1}{e_K}).$$

(2) For $j \leq 0$, the filtration of $\varphi(\alpha)$ is higher than that of α.

Proof. For (1), by Proposition 8.4, $\varphi(\alpha)$ has filtration

$$m' = \frac{p(e_K(m-1)+1-e_K(a-1))-1}{e_K} + 1 = p(m-j) + \frac{p-1}{e_K} + 1.$$

A short computation shows the desired result. For (2), since dz has filtration 1, we may assume $m \geq 1$. Then we may write $\alpha = \beta v^{-j}$ with $\beta \in \mathcal{N}^{\geq m}E_2^{1,0}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$. It follows that $\varphi(\alpha)$ is divisible by $\varphi(\beta)$, which belongs to $\mathcal{N}^{\geq pm}E_2^{1,0}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$. Now the desired result follows as $pm > m$. □

Now we prove Proposition 8.4.

Proof. Regard $z_1^{n-1}\sigma_1^j dz$ as an element of the cobar complex of $\mathrm{TC}^-_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z];\mathbb{F}_p)$. Note that $d(z_1), d(\sigma_1) \in \mathcal{N}^{\geq 1}\mathrm{TC}^-_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2,z_3];\mathbb{F}_p)$. Thus by Leibniz rule, we deduce that

$$d(z_1^{n-1}\sigma_1^j dz) \in \mathcal{N}^{\geq \frac{n-2}{e_K}+2}\mathrm{TC}^-_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2,z_3];\mathbb{F}_p).$$

Using Lemma 6.10, we deduce that there exists $\beta \in \mathcal{N}^{\geq \frac{n-2}{e_K}+2}\mathrm{TC}^-_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1,z_2];\mathbb{F}_p)$ such that $d(\beta) = d(z_1^{n-2}\sigma_1^j dz)$; hence $d(z_1^{n-1}\sigma_1^j dz - \beta) = 0$. Therefore, by induction on n, it reduces to treat the case

$$\alpha \equiv z_1^{n-1}\sigma_1^j(z_1 - z_2) \mod \mathcal{N}^{\geq \frac{n-2}{e_K}+2}.$$

By Lemma 8.3, we have

$$\varphi(\alpha) \equiv \varphi(z_1^{n-1}\sigma_1^{j-1})\varphi(\sigma_1(z_1 - z_2)) \mod \mathcal{N}^{p(\frac{n-2}{e_K}+2-j)}.$$

By Lemma 8.2 and Lemma 8.1, we have

$$\varphi(z_1^{n-1}\sigma_1^{j-1})\varphi(\sigma_1(z_1-z_2)) \equiv -\bar{\mu}^{-p(j-1)} z_1^{p(n-e_K(j-1))-1}\sigma_1^j(z_1-z_2) \mod \mathcal{N}^{\geq \frac{p(n-e_K(j-1))}{e_K}+1}.$$

Note that if $e_K > 3$, then

$$p(\frac{n-2}{e_K} + 2 - j) \geq \frac{p(n - e_K(j-1))}{e_K} + 1,$$

yielding the desired result for $e_K > 3$.

For the case of $e_K \leq 3$, let m, K', T_m and $\tilde{E}'_r(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$ be as in the proof of Proposition 6.31. Let $\tilde{E}'_r(\mathrm{TC}^-(\mathcal{O}_{K'}); \mathbb{F}_p)$ be the "less refined" algebraic homotopy fixed point spectral sequence, which is a truncation of $\tilde{E}'_r(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$. First note that Remark 6.35 [1] implies that

$$T_m : \tilde{E}_\infty^{1,*,*}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p) \to \tilde{E}_\infty'^{1,*,*}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$$

is injective. Thus it restricts to an injective map $\tilde{E}_\infty(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \to \tilde{E}'_\infty(\mathrm{TC}^-(\mathcal{O}_{K'}); \mathbb{F}_p)$. Since $z_1^{n-1}dz$ is non-zero in $\tilde{E}_\infty(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p)$, using Remark 6.35, we may deduce that $T_m(\alpha)$ is detected by $mz_1^{mn-1}\sigma^j dz$. Then by the case of $e_K > 3$, we get that

$$-m\bar{\mu}^{-p(j-1)} z_1^{p(mn-me_K(j-1))-1}\sigma_1^j dz$$

detects $T_m(\varphi(\alpha)) = \varphi(T_m(\alpha))$ in $\tilde{E}_\infty(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$. Now suppose $\varphi(\alpha)$ is detected by $\lambda z^{l-1}\sigma^j dz$. Then $T_m(\varphi(\alpha))$ is detected by $T_m(\lambda z^{l-1}\sigma^j dz) = \lambda m z^{mj-1}\sigma^j dz$. Comparing the two expressions, we get

$$l = p(n - me_K(j-1)), \quad \lambda = -\bar{\mu}^{-p(j-1)}$$

by Remark 6.35 again. This completes the proof. □

Lemma 8.7. *The canonical map induces*

- *for $j > 0$, a surjection*

$$E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \to \mathcal{N}^{\geq j} E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p);$$

- *for $j \leq 0$, an isomorphism*

$$E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \to E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p);$$

- *for $m \geq j$, a surjection*

$$\mathcal{N}^{\geq m} E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \to \mathcal{N}^{\geq m} E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p).$$

[1] Using Proposition 6.40, the argument of Lemma 6.33 (hence Remark 6.35) adapts to $\tilde{E}'^{1,j,*}(\mathrm{TP}(\mathcal{O}_{K'}); \mathbb{F}_p)$ for all $j \in \mathbb{Z}$.

Proof. These follow from the corresponding result on

$$\mathrm{can} : \mathrm{TC}^-_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}) \to \mathrm{TP}_{2j}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]^{\otimes \bullet}).$$

□

Combining Corollary 8.6 and Lemma 8.7, we deduce the following results immediately.

Corollary 8.8. *For $j \geq 1$ and $m \geq \frac{pj-1}{p-1}$, the map*

$$\mathrm{can} - \varphi : \mathcal{N}^{\geq m} E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \to \mathcal{N}^{\geq m} E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K); \mathbb{F}_p)$$

is surjective.

Corollary 8.9. *For $j \leq 0$, the map*

$$\mathrm{can} - \varphi : E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \to E_2^{1,2j}(TP(\mathcal{O}_K); \mathbb{F}_p)$$

is an isomorphism.

Now we are ready to compute $E_2(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p)$. Let d be the minimal number such that

$$p - 1 \mid e_K d, \quad \mathrm{N}_{\mathbf{k}/\mathbb{F}_p}(\bar{\mu})^d = 1,$$

where $\mathrm{N}_{\mathbf{k}/\mathbb{F}_p} : \mathbf{k} \to \mathbb{F}_p$ is the norm map.

The following lemma is a reformulation of Hilbert 90 for $\mathbf{k}/\mathbb{F}_p$.

Lemma 8.10. *For $b \in \mathbf{k}^\times$, the map*

$$b\varphi - \mathrm{id} : \mathbf{k} \to \mathbf{k}$$

is bijective if $\mathrm{N}_{k/\mathbb{F}_p}(b) \neq 1$, otherwise both the kernel and cokernel are isomorphic to $\mathbb{F}_p$.

Using Lemma 8.10, we may choose a $(p-1)$-th root $\bar{\mu}^{\frac{pd}{p-1}}$ of $\bar{\mu}^{pd}$ in $\mathbf{k}$. Denote by β the element in $\tilde{E}_2^{0,0,2d}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p) \subseteq E_2^{0,2d}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p)$ detected by $\bar{\mu}^{\frac{pd}{p-1}} z^{\frac{pe_K d}{p-1}} \sigma^d$.

Proposition 8.11. *We have*

$$\tilde{E}_2^{0,0,*}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbb{F}_p[\beta].$$

Proof. By Proposition 7.1, we first have

$$E_2^{0,*}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbf{k}[z^{\frac{pe_K j}{p-1}} \sigma^j],$$

where j is the smallest positive integer such that $p - 1 \mid e_K j$.

On the other hand, by Lemma 8.1,

$$\varphi(z^{\frac{pe_K j}{p-1}} \sigma^j) = \bar{\mu}^{-pj} z^{\frac{pe_K j}{p-1}} \sigma^j.$$

Thus $\lambda z^{\frac{pe_K j}{p-1}} \sigma^j \in \tilde{E}_2^{0,0,*}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p)$ if and only if $\bar{\mu}^{pj} = \lambda^{-1}\varphi(\lambda) = \lambda^{p-1}$ for some $\lambda \in \mathbf{k}$. In this case, it follows that $\mathrm{N}_{\mathbf{k}/\mathbb{F}_p}(\bar{\mu})^j = 1$. Hence $d \mid j$. Conversely, if $d \mid j$, then such λ is of the form $\lambda' \bar{\mu}^{\frac{pd}{p-1}}$ with $\lambda' \in \mathbb{F}_p$. Now the proposition follows. □

It turns out that $\tilde{E}_2^{i,j,*}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p)$ is a free $\mathbb{F}_p[\beta]$-module of finite rank for all i,j. In the following, we will find out their generators over $\mathbb{F}_p[\beta]$. Firstly, combing the proof of Proposition 8.11, Lemma 8.1 and Lemma 8.10, we obtain the following result.

Proposition 8.12. *We have that $\tilde{E}_2^{0,1,*}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p)$ is a free $\mathbb{F}_p[\beta]$-module of rank 1 generated by $1 \in E_2^{0,0}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$.*

Lemma 8.13. *There exists $\gamma \in \ker(\mathrm{can}-\varphi)$ detected by $\bar{\mu}^{\frac{pd}{p-1}} z^{\frac{pe_K d}{p-1}-1}\sigma^{d+1}dz$.*

Proof. Let $\gamma_0 \in E_2^{1,2(d+1)}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ be detected by $z^{\frac{pe_K d}{p-1}-1}\sigma^{d+1}dz$. By Proposition 8.4, $\varphi(\gamma_0)$ is detected by $\mu^{-pd}z^{\frac{pe_K d}{p-1}-1}\sigma^{d+1}dz$. It follows that

$$(\mathrm{can}-\varphi)(\mu^{\frac{pd}{p-1}}\gamma_0) \in \mathcal{N}^{\geq \frac{pd}{p-1}+1}E_2^{1,2(d+1)}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p).$$

By Corollary 8.8, $\mathrm{can}-\varphi$ is surjective on $\mathcal{N}^{\geq \frac{pd}{p-1}+1}E_2^{1,2(d+1)}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$. Hence we may modify γ_0 with higher terms to construct the desired element. □

In the following, let γ be as in Lemma 8.13.

Proposition 8.14. *We have that $\tilde{E}_2^{1,1,*}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p)$ is a free $\mathbb{F}_p[\beta]$-module of rank 1 generated by $\mathrm{can}(\gamma) \in E_2^{1,2(d+1)}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$.*

Proof. Let $\alpha \in E_2^{1,2j}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p)$ represents a non-trivial class in the cokernel of $\mathrm{can}-\varphi$ such that it has the highest leading term in that class. By Corollary 8.8 and Corollary 8.9, we see that $j \geq 1$ and the leading degree of α lies in $[1, \frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}]$.

On the other hand, if the leading degree of α is less than $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$, by Corollary 8.5 and Corollary 8.6, then we may find some α' with higher leading degree such that $\alpha = \varphi(\alpha')$. Note that $\mathrm{can}(\alpha')$ represents the same class as α, yielding a contradiction.

Therefore α must have leading degree $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$. That is, α is detected by some $\lambda z_1^{\frac{pe_K(j-1)}{p-1}-1}\sigma^j dz$. Using Lemma 8.10 and Lemma 8.13, we conclude that $d \mid j-1$ and $\alpha \in \mathbb{F}_p\beta^{\frac{j-1}{d}-1}\mathrm{can}(\gamma)$.

□

Proposition 8.15. *We have that $\tilde{E}_2^{1,0,*}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p)$ is a free $\mathbb{F}_p[\beta]$-module with a basis given by γ and the set of cocycles detected respectively by*

$$cz^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz \in E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$$

with $l \geq 0$ and

$$0<b<pe_K, \quad p\nmid b, \quad b\equiv -e_K(j-1) \mod p-1, \quad p^{l-1} \leq \frac{e_K j}{pe_K-b} < p^l, \quad 1\leq j\leq d, \tag{8.16}$$

and c runs over a basis of $\mathbf{k}$ over $\mathbb{F}_p$.

Proof. By Corollary 8.9, $\ker(\mathrm{can}-\varphi)$ is trivial for $j \le 0$. Now suppose $j \ge 1$, and let $0 \ne \alpha \in \tilde{E}_2^{1,0,*}(\mathrm{TC}(\mathcal{O}_K);\mathbb{F}_p)$. By Corollary 8.6, φ lowers the filtration if the filtration is less than $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$. Thus the leading degree of α is at least $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$.

If the leading degree of α is $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$, the by Lemma 8.10 and the argument of Proposition 8.14, there exists some $\beta' \in \mathbb{F}_p[\beta]\gamma$ such that $\alpha-\beta'$ has leading degree higher than $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$.

Now suppose α has leading degree higher than $\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}$. First note that for a cocycle given in Propositions 7.2, 7.6, it lies in $\mathcal{N}^{>\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}}$ if and only if $b>0$. Then it is straightforward to see that

$$\mathrm{can}:\mathcal{N}^{>\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}}E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)\to\mathcal{N}^{>\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}}E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p).$$

is surjective, and the cocyles given in the statement of the proposition form an $\mathbb{F}_p$-basis of $\ker(\mathrm{can})$. Let S be the $\mathbf{k}$-vector space generated by the remaining cocycles in $\mathcal{N}^{>\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}}E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$. It follows that can induces a filtration preserving isomorphism between S and $\mathcal{N}^{>\frac{pj}{p-1}-\frac{1}{p-1}-\frac{1}{e_K}}E_2^{1,2j}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$.

Now we may write $\alpha=\alpha_1+\alpha_2$ with $\alpha_1\in\ker(\mathrm{can}),\alpha_2\in S$. It follows that

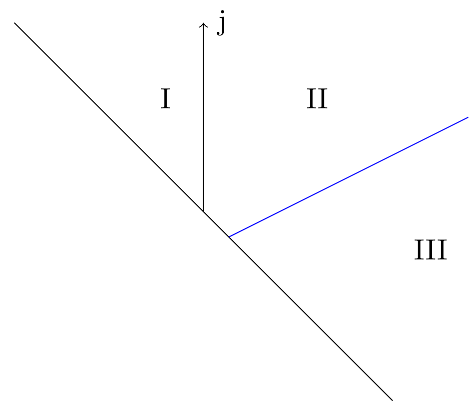
$$(\mathrm{can}-\varphi)(\alpha_2)=\varphi(\alpha_1).$$

Since φ raises the filtration, it follows that

$$\alpha_2=(1-\mathrm{can}^{-1}\varphi)^{-1}(\mathrm{can}^{-1}\varphi(\alpha_1))=\sum_{i\ge 1}(\mathrm{can}^{-1}\varphi)^i(\alpha_1).$$

Hence α_2 is uniquely determined by α_1 and has higher filtration than α_1. Thus the map $\alpha\mapsto\alpha_1$ induces an isomorphism between $\ker(\mathrm{can})$ and $\ker(\mathrm{can}-\varphi)$ preserving the leading term. This completes the proof. □

Remark 8.17. The above argument can be summarized by the following picture. Put $a=m-j$. The cocycles of $E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K);\mathbb{F}_p)$ with leading degree m is represented by the point $(a=m-j,j)$. Then we may divide the area of cocyles into three regions, bounded by the lines $j+a=0$, $a=0$ and $\frac{j}{p-1}-a-\frac{1}{p-1}-\frac{1}{e_K}=0$. The blue line is the "critical line" for the Frobenius action. In region I, the canonical map is an isomorphism, and the Frobenius raises filtration; thus $\mathrm{can}-\varphi$ is an isomorphism (Corollary 8.9). In region II, the Frobenius raises the filtration. One may produce an isomorphism between $\ker(\mathrm{can})$ and $\ker(\mathrm{can}-\varphi)$ preserving the leading term. In region III, the Frobenius lowers the filtration; thus $\ker(\mathrm{can}-\varphi)=0$. Along the critical line, the Frobenius differs from the canonical map by a certain power of $\bar{\mu}$.



Note that for $1 \le i \le e_K$ and $1 \le j \le d$, there is exactly one

$$b \in [(p-1)i+1, pi],$$

and hence one pair (b, l), satisfying (8.16). Denote by $\alpha_i^{(j)}$ the cocycle detected by

$$z^{\frac{pe_K(j-1)+bp^l}{p-1}-1}\sigma^j dz \in E_2^{1,2j}(\mathrm{TC}^-(\mathcal{O}_K); \mathbb{F}_p).$$

given in Proposition 8.15. Let λ denote the cocycle given by Proposition 8.12. Combining Propositions 8.11, 8.12, 8.15, 8.14, and Corollary 6.13, we conclude:

Theorem 8.18. *As $\mathbb{F}_p[\beta]$-modules, we have*

$$E_2^{0,*}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbb{F}_p[\beta],$$

$$E_2^{1,*}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbb{F}_p[\beta]\{\lambda, \gamma\} \oplus \mathbf{k}[\beta]\{\alpha_i^{(j)} | 1 \le i \le e_K, 1 \le j \le d\},$$

and

$$E_2^{2,*}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbb{F}_p[\beta]\{\lambda\gamma\}$$

with $|\lambda| = (1, 0)$, $|\gamma| = (1, 2(d+1))$, $|\alpha_i^{(j)}| = (1, 2j)$. Moreover, for $i \neq 0, 1, 2$,

$$E_2^{i,*}(\mathrm{TC}(\mathcal{O}_K); \mathbb{F}_p) = 0.$$

Corollary 8.19. *The descent spectral sequence computing $\mathrm{TC}(\mathcal{O}_K; \mathbb{F}_p)$ collapses at the E_2-term.*

Proof. There is no room for higher differentials in consideration of degrees. ☐

To complete the proof of Theorem 1.1, it remains to show $d = [K(\zeta_p) : K]$. This will be proved in Proposition 9.4.

9 Constant term of the Eisenstein polynomial

Recall that we may arrange the constant term of $E_K(z)$ to be $p\tau$ for some $\tau \in \mathbb{Z}_p^\times$, which is independent of K (Theorem 2.15(6)). In the following, we will show that $\tau = 1$. To proceed, recall that $\epsilon = \sigma_1\sigma_2^{-1}$ lies in $1 + \mathcal{N}^{\geq 1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2])$, and it satisfies

$$\frac{\epsilon}{\varphi(\epsilon)} = \frac{\varphi(E_K(z_1))}{\varphi(E_K((z_2))}$$

by (6.36).

Lemma 9.1. *We have that $\bar{\epsilon}$ is the unique element in $1+\mathcal{N}^{\geq 1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{Z}/p^n)$ satisfying*

$$\frac{\bar{\epsilon}}{\varphi(\bar{\epsilon})} = \frac{\varphi(E_K(z_1))}{\varphi(E_K((z_2))}.$$

Proof. For the uniqueness, suppose

$$\bar{\epsilon}' \in 1 + \mathcal{N}^{\geq 1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{Z}/p^n)$$

satisfying the same equation. Thus $\frac{\bar{\epsilon}}{\bar{\epsilon}'} = 1+\alpha$ for some $\alpha \in \mathcal{N}^{\geq 1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{Z}/p^n)$ satisfying $\alpha = \varphi(\alpha)$. Since $\varphi(\alpha) \equiv \alpha^p$ modulo p, we get

$$\alpha = \varphi^k(\alpha) \in (p) + \mathcal{N}^{p^k}$$

for any $k \geq 0$, concluding $\alpha \in (p)$. By Corollary 3.4, we deduce that

$$\alpha/p \in \mathcal{N}^{\geq 1}\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z_1, z_2]; \mathbb{Z}/p^{n-1})$$

is φ-invariant as well. Iterating this argument, we conclude $\alpha = 0$, yielding $\bar{\epsilon} = \bar{\epsilon}'$. □

Proposition 9.2. *We have $\tau = 1$.*

Proof. Fix $n \geq 1$ and put $K = \mathbb{Q}_p(\zeta_{p^n})$. Recall that $\mathbb{K}_2(\mathcal{O}_K; \mathbb{Z}/p^n)$ is non-nilpotent due to the existence of Bott elements. Using cyclotomic trace map, we deduce that $\mathrm{TC}_2(\mathcal{O}_K; \mathbb{Z}/p^n)$ is non-nilpotent as well. We may apply the same strategy to compute $\mathrm{TC}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}; \mathbb{Z}/p^n)$ as for $\mathrm{TC}_*(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}; \mathbb{F}_p)$. Namely we employ the descent spectral sequences and use Nygaard filtrations to compute their E_2-terms. By similar arguments, we conclude that $\tilde{E}_2^{i,k,j}(\mathrm{TC}(\mathcal{O}_K); \mathbb{Z}/p^n) = 0$ unless $i, k \in \{0, 1\}$. Since the spectral sequence $\tilde{E}(\mathrm{TC}(\mathcal{O}_K); \mathbb{Z}/p^n)$ is multiplicative, we get that $\tilde{E}_2^{i,k,j}(\mathrm{TC}(\mathcal{O}_K); \mathbb{Z}/p^n)$ is nilpotent unless $i = k = 0$. Hence $\tilde{E}_2^{0,0,2}(\mathrm{TC}(\mathcal{O}_K); \mathbb{Z}/p^n)$ is non-nilpotent.

Put $f(z) = (1 + z)^{p^n} - 1$, and consider

$$\beta = ((1 + z)^{p^n} - 1)\sigma \in \mathrm{TP}_2(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})}[z]; \mathbb{Z}/p^n).$$

We claim that β lies in $E_2^{0,2}(\mathrm{TP}(\mathcal{O}_K); \mathbb{Z}/p^n)$. This amounts to show

$$\epsilon \equiv \frac{f(z_2)}{f(z_1)} \mod p^n.$$

By previous lemma, this reduces to show

$$\frac{f(z_1^p)f(z_2)}{f(z_1)f(z_2^p)} \equiv \frac{E_K(z_1^p)}{E_K(z_2^p)} \mod p^n.$$

Since $E_K(z) = \tau\frac{(1+z)^{p^n}-1}{(1+z)^{p^{n-1}}-1}$, this is equivalent to

$$\left(\frac{(z_1^p+1)^{p^{n-1}}-1}{(z_1+1)^{p^n}-1}\right)\left(\frac{(z_2^p+1)^{p^{n-1}}-1}{(z_2+1)^{p^n}-1}\right) \equiv 1 \mod p^n.$$

This in turn follows from the fact that

$$(z+1)^{p^n} \equiv (z^p+1)^{p^{n-1}} \mod p^n. \tag{9.3}$$

Note that β maps to $z^{p^n}\sigma$ in $E_2^{0,2}(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$. Using Proposition 7.1 and the fact that the mod p reduction of $E_1(\mathrm{TP}(\mathcal{O}_K);\mathbb{Z}/p^n)$ is $E_1(\mathrm{TP}(\mathcal{O}_K);\mathbb{F}_p)$, we deduce that $E_2^{0,2}(\mathrm{TP}(\mathcal{O}_K);\mathbb{Z}/p^n)$ is a free $\mathbb{Z}/p^n$-module generated by β. Since $\tilde{E}_2^{0,0,2}(\mathrm{TC}(\mathcal{O}_K);\mathbb{Z}/p^n)$ is non-nilpotent, we conclude that the natural map

$$\tilde{E}_2^{0,0,2}(\mathrm{TC}(\mathcal{O}_K);\mathbb{Z}/p^n) \hookrightarrow E_2^{0,2}(\mathrm{TC}(\mathcal{O}_K);\mathbb{Z}/p^n)$$

is an isomorphism. In particular, β is invariant under the φ-action. Using (9.3) and

$$\varphi(\sigma^{-1})\sigma = \varphi(v)\varphi(u) = \varphi(E_K(z)) = \tau\frac{(1+z^p)^{p^n}-1}{(1+z^p)^{p^{n-1}}-1},$$

we deduce $\tau \equiv 1$ modulo p^n. Let $n \to \infty$, we conclude $\tau = 1$. □

Let d as in Theorem 8.18. That is, d is the minimal positive integer such that $p-1 \mid e_K d$ and $\mathrm{N}_{\mathbf{k}/\mathbb{F}_p}(\bar{\mu})^d = 1$. The following proposition completes the proof of Theorem 1.1.

Proposition 9.4. *We have $d = [K(\zeta_p):K]$.*

Proof. Put $d' = [K(\zeta_p):K]$. We first have

$$p-1 = [K_0(\zeta_p):K_0] \mid [K(\zeta_p):K_0] = d'e_K.$$

Secondly, we have

$$\begin{aligned}\mathrm{N}_{K_0/\mathbb{Q}_p}(\mu)^{d'} &= \mathrm{N}_{K_0/\mathbb{Q}_p}\big(\frac{p}{\mathrm{N}_{K/K_0}(-\varpi_K)}\big)^{d'} = \frac{p^{f_K d'}}{\mathrm{N}_{K_0(\zeta_p)/\mathbb{Q}_p}(-\varpi_K)}\\ &= \frac{\mathrm{N}_{K(\zeta_p)/\mathbb{Q}_p}((1-\zeta_p)^{d'})}{\mathrm{N}_{K(\zeta_p)/\mathbb{Q}_p}(-\varpi_K)} = \mathrm{N}_{K_0(\zeta_p)/\mathbb{Q}_p}\big(\frac{(1-\zeta_p)^{d'}}{\mathrm{N}_{K(\zeta_p)/K_0(\zeta_p)}(-\varpi_K)}\big).\end{aligned}$$

This yields $\mathrm{N}_{\mathbf{k}/\mathbb{F}_p}(\bar{\mu})^{d'} = \overline{\mathrm{N}_{K_0/\mathbb{Q}_p}(\mu)^{d'}} = \overline{\mathrm{N}_{K_0(\zeta_p)/\mathbb{Q}_p}\big(\frac{(1-\zeta_p)^{d'}}{\mathrm{N}_{K(\zeta_p)/K_0(\zeta_p)}(-\varpi_K)}\big)} = 1$. Hence $d|d'$.

It remains to show $d'|d$. The strategy is to construct a suitable degree d extension of K containing $K(\zeta_p)$ as a subfield. Let d_1 be the minimal positive integer such that $p-1 \mid e_K d_1$, and write $d = d_1 d_2$. Firstly, replacing K with its tamely ramified subextension over K_0, we reduce to the case $(e,p)=1$. Secondly, replacing K with its degree d_2 unramified extension, we reduce to the case $d = d_1$.

Note that $\mathrm{N}_{\mathbf{k}/\mathbb{F}_p}(\bar{\mu})^{d_1} = 1$ implies that

$$\mu \equiv \lambda^{e_K} \mod p$$

for some $\lambda \in K_0$. Consider the degree d_1 totally ramified extension $K(\sqrt[d_1]{\lambda\varpi_K})$ over K. It is clear that the minimal polynomial of $\sqrt[d_1]{\lambda\varpi_K}$ over K_0 is $E_1(z) = E_K(\lambda^{-1}z^{d_1})$, which has leading coefficient

$$\mu' = \mu\lambda^{-e_K} \equiv 1 \mod p. \tag{9.5}$$

On the other hand, consider $\sqrt[\frac{ed_1}{p-1}]{\zeta_p - 1}$, whose minimal polynomial over K_0 is

$$E_2(z) = z^{-\frac{ed_1}{p-1}}((z^{\frac{ed_1}{p-1}}+1)^p - 1).$$

Denote the roots of $E_2(z)$ by $\alpha_i, 1 \le i \le ed_1$. Since $(ed_1, p) = 1$, it is straightforward to check

$$v_p(\alpha_i - \alpha_j) = \frac{1}{ed_1}, \quad i \neq j.$$

Note that both $E_1(z)$ and $E_2(z)$ are Eisenstein polynomials of degree ed_1 and have the same constant term p. Moreover, by (9.5), their leading terms are congruent modulo p. It follows that

$$\prod_{i=1}^{ed_1}(\sqrt[d_1]{\lambda\varpi_K} - \alpha_i) = E_2(\sqrt[d_1]{\lambda\varpi_K}) = (E_2 - E_1)(\sqrt[d_1]{\lambda\varpi_K})$$

has p-adic valuation bigger than 1, yielding $v_p(\sqrt[d_1]{\lambda\varpi_K} - \alpha_i) > \frac{1}{ed_1}$ for some i. By Krasner's Lemma, this implies that $K_0(\sqrt[\frac{ed_1}{p-1}]{\zeta_p - 1}) \subseteq K(\sqrt[d_1]{\lambda\varpi_K})$. Hence $K(\zeta_p) \subseteq K(\sqrt[d_1]{\lambda\varpi_K})$. □

Remark 9.6. One can further show that if $\xi_{p^n} \in K$, then

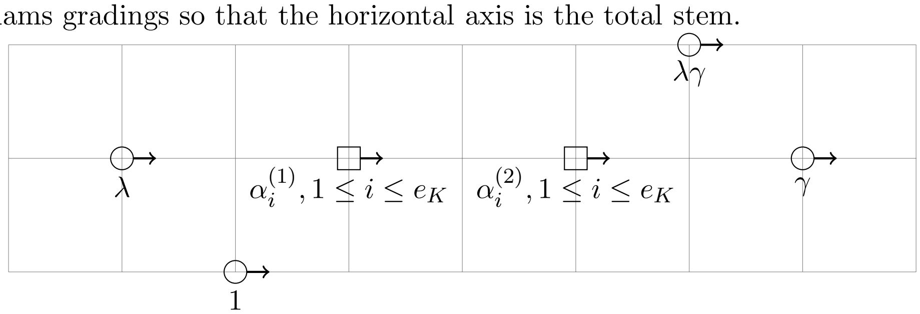
$$\mathrm{TC}_*(\mathcal{O}_K; \mathbb{Z}/p^n) \to \mathrm{TC}_*(\mathcal{O}_K; \mathbb{F}_p)$$

is surjective. In fact, it suffices to show that the generators given in Theorem 8.18 can be lifted to $\mathrm{TC}_*(\mathcal{O}_K; \mathbb{Z}/p^n)$. This is clear except for γ. For γ, we may use the fact that $\beta^{-1}\gamma$ is the mod p reduction of the trace of $\varpi_K \in K^\times \cong \mathbb{K}_1(K)$. This is compatible with results of [5].

10 Comparison with motivic cohomology

In this section we compare the descent spectral sequence computing $\mathrm{TC}_*(\mathcal{O}_K;\mathbb{F}_p)$ with the motivic spectral sequence computing $\mathbb{K}_*(K;\mathbb{F}_p)$. We take $d=2$ for the illustration.

By Theorem 8.18, the E_2-term of the spectral sequence computing $\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_{W(\mathbf{k})};\mathbb{F}_p)$ may be pictured as follows, in which a circle (resp. box) with one arrow means a $\mathbb{F}_p[\beta]$-module (resp. $\mathbf{k}[\beta]$-module) freely generated by the elements below it. We use the Adams gradings so that the horizontal axis is the total stem.



Let β be a generator of μ_p^d, which is isomorphic to $\mathbb{Z}/p$ as a $\mathrm{Gal}(\overline{K}/K)$-module. Let $\alpha^{(1)}$ be a generator of the $\mathcal{O}_K/p$-module

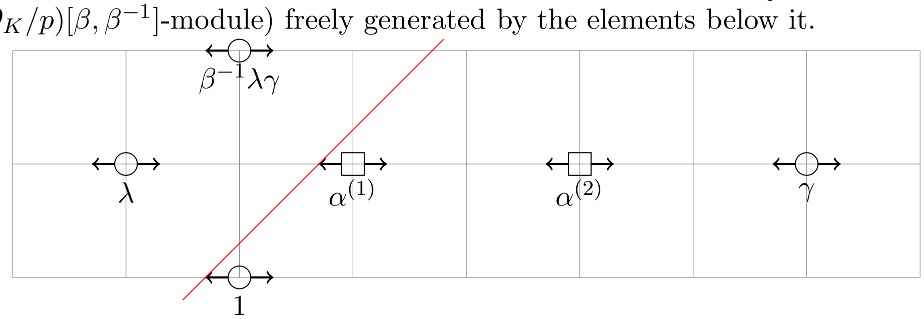
$$U_K/U_K^p \subset K^\times/(K^\times)^p \cong H^1_{\text{ét}}(K,\mu_p),$$

where U_K is the torsion free part of $\mathcal{O}_K^\times$. Let $\beta^{-1}\gamma \in H^1_{\text{ét}}(K,\mu_p)$ be the class represented by $\overline{\varpi_K} \in K^\times/(K^\times)^p$. Let λ be the element of

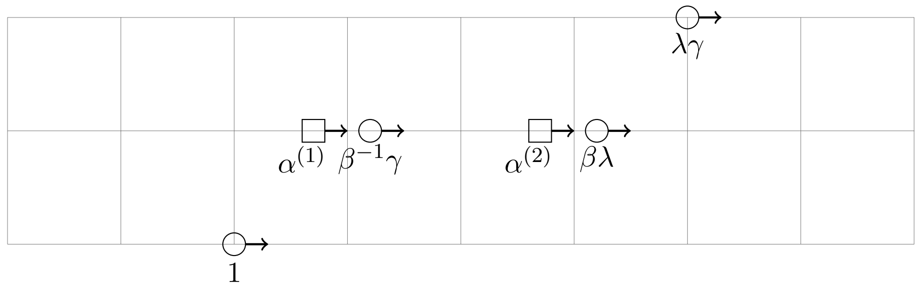
$$H^1_{\text{ét}}(K,\mathbb{Z}/p) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K),\mathbb{Z}/p) \cong \mathrm{Hom}(K^\times/(K^\times)^p,\mathbb{Z}/p)$$

corresponding to the unramified character sending Frobenius to 1. Let $\beta^{-1}\alpha^{(2)} \in H^1_{\text{ét}}(K,\mathbb{Z}/p)$ be a generator of the $\mathcal{O}_K/p$-module $\mathrm{Hom}(U_K/U_K^p,\mathbb{Z}/p)$. It follows that $\beta^{-1}\lambda\gamma \in H^2_{\text{ét}}(K,\mu_p)$ corresponds to the division algebra of invariant $\frac{1}{p}$ in the Brauer group.

The étale spectral sequence $E_2^{i,j} = H^i_{\text{ét}}(K,\mu_p^{\otimes j}) \Rightarrow \mathbb{K}^{\text{ét}}_{2j-i}(K,\mathbb{F}_p)$ may be pictured as follows, where a circle (resp. box) with two arrows means a $\mathbb{F}_p[\beta,\beta^{-1}]$-module (resp. $(\mathcal{O}_K/p)[\beta,\beta^{-1}]$-module) freely generated by the elements below it.



Using the Bloch-Kato conjecture proved by Voevodsky [14], the E_2-term of the motivic spectral sequence computing $\mathbb{K}_*(K;\mathbb{F}_p)$ may be identified with the part to the right of the red line of the étale spectral sequence:



One may show that λ generates the cokernel of the cyclotomic trace map

$$\mathbb{K}(\mathbb{F}_p;\mathbb{Z}_p) \to \mathrm{TC}(\mathbb{F}_p;\mathbb{Z}_p).$$

We thus see the similarity between the descent spectral sequence and motivic spectral sequence. We expect that, for certain algebraic varieties over $\mathcal{O}_K$, one would be able to construct some analogue of the motivic spectral sequence computing algebraic K-theory with $\mathbb{F}_p$-coefficients.

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