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A REDUCTION OF THE SEGAL CONJECTURE

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In this note, we show that the Segal conjecture holds for a given finite group G if it holds for all subgroups of G which have prime power order. We also point out that the p-group case reduces to a question about p-adic completions.

First we must say what we mean by the Segal conjecture. Various forms are current, but our methods apply equally well to any of them. Let π_G^* denote equivariant stable cohomotopy and recall that $\pi_G^0(pt)$ is canonically isomorphic to the Burnside ring $A(G)$ [7, 17]. In particular, $\pi_G^*(X)$ is a (Z-graded) module over $A(G)$. Let $I(G)$ denote the augmentation ideal of $A(G)$ and let $\hat{\pi}_G^*(X)$ denote the $I(G)$-adic completion of $\pi_G^*(X)$. Let EG be a free contractible G-space and let ε denote the map $EG \to pt$. As we shall recall in section 2, $\varepsilon^*: \pi_G^*(pt) \to \pi_G^*(EG)$ factors to give a homomorphism $\hat{\varepsilon}^*: \hat{\pi}_G^*(pt) \to \pi_G^*(EG)$, and the target here is isomorphic to the ordinary stable cohomotopy $\pi^*(BG)$. The Segal conjecture asserts that $\hat{\varepsilon}^*$ is an isomorphism. As Segal understood and we shall explain in section 2, this is equivalent to a more elaborate statement involving $RO(G)$-graded cohomology.

We shall prove the following results.

<u>THEOREM A</u>. The Segal conjecture is true for G if it is true for all subgroups of G having prime power order.

<u>PROPOSITION B</u>. The Segal conjecture is true for a p-group G if and only if $\varepsilon^*: \pi_G^*(pt) \to \pi^*(BG)$ induces an isomorphism on passage to p-adic completion.

If, as seems not unlikely, the Segal conjecture turns out to be true for Abelian p-groups but false in full generality, then Theorem A will imply the Segal conjecture for groups with Abelian p-Sylow subgroups.

In fact, we shall see that these reductions apply not just to equivariant cohomotopy but to the analogous completion map for

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quite general RO(G)-graded equivariant cohomology theories. For example, these reductions shed some light on Atiyah's original calculation of $K^*(BG)$ [2]. We shall emphasize this generic aspect of our work throughout.

Theorem A is actually an easy consequence of the following purely algebraic fact. It will be proven (and its undefined terms explained) in section 1. The application to cohomology theories will be discussed in section 2.

THEOREM C. The completed Burnside ring functor $\hat{\underline{A}}$ is a Green functor which satisfies induction with respect to its set of subgroups of prime power order.

This result sheds considerable light on the structure of $\hat{A}(G)$ and should have other uses.

We wish to acknowledge the earlier work of Laitinen [11, §1], which gave enough information to deduce the monomorphism part of the reduction of Theorem A, and of Madsen [15, §1], which led us to the idea of deducing Theorem A from Theorem C. Quite recently, Segal gave a different proof of Theorem A in a letter to Adams.

§1 INDUCTION FOR THE COMPLETED BURNSIDE RING

Let G be a finite group. We recall some terminology from [7] or [9]. A Mackey functor M consists of a covariant and a contravariant functor, with the same object function, from the category of finite G-sets to the category of Abelian groups. For a G-map $f: S \to T$, we think of the contravariant map $f^*: M(T) \to M(S)$ as restriction and the covariant map $f_*: M(S) \to M(T)$ as induction (or transfer). These are to be related in a suitable way, and the axioms imply that the entire Mackey functor is determined by its restriction to the full subcategory of orbits G/H. The category of Mackey functors admits a tensor product, and there is a resulting notion of a ring object, or Green functor.

In particular, there is a Green functor $\underline{A}$ whose value on the orbit G/H is the Burnside ring $A(H)$ of finite H-sets. A G-map $f: G/J \to G/K$ is given by a subconjugacy relation $gJg^{-1} \subset K$, and we use the letter i generically for inclusions $J \subset K$ (or π for the corresponding projection $G/J \to G/K$ of G-sets when we prefer to emphasize that point of view). Restriction $i^*: A(K) \to A(J)$ assigns to a K-set T the same set regarded as a J-set. Induction $i_*: A(J) \to A(K)$ assigns to a J-set S the K-set $K \times_J S$; i^* is a morphism of rings and i_* is a morphism of

A(J)-modules (Frobenius reciprocity).

There is a notion of a module Mackey functor over a Green functor, and the Green functor $\underline{A}$ acts universally.

LEMMA 1. Any Mackey functor has a natural structure of $\underline{A}$-module Mackey functor.

Let $\hat{A}(H)$ be the completion of $A(H)$ in the $I(H)$-adic topology. The restriction maps are evidently continuous and it will follow from Lemma 6 below that the induction maps are also continuous. It follows easily that $\hat{\underline{A}}$ inherits a structure of Green functor from $\underline{A}$. Similarly, any Mackey functor gives rise to an $\hat{\underline{A}}$-module Mackey functor upon completion.

Choose a p-Sylow subgroup G_p for each prime dividing the order of G (to be denoted $|G|$). Theorem C can then be restated as follows.

THEOREM 2. The sum $\sum_p i_*: \sum_p \hat{A}(G_p) \to \hat{A}(G)$ is an epimorphism.

By a basic result in induction theory [7; 9, §6], we have the following immediate consequence.

COROLLARY 3. Let M be an $\hat{\underline{A}}$-module Mackey functor and let $S = \coprod_p G/G_p$. Then the following sequence is exact:

$$0 \longrightarrow M(pt) \xrightarrow{\pi^*} M(S) \xrightarrow{\pi_1^* - \pi_2^*} M(S\times S) .$$

Here $\pi: S \to pt$ and $\pi_i: S \times S \to S$ are the evident projections.

Any Mackey functor converts disjoint unions to direct sums, and $S \times S$ can be written as a disjoint union of orbits G/H where the H are p-groups for varying primes p. Thus, by naturality and a comparison of exact sequences, the previous corollary implies the following one.

COROLLARY 4. Let $\sigma: M \to N$ be a morphism of $\hat{\underline{A}}$-module Mackey functors, for example the completion of any morphism of Mackey functors. If $\sigma: M(G/H) \to N(G/H)$ is an isomorphism for all subgroups H of G of prime power order, then $\sigma: M(pt) \to N(pt)$ is an isomorphism.

Of course, this is the result we shall use to prove Theorem A.

We now turn to the proof of Theorem 2, and we need some preliminary recollections and observations. Embed Z in $A(G)$ by sending $n > 0$ to the trivial n-pointed G-set. For $H \subset G$, define a ring homomorphism $\chi_H: A(G) \to Z$ by sending a G-set S to the cardinality of S^H. Then $I(G) = \text{Ker}\ \chi_e$, where e is the trivial group. Let $\hat{I}(G)$ be the completion of $I(G)$ in the

$I(G)$-adic topology. We have an obvious induced splitting $\hat{A}(G) = Z \oplus \hat{I}(G)$, and the natural composite

$$Z \subset \hat{A}(H) \xrightarrow{i_*} \hat{A}(G) \xrightarrow{\chi_e} Z$$

is multiplication by $|G/H|$. Since the greatest common divisor of the numbers $|G/G_p|$ is one, we see that it suffices to prove Theorem 2 with $\hat{A}$ replaced by $\hat{I}$.

Let $C(G)$ be the ring direct product of one copy of Z, denoted Z_H, for each conjugacy class (H) of subgroups of G and let $IC(G)$ be the ideal of elements with $e^{\underline{th}}$ coordinate zero. Let $\chi: A(G) \to C(G)$ be the ring homomorphism with $H^{\underline{th}}$ coordinate χ_H. Then χ is a monomorphism with finite cokernel, $|G|\cdot C(G)$ being contained in the image of χ [7, §1]. The following observation is essentially well-known and will imply Proposition B; compare [7, 4.1.1] and [11, 1.12].

<u>LEMMA 5</u>. Let M be a Mackey functor and let $\pi^*: M(pt) \to M(G)$ be induced by the projection $\pi: G \to pt$. Then $M(pt)$ is an $A(G)$-module and π^* is a morphism of $A(G)$-modules, where $A(G)$ acts on $M(G)$ through χ_e. Further, $|G|\cdot \mathrm{Ker}\ \pi^*$ is contained in $I(G)\cdot \mathrm{Ker}\ \pi^*$. If G is a p-group, then the p-adic topology and $I(G)$-adic topology on $\mathrm{Ker}\ \pi^*$ coincide.

<u>PROOF</u>. The first statement is part of Lemma 1, multiplication by a G-set S being the composite

$$M(pt) \xrightarrow{\pi^*} M(S) \xrightarrow{\pi_*} M(pt).$$

Taking $S = G$, we see immediately that $G\cdot \mathrm{Ker}\ \pi^* = 0$, hence

$$|G|\cdot \mathrm{Ker}\ \pi^* = (|G| - G)\cdot \mathrm{Ker}\ \pi^* \subset I(G)\cdot \mathrm{Ker}\ \pi^*.$$

Now let $|G| = p^n$. We obviously have $|G|^m\cdot \mathrm{Ker}\ \pi^* \subset I(G)^m\cdot \mathrm{Ker}\pi^*$. We claim that $I(G)^{n+1} \subset pI(G)$. This will imply that

$$I(G)^{m(n+1)}\cdot \mathrm{Ker}\ \pi^* \subset p^m\cdot \mathrm{Ker}\ \pi^*$$

and so complete the proof of the last statement. For $H \subset G$, $H \neq e$, and $K \subset G$, $\chi_H(G/K - |G/K|)$ is congruent to zero mod p since $G/K - (G/K)^H$ is a disjoint union of non-trivial H-orbits and thus has cardinality divisible by p. Therefore $\chi_H I(G) \subset pZ_H$, hence $\chi I(G) \subset pIC(G)$ and

$$\chi I(G)^{n+1} \subset p^{n+1}IC(G) \subset p\chi I(G).$$

Of course, the last statement fails if we replace $\mathrm{Ker}\ \pi^*$ by $M(pt)$. For example, $\hat{A}(G) = Z \oplus \hat{I}(G)$, whereas the completion of $A(G)$ at p would have Z replaced by its p-adic completion.

We shall need to know the prime ideal spectrum of $A(G)$ [8; 7, §1]. Let $q(H,0)$ be the kernel of χ_H and let $q(H,p)$ be the kernel of the composite of χ_H and reduction mod p. These are all of the prime ideals of $A(G)$, and the lattice of prime ideals is determined by the relations

$$q(H,0) \subset q(H,p),$$

$$q(H,0) = q(K,0) \quad \text{if } H \text{ is conjugate to } K,$$

and

$$q(H,p) = q(K,p) \quad \text{if } H^p \text{ is conjugate to } K^p,$$

where H^p is the smallest normal subgroup of H such that H/H^p is a p-group. Note in particular that $q(e,p) = q(H,p)$ if and only if H is a p-group. Write $q(H,p) = q(H,p;G)$ when necessary for clarity.

We need three lemmas. In all of them, we focus attention on a fixed given subgroup H of G. Via $i^*: A(G) \to A(H)$, any $A(H)$-module is an $A(G)$-module. Via $\chi: A(H) \to C(H)$, any $C(H)$-module is an $A(H)$-module.

<u>LEMMA 6</u>. The following topologies on $A(H)$ and $I(H)$ coincide.

(1) The $I(G)$-adic topology.

(2) The $I(H)$-adic topology.

(3) The subspace topology induced from the $I(H)$-adic topology on $C(H)$.

<u>PROOF</u>. The agreement of the first two topologies is due to Laitinen [11, 1.14]. Since $i^*I(G) \subset I(H)$, it is enough to show that $I(H)^n \subset i^*(IG)$ for some n, and this holds provided that any prime ideal of $A(H)$ which contains $i^*I(G)$ also contains $I(H)$. Since $(i^*)^{-1}q(K,p;H) = q(K,p;G)$ for $K \subset H$ and any p (including 0), because $\chi_K i^* = \chi_K$, this is a simple check of cases from the facts just recorded. The agreement of the last two topologies is a standard consequence of the Artin-Rees lemma [3, 10.11].

For any $A(H)$-module N and any $n \geq 1$, define

$$P_n(N,H) = P_nN = N/I(H)^nN.$$

Observe that we have induced homomorphisms

$$i^*: P_n(A(G),G) \to P_n(A(H),G) \quad \text{and} \quad i_*: P_n(A(H),G) \to P_n(A(G),G).$$

By the lemma, $P_n(A(H),G)$ is a quotient of some $P_m(A(H),H)$. Define

$$J^n(H) = \chi^{-1}(I(H)^nC(H)) \subset A(H)$$

and

$$Q_nN = N/J^n(H)N.$$

Since $I(H)^n$ is contained in $J^n(H)$, we have a natural epimorphism $P_nN \to Q_nN$, and this is evidently the identity when $N = C(H)$. Further, χ induces a monomorphism

$$Q_nA(H) \to Q_nC(H) = P_nC(H) = \prod_{(K)} P_nZ_K,$$

where Z_K is Z regarded as an $A(H)$-module via χ_K.

LEMMA 7. (i) If $K = e$, then $P_nZ_K = Z$ for all n.

(ii) If K is not a p-group for any p, then $P_nZ_K = 0$ for all n.

(iii) If K is a p-group, then P_nZ_K is a p-group for all n.

PROOF. For any prime p, the kernel of the composite

$$I(H) = q(e,0) \subset A(H) \xrightarrow{\chi_K} Z \longrightarrow Z_p = Z/pZ$$

is $q(K,p) \cap q(e,0)$, and $q(K,p)$ contains $q(e,0)$ if and only if K is e or a p-group. If $K = e$, the composite is always zero and $I(H)Z_e = 0$. If K is not a p-group for any p, the composite is always non-zero and therefore $I(H)Z_K = Z_K$. If K is a p-group, the composite is non-zero for all primes other than p and is zero for p, hence $I(H)Z_K = p^rZ_K$ for some $r \geq 1$.

LEMMA 8. The group $P_nI(H)$ is finite for all $n \geq 1$.

PROOF. The agreement of the last two topologies in Lemma 6 implies that $P_nI(H)$ is a quotient of $Q_mI(H)$ for some m. Since $Q_mI(H)$ injects into $P_mIC(H) = \prod_{(K)\neq e} P_mZ_K$ and the latter is finite by the previous lemma, the conclusion follows.

In view of the agreement of the first two topologies in Lemma 6, we have the following commutative diagram:

$$\begin{array}{ccc} \sum_p \hat{I}(G_p) & \xrightarrow{\sum_p i_*} & \hat{I}(G) \\ \wr\!\!\parallel & & \wr\!\!\parallel \\ \lim_n \sum_p P_n(I(G_p),G) & \xrightarrow{\lim_n \sum_p i_*} & \lim_n P_nI(G) \end{array}$$

Thus Theorem 2 will hold if the bottom arrow is an epimorphism. Since the groups $P_n(I(G_p),G)$ are finite, the usual $\lim^1$ exact sequence shows that it suffices to prove that each map

$$\sum_p i_*: \sum_p P_n(I(G_p),G) \to P_nI(G)$$

is an epimorphism. Again by finiteness, this will hold provided the $p^{\underline{th}}$ map i_* is surjective on p-primary components. We claim that the composite

$$P_nI(G) \xrightarrow{i^*} P_n(I(G_p),G) \xrightarrow{i_*} P_nI(G)$$

becomes an isomorphism when localized at p. This composite is multiplication by the G-set G/G_p regarded as an element of $A(G)$ or of its quotient ring $P_nA(G)$. By Lemma 6 again, the latter ring is a quotient of $Q_mA(G)$ for some m. Thus it suffices to check that G/G_p is a unit in $Q_mA(G)_{(p)}$. Since $Q_mC(G)/Q_mA(G)$ is finite, $Q_mC(G)_{(p)}$ is an integral extension of $Q_mA(G)_{(p)}$. It therefore suffices to check that G/G_p is a unit in $Q_mC(G)_{(p)}$; see e.g. [3, 5.10]. Lemma 7 shows that

$$Q_mC(G)_{(p)} = Z_{(p)} \times \prod_{(K)} Q_mZ_K,$$

where the product is restricted to the conjugacy classes of p-groups $K \subset G$. Since $q(K,p) = q(e,p)$ in $A(G)$ and since $\chi_e(G/G_p) = |G/G_p|$ is prime to p, we must have that $\chi_K(G/G_p)$ is also prime to p. Thus G/G_p is a unit in $Q_mC(G)_{(p)}$. This completes the proof of our claim and thus the proof of Theorem 2.

§2. APPLICATIONS TO EQUIVARIANT COHOMOLOGY THEORIES

Let k_G^* be an RO(G)-graded cohomology theory on G-spaces Y. Thus we are given groups $k_G^\alpha Y$ for $\alpha \in RO(G)$ such that the $k_G^{\alpha+n}Y$ for fixed α and varying $n \in Z \subset RO(G)$ comprise a Z-graded cohomology theory and there are coherent natural isomorphisms

$$\tilde{k}_G^\alpha(X) \simeq \tilde{k}_G^{\alpha+V}(X \wedge S^V)$$

for based G-spaces X (with G-fixed basepoint), where S^V denotes the one-point compactification of a representation V. Such theories were introduced by Segal [17] and have been studied by Kosniowski [10] and others. Ordinary RO(G)-graded theories were introduced in [12], and a comprehensive treatment will appear in [14]. Some of our details here will be more intuitive than precise, and we will take for granted various facts from [13] and [14] about equivariant spectra and cohomology theories.

Our reason for considering RO(G)-graded theories is that the following result is false for general Z-graded theories. Let Y^+ denote the union of a G-space Y and a disjoint G-fixed basepoint.

PROPOSITION 9. Let k_G^* be an RO(G)-graded cohomology theory, let $\alpha \in RO(G)$, and let Y be a G-space. Then the correspondence $G/H \to k_G^\alpha(G/H \times Y)$ determines the object function of a Mackey functor, and this Mackey functor structure is functorial in Y.

PROOF. This is folklore; compare [5] and [10]. We give a sketch of our favorite argument. There is an equivariant stable category of G-spectra, constructed by Lewis and May [13], and there is a functor Σ^∞ from based G-spaces to G-spectra. The theory k_G^* is represented by a G-spectrum k_G. More precisely, k_G represents an RO(G)-graded cohomology theory on G-spectra such that

$$k_G^*(\Sigma^\infty X) = \tilde{k}_G^*(X) \quad \text{and} \quad \tilde{k}_G^*(Y^+) = k_G^*(Y).$$

Let $\mathcal{O}$ be the full subcategory of the stable category whose objects are the G-spectra $\Sigma^\infty(G/H)^+$ for $H \subset G$. As we observed in work with Lewis [12, 14], a Mackey functor determines and is determined by an additive contravariant functor $\mathcal{O} \to \mathrm{Ab}$. With this homotopical interpretation of the algebraic notion of a Mackey functor, the conclusion becomes tautologically obvious. We have identifications of G-spectra

$$\Sigma^\infty((G/H^+ \wedge X) \simeq \Sigma^\infty(G/H)^+ \wedge X \simeq \Sigma^\infty(G/H)^+ \wedge \Sigma^\infty X.$$

For any G-spectrum E, such as $\Sigma^\infty Y^+$, the Abelian groups $k_G^\alpha(\Sigma^\infty(G/H)^+ \wedge E)$ and homomorphisms $(f \wedge 1)^*$ for morphisms $f \in \mathcal{O}$ specify an additive contravariant functor $\mathcal{O} \to \mathrm{Ab}$.

For $H \subset G$ with inclusion i, a G-spectrum k_G determines an H-spectrum $k_H = i^* k_G$ and thus an RO(H)-graded cohomology theory k_H^*. The precise definition implies

$$k_H^{i^*\alpha}(Y) = k_G^\alpha(G \times_H Y)$$

for $\alpha \in RO(G)$ and an H-space Y. We abbreviate $k_e^* = k^*$, this being the underlying nonequivariant cohomology theory associated to k_G^*. In practice, as for K-theory and stable cohomotopy, we are given an RO(G)-graded cohomology theory k_G^* for every G and must check that $k_H^* = (i^* k_G)^*$. We shall need a bit of extra structure; compare [10, §2].

DEFINITION 10. A cohomology theory k_G^* is said to be split if there is a morphism of Z-graded nonequivariant cohomology theories $\zeta\colon k^*Y \to k_G^*Y$ (where spaces Y are given trivial G-action on the right) such that the composite

$$k^*Y \xrightarrow{\ \zeta\ } k_G^*Y \xrightarrow{\ \pi^*\ } k_G^*(G \times Y) = k^*Y,$$

$\pi\colon G \times Y \to Y$, is a natural isomorphism. Of course, since π^* is also a morphism of cohomology theories, this will hold provided that the composite is an isomorphism when Y is a point. We say that k_G^* is a split ring theory if k_G^* is ring-valued (from which it follows that all k_H^* for $H \subset G$ are ring-valued) and

ζ is a morphism of ring valued cohomology theories.

REMARK 11. In terms of spectra, a G-spectrum k_G determines a nonequivariant fixed point spectrum $(k_G)^G$ which represents the Z-graded theory k_G^*Y on spaces Y with trivial G-action. There is a natural map $(k_G)^G \to k$ which represents π^*, and the definition requires this map to be a retraction in the stable category. If k_G^* is ring-valued, then, ignoring $\lim^1$ questions, k_G is a ring G-spectrum and induces ring structures on the k_H and on $(k_G)^G$ such that $(k_G)^G \to k$ is a ring map. The last part of the definition requires a ring map $k \to (k_G)^G$ such that the composite $k \to (k_G)^G \to k$ is the identity.

Let S_G be the 0-sphere G-spectrum, so that $S_G^* = \pi_G^*$. Then S_G^* is a split ring theory, the unit $S \to (S_G)^G$ providing the required splitting map.

One reason for introducing split theories is the following observation, which is due to Kosniowski [10, p. 92].

LEMMA 12. If k_G^* is a split cohomology theory, then the composite

$$k^*(Y/G) \xrightarrow{\zeta} k_G^*(Y/G) \xrightarrow{\pi^*} k_G^*(Y),$$

$\pi: Y \to Y/G$, is an isomorphism for all free G-CW complexes Y.

PROOF. We have assumed the result for G, and it follows by suspension that the evident reduced analog holds for $G^+ \wedge S^n$ for all $n \geq 1$. By induction and the five lemma, the result holds for the skeleta of Y. By the $\lim^1$ exact sequence, it holds for Y.

With these preliminaries, we return to the study of completions. We say that the completion conjecture holds for the theory k_G^* if $\varepsilon^*: k_G^n(pt) \to k_G^n(EG)$ induces an isomorphism on passage to $I(G)$-adic completion for all integers n, $\varepsilon: EG \to pt$. There are theories for which this is false; it is true for real and complex K-theory by Atiyah and Segal [4]. Corollary 4 immediately implies the following reduction, which in fact applies separately to each grading n.

THEOREM 13. The completion conjecture holds for the theory k_G^* if it holds for the theories k_H^* for all subgroups H of prime power order.

Lemma 5 leads to the following further reduction.

PROPOSITION 14. If G is a p-group and k_G^* is split, then the completion conjecture holds for k_G^* if and only if $\varepsilon^*: k_G^n(pt) \to k_G^n(EG)$ induces an isomorphism upon passage to p-adic completion for all integers n.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & k_G^*(pt) & \xrightarrow{\pi^*} & k^*(pt) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varepsilon^*} & & \downarrow{\scriptstyle \varepsilon^*} & & \cong\downarrow{\scriptstyle \varepsilon^*} & & \\ 0 & \longrightarrow & L & \longrightarrow & k_G^*(EG) & \xrightarrow{\pi^*} & k^*(EG) & \longrightarrow & 0 \end{array}$$

where K and L denote the respective kernels. By assumption, the top row is split exact, and there results a compatible splitting of the bottom row. By Lemma 5, this is a diagram of $A(G)$-modules, where $A(G)$ acts trivially on $k^*(pt)$ and $k^*(EG)$, and the $I(G)$-adic and p-adic topologies on K and on L coincide. The conclusion follows.

The interest in the completion conjecture lies in the algebraic computability of $k_G^*(pt)$ and the homotopical interest of $k_G^*(EG)$. We quickly review the latter, following Atiyah and Segal [4]. We assume that k_G^* is a split ring theory. Then, by the proof of Proposition 9, $A(G)$ acts on k_G^*Y by pullback of the natural $k_G^0(pt)$-module structure along the unit

$$\eta_*: A(G) = \pi_0^G(S_G) \to \pi_0^G(k_G) = k_G^0(pt).$$

By Lemma 12, we have an isomorphism of Z-graded rings

$$k_G^*(EG) \cong k^*(BG).$$

Let B^qG be the q-skeleton of BG. It is the union of $q+1$ contractible subsets, hence all $(q+1)$-fold products are zero in $\tilde{k}^*(B^qG)$, hence the composite

$$k_G^*(pt) \to k_G^*(EG) \cong k^*(BG) \to k^*(B^qG)$$

factors through $k_G^*(pt)/I(G)^{q+1}k_G^*(pt)$. Passing to limits, we obtain $\hat{k}_G^*(pt) \to \lim_q k^*(B^qG)$. If $\lim_q{}^1 k^*(B^qG) = 0$, then the target here is $k^*(BG)$, and this is complete. Thus the completion conjecture asserts that

$$\hat{\varepsilon}^*: \hat{k}_G^*(pt) \to k^*(BG)$$

is an isomorphism. It is this form of the assertion that motivated our original Z-graded formulation. However, there is an easy generalization to an $RO(G)$-graded formulation. Recall that EG/H is a model for BH for any $H \subset G$.

PROPOSITION 15. Let k_G^* be an $RO(G)$-graded split ring theory such that each $k_G^\alpha(pt)$ is a finitely generated $A(G)$-module. Assume that $\lim_q{}^1 k^n(B^qH) = 0$ and the completion map

$$\hat{\varepsilon}^*: \hat{k}_H^n(pt) \to k_H^n(EG) \cong k^n(BH)$$

is an isomorphism for all integers n and subgroups H of G. Then, for any finite G-CW complex Y, the projection $\varepsilon\colon Y \times EG \to Y$ induces an isomorphism

$$\hat{\varepsilon}^*\colon \hat{k}_G^{\alpha}(Y) \to k_G^{\alpha}(Y \times EG)$$

for all $\alpha \in RO(G)$.

PROOF. Rather than consider $\lim^1$ terms, we observe that the following diagram commutes and define $\hat{\varepsilon}^*$ to be the displayed composite extension of ε^*:

$$\begin{array}{ccccc}
k_G^{\alpha}(Y) & \xrightarrow{\qquad\qquad\qquad\varepsilon^*\qquad\qquad\qquad} & & & k_G^{\alpha}(Y \times EG) \\
\downarrow & & & & \uparrow \phi \\
\hat{k}_G^{\alpha}(Y) \cong k_G^{\alpha}(Y) \otimes_{A(G)} \hat{A}(G) & \xrightarrow{1 \otimes \hat{\eta}} & k_G^{\alpha}(Y) \otimes_{A(G)} \hat{k}_G^0(pt) & \xrightarrow{1 \otimes \hat{\varepsilon}^*} & k_G^{\alpha}(Y) \otimes_{A(G)} k_G^0(EG)
\end{array}$$

Here ϕ is the external product. Our finiteness assumptions ensure that $\hat{k}_G^{\alpha}(Y)$ is finitely $A(G)$-generated and so give the unlabeled isomorphism [3, 10.13], and of course $\hat{A}(G)$ is $A(G)$-flat [3, 10.14]. Thus $\hat{k}_G^*$ is an $RO(G)$-graded cohomology theory on finite G-CW complexes and is represented by a G-spectrum $\hat{k}_G$; $k_G^*(? \times EG)$ is also such a theory and is represented by the function G-spectrum $F(EG^+, k_G)$. Since $\hat{\varepsilon}^*$ is a morphism of cohomology theories, it is represented by a map of G-spectra $\hat{\varepsilon}\colon \hat{k}_G \to F(EG^+, k_G)$. By hypothesis (and Lemma 6), $\hat{\varepsilon}^*$ is an isomorphism when $Y = G/H$ and $\alpha = n$. This means that $\hat{\varepsilon}$ induces an isomorphism on equivariant homotopy groups where, for a G-spectrum k_G, $\pi_n^H(k_G) = k_H^{-n}(pt)$. By the Whitehead theorem in the equivariant stable category [13], it follows that $\hat{\varepsilon}$ is an isomorphism in that category and so induces an isomorphism of cohomology theories.

The cited Whitehead theorem asserts that a map $k_G \to k_G'$ of G-spectra is an isomorphism in the equivariant stable category if and only if its fixed point maps $(k_G)^H \to (k_G')^H$ are isomorphisms in the nonequivariant stable category, and, as one would expect, $(k_G)^H = (k_H)^H$.

Returning to the situation of the proposition, the isomorphism $k_G^*(EG) \cong k^*(BG)$ of Lemma 12 implies an isomorphism

$$F(EG^+, k_G)^G \simeq F(BG^+, k)$$

in the stable category. Thus the completion conjecture asserts that

$$\hat{\varepsilon}\colon (\hat{k}_G)^G \to F(BG^+, k)$$

is an isomorphism in the stable category. If G is a p-group, we may replace $I(G)$-adic completion by p-adic completion provided we also p-adically complete the canonical sphere wedge summand of $F(BG^+,k)$, $F(BG,k)$ already being p-complete. Since passage to fixed point spectra commutes with p-adic completion, the completion conjecture here asserts that

$$\varepsilon: (k_G)^G \to F(BG^+,k)$$

becomes an isomorphism in the stable category upon completion at p.

In the case $k_G = S_G$, there is an equivalence of spectra

$$\bigvee_{(H)} \xi_H: \bigvee_{(H)} \Sigma^\infty BWH^+ \to (S_G)^G,$$

where $WH = N_GH/H$ and the wedge is taken over the conjugacy classes of subgroups H of G; see Segal [17], Kosniowski [10], and tom Dieck [6]. In a sequel, we shall verify that the composite $\varepsilon \cdot \xi_H: \Sigma^\infty BWH^+ \to F(BG^+,S)$ has adjoint

$$\tau(1): \Sigma^\infty BWH^+ \wedge BG^+ = \Sigma^\infty(BWH \times BG)^+ \to S,$$

where τ is the transfer associated to the natural cover $E \to BWH \times BG$ with fibre G/H and $1 \in \pi^0 E$ is the unit. This will recover the formulation of the Segal conjecture preferred by those engaged in its study by Adams spectral sequence techniques. We refer the reader to Adams [1] for a summary of work in that direction.

We close with the homology analog of Lemma 12, which will be needed in the sequel.

<u>LEMMA 16</u>. If k_G^* is a split cohomology theory, then the composite

$$k_*(Y/G) \xrightarrow{\zeta} k_*^G(Y/G) \xrightarrow{\tau} k_*^G(Y)$$

is an isomorphism for all free G-CW complexes Y, where τ is the equivariant transfer associated to $\pi: Y \to Y/G$.

<u>PROOF</u>. Here ζ is induced by $k \to (k_G)^G$; see Remark 11. The map π is a G-fibration, in fact a (G,A)-bundle where A is the group of automorphisms of G regarded as a discrete set. The transfer is induced by a map of G-spectra $\tau: \Sigma^\infty(Y/G)^+ \to \Sigma^\infty Y^+$. It must not be confused with the obvious nonequivariant transfer associated to π, which is induced by the map of nonequivariant spectra obtained from τ by neglect of G action. See [16, 18, 13]. There is a relative equivariant transfer compatible with connecting homomorphisms [13], and, as in Lemma 12, it suffices to prove the result for $Y = G$. Here the conclusion holds by hypothesis since we have commutative diagrams

$$\begin{array}{ccc} k_n^G(pt) & \xrightarrow{\tau} & k_n^G(G), \\ \| & & \downarrow \delta \\ k_G^{-n}(pt) & \xrightarrow{\pi^*} & k_G^{-n}(G) \end{array}$$

where δ is an equivariant Spanier-Whitehead duality isomorphism [19, 13, 14].

BIBLIOGRAPHY

1. J. F. Adams. Graeme Segal's burnside ring conjecture. Bull. Amer. Math. Soc. To appear.

2. M. F. Atiyah. Characters and cohomology of finite groups. Inst. Hautes Etudes Sci. Publ. Math. No. 9 (1961), 23-64.

3. M. F. Atiyah and J. G. Macdonald. Introduction to commutative algebra. Addison-Wesley Publishing Co. Reading, Mass. 1969.

4. M. F. Atiyah and G. B. Segal. Equivariant K-theory and completion. J. Diff. Geometry 3 (1969), 1-18.

5. T. tom Dieck. Equivariant homology and Mackey functors. Math. Ann. 206 (1973), 67-78.

6. T. tom Dieck. Orbittypen und aquivariante Homologie II. Arch. Math 26 (1975), 650-662.

7. T. tom Dieck. Transformation groups and representation theory. Springer Lecture Notes in Mathematics Vol 766. 1979.

8. A. Dress. A characterization of solvable groups. Math. Z. 110 (1969), 213-217.

9. A. Dress. Contributions to the theory of induced representations. Springer Lecture Notes in Mathematics Vol 342, 1973, pp. 183-240.

10. C. Kosniowski. Equivariant cohomology and stable cohomotopy. Math. Ann. 210 (1974), 83-104.

11. E. Laitinen. On the Burnside ring and stable cohomotopy of a finite group. Math. Scand. 44 (1979), 37-72.

12. L. G. Lewis, J. P. May, and J. E. McClure. Ordinary RO(G)-graded cohomology. Bull. Amer. Math. Soc. 4 (1981), 208-212.

13. L. G. Lewis, J. P. May, J. E. McClure, and M. Steinberger. Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics. To appear.

14. L. G. Lewis, J. P. May, J. E. McClure, and S. Waner. Equivariant cohomology theory. To appear.

15. I. Madsen. Smooth spherical space forms. Springer Lecture notes in Mathematics Vol. 657, 1978, 303-352.

16. G. Nishida. The transfer homomorphism in equivariant generalized cohomology theories. J. Math Kyoto 18(1978), 435-451.

17. G. Segal. Equivariant stable homotopy theory. Actes Congrès Intern. Math. 1970. Tome 2, 59-63.

18. S. Waner. Equivariant fibrations and transfer. Trans. Amer. Math. Soc. 258 (1980), 369-384.

19. K. Wirthmuller. Equivariant S-duality. Arch. Math. 26 (1975), 427-431.

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