ON THE TOPOLOGICAL HOCHSCHILD HOMOLOGY OF *bu*, I

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1. Introduction. The purpose of this paper and its sequel is to determine the homotopy groups of the spectrum $THH(\ell)$. Here p is an odd prime, ℓ is the Adams summand of p-local connective K-theory (see for example [1]) and THH is the topological Hochschild homology construction introduced by Bökstedt in [6]. In the present paper we will determine the mod p homotopy groups of $THH(\ell)$ and also the homotopy type of $THH(L)$ (where L denotes the periodic Adams summand). In the sequel we will investigate the integral homotopy groups of $THH(\ell)$ using our present results as a starting point.

The THH construction appears to be of basic importance in algebraic K-theory because it combines two useful properties: it can be used to construct good approximations to the algebraic K-theory functor, and it is very accessible to calculation. We shall review what is known about the first property in a moment; the second property was demonstrated by Bökstedt's calculation, in his paper [7], of the homotopy groups of $THH(H\mathbf{Z}/p)$ and $THH(H\mathbf{Z})$ (here $H\mathbf{Z}/p$ and $H\mathbf{Z}$ denote the evident Eilenberg-Mac Lane spectra). It is natural to ask about $THH(R)$ for other popular ring spectra R, and our work is a first step in this direction. We pay special attention to the connective case because this is the case which is likely to be relevant in applications (see Subsection 1.4 below).

The calculation which we present in this paper is a homotopy-theoretic one which uses the Adams spectral sequence. This calculation has several interesting features; in particular it is a pleasing example of an Adams spectral sequence calculation in which, although there are infinitely many differentials, it is still possible to get the complete answer.

Here is a summary of the contents of the paper. In Section 2 we review the facts we need to know about ordinary Hochschild homology. In Section 3 we do the same for topological Hochschild homology. In Section 4 we calculate the mod p homology of $THH(\ell)$ and use it to find the E_2 term of the Adams spectral sequence converging to $\pi_*(THH(\ell); \mathbf{Z}/p)$. This section also contains a quick calculation, which was pointed out to us by Larry Smith and Andy Baker,

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of the homotopy groups of $THH(MU)$ and $THH(BP)$, where MU is the complex cobordism spectrum and BP is the Brown-Peterson summand of MU (but we must assume for the latter calculation that BP has an E_∞ structure). In Section 5 we calculate the mod p K-theory of $THH(\ell)$ and use it to determine the "v_1-inverted" homotopy of $THH(\ell)$. In Section 6 we work backwards from this result to determine the behavior of the v_1-inverted Adams spectral sequence for $THH(\ell)$. In Section 7 we show that the behavior of the v_1-inverted Adams spectral sequence completely determines that of the Adams spectral sequence itself, thereby completing the calculation of $\pi_*(THH(\ell); \mathbf{Z}/p)$. In Section 8, which depends only on Sections 2, 3, and 5, we determine the homotopy type of $THH(L)$. In Section 9 we confess that our definition of the spectrum ℓ is not the usual one; on the other hand we show that it agrees with the usual one up to p-adic completion. Our definition has the advantage that it provides an E_∞ structure for ℓ; this implies that ℓ has an A_∞ structure, which is necessary in order for $THH(\ell)$ to be defined, and it also provides extra structure for $THH(\ell)$ which will be used in the sequel to determine differentials and extensions in the Adams spectral sequence converging to $\pi_*THH(\ell)$.

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In the remainder of the introduction we shall give a short summary of some things which are known or suspected about algebraic K-theory; these provide motivation for the THH construction, but none of what follows will actually be used in our work.

1.1. The Dennis trace map. The simplest way in which Hochschild homology is related to algebraic K-theory is via the Dennis trace map, which is a natural transformation

$$\tau : K_*S \to \mathbf{HH}_*S;$$

here S is a discrete ring and $\mathbf{HH}_*S$ denotes ordinary Hochschild homology. (See [20, pages 106–114] or [17, Section II.1] for the construction of τ). Unfortunately, this map does not usually give much information about $K_*(S)$, (although it can be useful for low-dimensional calculations; see [13]). It is, however, possible to improve τ by factoring it through one of the variants of cyclic homology: that is, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{HC}^-_*S \arrow[dr, "\pi"] & \\
K_*S \arrow[u, "\alpha"] \arrow[r, "\tau"] & \mathbf{HH}_*S
\end{tikzcd}$$

(see [17, page 364] for the definitions of $\mathbf{HC}^-$ and π and [17, Section 2.3] for the definition of α). The following basic theorem, due to Goodwillie [17, Theorem II.3.4], says that the map α can be used to calculate *rationalized* relative algebraic K-theory in certain situations.

THEOREM 1.1. *If $S_1 \to S_2$ is a surjection with nilpotent kernel then*

$$\alpha \otimes \mathbf{Q} : K_*(S_1 \to S_2) \otimes \mathbf{Q} \to \mathbf{HC}_*^-(S_1 \otimes \mathbf{Q} \to S_2 \otimes \mathbf{Q})$$

is an isomorphism.

See [17, pages 365 and 373] for the definitions of $\mathbf{HC}^-$ and α in the relative situation.

The most important application of Theorem 1.1 is to Waldhausen's functor $A(X)$. For this, one needs to generalize Theorem 1.1 to apply to simplicial rings S. This can be done (see [17]), and in this generality the hypothesis of Theorem 1.1 is replaced by the much less stringent hypothesis that the map

$$\pi_0 S_1 \to \pi_0 S_2$$

be a surjection with nilpotent kernel (see [17]). Now given a space X, it is easy to construct a simplicial ring whose K-theory agrees rationally with $A(X)$, and thus Theorem 1.1 can be applied to calculate $\pi_* A(X \to Y) \otimes \mathbf{Q}$ whenever $X \to Y$ is a 2-connected map (see [17, pages 348–349]).

1.2. Algebraic K-theory of ring spectra. The reason for introducing topological Hochschild homology is to try to formulate and prove an analog of Theorem 1.1 which holds *integrally* and not just rationally. One can get a hint as to how to do this by recalling that one of the basic principles of Waldhausen's work on algebraic K-theory is that the K-functor should be applied not just to rings but to ring *spectra* (also called "brave new rings"). Waldhausen gave a sketch of how to do this in [35], and a precise construction was given by the combined work of May, Steinberger, and Steiner (see [25], [33] and [34], and also [31]). For technical reasons one must restrict to A_∞ ring spectra, but in practice this is not an inconvenience. We shall refer to this functor as Waldhausen K-theory and denote it by K^W; when R is an A_∞ ring spectrum, $K^W(R)$ is a spectrum whose homotopy groups will be denoted by $K_*^W(R)$. The functor K_*^W generalizes both K_* and $A(X)$, for when R is the Eilenberg-Mac Lane spectrum HS associated to a discrete ring S one has the equation

(1) $$K_*^W HS = K_* S;$$

and when R is the sphere spectrum S^0, or more generally the suspension spectrum $\Sigma^\infty(\Omega X)_+$, one has

$$K^W(S^0) = A(*)$$

and

$$K^W(\Sigma^\infty(\Omega X)_+) = A(X)$$

(here $(\Omega X)_+$ denotes the space obtained by adding a disjoint basepoint to the loop space of X).

1.3. Topological Hochschild homology. In view of what has been said so far, it is natural to try to approximate $K_*^W R$ by means of a Hochschild homology construction which can be applied to A_∞ ring spectra R. This is what topological Hochschild homology $THH(R)$ is. It is clear enough in principle how one should construct $THH(R)$ (see Section 3), although the technical details are quite complicated (see [6] and [15]). $THH(R)$ is a spectrum and we shall denote its homotopy $\pi_* THH(R)$ by $\mathbf{THH}_*(R)$.

There is a natural transformation

$$\tau' : K_*^W R \to \mathbf{THH}_* R$$

which is analogous to the Dennis trace map. (See [6, Section 2] for the construction of τ').

In the special case $R = HS$ it is important to note that the analog of equation (1) does *not* hold for $\mathbf{THH}_*$; that is, it is not true that $\mathbf{THH}_*(HS)$ agrees with $\mathbf{HH}_*(S)$ for a discrete ring S. Instead, there is a commutative diagram which shows that τ' gives a second way of lifting the Dennis trace map:

$$\begin{tikzcd} K_*S \arrow[r, "\tau"] \arrow[d, "\tau'"'] & \mathbf{HH}_*S \\ \mathbf{THH}_*HS \arrow[ur, "\phi"'] & \end{tikzcd}$$

(See Remark 3.5 for a hint about the construction of the map ϕ). This diagram shows that τ' can detect elements of K_*S which are not detected by τ. For example, if $S = \mathbf{Z}$ then τ must be zero in all positive dimensions (because $\mathbf{HH}_*\mathbf{Z}$ is zero in all positive dimensions), but Bökstedt has shown that τ' is nonzero in infinitely many dimensions (see [8]); more precisely, what he shows is that for each prime p the localization of τ' at p is nontrivial (and epimorphic) in dimension $2p - 1$.

In the cases $R = S^0$ and $R = \Sigma^\infty(\Omega X)_+$ mentioned above one can give explicit descriptions of $THH(R)$:

$$THH(S^0) = S^0$$

and

$$THH(\Sigma^\infty(\Omega X)_+) = \Sigma^\infty(\Lambda X)_+,$$

where Λ denotes the free loop space; the first equation is obvious from the definition in Section 3 and the second follows from that definition and [21, Theorem 6.2].

Probably the most important fact about τ' is that it can be identified with the map from K^W to its first Goodwillie derivative; more precisely we mean the derivative "at $X = S^0$" of the functor

$$X \mapsto K^W(R \wedge (\Omega X)_+)$$

from pointed spaces to spectra[1] (see [18] for the definition of the derivative and the proof of this fact in the special case $R = S^0$). This fact is significant in two ways: it implies that THH is a "first order" approximation to K^W in much the same way that stable homotopy is a first order approximation to unstable homotopy, and it can be used to obtain a "higher order" approximation, as we explain in the next subsection.

1.4. Topological cyclic and epicyclic homology. The next step is to consider functors which combine the desirable properties of $\mathbf{HC}^-$ and THH. For example, one can define topological *cyclic* homology THC^- by observing that the spectrum $THH(R)$ has a natural cyclic structure and therefore has an S^1 action (at least if everything works as in the category of spaces— cf. [21] and [15]), and letting

$$\mathbf{THC}_*^- R = \pi_* THH(R)^{hS^1},$$

where hS^1 denotes the homotopy fixed-point spectrum (cf. Remark 3.5). Unfortunately it is known that this functor cannot satisfy an integral version of Theorem 1.1.[2] On the other hand, there is considerable evidence for the following conjecture

[1]Lecture on "Brave new rings" by F. Waldhausen at Northwestern University, Springer 1988.
[2]Letter from T. G. Goodwillie to F. Waldhausen, August 10 1987.

Conjecture 1.2. It is possible to construct a functor THE, related to THH and THC^{-}, and a natural transformation

$$\tau'' : K^W(R) \to THE(R)$$

which induces an equivalence of derivatives.

The notation THE stands for "topological epicyclic homology".[3] Goodwillie has recently given a specific candidate for THE which is closely related to the functor defined in [9] (but we warn the reader that Goodwillie uses the notation TC for his functor instead of THE) and he has proved the conjecture in the special case $R = \Sigma^\infty(\Omega X)_+$.

If the conjecture is true then the calculus of functors will imply the following integral version of Theorem 1.1: the induced map

$$\tau'' : K_*^W(R_1 \to R_2) \to \mathbf{THE}_*(R_1 \to R_2)$$

is an isomorphism for any map $R_1 \to R_2$ of A_∞ ring spectra such that

$$\pi_i R_1 \to \pi_i R_2$$

is an isomorphism for $i \leq 0$.

This explains the statement we made earlier that connective spectra are of particular importance for the potential applications.

We conclude with one further remark about the potential applications of THH. In [36], Waldhausen has proposed an interesting program for studying the relative K^W theory of the map

$$S^0 \to H\mathbf{Z}$$

by means of the intermediate spectra $K^W(L_n(S^0))$ and $K^W(L_n(S^0)_c)$, where $L_n(S^0)$ denotes the L_n-localization of the sphere (see [27]) and $L_n(S^0)_c$ is the associated connective spectrum. When $n = 1$ the spectrum $L_1(S^0)_c$ is the connective image-of-J spectrum j. It seems likely that the results and methods of our work are a good way to obtain information about $THH(j)$.

2. A brief review of Hochschild homology. In this section we recall the facts we need about ordinary (algebraic) Hochschild homology. Our basic reference for this subject is [12, Chapters IX and X].

[3]Letter from T. G. Goodwillie to F. Waldhausen, August 10 1987.

If S is a graded algebra over a ground field k, its Hochschild homology $\mathbf{HH}_*(S)$ is defined to be the homology of the *Hochschild complex* [12, page 175]

$$\begin{array}{c} \vdots \\ \downarrow \\ S \otimes S \otimes S \\ \downarrow \\ S \otimes S \\ \downarrow \\ S, \end{array} \tag{2}$$

in which the differential is given by the formula

$$\begin{aligned} d(t_0 \otimes \cdots \otimes t_n) = & \sum_{i=0}^{n-1} (-1)^i t_0 \otimes \cdots \otimes t_i t_{i+1} \otimes \cdots \otimes t_n \\ & + (-1)^n (-1)^{|t_n|(|t_0|+\cdots+|t_{n-1}|)} t_n t_0 \otimes t_1 \otimes \cdots \otimes t_{n-1}. \end{aligned}$$

As one might expect, $\mathbf{HH}_*(S)$ can also be described in terms of Tor; it is

$$\mathrm{Tor}^{S \otimes S^{\mathrm{op}}}(S, S),$$

where the first factor of $S \otimes S^{\mathrm{op}}$ acts on S by multiplication on the left and the second factor by multiplication on the right [12, page 169]. The reader may perhaps wonder why one uses this definition for the homology of S instead of the "obvious" definition $\mathrm{Tor}^S(k, k)$. For our purposes, the answer is that the latter is the appropriate definition for the category of *augmented* algebras, but we need to work more generally; the functor $\mathbf{HH}_*(S)$ is closely related to $\mathrm{Tor}^S(k, k)$, but it is defined for arbitrary algebras S.

There is an evident natural map

$$\iota : S \to \mathbf{HH}_0(S),$$

which is an isomorphism when S is commutative. There is also a "suspension" map

$$\sigma : S \to \mathbf{HH}_1(S)$$

which takes $t \in S$ to the class of $1 \otimes t$. If S is commutative there is a product

$$\mathbf{HH}_i(S) \otimes \mathbf{HH}_j(S) \to \mathbf{HH}_{i+j}(S),$$

which gives $\mathbf{HH}_*(S)$ the structure of a commutative graded S-algebra (see [12, page 217]); moreover ι is a ring homomorphism and σ is a derivation:

$$\sigma(st) = s\sigma(t) + (-1)^{|s||t|} t\sigma(s). \tag{3}$$

It will not surprise the reader to find that there are times when we actually need to compute the ring $\mathbf{HH}_*(S)$. The following result is sufficient for our purposes.

PROPOSITION 2.1. *If S has the form*

$$\mathbf{Z}/p\,[x_1, x_2, \ldots] \otimes \Lambda(y_1, y_2, \ldots),$$

then $\mathbf{HH}_*(S)$ has the form

$$S \otimes \Lambda(\sigma(x_1), \sigma(x_2), \ldots) \otimes \Gamma(\sigma(y_1), \sigma(y_2), \ldots),$$

where the inclusion of the first factor is the natural map

$$\iota : S \xrightarrow{\cong} \mathbf{HH}_0(S)$$

and σ is the suspension map

$$S \to \mathbf{HH}_1(S).$$

Proof. Let

$$\varphi : S \to S \otimes S$$

be the ring map which takes x_i and y_j to

$$x_i \otimes 1 - 1 \otimes x_i$$

and

$$y_j \otimes 1 - 1 \otimes y_j,$$

respectively. By [12, Theorem X.6.1], φ induces an isomorphism

$$\mathrm{Tor}^S(S_\varphi, \mathbf{Z}/p) \to \mathrm{Tor}^{S \otimes S}(S, S) = \mathbf{HH}_*(S),$$

where S_φ denotes the S-module structure on S obtained by pulling back its $S \otimes S$-module structure along φ. In our case S is commutative, so that S_φ has the trivial

S-module structure, and we conclude that there is an isomorphism

$$S \otimes \mathrm{Tor}^S(\mathbf{Z}/p, \mathbf{Z}/p) \cong \mathbf{HH}_*(S). \tag{4}$$

But it is well known that

$$\mathrm{Tor}^S(\mathbf{Z}/p, \mathbf{Z}/p) \cong \Lambda(\sigma'(x_1), \sigma'(x_2), \ldots) \otimes \Gamma(\sigma'(y_1), \sigma'(y_2), \ldots),$$

where Γ denotes a divided polynomial algebra and σ' is the suspension map

$$S \to \mathrm{Tor}_1^S(\mathbf{Z}/p, \mathbf{Z}/p);$$

and it is not hard to check that the isomorphism (4) takes $\sigma'(x_i)$ to $\sigma(x_i)$ and $\sigma'(y_j)$ to $\sigma(y_j)$. $\square$

3. Introduction to topological Hochschild homology. In this section we turn to the topological version of Hochschild homology. We will give an overview of the construction and some of its properties, without any attempt at technical completeness (those who are interested in knowing the details should consult [6] and [15]). Our goal in this section is simply to provide the reader with the intuition which is needed for the rest of the paper (and for the other papers in this area).

Roughly speaking, topological Hochschild homology is constructed by replacing the algebra S in the Hochschild complex (2) by an A_∞ ring spectrum R. As a first step in making this idea more precise we must reformulate the definition of $\mathbf{HH}_*(S)$. Let $HH_\bullet(S)$ be the simplicial abelian group

$$\begin{gathered} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ S \otimes S \otimes S \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ S \otimes S \\ \downarrow\uparrow\downarrow \\ S. \end{gathered}$$

Here the face maps ∂_i and degeneracy maps s_i are given by the formulas

$$\partial_i(t_0 \otimes \cdots \otimes t_n) = \begin{cases} t_0 \otimes \cdots \otimes t_i t_{i+1} \otimes \cdots \otimes t_n & \text{if } 0 \le i < n \\ (-1)^{|t_n|(|t_0| + \cdots + |t_{n-1}|)} t_n t_0 \otimes t_1 \otimes \cdots \otimes t_{n-1} & \text{if } i = n, \end{cases}$$

and

$$s_i(t_0 \otimes \cdots \otimes t_n) = t_0 \otimes \cdots \otimes t_i \otimes 1 \otimes t_{i+1} \otimes \cdots \otimes t_n.$$

Clearly the Hochschild complex is the chain complex associated to this simplicial abelian group. But for any simplicial abelian group, the homology of its associated chain complex is the same as the homotopy of its geometric realization (see [22, Theorem 22.1]), so in our case we conclude

$$\mathbf{HH}_*(S) = \pi_*|HH_\bullet(S)|.$$

Next we want to show how to modify this definition to obtain the *topological Hochschild homology* spectrum $THH(R)$ associated to an A_∞ ring spectrum R. If R were a strictly associative ring spectrum we could define $THH(R)$ to be the geometric realization of the simplicial spectrum

$$THH_\bullet(R) = \begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ R\wedge R\wedge R \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ R\wedge R \\ \downarrow\uparrow\downarrow \\ R. \end{array}$$

Here the face map

$$\partial_i : \overbrace{R\wedge\cdots\wedge R}^{n+1} \to \overbrace{R\wedge\cdots\wedge R}^{n}$$

is defined by the following equation (suitably interpreted):

$$\partial_i(r_0\wedge\cdots\wedge r_n) = \begin{cases} r_0\wedge\cdots\wedge r_i r_{i+1}\wedge\cdots\wedge r_n & \text{if } 0\le i<n \\ r_n r_0\wedge r_1\wedge\cdots\wedge r_{n-1} & \text{if } i=n, \end{cases}$$

and the degeneracy map

$$s_i : \overbrace{R\wedge\cdots\wedge R}^{n+1} \to \overbrace{R\wedge\cdots\wedge R}^{n+2}$$

is defined to be the composite

$$\overbrace{R\wedge\cdots\wedge R}^{n+1} \xrightarrow{\cong} \overbrace{R\wedge\cdots\wedge R}^{i+1}\wedge S^0\wedge \overbrace{R\wedge\cdots\wedge R}^{n-i} \xrightarrow{1\wedge e\wedge 1} \overbrace{R\wedge\cdots\wedge R}^{n+2},$$

where e is the unit map $S^0 \to R$; thus the i-th degeneracy inserts a unit in the $(i+1)$-st position. (Our assumption that the multiplication in R is strictly associative is necessary in order that the maps ∂_i and s_i defined in this way satisfy the simplicial identities).

Unfortunately we must now admit that there is no known category of spectra in which strictly associative ring spectra can even exist! (Thus the use of A_∞ structures in this context cannot be avoided). Nevertheless the description just given is still the best guide to intuition and we shall proceed in the remainder of the section as if it were literally correct. (In fact it *is* possible to define, for each A_∞ ring spectrum R, a simplicial spectrum $THH_\bullet(R)$ which agrees up to homotopy with the object denoted $THH_\bullet(R)$ in the previous discussion; see [15] for the details. One then defines $THH(R)$ to be the geometric realization of this simplicial spectrum).

We shall write

$$\tilde{\iota} : R \to THH(R)$$

for the inclusion of the 0-th simplicial filtration in $THH(R)$. If the multiplication in R is sufficiently commutative then $THH(R)$ inherits a ring-spectrum structure and $\tilde{\iota}$ is a ring map (see [6, Section 2]). If R is an E_∞ ring spectrum then $THH(R)$ inherits an E_∞ ring structure and $\tilde{\iota}$ is an E_∞ ring map (see [15]). Our definition of the spectrum ℓ, which is given in Section 9, automatically implies that ℓ is an E_∞ ring spectrum, so we conclude that $THH(\ell)$ is also.

Now suppose that we are given a homology theory h_* with a multiplication and that we want to know $h_*(THH(R))$. In [7], Bökstedt introduced the following spectral sequence for this sort of calculation.

PROPOSITION 3.1. *If h_* satisfies the strict Künneth formula*

$$h_*(X \wedge Y) \cong h_*X \otimes_{h_*S^0} h_*Y$$

then there is a spectral sequence

$$\mathbf{HH}_*(h_*(R)) \Rightarrow h_*(THH(R)), \tag{5}$$

where $\mathbf{HH}_*$ is defined with respect to the ground ring h_*S^0. For each $x \in h_*(R)$ the element

$$\iota_*(x) \in \mathbf{HH}_0(h_*(R))$$

survives to

$$\tilde{\iota}_*(x) \in h_*(THH(R)).$$

We warn the reader that there is in general no Hopf algebra structure in this spectral sequence (but see [19] for a situation in which a Hopf algebra structure does exist).

If h_* does not satisfy the strict Künneth formula then the spectral sequence still exists and converges; moreover it is likely, although we shall not attempt to prove it, that the E^2 term in this generality can be described explicitly as $\mathrm{Tor}^{h_*(R\wedge R)}(h_*(R), h_*(R))$ (cf. [29, Theorem 13.1]).

Outline of the proof of Proposition 3.1. For any simplicial spectrum $X_\bullet$, we may apply the theory h_* to the simplicial filtration of $|X_\bullet|$ in the usual way to obtain a spectral sequence converging to $h_*(|X_\bullet|)$. One would like to say that the E_2 term of this spectral sequence is the homology of the complex

$$\cdots \to h_*(X_n) \to \cdots \to h_*(X_1) \to h_*(X_0), \tag{6}$$

with differential

$$d = \sum (-1)^i (\partial_i)_*,$$

but for this one must assume that $X_\bullet$ is "proper" (meaning that the degeneracy maps are cofibrations—cf. [23, Theorem 11.14]). On the other hand, Elmendorf has shown that there is a "properization" functor which replaces each simplicial spectrum by a weakly equivalent proper one (see [14]). Thus if we *redefine* geometric realization to be properization followed by geometric realization in the usual sense (as we have implicitly done in the definition of THH), then we can conclude that the spectral sequence always has the desired E_2 term. In particular when $X_\bullet$ is $THH_\bullet(R)$ and h_* satisfies the strict Künneth formula the complex given above is just the Hochschild complex for $h_*(R)$, and we conclude that

$$E_2 \cong \mathbf{HH}_*(h_*(R))$$

as required. □

At the end of the next section we shall need to have somewhat tighter control of the spectral sequence (5). The information we need is provided by our next result.

PROPOSITION 3.2. *There is a natural transformation*

$$\tilde{\sigma} : \Sigma R \to THH(R)$$

such that the element

$$\sigma_*(x) \in \mathbf{HH}_1(h_*(R))$$

survives to

$$\tilde{\sigma}_*(\Sigma x) \in h_*(THH(R))$$

for each $x \in h_*(R)$.

The transformation $\tilde{\sigma}$ may seem somewhat mysterious—we give some motivation for it in Remark 3.6.

Outline of the proof. Before we can define $\tilde{\sigma}$ we need some preliminary constructions. Let $S_\bullet(R)$ be the simplicial spectrum obtained by "replacing all $\wedge$'s in $THH_\bullet(R)$ by $\vee$'s." More precisely, the n-th simplicial degree of $S_n(R)$ is

$$\overbrace{R \vee \ldots \vee R}^{n+1}.$$

The i-th face operator

$$\partial_i : \overbrace{R \vee \ldots \vee R}^{n+1} \to \overbrace{R \vee \ldots \vee R}^{n}$$

is defined by the equation

$$\partial_i \circ I_j = \begin{cases} I_{j-1} & \text{if } i < j \\ I_j & \text{if } i \geq j \text{ and } j < n \\ I_0 & \text{if } i = j = n; \end{cases}$$

here

$$I_j : R \to R \vee \ldots \vee R$$

is the inclusion of the j-th wedge summand. The i-th degeneracy map s_i is defined by the equation

$$s_i \circ I_j = \begin{cases} I_{j+1} & \text{if } i < j \\ I_j & \text{if } i \geq j. \end{cases}$$

We pause to determine the homotopy type of $|S_\bullet(R)|$.

LEMMA 3.3. $|S_\bullet(R)| \simeq R \vee \Sigma R$.

Proof of Lemma 3.3. Clearly we have

$$|S_\bullet(R)| \cong |S_\bullet(S^0)| \wedge R.$$

Now $S_\bullet(S^0)$ can be obtained by adding a disjoint basepoint to the standard simplicial decomposition of S^1 (see [6, page 20]), and so we have

$$|S_\bullet(R)| \cong (S^1)^+ \wedge R.$$

But for any space X, the space X^+ obtained by adding a disjoint basepoint splits stably as $S^0 \vee X$, so finally we have

$$|S_\bullet(R)| \simeq (S^0 \vee S^1) \wedge R \simeq R \vee \Sigma R$$

as required. □

For each n we can define a map

$$\omega_n : \overbrace{R \vee \ldots \vee R}^{n+1} \to \overbrace{R \wedge \ldots \wedge R}^{n+1}$$

by letting the restriction of ω_n to the j-th wedge summand be the composite

$$R \xrightarrow{\cong} \overbrace{S^0 \wedge S^0}^{j} \wedge R \wedge \overbrace{S^0 \wedge S^0}^{n-j} \xrightarrow{e \wedge 1 \wedge e} \overbrace{R \wedge \ldots \wedge R}^{n+1}.$$

Taken together, the ω_n give a map

$$\omega_\bullet : S_\bullet(R) \to THH_\bullet(R).$$

By passing to geometric realizations and using Lemma 3.3 we obtain a map[4]

$$\omega : R \vee \Sigma R \to THH(R)$$

The restriction of ω to the R summand is the map

$$\tilde{\iota} : R \to THH(R)$$

[4]In [6, Section 3] and [7, Section 2] this map is denoted by λ.

defined earlier. We can now define

$$\tilde{\sigma} : \Sigma R \to THH(R)$$

to be the restriction of ω to the ΣR summand.

To complete the proof of Proposition 3.2 it only remains to show that the transformation $\tilde{\sigma}$ has the desired relation to the spectral sequence (5). Let $C_*(X_\bullet)$ denote the chain complex (6). A straightforward calculation shows that the homology of $C_*(S_\bullet(R))$ vanishes in all dimensions except 0 and 1, and in particular the spectral sequence associated to $S_\bullet(R)$ collapses. For each $x \in h_*(R)$ the element

$$I_{1*}x \in h_*(R \vee R)$$

is a 1-dimensional cycle in $C_*(S_\bullet(R))$ which represents a class $\bar{x}$ in $E_2(S_\bullet(R))$. If we write J for the inclusion of ΣR as a wedge summand in $|S_\bullet(R)|$, then $\bar{x}$ survives to

$$J_*(\Sigma x) \in h_*(|S_\bullet(R)|).$$

It follows that the image of $\bar{x}$ in $E_2(THH_\bullet(R))$ survives to $\omega_* J_*(\Sigma x)$, which by definition is $\tilde{\sigma}(\Sigma x)$, in $h_*(THH_\bullet(R))$. But the image of $I_{1*}x$ in the Hochschild complex $C_*(THH_\bullet(R))$ is $1 \otimes x$, and so the image of $\bar{x}$ in $E_2(THH_\bullet(R))$ is σx. We have now shown that σx survives to $\tilde{\sigma}(\Sigma x)$, as required to finish the proof of Proposition 3.2. □

We conclude this section with some remarks which will not be used in the rest of the paper.

Remark 3.4. Alan Robinson has defined a "topological" analog of $\mathrm{Tor}^S(M,N)$, which he denotes by $E \wedge_R F$ (see [29]). In analogy with the equation

$$\mathbf{HH}_* S = \mathrm{Tor}^{S \otimes S^{\mathrm{op}}}(S,S)$$

one presumably has

$$THH(R) \simeq R \wedge_{R \wedge R^{\mathrm{op}}} R.$$

Remark 3.5. There is another way to relate Robinson's work to THH. If T is an E_∞ ring spectrum and $T \to R$ is an A_∞ map, it should be possible to define a spectrum

$$THH_T(R)$$

(i.e., "topological Hochschild homology over the ground ring T") by replacing all the smash products in the definition of $THH(R)$ by $\wedge_T$ products. In particular, if $R = HS$ for a discrete ring spectrum S one should have a formula

$$\pi_* THH_{H\mathbf{Z}} HS = \mathbf{HH}_* S$$

relating topological Hochschild homology to ordinary Hochschild homology; this would give one way to construct the map ϕ mentioned in Subsection 1.3 of the introduction. It should also be the case that

$$\pi_*(THH_{H\mathbf{Z}} HS)^{hS^1} = \mathbf{HC}_*^- S;$$

cf. Subsection 1.4.

Remark 3.6. Here is one more way to motivate the THH construction. If G is a topological group, its classifying space BG is the geometric realization of the following simplicial space.

$$\begin{array}{c} \vdots \\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \\ G \times G \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ G \\ \downarrow\uparrow\downarrow \\ * \end{array}$$

The face and degeneracy maps are given by the equations

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

and

$$s_i(g_1, \ldots, g_n) = (\ldots, g_i, 1, g_{i+1}, \ldots).$$

The same construction can be applied when G is merely an associative monoid.

It would be natural to try to apply this construction to an associative ring spectrum R, replacing the G's by R's, the $\times$'s by $\wedge$'s, and $*$ by S^0. If one attempts to carry this out, however, it becomes apparent that there is no sensible way to define the first and last degeneracy maps ∂_0 and ∂_n. Further reflection shows that this is because there is no sensible way to define an augmentation map $R \to S^0$. This brings us back to the observation made at the beginning of Section 2: in the analogous algebraic situation, one compensates for the lack of an augmentation by using the Hochschild complex instead of the bar construction. Thus one can think of the THH construction as being the closest one can come to imitating the classifying space construction for a ring spectrum R. From this point of view the map $\tilde{\sigma}$ described in Proposition 3.2 is the analog of the standard map from the suspension of a group to its classifying space. (The analogy is not precise, however, and in particular if the analog of the Hochschild construction is applied to a topological group G the result is not BG but instead is the free loop space $\mathrm{Map}(S^1, BG)$; see [16]).

4. Calculation of the E_2-term of the Adams spectral sequence. We remind the reader that p denotes an odd prime.

Let $\mathbf{M}$ denote the Moore spectrum $S^0 \cup_p e^1$. By definition we have

$$\pi_*(X; \mathbf{Z}/p) = \pi_*(X \wedge \mathbf{M}))$$

for any spectrum X, and we accordingly write

$$E_r(X; \mathbf{Z}/p)$$

for the classical Adams spectral sequence converging to $\pi_*(X \wedge \mathbf{M})$. We will index this spectral sequence as usual, so that

$$E_2^{t-s,s}(X; \mathbf{Z}/p) = \mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/p, H_*(X \wedge \mathbf{M}; \mathbf{Z}/p)).$$

The differentials d_r have bidegree $(-1, r)$, and $E_\infty^{t-s,*}$ is the associated graded of a filtration on $\pi_{t-s}(X; \mathbf{Z}/p)$. The Moore spectrum $\mathbf{M}$ is a ring spectrum (since p is odd; cf. [4]), and it follows that the spectral sequence has a multiplicative structure (see [28, Theorem 2.3.3]).

The case $X = \ell$ is of particular importance for our work. In this case it is well known (cf. [28, page 75], and also see the proof of Theorem 4.1 below) that

$$E_2(\ell; \mathbf{Z}/p) \cong \mathbf{Z}/p\,[a_1] \cong E_\infty(\ell; \mathbf{Z}/p),$$

where a_1 is an element in bidegree $(2p-2, 1)$ representing $v_1 \in \pi_{2p-2}(\ell; \mathbf{Z}/p)$. We shall also write a_1 for the image of this element under the map

$$E_2(\ell; \mathbf{Z}/p) \to E_2(THH(\ell); \mathbf{Z}/p)$$

induced by the inclusion $\tilde{\iota} : \ell \to THH(\ell)$.

The purpose of this section is to prove the following theorem.

THEOREM 4.1. *$E_2(THH(\ell); \mathbf{Z}/p)$ has the form*

$$\mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu],$$

where λ_1 *is in bidegree* $(2p-1,0)$*,* λ_2 *is in bidegree* $(2p^2-1,0)$*, and* μ *is in bidegree* $(2p^2,0)$.

This, in turn, is a consequence of our next result. As usual, we write A_* for the dual of the Steenrod algebra.

PROPOSITION 4.2. *As an algebra,* $H_*(THH(\ell);\mathbf{Z}/p)$ *has the form*

$$H_*(\ell;\mathbf{Z}/p)\otimes\Lambda(\lambda_1,\lambda_2)\otimes\mathbf{Z}/p\,[\mu],$$

where λ_1 *is in degree* $2p-1$*,* λ_2 *is in degree* $2p^2-1$*,* μ *is in degree* $2p^2$*, and the inclusion of the first factor is the natural map*

$$\bar{\iota}_*:H_*(\ell;\mathbf{Z}/p)\to H_*(THH(\ell);\mathbf{Z}/p).$$

The A_**-coaction*

$$\nu:H_*(THH(\ell);\mathbf{Z}/p)\to A_*\otimes H_*(THH(\ell);\mathbf{Z}/p)$$

is determined by the equations

$$\nu(\lambda_i)=1\otimes\lambda_i$$

and

$$\nu(\mu)=1\otimes\mu+\tau_0\otimes\lambda_2.$$

Proof of Theorem 4.1. We need to calculate

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/p,H_*(THH(\ell)\wedge\mathbf{M};\mathbf{Z}/p)).$$

We shall do this by using a standard change-of-rings theorem:

$$\mathrm{Ext}_\Gamma(\mathbf{Z}/p,\Gamma\square_\Sigma N)\cong\mathrm{Ext}_\Sigma(\mathbf{Z}/p,N) \tag{7}$$

(see [28, Theorem A1.3.12]). Here Γ denotes a Hopf algebra, Σ a quotient Hopf algebra, and N a Σ-comodule; the $\square$-product is defined on page 311 of [28]. (See pages 337–339 of [2] for a dual version of the following argument which avoids the $\square$-product).

First we observe that if N is actually a Γ-comodule (more precisely if the Σ-comodule structure on N is induced by a Γ-comodule structure) then the map

$$\Gamma\otimes N\to\Gamma\otimes N$$

which takes $g \otimes n$ to

$$\sum_i gg_i \otimes n_i$$

(where, as usual, we have written

$$n \mapsto \sum_i g_i \otimes n_i$$

for the Γ-coaction on N) induces an isomorphism

$$(\Gamma \Box_\Sigma \mathbf{Z}/p) \otimes N \to \Gamma \Box_\Sigma N$$

(where the domain has the diagonal Γ-coaction). We can therefore rewrite (7) in this situation as follows:

$$\text{(8)} \qquad \mathrm{Ext}_\Gamma(\mathbf{Z}/p, (\Gamma \Box_\Sigma \mathbf{Z}/p) \otimes N) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, N).$$

Now let Γ be A_*, let Σ be the Hopf-algebra quotient of the inclusion

$$H_*(\ell \wedge \mathbf{M}; \mathbf{Z}/p) \to H_*(H\mathbf{Z}/p; \mathbf{Z}/p) = A_*,$$

and let N be

$$\Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu].$$

Proposition 4.2 implies

$$H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p) \cong H_*(\ell \wedge \mathbf{M}; \mathbf{Z}/p) \otimes N,$$

and we have

$$\Gamma \Box_\Sigma \mathbf{Z}/p \cong H_*(\ell \wedge \mathbf{M}; \mathbf{Z}/p)$$

by [28, Lemma A1.1.16], so finally we have

$$H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p) \cong (\Gamma \Box_\Sigma \mathbf{Z}/p) \otimes N.$$

We can therefore conclude from equation (8) that

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/p, H_*(THH(\ell) \wedge \mathbf{M}; \mathbf{Z}/p)) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, N).$$

But

$$\Sigma = \Lambda(\chi\tau_1),$$

and in particular Proposition 4.2 shows that the Σ-coaction on N is trivial. Thus we have

$$\mathrm{Ext}_\Sigma(\mathbf{Z}/p, N) \cong \mathrm{Ext}_\Sigma(\mathbf{Z}/p, \mathbf{Z}/p) \otimes N,$$

and by [28, Lemma 3.1.9] this is

$$\mathbf{Z}/p\,[a_1] \otimes N$$

as required. □

Proof of Proposition 4.2. Of course, the first step in the proof of Proposition 4.2 is to calculate $H_*(THH(\ell); \mathbf{Z}/p)$ as an algebra by using the spectral sequence of Proposition 3.1. We shall denote this spectral sequence by

$$\widehat{E}^r(R)$$

in order to distinguish it from the Adams spectral sequence. To carry out the spectral sequence calculation, all we have to do is modify the proof of [7, Theorem 1.1].

Since

$$\widehat{E}^2(\ell) \cong \mathbf{HH}_*(H_*(\ell; \mathbf{Z}/p)),$$

we must begin by remembering what $H_*(\ell; \mathbf{Z}/p)$ is. In order to describe it we recall from [2, Lemma 16.8] that the canonical map

$$\varepsilon : \ell \to H\mathbf{Z}/p$$

(i.e., the map which represents the generator of $H^0(\ell; \mathbf{Z}/p)$) induces a monomorphism

$$H_*(\ell; \mathbf{Z}/p) \to H_*(H\mathbf{Z}/p; \mathbf{Z}/p) = A_*$$

with image

$$\mathbf{Z}/p\,[\chi\xi_1, \chi\xi_2, \ldots] \otimes \Lambda(\chi\tau_2, \chi\tau_3, \ldots);$$

here χ is the canonical anti-automorphism of A_*[5].

We can now apply Proposition 2.1 of Section 2 to conclude that

$$\widehat{E}^2(\ell) \cong H_*(\ell; \mathbf{Z}/p) \otimes \Lambda(\sigma(\chi\xi_1), \sigma(\chi\xi_2), \ldots) \otimes \Gamma[\sigma(\chi\tau_2), \sigma(\chi\tau_3), \ldots],$$

[5]We need to use χ in this description where Adams does not because we are thinking of $H_*(\ell; \mathbf{Z}/p)$ as $\pi_*(H\mathbf{Z}/p \wedge \ell)$ instead of $\pi_*(\ell \wedge H\mathbf{Z}/p)$.

where $H_*(\ell;\mathbf{Z}/p)$ is in filtration-degree 0, $\sigma(\chi\xi_i)$ has bidegree $(2p^i-2,1)$, and $\sigma(\chi\tau_i)$ has bidegree $(2p^i-1,1)$.[6] The next step is to determine the differentials in the spectral sequence $\widehat{E}^r(\ell)$. This is easily done by comparing it with the spectral sequence $\widehat{E}^r(H\mathbf{Z}/p)$, whose behavior has been completely determined in [7]. It follows from [31, Proposition 4.1] that the map ε is an E_∞ ring map (more precisely, we should say that there is an E_∞ ring map in its homotopy class). In particular, it is an A_∞ ring map, and thus it induces a map

$$\varepsilon_*:\widehat{E}^r(\ell)\to\widehat{E}^r(H\mathbf{Z}/p).$$

A calculation similar to that for $\widehat{E}^2(\ell)$ shows that

$$\widehat{E}^2(H\mathbf{Z}/p)\cong A_*\otimes\Lambda(\sigma(\chi\xi_1),\sigma(\chi\xi_2),\dots)\otimes\Gamma[\sigma(\chi\tau_0),\sigma(\chi\tau_1),\dots].$$

By [7, Lemma 1.3], the only nontrivial differential in $\widehat{E}^r(H\mathbf{Z}/p)$ is given by the formula

$$d^{p-1}\gamma_j(\sigma(\chi\tau_i))=\sigma(\chi\xi_{i+1})\cdot\gamma_{j-p}(\sigma(\chi\tau_i))\quad\text{if } j\geq p;$$

here we have written $\gamma_j(\sigma(\chi\tau_i))$ for the j-th divided power of $\sigma(\chi\tau_i)$. The same formula therefore holds in $\widehat{E}^{p-1}(\ell)$ for $i\geq 2$, and we conclude that

$$\widehat{E}^p(\ell)\cong H_*(\ell;\mathbf{Z}/p)\otimes\Lambda(\sigma(\chi\xi_1),\sigma(\chi\xi_2))\otimes TP_p[\sigma(\chi\tau_2),\sigma(\chi\tau_3),\dots],$$

where TP_p denotes a truncated polynomial algebra of height p (cf. [7, page 6]). Since all indecomposables in $\widehat{E}^p(\ell)$ are in filtrations 0 and 1 we can further conclude that

$$\widehat{E}^p(\ell)=\widehat{E}^\infty(\ell).$$

Proposition 3.2 implies that the elements $\sigma(\chi\xi_i)$ and $\sigma(\chi\tau_i)$ in $\widehat{E}^\infty$ are represented in $H_*(THH(\ell))$ by $\tilde{\sigma}_*(\Sigma(\chi\xi_i))$ and $\tilde{\sigma}_*(\Sigma(\chi\tau_i))$ respectively.

Next we need to determine the multiplicative extensions in $H_*(THH(\ell);\mathbf{Z}/p)$. For this we use Dyer-Lashof operations. As we have seen in the previous section, $THH(\ell)$ is an E_∞ ring spectrum, and so its homology supports Dyer-Lashof operations

$$Q^i:H_n(THH(\ell);\mathbf{Z}/p))\to H_{n+2i(p-1)}(THH(\ell);\mathbf{Z}/p)),$$

[6] We can now explain why it is necessary to use the Adams summand ℓ instead of bu itself in our work. The mod p homology of bu is related to that of ℓ by the equation

$$H_*(bu;\mathbf{Z}/p)\cong H_*(\ell;\mathbf{Z}/p)\otimes\mathbf{Z}/p\,[x]/(x^{p-1}).$$

The Hochschild homology of the factor $\mathbf{Z}/p\,[x]/(x^{p-1})$, while it is not difficult to compute, is rather complicated, and its presence would make it difficult if not impossible to apply the methods of this paper.

(see [32]). If x is an element of dimension $2s$ then $Q^s x = x^p$, ([32, Theorem 1.1(4)]) so in particular we have

$$(\tilde{\sigma}_*(\Sigma(\chi\tau_i)))^p = Q^{p^i}\tilde{\sigma}_*(\Sigma(\chi\tau_i)).$$

But Bökstedt shows that the map

$$\tilde{\sigma}_*\Sigma : H_n(R; \mathbf{Z}/p) \to H_{n+1}(THH(R); \mathbf{Z}/p)$$

commutes with Dyer-Lashof operations (see [7, Lemma 2.9]), and Steinberger has calculated the action of the Q^i in $H_*(\ell; \mathbf{Z}/p)$:

$$Q^{p^i}\chi\tau_i = \chi\tau_{i+1}$$

(see [32, Theorem 2.3]). We conclude that

$$(\tilde{\sigma}_*(\Sigma\chi\tau_i))^p = \tilde{\sigma}_*(\Sigma\chi\tau_{i+1})$$

for all $i \geq 2$, and hence that

$$(\tilde{\sigma}_*(\Sigma\chi\tau_2))^{p^i} = \tilde{\sigma}_*(\Sigma\chi\tau_{i+2})$$

for all $i \geq 0$. If we denote $\tilde{\sigma}_*\Sigma(\chi\xi_1)$ by λ_1, $\tilde{\sigma}_*\Sigma(\chi\xi_2)$ by λ_2, and $\tilde{\sigma}_*\Sigma(\chi\tau_2)$ by μ, we have now shown that

$$H_*(THH(\ell); \mathbf{Z}/p) \cong H_*(\ell; \mathbf{Z}/p) \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu]$$

as an algebra.

To complete the proof of Proposition 4.2 we need to determine the A_*-coaction on λ_1, λ_2, and μ. We shall give the calculation of $\nu(\lambda_2)$; the others are similar.

Since the map $\tilde{\sigma}_*\Sigma$ commutes with ν, we have

$$\nu(\lambda_2) = (1 \otimes \tilde{\sigma}_*\Sigma)\nu(\chi\xi_2).$$

Now $\nu(\chi\xi_2)$ is determined by Milnor's calculations: it is

$$1 \otimes \xi_2 + \xi_1 \otimes \xi_1^p + \xi_2 \otimes 1$$

(see [28, Theorem 3.1.1]). We therefore conclude that

$$\nu(\lambda_2) = 1 \otimes \lambda_2 + \xi_1 \otimes \tilde{\sigma}_*\Sigma(\xi_1^p) + \xi_2 \otimes \tilde{\sigma}_*\Sigma(1)$$

and it remains to show that the second and third terms are zero. But $\tilde{\sigma}_*\Sigma(\xi_1^p)$ represents the element $\sigma(\xi_1^p)$ in the spectral sequence, and this element is zero because σ is a derivation (equation (3) of Section 2). It follows that $\tilde{\sigma}_*\Sigma(\xi_1^p)$ is an element in filtration 0 with dimension $2p^2 - 2p + 1$, and an inspection of the spectral sequence shows that the only such element is 0. Similarly, $\tilde{\sigma}_*\Sigma(1)$ is an element in filtration 0 with dimension 1, and again the only such element is 0. This completes the proof of Proposition 4.2. □

Remark 4.3. (Andy Baker and Larry Smith) The methods we have used in this section can also be used to give simple calculations of $\pi_*THH(BP)$ and $\pi_*THH(MU)$.[7] We shall use a result of Brown and Peterson [11, Theorem 1.3] which says that if a p-local spectrum Y satisfies:

(i) H_*Y is torsion free, and

(ii) $H_*(Y;\mathbf{Z}/p)$ is isomorphic, as a comodule over A_*, to $H_*(BP;\mathbf{Z}/p)\otimes PH_*(Y;\mathbf{Z}/p)$ (where P denotes the primitives with respect to the A_* coaction),

then it also satisfies

(iii) Y is a wedge of suspensions of BP,

(iv) $\pi_*(Y;\mathbf{Z}/p)\cong\pi_*(BP;\mathbf{Z}/p)\otimes PH_*(Y;\mathbf{Z}/p)$, and

(v) the Hurewicz homomorphism $\pi_*Y\to H_*Y$ is a monomorphism.

We begin by applying this result to $Y = THH(BP)$. Since H_*BP is torsion free, the Künneth theorem implies that there is a spectral sequence

$$\mathbf{HH}_*(H_*BP)\Rightarrow H_*THH(BP).$$

The integral homology of BP is given by

$$H_*(BP)\cong\mathbf{Z}_{(p)}[b_1,b_2,\ldots],$$

where the degree of b_i is $2p^i-2$, and thus the E^2 term of this spectral sequence is

$$H_*BP\otimes\Lambda(\sigma(b_1),\sigma(b_2),\ldots);$$

the spectral sequence collapses for dimensional reasons and we conclude

$$H_*THH(BP)\cong H_*BP\otimes\Lambda(\tilde{\sigma}_*(\Sigma b_1),\tilde{\sigma}_*(\Sigma b_2),\ldots).$$

[7]In order to make the calculation of $\pi_*THH(BP)$ rigorous we must *assume* that BP is an A_∞ ring spectrum (otherwise $THH(BP)$ isn't even defined) and also that $THH(BP)$ is a ring spectrum (for which it would suffice to know that BP is E_∞). On the other hand, the calculation of $\pi_*THH(MU)$ is rigorous since MU is known to be E_∞ (see [24, Section IV.2]).

In particular, $THH(BP)$ satisfies condition (i) above. Next we consider the spectral sequence

$$\mathbf{HH}_*(H_*(BP;\mathbf{Z}/p)) \Rightarrow H_*(THH(BP);\mathbf{Z}/p).$$

From the equation

$$H_*(BP;\mathbf{Z}/p) = \mathbf{Z}/p\,[\chi\xi_1, \chi\xi_2, \ldots]$$

we see that E^2 has the form

$$H_*(BP;\mathbf{Z}/p) \otimes \Lambda(\sigma(\chi\xi_1), \sigma(\chi\xi_2), \ldots).$$

The spectral sequence collapses, and thus we have

$$H_*(THH(BP);\mathbf{Z}/p) = H_*(BP;\mathbf{Z}/p) \otimes \Lambda(\tilde{\sigma}_*\Sigma(\chi\xi_1), \tilde{\sigma}_*\Sigma(\chi\xi_2), \ldots).$$

As in the proof of Proposition 4.2, it is easy to check that the elements $\tilde{\sigma}_*\Sigma(\chi\xi_i)$ are primitives of the A_* coaction, hence condition (ii) above is satisfied and we conclude that $THH(BP)$ is a wedge of suspensions of BP and that

$$\pi_*THH(BP) \cong \pi_*BP \otimes \Lambda(\lambda_1, \lambda_2, \ldots),$$

where the mod p reduction of the Hurewicz image of λ_i is $\tilde{\sigma}_*\Sigma(\chi\xi_i)$.

The case where $Y = THH(MU)_{(p)}$ is handled similarly; the only new ingredient is Lemma 3.1.7 of [28] which gives the structure of $H_*(MU;\mathbf{Z}/p)$ as an A_*-comodule. The conclusion is that

$$\pi_*THH(MU) \cong \pi_*MU \otimes \Lambda(\lambda_1', \lambda_2', \ldots),$$

where the dimension of λ_i' is $2i+1$; it follows from this that $THH(MU)$ is a wedge of suspensions of MU.

5. The mod p K-theory of $THH(\ell)$. In the previous section we determined the mod p homology of $THH(\ell)$. In this section we consider its mod p K-theory.

THEOREM 5.1. *The inclusion*

$$\tilde{\iota} : \ell \longrightarrow THH(\ell)$$

induces an isomorphism

$$\tilde{\iota}_* : K(1)_*\ell \xrightarrow{\cong} K(1)_*(THH(\ell)).$$

Here $K(1)$ denotes the first Morava K-theory. When $p = 2$ this is the same as mod 2 K-theory, and for odd primes it is the Adams summand of mod p K-theory. Its coefficient ring is given by

$$\pi_*(K(1)) = \mathbf{Z}/p\,[v_1, v_1^{-1}],$$

where the dimension of v_1 is $2(p-1)$.[8]

Theorem 5.1 will be used in two ways in our work. In Section 8 it will be the main step in determining the homotopy type of $THH(L)$. In the present section we will use it to determine the v_1-periodic homotopy of $THH(\ell)$. We define the v_1-periodic mod p homotopy $v_1^{-1}\pi_n(X;\mathbf{Z}/p)$ of an ℓ-module X to be the direct limit of the system

$$\pi_n(X;\mathbf{Z}/p) \to \pi_{2(p-1)+n}(X;\mathbf{Z}/p) \to \pi_{4(p-1)+n}(X;\mathbf{Z}/p) \to \cdots,$$

where each of the maps is multiplication by $v_1 \in \pi_{2(p-1)}\ell$. (We shall show at the end of the section that this is a special case of the usual definition of v_1-periodic homotopy given, for example, on page 271 of [10].)

COROLLARY 5.2. *The inclusion*

$$\iota : \ell \longrightarrow THH(\ell)$$

induces an isomorphism

$$\iota_* : v_1^{-1}\pi_*(\ell;\mathbf{Z}/p) \xrightarrow{\cong} v_1^{-1}\pi_*(THH(\ell);\mathbf{Z}/p).$$

Corollary 5.2 is an immediate consequence of Theorem 5.1 and [10, Theorem 4.11(ii)] once one knows that our definition of v_1-periodic homotopy agrees with that in [10]. We shall also give an elementary proof which does not depend on the deep results in Section 4 of [10].

Corollary 5.2 will be used in later sections to determine the differentials in the Adams spectral sequence $E_r(THH(\ell);\mathbf{Z}/p)$ converging to the mod p homotopy of $THH(\ell)$. It is in some sense analogous to Bökstedt's results on $THH(\mathbf{Z}_{(p)})$ in [7]. (Actually he discusses $THH(\mathbf{Z})$ and not $THH(\mathbf{Z}_{(p)})$, but it is clear that the results in [7] have parallels for $THH(\mathbf{Z}_{(p)})$.) It is an easy consequence of his computations that the inclusion of spectra

$$\iota : \mathbf{Z}_{(p)} \longrightarrow THH(\mathbf{Z}_{(p)})$$

[8] If $K(1)$ is replaced in Theorem 5.1 by $K(n)$ and ℓ by $BP\langle n\rangle$ (see [28, page 132]), and if we *assume* that $BP\langle n\rangle$ has an A_∞ structure, so that $THH(BP\langle n\rangle)$ is defined, then the theorem remains true with essentially the same proof.

induces an isomorphism in homotopy tensored with $\mathbf{Z}[1/p]$. In other words, inverting p in homotopy kills the difference between the Eilenberg-MacLane spectrum $\mathbf{Z}_{(p)}$ and $THH(\mathbf{Z}_{(p)})$. Corollary 5.2 states that something similar happens for $THH(\ell)$ if we consider multiplication by v_1 in mod-p homotopy instead of multiplication by p in p-local homotopy.

The rest of this section will be devoted to the proofs of 5.1 and 5.2. For the proof of 5.1 we recall that $K(1)$ satisfies the strict Künneth formula (see [28, page 133]), and so Proposition 3.1 gives a spectral sequence

$$\mathbf{HH}_*(K(1)_*\ell) \Rightarrow K(1)_*(THH(\ell)).$$

We therefore consider the structure of the ring $K(1)_*\ell$.

PROPOSITION 5.3.

(a) $K(1)_*\ell \cong K(1)_* \otimes K(1)_0\,\ell$.

(b) *$K(1)_0\,\ell$ has the form $\varinjlim A_n$, where each A_n is a direct sum of finite fields.*

(c) *The Hochschild homology of $K(1)_0\,\ell$ with respect to the ground ring $\mathbf{Z}/p$ is given by the formula*

$$\mathbf{HH}_i(K(1)_0\,\ell) = \begin{cases} K(1)_0\,\ell & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

(d) *The Hochschild homology of $K(1)_*\ell$ with respect to the ground ring $K(1)_*$ is given by the formula*

$$\mathbf{HH}_i(K(1)_*\ell) = \begin{cases} K(1)_*\ell & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

Of course, part (d) of this result, and the spectral sequence given above, immediately imply Theorem 5.1. (We should note that Proposition 5.3 is implicitly contained in Robinson's paper [30], where it is used for a different purpose; cf. [30, Theorem 2.2]).

Proof of Proposition 5.3. We begin by calculating $K(1)_*\ell$. First we recall the equation

$$BP_*BP = BP_*[t_1, t_2, \ldots] \tag{9}$$

(see [28, Theorem 4.1.18(b)]); here t_k has degree $2(p^k-1)$. Now $K(1)$ is a module spectrum over BP, and equation 9 implies that BP_*BP is a free left module over BP_*, so we have

$$K(1)_*BP \cong K(1)_* \otimes_{BP_*} BP_*BP \cong \mathbf{Z}/p\,[v_1, v_1^{-1}][t_1, t_2, \ldots]. \tag{10}$$

We shall write η_R, as usual, for the right unit map $BP_* \to BP_*BP$, and $\bar{\eta}_R$ for the projection of this map to $K(1)_*BP$. Then we have

$$\bar{\eta}_R(v_{k+1}) \equiv v_1 t_k^p - v_1^{p^k} t_k \quad \text{modulo } (\bar{\eta}_R(v_2), \ldots, \bar{\eta}_R(v_k)) \tag{11}$$

in $K(1)_*(BP)$ for all $k \geq 1$ (see formula 6.1.13 of [28], but note that the equation there differs from ours by a typographical error). Next we recall that ℓ can be constructed from the spectrum BP by an iterated Sullivan-Baas construction

$$BP \longrightarrow \ell$$

which realizes the quotient map

$$\mathbf{Z}_{(p)}[v_1, \ldots, v_n, \ldots] \longrightarrow \mathbf{Z}_{(p)}[v_1, \ldots, v_n, \ldots]/(v_2, v_3, \ldots)$$

on homotopy groups (see [28, pages 131–132]; in the notation used there one has $\ell = BP\langle 1\rangle$). Now equation (11) implies for each $k \geq 1$ that $\bar{\eta}_R(v_{k+1})$ is not a zero divisor mod $(\bar{\eta}_R(v_2), \ldots, \bar{\eta}_R(v_k))$. Combining this fact with the standard cofiber sequence associated to the Sullivan-Baas construction (cf. [28, page 131]) and equation (10) we obtain

$$\begin{aligned} K(1)_*\ell &\cong K(1)_*BP/(\bar{\eta}_R(v_2), \bar{\eta}_R(v_3), \ldots) \\ &\cong \mathbf{Z}/p\,[v_1, v_1^{-1}][t_1, t_2, \ldots]/(v_1 t_k^p - v_1^{p^k} t_k : k \geq 1); \end{aligned}$$

in particular, this implies that $K(1)_*\ell$ is zero in degrees not divisible by $2(p-1)$, and this in turn implies part (a) of the proposition.

Next we define subalgebras A_n of $K(1)_0\,\ell$ for each $n \geq 1$ by

$$A_n = \mathbf{Z}/p\,[u_1, u_2, \ldots, u_n]/(u_k^p - u_k : 1 \leq k \leq n),$$

where the elements u_k are defined by

$$u_k = v_1^{-p^{k-1} - \cdots - 1} t_k.$$

We clearly have $K(1)_0\,\ell \cong \varinjlim A_n$. Each element of A_n is equal to its own p-th power, so A_n certainly contains no nilpotent elements. We also know that A_n is finite dimensional over $\mathbf{Z}/p$ so we conclude that it is semisimple. Since it is commutative and finite it must therefore be a direct sum of finite fields (in fact it is a direct sum of copies of $\mathbf{Z}/p$). This proves part (b).

For part (c), we first recall that the 0-dimensional Hochschild homology of any commutative algebra A is equal to A (see formula (3) on page 170 of [12]). Next we observe that Hochschild homology commutes with direct sums [12, Theorem IX.5.3] and direct limits. The result now follows from part (b) and the fact that the Hochschild homology of a finite field, with $\mathbf{Z}/p$ as the ground ring, vanishes in positive dimensions (see [12, Theorem IX.7.10]).

To show part (d) we need only observe that part (a) implies the formula

$$\mathbf{HH}_*(K(1)_*\ell) \cong K(1)_* \otimes \mathbf{HH}_*(K(1)_0\,\ell)$$

relating the Hochschild homology of $K(1)_*\ell$ with respect to the ground ring $K(1)_*$ to that of $K(1)_0\,\ell$ with respect to the ground ring $\mathbf{Z}/p$. □

Our next result, together with Theorem 5.1, will immediately imply Corollary 5.2.

LEMMA 5.4. *Let X and Y be ℓ-module spectra and let $f : X \to Y$ be an ℓ-module map which induces an isomorphism $K(1)_*X \to K(1)_*Y$. Then f also induces an isomorphism $v_1^{-1}\pi_*(X;\mathbf{Z}/p) \to v_1^{-1}\pi_*(Y;\mathbf{Z}/p)$.*

Proof. Let $\mu_X : \ell\wedge X \to X$ and $\mu_Y : \ell\wedge Y \to Y$ define the ℓ-module structures on X and Y. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_*(X;\mathbf{Z}/p) & \xrightarrow{f_*} & \pi_*(Y;\mathbf{Z}/p) \\
\downarrow h & & \downarrow h \\
\pi_*(\ell\wedge X;\mathbf{Z}/p) & \xrightarrow{f_*} & \pi_*(\ell\wedge Y;\mathbf{Z}/p) \\
\downarrow \mu_{X*} & & \downarrow \mu_{Y*} \\
\pi_*(X;\mathbf{Z}/p) & \xrightarrow{f_*} & \pi_*(Y;\mathbf{Z}/p)
\end{array}$$

where the h's are Hurewicz maps induced by the unit map $S^0 \to \ell$. The composites $\mu_{X*}\circ h$ and $\mu_{Y*}\circ h$ are identity maps, and in particular μ_{Y*} is a surjection. The maps in the lower square (but not those in the upper square!) are $\pi_*\ell$-module maps and so we can localize the lower square to obtain the diagram

$$\begin{array}{ccc}
v_1^{-1}\pi_*(\ell\wedge X;\mathbf{Z}/p) & \xrightarrow{f_*} & v_1^{-1}\pi_*(\ell\wedge Y;\mathbf{Z}/p) \\
\downarrow \mu_{X*} & & \downarrow \mu_{Y*} \\
v_1^{-1}\pi_*(X;\mathbf{Z}/p) & \xrightarrow{f_*} & v_1^{-1}\pi_*(Y;\mathbf{Z}/p)
\end{array}$$

Now we have

$$K(1)_*(Z) = v_1^{-1}\pi_*(\ell \wedge Z; \mathbf{Z}/p)$$

for any Z and so the hypothesis states that the upper arrow here is an isomorphism. As the localization of an epimorphism is an epimorphism, the arrow μ_{Y*} in the new diagram is an epimorphism, so we conclude that the lower f_*,

$$f_* : v_1^{-1}\pi_*(X; \mathbf{Z}/p) \longrightarrow v_1^{-1}\pi_*(Y; \mathbf{Z}/p),$$

is also surjective.

Now we must prove the lower f_* is injective. Suppose that $x \in \ker(f_*)$. By definition of localization we can find an integer m such that $v_1^m x = x' \in \pi_*(X; \mathbf{Z}/p)$. By choosing m larger if necessary we can arrange that

$$f_* : \pi_*(X; \mathbf{Z}/p) \longrightarrow \pi_*(Y; \mathbf{Z}/p)$$

carries x' to zero. By commutativity of the upper square of the first diagram,

$$f_*(h(x')) = 0,$$

so that injectivity of the localized f_* implies there is m' such that

$$v_1^{m'} h(x') = 0.$$

Then

$$\begin{aligned} 0 &= \mu_{X*}(v_1^{m'} h(x')) \\ &= v_1^{m'} \mu_{X*}(h(x')) \\ &= v_1^{m'} x' \end{aligned}$$

in $\pi_*(X; \mathbf{Z}/p)$ so that in $v_1^{-1}\pi_*(X; \mathbf{Z}/p)$

$$0 = v_1^{m+m'} x.$$

We conclude that $x = 0$, so that

$$f_* : v_1^{-1}\pi_*(X; \mathbf{Z}/p) \longrightarrow v_1^{-1}\pi_*(Y; \mathbf{Z}/p)$$

is also injective. □

We conclude this section by showing that our definition of $v_1^{-1}\pi_n(X; \mathbf{Z}/p)$ for ℓ-modules X is a special case of the usual definition of v_1-periodic homotopy.

PROPOSITION 5.5. *If X is any ℓ module, then its v_1-periodic mod p homotopy $v_1^{-1}\pi_n(X;\mathbf{Z}/p)$ as defined above is the direct limit of the system*

$$(12)\ \pi_n(X\wedge \mathbf{M}) \xrightarrow{(1\wedge A)_*} \pi_n(X\wedge \Sigma^{-2(p-1)}\mathbf{M}) \xrightarrow{(1\wedge A)_*} \pi_n(X\wedge \Sigma^{-4(p-1)}\mathbf{M}) \xrightarrow{(1\wedge A)_*} \cdots,$$

where $\mathbf{M}$ is the Moore spectrum $S^0\cup_p e^1$ and

$$A:\mathbf{M}\to\Sigma^{-2(p-1)}\mathbf{M}$$

is the Adams map.

Let us write $\pi_n^{\mathrm{per}}X$ for the direct limit of the system (12). We begin with a lemma which will also be needed in Section 8.

LEMMA 5.6. *Let X be an ℓ-module and let v_X denote the composite*

$$\Sigma^{2(p-1)}X \xrightarrow{v_1\wedge 1} \ell\wedge X\to X.$$

Then v_X induces a $K(1)$-isomorphism.

Proof. Consider the diagram

$$\begin{array}{ccc} \Sigma^{2(p-1)}\ell\wedge X & \xrightarrow{v_\ell\wedge 1} & \ell\wedge X\\ \downarrow & & \downarrow\\ \Sigma^{2(p-1)}X & \xrightarrow{v_X} & X \end{array}$$

The vertical maps are split epimorphisms, and the diagram commutes, so it suffices to prove the lemma for $X=\ell$. Next recall that the map

$$(1\wedge A)_*:\pi_n(\ell\wedge\mathbf{M})\to\pi_{n+2(p-1)}(\ell\wedge\mathbf{M})$$

is multiplication by v_1. Since v_ℓ also induces multiplication by v_1, it is clearly an isomorphism on π_*^{per}. But then it is also a $K(1)$-isomorphism by [10, Theorem 4.11(ii)]. $\square$

Proof of Proposition 5.5. Consider the following double direct system.

$$\begin{array}{ccccc} \pi_*(X\wedge\mathbf{M}) & \xrightarrow{(1\wedge A)_*} & \pi_*(X\wedge\mathbf{M}) & \xrightarrow{(1\wedge A)_*} & \cdots\\ \downarrow (v_X\wedge 1)_* & & \downarrow (v_X\wedge 1)_* & & \\ \pi_*(X\wedge\mathbf{M}) & \xrightarrow{(1\wedge A)_*} & \pi_*(X\wedge\mathbf{M}) & \xrightarrow{(1\wedge A)_*} & \cdots\\ \downarrow (v_X\wedge 1)_* & & \downarrow (v_X\wedge 1)_* & & \\ \vdots & & \vdots & & \end{array}$$

Of course, the direct limit of this system is the same whether one first passes to the limit horizontally and then vertically or first vertically and then horizontally. We will prove the proposition by showing that the first way of passing to the limit gives $\pi_*^{\mathrm{per}}X$ and the second way gives $v_1^{-1}\pi_*(X;\mathbf{Z}/p)$.

The limit along each row is, by definition, $\pi_*^{\mathrm{per}}X$, so passage to the limit along the rows gives the direct system

$$\begin{array}{c} \pi_*^{\mathrm{per}}X \\ \downarrow (v_X)_* \\ \pi_*^{\mathrm{per}}X \\ \downarrow (v_X)_* \\ \vdots \end{array}$$

But the map v_X is a $K(1)$-isomorphism by Lemma 5.6., and hence it is a π_*^{per}-isomorphism by [10, Theorem 4.11(ii)]. Thus the direct limit of this system is $\pi_*^{\mathrm{per}}X$ as asserted.

On the other hand, passage to the limit along the columns gives the direct system

$$v_1^{-1}\pi_*(X \wedge \mathbf{M}) \xrightarrow{(1\wedge A)_*} v_1^{-1}\pi_*(X \wedge \mathbf{M}) \xrightarrow{(1\wedge A)_*} \cdots$$

But the map $1\wedge A$ is a $K(1)$-isomorphism and hence (by Lemma 5.4) it induces an isomorphism of $v_1^{-1}\pi_*$. Thus the direct limit of this system is equal to $v_1^{-1}\pi_*(X \wedge \mathbf{M})$, and this in turn is (by definition) equal to $v_1^{-1}\pi_*(X;\mathbf{Z}/p)$. This completes the proof. □

6. The localized Adams spectral sequence. This section contains the last of the preliminary computations we need before we compute $\pi_*(THH(\ell);\mathbf{Z}/p)$ with the classical Adams spectral sequence.

Until now we have used the notation $E_r(X;\mathbf{Z}/p)$ for the Adams spectral sequence converging to $\pi_*(X;\mathbf{Z}/p)$, but from now on we shall use the somewhat simpler notation E_rX. Let X be an ℓ module spectrum. Since $E_r\,\ell$ is isomorphic to $\mathbf{Z}/p\,[a_1]$ for all r, we know that E_rX is a spectral sequence of $\mathbf{Z}/p\,[a_1]$ modules, and since localization is an exact functor we may invert a_1 in this situation to obtain a spectral sequence of $\mathbf{Z}/p\,[a_1, a_1^{-1}]$ modules which we will denote by $a_1^{-1}E_rX$. We do *not* assert in general that this spectral sequence converges to $v_1^{-1}\pi_*(X;\mathbf{Z}/p)$.

Our main result in this section (Theorem 6.2) will determine all differentials in the spectral sequence $a_1^{-1}E_rTHH(\ell)$.

First recall that Theorem 4.1 gives an isomorphism

$$E_2THH(\ell) \cong \mathbf{Z}/p\,[a_1] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu], \tag{13}$$

where bidegree $a_1 = (2(p-1),1)$, bidegree $\lambda_i = (2p^i - 1, 0)$, and bidegree $\mu = (2p^2, 0)$; we therefore have an isomorphism

$$a_1^{-1}E_2THH(\ell) \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbf{Z}/p\,[\mu]. \tag{14}$$

Next we consider the E_∞ term.

PROPOSITION 6.1. *The map $\tilde{\iota} : \ell \to THH(\ell)$ induces an isomorphism*

$$a_1^{-1}\tilde{\iota}_* : \mathbf{Z}/p\,[a_1, a_1^{-1}] \xrightarrow{\cong} a_1^{-1}E_\infty THH(\ell).$$

Proof. Essentially, the proposition is a consequence of the isomorphism

$$v_1^{-1}\tilde{\iota}_* : v_1^{-1}\pi_*(\ell; \mathbf{Z}/p) \xrightarrow{\cong} v_1^{-1}\pi_*(THH(\ell); \mathbf{Z}/p) \tag{15}$$

given by Theorem 5.2, but we need to be a bit careful with the details since we are not assuming any relationship between $a_1^{-1}E_\infty(THH(\ell))$ and $v_1^{-1}\pi_*(THH(\ell); \mathbf{Z}/p)$.

First of all, suppose that a_1^n is in the kernel of $a_1^{-1}\tilde{\iota}_*$. We may assume that n is positive and that a_1^n is zero in $E_\infty THH(\ell)$. We know by (15) that the image of v_1^n in $\pi_*(THH(\ell); \mathbf{Z}/p)$ must be nonzero, and it follows that this image must represent a nonzero element of $E_\infty THH(\ell)$ with dimension $2n(p-1)$ and filtration higher than that of a_1^n. But an inspection of equation (13) above shows that there is no such element, so a_1^n must be nonzero in $E_\infty THH(\ell)$ after all. This shows that the map $a_1^{-1}\tilde{\iota}_*$ is a monomorphism.

Next suppose that $x \in a_1^{-1}E_\infty(THH(\ell))$ is a nonzero element in the cokernel of $a_1^{-1}\tilde{\iota}_*$. Without loss of generality we may assume that x is actually an element of $E_\infty(THH(\ell))$. Let $\xi \in \pi_*(THH(\ell); \mathbf{Z}/p)$ be an element representing x. By equation (15), there must be an n such that $\xi - v_1^n$ is killed by a power of v_1. But we have seen above that x must be in lower filtration than a_1^n, so $\xi - v_1^n$ must represent x, and it follows that some power of a_1 must kill x. This contradicts the assumption that x is nonzero in $a_1^{-1}E_\infty(THH(\ell))$, and we conclude that $a_1^{-1}\tilde{\iota}_*$ is an epimorphism. □

We will see in a moment that there is only one pattern of differentials in the spectral sequence $a_1^{-1}E_r(THH(\ell)$ which is consistent with Proposition 6.1. Before giving a formal statement, let us consider the first few differentials. Equation 14 implies that λ_1 and λ_2 are permanent cycles for dimensional reasons (note that λ_2 cannot hit a_1^{p+1} because of Proposition 6.1), and that μ survives at least to $a_1^{-1}E_p$. On the other hand, if $d_p(\mu) = 0$ then μ is a permanent cycle and the multiplicative structure implies that there are no differentials in the spectral sequence. Since this would contradict Proposition 6.1, μ must hit the only thing which is dimensionally possible, and thus we have

$$d_p(\mu) \doteq a_1^p \lambda_1;$$

here and from now on we use the symbol $\doteq$ to denote equality up to multiplication by a nonzero element of $\mathbf{Z}/p$. We can now conclude that

$$a_1^{-1}E_{p+1} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_2, \lambda_1\mu^{p-1}) \otimes \mathbf{Z}/p\,[\mu^p],$$

where λ_2, $\lambda_1\mu^{p-1}$, and μ^p are in bidegrees $(2p^2-1, 0)$, $(2p^3-2p^2+2p-1, 0)$, and $(2p^3, 0)$ respectively. For dimensional reasons μ^p must survive at least to $a_1^{-1}E_{p^2}$, so we have $a_1^{-1}E_{p^2} \cong a_1^{-1}E_{p+1}$, but as before we must have $d_{p^2}(\mu^p) \neq 0$, and this implies

$$d_{p^2}(\mu^p) \doteq a_1^{p^2}\lambda_2$$

and

$$a_1^{-1}E_{p^2+1} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_1\mu^{p-1}, \lambda_2\mu^{p^2-p}) \otimes \mathbf{Z}/p\,[\mu^{p^2}].$$

Similarly, we have

$$d_{p^3+p}(\mu^{p^2}) \doteq a_1^{p^3+p}\lambda_1\mu^{p-1}$$

and

$$a_1^{-1}E_{p^3+p+1} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_2\mu^{p^2-p}, \lambda_1\mu^{p^3-p^2+p-1}) \otimes \mathbf{Z}/p\,[\mu^{p^3}].$$

The general pattern is given by our next result.

THEOREM 6.2. *Define a sequence of numbers $r(n)$ by the equations*

$$r(n) = p^n + p^{n-2} + \cdots + p$$

if n is odd, and

$$r(n) = p^n + p^{n-2} + \cdots + p^2$$

if n is even. Then

(a) *the only nonzero differentials in the spectral sequence $a_1^{-1}E_r THH(\ell)$ occur in the terms $a_1^{-1}E_{r(n)}$,*

(b) *$a_1^{-1}E_{r(n)}$ has the form*

$$\mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_n, \lambda_{n+1}) \otimes \mathbf{Z}/p\,[\mu^{p^{n-1}}],$$

where the λ_n *are defined for* $n \geq 3$ *by* $\lambda_n = \lambda_{n-2}\mu^{p^{n-3}(p-1)}$, *and*

(c) $d_{r(n)}$ *is determined by the multiplicative structure and the formulas*

$$d_{r(n)}(\lambda_n) = 0,\ d_{r(n)}(\lambda_{n+1}) = 0,\ \textit{and}\ d_{r(n)}(\mu^{p^{n-1}}) \doteq a_1^{r(n)}\lambda_n.$$

Proof. We have already verified the assertions for $n \leq 3$. Let us assume inductively that

$$a_1^{-1}E_{r(n)} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_n, \lambda_{n+1}) \otimes \mathbf{Z}/p\,[\mu^{p^{n-1}}],$$

with λ_n and λ_{n+1} defined as above. The bidegree of $\mu^{p^{n-1}}$ is $(2p^{n+1}, 0)$, and an easy inductive argument shows that the bidegree of λ_m is $(2p^m - 2p^{m-1} + \cdots + 2p - 1, 0)$ for m odd and $(2p^m - 2p^{m-1} + \cdots + 2p^2 - 1, 0)$ for m even. The determination of the next differential follows the same pattern as the argument given for the first three differentials. By inductive hypothesis one knows that λ_n is an infinite cycle. The only way λ_{n+1} could fail to be an infinite cycle is if $d_{r(n)}(\lambda_{n+1}) \doteq a_1^{r(n)+1}$, which is impossible by Proposition 6.1. On the other hand, $\mu^{p^{n-1}}$ must fail to be an infinite cycle, and the only way this can happen is for the equation

$$d_{r(n)}(\mu^{p^{n-1}}) \doteq a_1^{r(n)}\lambda_n$$

to hold. We now have

$$a_1^{-1}E_{r(n)+1} \cong \mathbf{Z}/p\,[a_1, a_1^{-1}] \otimes \Lambda(\lambda_{n+1}, \lambda_{n+2}) \otimes \mathbf{Z}/p\,[\mu^{p^n}],$$

and for dimensional reasons we must have $a_1^{-1}E_{r(n)+1} = a_1^{-1}E_{r(n+1)}$. This completes the proof of the inductive step and thereby of the theorem. □

7. The mod p homotopy of $THH(\ell)$. We can now state a theorem which, together with Theorem 6.2, completely determines the differentials in the Adams spectral sequence $E_rTHH(\ell)$ converging to $\pi_*(THH(\ell); \mathbf{Z}/p)$. From now on we will write S for the graded ring $\mathbf{Z}/p\,[a_1]$.

THEOREM 7.1.

(a) *Let* r *be at least* 2. *As an* S *module,* $E_rTHH(\ell)$ *is generated by elements in filtration 0, and it is a direct sum of copies of* S *and of* $S/(a_1^i)$ *for* $i < r$.

(b) *For each* $r \geq 2$, *the localization map*

$$E_rTHH(\ell) \to a_1^{-1}E_rTHH(\ell)$$

is a monomorphism in filtrations $\geq r - 1$.

(c) *For each r, the differentials in* $E_rTHH(\ell)$ *are determined by those in* $a_1^{-1}E_rTHH(\ell)$*; more precisely, each element of* $E_rTHH(\ell)$ *has the same differential as its image in* $a_1^{-1}E_rTHH(\ell)$.

As a consequence, we obtain an explicit description of $\pi_*(THH(\ell); \mathbf{Z}/p)$. Recall the numbers $r(n)$ and the elements λ_n introduced in the statement of Theorem 6.2.

COROLLARY 7.2.

(a) *For each* $n \geq 1$ *and each nonnegative integer m with* $m \not\equiv p-1 \bmod p$ *there are elements* $x_{n,m}$ *and* $x'_{n,m}$ *in* $\pi_*(THH(\ell); \mathbf{Z}/p)$ *such that*

(i) $x_{n,m}$ *projects to* $\lambda_n \mu^{mp^{n-1}}$ *in* $E_\infty^{*,0}$.
(ii) $x'_{n,m}$ *projects to* $\lambda_n \lambda_{n+1} \mu^{mp^{n-1}}$ *in* $E_\infty^{*,0}$.
(iii) $v_1^{r(n)} x_{n,m} = v_1^{r(n)} x'_{n,m} = 0$.

(b) *As an S module,* $\pi_*(THH(\ell); \mathbf{Z}/p)$ *is generated by the unit element* $1 \in \pi_0$ *and the elements* $x_{n,m}$ *and* $x'_{n,m}$*. The only relations are* $v_1^{r(n)} x_{n,m} = 0$ *and* $v_1^{r(n)} x'_{n,m} = 0$.

Proof of Theorem 7.1. We begin with part (a). Suppose inductively that this result holds for some r (Theorem 4.1 implies that it holds for $r = 2$). Choose a basis $\{\alpha_i\}$ for the $\mathbf{Z}/p$-vector space

$$\{ x \in E_r^{*,0} \mid a_1^{r-1} x = 0 \}.$$

The differential d_r of any element α_i must be an element in filtration r which is killed by a_1^{r-1}. But the inductive hypothesis implies that all nontrivial elements in filtration r have infinite a_1-order, and it follows that all elements α_i are d_r-cycles. Let us next choose a set $\{\beta_j\} \subset E_r^{*,0}$ so that the set $\{d_r\beta_j\}$ is a basis for the image of

$$d_r : E_r^{*,0} \to E_r^{*,r},$$

and let us choose elements $\gamma_j \in E_r^{*,0}$ with $a_1^r \gamma_j = d_r \beta_j$. Then the α_i and γ_j are linearly independent d_r-cycles, so we can choose elements δ_k in $E_r^{*,0}$ such that the set

$$\{\alpha_i\} \cup \{\gamma_j\} \cup \{\delta_k\}$$

is a basis for the d_r-cycles in $E_r^{*,0}$. It is now clear that the set

$$\{\alpha_i\} \cup \{\beta_j\} \cup \{\gamma_j\} \cup \{\delta_k\}$$

is a basis for $E_r^{*,0}$ and that the differential d_r is determined by the formulas

$$d_r\alpha_i = 0, \qquad d_r\beta_j = a_1^r\gamma_j, \qquad d_r\gamma_j = 0, \text{ and } \qquad d_r\delta_k = 0.$$

This in turn implies that E_{r+1} is generated as an S module by the elements α_i (each of which is killed by a_1^{r-1}), γ_j (killed by a_1^r) and δ_k (each of which has infinite a_1-order). This completes the inductive step for part (a).

Part (b) follows from part (a) since all elements of E_r in filtration $\geq r-1$ have infinite a_1-order.

For part (c), we need only show that if an element x of E_r maps to a cycle in $a_1^{-1}E_r$ then it is already a cycle in E_r. By part (a) we know that x has the form $a_1^n y$ with $y \in E_r^{*,0}$, and by part (b) we know that $d_r y = 0$; the result follows. □

Remark. An inspection of the proof of Theorem 7.1 shows that the result remains valid with $THH(\ell)$ replaced by any ℓ module X for which E_2X is a direct sum of copies of S and of $S/(a_1)$ generated by elements in filtration 0.

Proof of Corollary 7.2. It suffices to show that the corresponding statement about E_∞ holds, i.e, that the elements $\lambda_n\mu^{mp^{n-1}}$ and $\lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ are infinite cycles, that together with the unit element they form a basis for $E_\infty^{*,0}$, and that the a_1-order of each $\lambda_n\mu^{mp^{n-1}}$ and $\lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ is $r(n)$. Let us write $\xi_{n,m}$ and $\xi'_{n,m}$ for $\lambda_n\mu^{mp^{n-1}}$ and $\lambda_n\lambda_{n+1}\mu^{mp^{n-1}}$ respectively. We shall show by induction on n that $E_{r(n)}(THH(\ell))$ has the form

$$M_n \oplus \left(S \otimes \Lambda(\lambda_n, \lambda_{n+1}) \otimes \mathbf{Z}/p\,[\mu^{p^{n-1}}]\right),$$

where M_n is generated by the set

$$\{\, \xi_{k,m}, \xi'_{k,m} \mid k < n \,\}$$

with relations

$$a_1^{r(k)}\xi_{k,m} = a_1^{r(k)}\xi'_{k,m} = 0,$$

and this will imply the corollary. So let us assume that this statement is true for some n (it clearly holds for $n = 1$). Then Theorems 7.1(c) and 6.2 imply that the only nontrivial differentials on $E_{r(n)}^{*,0}$ are given by the formulas

$$d_{r(n)}\mu^{(m+1)p^{n-1}} = (m+1)a_1^{r(n)}\lambda_n\mu^{mp^{n-1}} \doteq a_1^{r(n)}\xi_{n,m}$$

and

$$d_{r(n)}\lambda_{n+1}\mu^{(m+1)p^{n-1}} = (m+1)a_1^{r(n)}\lambda_n\lambda_{n+1}\mu^{mp^{n-1}} \doteq a_1^{r(n)}\xi'_{n,m}$$

for $m \not\equiv (p-1) \bmod p$. Together with Theorem 7.1(a), these formulas imply that $E_{r(n)+1}$ has the form

$$M_n \oplus N_{n+1} \oplus \left(S \otimes \Lambda(\lambda_{n+1}, \lambda_n \mu^{(p-1)p^{n-1}}) \otimes \mathbf{Z}/p\,[\mu^{p^n}]\right),$$

where N_{n+1} has generators $\xi_{n,m}$ and $\xi'_{n,m}$ and relations

$$a_1^{r(n)} \xi_{n,m} = a_1^{r(n)} \xi'_{n,m} = 0.$$

To complete the proof of the inductive step, we need only observe that $E_{r(n)+1} = E_{r(n+1)}$ (by Theorems 7.1(c) and 6.2(a)), and that $M_n \oplus N_{n+1} = M_{n+1}$ and $\lambda_n \mu^{(p-1)p^{n-1}} = \lambda_{n+2}$ (by the definitions). □

8. The homotopy type of $THH(L)$. Let us fix a prime p and write L for the p-adic completion of the Adams summand of complex K-theory. In this section we *assume* that L has an A_∞ structure, so that $THH(L)$ is defined,[9] and we prove the following theorem. We shall denote the rationalization of a spectrum X by $X_{\mathbf{Q}}$.

THEOREM 8.1. $THH(L) \simeq L \vee (\Sigma L)_{\mathbf{Q}}$.

For the proof of Theorem 8.1 we shall use Bousfield's theory of localization. First we recall that Proposition 2.9 of [10] gives a homotopy pullback diagram

$$\begin{array}{ccc} THH(L) & \longrightarrow & \prod_q THH(L)_{S\mathbf{Z}/q} \\ \downarrow & & \downarrow \\ THH(L)_{\mathbf{Q}} & \longrightarrow & \left(\prod_q THH(L)_{S\mathbf{Z}/q}\right)_{\mathbf{Q}} \end{array}, \tag{16}$$

where q runs over all primes and $S\mathbf{Z}/q$ denotes a mod q Moore spectrum; note that the $S\mathbf{Z}/q$-localization of a spectrum is its q-adic completion.

We begin by determining the spectra $THH(L)_{S\mathbf{Z}/q}$ when q is not equal to the prime p which we fixed at the beginning of the section. The mod q homotopy of L is trivial for every $q \neq p$ (see for example Proposition 2.5 of [10]) and the same holds for $THH(L)$, since $THH(L)$ can be obtained from L by smash products, passage to cofibers, and a direct limit. In the language of [10], this says that $THH(L)$ is $S\mathbf{Z}/q$-acyclic, and hence $THH(L)_{S\mathbf{Z}/q}$ is trivial for each $q \neq p$.

[9] Andy Baker has proved a result in [5] which is expressed in somewhat different language but probably implies what we need. Moreover, the first author has an unpublished proof that L is actually E_∞, a result which has been simplified and greatly extended (to a certain version of $E(n)$) in recent work of Mike Hopkins and Haynes Miller.

Next we consider $THH(L)_{S\mathbf{Z}/p}$.

PROPOSITION 8.2. *The inclusion* $\tilde{\iota} : L \to THH(L)$ *induces an equivalence*

$$L \xrightarrow{\simeq} THH(L)_{S\mathbf{Z}/p}.$$

Proof. First consider the following diagram

$$\begin{array}{ccc} K(1)_*\ell & \xrightarrow{\tilde{\iota}_*} & K(1)_*THH(\ell) \\ \downarrow & & \downarrow \\ K(1)_*L & \xrightarrow{\tilde{\iota}_*} & K(1)_*THH(L) \end{array}$$

The left-hand vertical arrow is an isomorphism by Lemma 5.6., since in the notation of that Lemma L is the direct limit of the system

$$\ell_{S\mathbf{Z}/p} \xrightarrow{v_\ell} \Sigma^{-2(p-1)}\ell_{S\mathbf{Z}/p} \xrightarrow{v_\ell} \Sigma^{-4(p-1)}\ell_{S\mathbf{Z}/p} \cdots;$$

(note that $\ell_{S\mathbf{Z}/p}$ has the same mod p K-homology as ℓ since already the smash product of $\ell_{S\mathbf{Z}/p}$ with the mod p Moore spectrum $S\mathbf{Z}/p$ is homotopy equivalent to $\ell \wedge S\mathbf{Z}/p$). The spectral sequence of Proposition 3.1 now implies that the right-hand vertical arrow is also an isomorphism. Furthermore, Theorem 5.1 implies that the upper $\tilde{\iota}_*$ is an isomorphism, so we conclude that the lower $\tilde{\iota}_*$ is an isomorphism. Since mod p K-theory is a direct sum of copies of $K(1)$, we have shown that $\tilde{\iota}$ induces an isomorphism

$$K_*(L; \mathbf{Z}/p) \xrightarrow{\tilde{\iota}_*} K_*(THH(L); \mathbf{Z}/p)$$

If we write $K\mathbf{Z}/p$ for the spectrum representing mod p K-theory, then the definition of localization now gives an equivalence

$$L_{K\mathbf{Z}/p} \xrightarrow{\simeq} THH(L)_{K\mathbf{Z}/p},$$

and by [10, Proposition 2.11] we can rewrite this in the form

$$(L_K)_{S\mathbf{Z}/p} \xrightarrow{\simeq} (THH(L)_K)_{S\mathbf{Z}/p}.$$

But L is K-local (by [10, Theorem 4.11(i)]). Moreover, K-localization commutes with smash products, cofiber sequences, and direct limits (see [10, Corollary 4.7]), and since $THH(L)$ can be built up from L by these operations, we see that $THH(L)$ is also K-local. So we can now rewrite our last equivalence as follows:

$$L_{S\mathbf{Z}/p} \xrightarrow{\simeq} THH(L)_{S\mathbf{Z}/p}.$$

But $L_{S\mathbf{Z}/p}$ is equivalent to L, since we have assumed that L is p-adically complete. This concludes the proof. □

Our next result, which is an immediate consequence of what has been shown so far, determines the lower right-hand corner of diagram (16).

COROLLARY 8.3. *The map* $\tilde{\iota} : L \to THH(L)$ *induces an equivalence*

$$L_{\mathbf{Q}} \xrightarrow{\simeq} \left(\prod_q THH(L)_{S\mathbf{Z}/q} \right)_{\mathbf{Q}} .$$

To complete our analysis of diagram (16), it remains to determine the lower left corner. Recall the natural transformation

$$\tilde{\sigma} : \Sigma R \to THH(R)$$

defined in Section 3.

PROPOSITION 8.4. *The map*

$$\tilde{\iota} \vee \tilde{\sigma} : L \vee \Sigma L \to THH(L)$$

induces an equivalence

$$L_{\mathbf{Q}} \vee (\Sigma L)_{\mathbf{Q}} \xrightarrow{\simeq} THH(L)_{\mathbf{Q}}.$$

Proof. Consider the spectral sequence

$$\mathbf{HH}_*(H_*(L; \mathbf{Q})) \Rightarrow H_*(THH(L); \mathbf{Q})$$

provided by Proposition 3.1. Since rational homology is the same thing as rational stable homotopy, we have

$$H_*(L; \mathbf{Q}) \cong \mathbf{Q}[v_1, v_1^{-1}].$$

Since Hochschild homology commutes with algebraic localization, we have

$$\mathbf{HH}_*(\mathbf{Q}[v_1, v_1^{-1}]) \cong \mathbf{Q}[v_1, v_1^{-1}] \otimes_{\mathbf{Q}[v_1]} \mathbf{HH}_*(\mathbf{Q}[v_1]),$$

and thus the E_2-term of the spectral sequence is

$$\mathbf{Q}[v_1, v_1^{-1}] \otimes \Lambda(\sigma(v_1))$$

with v_1 in bidegree $(0, 2(p-1))$ and $\sigma(v_1)$ in bidegree $(1, 2(p-1))$. In particular the spectral sequence collapses. Propositions 3.1 and 3.2 now imply that $\tilde{\iota} \vee \tilde{\sigma}$ induces a rational homology isomorphism, and hence an equivalence of rationalizations as required. □

To sum up, we have now shown that diagram (16) has the following form:

$$\begin{array}{ccc} THH(L) & \longrightarrow & L \\ \downarrow & & \downarrow \\ L_{\mathbf{Q}} \vee (\Sigma L)_{\mathbf{Q}} & \longrightarrow & L_{\mathbf{Q}}. \end{array}$$

Furthermore, the restriction of the lower horizontal arrow to $(\Sigma L)_{\mathbf{Q}}$ must be trivial, since $L_{\mathbf{Q}}$ is a wedge of even-dimensional rational Eilenberg-MacLane spectra and $(\Sigma L)_{\mathbf{Q}}$ is a wedge of odd-dimensional Eilenberg-MacLane spectra. The definition of homotopy pullback now gives a map

$$L \vee (\Sigma L)_{\mathbf{Q}} \to THH(L)$$

and the Mayer-Vietoris sequence of the homotopy pullback diagram shows that this map is a homotopy equivalence. This completes the proof of Theorem 8.1.

9. An E_∞-structure for the Adams summand. In [1] J.F. Adams constructed an idempotent cohomology operation ϵ in the theory

$$X \longmapsto K(X; \mathbf{Z}_{(p)}) = [X, BU_{(p)}]$$

such that

$$\epsilon(S^n) : K(S^n; \mathbf{Z}_{(p)}) \longrightarrow K(S^n; \mathbf{Z}_{(p)})$$

is the identity if $n \equiv 0 \bmod 2(p-1)$ and the zero map otherwise. Since $\epsilon(X)$ is idempotent, the image groups $\epsilon(X)K(X)$ form a representable functor. The representing space W satisfies a variant of Bott periodicity (but with 2-fold periodicity replaced by $2(p-1)$-fold periodicity) and in the usual way W gives rise to a a periodic spectrum. Killing the negative dimensional homotopy groups of this spectrum gives the *connective Adams summand* which we shall denote by ℓ_{Adams}. There are maps

$$\theta : \ell_{\text{Adams}} \to bu_{(p)}$$

and

$$\phi : bu_{(p)} \to \ell_{\text{Adams}}$$

such that $\phi \circ \theta$ is the identity and $\theta \circ \phi$ represents the idempotent ϵ; in particular ℓ_{Adams} is indeed a summand of $bu_{(p)}$. The notation ℓ_{Adams} has been chosen in order to distinguish this object from that which we define next.

Definition 9.1. Let $p \neq 2$ be a prime and choose a prime q such that the residue class of q generates the group of units $\mathbf{Z}/p^*$. Let $\mathbf{F}_{q^{p^i}}$ be the finite field of order q^{p^i} and let k' be the field $\bigcup_{i=0}^{\infty} \mathbf{F}_{q^{p^i}}$. Then ℓ is the algebraic K-theory spectrum $K(k')$.

This is the definition of ℓ which we have been using throughout the paper. Its advantage for us lies in the fact that algebraic K-theory spectra have E_∞ structures (see [24, Section 8.1]), whereas there is no known way to obtain such a structure from Adams' original construction. The idea of using the field k' to define ℓ was suggested by Section 10 of Aguadé's paper [3].

We warn the reader that ℓ and ℓ_{Adams} are not equivalent spectra. Our next result, which is intended to provide motivation for Definition 9.1, shows that there is nevertheless a close relationship between ℓ and ℓ_{Adams}. Here, and for the rest of the section, we revert to the traditional notation $X_p^\wedge$ for the p-adic completion of X (instead of the notation $X_{S\mathbf{Z}/p}$ which was used in the previous section).

PROPOSITION 9.2. *There is a map of spectra*

$$\lambda' : K(k') \longrightarrow \ell_{\mathrm{Adams}}$$

such that the induced map of p-adic completions

$$(\lambda')_p^\wedge : K(k')_p^\wedge \longrightarrow (\ell_{\mathrm{Adams}})_p^\wedge$$

is an equivalence.

This result says that ℓ is a "discrete model" for ℓ_{Adams} in the same way that the K-theory spectrum of the algebraic closure $\bar{\mathbf{F}}_q$ is a discrete model for bu (see [24, page 217]).

Proof. First recall that according to [24, Theorem 2.8, page 218] the Brauer lifting of characters induces a map of spectra

$$K(\bar{\mathbf{F}}_q) \longrightarrow bu$$

whose completion is an equivalence whenever q is a prime different from p.

Next note that it is not necessary to go all the way to the algebraic closure $\bar{\mathbf{F}}_q$ to obtain such an equivalence. By [26, page 577] the cohomology $H^*(GL(k); \mathbf{Z}/p)$

for any algebraic extension k of $\mathbf{F}_q$ depends only on the image of

$$Gal(\bar{\mathbf{F}}_q/k) \longrightarrow Aut(\mu_{p^\infty}),$$

where μ_{p^∞} is the group of pth roots of unity in $\bar{\mathbf{F}}_q$. So, for instance, if we take

$$k = \mathbf{F}_q\,[\mu_{p^\infty}] = \bigcup_{i=0}^{\infty} \mathbf{F}_{q^{(p-1)p^i}},$$

then we have a chain of cohomology isomorphisms

$$H^*(BU; \mathbf{Z}/p) \to H^*(GL(\bar{\mathbf{F}}_q); \mathbf{Z}/p) \to H^*(GL(k); \mathbf{Z}/p)$$

and the proof of [24, Theorem VIII.2.8] shows that Brauer lifting induces a map

$$\lambda: \ K(k) \longrightarrow bu$$

whose p-adic completion is an equivalence.

We can now define the map λ' by means of the diagram

$$\begin{array}{ccc} K(k') & \xrightarrow{\lambda'} & (\ell_{\mathrm{Adams}}) \\ {\scriptstyle K(j)}\downarrow & & \uparrow \phi \\ K(k) & \xrightarrow{\lambda} & bu, \end{array}$$

where we write $K(j)$ for the map of K-theory spectra induced by the inclusion of fields $j: \ k' \subset k$. To prove the proposition, it suffices to show that λ' induces an isomorphism on mod p homotopy groups. What we already know is that λ is an isomorphism of mod p homotopy groups in all dimensions, and that ϕ is an isomorphism of mod p homotopy in dimensions $i \equiv 0 \bmod 2(p-1)$ and is otherwise zero. Therefore, what we actually have to do is to verify that

$$\pi_n(K(k'); \mathbf{Z}/p) = \begin{cases} \mathbf{Z}/p & \text{if } n \equiv 0 \bmod 2(p-1) \\ 0 & \text{otherwise} \end{cases}$$

and to prove that $K(j)$ induces an isomorphism in mod p homotopy in dimensions $i \equiv 0 \bmod 2(p-1)$. To calculate the homotopy groups we use Quillen's results

$$K_{2j}(\mathbf{F}_{q^r}) = 0$$

and

$$K_{2j-1}(\mathbf{F}_{q^r}) \cong \mathbf{Z}/(q^{r \cdot j} - 1).$$

The tower

$$\mathbf{F}_q \subset \mathbf{F}_{q^p} \subset \cdots \subset \mathbf{F}_{q^{p^i}} \subset \cdots \subset k'$$

implies

$$K_n(k') = \lim_i K_n(\mathbf{F}_{q^{p^i}}),$$

so obviously

$$K_0(k') \cong \mathbf{Z}$$

and

$$K_{2j}(k') = 0.$$

Since we will eventually take homotopy with $\mathbf{Z}/p$-coefficients, we just have to compute the p-torsion parts of the groups $K_{2j-1}(k')$. Since we assumed that q generates $\mathbf{Z}/p^*$ we see that p divides $q^{p^i \cdot j} - 1$ if and only if $j \equiv 0 \bmod (p-1)$, and that if $j \equiv 0 \bmod (p-1)$, then p^i divides $q^{p^i \cdot j} - 1$. Going back to the formula

$$K_{2j-1}(\mathbf{F}_{q^{p^i}}) \cong \mathbf{Z}/(q^{p^i \cdot j} - 1),$$

and using the fact that each higher K-group of a finite field is cyclic, we conclude that

$$K_{2j-1}(k')_{(p)} = \begin{cases} \mathbf{Z}/p^\infty & \text{if } j \equiv 0 \bmod (p-1) \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\pi_n(K(k'); \mathbf{Z}/p) = \begin{cases} \mathbf{Z}/p & \text{if } n \equiv 0 \bmod 2(p-1) \\ 0 & \text{otherwise} \end{cases}$$

as desired. To calculate the map

$$K(j)_* : \ \pi_{2n}(K(k); \mathbf{Z}/p) \longrightarrow \pi_{2n}(K(k'); \mathbf{Z}/p)$$

we appeal to the remark on page 585 of [26]: if k/k' is any extension of fields algebraic over $\mathbf{F}_q$ with Galois group G, then

$$K_*(k') \longrightarrow K_*(k)$$

induces an isomorphism

$$K_*(k') \xrightarrow{\cong} K_*(k)^G,$$

where the superscript G denotes G-invariant elements. The remark is also true for K-groups with coefficients, and in our situation it is easy to see that the extension k/k' is algebraic with Galois group $G \cong \mathbf{Z}/(p-1)$, so the remark applies. But the action of G on the cyclic groups $K_{2j}(k; \mathbf{Z}/p)$ is either trivial, or it has trivial invariants, so we conclude that

$$K(j)_* : \ \pi_{2n}(K(k'); \mathbf{Z}/p) \longrightarrow \pi_{2n}(K(k'); \mathbf{Z}/p)$$

is an isomorphism when $n \equiv 0 \bmod 2(p-1)$, as required. □

We conclude this section with some technical remarks. Since the map ϕ used in the construction of λ' is not a ring map, the proof just given does not provide an equivalence of ring spectra between $K(k')$ and ℓ_{Adams}. It is possible to obtain such an equivalence by giving a more elaborate argument (essentially one must show that the composite $\lambda_p^\wedge \circ K(j)_p^\wedge$ factors through the map $\theta : (\ell_{\mathrm{Adams}})_p^\wedge \to bu_p^\wedge$).

It is also not true that Proposition 9.2 provides an E_∞ ring spectrum which is homotopy equivalent to $(\ell_{\mathrm{Adams}})_p^\wedge$. This would be true however if, as seems likely, the p-adic completion of a connective E_∞ ring spectrum is always E_∞.

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