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The Morava K-Theory of Algebraic K-Theory Spectra

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Abstract. If $n \geqslant 2$ the Morava K-theory $K(n)_*$ of an algebraic K-theory spectrum KX vanishes for any ring or scheme X. This is proved using the v_n-complexes of Hopkins and Smith, together with the following theorem. The natural map $f: Q_0S^0 \to BGL\mathbb{Z}^+$ factors through the space $\operatorname{Im} J$. In particular $f_*: \pi_*^s \to K_*\mathbb{Z}$ annihilates Coker J. These results are closely related to the Lichtenbaum–Quillen conjectures.

Keywords. Algebraic K-theory, Morava K-theory.

0. Introduction

The K-groups of a ring R are the homotopy groups of the space $BGLR^+ \times K_0R$, where $BGLR^+$ is Quillen's plus construction. The machinery of May [18] or Segal [30] shows that $BGLR^+ \times K_0R$ is an infinite loop space, and so defines a spectrum KR. Similarly, any scheme X determines a spectrum KX whose homotopy groups are the K-groups of the category of locally free sheaves on X, by definition. There are also spectra GR (resp. GX) obtained from the category of all finitely generated R-modules (resp. coherent sheaves). Algebraic K-theory thus enters the realm of stable homotopy theory. Over the last few years, there has been dramatic progress toward understanding stable homotopy theory from a global, or qualitative, point of view, due to Mike Hopkins, Ethan Devinatz and Jeff Smith [7, 13, 14]. It is natural to ask what implications these developments have for algebraic K-theory. For example, in [14] (see [13]) it is shown that the 'prime fields' of the stable homotopy category are precisely the Morava K-theories $K(n)$, for the various primes p; here mod p homology must be included as $K(\infty)$. We recall here that $K(0)$ is rational homology and $K(1)$ is a summand of mod p complex K-theory. The 'higher' Morava K-theories $(2 \leqslant n < \infty)$ are somewhat more mysterious, although well understood algebraically. In any case, the work of Hopkins *et al.* increases the significance of the natural question: What happens when higher Morava K- theory meets higher algebraic K-theory? Answer: Nothing, the two theories are orthogonal.

THEOREM A. *For all primes p and all $n \geqslant 2$, $K(n)_*K\mathbb{Z} = 0$.*

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COROLLARY. *Let E be any module spectrum over $K\mathbb{Z}$. For example $E = KR$ (R any ring), $E = KX$ (X any scheme), or $E = GX$. Then for all p and all $n \geqslant 2$, $K(n)_* E = 0$.*

Here we recall that the K-theory spectrum of a commutative ring R is a commutative ring spectrum, and if X is a scheme over spec R then KX is a module spectrum over KR [18, 19, 38]. The corollary then follows for trivial reasons. By definition, there is a map $K\mathbb{Z} \wedge E \to E$ such that the composite $S^0 \wedge E \xrightarrow{i \wedge 1} K\mathbb{Z} \wedge E \to E$, where i is the unit map of the ring spectrum $K\mathbb{Z}$, is the identity. In particular, E is a retract of $K\mathbb{Z} \wedge E$, so if $X \wedge K\mathbb{Z} \cong *$ for some spectrum X, $X \wedge K\mathbb{Z} \wedge E \cong *$ and, hence, $X \wedge E \cong *$.

Theorem A is proved by showing the 'v_n-periodic homotopy' of $K\mathbb{Z}$ is zero, $n \geqslant 2$. This in turn is deduced from a theorem on the space level. Let J_p denote the 'Image of J' space at the prime p (see Section 2 for the precise definition). Let $QS^0 = \varinjlim_n \Omega^n S^n$. There is a natural map $r: QS^0 \to J_p \times \mathbb{Z}$. We also have $f: QS^0 \to BGL\mathbb{Z}^+ \times \mathbb{Z}$, obtained by applying the functor Ω^∞: (spectra) $\to$ (infinite loop spaces) to the unit map $S^0 \xrightarrow{i} K\mathbb{Z}$.

THEOREM B. *After localization at p, there is a homotopy commutative diagram*

$$\begin{tikzcd} QS^0 \arrow[r, "f"] \arrow[d, "r"'] & BGL\mathbb{Z}^+ \times \mathbb{Z} \\ J_p \times \mathbb{Z} \arrow[ur, "h"'] & \end{tikzcd}$$

COROLLARY. *There is a commutative diagram*

$$\begin{tikzcd} \pi_*^s \arrow[r, "f_*"] \arrow[d, "r_*"'] & K_*\mathbb{Z} \\ \pi_*(J_p \times \mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

Here $\pi_*(J_p \times \mathbb{Z})$ is precisely Im J, the image of the classical J-homomorphism, if p is odd. If $p = 2$, $\pi_* J_p$ consists of Im J together with the μ-family constructed by Adams. The μ-classes are elements of order two in $\pi_{8k+1}^s, \pi_{8k+2}^s$, which must be regarded as honorary members of Im J. Combining the corollary with earlier results, the map f_* is completely determined. Quillen [25] showed that the restriction of $f_*: \pi_n^s \to K_n\mathbb{Z}$ to Im J is split injective at odd primes or if $n = 7 \bmod 8$ and injective if $n = 3 \bmod 8$. He also showed that the μ-family injects onto a direct summand and conjectured that f_* is zero on the remaining Im J elements in degrees $n = 0$, $1 \bmod 8$. This conjecture was proved by Waldhausen [36]. The only remaining question is whether f_* is split injective on the 2-component $\mathbb{Z}/8$ of Im J when $n = 3 \bmod 8$. But Lee and Szczarba [17] showed $K_3\mathbb{Z} \cong \mathbb{Z}/48$, and Browder [5] showed that the $\mathbb{Z}/16$ propagates as a direct summand in $K_{8k+3}\mathbb{Z}$ for all k. In particular, f_* does not split at 2 when $n = 3 \bmod 8$, but embeds Im J in a cyclic summand of order 16.

Theorems A and B are closely related to one formulation of the Lichtenbaum–Quillen conjectures. Let $M = M\mathbb{Z}/p^n$ denote a mod p^n Moore spectrum. A remarkable theorem of Thomason [34] shows that for suitable schemes X, the localized mod p^n K-groups $\beta^{-1}\pi_* KX \wedge M$ admit a descent spectral sequence, thereby establishing the Lichtenbaum–Quillen conjectures for these localized groups. Here $\beta \in \pi_* KX \wedge M$ is the 'Bott element'. However, Thomason and also Waldhausen [37] observed that the mapping telescope $\beta^{-1}KX \wedge M$ can be identified with $L_1(KX \wedge M)$ – the Bousfield localization of $KX \wedge M$ with respect to ordinary complex K-theory. (This observation depends on work of Snaith, Miller, Bousfield, and others; it would require too lengthy a digression to explain the dependence here.) This allows one to reformulate the Lichtenbaum–Quillen conjectures as [37]:

CONJECTURE C. *The natural map $j: KX \wedge M \to L_1(KX \wedge M)$ induces an isomorphism on π_n for n sufficiently large.*

Here we have suppressed the hypotheses on X needed for Thomason's theorem. There are also specific conjectures about just how large 'sufficiently large' is, but we will ignore this point as well. We view the conjecture as asserting that algebraic K-theory spectra should be 'local', in some weak sense, with respect to topological complex K-theory. For example, it follows easily from the work of Quillen [23] and Suslin [32, 33] that the conjecture holds for $X = \operatorname{Spec} K$, where K is a field which is either finite or separably closed. Now let F denote the fibre of j. The conjecture then asserts that $\pi_* F$ is bounded above. But any torsion spectrum F bounded above is 'dissonant' in the sense of Ravenel [26]; that is, $K(n)_* F = 0$ for all $n < \infty$. It follows [see Section 3] from this that $K(n)_* KX = 0$ for all $n \geqslant 2$. Conversely, it follows from Theorem A that F is dissonant, but not that F is bounded above. Indeed, Theorem A involves no assumptions at all on X or R, or special considerations when $p = 2$, (compare [34]), so it is clearly a much cruder statement than Conjecture C. However, it does imply that the Lichtenbaum–Quillen conjectures hold for the 'harmonic' localization of KX (see Section 3). One can also show that Conjecture C for Spec $\mathbb{Z}$ implies Theorem B, provided that the 'sufficiently large' n can be taken sufficiently small.

The proof of Theorem B can be briefly explained as follows, taking p odd for simplicity. We have $Q_0 S^0 = B\Sigma_\infty^+$ (Barratt–Priddy–Quillen) and $J_p = BGL\mathbb{F}_q^+$ (Quillen), where q is a prime which generates $(\mathbb{Z}/p^2)^*$. Thus, all three spaces in the diagram are defined in terms of classifying spaces of groups. In Section 1 we prove, in effect, an algebraic analogue of Theorem B. If G is a p-group, there is a commutative diagram

$$\begin{tikzcd} AG \arrow[r] \arrow[d] & R_{\mathbb{Z}}G \\ R_{\mathbb{F}_q}G \arrow[ur, dashed] & \end{tikzcd}$$

where AG is the Burnside ring and the solid arrows are the natural maps to the representation rings. Actually in Section 1 we replace $R_{\mathbb{F}_q}G$ by $R_{\mathbb{Q}}G$ and $R_{\mathbb{Z}}G$ by $R_{\mathbb{Z}[1/p]}G$. But with our hypotheses on p and q, the decomposition map $R_{\mathbb{Q}}G \to R_{\mathbb{F}_q}G$ is an isomorphism, and $R_{\mathbb{Z}}G \to R_{\mathbb{Z}[1/p]}G$ is an isomorphism by a standard localization argument, since G is a p-group. So the effect is the same. In Section 2 we use some by now rather standard homotopy theoretic methods to reduce Theorem B to the algebraic analogue.

In Section 3 we prove Theorem A. The key ingredients are Theorem B, and the existence of v_n-self-maps of finite spectra [12]. For the convenience of the reader, in Section 4 we provide short proofs of two standard theorems on QS^0 and $\mathrm{Im}\, J$ which are needed for Theorem B.

1. Permutation Representations of Finite p-Groups

Throughout this section G is a finite p-group, $R = \mathbb{Z}[1/p]$, and $F_n = \mathbb{Q}[\xi_{p^n}]$, where $\xi_{p^n} = e^{2\pi i/p^n}$. Our main goal is to prove Theorem 1.12 on permutation representations of G over R. We begin by reviewing the representation theory of G over $\mathbb{Q}$, following Roquette [29].

The p-group G has *normal rank one* if every normal Abelian subgroup is cyclic. The classification of such groups is a rather lengthy but elementary exercise, or can be extracted from [9]; for the convenience of the reader we provide a short proof of Theorem 1.1 below in Section 5.

(1.1) THEOREM. *If G has normal rank one, then either G is cyclic, or else $p = 2$ and G is dihedral, semi-dihedral or quaternionic.* □

Now let V be a simple $\mathbb{Q}G$-module, $D = \mathrm{End}_{\mathbb{Q}G} V$. Regard V as a $\mathbb{Q}G - D$ bimodule. V is *primitive* if there is no nontrivial splitting $V = V_1 \oplus \cdots \oplus V_k$ as D-modules such that G permutes the V_i. We will call V *strictly primitive* if there is no such splitting as $\mathbb{Q}$-modules.

(1.2) THEOREM. (a) *G admits a faithful primitive representation over $\mathbb{Q}$ if and only if G has normal rank one.*

(b) *G admits a faithful strictly primitive representation over $\mathbb{Q}$ if and only if $G = \mathbb{Z}/p$.*

We sketch the proof, since we will need some of the details. If V is a primitive $\mathbb{Q}G$-module any normal Abelian subgroup, A must act isotypically on V. If V is faithful, this forces A cyclic. Conversely, suppose G has normal rank one. Then, using Theorem 1.1, one can easily check that G has a unique faithful irreducible representation over $\mathbb{Q}$, and that V is primitive. V can be described explicitly in the various cases as follows:

(1.3) $G = \mathbb{Z}/p^n$. Then $V = F_n$, with G acting via any monomorphism $G \to F_n^*$. Clearly $D = F_n$.

(1.4) $G = D_n$ or S_n, where D_n (resp. S_n) is the dihedral (resp. semi-dihedral) group of order 2^{n+1}, $n \geqslant 3$. Then $\mathrm{Gal}(F_n/\mathbb{Q})$ has three elements σ of order

2. The roots of unity together with σ generate a subgroup $G = \mathbb{Z}/2^n \rtimes \mathbb{Z}/2$ of $\mathrm{Aut}_{\mathbb{Q}} V(V = F_n)$, isomorphic to D_n, S_n or M_n depending on the choice of σ. When $G = D_n$ or S_n, V is primitive, with $D = F_n^\sigma$.

(1.5) $G = Q_n$, where Q_n is the generalized quaternion group of order 2^{n+1}. Let $\mathbb{H} = \mathbb{R} \cdot \{1, i, j, k\}$ denote the quaternions. Let $\mathbb{H}_n$ denote the subdivision algebra over $\mathbb{Q}$ given by $F_n \cdot 1 \oplus F_n \cdot j$. Then $\mathbb{H}_n^*$ contains Q_n as the subgroup generated by ξ_n and j, and $V = \mathbb{H}_n$. Clearly $D = \mathbb{H}_n$.

Part (b) was observed by Tornehave in [35]. A proof can be sketched as follows: By part (a), we can assume G has normal rank one. Then inspection of the faithful irreducible V described above shows that it is induced from a proper subgroup unless $G = \mathbb{Z}/p$. □

Now recall that over $\mathbb{C}$, any representation of G is induced from a one-dimensional representation. This generalizes to $\mathbb{Q}$ in two distinct ways:

(1.6) THEOREM. *Let V be a simple $\mathbb{Q}G$-module. Then*

(a) $V = \mathbb{Q}G \otimes_{\mathbb{Q}H} W$, *where* (i) *$W$ is the faithful irreducible representation of a group K of normal rank one, pulled back along some surjective homomorphism $H \to K$, and* (ii) $\mathrm{End}_{\mathbb{Q}G} V = \mathrm{End}_{\mathbb{Q}K} W$

(b) $V = QG \otimes_{\mathbb{Q}H'} W'$, *where W' is the faithful irreducible representation of $\mathbb{Z}/p$, pulled back along some surjective homomorphism $H' \to \mathbb{Z}/p$.*

Remark. All of our modules are left modules except when stated otherwise. But of course the obvious analogue of (1.6) for right modules holds as well.

The proof of (a) is a simple induction on the order of G, see [29]. Note condition (ii) shows $\mathbb{Q}G$ is quasisplit for p odd; i.e. the division rings $\mathrm{End}_{\mathbb{Q}G} V$ are all fields – in fact equal to F_n for some n. When $p = 2$ we conclude that $\mathrm{End}_{\mathbb{Q}G} V$ is either F_n, F_n^σ as in (1.4), or $\mathbb{H}_n$. Part (b) follows from (a) and Theorem 1.2b; cf. [35].

We next turn to the representation theory over R. For each simple $\mathbb{Q}G$-module V, fix once and for all an isomorphism $V \cong \mathbb{Q} \oplus_{\mathbb{Q}H} W$ as in (1.6a). Thus, W is the faithful irreducible representation of a group K of normal rank one, regarded as an H-module via a fixed surjective homomorphism $H \to K$. We can and do assume that W is the underlying $\mathbb{Q}$-vector space of a field F_n or division algebra $\mathbb{H}_n$ as in (1.3)–(1.5). Thus, in all cases W is in fact a division algebra. Furthermore if $D_V = \mathrm{End}_{\mathbb{Q}G} V$, then either $W = D_V$ or W is a field extension of D_V, and the isomorphism is as $\mathbb{Q}G - D_V$ bimodules. In each V we choose a (full) RG-lattice of the following type: first, choose a maximal R-order Λ in W (for background on maximal orders, see [28]). When $W = F$ is a field, there is no choice; $\Lambda = \mathcal{O}_F[1/p]$, where $\mathcal{O}_F$ is the ring of integers. If $W = \mathbb{H}_n$ we take $\Lambda = \mathcal{O}_{F_n}[\frac{1}{2}] \oplus \mathcal{O}_{F_n}[\frac{1}{2}] \cdot j$. Then $\Gamma_V = RG \oplus_{RH} \Lambda$ is the desired lattice. Now let $\mathcal{O}_V$ be a maximal order in D_V. Thus, $\mathcal{O}_V = \Lambda$ except in case (1.4), when $\mathcal{O}_V = \Lambda^\sigma$. Then corresponding to the Artin–Wedderburn decomposition $\mathbb{Q}G \cong \Pi M_{n_i} D_i$ we have

(1.7) THEOREM. $RG \cong \Pi M_{n_i} \mathcal{O}_i$.

Proof. Note each Λ_i is a free right $\mathcal{O}_i$-module. This is trivial except when $\mathcal{O}_i = \Lambda_i^\sigma$ as in (1.4). Then Λ_i is a projective module of rank two over $\mathcal{O}_i$. But one easily checks Λ_i is generated by 1 and ξ_{2^n} as $\mathcal{O}_i$-module, and hence is free. We conclude:

(1.8) LEMMA. *Γ_i is a free right $\mathcal{O}_i$-module.*

It follows that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}G & \xrightarrow{\alpha} & \Pi M_{n_i} D_i \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ RG & \xrightarrow[\beta]{} & \Pi M_{n_i} \mathcal{O}_i \end{array}$$

where i and j are inclusions; α is an isomorphism, and β is injective. But RG is a maximal R-order in $\mathbb{Q}G$, since $1/|G| \in R$, and hence $\alpha(RG)$ is a maximal R-order in $\Pi M_{n_i} D_i$. Since $\Pi M_{n_i} \mathcal{O}_i$ is also a (maximal) R-order, β must be an isomorphism. □

There are several Grothendieck groups associated to RG: G_0RG, $G_0^R RG$, $K_0 RG$, and $G_0' RG$. These are defined using finitely generated R-modules which are in the first three cases respectively, arbitrary, R-free, and RG-projective. $G_0'RG$ is the 'Green ring' defined using R-free RG-modules but only the RG-split exact sequences. The relevant group for us is $G_0^R RG$. However, it is worth noting that since G is a p-group and R is a principal ideal domain with $p^{-1} \in R$, all four groups are isomorphic. We define the class group $ClRG$ as the kernel of $G_0^R RG \to G_0 \mathbb{Q}G$.

(1.9) COROLLARY. *There is a split exact sequence*

$$0 \to \oplus_i Cl(\mathcal{O}_i) \to G_0^R(RG) \to G_0(\mathbb{Q}G) \to 0$$

where $Cl\mathcal{O}_i$ is the class group of $\mathcal{O}_i$.

Proof. Since $G_0^R(RG) \cong \oplus\, G_0 M_{n_i} \mathcal{O}_i$, etc., we reduce to showing

$$0 \to Cl\mathcal{O}_i \to G_0 M_{n_i} \mathcal{O}_i \to G_0 M_{n_i} D_i \to 0$$

is split exact. But this follows from Morita equivalence and the exact sequence $0 \to Cl\mathcal{O}_i \to G_0^R \mathcal{O}_i \to G_0 D_i \to 0$. □

(1.10) *Remarks.* (a) The groups $Cl\mathcal{O}_i$ are finite Abelian. In particular G_0RG is finitely generated. $Cl\mathcal{O}_i$ can be nonzero and in fact can have nontrivial p-component-for example, when $G = \mathbb{Z}/p$, p an irregular prime.

(b) It will be convenient to display the Morita equivalence used in (1.8) in an explicit form, without using the matrix algebra decomposition. Let $V_1', \ldots, V_r'$ be a complete set of simple *right* $\mathbb{Q}G$-modules. Write each V_0' in the form $W_i' \otimes_{\mathbb{Q}H} \mathbb{Q}G$, as in (1.6)a, let $\Gamma_i' = \Lambda_i' \otimes_{RH} RG$, and so on. Thus, if $\Gamma' = \oplus_i \Gamma_i'$ and $\mathcal{O} = \oplus\, \mathcal{O}_i$, Γ' is an $\mathcal{O} - RG$ bimodule, and $M \mapsto \Gamma' \otimes_{RG} M$ defines a Morita equivalence $\varphi : RG - \text{mod} \to \mathcal{O} - \text{mod}$. In particular, φ induces the isomorphism $G_0^R(RG) \cong G_0^R \mathcal{O}$.

Let AG denote the Burnside ring of G. There are natural ring homomorphisms $AG \underset{\eta}{\rightarrow} G_0RG$ and $AG \underset{\gamma}{\rightarrow} G_0\mathbb{Q}G$ defined by sending a G-set S to the corresponding permutation representation of G. As noted in [35], Theorem 1.6 (b) yields immediately the theorem of Segal:

(1.11) THEOREM. *If G is a p-group, $\gamma: AG \rightarrow G_0\mathbb{Q}G$ is surjective.* □

The main result of this section is the following theorem.

(1.12) THEOREM. *The kernel of $\eta: AG \rightarrow G_0^R(RG)$ coincides with the kernel of $\gamma: AG \rightarrow G_0(\mathbb{Q}G)$.*

(1.13) LEMMA. *Let M be a permutation module over RG. Then for all i, $\Gamma_i \otimes_{RG} M$ is a free $\mathcal{O}_i$-module.*

Proof. Let $\Gamma_i = \Lambda_i \otimes_{RK} RG$ as in Remark 1.10b. It suffices to take M of the form $RG \otimes_{RH} R$. Then $\Gamma_i \otimes_{RG} M = \Lambda_i \otimes_{RK} (RG \otimes_{RH} R)$ as $\mathcal{O}_i$-modules. Since $RG \otimes_{RH} R$ is a permutation module over RK, it splits as a direct sum of modules of the form $RK \otimes_{RL} R$, $L \subseteq K$. Hence, we are reduced to showing $\Lambda_i \otimes_{RK} (RK \otimes_{RL} R) = \Lambda_i \otimes_{RL} R$ is $\mathcal{O}_i$-free. But this is precisely the module of invariants Λ_i^L, which is a direct summand of Λ_i as we have inverted p. If $\mathcal{O}_i = \Lambda_i$, $\Lambda_i^L = \mathcal{O}_i$ or zero as $\mathcal{O}_i$-module. The remaining case is when $p = 2$, $\Lambda_i = \mathcal{O}_{F_n}[1/p]$ and $\mathcal{O}_i = \Lambda_i^\sigma$ for some n and suitable σ of order 2 in $\mathrm{Gal}(F_n/\mathbb{Q})$. Here L is acting via some homomorphism φ to $H = D_n$ or S_n in $C \rtimes \mathrm{Gal}(F_n/\mathbb{Q})$, where C is the group of 2^nth roots of unity. If $\varphi(L) \cap C \neq \{1\}$, clearly $\Lambda_i^L = 0$. Otherwise $\varphi(L) = \mathbb{Z}/2$, generated by some $x = \xi^k\sigma$. But obviously Λ_i^x as $\mathcal{O}_i$-module depends only on the conjugacy class of x in H, so we can assume $x = \sigma$ or $H = D_n$ and $k = 1$. If $x = \sigma$, $\Lambda_i^x = \mathcal{O}_i$. If $H = D_n$ and $k = 1$, one easily checks that $\mathcal{O}^x$ is the free $\mathcal{O}_i$-module generated by $1 + \xi$.

Proof of Theorem 1.11. For each i, the exact sequence

$$0 \rightarrow Cl\mathcal{O}_i \rightarrow G_0^R\mathcal{O}_i \rightarrow G_0D_i \cong \mathbb{Z} \rightarrow 0$$

has a canonical splitting, defined by sending the free D_i-module of rank one to the free $\mathcal{O}_i$-module of rank one. Thus if we let $S \subseteq G_0^R\mathcal{O}$ denote the subgroup generated by the free $\mathcal{O}_i$-modules, $\xi: G_0^R\mathcal{O} \rightarrow G_0D$ maps S isomorphically onto G_0D. Now consider the commutative diagram

$$\begin{tikzcd} G_0^R RG \arrow[r, "\varphi", "\cong"'] \arrow[d] & G_0^R\mathcal{O} \arrow[d] \\ G_0\mathbb{Q}G \arrow[r, "\cong", "\varphi'"'] & G_0D \end{tikzcd}$$

where φ and φ' are induced by Morita equivalences as in Remark 1.9b. It is enough to show that if M is a permutation module over RG, then $\varphi[M] \in S$. But this is immediate from the lemma. □

In fact we will need a completed version of (1.12). The rings AG, $G_0^R RG$, etc., all come equipped with augmentations ε into $\mathbb{Z}$. We can then form the I-adic completion, where $I = \operatorname{Ker} \varepsilon$. In the case of AG, G a p-group, it is known that the I-adic completion is just $AG^\wedge = \mathbb{Z} \oplus I^{\wedge_p}$, where $\wedge_p$ denotes p-adic completion. However, this is probably false for $G_0^R RG$, so we adopt the p-adic version as a definition. Let B be any commutative ring with augmentation $\varepsilon : B \to \mathbb{Z}$, $I = \operatorname{Ker} \varepsilon$. Thus $B = I \oplus \mathbb{Z}$ as groups (canonically) and we define $B^\wedge = I^{\wedge_p} \oplus \mathbb{Z}$. Clearly $B^\wedge$ is functorial for ring homomorphisms preserving the augmentation. If $\alpha : B \to C$ is such a homomorphism we let $\alpha^\wedge : B^\wedge \to C^\wedge$ be the induced homomorphism.

(1.14) THEOREM. *The kernel of* $\eta^\wedge : AG^\wedge \to G_0^R RG^\wedge$ *coincides with the kernel of* $\gamma^\wedge : AG^\wedge \to G_0 \mathbb{Q} G^\wedge$.

Proof. From (1.9), (1.11) and (1.12) we see that the commutative diagram

$$\begin{tikzcd} & G_0^R RG \arrow[d] \\ AG \arrow[ur] \arrow[dr] & \\ & G_0 \mathbb{Q} G \end{tikzcd}$$

is abstractly isomorphic to a diagram of Abelian groups of the form

$$\begin{tikzcd} & A \oplus B \arrow[d] \\ A \oplus A' \arrow[ur] \arrow[dr] & \\ & A \end{tikzcd}$$

in which each map is the obvious projection. □

Remark. A remarkable theorem of Tornehave [35] identifies the kernel of γ explicitly in terms of virtual G-sets induced from three specified types of small subgroups of G, e.g., subgroups $\mathbb{Z}/p \times \mathbb{Z}/p$. This leads to another proof of (1.12): it is enough to check the three cases explicitly.

We conclude this section with a well-known fact that will be needed in Section 4. Let q be a prime, $q \neq p$.

(1.15) THEOREM. *The natural homomorphism* $\theta : AG \to G_0 \mathbb{F}_q G$ *is surjective for all* p*-groups* G *if and only if* p *is odd and* q *generates the group of units of* $\mathbb{Z}/p^2$.

Of course there are infinitely many such q, by Dirichlet's theorem.

Proof. Suppose p is odd and q generates $(\mathbb{Z}/p^2)^*$. Since θ commutes with induction, by a simple induction argument as in [29] we reduce to the case where G has a faithful strictly primitive representation over $\mathbb{F}_q$. In particular, G has normal rank one and, hence, is cyclic since p odd; say $G = \mathbb{Z}/p^k$. With our hypotheses on G the simple $\mathbb{F}_q G$-modules have dimensions 1, $(p-1), \ldots, p^{k-1}(p-1)$, and only the simple of highest dimension is faithful. But one easily checks that this module is an induced module – i.e., not strictly primitive – unless $k = 1$. So we reduce to the case $G = \mathbb{Z}/p$ where the conclusion is obvious.

Conversely, suppose θ is surjective for all p-groups G, p odd. Taking $G = \mathbb{Z}/p$ and counting ranks we see that $\mathbb{F}_q\mathbb{Z}/p$ has one nontrivial simple module, which forces $q^a \neq 1 \bmod p$ for $a < p - 1$. Now take $G = \mathbb{Z}/p^2$. Then $\mathbb{F}_qG$ has a unique faithful simple module, of dimension $p(p-1)$, which forces $p^2 \nmid q^{p-1} - 1$ (otherwise we get one of dimension $p - 1$), and hence q generates $(\mathbb{Z}/p^2)^*$. If $p = 2$, one easily checks that θ is not onto for $\mathbb{Z}/4$ if $q = 1 \bmod 4$, and is not onto for $\mathbb{Z}/8$ when $q = 3 \bmod 4$. $\square$

2. On the Unit Map $Q_0S^0 \to BGL\mathbb{Z}^+$

We begin by reviewing some basic facts about the plus construction and group completion. For further details see [1] and the references cited there. The following context will suffice for our purposes: We are given a sequence of groups $\{1\} = G_0, G_1, \ldots$ with injective homomorphisms $G_m \times G_n \to G_{m+n}$ that are strictly associative with G_0 as strict identity and commutative up to conjugation. The space $MG = \amalg BG_n$ then becomes a strictly associative and homotopy commutative monoid. The group completion of MG is by definition $\Omega B(MG)$, where $B(MG)$ is the classifying space. There is a canonical map $MG \xrightarrow{i} \Omega B(MG)$, whose effect on homology is to localize H_*MG by inverting the element $x \in \pi_0 MG \subseteq H_0 MG$ corresponding to BG_1. Furthermore, if $G_\infty = \varinjlim G_n$, where the direct limit is taken over the inclusions $G_n = G_n \times \{1\} \subseteq G_n \times G_1 \to G_{n+1}$, then i induces a map $BG_\infty \xrightarrow{j} \Omega_0 B(MG)$ which is an isomorphism on homology. If G_∞ has perfect commutator subgroup, it follows that j factors through a homotopy equivalence $BG_\infty^+ \to \Omega_0 B(MG)$, where BG_∞^+ is Quillen's plus construction. Finally, suppose the sequence G_n is also equipped with suitable homomorphisms $\Sigma_n \int G_m \to G_{nm}$: that is, the category with objects the nonnegative integers and morphisms $n \to n$ given by G_n is a permutative category. Then the machinery of May or Segal shows $\Omega(B(MG))$ is in fact an infinite loop space, and so defines a spectrum $\mathbf{G}$. It is worth emphasizing that this spectrum is always connective. For an overview and references the reader should consult [1], and [8].

We have the following basic examples: (1) $G_n = \Sigma_n$. Then $\Omega BMG = QS^0$ by the Barratt–Priddy–Quillen theorem. The spectrum is S^0. (2) $G_n = GL_nR$, R a ring. The associated spectrum will be denoted KFR; its homotopy groups are the algebraic K-groups of the category of finitely generated *free* R-modules. It differs from KR only in π_0. (3) $G_n = NO_n\mathbb{F}_3$. Here $NO_n\mathbb{F}_3$ is the subgroup of $GL_n\mathbb{F}_3$ consisting of matrices A which are orthogonal ($AA^T = I$) and such that $\det A = N(A)$, where $N(A)$ is the spinor norm. The associated spectrum, localized at 2, is the connective J spectrum for $p = 2$ (see [8]). (4) $G_n = U(n)$. Then $\Omega BM(G) = BU \times \mathbb{Z}$ and the spectrum is connective complex K-theory (May [20]). If we instead take $G_n = GL_n\mathbb{C}$, with the classical topology, the result is the same up to homotopy equivalence.

If X is a space then the set of based homotopy classes $[X_+, M(G)]$ is an Abelian monoid, and we have a transformation $[X_+, M(G)] \xrightarrow[\varphi]{} [X_+, \Omega BM(G)]$ which is natural in X and G. Since $[X_+, \Omega BM(G)]$ is an Abelian group, φ factors through the group completion (in the usual algebraic sense) $[X_+, M(G)]'$ of $[X_+, M(G)]$. Consider the case $X = BH$, H a finite group. Let $\mathrm{Rep}(H, G_n)$ denote conjugacy classes of homomorphisms $H \to G_n$. Then there is a natural map $\mathrm{Rep}(H, G_n) \to [BH_+, BG_n]$, which is in fact bijective if G_n is discrete. Furthermore, $\mathrm{Rep}(H, G_*) \equiv \amalg \mathrm{Rep}(H, G_n)$ is itself an Abelian monoid with a natural map to $[BH_+, M(G)]$. We thus obtain a homomorphism $\mathrm{Rep}(H, G_*)' \to G^0(BH_+)$, natural in H and G.

EXAMPLES. (1) $G_n = \Sigma_n$. Then $AH \to \pi_s^0(BH_+)$ is the usual map from the Burnside ring to stable cohomotopy.

(2) $G_n = GL_n R$, $R = \mathbb{Z}[1/p]$. Then $\mathrm{Rep}(H, G_+)'$ is precisely the Green ring $G_0'(RH)$ mentioned in Section 1. As noted there, $G_0' RH \cong G_0^R RH$ when H is a p-group, so we get a homomorphism $G_0^R RH \to KR^0 BH_+$. (In fact this homomorphism can be defined for any finite H and commutative R, by appealing to a theorem of Quillen – see e.g. [12].)

Remark. In all the examples we consider the permutative categories are bipermutative, the spectra are ring spectra, and the homomorphisms just defined are ring homomorphisms; cf. [20].

To prove Theorem B we need two theorems about the 'Image of J' spaces. We define J_p in the following way: if p is odd, choose another prime q such that q generates the group of units of $(\mathbb{Z}/p^2)$. Then J_p is the localization at p of $BGL\mathbb{F}_q^+$. We define J_2 to be the localization at 2 of $BNO\mathbb{F}_3^+$, where $NO\mathbb{F}_3$ is the group defined above. By Quillen's technique of Brauer lifting, J_p is known to be equivalent to the localized fibre of $\psi^q - 1 : BU \to BU$ (p odd) or $\psi^3 - 1 : BO \to B\,\mathrm{Spin}$ ($p = 2$); see [23], [8]. Although this description would usually be the definition, and ultimately is of course essential to our program, we make no use of it in the present section. The following theorem and corollary is due in various guises to Sullivan and Tornehave [20]. For the convenience of the reader, we give a short proof for p odd in Section 4.

(2.1) THEOREM. *Let p, q be distinct primes. Then*

(a) *The natural map $B\Sigma_\infty^+ \xrightarrow{r} BGL\mathbb{F}_q^+$ is a homotopy retraction at p if and only if p is odd and q generates $(\mathbb{Z}/p^2)^*$.*
(b) *The natural map $B\Sigma_\infty^+ \xrightarrow{r} BNO\mathbb{F}_3^+$ is a homotopy retraction at 2.*

(2.2) COROLLARY. *$B\Sigma_\infty^+ \cong J_p \times F$ at p, where F is the fibre of r.*

We now come to the main result of this section. It is equivalent to Theorem B, by the preceding discussion.

(2.3) THEOREM. *There is a map $g: J_p \to BGL\mathbb{Z}^+$ such that the diagram*

$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r, "f"] \arrow[d, "r"'] & BGL\mathbb{Z}^+ \\ J_p \arrow[ur, "g"'] & \end{tikzcd}$$

is homotopy commutative.

Proof. We first note that such a factorization exists with $BGL\mathbb{Z}^+$ replaced by BU. That is, let $m: BGL\mathbb{Z}^+ \to BGL\mathbb{C}^{\mathrm{top}} \cong BU$ denote the natural map. Then

(2.4) LEMMA. *There is a map $\theta: J_p \to BU$ such that the diagram*

$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r, "mf"] \arrow[d, "r"'] & BU \\ J_p \arrow[ur, "\theta"'] & \end{tikzcd}$$

is homotopy commutative.

In fact θ is unique and is given by the usual Brauer lift – see (4.3). Furthermore, by a theorem of Snaith [31] r^* is, in fact, an isomorphism on K-theory. For yet another proof of (2.4), one can use the fact that the unit map $S^0 \to j$ of the connective j-spectrum is an isomorphism on K-theory and, hence, the diagram (2.4) exists on the spectrum level. However, all we need is the lemma as stated.

Since r is a retraction up to homotopy, there is a map $s: J_p \to B\Sigma_\infty^+$ such that $rs \sim 1$. let $e = sr$. Then $e^2 \sim e$ and clearly g exists if and only if $f = fe$; i.e., $g = fs$. To show that $f = fe$, we consider the following composite

$$BG_n \xrightarrow{Bi} B\Sigma_n \xrightarrow{Bj} B\Sigma_\infty \xrightarrow{k} B\Sigma_\infty^+ \xrightarrow{f} BGL\mathbb{Z}^+.$$

Here k is the canonical map, j is the inclusion $\Sigma_n \subset \Sigma_\infty$ and i is the inclusion of a p-Sylow subgroup G_n of Σ_n. Let $h = k \cdot Bj \cdot Bi$. Think of $f - fe$ as an element of $K\mathbb{Z}^0(B\Sigma_\infty^+)$. Then it is enough to show $h^*(f - fe) = 0$, because: (i) k is a homology isomorphism, and hence a stable equivalence; (ii) $K\mathbb{Z}^*(B\Sigma_\infty) \to \varprojlim K\mathbb{Z}^+ B\Sigma_n$ is an isomorphism, and (iii) Bi is a stable retraction. Here (ii) follows from the Milnor sequence, provided $\lim^1 K\mathbb{Z}^* B\Sigma_n = 0$. But this is true for any cohomology theory, since $B\Sigma_{n-1}$ is stably a direct summand of $B\Sigma_n$, by a theorem of Kahn and Priddy [15]. (Alternatively, one can use a standard argument based on the fact that $B\Sigma_n$ is a direct limit of finite torsion complexes and $K\mathbb{Z}$ has finite type; see 4.2.) Assertion (iii) follows from a standard transfer argument, since we have localized at p. To show $h^*(f - fe) = 0$ we will think of $\alpha = (1 - e) \circ h$ as an element of $\pi_s^0 BG_+$ and think of f as a natural transformation $\pi_s^0 \xrightarrow{f_*} K\mathbb{Z}^0$. It is then sufficient to prove:

(2.5) LEMMA. *Let G be a finite p-group, $\beta \in \pi_s^0 BG_+$. Then $f_*((1 - e)\beta) = 0$.*

Now Lemma 2.4 implies $(mf)_*((1 - e)_\beta) = 0$. Lemma (2.5) then follows from

(2.6) LEMMA. *Let G be a finite p-group, $\alpha \in \pi_s^0 BG_+$. If $m_* f_* \alpha = 0$, then $f_* \alpha = 0$.*

By a theorem of Quillen [24], there is a fibre sequence

$$BGL\mathbb{F}_p^+ \times \mathbb{Z} \to BGL\mathbb{Z}^+ \to BFL\mathbb{Z}\left[\frac{1}{p}\right]^+.$$

Actually Quillen constructs a fibration which is a delooping of (1.7); the connection between the two is provided by his '$Q = +$' theorem [10]. But Quillen also showed in [23] that $BGL\mathbb{F}_p^+$ is acyclic and hence contractible at p. Hence $BGL\mathbb{Z}^+$ is a principal covering of $BGL\mathbb{Z}[1/p]^+$, with group $\mathbb{Z}$, and since G is a finite p-group we may therefore replace $\mathbb{Z}$ by $R = \mathbb{Z}[1/p]$ in the proof of Lemma 1.6. Consider the commutative diagram

$$\begin{array}{ccccc} AG & \longrightarrow & G_0^R(RG) & \longrightarrow & G_0(\mathbb{C}G) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_s^0 BG_* & \xrightarrow[f_*]{} & KR^0(BG_+) & \xrightarrow[m_*]{} & K^0_{\text{top}}BG_+ \end{array}$$

where the unlabelled maps were defined at the beginning of this section. The groups in the top row are all augmented rings of the form $B \xrightarrow{\varepsilon} \mathbb{Z}$ and, hence, we can define the (essentially p-adic) completion $B^\wedge$ as in Section 1. Now for any spectrum E of finite type, E^*BG is a p-complete Abelian group. Hence, we obtain a commutative diagram

$$\begin{array}{ccccc} AG^\wedge & \xrightarrow{d} & G_0^R RG^\wedge & \xrightarrow{e} & G_0\mathbb{C}G^\wedge \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ \pi_s^0 BG_+ & \xrightarrow[f_*]{} & KR^0 BG_+ & \xrightarrow[m_*]{} & K^0_{\text{top}}BG_+ \end{array}$$

Furthermore c is an isomorphism by a theorem of Atiyah [3], and a is an isomorphism by the affirmed Segal conjecture (Carlsson [6], Adams–Miller–Gunnawardena [2]). Hence, we may regard α as an element of the completed Burnside ring, and $ed(\alpha) = 0$. It remains to show $d(\alpha) = 0$. Now e factors through the natural map $G_0(\mathbb{Q}G)^\wedge \xrightarrow{i^\wedge} G_0\mathbb{C}G^\wedge$. Since $i : G_0\mathbb{Q}G \to G_0\mathbb{C}G$ is an injective homomorphism of finitely generated free Abelian groups, $i^\wedge$ is also injective. By Theorem 1.13, we conclude $d(\alpha) = 0$. This proves Lemma 2.6, and completes the proof of the theorem. □

Remark. The use of the Segal conjecture could probably be avoided, since it is not necessary in the proof to consider an arbitrary $\alpha \in \pi_s^0 BG_+$; the relevant α actually has a very special form.

3. Vanishing of v_n-Periodic Homotopy Groups of Algebraic K-Theory Spectra

In this section we prove Theorem A. For a general discussion of the philosophy lurking behind this section, the reader should consult [13] and [27], as well as Ravenel's seminal paper [26]. We recall from [13] that a finite p-torsion spectrum

X has type n if $K(i)_*X = 0$ for $i < n$ and $K(n)_*X \neq 0$, where $K(n)$ is the nth Morava K-theory. A self-map $f: X \to X$ is a v_n-map if f_* is an isomorphism on $K(i)_*$ for $i = n$ and nilpotent for $i \neq n$.

(3.1) THEOREM [21]. *Finite spectra of type n exist for all primes p and $n \geqslant 0$.*

(3.2) THEOREM [13, 14]. *Every finite spectrum X of type n admits a v_n-map f. Furthermore, for some $k > 0$ f^k is central in $\{X, X\}$. If g is another such map, then $f^i = g^j$ for some i, j.*

Remark. Another construction of finite spectra of type n was given by Jeff Smith (still unpublished, unfortunately). Smith's construction is more flexible than that of the author [21], and yields spectra with better cohomological properties. Hopkins and Smith [13] use an Adams spectral sequence argument to construct v_n-self maps on the Smith complexes. An ingenious argument applying Ravenel's nilpotence conjecture (as proved in [7]) then yields (3.2); see [13]. We note, however, that all we will need is the existence of a spectrum of type n with a self-map inducing an isomorphism on $K(n)_*$. In particular, we make no use of the nilpotence theorem itself.

Let X be a finite spectrum with v_n-map f. To save space we will simply refer to the pair (X, f) as a v_n-complex. Given a spectrum E, the v_n-periodic homotopy of E with coefficients in X is defined to be $f^{-1}[X, E]$. Here $[X, E]$ is regarded as a right $[X, X]$-module and $f^{-1}[X, E]$ is the direct limit of the sequence of left $[E, E]$-modules

$$[X, E] \xrightarrow{f^*} [X, E] \xrightarrow{f^*} [X, E] \to \cdots$$

Of course by (3.2) we can even assume f central in $[X, X]$, so that $f^{-1}[X, E]$ is also an $[X, X]$-module. Note $f^{-1}[X, E]$ is independent of f by (3.2), although it may well depend on X.

Remark. For any finite spectrum X, and arbitrary spectrum Y, $[X, Y]_* \cong \pi_* Y \wedge DX$, natural in X and Y. Here DX is the Spanier–Whitehead dual of X. Explicitly, the naturality in Y means that for any map $g: Y \to Z$ there is a commutative diagram

$$\begin{tikzcd} {[X, Y]_*} \arrow[r, "g_*"] \arrow[d, "\cong"'] & {[X, Z]_*} \arrow[d, "\cong"] \\ \pi_* Y \wedge DX \arrow[r, "(g \wedge 1)_*"] & \pi_* Z \wedge DX \end{tikzcd}$$

The naturality in X implies in particular that if $f: X \to X$ is a self-map $f^*: [X, Y]_* \to [X, Y]_*$ corresponds to $(1 \wedge Df)_*: \pi_* Y \wedge DX \to \pi_* Y \wedge DX$. Hence, there is a natural isomorphism $f^{-1}[X, Y] \cong (1 \wedge Df)^{-1} \pi_* E \wedge DX$.

(3.3) PROPOSITION. *Suppose $f^{-1}[X, E] = 0$ for some v_n-complex (X, f). Then $K(n)_* E = 0$.*

Proof. The mapping telescope T of $E \wedge DX \xrightarrow{1 \wedge Df} E \wedge DX \xrightarrow{1 \wedge Df} \cdots$ is contractible by assumption, so certainly $K(n)_*T = 0$. But (DX, Df) is also a v_n-complex, and $K(n)_*(E \wedge DX) \cong K(n)_*E \otimes_{K(n)_*} K(n)_*DX$. Hence, $(1 \wedge Df)_*$ is an isomorphism on $K(n)_*(E \wedge DX)$, forcing $K(n)_*E = 0$. □

Remark. The converse of (3.3) is Ravenel's 'Telescope Conjecture', which is the only major conjecture from [26] which is still unsolved. If true, it would imply that the condition $f^{-1}[X, E] = 0$ depends on p and n, and not on f or X.

(3.4) PROPOSITION. *Let E be a spectrum whose homotopy is bounded above, and let (X,f) be a v_n-complex, $1 \leqslant n < \infty$. Then $K(n)_*E = 0 = f^{-1}[X, E]$.*

Proof. Obviously $f^{-1}[X, E] = 0$, since any map of a highly connected spectrum into E is null by obstruction theory. Hence, $K(n)_*E = 0$ by (3.3); this also follows easily from the fact that $K(n)_*$ of any Eilenberg–Maclane spectrum is zero ($n \geqslant 1$) ([26], 4.7 and 4.8). □

(3.5) COROLLARY. *Let E be any spectrum, $E[k]$ its $(k-1)$-connected cover for $k \in \mathbb{Z}$. Then the canonical map $E[k] \to E$ is an isomorphism on $K(n)_*$ and on $f^{-1}[X, -]$, $1 \leqslant n < \infty$.* □

Let R be a finite associative ring spectrum of type n. An element $\alpha \in \pi_*R$ is a *v_n-element* if its Hurewicz image $K(i)_*\alpha \in K(i)_*R$ is a unit for $i = n$ and nilpotent otherwise. As in Theorem 3.2, replacing α by some α^k if necessary we can assume α is central, $K(i)_*\alpha = 0$ for $i \neq n$, and α is essentially unique (see [13]). Clearly right (or left) multiplication r_α by α makes (R, r_α) a v_n-complex: explicitly, r_α is the composite

$$R \cong R \wedge S \xrightarrow[1_R \wedge \alpha]{} R \wedge R \xrightarrow{m} R$$

where m is the multiplication.

(3.6) PROPOSITION. *For all p, n there exists a finite associative ring spectrum R and a v_n-element $\alpha \in \pi_*R$.*

Proof. Let (X,f) be a v_n-complex. Then $X \wedge DX$ is a finite associative ring spectrum of type n. If α corresponds to f under the canonical isomorphism $[X, X]_* \cong \pi_*X \wedge DX$, then α is a v_n-element. □

Next, observe that for any finite spectrum X with self map $f: \Sigma^r X \to X (r \geqslant 1)$, $f^{-1}[X, Y]_*$ makes perfectly good sense for a *space* Y: if Y is a spectrum, we regard $f^{-1}[X, Y]_*$ as periodically graded, with period r. But $\Sigma^k X \cong \Sigma^\infty X'$ for some $k \geqslant 0$, where X' is a finite complex, and f is defined as a map of spaces $\Sigma^r X' \xrightarrow[g]{} X'$. Hence, when Y is a space we can set

$$f^{-1}[X, Y]_j = \varinjlim \left([\Sigma^j X', Y] \xrightarrow[\Sigma^j g]{} [\Sigma^{r+j} X', Y] \xrightarrow[\Sigma^{r+j} g]{} [\Sigma^{2r+j} X', Y] \to \cdots\right).$$

It is easy to check that this definition is independent of the choice of k, X', and g. Hence, we can define the v_n-periodic homotopy of a space Y with coefficients in a finite spectrum X of type n, denoted $f^{-1}[X, Y]_*$ as before. Now consider the adjoint functors Σ^∞ : spaces $\to$ spectra and Ω^∞ : spectra $\to$ spaces. Thus if Y is a space and E

is a spectrum we have a bijection $[\Sigma^\infty Y, E] = [Y, \Omega^\infty E]$, natural in Y and E. From the naturality we clearly have:

(3.7) PROPOSITION. *Let (X,f) be a v_n-complex. Then for any spectrum E there is an isomorphism $f^{-1}[X, E]_* \cong f^{-1}[X, \Omega^\infty E]$, natural in E.* □

Theorem A follows at once from Theorem B and:

(3.8) THEOREM. *Let E be any ring spectrum, with unit map $S^0 \xrightarrow{i} E$. Suppose $\Omega_0^\infty i : Q_0 S^0 \to \Omega_0^\infty E$ factors through J_p. Then, for all $n \geqslant 2$, there exists a v_n-complex (X,f) with $f^{-1}[X, E]_* = 0$. Hence, $K(n)_* E = 0$ for all $n \geqslant 2$.*

(3.9) LEMMA. *For any v_n-complex (X,f) with $n \geqslant 2$, $f^{-1}[X, J_p]_* = 0$.*

Proof. Fix (X,f) and call a spectrum or space $F(X,f)$-acyclic if $f^{-1}[X, F]_* = 0$. The periodic K-theory spectrum KU is obviously (X,f) acyclic; indeed $[X, KU]$ is already zero since X is KU-acyclic and KU, being a ring spectrum, is local with respect to itself ([26], 1.17). Hence, the fibre F of $\psi^q - 1 - KU \to KU$ is (x,f)-acyclic and, hence, by (3.7) so is $\Omega_0^\infty F = F\psi^q$, where $F\psi^q$ is the fibre of $\psi^q - 1 : BU \to BU$. (Here q is a prime different from p.) By Quillen's theorem [23] $F\psi^q \cong BGL\mathbb{F}_q^+$. If p is odd then $J_p = BGL\mathbb{F}_q^+$ for suitable q, and the lemma is proved. Now suppose $p = 2$. Since KU and KO are Bousfield equivalent, $KO_* X = 0$ and, hence, $[X, KO] = 0$ as before. Let bo (resp. b Spin) denote the (-1)-connected (resp. 3-connected) cover of KO. Then bo and b Spin are (X,f)-acyclic by (3.5). Hence, so is the fibre F of $\psi^3 - 1 : bo \to b$ Spin. But $\Omega_0^\infty F$ is J_2 (see [8]); hence J_2 is also $[X,f]$-acyclic. □

Proof of Theorem 3.8. Let (X,f) be any v_n-complex, $n \geqslant 2$. By the lemma, $(\Omega_0^\infty i)_*$ is zero on $f^{-1}[X, -]$. Hence, $(\Omega^\infty i)_*$ is zero on $f^{-1}[X, -]$ and, hence, by (3.7) i_* is zero on $f^{-1}[X, -]$. By our earlier remarks there is a commutative diagram

$$\begin{array}{ccc} [X, S^0]_* & \xrightarrow{i_*} & [X, E]_* \\ \cong\downarrow & & \downarrow\cong \\ \pi_* DX & \xrightarrow[h_*]{} & \pi_* E \wedge DX \end{array}$$

where $h : DX \to E \wedge DX$ is the Hurewicz map for E-homology. Reversing the roles of X and DX, we conclude that for any v_n-complex $(X,f), f^{-1}h_* : f^{-1}\pi_* X \to (1 \wedge f)^{-1}\pi_* E \wedge X$ is zero. In other words, for each $\beta \in \pi_* X$ there is a k such that $(1 \wedge f)^k \circ h(\beta) = 0$. Now take $X = R$ to be a ring spectrum as in (3.6), with $f = r_\alpha$ for some v_n-element $\alpha \in \pi_* R$. Then $E \wedge R$ is itself a ring spectrum, and h is a map of ring spectra. Furthermore, for any $\alpha \in \pi_* R$, $(1 \wedge r_\alpha)_* : \pi_* E \wedge R \to \pi_* E \wedge R$ coincides with right multiplication by $h(\alpha)$. Hence, for each β in $\pi_* R$, there is a k such that $h(\beta)h(\alpha)^k = 0$. In particular, $h(1)h(\alpha)^k = 0$ for some k. But $h(1) = 1$, so for all $\gamma \in \pi_* E \wedge R$ we have $\gamma h(\alpha)^k = \gamma \cdot (1 \cdot h(\alpha)^k) = 0$. This completes the proof. □

(3.10) COROLLARY. *If E is a module spectrum over $K\mathbb{Z}$, and F is the fibre of $j : E \to L_1 E$, F is dissonant.*

Proof. L_1E is Bousfield localization with respect to $E(1)$, the Adams summand of p-local complex K-theory. $E(1)$ is Bousfield equivalent to $K(0) \vee K(1)$, so by definition j is an isomorphism on $K(0)$ and $K(1)$. Moreover, $K(n)_*L_1E = 0$ for any $E, n \geqslant 2$. This is immediate from (3.3), or one can show directly that it is true for S^0 and then use the fact that $L_1E = E \wedge L_1S^0$ [26]. □

Remarks. (1) it also follows that the chromatic tower of E ([26]) collapses to L_1E; that is, $L_nE \to L_1E$ is an equivalence for $n > 1$. However, this is still not sufficient for conjecture C of the introduction.

(2) Note that if conjecture C holds for a given E – i.e. $j : E \wedge M\mathbb{Z}/p^m \to L_1(E \wedge M\mathbb{Z}/p^m)$ induces an isomorphism on π_k for $k \gg 0$ – then $K(n)_*E = 0$ for $n \geqslant 2$. For the fibre F of j is then a bounded above torsion spectrum and so is dissonant. As in (3.10), it follows that $K(n)_*(E \wedge M\mathbb{Z}/p^m) = 0$ for $n \geqslant 2$ and, hence, $K(n)_*E = 0$ for $n \geqslant 2$.

(3) If E is any spectrum with $K(n)_*E = 0$ for $n \geqslant 2$, then its K-theory localization L_1E coincides with its *harmonic* localization $L_\infty E$. Here L_∞ is Bousfield localization with respect to $V_{0 \leqslant n < \infty} K(n)$. Taking $E = KX$, where X is a scheme satisfying the hypotheses of Thomason's theorem [34], it follows that the Lichtenbaum–Quillen conjectures hold for the p-completion of $L_\infty KX$. We plan to discuss the implications of this fact in a future paper.

(4) In a very interesting paper [16], N. Kuhn has shown that the $K(n)$-localization functor factors through Ω^∞ (generalizing an earlier result of Bousfield for $n = 1$). This provides an alternative approach to Theorem 3.8.

4. On $B\Sigma_\infty^+ \to BGL\mathbb{F}_q^+$

The purpose of this section is to provide short proofs of Theorems (4.1) and (4.3) below, which were crucial ingredients in Theorem B. No particular claim to originality is made; the point is to save the reader the trouble of having to extract these arguments from the literature. The method used in Theorem (4.1) appears, for example, in [11]. The first theorem is due in various guises to Sullivan and Tornehave. Fix primes $p, q, p \neq q$.

(4.1) THEOREM. *After localization at p, the natural map $B\Sigma_\infty^+ \to BGL\mathbb{F}_q^+$ is a homotopy retraction if and only if p is odd and q generates $(\mathbb{Z}/p^2)^*$.*

There is a standard argument for getting around not only 'lim^1 problems' but also 'lim^0 problems' in this context. It seems worthwhile to state one version of the argument explicitly.

(4.2) LEMMA. *Let X be a CW-complex, $X_1 \subset X_2 \subset X_3 \cdots$ a filtration by subcomplexes of finite type with $UX_i = X$. Suppose $\tilde{H}_kX_n$ is finite for all k, n. Then*

(a) *If Y is any loop space of finite type, $[X, Y] \to \lim^0[X_n, Y]$ is an isomorphism.*
(b) *If Z is another loop space of finite type, and $g : Y \to Z$ a map, then a map $f : X \to Z$ lifts to Y if and only if $f|_{X_n}$ lifts for all n.* □

Proof of Theorem 4.1. Since $B\Sigma_\infty^+$ and $BGL\mathbb{F}_q^+$ are infinite loop spaces, and $BGL\mathbb{F}_q \to BGL\mathbb{F}_q^+$ is a stable equivalence, it is sufficient to produce a lifting

$$\begin{tikzcd} & B\Sigma_\infty^+ \arrow[d] \\ BGL\mathbb{F}_q \arrow[r] \arrow[ur] & BGL\mathbb{F}_q^+ \end{tikzcd}$$

By Lemma 4.2, it is enough to produce such a lifting on each $BGL_n\mathbb{F}_q$ separately. Let G be a p-Sylow subgroup of $GL_n\mathbb{F}_q$. Since $BGL_n\mathbb{F}_q$ is a stable retract of BG (at p), it is enough to lift the composite $\alpha : BG \to BGL_n\mathbb{F}_q \to BGL\mathbb{F}_q^+$. But we may regard α as an element of $G_0(\mathbb{F}_qG)$ (cf. Section 2). If p is odd and q generates $(\mathbb{Z}/p^2)^*$, $\theta : AG \to G_0\mathbb{F}_qG$ is onto by Theorem (1.15). Hence, if $\theta(\beta) = \alpha$, β defines an element of $[BG, B\Sigma_\infty^+]$ which is the desired lift (here α, β lie in the augmentation ideals).

For the converse, suppose first p odd. Then $\pi_{2n-1}BGL\mathbb{F}_q^+ = \mathbb{Z}/(q^n-1)$, but $\pi_{2n-1}B\Sigma_\infty^+$ is zero for $n < p-1$ and is $\mathbb{Z}/p$ for $n = p-1$. The converse follows easily from this. If $p = 2$, then $|\pi_5 BGL\mathbb{F}_q^+| \geqslant 2$ but $\pi_5 B\Sigma_\infty^+ = 0$, preventing the existence of a retraction. □

Let $\theta : J_p \to BU$ denote the Brauer lift of Quillen [23]. When $p = 2$, this means the composite $BNO\mathbb{F}_3^+ \to BGL\mathbb{F}_3^+ \xrightarrow{\theta} BU$.

(4.3) THEOREM. *The following diagram is commutative*:

$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r, "mf"] \arrow[d, "r"'] & BU \\ J_p \arrow[ur, "\theta"'] & \end{tikzcd}$$

Proof. As in the proof of (4.2), we reduce to checking that $\theta r = mf$ when restricted to BG, G a p-Sylow subgroup of Σ_n. On $B\Sigma_n$, θr corresponds to the virtual character χ defined as follows: Let χ' be the character of the standard representation i_n. Each $g \in \Sigma_n$ can be written uniquely in the form $g = h_1h_2$, where h_1 has order q^k and h_2 has order prime to q. Then $\chi(g) = \chi'(h_2)$. In particular, $\chi = \chi'$ on any subgroup of order prime to q, and the theorem follows. □

5. Appendix: Finite p-Groups of Normal Rank 1

In this section we give a quick proof of Theorem 1.1. We first consider non-Abelian extensions of the form

(5.1) $C_n \to G \to \mathbb{Z}/p$.

where C_n is a cyclic group of order $p^n, n \geqslant 2$. Let x (resp. y) denote a fixed generator of C_n (resp. $\mathbb{Z}/p$), and let σ denote the automorphism of C_n induced by conjugation by y. There are the following obvious examples:

(i) $M_n(p)$, the split extension with $\sigma(x) = x^{1+p^{n-1}}$ ($n \geqslant 3$ if $p = 2$),
(ii) D_n, the dihedral group with $p = 2$ and $\sigma(x) = x^{-1}$ ($n \geqslant 2$),
(iii) S_n, the semi-dihedral group with $p = 2$ and $\sigma(x) = x^{-1+2^{n-1}}$ ($n \geqslant 3$),
(iv) Q_n, the quaternion group with $p = 2$ and $\sigma(x) = x^{-1}$ ($n \geqslant 2$).

Here examples D_n and S_n are split but Q_n is not.

(5.2) LEMMA. *Let G be a non-Abelian p-group with a cyclic subgroup of order p^n and index p. Then G is isomorphic to exactly one of the groups $M_n(p)$, D_n, S_n, or Q_n.*

Proof. A subgroup of index p is necessarily normal and, hence, G has the form (5.1). The classification of split extensions is trivial and is left to the reader, as is the proof that the listed groups are nonisomorphic. The key point is to show that if the extension does not split, then $p = 2$, $\sigma(x) = x^{-1}$ and $G \cong Q_n$. Let $\tilde{y} \in G$ denote a fixed lift of y. The extension split if and only if a lift can be chosen with order p – i.e. the equation $(a\tilde{y})^p = 1$ has a solution with $a \in C_n$. But $(a\tilde{y})^p = N(a)\tilde{y}^p$, where N is the norm homomorphism $C_n \to C_n$ given by $N(a) = a\sigma(a) \cdots \sigma^{p-1}(a)$. Since $N(C_n) \subseteq Z(G) \cap C_n = Z(G)$ and $\tilde{y}^p$ is evidently central, we conclude that the extension will split provided $N: C_n \to Z(G)$ is onto. If p is odd, we can assume $\sigma(x) = x^{1+p^{n-1}}$ and, hence, $Z(G) = C_{n-1} \subset C_n$. Then $N(x) = x^p$ and, hence, N is onto as desired. Now let $p = 2$. If $\sigma(x) = x^{1+2^{n-1}}$, then $Z(G) = C_{n-1}$ and $N(x) = x^{2+2^{n-1}}$, so N is onto. If $\sigma(x) = x^{-1+2^{n-1}}$, then $Z(G) = C_1$, and $N(x) = x^{2^{n-1}}$, so N is onto. However, if $\sigma(x) = x^{-1}$ then N is trivial. In this case $Z(G) = C_1$. Since $\tilde{y}^2$ is central, if the extension is nonsplit we must have $\tilde{y}^2 = x^{2^{n-1}}$. But then the standard presentation of Q_n leads to a surjective homomorphism $Q_n \to G$, which is necessarily an isomorphism.

The proof of the next lemma is left to the reader. Note the dihedral group of order eight must be excluded in (a).

(5.3) LEMMA. (a) *The groups $D_n (n \geqslant 3)$, S_n and Q_n have normal rank one.* (b) *The elements of order p in $M_n(p)$ form a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Hence, $M_n(p)$ does not have normal rank one, and cannot even be a normal subgroup of a group of normal rank one.*

Proof of Theorem 1.1. Let G have normal rank one, and let A be a maximal normal Abelian subgroup. Then $A \cong C_n$ for some n, with generator x. If $A = G$ we are done, so assume $A \neq G$. Then there exists a normal subgroup H in G, with $A \subset H$ and $[H:A] = p$. By the maximality of A, H is non-Abelian and in particular $n \geqslant 2$. If p is odd, $H \cong M_n(p)$ by (5.2), which is a contradiction by (5.3b). Hence G is cyclic if p odd. If $p = 2$, it again follows from (5.2) and 5.3b that $H \cong D_n$, S_n, or Q_n, and it remains to show $H = G$. Suppose $H \neq G$. Then there is a normal subgroup H' in G, with $H \subset H'$ and $[H':H] = 2$; thus H'/A is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$. Let $\sigma \in \operatorname{Aut} A$ be induced by a generator of H/A. If H'/A is $\mathbb{Z}/4$ then σ is divisible by 2 in $\operatorname{Aut} A$. This forces $n \geqslant 3$ and $\sigma(x) = x^{1+2^{n-1}}$. But then

$H \cong M_n(2)$, contradicting (5.3b). Now suppose $H'/A \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Note that H' has exactly three subgroups H'' of index 2 containing A (one of which is H) and these are permuted by the conjugation action of G. If the natural homomorphism $\varphi : H'/A \to \operatorname{Aut} A$ is injective, clearly the three subgroups are nonisomorphic and hence normal in G. Moreover, one of them is isomorphic to $M_n(2)$, contradicting 5.3b. If φ is not injective, then exactly one of the subgroups H'' is Abelian, and hence also normal, contradicting the maximality of A. Hence, $H = G$ and the proof is complete. □

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