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Algebraic cobordism and K-theory

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ABSTRACT

A decomposition is given of the S-type of the classifying spaces of the classical groups. This decomposition is in terms of Thom spaces and by means of it cobordism groups are embedded into the stable homotopy of classifying spaces. This is used to show that each of the classical cobordism theories, and also complex K-theory, is obtainable as a localisation of the stable homotopy ring of a classifying space. Similar decompositions are given for classical groups over $\mathbb{F}_q$. The new construction of cobordism generalises immediately to define the algebraic cobordism of any ring. The familiar properties of cobordism are described in terms of the new formulation. Also the (p-adic) algebraic cobordism is computed for several $\overline{\mathbb{F}}_q$-algebras and schemes.

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TABLE OF CONTENTS

PROLOGUE

Each of the four parts of this paper has its own introduction. In this prologue I wish, therefore, to describe the global mathematical and philosophical theme which will be pursued in this paper.

We seek simultaneously to satisfy the following motivating demands.

(i) To find invariants for use in algebraic geometry which are at least as powerful as (and related to) Quillen K-theory.

(ii) To achieve (i) within the framework of stable homotopy theory.

(iii) To achieve (i) by a method which recogniseably generalises classical cobordism theories.

A few remarks on (i)-(iii) are in order. Recall first that the Chow ring of a smooth variety $A^*(X)$ may be obtained from algebraic K-theory as the sheaf cohomology $\bigoplus_n H^n_{Zar}(X;\underline{K}_n)$ [Q4]. This recommends K-theory and its more powerful relatives as a source of suitable invariants to study. Secondly the computational machinery available in stable homotopy theory is superior to that of ordinary homotopy theory, whence (ii). Also in the topologists' natural area of geometry - manifold theory - cobordism theories have been very important invariants, whence (iii).

Let us for the moment restrict attention to a commutative ring A, rather than the general form of (i). One way in which to satisfy (i) and (ii) in this case is to look at $\pi^S_*(BGLA^+)(\cong \pi^S_*(BGLA))$ since this ring contains the K-theory of A, $\bigoplus_{0<i} K_iA$, as a summand. However $\pi^S_*(BGLA)$ will be too difficult to compute in general. Therefore in practice we will localise it, while attempting to achieve (iii).

Here then is the main result, extracted from Part II and Part III. We state it in terms of localisation.

If Y is a homotopy commutative and associative H-space then the set of stable homotopy classes

$$\{\Sigma^* X, Y\} = \bigoplus_{0 \le n} \{\Sigma^n X, Y\}$$

is a graded, commutative ring which is a $\pi_*^S(Y)$-module. Let $B \in \pi_2^S(\mathbb{C}P^\infty) \subset \pi_2^S(BU)$, $B' \in \pi_4^S(BSp)$ and $\eta \in \pi_1^S(BO)$ be generators.

Theorem

If $\dim X < \infty$ then there are ring isomorphisms

(a) $\{\Sigma^* X, BSp\}[1/B'] \cong MSp^*(X)[u_4, u_4^{-1}]$

(b) $\{\Sigma^* X, BU\}[1/B] \cong MU^*(X)[u_2, u_2^{-1}]$

(c) $\{\Sigma^* X, \mathbb{C}P^\infty\}[1/B] \cong KU^*(X)$

and (d) $\{\Sigma^* X, BO\}[1/\eta] \cong MO^*(X)[u_1, u_1^{-1}]$

where u_i is a shift operator of dimension i.

This result indicates our approach to accomplishing (iii). Namely, in Part III and Part IV, we define p-adic algebraic cobordism by means of localisations of $\pi_*^S(BGLA^+)^\wedge_p)$, where $(\_)^\wedge_p$ denotes p-completions. The definitions make sense for any scheme over Spec $\overline{\mathbb{F}}_q$ and the above theorem enables us to make some computations.

In fact our result above has some new things to say about classical cobordism theory. For example for unitary cobordism we obtain new constructions of the Adams operations and Adams idempotents in MU-theory and a new proof of the Conner-Floyd theorem (see Part II). A spectral sequence derived in Part III, §1 when applied to the fibration sequence of infinite loopspaces

$$JSp \to BSp \xrightarrow{\psi^3 - 1} BSp \to BJSp$$

yields a new homology-type spectral sequence of $Z \times Z/4$-bigraded algebras (note the bi-grading!)

$$E^2_{p,q} = H_p(BJSp;\ \pi_q(\underline{MSp};\ Z_{(2)})) \Longrightarrow \pi_{p+q}(\underline{MSp};\ \mathbb{Q}).$$

There is a similar spectral sequence for MU-theory. In the text we make no use of there spectral sequences since they are, as far as I can see, of limited computational use for MU- or MSp-theory.

I have now sketched the philosophy and ideas that are at stake. I suggest that the reader browse through the introduction to Parts I - IV and through the list of contents in order to obtain quickly an overall picture.

During the three years of writing and re-writing this paper I have been helped by discussions and correspondence with many people. In particular I wish to thank Frank Adams, David Cox, Eric Friedlander, Ian Hambleton, Gerhard Harder, Stanley Kochman, Ib Madsen, Peter May and Jorgen Tornehave. Also I thank the team of referees for their helpful suggestions and the University of Western Ontario for financial support. Finally, for typing the various versions of this paper, Charine Haist and Janet Williams deserve my deepest thanks.

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ALGEBRAIC COBORDISM AND K-THEORY

Victor P. Snaith*

PART I: COBORDISM AND THE STABLE HOMOTOPY OF CLASSIFYING SPACES

§1. INTRODUCTION

Homologically the space BU appears to have each BU(n) as a summand. BSp behaves in a similar fashion and so does BO and BSO when suitably localised. I show in section 4 that these splittings are actually realised geometrically in the stable category. This splitting can be used to embed, for example, $\pi_*(MU)$ in $\pi_*^S(BU)$. These "exotic" elements are used in section 6 to compute $\pi_j^S(BU(n))$ for $j \leq 10$, modulo odd torsion.

Now let me describe the results of Part I in more detail. The reference numbers refer to those used in the body of the text in Part I.

Let $G_n = U(n)$, $Sp(n)$, $O(2n)$ or $SO(2n+1)$ then BG_∞ is filtered by $\{BG_n; n \geq 1\}$. If $H_n \subset G_n$ is the subgroup $\Sigma_n \int U(1)$, $\Sigma_n \int Sp(1)$, $\Sigma_n \int O(2)$ or NT^n the Becker-Gottlieb transfer [B-G1] is a map $\tau : BG_n \to QBH_n$. In section 2 the following result is proved:

2.1: Theorem. The Becker-Gottlieb transfer gives a filtered map $\tau : BG_\infty \to QBH_\infty$.

This amounts to showing that certain transfers fit together. This is accomplished by some generally applicable tricks with smooth fibre bundles (Propositions 2.2 - 2.4). In Examples 2.5 - 2.8 and 2.10 - 2.15 this trick is applied to a number of examples by means of which homology calculations are made in section 3 to prove the following result. When $G_n = U(n)$, $Sp(n)$ or $O(2n)$ there is a fibring of infinite loop maps

$$F_{G_\infty} \to QBG_1 \to BG_\infty .$$

3.2: Theorem. (i) There are equivalences of H-spaces

$$QBU(1) \simeq BU \times F_U$$

and

$$QBSp(1) \simeq BSp \times F_{Sp} .$$

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(ii) There is a homotopy equivalence

$$QBO(2) \simeq BO \times F_0 .$$

As a corollary we obtain (Corollary 3.6.1) that the Becker-Gottlieb solution of the unitary Adams conjecture is an H-map. This is not so in the case of the orthogonal Adams conjecture. In fact in §7 we evaluate the deviation from additivity of the transfer $\tau : BO \to QBO(2)$ and hence the deviation from additivity of the Becker-Gottlieb solution of the real Adams conjecture.

From [Sn 1] we know that QBG_1 splits as a wedge in the stable category. This splitting must split BU, BSp and BO. Using the homology calculations of section 3 it is shown in section 4 that BG_∞ splits into a wedge of factors $\frac{BG_n}{BG_{n-1}}$.

4.2: Theorem. In the stable category there are equivalences

$$BU \simeq \bigvee_{1\le k} MU(k)$$

$$BSp \simeq \bigvee_{1\le k} MSp(k)$$

$$BO \simeq \bigvee_{1\le k} \frac{BO(2k)}{BO(2k-2)}$$

and

$$BSO \simeq \bigvee_{1\le k} \frac{BSO(2k+1)}{BSO(2k-1)} \quad \text{(at odd primes)}.$$

Theorem 4.2 implies that the stable homotopy classes $\{X,BU\}$, for a finite dimensional X, has part of the unitary cobordism of X as a summand. In section 5 we identify an embedding of this factor by means of the following result.

5.1: Theorem. If $\dim X \le 4n$ there is an isomorphism

$$\Phi_U(n) : \left\{X, \frac{BU}{BU(n-1)}\right\} \to \prod_{n\le k} MU^{2k}(X)$$

defined by $\Phi_U(n)(f) = \Pi f^*(c_k')$. $\Phi_U(n)$ is an epimorphism when $\dim X = 4n+1$.

The classes of c_k' are defined in section 5 and the analogous result for $MSp^*(X)$ is proved in Theorem 5.2. The analogous result for $MO^*(X)$ is proved in [Part III, §3].

In addition to being a $\pi_*^S(S^\circ)$-module $\pi_*^S(BU)$ has a "tensor product" pairing and a "Whitney sum" pairing. This rich structure facilitates the generation of elements. By way of illustration

$$\frac{\pi_*^S(BU)}{(\text{odd torsion})} \quad \text{is computed in section 6}$$

in dimensions ≤ 10.

In §8 we will obtain stable decompositions of $BGL\mathbb{F}_q$ and $BO\mathbb{F}_3$ (suitably localised) which are analogous to those of Theorem 4.2. The S-type of $BO\mathbb{F}_3$ is particularly interesting because it is the same as the S-type of imJ, a factor in the space SG. SG is the 1-component of $QS°$. Hence $\pi_*^S(BO\mathbb{F}_3)$ maps injectively to $\pi_*^S(SG)$ and thence to $\pi_*(SG) \cong \pi_*^S$. This is explained in Part III where an infinity of homotopy elements in $\pi_*^S(BO\mathbb{F}_3)_{(2)}$ are constructed.

Finally I would like to express my gratitude to the algebraic topologists of Purdue University and of the Centro de Investigacion del I.P.N. for their hospitality shown to me during the preparation of Part I of this paper.

The paper would not get off the ground without the all-important technical results of section 2.2-2.3 relating the transfer of vector fields. The technique was first used in this form by Brumfiel-Madsen, whose work has now appeared [B-M]. Becker used some particular cases of the technique in [Be]. The presentation in section 2 is my version of an outline given by Ib Madsen in a Chicago lecture in August 1975.

The material in Part I dates from late 1975. I am deeply indebted to Peter May and Ib Madsen for their comments and suggestions concerning the obscurity of my earlier expositions of the material in Part I.

§2: COMPUTATIONS WITH THE TRANSFER

Let F and Y be G-spaces where the action of G on Y is free. Set $E = Y \underset{G}{\times} F$ and $X = \frac{Y}{G}$ and let $\pi:E \to X$ be induced by left projection. Suppose

$$F \to E \to X$$

is a differentiable fibre bundle in which Y is the limit of compact G-spaces and F has the equivariant homotopy type of a compact G-manifold. From this data we are entitled by [B-G1; B-G2; C-G] to an S-map $\tau(\pi):X \to E$ called the transfer. $\tau(\pi)$ is equivalent (by taking adjoints) to a map $\tau(\pi):X \to Q(E)$ where $Q(E) = \Omega^\infty\Sigma^\infty E = \varinjlim_n \Omega^n\Sigma^n E$. For example if $F = \frac{U(n)}{\Sigma_n \int U(1)}$ the fibring

$$F \to B\Sigma_n \int U(1) \xrightarrow{\pi_n} BU(n)$$

yields

$$\tau(\pi_n):BU(n) \to Q(B\Sigma_n \int U(1)).$$

Here $\Sigma_n \int U(1)$ is the canonical wreath product which is the normalizer of the maximal torus of diagonal matrices. The main result of this section concerns the coherence of $\{\tau(\pi_n);\ n \geq 1\}$ in the above fibring and in several similar fibrings involving the classical groups.

When F is a smooth manifold (not necessarily compact) which admits a G-embedding into a finite dimensional G-module the results of [B-G2] assure us that we may use the construction of [B-G1] to obtain the transfer without ambiguity.

2.1: Theorem. Let G_n be one of the Lie groups $U(n), Sp(n), O(2n)$ or $SO(2n+1)$. Let H_n be the subgroup $\Sigma_n \int U(1)$, $\Sigma_n \int Sp(1)$, $\Sigma_n \int O(2)$ or NT^n respectively. Then the following diagram is homotopy commutative.

$$\begin{array}{ccc} BG_n & \xrightarrow{\tau(\pi_n)} & QBH_n \\ \downarrow & & \downarrow \\ BG_{n+1} & \xrightarrow[\tau(\pi_{n+1})]{} & QBH_{n+1} \end{array} .$$

Here the vertical maps are induced by inclusions of subgroups while $\tau(\pi_n)$ is the transfer associated with

$$\frac{G_n}{H_n} \to BH_n \to BG_n .$$

NT^n is the normalizer of the standard maximal torus in $SO(2n + 1)$.

This result establishes a filtered S-map $BG_\infty \to BH_\infty$. However the filtered S-type of BH_∞ was the subject of the decomposition theorem of [Sn 1]. Here I am using terms such as S-map, S-type, etc. in the sense of the Adams stable homotopy category [Ad 1, Part III]. That is, a space (for example, QBH_∞) is considered as an object in the Adams category by means of its suspension spectrum. This usage is consistent with that of [Sn 1].

In section 4 I will combine these results to obtain a decomposition of the S-type of BG_∞. Theorem 2.1 is established according to the following programme. Proposition 2.2 gives a result which enables one to reinterpret a transfer map with the aid of an equivariant vector field on the fibre. Then Proposition 2.4 gives a general construction, due to Samelson and outlined to me by Ib Madsen, of equivariant vector fields on homogeneous spaces. Propositions 2.2 and 2.4 are then applied to several examples, including those referred to in Theorem 2.1.

2.2: Proposition. Let $F \to Y \times_G F = E \xrightarrow{\pi} \frac{Y}{G} = X$ be a differentiable fibre bundle as described above in which $\partial F = \phi$. Let F_1 be a G-submanifold of F with equivariant tubular neighbourhood N. Suppose that ρ is an equivariant vector field on F which, on ∂N, is homotopic through nowhere zero vector fields to an outward normal and satisfies $|\rho(x)| = 1$ for $x \notin N$.

Then the following diagram is homotopy commutative.

$$\begin{tikzcd} & Q(Y \times_G F_1) \arrow[d, "Q(i)"] \\ X \arrow[ur, "\tau(\pi')"] \arrow[r, "\tau(\pi)"'] & Q(E) \end{tikzcd}$$

Here $\tau(\pi')$ is the transfer for $F_1 \to Y \times_G F_1 \xrightarrow{\pi'} X$ and $Q(i)$ is induced by the inclusion $i: F_1 \subset F$.

Proof: In [B-G 2, §5.3] it is shown that the homotopy class of the transfer is invariant under fibre homotopy equivalence. Consider the fibre bundle $N \to Y \times_G N \xrightarrow{\pi''} X$. This is fibre homotopy equivalent to $F_1 \to Y \times_G F_1 \xrightarrow{\pi'} X$. Hence the composition

$$X \xrightarrow{\tau(\pi'')} Q(Y \times_G N) \to Q(E)$$

is homotopic to $Q(i) \circ \tau(\pi')$. I will show that $\tau(\pi)$ is homotopic to this composition.

The transfer is defined as follows. Take an equivariant embedding $F \subset V$ into a finite dimensional G-module. Let N_1 be the normal bundle. If ξ is a vector bundle let $Th(\xi)$ denote its Thom space. If $\gamma: Th(V) \to Th(N_1)$ is the Pontrjagin-Thom map then we have a map

$$Th(V) \xrightarrow{\gamma} Th(N_1) \xrightarrow{i} Th(N_1 \oplus TF)$$

where TF is the tangent bundle and i is induced by the inclusion $N_1 \subset N_1 \oplus TF \simeq F \times V$. The transfer is obtained by taking the product of this G-map

$$Th(V) \to Th(N_1 \oplus TF) = F^+ \wedge Th(V)$$

with the identity map of Y, dividing out by the G-action and then stabilizing. Details are to be found in [B-G 1; Section 3].

Now define for $0 \le s \le 1$

$$i_s: Th(N_1) \to Th(N_1 \oplus TF)$$

by

$$i_s(v) = \begin{cases} \dfrac{1}{(1 - s\,|\rho(x)|)}\,(v, s\rho(x)) & \text{if } |s\rho(x)| < 1 \\ \infty & \text{otherwise} \end{cases}$$

where v belongs to the fibre of N_1 at x. This homotopy takes $i_0 \circ \gamma = i \circ \gamma$ to $i_1 \circ \gamma$. However, if ρ were an outward normal on ∂N then $i_1 \circ \gamma$ would be the map used in [B-G 1, section 2.8] to define $\tau(\pi'')$. Therefore, up to homotopy, our map is $\tau(\pi'')$.

2.2.1: Remark. Let me describe an example, commonly occurrent, satisfying the hypotheses of Proposition 2.2. We are considering the fibring $F \to Y \times_G F = E \overset{\pi}{\to} \frac{Y}{G} = X$. Here I remind the reader that F is a differential manifold having the homotopy type of a compact manifold. Also F is G-embeddable into a finite dimensional G-module. Now let ρ be a G-equivariant vector field F which is non-generate on its singular set, F_1. Suppose, in addition, that F_1 is connected and that G acts transitively on F_1. Of course we assume G acts isometrically with respect to the Riemannian metric on F. Now choose a very small $\varepsilon > 0$ and consider $N = \{f \in F \mid |\rho(f)| \leq \varepsilon\}$. Since ρ is non-degenerate at $f_0 \in F_1$ by choosing ε small enough we may ensure that on ∂N near f_0 ρ has a non-zero component in the direction normal to ∂N. The transitive action of G allows us to translate the picture near f_0 all over F_1. From this we see that, for sufficiently small $\varepsilon > 0$, N is an equivariant tubular neighbourhood of F_1 in F and at each point of ∂N ρ has a non-zero normal component. The connectivity of F_1 assures us that if ρ has an outward normal component on ∂N near f_0 it has an outward normal component at each point of ∂N. If this is the case then ρ on ∂N is homotopic to an outward normal field, through non-zero vector fields, by linearly shrinking the tangential component to zero.

Now let G be a compact Lie group with Lie algebra $\mathcal{G}$. For $v \in \mathcal{G}$ define a vector field, ϕ_v, on G as follows. For $z \in G$ let r_z and ℓ_z denote respectively right and left transformation by z. Then

$$\phi_v(z) = (Dr_z)_e(v) \in T_zG .$$

Here Dr_z is the derivative of r_z. The following result is straight forward and will be left to the reader. It suffices to treat the case $G = GL_n(C)$ which requires only the very basic information which is to be found in [Ad 2, Chapter 4; Mi 1, section 1].

2.3: Lemma. (i) If $w \in G$ is in the centralizer of $\exp v \in G$ then

$$D(\ell_w)(\phi_v(z)) = \phi_v(wz) \in T_{wz}G .$$

(ii) If $w \in G$ is any element then

$$D(r_x)(\phi_v(z)) = \phi_v(zw) \in T_{zw}G .$$

2.4: Proposition. Let G be a compact Lie group and let H be a closed subgroup. Let $0 \neq v \in \mathcal{G}$. There exists a vector field, ρ_v, on $\frac{G}{H}$ with the following properties.

(i) $\rho_v(gH) = 0$ if and only if $g^{-1}(\exp v)g \in H$.

(ii) If $w \in C(\exp v)$, the centralizer of $\exp v$,

then

$$D(\ell_w)(\rho_v(gH)) = \rho_v(wgH) \quad \text{for all cosets } gH.$$

<u>Proof</u>: ρ_v is induced from ϕ_v of Lemma 2.3 by taking quotients by the right H-action. It is a very basic computation to show that

$$\phi_v(g) \in TH \quad \text{if and only if} \quad g^{-1}(\exp v)g \in H.$$

The reader is again referred to [Ad 2, Chapter 4; Mi 1, section 1] for the basic information required in this computation. Hence $\rho_v(gH) = 0$ if and only if $g^{-1}(\exp v)g \in H$.

The remainder of this section will be devoted to the applications of Propositions 2.2 and 2.4 which will be needed later.

<u>2.5</u>: <u>Example</u>. Take $G = U(n)$, $H = \Sigma_n \int U(1)$ and $v \in \mathcal{U}(n)$ such that

$$w = \exp v = \begin{pmatrix} x & & & & \\ & x & & & \\ & & \ddots & & \\ & & & x & \\ & & & & y \end{pmatrix} \in T^n$$

$(x \neq y)$. Then $C(w) = U(n-1) \times U(1)$. Also if $g^{-1}wg \in \Sigma_n \int U(1)$ there exists $\sigma \in \Sigma_n$ such that $\sigma^{-1}g^{-1}wg\sigma \in T^n$. Hence $g\sigma T^n\sigma^{-1}g^{-1}$ is a maximal torus containing w. We know from [Ad 2, p. 97] that the identity component of C(w) equals the union of all maximal tori containing w. Hence there is $b \in U(n-1) \times U(1)$ satisfying $g\sigma T^n\sigma^{-1}g^{-1} = bT^nb^{-1}$. Therefore $g \in (U(n-1) \times U(1))\Sigma_n \int U(1)$. Conversely all such g are singular. Therefore ρ_v is a (left) $U(n-1) \times U(1)$-equivariant vector field on $\dfrac{U(n)}{\Sigma_n \int U(1)}$ whose singular set is equivariantly homeomorphic to $\dfrac{U(n-1)}{\Sigma_{n-1} \int U(1)}$.

<u>2.6</u>: <u>Example</u>. Similar to Example 2.5 is the case when $G = Sp(n)$ and $H = \Sigma_n \int Sp(1)$. We obtain an $Sp(n-1) \times Sp(1)$-equivariant vector field on $\dfrac{Sp(n)}{\Sigma_n \int Sp(1)}$ whose singular set is equivariantly homeomorphic to $\dfrac{Sp(n-1)}{\Sigma_{n-1} \int Sp(1)}$.

<u>2.7</u>: <u>Example</u>. Take $G = O(2n)$, $H = \Sigma_n \int O(2)$ and choose $v \in \mathcal{O}(2n)$ such that

$$w = \exp v = \begin{pmatrix} I_{2n-2} & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix} \quad (\theta \neq n\pi).$$

Then $C(w) = O(2n-2) \times SO(2)$. If $g^{-1}wg \in \Sigma_n \int O(2)$ there exists $x \in \Sigma_n \subset \Sigma_n \int O(2)$ such that $x^{-1}g^{-1}wgx \in O(2)^n \cap SO(2n)$. Now choose $y \in O(2)^n$ such that

$$y^{-1}x^{-1}g^{-1}wgxy = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}$$

where

$$\alpha_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

or

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence there exists $\sigma \in \Sigma_{2n}$ such that $\sigma^{-1}y^{-1}x^{-1}g^{-1}wgxy\sigma \in SO(2)^n$. Arguing as in Example 2.5

$$gxy\sigma SO(2)^n\sigma^{-1}y^{-1}x^{-1}g^{-1} = zSO(2)^n z^{-1}$$

for some $z \in O(2n-2) \times SO(2)$. The normalizer of $SO(2)^n$ in $O(2n)$ is $\Sigma_n \int O(2)$. Hence $z^{-1}gxy \in \Sigma_n \int O(2)$ and

$$g \in (O(2n-2) \times SO(2))(\Sigma_n \int O(2))\Sigma_{2n}(\Sigma_n \int O(2)) = U,$$

say. Conversely if $g \in U$ we may write $g = abc$ where $a \in O(2n-2) \times O(2)$, $b \in \Sigma_{2n}$ and $c \in \Sigma_N \int O(2)$ implies

$$b^{-1}wb \in \Sigma_n \int O(2) \text{ or } b^{-1}w'b \in \Sigma_n \int O(2)$$

where

$$w' = \begin{pmatrix} I_{2n-2} & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix}.$$

It is easy to see that this can only happen if $b \in \Sigma_{2n-2} \times \Sigma_2$. Hence $g \in (O(2n-2) \times SO(2))\Sigma_n \int O(2)$. Therefore ρ_v is an $O(2n-2) \times SO(2)$-equivariant field on $\dfrac{O(2n)}{\Sigma_n \int O(2)}$ whose singular set is equivariantly homeomorphic to

$$\frac{O(2n-2)}{\Sigma_{n-1} \int O(2)}.$$

2.8: Example. Similar to Example 2.7 is the case $G = SO(2n+1)$ and $H = NT^n$ the normalizer of $SO(2)^n$ in G. We obtain an $SO(2n-1) \times SO(2)$-equivariant vector field on $\frac{SO(2n+1)}{NT^n}$ whose singular set is equivariantly homeomorphic to $\frac{SO(2n-1)}{NT^{n-1}}$.

2.9: Proof of Theorem 2.1. In Examples 2.5 - 2.8 G_{n-1}-equivariant vector fields were constructed on $\frac{G_n}{H_n}$ with singular sets equal to $\frac{G_{n-1}}{H_{n-1}}$. The transfer is natural for pullbacks [B-G 1, Section 3.2]. Consider the pullback diagram

$$\begin{tikzcd}
 & \frac{G_n}{H_n} \arrow[dl] \arrow[dr] & \\
EG_n \times_{G_{n-1}} \frac{G_n}{H_n} \arrow[rr, "j'"] \arrow[d, "\pi_n'"] & & EG_n \times_{G_n} \frac{G_n}{H_n} \arrow[d, "\pi_n"] \\
BG_{n-1} \arrow[rr, "j"] & & BG_n
\end{tikzcd}$$

By naturality $\tau(\pi_n) \circ j$ is homotopic to the composition $Q(j') \circ \tau(\pi_n')$. The vector fields of Examples 2.5-2.8 have connected singular sets which are acted transitively upon by G_{n-1}. The fields are well-known to be non-degenerate in the sense of the discussion in Section 2.2.1. In Proposition 2.2 we may take N to be the neighbourhood of $\frac{G_{n-1}}{H_{n-1}}$ consisting of points x where $|\rho_v(x)| \leq \varepsilon$ for suitably small $\varepsilon > 0$. Since N is a level surface for $|\rho_v(x)|$ ρ_v is a normal vector which is an outward normal near the identity coset. Hence, by Proposition 2.2 and Section 2.2.1, $\tau(\pi_n')$ is homotopic to $Q(i) \circ \tau(\pi_{n-1})$ where

$$i : BH_{n-1} = EG_n \times_{G_{n-1}} \frac{G_{n-1}}{H_{n-1}} \to EG_n \times_{G_{n-1}} \frac{G_n}{H_n}$$

is induced by the inclusion of the singular set of the vector field.

The remaining examples of equivariant vector fields which are collected in this section will be used in Part II and in Sections 3-4 where we determine the filtered S-types of BU, BSp, BO and BSO and factorise QBU(1), QBSp(1) and QBO(2).

2.10: Example. Take $G = SO(2n+1)$, $H = NT^n$ and choose $v \in \mathit{SO}(2n+1)$ such that

$$w = \exp v = \begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_n & \\ & & & & 1 \end{pmatrix} \in T^n$$

with $I_2 \neq \alpha_i \neq \alpha_j \in SO(2)$ if $i \neq j$. Then $C(w) = T^n$. Arguing as in Example 2.7 we find that $g^{-1}wg \in NT^n$ if and only if $g \in NT^n$. Therefore ρ_v is a T^n-equivariant vector field whose singular set is a point.

The next example is a cautionary one. It is included to show that not every ρ_v constructed by means of Proposition 2.4 can be used in Proposition 2.2. When the singular set is disconnected ρ_v may not be an outward normal to its tubular neighbourhood.

<u>2.11</u>: <u>Example</u>. Take $G = GL_{2n}(\mathbb{R}) \simeq O(2n)$ and $H = \Sigma_n \int GL_2(\mathbb{R}) \simeq \Sigma_n \int O(2)$. Choose $v \in GL_{2n}(\mathbb{R})$ such that $w = \exp v$ is a diagonal matrix with distinct entries. Then $C(w) = GL_1(\mathbb{R})^{2n}$. If $g^{-1}wg \in \Sigma_n \int GL_2(\mathbb{R})$ then there exists $x \in \Sigma_n \subset \Sigma_n \int GL_2(\mathbb{R})$ such that $x^{-1}g^{-1}wgx \in GL_2(\mathbb{R})^n$. Having distinct real eigenvalues this matrix is diagonalizeable and there exists $y \in GL_2(\mathbb{R})^n$ such that $y^{-1}x^{-1}g^{-1}wgxy \in GL_1(\mathbb{R})^{2n}$. This must be a permutation of w so there is $z \in \Sigma_{2n}$ such that $gxyz \in C(w)$. Hence

$$g \in (\Sigma_{2n} \int GL_1(\mathbb{R}))\Sigma_n \int GL_2(\mathbb{R}) .$$

Conversely all such g are singular. Thus there is an $O(1)^{2n}$-equivariant vector field, ρ_v, on $\dfrac{GL_{2n}(\mathbb{R})}{\Sigma_n \int GL_2(\mathbb{R})}$ whose singular set is equivariantly homeomorphic to $\dfrac{\Sigma_{2n}}{\Sigma_n \int \Sigma_2}$. Caution is required in applying Proposition 2.2 when $n \neq 1$ since ρ_v is not the outward normal at each point of a tubular neighbourhood of $\dfrac{\Sigma_{2n}}{\Sigma_n \int 2}$. Since the Euler characteristic of $\dfrac{GL_{2n}(\mathbb{R})}{\Sigma_n \int GL_2(\mathbb{R})}$ is one, the Poincare-Hopf theorem [Mi 1, p. 35] implies that the sum of the indices of ρ_v at the singular points of $\dfrac{\Sigma_{2n}}{\Sigma_n \int \Sigma_2}$ is equal to one.

<u>2.11.1</u>: <u>Remark</u>. In order that I may later make use of Example 2.11 in conjunction with Proposition 2.2 and section 2.2.1, I must give some explanation of the tricky technical points and how they are overcome. Example 2.11 will be used in sections 3.7-3.8 to make a homology computation similar to that of [B-M]. In [B-M] this sort of computation is made using algebraic subgroups of

the general linear groups in order to consider only fibrings with compact manifolds as fibre. As explained in section 2.2.1 this is not necessary for the following reason. In [B-G 2] two equivalent constructions of the transfer are given for smooth fibrings of the type we are considering. The S-duality definition clearly requires only data up to fibre homotopy equivalence. However the proof of the equivalence of the two definitions, in the smooth case [B-G 2], requires only that the fibre be G-equivariantly embeddable in Euclidean space. When G is a finite group this can be accomplished for non-compact, smooth manifolds such as those occurring in Example 2.11.

The above discussion would permit us to apply arguments like that of Proposition 2.2 and section 2.9 using vector fields, ρ_v, which are homotopic to outward normals on their singular set. However there is one case when we can drop the normality condition. Namely if the singular set is a finite set with trivial G-action. In this case it is possible to obtain formulae for restrictions of transfers but not solely in terms of other transfers. The individual indices of ρ_v at the points of the singular set enter into the formula, (see section 3.7).

2.12: Example. Take $G = U(2n)$, $H = \Sigma_n \int U(1)$ and choose $v \in \mathcal{U}(n)$ such that

$$w = \exp v = \begin{pmatrix} x_1 I_n & 0 \\ 0 & x_2 I_n \end{pmatrix} \qquad (x_1 \neq x_2)\,.$$

Then $C(w) = U(n) \times U(n)$. Arguing as in Example 2.5 we find that $g^{-1}wg \in \Sigma_n \int U(1)$ if and only if

$$g \in (U(n) \times U(n))\Sigma_{2n} \int U(1)\,.$$

Thus ρ_v is a $U(n) \times U(n)$-equivariant vector field on $\dfrac{U(2n)}{\Sigma_{2n}\int U(1)}$ whose singular set is equivariantly homeomorphic to $\left(\dfrac{U(n)}{\Sigma_n \int U(1)}\right)^2$.

The following example is similar to Example 2.12.

2.13: Example. Take $G = Sp(n)$ and $H = \Sigma_{2n} \int Sp(1)$. Then there is an $Sp(n) \times Sp(n)$-equivariant vector field on $\dfrac{Sp(n)}{\Sigma_{2n}\int Sp(1)}$ whose singular set is equivariantly homeomorphic to $\left(\dfrac{Sp(2n)}{\Sigma_{2n}\int Sp(1)}\right)^2$.

2.14: Example. Take $G = U(n)$ and $H = \Sigma_n \int U(1)$. Choose $v \in \mathcal{U}(n)$ such that $w = \exp v$ is a diagonal matrix with distinct entries. Arguing as in Example 2.5 we find that ρ_v is a T^n-equivariant vector field on $U(n)/\Sigma_n \int U(1)$ whose singular set is a point.

The following example is similar to Example 2.13.

2.15: Example. Take $G = Sp(n)$ and $H = \Sigma_n \int Sp(1)$. Then there exists a T^n-equivariant vector field, ρ_V, on $\frac{Sp(n)}{\Sigma_n \int Sp(1)}$ whose singular set is a point. Here T^n is the canonical maximal torus of $Sp(n)$.

§3. FACTORIZATIONS OF QBU(1), QBO(2) AND QBSp(1)

A map $X \to Y$ into an infinite loopspace, Y, extends to an infinite loop map

$$QX = \Omega^\infty \Sigma^\infty X \to Y .$$

This extension is unique up to homotopy. Details will be given in section 3.4. Hence there are infinite loopspace fibrings

$$F_U \xrightarrow{j_U} QBU(1) \xrightarrow{\lambda_U} BU$$
$$F_O \xrightarrow{j_O} QBO(2) \xrightarrow{\lambda_O} BO$$
$$F_{Sp} \xrightarrow{j_{Sp}} QBSp(1) \xrightarrow{\lambda_{Sp}} BSp$$

in which λ_U, λ_O and λ_{Sp} extend the canonical maps. For example, λ_U extends the map which classifies the reduced Hopf bundle in $\widetilde{KU}(BU(1))$.

By Theorem 2.1 the transfer yields maps

$$\tau_U : BU \longrightarrow QBU(1)$$
$$\tau_O : BO \longrightarrow QBO(2) \quad \text{and}$$
$$\tau_{Sp} : BSp \longrightarrow QBSp(1)$$

in the manner explained in section 3.4.

The main result of this section is the following:

3.2: Theorem. (i) τ_U and τ_{Sp} are H-maps.

(ii) The compositions $\lambda_U \circ \tau_U$, $\lambda_O \circ \tau_O$ and $\lambda_{Sp} \circ \tau_{Sp}$ are homotopy equivalences.

(iii) $j_O + \tau_O : F_O \times BO \to QBO(2)$ is a homotopy equivalence.

(iv) $j_U + \tau_U : F_U \times BU \to QBU(1)$ and $j_{Sp} + \tau_{Sp} : F_{Sp} \times BSp \to QBSp(1)$ are equivalences of H-spaces.

Remark. In (iii) the equivalence is not an equivalence of H-spaces. However τ_O deviates from an H-map in a very subtle manner. This topic is taken up in [Sn 2 and in §7].

J. C. Becker [Be] and G. B. Segal [Se] have proved parts of Theorem 3.2, by different arguments.

3.3: Sketch of Proof. In the notation of Theorem 2.1, part (i) is a matter of evaluating

$$BG_n \times BG_n \longrightarrow BG_{2n} \xrightarrow{\tau(\pi_{2n})} QBH_{2n} .$$

This is accomplished in Proposition 3.6 with the aid of Proposition 2.2 and Examples 2.12 and 2.13. Parts (iii) and (iv) follow immediately from part (ii). There are several cases to be considered in (ii). The spaces QBU(1), QBO(2) and QBSp(1) are homologically like BH_∞ for H_∞ chosen appropriately from Theorem 2.1. If in homology $(\lambda_{G_\infty})_* = (\pi_\infty)_*$ it is easy to show that $\lambda_{G_\infty} \circ \tau_G$ is a homology isomorphism. In Proposition 3.9 this is shown to be true for most of the cases. The remaining case is BO mod 2. In Proposition 3.7 and Corollary 3.8 we analyze the restriction of $\lambda_0 \circ \tau_0$ to $BO(1)^{2n}$ in mod 2 homology. The proof of Theorem 3.2 is given in section 3.10.

3.4: QX. Let us recall a few facts about QX. [Ma 1] is a suitable reference for further details.

If X is a nice space (a CW complex which is compactly generated, for example) then $QX = \Omega^\infty\Sigma^\infty X$ may be considered as a filtered space. To be precise there exist filtered spaces [Ba; Ma 1] which are homotopy equivalent to QX. I will use the filtered spaces, $C_\infty X$, of [Ma 1, section 6]. The filtered space

$$X = F_1C_\infty X \subset \cdots \subset F_nC_\infty X \subset F_{n+1}C_\infty X \subset \cdots \subset C_\infty X$$

is equipped with a map

$$\alpha_\infty : C_\infty X \to QX$$

which is homotopy equivalence for connected X [Ma 1, section 6.3]. There exist maps

$$i_n : E\Sigma_n \times_{\Sigma_n} X^n \to F_nC_\infty X$$

[Ma 1, sections 2.4 and 4.8] such that $\alpha_\infty \circ i_n$ is the restriction of the structure map $d : QQX \to QX$ of the free functor Q. For details the reader is referred to [Ma 1, section 5] or [Ma 2, Chapter II, section 1].

Note that $i_1 : X \to F_1C_\infty X$ is a homeomorphism and the composite

$$X \xrightarrow{i_1} F_1C_\infty X \subset C_\infty X$$

corresponds to the "suspension" map $\Sigma : X \to QX$.

Also
$$i_n|E_{n-1} \times_{\Sigma_{n-1}} X^{n-1} = i_{n-1} .$$

If Y is an infinite loopspace and $f':X \to Y$ is a map then there is an infinite loop map $f:QX \to Y$ such that $f|X = f'$. f is unique up to homotopy and equal to the composite $d_\circ Q(f')$ where $d:QY \to Y$ is the structure map of the infinite loopspace Y [Ma 1; Ma 2 ibid.].

By definition the restriction of f to $E\Sigma_n \times_{\Sigma_n} X^n$ is equal to the image of $X^n \xrightarrow{\text{proj}} X \xrightarrow{f'} Y$ under the Kahn-Priddy transfer [K-P] associated with the covering $X^n \to E\Sigma_n \times_{\Sigma_n} X^n$.

By Theorem 2.1 the transfers

$$\tau(\pi_n) : BG_n \to Q(BH_n)$$

fit together to give

$$\tau(\pi_\infty) : BG_\infty \to Q(BH_\infty) .$$

When $G_\infty = U$, O or Sp we may form the composition

$$(3.5) \qquad \tau_{G_\infty} : BG_\infty \xrightarrow{\tau(\pi_\infty)} Q(BH_\infty) \xrightarrow{Q(i_\infty)} QQBG_1 \xrightarrow{d} QBG_1 .$$

Here $i_\infty = \varinjlim_n i_n$.

3.6: Proposition. If $G_\infty = U$ or Sp then τ_{G_∞} is an H-map.

Proof: Let $m: QBG_1^2 \to QBG_1$ be the H-space addition. If we can show that the diagram

$$\begin{array}{ccc} BG_n \times BG_n & \longrightarrow & BG_{2n} \\ (\tau_{G_\infty})^2 \downarrow & & \downarrow \tau_{G_\infty} \\ QBG_1 \times QBG_1 & \xrightarrow{m} & QBG_1 \end{array}$$

is a homotopy commutative then the result will follow. This involves a $\varprojlim^1$ argument. By Theorem 3.2 (ii), (which is proved independently of part (i)) $QBG_1 = F_{G_\infty} \times BG$ where F_{G_∞} has finite homotopy groups. Hence in the Milnor exact sequence [Mi 2] the $\varprojlim^1$-term is

$$\varprojlim_n{}^1[\Sigma BG_n^2, QBG_1] \simeq \varprojlim_n{}^1[\Sigma(BG_n^2), BG_\infty] = 0 .$$

Thus

$$[BG_\infty \times BG_\infty, QBG_1] \simeq \varprojlim_n[BG_n \times BG_n, QBG_1] .$$

Now consider $(\tau(\pi_{2n})|BG_n^2)$. Example 2.12 and 2.13 together with Proposition 2.2 assures that $(\tau_{G_\infty}|BG_n^2)$ is equal to the composite

$$BG_n^2 \xrightarrow{\tau'} Q(BH_n^2) \xrightarrow{j} QBH_{2n} \xrightarrow{d \circ Q(i_{2n})} QBG_1 .$$

The argument is an application of the "vector field trick" used in section 2.9 and elaborated upon in sections 2.2 - 2.2.1. Here τ' is the transfer associated with $BH_n^2 \to BG_n^2$ and j is induced by the inclusion of H_n^2 in H_{2n}. From [B-G 2, section 5.6] $j \circ \tau'$ is equal to the composition

$$BG_n^2 \xrightarrow{\tau(\pi_n)^2} (QBH_n)^2 \xrightarrow{j'} QBH_{2n} .$$

The following commutative diagram now yields the result.

$$\begin{tikzcd} BG_n^2 \arrow[r, "\tau(\pi_n)^2"] & (QBH_n)^2 \arrow[r, "j'"] \arrow[d, "Q(i_n)^2"] & QBH_{2n} \arrow[d, "Q(i_{2n})"] \\ & (QQBG_1)^2 \arrow[r] \arrow[d, "d^2"] & QQBG_1 \arrow[d, "d"] \\ & (QBG_1)^2 \arrow[r, "m"] & QBG_1 \end{tikzcd}$$

3.6.1: Corollary. The Becker-Gottlieb solution to the complex Adams conjecture

$$BU \to G/U$$

is an H-map. (Compare this with Lemma 7.2.)

Proof: The Becker-Gottlieb solution to the Adams conjecture, as presented in [Be], is equal to a composite of the form

$$BU \xrightarrow{\tau_U} QBU(1) \xrightarrow{Q(\alpha)} QG/U \xrightarrow{d} G/U.$$

Here $\alpha : BU(1) \to G/U$ is Adams' solution to the Adams conjecture for U(1)-bundles [Ad 5]. Since $Q(\alpha)$ and d are H-maps the result follows from Proposition 3.6.

Now let $h : BO(1)^{2n} \to BO(2n)$ and $k : BO(1)^{2n} \to B\Sigma_n \int O(2)$ be the natural maps. For $g \in \Sigma_{2n}$ denote by k_g the composite

$$BO(1)^{2n} \xrightarrow{\sigma(g)} BO(1)^{2n} \xrightarrow{k} B\Sigma_n \int O(2) \xrightarrow{\Sigma} QB\Sigma_n \int O(2)$$

where $\sigma(g)$ is conjugation by g.

3.7: Proposition. The composition

$$BO(1)^{2n} \xrightarrow{h} BO(2n) \xrightarrow{\tau(\pi_n)} QB\Sigma_n \int O(2)$$

is equal to $\Sigma_g I(g)k_g$. The sum, taken in the group $[BO(1)^{2n}, QB\Sigma_n \int (2)] \simeq \{BO(1)^{2n}, B\Sigma_n \int O(2)\}$, is taken over a set of coset representatives of $\Sigma_{2n} / \Sigma_n \int \Sigma_2$. I(g) is the index at $g\Sigma_n \int \Sigma_2$ of the vector field, ρ_v, in Example 2.11.

Proof: In Example 2.11 $O(1)^{2n}$ acts trivially on the singular set $\dfrac{\Sigma_{2n}}{\Sigma_n \int \Sigma_2}$.

I propose to proceed as if applying Proposition 2.2, in the situation described in section 2.2.1, to this vector field. It is non-degenerate at its singular set (F_1 in the notation of sections 2.2 - 2.2.1). However F_1 is disconnected and the $O(1)^{2n}$-action is not transitive. Nevertheless we obtain, by the argument of Proposition 2.2, a homotopy commutative diagram.

$$\begin{tikzcd} BO(1)^{2n} \arrow[r, "h"] \arrow[d, "\tau'"'] & BO(2n) \arrow[d, "\tau(\pi_n)"] \\ Q\left(BO(1)^{2n} \times \dfrac{\Sigma_{2n}}{\Sigma_n \int \Sigma_2}\right) \arrow[r] & Q(B\Sigma_n \int O(2)). \end{tikzcd}$$

However, as pointed out in Example 2.11 and section 2.11.1, τ' is not the transfer in the sense of [B-G 1]. If τ' were the transfer it would be a sum of a number of copies of the adjoint of the identity map. Nevertheless τ' is still the sum of a number of maps, one for each point in the singular set. By definition of the index of ρ_V at $g\Sigma_n \int \Sigma_2$ [Mi 1, p. 32] the map at $g\Sigma_n \int \Sigma_2$ is I(g) times the adjoint of the identity map, $\Sigma \circ k$. The map

$$Q\left(BO(1)^{2n} \times \frac{\Sigma_{2n}}{\Sigma_n \int \Sigma_2}\right) \to Q(B\Sigma_n \int O(2))$$

is induced by the inclusion of the singular set into $\dfrac{O(2n)}{\Sigma_n \int O(2)}$. Hence it is clear that $BO(1)^{2n} \times g\Sigma_n \int \Sigma_2$ is mapped in by means of the conjugate by g of the canonical map, $\Sigma \circ k$.

<u>3.8</u>: <u>Corollary</u>. The maps $\lambda_0 \circ \tau_0 \circ h$ and $h : BO(1)^{2n} \to BO$ induce the same maps in homology.

<u>Proof</u>: Consider the following commutative diagram.

$$\begin{tikzcd} BO(1)^{2n} \arrow[r, "h"] & BO(2n) \arrow[r, "\tau_0"] \arrow[d, "\tau(\pi_n)"'] & QBO(2) \arrow[r, "\lambda_0"] & BO \\ & QB\Sigma_n \int O(2) \arrow[r, "Q(i_n)"'] & QQBO(2) \arrow[u, "d"'] \arrow[r, "Q(\lambda_0)"'] & QBO . \arrow[u, "d"'] \end{tikzcd}$$

We will apply Proposition 3.7 to this diagram in order to evaluate $\lambda_0 \circ \tau_0 \circ h$ in homology. Firstly we need some notation.

Let $m = |\Sigma_{2n} : \Sigma_n \int \Sigma_2|$ and let $g_1, \ldots, g_m$ be the coset representatives in the statement of Proposition 3.7. Let $h_i : BO(1)^{2n} \to BO(2n)$ be the conjugate of h by g_i. Also denote by h_i the composition of h_i with the natural map into BO. Then $(h_i)_* = h_* : H_*(BO(1)^{2n}) \to H_*(BO)$ for $1 \le i \le m$; since the inner automorphisms induce the identity.

If $I(g_i) = 1$ let $\chi_i : BO \to BO$ be the identity map. If $I(g_i) = -1$ let $\chi_i : BO \to BO$ be the inverse map of the H-space, BO.

The maps d, $Q(i_n)$ and $Q(\lambda_0)$ are H-maps. Therefore Proposition 3.7 implies that

$$d \circ Q(\lambda_0) \circ Q(i_n) \circ \tau(\pi_n) \circ h$$

is homotopic to the composition $BO(1)^{2n} \xrightarrow{\Delta} (BO(1)^{2n})^m \to BO^m \to BO$ of the diagonal followed firstly by $\pi_i\chi_i \circ h_i$ and then by the iterated H-space addition.

Suppose $x \in H_*(BO(1)^{2n})$ satisfies $\Delta_*(x) = \Sigma x_1 \otimes \cdots \otimes x_m$. Then, from our diagram,

$$\begin{aligned}(\lambda_0 \circ \tau_0 \circ h)_*(x) &= \Sigma(\chi_1 \circ h_1)_*(x_1) \ldots (\chi_m \circ h_m)_*(x_m)\\ &= \Sigma(\chi_1)_* h_*(x_1) \ldots (\chi_m)_* h_*(x_m)\\ &= (\prod_{i=1} \chi_i)_* \Delta_*(h_*(x)).\end{aligned}$$

However, to complete the proof, we observe that

$$\left(\prod_{i=1}^{m} \chi_i\right) \circ \Delta \simeq 1_{BO} : BO \xrightarrow{\Delta} BO^m \to BO.$$

For it is a sum in the group [BO,BO] of t copies of 1_{BO} and s copies of (-1_{BO}) where $s + t = \sum_g I(g) = 1$, as explained in Example 2.11.

3.9: Proposition. In the notation of Theorem 2.1, let $G_n = U(n)$, $Sp(n)$ or $O(2n)$. Let R be a torsion free commutative ring. If $H_*(BG_\infty;R)$ has no torsion then

$$BH_\infty \xrightarrow{i_\infty} QBG_1 \xrightarrow{\lambda_{G_\infty}} BG_\infty$$

and π_∞ induce the same map in $H_*(-;R)$. Here i_∞ is the map which was introduced in section 3.4.

Proof: By definition $\lambda_{G_\infty} \circ i_n : B\Sigma_n \int G_1 \to BG_\infty$ is the Kahn-Priddy transfer, $tr(\theta \circ \pi_1)$, [K-P] of

$$BG_1 \times BH_{n-1} \xrightarrow{\pi_1} BG_1 \xrightarrow{\theta} BG_\infty$$

with respect to the covering $BG_1 \times BH_{n-1} \to BH_n$. Here θ is the canonical map considered as representing a class in reduced K-theory. Hence $\theta = E - \dim E$ where E is either the Hopf bundle over BU(1) and BSp(1) or the canonical 2-plane over BO(2). Here dim E refers to the complex, quaternionic or real dimension of E, as appropriate.

The Kahn-Priddy transfer is additive [K-P, section 1.8]. Hence in homology $(\lambda_{G_\infty} \circ i_n)_*$ is the Pontrjagin quotient of $tr(E)_*$ by $tr(\dim E)_*$. Explicitly in homology this has the following meaning. Let $\chi : BG_\infty \to BG_\infty$ be the inverse map of the H-space, BG_∞. Suppose that the diagonal on $x \in \tilde{H}_*(BH_n)$ is given by

$$\Delta_*(x) = \Sigma x_1 \otimes x_2$$

Then

$$(\lambda_{G_\infty} \circ i_n)_*(x) = \Sigma tr(E)_*(x_1)\chi_* tr(\dim E)_*(x_2).$$

Hence the result will follow if $tr(\dim E)_*$ kills

$$\tilde{H}_*(B\Sigma_n \int G_1) = \tilde{H}_*(BH_n).$$

$\dim E$ is a trivial bundle over BG_1. Therefore, by the naturality properties of the Kahn-Priddy transfer [K-P]

$$tr(\dim E) : B\Sigma_n \int G_1 \to BG_\infty$$

factors through the map $B\Sigma_n \int G_1 \to B\Sigma_n$ induced by the homomorphism $G_1 \to \{1\}$. In fact $tr(\dim E)$ is equal to the composite of $B\Sigma_n \int G_1 \to B\Sigma_n$ with the canonical map $B\Sigma_n \to BG_n \to BG_\infty$.

Since Σ_n is a finite group $\tilde{H}_*(B\Sigma_n;R)$ is torsion. Therefore the composite

$$tr(\dim E)_* : \tilde{H}_*(B\Sigma_n \int G_1;R) \to \tilde{H}_*(B\Sigma_n;R) \to \tilde{H}_*(BG_\infty;R)$$

is zero. Finally it is shown in [Ma 2, chapter VIII, Proposition 1.1; see also K-P] that $tr(E)$ is equal to the composite $BH_n \xrightarrow{\pi_n} BG_n \to BG$. Therefore $\lambda_{G_\infty} \circ i_n$ equals π_n in homology and the result follows by letting n tend to infinity.

<u>3.10</u>: <u>Proof of Theorem 3.2</u>. I proved (i) in Proposition 3.6. Given (ii) it is immediate from the homotopy exact sequences of (3.1) that the sum of τ_{G_∞} and j_{G_∞} is a homotopy equivalence. Also j_{G_∞} is an infinite loop may so that $\tau_{G_\infty} + j_{G_\infty}$ is an H-map if τ_{G_∞} is. BG_∞ is simple so, to prove (ii), it suffices by the universal coefficient theorem [Sp, p. 246] and a theorem of J. H. C. Whitehead [Sp, p. 399] to show the following.

(a) $\lambda_U \circ \tau_U$ and $\lambda_{Sp} \circ \tau_{Sp}$ induce integral homology isomorphisms and

(b) $\lambda_0 \circ \tau_0$ induces isomorphisms in homology with coefficients in $Z[\frac{1}{2}]$ and $Z/2$.

Now, with any coefficients, $(\pi_n)_* \circ \tau(\pi_n)_*$ is multiplication by the Euler characteristic $\chi\left(\frac{G_n}{H_n}\right) = 1$ we obtain $(\pi_n)_* \circ \tau(\pi_n)_* = 1$. Hence by Proposition 3.9 $(\lambda_{G_\infty} \circ \tau_{G_\infty})_* = 1$ on $H_*(BU;Z), H_*(BSp;Z)$ and $H_*(BO;Z[\frac{1}{2}])$. The remaining case follows from Corollary 3.8 since $(k)_* : H_*(BO(1)^\infty;Z/2) \to H_*(BO;Z/2)$ is onto.

§4. STABLE DECOMPOSITIONS OF BU, BSp, BO AND BSO

Let $F_kC_\infty X$ be as in section 3.4. In [Sn 1] stable equivalences

$$\alpha_k : F_kC_\infty X \to \bigvee_{t\le k} \frac{F_tC_\infty X}{F_{t-1}C_\infty X}$$

and

$$\beta_k : \bigvee_{t\le k} \frac{F_tC_\infty X}{F_{t-1}C_\infty X} \to F_kC_\infty X$$

are constructed. These equivalences fit together coherently. Details are given below in section 4.5. However $\frac{F_nC_\infty X}{F_{n-1}C_\infty X}$ is homeomorphic [Ma 1, Proposition 2.6 (ii)] to an equivariant half-smash product. Suffice it to say this space is a quotient of $E \underset{\Sigma_n}{\times} X^n$ where E is a contractible space with a free Σ_n-action. Then

$$B\Sigma_n \int U(1) = E \underset{\Sigma_n}{\times} BU(1)^n$$

and

$$B\Sigma_{n-1} \int U(1) = E \underset{\Sigma_{n-1}}{\times} BU(1)^{n-1} .$$

The inclusion of the first n-1 factors $BU(1)^{n-1} \to BU(1)^n$ induces a map

$$B\Sigma_{n-1} \int U(1) \to B\Sigma_n \int U(1)$$

which is homotopic to the canonical map. The quotient map

$$E \underset{\Sigma_n}{\times} BU(1)^n \to \frac{F_nC_\infty BU(1)}{F_{n-1}C_\infty BU(1)}$$

sends $B\Sigma_{n-1} \int U(1)$ to a point. This map and its quaternionic and real analogues induce the following homeomorphisms.[1]

$$\frac{F_nC_\infty BU(1)}{F_{n-1}C_\infty BU(1)} = \frac{B\Sigma_n \int U(1)}{B\Sigma_{n-1}\int U(1)}$$

$$\frac{F_nC_\infty BSp(1)}{F_{n-1}C_\infty BSp(1)} = \frac{B\Sigma_n \int Sp(1)}{B\Sigma_{n-1}\int Sp(1)} \qquad (4.1)$$

and

$$\frac{F_nC_\infty BO(2)}{F_{n-1}C_\infty BO(2)} = \frac{B\Sigma_n \int O(2)}{B\Sigma_{n-1}\int O(2)} .$$

In the notation of Theorem 2.1 let $G_n = U(n)$, $Sp(n)$ or $O(2n)$. Define stable maps

$$\nu_{G_n} : BG_n \to \bigvee_{t\le n} \frac{BG_t}{BG_{t-1}}$$

by means of the following composition

$$BG_n \xrightarrow{\tau(\pi_n)} B\Sigma_n \int G_1 \xrightarrow{\alpha_n} \bigvee_{t\leq n} \frac{B\Sigma_t \int G_1}{B\Sigma_{t-1}\int G_1} \xrightarrow{\gamma} \bigvee_{t\leq n} \frac{BG_t}{BG_{t-1}}$$

where $\gamma = \bigvee_{t\leq n} \pi_t$.

The first result of this section is the following which is proved in section 4.6.

4.2: Theorem.

(i) If $G_n = U(n)$ or $Sp(n)$ then

$$\nu_{G_n} : BG_n \to \bigvee_{t\leq n} \frac{BG_t}{BG_{t-1}}$$

is a stable equivalence for $1 \leq n \leq \infty$.

(ii) $$\nu_{O(2n)} : BO(2n) \to \bigvee_{t\leq n} \frac{BO(2t)}{BO(2t-2)}$$

is a stable equivalence for $1 \leq n \leq \infty$.

(iii) For $G_n = U(n)$, $Sp(n)$ or $O(2n)$

$$\nu_{G_n}\Big|BG_{n-1} \simeq \nu_{G_{n-1}} .$$

The other result of this section gives stable equivalences in the opposite direction and includes the case $BO(= BSO)$ at odd primes.

By Theorem 2.1 the transfer induces stable maps

$$\tau(\pi_n) : \frac{BG_n}{BG_{n-1}} \to \frac{BH_n}{BH_{n-1}} .$$

Here G_n and H_n are as in Theorem 2.1. Hence if $G_n = U(n)$, $Sp(n)$ or $O(2n)$ we have a stable map

$$\mu_{G_n} : \bigvee_{t\leq n} \frac{BG_t}{BG_{t-1}} \to BG_\infty$$

given by the composition

$$\bigvee_{t\leq n} \frac{BG_t}{BG_{t-1}} \xrightarrow{\delta} \bigvee_{t\leq n} \frac{B\Sigma_t\int G_1}{B\Sigma_{t-1}\int G_1} \xrightarrow{\beta_n} F_n C_\infty BG_1 \xrightarrow{i_n} QBG_1 \xrightarrow{\lambda_{G_\infty}} BG_\infty .$$

Here $\delta = \bigvee_{t\leq n} \tau(\pi_t)$. Also the normalizer $NT^n \subset SO(2n + 1)$ lies in $\Sigma_n \int O(2) \times O(1) \subset O(2n + 1)$. Thus we have a map

$$\frac{BNT^n}{BNT^{n-1}} \to \frac{B\Sigma_n\int O(2)}{B\Sigma_{n-1}\int O(2)} .$$

Define a stable map

$$\mu_{SO(2n+1)} : \bigvee_{t\le n} \frac{BSO(2n+1)}{BSO(2n-1)} \to BO$$

by means of the composition

$$\bigvee_{t\le n} \frac{BSO(2t+1)}{BSO(2t-1)} \xrightarrow{\delta} \bigvee_{t\le n} \frac{BNT^t}{BNT^{t-1}} \to \bigvee_{t\le n} \frac{B\Sigma_t\int O(2)}{B\Sigma_{t-1}\int O(2)} \xrightarrow{\beta_n} F_nC_\infty BO(2) \xrightarrow{i_n} QBO(2) \xrightarrow{\lambda_0} BO.$$

Again $\delta = \bigvee_{t\le n} \tau(\pi_t)$.

4.3: Theorem. (i) If $G_n = U(n)$ or $Sp(n)$

$$\mu_{G_\infty} : \bigvee_{1\le t} \frac{BG_t}{BG_{t-1}} \to BG_\infty$$

is a stable equivalence.

(ii) $$\mu_0 : \bigvee_{1\le t} \frac{BO(2t)}{BO(2t-2)} \to BO$$

is a stable equivalence.

(iii) $$\mu_{SO} : \bigvee_{1\le t} \frac{BSO(2t+1)}{BSO(2t-1)} \to BO$$

is a stable equivalence at odd primes.

4.4. Theorems 4.2 and 4.3 will be proved according to the following programme. We wish to show that a number of maps induce isomorphisms in homology. These maps are compositions. Part of these compositions are the stable maps α_k and β_k. In section 4.5 we recall the salient facts about α_k and β_k, namely how they behave with respect to the filtration. These facts together with the homology information, which was garnered in section 3, about the other maps in the composition will be used in sections 4.6 and 4.7 to prove the theorems. In section 8, a decomposition theorem for $BGL\mathbb{F}_q$ and $BO\mathbb{F}_3$ is proved.

4.5: Properties of α_k and β_k. Recall [Ma 1, section 6] there are filtered spaces, C_nX, equipped with maps

$$C_nX \to \Omega^nS^nX$$

and satisfying $C_\infty X = \varinjlim_n C_nX$ as a filtered space. In [Sn 1, Theorem 1.1] explicit stable equivalences

$$\alpha_k(n) : F_kC_nX \to \bigvee_{t\le k} \frac{F_tC_nX}{F_{t-1}C_nX}$$

were constructed. These stable equivalences enjoy the following properties:

(a) $\alpha_k(n) \simeq \alpha_{k+1}(n)|F_kC_nX$ [Sn 1, section 3.2]

(b) The composition of $\alpha_k(n)$ with the collapsing map onto the factor $\frac{F_kC_nX}{F_{k-1}C_nX}$ is homotopic to the canonical collapsing map

$$F_kC_nX \to \frac{F_kC_nX}{F_{k-1}C_nX} \qquad \text{[Sn 1, section 3.2]}$$

(c) $\alpha_k = \varinjlim_n \alpha_k(n)$ exists and is a stable equivalence [Sn 1, section 3.6].

Take the stable equivalence β_k to be a homotopy inverse of α_k. The β_k may be chosen so that

$$\beta_{k+1} \Big| \bigvee_{t \le k} \frac{F_tC_\infty X}{F_{t-1}C_\infty X} \simeq \beta_k$$

and I shall assume this done.

4.6: Proof of Theorem 4.2. (iii) The coherence of the ν_{G_n} follows from the coherence of the $\alpha_k(n)$ (section 4.5(a)) and of the transfer, $\tau(\pi_n)$ (Theorem 2.1).

For the rest we have, by a theorem of J. H. C. Whitehead [Sp, p. 399], to show that ν_{G_n} induces isomorphisms in homology.

(i) Suppose that $\nu_{G_{n-1}}$ is a stable equivalence and commence induction with n = 1. Consider the composition

$$BG_n \xrightarrow{\nu_{G_n}} \bigvee_{t \le n} \frac{BG_t}{BG_{t-1}} \to \frac{BG_n}{BG_{n-1}} .$$

By 4.5(b) this is stably homotopic to the composite

$$BG_n \to \frac{BG_n}{BG_{n-1}} \xrightarrow{\tau'} \frac{B\Sigma_n \int G_1}{B\Sigma_{n-1} \int G_1} \xrightarrow{\pi'} \frac{BG_n}{BG_{n-1}}$$

where τ' and π' are induced by $\tau(\pi_n)$ and π_n respectively. However (cf. section 3.10) $\pi_n \circ \tau(\pi_n)$ induces the identity in homology by [B-G 1, Theorem 2.4]. Also $BG_n \to \frac{BG_n}{BG_{n-1}}$ is onto in homology with kernel given by the image of $H_*(BG_{n-1})$. Therefore by induction on n, it is clear that ν_{G_n} induces a homology isomorphism.

(ii) This case is similar to case (i).

4.7: Proof of Theorem 4.3. The proof is very similar to that of Theorem 4.2. The basic ingredients are Proposition 3.9 and Theorem 3.2(ii) for BO at p = 2. Details are left to the reader.

§5. CONNECTIONS WITH COBORDISM

In Theorems 4.2 and 4.3 I showed that BU and BSp are stably equivalent to a wedge of Thom spaces. The stable maps of X into a Thom space, MU(n) or MSp(n), are related to the cobordism of X (unitary or symplectic respectively). Suitable references for the cobordism material of this section are [Ad 1; St 1].

We will work in Adams category of CW spectra [Ad 1; p. 146). Let $c_n' \in MU^{2n}\left(\frac{BU}{BU(n-1)}\right)$ and $p_n' \in MSp^{4n}\left(\frac{BSp}{BSp(n-1)}\right)$ be cobordism classes which restrict to the Conner-Floyd classes

$$c_n \in MU^{2n}(BU) \quad \text{and} \quad p_n \in MSp^{4n}(BSp)$$

respectively. Let $\{X,Y\}$ denote stable maps of degree zero from X to Y. The main results of this section are the following:

5.1: Theorem. Define $\Phi_U(n) : \left\{X, \frac{BU}{BU(n-1)}\right\} \to \prod_{n\le k} MU^{2k}(X)$ by

$$\Phi_U(n)(f) = \prod_{n\le k} f^*(c_k') .$$

Then $\Phi_U(n)$ is an isomorphism if $\dim X \le 4n$ and is a surjection of $\dim X = 4n + 1$.

5.2: Theorem. Define $\Phi_{Sp}(n) : \left\{X, \frac{BSp}{BSp(n-1)}\right\} \to \prod_{n\le k} MSp^{4k}(x)$ by

$$\Phi_{Sp}(n)(f) = \prod_{n\le k} f^*(p_k') .$$

Then $\Phi_{Sp}(n)$ is an isomorphism if $\dim X \le 8n + 2$ and is a surjection if $\dim X = 8n + 3$.

5.3: Sketch of Proof. Theorems 5.1 and 5.2 will be proved by appealing to Theorems 4.2 and 4.3. The role of the Thom spaces

$$\frac{BU(n)}{BU(n-1)} \simeq MU(n)$$

and

$$\frac{BSp(n)}{BSp(n-1)} \simeq MSp(n)$$

makes it clear that Theorems 4.2 and 4.3 have, as a corollary, an isomorphism of the desired type. In section 5.7 we verify by means of the facts collected in section 4.5 that the isomorphisms are equal modulo the skeletal filtration to $\Phi_U(n)$ and $\Phi_{Sp}(n)$. This verification is very simple.

5.4: Remark. Theorems 5.1 and 5.2 might, of course, be proved by computing

$$(c_k)_* : H_*(BU) \to H_*(MU) \qquad (k \ge 1)$$

and

$$(p_k)_* : H_*(BSp) \to H_*(MSp) .$$

Here BU and BSp are the suspension spectra of BU and BSp respectively and the homology groups refer to homology of spectra [Ad 1, p. 196]. This is not difficult and I leave it as an exercise to the interested reader to accomplish this. In fact $(c_k)_*$ is computed in section 6.14 in order to relate the Boardman-Hurewicz maps

$$\pi_*(\underline{BU}) \to H_*(BU)$$

and

$$\pi_*(MU) \to H_*(MU).$$

Firstly, for completeness, here are two observations which are required in order to prove Theorems 5.1 and 5.2.

5.5: Lemma. Let

$$\varepsilon_n : \Sigma^2 MU(n) \to MU(n+1)$$
$$\delta_n : \Sigma^4 MSp(n) \to MSp(n+1)$$

be the structure maps of the MU- and MSp-spectra respectively. Then ε_n is a $(4n+3)$-equivalence and δ_n is an $(8n+7)$-equivalence.

5.6: Corollary. (i) The canonical map $\varepsilon : \{X, MU(n)\} \to MU^{2n}(X)$ is an isomorphism if $\dim X \le 4n$ and is a surjection if $\dim X = 4n+1$.

(ii) The canonical map $\delta : \{X, MSp(n)\} \to MSp^{4n}(X)$ is an isomorphism if $\dim X \le 8n+2$ and is a surjection if $\dim X = 8n+3$.

Proof. (i) By Lemma 5.5 and [Sp, pp. 399-405] $(\varepsilon_n)_* : [\Sigma^m X, \Sigma^m MU(n)]$ $[\Sigma^m X, \Sigma^{m-2} MU(n+1)]$ is an isomorphism if $\dim X \le 4n$ and is an epimorphism if $\dim X = 4n+1$. Hence, taking limits, $\varepsilon : \{X, MU(n)\} \to \varinjlim_m [\Sigma^{2m} X, MU(n+m)]$ $= MU^{2n}(X)$ is an isomorphism if $\dim X \le 4n$ and an epimorphism if $\dim X = 4n+1$.

Case (ii) is similar to case (i).

5.7: Proof of Theorems 5.1 and 5.2. The argument is the same in both cases so I will deal with the unitary case.

From Theorem 4.2 and Corollary 5.6 we know that

$$\left\{X, \frac{BU}{BU(n-1)}\right\} \simeq \left\{X, \prod_{n\le k} MU(k)\right\} \xrightarrow{\varepsilon} \prod_{n\le k} MU^{2k}(x) \tag{5.8}$$

is an isomorphism if $\dim X \le 4n$ and an epimorphism if $\dim X \le 4n+1$. If MU_{2k} classifies $MU^{2k}(\_)$ then (5.8) is induced by a stable $(4n+1)$-equivalence

$$\gamma : \frac{BU}{BU(n-1)} \to \prod_{n\le k} MU_{2k} .$$

The homomorphism (5.8) assigns $f^*(\gamma)$ to a stable class, f. Any map which in homology induces γ_* would give a $(4n+1)$-equivalence. I will show that this is true for $\prod_{n\le k} c'_k$. Since $BU \to \frac{BU}{BU(n-1)}$ is stably a split surjection it suffices to show that the natural transformation

$$\{X, BU\} = \{X, \bigvee_{1\le k} MU(k)\} \xrightarrow{\varepsilon} \prod_{1\le k} MU^{2k}(X) \tag{5.9}$$

corresponds to a class $\gamma' < \prod_{1\le k} MU^{2k}(BU)$ which coincides in homology with

$\prod_{1\le k} c_k$. To do this it suffices to show that

$$\gamma' \equiv \prod_{1\le k} c_k \pmod{MU^*(BU).I}$$

where $I = (x \in MU^*(\text{point});\ \deg x < 0)$. I will show by induction on n that this is true for γ' restricted to $BU(n)$. The result then follows by taking limits. From section 4.5(b) we see that if $n = 1$ the stable equivalence of Theorem 4.3 is the identity $BU(1) \to MU(1) \simeq BU(1)$. In general section 4.5(b) tells us that the stable equivalence $BU(n) \to \bigvee_{1\le k\le n} MU(k)$ is given by c_n on the wedge factor $MU(n)$. However

$$\ker(MU^{2k}(BU(n)) \to MU^{2k}(BU(n-1)))$$

is in $MU^*(BU(n)).I$ if $k < n$ and so the induction step is complete.

§6. APPLICATION TO $\pi_*^S(BU(n))$

This section presents a calculation of $\dfrac{\pi_j^S(BU(n))}{\text{(odd torsion)}}$ for $j \le 10$. Only the essential ingredients of the calculation are given here (section 6.14-6.16). The method, which is very simple, is explained in section 6.8. In addition to the results of this section the prerequisites for the reproduction of the calculation are the results of [Mo, section 6; T, pp. 189-190] and a large sheet of paper.

In fact (with an even larger sheet of paper!) one can perform these calculations without the use of the results of [Mo]. This is because of the interrelation of the $\pi_*^S(MU(n))$ which actually force the behaviour of the spectral sequences of (6.9) when $X = MU(n)$ for all n ($n = 1$ is the case of [Mo]) in dimensions ≤ 19. For details of this sort of calculation the reader is referred to [K-Sn] in which $\pi_*^S(MSp(n))$ and $\pi_*^S(BSp(n))$ are calculated for $* \le 26$.

The main result of this section is the following:

<u>6.1</u>: <u>Theorem</u>. The canonical map $BU(n) \to BU$ embeds $\pi_*^S(BU(n))$ as a direct summand of $\pi_*^S(BU)$ for $n \ge 1$.

If

$$\frac{\pi_j^S(BU(n))}{\text{(odd torsion)}} \simeq \frac{\pi_j^S(BU(n-1))}{\text{(odd torsion)}} \oplus A_j(n)$$

then $A_j(n)$ is given by Tables 6.2-6.6 below when $j \le 10$.

6.2: Table for $A_j(1)$.

j	$A_j(1)$	generators
2	Z	x
4	Z	$x*^2$
5	Z/2	$\nu \circ x$
6	Z	$x*^3$
7	Z/2	ν
8	Z ⊕ Z/2	$x*^4$, $\nu^2 \circ x$
9	Z/8	$\sigma \circ x$
10	Z	$x*^5$

6.3: Table for $A_j(2)$.

j	$A_j(2)$	generators
4	Z	x^2
6	Z	$x(x*^2) = a_{11}$
8	Z ⊕ Z	$(x*^2)^2$, a_{12}
9	Z/2	w
10	Z ⊕ Z ⊕ Z/2	$2a_{22}+a_{11}a_{12}$, $x*^3x*^2$, $\eta \circ w$

6.4: Table for $A_j(3)$.

j	$A_j(3)$	generators
6	Z	x^3
8	Z	$(x*^2)x^2$
10	Z ⊕ Z	$x(x*^2)^2$, xa_{12}

6.5: Table for $A_j(4)$.

j	$A_j(4)$	generators
8	Z	x^4
10	Z	$(X*^2)x^3$

6.6: Table for $A_j(5)$.

j	$A_j(5)$	generators
10	Z	x^5

6.7: Explanation of Tables 6.2-6.6. In the range $j \leq 10$ $A_j(n) = 0$ for $n \geq 6$ and the tables display only the non-zero groups in this range. The generators arise in the following manner. All the displayed groups are summands in $\pi_*^S(BU)$. $\pi_*^S(BU)$ is a module over stable homotopy of spheres, by means of composition, and for

$$a \in \pi_*^S(S^0), \qquad b \in \pi_*^S(BU)$$

$a \circ b$ denotes the composition product. Also BU has two structure maps $BU \times BU \to BU$ coming from sum and tensor product in K-theory. For $a_1, a_2 \in \pi_*^S(BU)$

$$a_1 * a_2 \text{ denotes "tensor product"}$$

and a_1a_2 denotes "Whitney sum". The notation for elements in $\pi_*^S(S^O)$ is taken from [T, pp. 189-190]. The element $x \in \pi_2^S(CP^\infty)$ is the inclusion of $CP^1 = S^2$. By Theorem 5.1 $\pi_*^S(BU(n))$ contains elements from MU*(point) which account for the presence of a_{11}, a_{12} and a_{22}. Here $a_{ij} \in \pi_{2(i+j-1)}(MU)$ is the coefficient in the formal group law for complex cobordism in the notation of [Ad 1, p. 40].

My calculations end rather arbitrarily at $j = 10$. The input for the calculations, as will be explained below, is knowledge of $\pi_*^S(CP^\infty)$ and $\pi_*^S(S^\circ)$. Given this information through a range it is relatively simple to compute $\pi_*^S(BU(n))$ for a comparable range. My computations were motivated by the speculation that $\pi_*^S(S^O)$ and $\pi_*^S(CP^\infty)$ might generate $\pi_*^S(BU)$ under composition, Whitney sum and tensor product. However this is not so, even in the range $j \leq 10$. The reader, equipped with the techniques of this section and the computations of [Ad 1, Part II, section 12] can soon verify that $a_{12} \in \pi_*^S(BU(n))$ $(n \geq 3)$ is not generated in this manner.

<u>6.8</u>: <u>Method of Calculation</u>. For a space X consider the spectral sequence of $\pi_*^S(S^O)$-modules

$$E^2_{p,q} = \tilde{H}_p(X;\pi_q^S(S^O)) \Rightarrow \pi_{p+q}^S(X) \tag{6.9}$$

$(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})$. The associated filtration is

$$\underline{(0)} = F_{-1,k} \subset F_{0,k} \subset F_{1,k} \subset \cdots \subset F_{k,k} = \pi_k^S(X)$$

where

$$E^\infty_{m,k-m} = \frac{F_{m,k}}{F_{m-1,k}} .$$

The spectral sequence was studied for $X = CP^\infty$ in [Mo] (note that my $E^r_{2p,q}$ would be $E^r_{p,p+q}$ in the notation of [Mo]). When X = BU the following properties hold:

(i) The products $a_1{*}a_2$ and a_1a_2 (see section 6.7) operate in the spectral sequence in such a way that the differentials are derivations.

(ii) By Theorem 4.2 the spectral sequence is additively the direct sum of the spectral sequences for MU(n); $n \geq 1$. Therefore the differentials respect the splitting.

(iii) $\pi_*(MU) = MU^*(\text{point})$ is known [Ad 1, Part II]. By Theorem 5.1 we have isomorphisms

$$\left\{S^{4n-2\varepsilon}, \frac{BU}{BU(n-1)}\right\} \xrightarrow{\Phi_{\underline{U}}(n)} \prod_{j=\varepsilon-n}^{0} MU^{2j}(\text{point})$$

($\varepsilon = 0$ or 1). A stable homotopy class on the left converges in the spectral sequence (6.9) to its image under the Hurewicz map H. The Boardman-Hurewicz map is an injection

$$B : \pi_*(MU) \to H_*(MU).$$

Hence we can discover the permanent cycles which represent elements of $\pi_*(MU)$ by means of the commutative diagram

$$\begin{array}{ccc} \{S^{4n-2\varepsilon}, BU\} & \xrightarrow{\Phi_U} & \prod_{j=\varepsilon-n}^{0} MU^{2j}(\text{point}) \\ \downarrow H & & \downarrow B \\ H_*(BU) & \xrightarrow[(\Phi_U)_* = (\prod_k c_k)_*]{} & H_*(MU) \end{array} \tag{6.10}$$

Using (i)-(iii) and the determination of the spectral sequence when $X = CP^\infty$ [Mo, section 6] all the remaining differentials, d_r, $r \leq 8$, may easily be obtained in total degree ≤ 11 when $X = BU$. Then Theorem 6.1 follows immediately.

The remainder of this section is devoted to the computation of $(\Phi_U)_*$ in (6.10) (see sections 6.14 and 6.15) and to the identification of cycles representing cobordism classes (see Proposition 6.16). Section 6.17 contains an illustrative example.

<u>6.11</u>: <u>Notation</u>. Recall $H_*(BU) = Z[\beta_1,\beta_2,\ldots]$ (deg $\beta_j = 2j$) [Ad 1, p. 49, Lemma 4.3] and $H_*(MU) = Z[b_1,b_2,\ldots]$ (deg $b_j = 2j$) [Ad 1, Part II, section 6]. If $a_{ij} \in \pi_{2(i+j-1)}(MU)$ is the coefficient in the formal group law [Ad 1, p. 40] then $B(a_{ij}) \in H_*(MU)$ is computed in [Ad 1, Part II, section 6]. The inductive formula is

$$B(a_{ij}) = \binom{i+j}{i} b_{i+j-1} - \sum_s \sum_t B(a_{st}) b_i^s b_j^t . \tag{6.12}$$

In (6.12) the sum is over $1 \leq s \leq i$, $1 \leq t \leq j$ and $s + t \neq i + j$. Also

$$b_z^i = \Sigma\, b_{k_1} \ldots b_{k_i} \quad \text{where} \quad \sum_{j=1}^{i} k_j = z - i$$

(summed over all partitions $(k_1,k_2,\ldots)$) and $b_0 = 1$. The first few examples are

$$\begin{aligned} B(a_{11}) &= 2b_1 \\ B(a_{12}) &= 3b_2 - 2b_1^2 \\ B(a_{13}) &= 4b_3 - 8b_1b_2 + 4b_1^3 \\ B(a_{22}) &= 6b_3 - 6b_1b_2 + 2b_1^3 \\ B(a_{23}) &= 10b_4 - 3b_2^2 - 4b_1^4 + 14b_1^2b_2 - 16b_1b_3 \end{aligned} \tag{6.13}$$

<u>6.14</u>: <u>Lemma</u>. For $c_k \in MU^{2k}(BU)$

$$(c_k)_*(\beta_{i_1+1} \cdots \beta_{i_t+1}) \in H_{2(i+t-k)}(MU)$$

$(i = \sum_{j=1}^{t} i_j)$ is equal to $b_{i_1} \ldots b_{i_t}$ if $k = t$ and zero otherwise.

Proof. In terms of the slant product [Ad 1, p. 229] $(c_k)_*(a) = c_k\backslash a$. By definition $b_i = c_1\backslash\beta_{i+1}$ by [Ad 1, p. 51, Lemma 4.5]. $\beta_{i_1+1}\cdots\beta_{i_t+1}$ originates in $H_*((CP^\infty)^t)$. The image of c_k in $MU^{2k}(CP^\infty)^t)$ is zero if $k > t$. If $k \le t$ this image is the sum of the translates of $c_1^{\otimes k}\otimes 1^{\otimes t-k}$ under the action of the permutation group Σ_t. Now

$$(c_1^{\otimes k}\otimes 1^{\otimes t-k})\backslash\beta_{i_1+1}\cdots\beta_{i_t+1} = (c_1\backslash\beta_{i_1+1})\cdots(1\backslash\beta_{i_t+1})$$

by [Ad 1, p. 229, Proposition 9.1] and $(1\backslash\beta_i) = 0$. Thus the expression is zero unless $t = k$ and by definition $(c_1\backslash\beta_{j+1}) = b_j$ so the result follows.

6.15: Corollary.

$$(\Phi_U)_* : H_*(BU) \to H_*(MU)$$

is given by

$$(\Phi_U)_*(\beta_{i_1+1}\cdots\beta_{i_t+1}) = b_{i_1}\cdots b_{i_t}\,.$$

6.16: Proposition. Let $z \in \prod_{j=-n}^{0} MU^{2j}(\text{point}) \simeq \left\{S^{4n}, \frac{BU}{BU(n-1)}\right\}$ satisfy $H(z) = y \in H_{4n}\left(\frac{BU}{BU(n-1)}\right)$. Let

$$z_1 \in \left\{S^{4n+2}, \frac{BU}{BU(n)}\right\} \simeq \prod_{j=-n}^{0} MU^{2j}(\text{point})$$

and

$$z_2 \in \left\{S^{4n+4}, \frac{BU}{BU(n)}\right\} \simeq \prod_{j=-n-1}^{0} MU^{2j}(\text{point})$$

be the elements corresponding to z. Then

$$H(z_1) = \beta_1 y \quad\text{and}\quad H(z_2) = \beta_1^2 y\,.$$

(Here $H_*\left(\frac{BU}{BU(n)}\right)$ is identified with the subgroup of $H_*(BU)$ generated by monomials of weight greater than n.)

Proof. We have $(\Phi_U)_*(Hz_1) = Bz$ with $H(z_1) \in H_{4n+2}\left(\frac{BU}{BU(n)}\right)$. However $(\Phi_U)_*(y) = Bz$ for $y \in H_{4n}\left(\frac{BU}{BU(n-1)}\right)$. Hence $\beta_1 y = H(z_1)$ by Corollary 6.15. Similarly $\beta_1^2 y = H(z_2)$.

6.17: Illustrative Examples. $x \in \pi_2^S(BU(1))$ is represented by $\beta_1 \in E^2_{2,0}$ in the spectral sequence. By [Mo, section 3] x^{*2} is represented by $2\beta_2$ so $2\beta_1\beta_2$ represents $(x_1^{*2})x \in \pi_6^S(BU(2))$. However

$$\pi_6^S\left(\frac{BU}{BU(1)}\right) \simeq MU^{-2}(\text{point}) \times MU^0(\text{point}) = Z \oplus Z.$$

The first summand is generated by a_{11} and $Ba_{11} = 2b_1$, by (6.13). Hence $H(a_{11}) = 2\beta_2\beta_1 \in E^2_{6,0}$ and $a_{11} = (x^{*2})x$.

Now consider

$$a_{12} \in MU^{-4}(\text{point}) \subset \left\{S^8, \frac{BU}{BU(1)}\right\} .$$

Since $Ba_{12} = 3b_2 - 2b_1^2$, by (6.13),

$$H(a_{12}) = 3\beta_3\beta_1 - 2\beta_2^2 \in H_8\left(\frac{BU}{BU(1)}\right)$$

and a_{12} converges to $3\beta_3\beta_1 - 2\beta_2^2$. Also a_{12} is in $\pi_8^S(BU(2))$. By Proposition 6.16 $xa_{12} \in \pi_{10}^S\,(BU(3))$ is represented by $3\beta_3\beta_1^2 - 2\beta_2^2\beta_1$.

§7: NON-ADDITIVITY OF $\tau : BO \to QBO(2)$

The Becker-Gottlieb solution of the real Adams conjecture, as presented in [Be], is equal to a composite of the form

$$BO \xrightarrow{\tau_0} QBO(2) \xrightarrow{Q(\alpha)} Q(G/O) \xrightarrow{d} G/O \tag{7.1}$$

Here τ_0 is the transfer of Theorem 3.2, $\alpha : BO(2) \to G/O$ is Adams' solution to the Adams conjecture for O(2)-bundles [Ad 5] and d is the structure map of the infinite loopspace G/O. Although $Q(\alpha)$ and d are H-maps the composite (7.1) is not. This is seen by means of the following argument, which is due to Ib Madsen.

<u>7.2</u>: <u>Lemma</u>. The composite of (7.1) is not an H-map at the prime two.

<u>Proof</u>. A solution of the Adams conjecture induces, in the manner described in [M-S-T; Sn 2], a diagram of 2-local spaces and maps

$$\begin{tikzcd} SO \arrow[rr, "J"] \arrow[dr, "w"'] & & SG \\ & (JO)_o \arrow[ur, "\mu"'] & \end{tikzcd} \tag{7.3}$$

factorising the J-homomorphism, $J : SO \to SG$. Here $(JO)_o$ is the base-point component of JO = fibre $(\psi^3 - 1 : BO \to BO)$. If (7.1) were an H-map then (7.3) would be a diagram of H-maps. In the notation of [Mad 1; Mad 2] we have Bockstein spectral sequences with E^2-terms as follows:

$$E^2(SO) \cong \wedge(u_1u_2, u_3u_4, \ldots) \quad ,$$
$$E^2(JO) \cong E^2(SO) \otimes P(b_2^2, b_4^2, \ldots)$$

where $H_*((JO)_o; Z/2) \cong \wedge(u_1, u_2, \ldots) \otimes P(b_1, b_2, \ldots)$. Also $d_2(b_2^2) = u_1u_2$. If SG is the identity component of G then (Q^i = i-th Dyer-Lashof operation)

$$E^2(SG) \cong \wedge(u_1u_2) \otimes P(Q^2Q^2[1]*[-3]) \otimes \ldots \text{ with } d_2(Q^2Q^2[1]*[-3]) = u_1u_2 .$$

Hence μ induces μ_* in mod 2 homology which satisfies

$$\mu_*(b_2^2) = Q^2Q^2[1]*[-3] \text{ which is indecomposable.}$$

In this section I will give a description of the deviation from additivity of τ_0. This gives a description of the deviation from additivity of (7.1). Define the deviation from additivity of τ_0 as γ: BO × BO → QBO(2) in the homotopy commutative diagram

$$\begin{tikzcd}
BO \times BO \arrow[rr, "\Delta"] \arrow[d, "m_1"'] & & BO \times BO \times BO \times BO \arrow[d, "\tau \times \tau \times \gamma"] \\
BO \arrow[dr, "\tau"'] & & QBO(2) \times QBO(2) \times QBO(2) \arrow[dl, "m_2"] \\
& QBO(2) &
\end{tikzcd}$$

Here m_1 and m_2 come from the H-space multiplication and $\Delta(x,y) = (x,y,x,y)$. The map γ may be explicitly described in terms of the geometry of the orthogonal groups. The result takes the following form. Let

$$X_n = O(4n)/H(2n) \text{ where } H(m) = \Sigma_m \int O(2) \subset O(2m).$$

Also write P(2n) for O(n) × O(n). If E is a free, contractible O(4n)-space set

$$Y_n = E \times_{P(4n)} X_n \ .$$

<u>7.5</u>: <u>Theorem</u>. Let γ: BO × BO → QBO(2) be the deviation from additivity of the transfer τ_0 : BO → QBO(2). Then, associated with the canonical fibring $X_n \to Y_n \to BP(4n)$, there is a transfer-like map, described in §7.9, ϕ_n : BP(4n) → QY_n such that γ restricted to BP(4n) = BO(2n) × BO(2n) is homotopic to a composite $BP(4n) \xrightarrow{\phi_n} QY_n \xrightarrow{h} QBO(2)$. Here P(4n), X_n and Y_n are the spaces introduced above and h is the map described in §7.10.

In Theorem 7.5 it would be preferable to have a description of γ in terms of genuine transfer maps. In principle this is possible from the description of ϕ_n given in §7.9. This would be accomplished using the technique explained in §2. However this would involve a lengthy analysis of the double coset spaces of the symmetric groups (cf. §7.12) which would be out of place here.

Here is the answer when n = 1.

<u>7.6</u>: <u>Theorem</u>. Let γ be as in Theorem 7.5. Then the restriction of γ to BP(4) = BO(2) × BO(2) is given by a composition of the following form, whose ingredients are described in §7.12,

$$BO(2)^2 \xrightarrow{\bar{\tau}^2} (QB\Sigma_2 \int \Sigma_2)^2 \xrightarrow{g} QBO(2).$$

Here $\Sigma_2 \int \Sigma_2$ is a subgroup of O(2) and $\bar{\tau}$ is the transfer associated with $B\Sigma_2 \int \Sigma_2 \to BO(2)$.

Theorems 7.5 and 7.6 will be proved in §7.10 and §7.12. Firstly we will need the following result.

7.7: Proposition. There is a non-degenerate, left $P(4n)$-equivariant vector field ρ, on X_n which is zero precisely on

$$P(4n)\Sigma_{4n}H(2n)/H(2n) = A_n, \text{ say.}$$

Here $\Sigma_{4n} \subset O(4n)$ permutes the standard basis of $\mathbb{R}^{4n}$.

Proof: Let $v \in \mathcal{O}(4n)$, the Lie algebra of $O(4n)$, satisfy $\exp(v) = w$ where

$$w = \begin{pmatrix} \lambda I_{2n} & 0 \\ 0 & \mu I_{2n} \end{pmatrix}$$

with $0 < \lambda,\mu$ distinct. When applied to v the construction described in §2.4 gives ρ. The details are entirely similar to those of Example 2.7.

We also have the following observation.

7.8: Lemma. Let A_n be as in Proposition 7.7. The subset $P(4n)H(2n)/H(2n)$ of A_n is homeomorphic to $(O(2n)/H(n))^2$. Using this homeomorphism A_n may be written as a disjoint union $A_n = C_n \cup (O(2n)/H(n))^2$.

7.9: Definition of ϕ_n. Let ρ be the vector field of Proposition 7.7 scaled down if necessary so that $|\rho(x)| \leq 1$ for all $x \in X_n$. Choose a finite dimensional $O(4n)$-module, V, together with an equivariant embedding of X_n in V. Let ν be an equivariant normal bundle of this embedding. Choose as follows N to be a $P(4n)$-invariant neighbourhood of A_n in X_n. Let N be a disjoint union of equivariant tubular neighbourhoods of the components of A_n in X_n. Suppose ρ and N satisfy $|\rho(X)| = 1$ if $x \notin N$ (this is just another scaling up).

Let Th(_) denote a Thom space and let $p : Th(V) \to Th(\nu)$ be the Pontrjagin-Thom map. If τ_1 is the tangent bundle of X_n define

$$q : Th(V) \to TH(\nu \oplus \tau_1) \cong Th(X \times V)$$

by

$$q(u) = \begin{cases} \infty & \text{if} \quad u \notin \nu , \\ \infty & \text{if } u \in \nu_x,\ x \notin N_2 , \\ \dfrac{1}{1-|\rho(x)|}(u,\rho(x)) & \text{if } u \in \nu_x,\ x \in N_2 . \end{cases}$$

Here $N = N_1 \cup N_2$ and N_1 is the tubular neighbourhood of $(O(2n)/H(n))^2$ in X_n.

Now let E_t be the t-fold join of $O(4n)$ with itself and set $B_t = E_t/P(4n)$. Set $E = \varinjlim_t E_t$ then $BP(4n) = \varinjlim_t B_t$. Applying the Becker-Gottlieb umkehr construction [B-G 1] to the map $1 \times_{P(4n)}(q \circ p)$ yields a stable map from B_t to $E_t \times_{P(4n)} X_n$. Set $Y_n(t) = E_t \times_{P(4n)} X_n$ and $Y_n = E \times_{P(4n)} X_n$. This stable map is equivalent to a map $B_t \to QY_n(t)$ and up to homotopy this map is independent of the choices made. Letting $t \to \infty$ we obtain an element of $\varprojlim_t [B_t, QY_n]$. Let $\phi_n : BP(4n) \to QY_n$ be any map restricting to this element. In fact there is essentially one choice for our purposes. From the Milnor exact sequence there may be several choices for ϕ_n and any two choices differ by an element in $\varprojlim^1 [\Sigma BP(4n), QY_n]$. However we are going to consider ϕ_n as part of a composite map $BP(4n) \to QBO(2)$. By §3.2 and the argument of §3.6 (proof) a map $BP(4n) \to QBO(2)$ is determined by its restrictions to the finite complexes, B_t.

7.10: Proof of Theorem 7.5. This will be just a sketch. The computation is very similar to those of §2.2.

By naturality of the transfer we have a homotopy commutative diagram of transfers [B-G 1; B-G 2]

$$\begin{tikzcd} BP(4n) \arrow[r, "\tau(\pi')"] \arrow[d, "k"'] & QY_n \arrow[d, "Q(j)"] \\ BO(4n) \arrow[r, "\tau(\pi_{2n})"'] & QBH(2n) \end{tikzcd} \tag{7.11}$$

Here k is the natural map and j is induced by passing from P(4n) to O(4n) orbits. The restriction of $\tau_0 : BO \to QBO(2)$ to BP(4n) is a composite of the form

$$BP(4n) \xrightarrow{\tau(\pi_{2n}) \circ k} QBH(2n) \xrightarrow{Q(i_{2n})} QQBO(2) \xrightarrow{d} QBO(2)$$

in which $\tau(\pi_{2n}) \circ k$ is as in (7.11). Details, in particular the definition of $i_n : BH_n \to QBO(2)$, are given in §3.4. Using the vector field of Proposition 7.7, the technique of §2.9 permits us to express $\tau(\pi')$ of (7.11) as a composite

$$BP(4n) \xrightarrow{\tau(\pi'')} Q(E \times_{P(4n)} N) \to QY_n$$

in which $\tau(\pi'')$ is the transfer associated with

$$N \to E \times_{P(4n)} N \to BP(4n)$$

and the second map is induced by the inclusion of N (defined in §7.9) into X_n. However [B-G 2] the transfer $\tau(\pi'')$ is the sum (in the H-space structure on QY_n) of the transfers associated with the components of N. That is, each P(4n)-invariant component defines a fibring over BP(4n) and hence a transfer map. Recall from §7.9 that $N = N_1 \cup N_2$ where N_1 is a tubular neighbourhood of $(O(2n)/H(n))^2$ while N_2 is a neighbourhood of C_n. The transfer associated with N_1, $BP(4n) \to Q(E \times_{P(4n)} N_1)$, when composed with $Q(E \times_{P(4n)} N_1) \to QY_n \to QBH(2n)$, is homotopic (cf. §3.6) to

$$BP(4n) = BO(2n)^2 \xrightarrow{\tau(\pi_n)^2} (QBH(n))^2 \to QBH(2n).$$

Chasing diagrams like that of §3.6 easily implies that N_1's contribution to the restriction $(\tau|BP(4n))$ is homotopic to the sum (in the H-space structure on $QBO(2)$) $(\tau|BP(4n))\circ\pi_1 + (\tau|BP(4n))\circ\pi_2$ where $\pi_i : BP(4n) = BO(2n)^2 \to BO(2n)$ is the i-th projection. Hence N_1 contributes the "additive" part of $(\tau|BP(4n))$ and N_2 contributes the deviation from additivity. However the transfer associated with

$$N_2 \to E \times_{P(4n)} N_2 \xrightarrow{\tau'''} BP(4n)$$

when composed with $Q(E\times_{P(4n)}N_2) \to QY_n$ is precisely ϕ_n. The map $h : QY_n \to QBO(2)$ is defined as the composite

$$QY_n \to QBH(2n) \xrightarrow{Q(i_{2n})} QQBO(2) \xrightarrow{d} QBO(2).$$

7.12: Proof of Theorem 7.6. In the notation of Theorem 7.5 we have to evaluate $h\circ\phi_1$. Let $\Sigma_2\int\Sigma_2$ be the subgroup of $O(2)$ generated by

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO(2) \quad\text{and}\quad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(2).$$

Let $G = \text{kernel}(\det : (\Sigma_2\int\Sigma_2)^2 \to Z/2$ where "det" is the usual determinant homomorphism. It is straightforward to show that C_1 is homeomorphic as a $P(4)$-space to $P(4)/G$. The homeomorphism can be chosen to carry the inclusion $C_1 \subset X_1$ into a map $\lambda : P(4)/G \to O(4)/H(2)$ defined by $\lambda((a,b)G) = ab\sigma H(2)$ where

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, by the argument of §2.9, the composite

$$BP(4) \xrightarrow{\phi_1} QY_1 \to QBH(1) \xrightarrow{Q(i_1)} QQBO(2) \xrightarrow{d} QBO(2)$$

is homotopic to the composite

$$BP(4) \xrightarrow{\tau(\pi)} QBG \xrightarrow{Q(\lambda')} QBH(1) \xrightarrow{Q(i_1)} QQBO(2) \xrightarrow{d} QBO(2)$$

where $\tau(\pi)$ is the transfer map of the canonical fibring

$$C_1 \to E\times_{P(4)}C_1 = BG \xrightarrow{\pi} BP(4)$$

and λ' is given by

$$BG = E\times_{P(4)}P(4)/G \xrightarrow{1\times\lambda} E\times_{O(4)}O(4)/H(1) = BH(1).$$

However the transfer is transitive on fibrings. For example, π above is the composite

$$BG \xrightarrow{\pi_1} (B\Sigma_2\int\Sigma_2)^2 \xrightarrow{(\pi_2)^2} BO(2)^2 = BP(4)$$

in which each map is induced by a group inclusion, so $\tau(\pi)$ is the "composite" of $\tau(\pi_2^2)$ and $\tau(\pi_1)$. Also $\tau(\pi_2^2)$ is the "product", $\tau(\pi_2)^2$, so that $\tau(\pi)$ equals the following composite.

$$\begin{tikzcd} BO(2) \arrow[r, "\tau(\pi_2)^2"] & (QB\Sigma_2\textstyle\int\Sigma_2)^2 \arrow[r, "m"] & Q(B(\Sigma_2\Sigma\textstyle\int_2)^2) \arrow[dll, "Q(\tau(\pi_1))"'] \\ QQBG \arrow[rr, "d"] & & QBG \end{tikzcd}$$

Here m is the H-space sum of the two canonical maps $QB\Sigma_2\int\Sigma_2 \to Q(B(\Sigma_2\int\Sigma_2)^2)$ and d is the structure map of the infinite loopspace QBG (cf. §3.4). Setting $\bar{\tau} = \tau(\pi_2)$ and $g = d \circ Q(i_1) \circ Q(\lambda') \circ d \circ Q(\tau(\pi_1)) \circ m$ completes the proof.

§8. STABLE DECOMPOSITIONS OF $BGL\mathbb{F}_q$ AND $BO\mathbb{F}_3$

8.1. Throughout this section let $\mathbb{F}_q$ denote the field with q elements. $GL_n\mathbb{F}_q$ is the linear group of invertible $n \times n$ matrices with entries in $\mathbb{F}_q$. In addition let O_m denote the orthogonal group of the quadratic form $\sum_{i=1}^{m} X_i^2$ over $\mathbb{F}_3$. Firstly we are going to decompose the S-type of $(BO\mathbb{F}_3)_{(2)}$. For this purpose everything will be 2-local and all cohomology and homology will be taken with mod 2 coefficients. Let $\Sigma_n\int O_2$ denote the wreath product of the permutation group, Σ_n, with O_2. Hence $\Sigma_n\int O_2$ sits inside O_{2n} in a canonical way as the subgroup obtained by "exploding" matrices consisting of 2×2 diagonal blocks. We note that $O_2 \cong \Sigma_2\int\Sigma_2$ has generators

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying $\beta^2 = I = \gamma^4$, $\beta\gamma\beta = \gamma^3$.

Let H_* denote mod 2 singular homology. The object of this section is to establish the following technical result, used in the proof of Theorem 8.2.1.

8.1.1: Theorem. Let

$$\pi : \Sigma_n\textstyle\int O_2 \to O_{2n} \quad \text{and} \quad i : O_2^n \to \Sigma_n\textstyle\int O_2$$

be the canonical inclusions. Let $\tau : BO_{2n} \to B\Sigma_n\int O_2$ be the S-map which is the transfer [B-Gl; K-P] associated with $B\pi$. Then the following diagram commutes

$$\begin{tikzcd} H_*(BO_2^n) \arrow[r, "B(\pi\circ i)_*"] \arrow[dr, "B(i)_*"'] & H_*(BO_{2n}) \arrow[d, "\tau_*"] \\ & H_*(B\Sigma_n\textstyle\int O_2) \end{tikzcd}$$

Theorem 8.1.1 will be proved by appealing to the well-known Double Coset formula.

8.1.2: Proposition (Double Coset formula) [E-C; Fe]. Let $j:H \to G$ and $k:K \to G$ be inclusions of finite groups. Let $\tau_* : H_*(BG) \to H_*(BH)$ denote the homology transfer.

Then

$$\tau_* \circ B(k)_* = \Sigma\sigma_g \tag{8.1.3}$$

where σ_g is the composition of the transfer

$$\tau_* : H_*(BK) \to H_*(B(K \cap gHg^{-1}))$$

with the map induced by

$$K \cap gHg^{-1} \subset gHg^{-1} \xrightarrow{g^{-1}(\_)g} H .$$

The sum is taken of double coset representatives of

$$K \backslash G / H .$$

Theorem 8.1.1 is (8.1.3) with the right hand expression replaced by σ_1. In a series of lemmas it will be shown below that all the other terms in the sum vanish in the context of Theorem 1.1.

8.1.4: Lemma. Suppose H is a proper subgroup of O_2^n of index greater than two. Then there exists H' such that $H \subsetneq H' \subsetneq O_2^n$.

Proof: Consider the projections

$$\phi : H \subset O_2^{n-1} \times O_2 \to O_2^{n-1} \text{ and } \Psi : H \subset O_2^{n-1} \times O_2 \to O_2 .$$

By induction on n we may assume both ϕ and Ψ are onto for each factorisation $O_2^n \cong O_2^{n-1} \times O_2$. Hence $|\ker \phi| = 1$ or 2.

Case (a): $|\ker \phi| = 1$. In this case $\Psi \circ \phi^{-1}$ induces a well-defined homomorphism $\lambda : O_2^{n-1} \to O_2$ identifying H as the subgroup $\left\{(x,\lambda(x)) \mid x \in O_2^{n-1}\right\}$. Taking $1 \neq z \in Z(O_2)$, the center of O_2 we may set $H' = \langle H, (1,z)\rangle$.

Case (b): $|\ker \phi| = 2$. Here $\Psi \circ \phi^{-1}$ induces $\lambda : O_2^{n-1} \to O_{2/\ker \phi}$. Here $\operatorname{Ker} \phi = H \cap ((1) \times O_2)$ is considered as a subgroup of O_2. Since Ψ is onto $\operatorname{Ker} \phi \triangleleft O_2$ which implies $\ker \phi = \langle \gamma^2 \rangle = Z(O_2)$ the center of O_2. Thus λ has range $O_{2/Z(O_2)} \cong Z/2 \times Z/2$. Hence $H = ((x,y) \in O_2^{n-1} \times O_2 \mid \lambda(x) = \pi(y))$ where π is the natural projection. Let det denote determinant or its induced map $O_{2/Z(O_2)} \to Z/2$. Set $H' = \ker((a,b) \to \det \lambda(a) \det b) \subset O_2^{n-1} \times O_2$. $H \subset H'$ since $(x,y) \in H$ is sent to $\det \lambda(x). \det \pi(y) = (\det \lambda(x))^2 = 1$.

8.1.5: Lemma. If G is a subgroup of index two in O_n^2 then the associated transfer

$$\tau_* : \tilde{H}_*(BO_2^n) \to \tilde{H}_*(BG)$$

is zero.

Proof: Recall that H_* means mod 2 homology. Observe firstly that the homomorphisms from O_2 to Z/2 are precisely the trivial map, the determinant, which has kernel $\langle\gamma\rangle \cong Z/4$, and the map with kernel $\langle\beta,\beta\gamma^2\rangle \cong Z/2 \times Z/2$. Call these maps h_1, h_2, h_3, respectively. Then G is the kernal of a map of the form

$$O_2^a \times O_2^b \times O_2^{n-b-a} \xrightarrow{h_1^a \times h_2^b \times h_3^{n-a-b}} (Z/2)^n \to Z/2$$

in which the last map is multiplication. By the product formula for transfers [B-G2] we may assume a = o. From [F-P] we know that $H_*(BO_2^n)$ has a basis consisting of classes in the image of induced homomorphisms associated with inclusions of the form $(Z/2)^{2n} \to O_2^n$. Applying the Double Coset formula (Proposition 8.1.2) to this situation shows that it suffices to show that the transfer

$$\tau_* : \tilde{H}_*(B(Z/2)^{2n}) \to \tilde{H}_*(B((Z/2)^{2n} \cap gGg^{-1})) \qquad (8.1.6)$$

is zero. If b ≠ o $(Z/2)^{2n} \cap gGg^{-1}$ is a proper subgroup of $(Z/2)^{2n}$ since no $Z/2 \times Z/2 \subset O_2$ is in the kernel of the determinant. Hence (8.1.6) is zero since composition with the injection

$$H_*(B((Z/2)^{2n} \cap gGg^{-1})) \to H_*(B(Z/2)^{2n})$$

is multiplication by an even integer. This argument disposes of (8.1.6) in all cases except b = o and $(Z/2)^{2n} = (\ker h_3)^n$. In this case (8.1.6) is the identity map. However, in the double coset formula for the evaluation of

$$H_*(B(\ker h_3)^n) \to H_*(BO_2^n) \xrightarrow{\tau_2} H_*(BG) \qquad (8.1.7)$$

(8.1.6) appears with multiplicity equal to the number of coset representatives g in

$$(\ker h_3)^n \backslash O_2^n / G \qquad \text{such that} \qquad (\ker h_3)^n \subset gGg^{-1}.$$

However, G is normal so either no such g exist or exactly two and in either case the contribution to the double coset formula is zero.

8.1.8: Proof of Theorem 8.1.1. Consider the double coset formula (8.1.3) for evaluating $\tau_* \circ B(\pi \circ i)_*$. Each term σ_g for which

$$O_2^n \cap g\Sigma_n \int O_2 g^{-1} \neq O_2^n$$

contributes zero. This follows from Lemmas 8.1.4 and 8.1.5 together with the transitivity of the transfer [B-G2]. It remains to show that there is only one double coset for which $O_2^n \subset g\Sigma_n \int O_2 g^{-1}$. We do this by induction on n. Suppose that

$$g^{-1}O_2^n g \subset \Sigma_n \int O_2 \quad \text{and set}$$

$$w = \begin{pmatrix} \gamma & & & & & \\ & \gamma & & & & \\ & & \gamma & & & \\ & & & \ddots & & \\ & & & & \gamma & \\ & & & & & I \end{pmatrix} \quad \text{where } \gamma \text{ is the}$$

2×2 matrix given at the beginning of the section and I is the 2×2 identity. The eigenvalues of γ are $\pm\sqrt{-1}$ in $\overline{\mathbb{F}}_3$, the algebraic closure of $\mathbb{F}_3$. The only other element in O_2 with these eigenvalues is $\gamma^3 = \beta\gamma\beta$. Hence there exists $\sigma \in \Sigma_n \int O_2$ such that $\sigma^{-1}g^{-1}wg\sigma = w$. Direct calculation shows that $g\sigma \in O_{2n-2} \times O_2$. Hence g may be taken to lie in $O_{2n-2} \times \{1\}$. The induction now proceeds by replacing w in the above analysis by the matrices

$$\begin{pmatrix} \gamma & & & & & & & \\ & \gamma & & & & & & \\ & & \ddots & & & & & \\ & & & \gamma & & & & \\ & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \end{pmatrix}$$

having k γ's and n-k I's.

8.2. The main result of this section concerns the S-type (stable homotopy type) of $BO_{2n}\mathbb{F}_3$ where $O_m\mathbb{F}_3$ is the finite orthogonal group introduced in §8.1.

8.2.1: Theorem: For each $1 \le n \le \infty$ there exists a 2-local S-equivalence

$$\phi_n : BO_{2n}\mathbb{F}_3 \to \bigvee_{1\le k\le n} \frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$$

Proof: Firstly suppose n is finite. Let $\tau : BO_{2n}\mathbb{F}_3 \to B\Sigma_n \int O_2\mathbb{F}_3$ be the S-map given by the transfer of Theorem 8.1.1. As explained in §§3.4, 4 there are maps

$$d_n : B\Sigma_n \int O_2\mathbb{F}_3 \to QBO_2\mathbb{F}_3$$

$(QW = \varinjlim_m \Omega^m\Sigma^m W)$ together with homeomorphisms

$$\frac{\mathrm{im}(d_n)}{\mathrm{im}(d_{n-1})} \cong \frac{B\Sigma_n \int O_2\mathbb{F}_3}{B\Sigma_{n-1}\int O_2\mathbb{F}_3} .$$

Also by [Sn 1] there are compatible S-equivalences

$$\mu_n : \mathrm{im}(d_n) \to \bigvee_{1\le k\le n} \frac{\mathrm{im}(d_k)}{\mathrm{im}(d_{k-1})}$$

such that the summand of μ_n with range $\frac{\mathrm{im}(d_n)}{\mathrm{im}(d_{n-1})}$ is the canonical collapsing map. Set ϕ_n equal to the composite

$$BO_{2n}\mathbb{F}_3 \xrightarrow{\tau} B\Sigma_n\int O_2\mathbb{F}_3 \xrightarrow{\mu_n \circ d_n} \bigvee_{1\le k\le n} \frac{B\Sigma_k\int O_2\mathbb{F}_3}{B\Sigma_{k-1}\int O_2\mathbb{F}_3}$$

$$\longrightarrow \bigvee_{1\le k\le n} \frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3} \qquad (8.2.2)$$

in which the last map is induced by the canonical inclusion of $\Sigma_k\int O_2\mathbb{F}_3$ in $O_{2k}\mathbb{F}_3$. To show that this composite is a 2-local equivalence it suffices, by a theorem of J. H. C. Whitehead [Sp, p. 399] and the universal coefficient theorems [Sp, p. 283], to show that ϕ_n induces an isomorphic in mod 2 singular homology. From [F-P] we know that the canonical map

$$BO_2\mathbb{F}_3^n \to BO_{2n}\mathbb{F}_3$$

is onto in mod 2 homology. The S-map above

$$B\Sigma_\ell\int O_2\mathbb{F}_3 \to B\Sigma_n\int O_2\mathbb{F}_3 \to \bigvee_{1\le k\le n} \frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$$

lands in the factor $\bigvee_{1\le k\le \ell} \frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$ for $\ell \le n$ ([Sn 1] or §4). The summand with range $\frac{BO_{2\ell}\mathbb{F}_3}{BO_{2\ell-2}\mathbb{F}_3}$ is the canonical map. Hence, by Theorem 8.1.1 we see by induction on ℓ that ϕ_n is a mod 2 homology isomorphism. Finally we must do the case $n = \infty$. If the transfers, τ, were compatible as n varies we could proceed as in §4. However I do not know this to be the case. Instead consider a cofinal family $\{X\gamma\}$ of finite subcomplexes of $BO_{2n}\mathbb{F}_3$. Let $P_{\gamma,n}$ be the subset of S-maps $f : X_\gamma \to B\Sigma_n\int O_2\mathbb{F}_3$ such that the induced map in mod 2 homology

$$H_*(X_\gamma) \to H_*(B\Sigma_n\int O_2\mathbb{F}_3) \to H_*(BO_{2n}\mathbb{F}_3)$$

is equal to that induced by the inclusion of X_γ. By Theorem 8.1.1 $P_{\gamma,n}$ is non-empty. $P_{\gamma,n}$ is also finite since $B\Sigma_n\int O_2\mathbb{F}_3$ has finite homotopy groups. Set $Q_{\gamma,n}$ equal to the image of $P_{\gamma,n}$ in $\{X_\gamma, B\Sigma_\infty\int O_2\mathbb{F}_3\}$ (where {_,_} denotes S-homotopy classes). The inverse limit of compact sets is non-empty so choosing an element of $\varprojlim_{\gamma,n} Q_{\gamma,n}$ yields

$$\tau \in \varprojlim_{\gamma} \{X_\gamma, B\Sigma_\infty \int O_2\mathbb{F}_3\} \cong \{BO_2\mathbb{F}_3, B\Sigma_\infty \int O_2\mathbb{F}_3\} .$$

Define ϕ_∞ by the composition (8.2.2) ($n=\infty$) using this τ instead of the transfer. The homology argument used above for ϕ_n now shows that ϕ_∞ is a 2-local s-equivalence.

We close this section with a stable decomposition theorem for $BGL\mathbb{F}_q$. Let q be a prime power, let ℓ be a prime not dividing q. Denote by r the order of q in the units $(Z/\ell)^*$.

<u>8.2.3</u>: <u>Theorem</u>. Let ℓ, q and r be as above. Then, localised at ℓ, there exists a stable equivalence

$$BGL\mathbb{F}_q \simeq \bigvee_{1\le k} \frac{BGL_{kM}\mathbb{F}_q}{BGL_{(k-1)M}\mathbb{F}_q} .$$

Here M is defined by $M = \begin{cases} r \text{ if } \ell \neq 2 \text{ or } \ell = 2 \text{ and } q \equiv 1(4) \\ 2 \text{ otherwise} \end{cases}$.

<u>Proof</u>: The proof, being similar to that of Theorem 8.2.1, will only be sketched. From [Q, p. 574] we obtain the following facts about Sylow ℓ-subgroups of $GL_n\mathbb{F}_q$. If $\ell \neq 2$ or $\ell = 2$ and $q \equiv 1(4)$ the wreath product $\Sigma_m \int GL_r\mathbb{F}_q$ contains a Sylow ℓ-subgroup of $GL_{mr}\mathbb{F}_q$. If $\ell = 2$ and $q \equiv 3(4)$ $\Sigma_m \int GL_2\mathbb{F}_q$ contains a Sylow 2-subgroup of $GL_{2m}\mathbb{F}_q$.

Consider the canonical maps

$$BGL_M\mathbb{F}_q \to BGL\mathbb{F}_q^+$$

where $BGL\mathbb{F}_q^+$ is the space described in [H-S; Q]. These maps extend to infinite loopmaps (cf. section 3)

$$\lambda(\mathbb{F}_q) : QBGL_M\mathbb{F}_q \to BGL\mathbb{F}_q^+ .$$

The facts about Sylow subgroups and the argument of [H-S, Theorem 3.1] imply that, at the prime ℓ, there exist maps $\tau(\mathbb{F}_q)$ splitting $\lambda(\mathbb{F}_q)$. That is

$$\lambda(\mathbb{F}_q) \circ \tau(\mathbb{F}_q) \simeq 1_{BGL\mathbb{F}_q^+} .$$

Since $BGL\mathbb{F}_q \to BGL\mathbb{F}_q^+$ induces a homology isomorphism we see that $BGL\mathbb{F}_q$ and $BGL\mathbb{F}_q^+$ are stably equivalent. Hence we may form S-maps

$$\begin{array}{ccc} BGL\mathbb{F}_q & \xrightarrow{\tau(\mathbb{F}_q)} & QBGL_M\mathbb{F}_q \\ & & \downarrow \\ & & \bigvee_{1\le k} \dfrac{B\Sigma_k \int GL_M\mathbb{F}_q}{B\Sigma_{k-1}\int GL_M\mathbb{F}_q} \to \bigvee_{1\le k} \dfrac{BGL_{kM}\mathbb{F}_q}{BGL_{(k-1)M}\mathbb{F}_q} \end{array}$$

Here the second map is the stable equivalence of [Sn 1] when we identify

$$\frac{F_k C_\infty BGL_M \mathbb{F}_q}{F_{k-1} C_\infty BGL_M \mathbb{F}_q} \quad \text{with} \quad \frac{B\Sigma_k \int GL_M \mathbb{F}_q}{B\Sigma_{k-1} \int GL_M \mathbb{F}_q}$$

as in section 4.1. The third map is induced by inclusion of subgroups. The proof may now be completed by arguments similar to those used in sections 4.6 and 8.2.1.

PART II: A NEW REPRESENTATION OF UNITARY AND SYMPLECTIC COBORDISM

§0. INTRODUCTION

In part I (Theorems 5.1 and 5.2) we saw that large slices of $MU^{2*}(X)$ and $MSp^{4*}(X)$ may be constructed solely in terms of the classifying spaces BU and BSp. That partial description was sufficient for our purpose at the time. In Part II I will give a sharper presentation of this connection between K-theory (via BU or BSp) and cobordism. I will show how, from KU- or KSp-theory to construct a spectrum, AU or ASp, whose associated cohomology theory is (total) unitary or symplectic cobordism.

Let me now describe in more detail the results of Part II. The reference numbers refer to those used in the body of the text in Part II.

The Whitney sum of the Bott class with the identity of BU induces a map $\Sigma^4 BU \to \Sigma^2 BU$. Using this we may define a ring (see section 2)

$$AU^\circ(X) = \varinjlim_N \{\Sigma^{2N}X, BU\} .$$

The homomorphism Φ_U of [Part I, Theorem 5.1] induces a natural ring homomorphism

$$\Phi_U : AU^\circ(X) \to MU^{2*}(X) .$$

<u>Theorem 2.1</u>. If $\dim X < \infty$ then $\Phi_U : AU^\circ(X) \xrightarrow{\sim} MU^{2*}(X)$.

There is an analogous result for $MSp^{4*}(X)$ [Part II, Theorem 2.2]. The proof of Theorem 2.2 is sketched in section 2.3. Essentially [Part I, Theorem 5.1] tells us Φ_U is onto. To prove Φ_U is injective requires a detailed analysis of the S-map $\Sigma^4 BU \to \Sigma^2 BU$ on each stable summand MU(k) in BU [Part I, Theorem 4.2]. This analysis is accomplished in section 1.

Using the map $\varepsilon : \Sigma^4 BU \to \Sigma^2 BU$ and its symplectic analogue spectra AU and ASp may be constructed. For example $AU_{2k} = \Sigma^2 BU$ and the structure map is given by ε.

<u>Theorem 3.1</u>. AU and ASp are commutative ring spectra.

These results may be generalised by replacing BU by BUΛ, representing space for $KU^\circ(\_;\Lambda)$. The resulting spectrum is AUΛ. Some KUΛ-operations induce AUΛ operations in the obvious manner.

<u>Theorem 4.2</u>. If $\frac{1}{k}\psi^k : BU\Lambda \to BU\Lambda$ is defined ($\psi^k \equiv$ an Adams operation) it induces a ring homomorphism

$$\Psi^k : AU\Lambda^*(\_) \to AU\Lambda^*(\_)$$

which corresponds under Φ_U to the Adams operation in $MU\Lambda^*(\_)$.

Also the Adams idempotent $E_1 : BUR(d) \to BUR(d)$ [Ad 3, p. 89] induces a ring homomorphism.

Theorem 5.1. The Adams idempotent $E_1 : BUR(d) \to BUR(d)$ induces an idempotent ring homomorphism

$$\varepsilon(d) : AUR(d)^*(\_) \to AUR(d)^*(\_).$$

Under Φ_U, $\varepsilon(d)$ corresponds with the Adams idempotent in $MUR(d)^*(\_)$.

Also if $p \equiv 1(d)$ is a prime then $\varepsilon(d)$ induces

$$\varepsilon(d) : AU\hat{Z}_p^*(\_) \to AU\hat{Z}_p^*(\_)$$

satisfying

$$[\varepsilon(d)(f)]^d = \Pi\, \Psi^{\alpha_j}(f).$$

Here the product runs over d-th roots of unity in $\hat{Z}_p$.

The K-theory operation of complexification, $c : BSp \to BU$, induces a complexification homomorphism of a rather unexpected type.

Theorem 6.1. Complexification $c : BSp \to BU$ induces a natural ring homomorphism

$$c : ASp^\circ(X) \to AU^\circ(X)[1 - \alpha_{11}]^{-1}.$$

Here $\alpha_{11} \in AU^\circ(S^\circ)$ satisfies $\Phi_U(\alpha_{11}) = a_{11} \in MU^2(S^\circ)$.

In MU-theory one has the Landweber-Novikov operations s_α. In AU-theory one would expect an operation corresponding to the total Landweber-Novikov operation, $\Sigma_\alpha s_\alpha$.

Theorem 7.1. The "super-total" Conner-Floyd class

$$c = \Sigma_\alpha c_\alpha : KU^\circ(X) \to MU^{2*}(X) \simeq AU^\circ(X)$$

induces a natural ring homomorphism

$$S : AU^*(X) \to AU^*(X)$$

which corresponds under Φ_U to $\Sigma_\alpha s_\alpha$, the total Landweber-Novikov operation.

Having seen the connections between KU-theory and AU-theory which are listed above the following result will come as no surprise to the reader.

In §8 the classical Pontrjagin-Thom construction for stably almost complex manifolds is given in terms of AU-theory. In fact, two equivalent descriptions are given with a view to generalising in Part IV the construction to the case of a smooth algebraic embedding.

In §9, as an application of AU-theory, we prove two theorems about the stable homotopy of CP^∞. The first (Theorem 9.1.1) shows how to construct $KU^\circ(X)$

as a limit of stable homotopy groups of the form

$$KU^\circ(X) \cong \varinjlim_N \{\Sigma^{2N}X,\ CP^\infty\} .$$

The second (Theorem 9.1.2) states that any torsion element $y \in \pi_*^S(CP^\infty)$ is annihilated by iterated product with $x \in \pi_2(CP^\infty)$, the generator. The product referred to comes from the H-space multiplication on CP^∞.

That completes the list of the main results in Part II. In consideration of the reader I have postponed until Part III the general construction which generalizes the construction AU and ASp. Also in Part III the representation of MO* is given. This material is described in the introduction to Part III.

§1. HOMOLOGY AND THE STABLE DECOMPOSITION OF $\Omega^t\Sigma^t BU(1)$

In [Sn 1] I constructed a stable decomposition of the space $\Omega^t\Sigma^t X$ for connected X and $t \geq 1$. Details will be given when they are needed (section 1.3). The decomposition has the following form. $\Omega^t\Sigma^t X$ is homotopy equivalent to a space C_tX[Ma 1] which has a filtration $\{F_nC_tX;\ n \geq 1\}$. The decomposition theorem asserts an equivalence in the stable category [Ad 1] of the form

$$F_nC_tX = \bigvee_{k=1}^{n} \frac{F_kC_tX}{F_{k-1}C_tX} \tag{1.1}$$

Here a stable equivalence between spaces means an equivalence in the Adams stable category between their suspension spectra. In this section I will prove the following:

<u>1.2</u>: <u>Theorem</u>. For $1 \leq i \leq n$ let $a_i \in \tilde{H}_*(BU(1);Z)$.

Then for all $t \geq 1$ and $1 \leq m < n$ the stable map

$$BU(1)^n \to F_nC_tBU(1) \to \bigvee_{k=1}^{n} \frac{F_kC_tBU(1)}{F_{k-1}C_tBU(1)} \to \frac{F_mC_tBU(1)}{F_{m-1}C_tBU(1)}$$

induces in homology a homomorphism which annihilates $a_1 \otimes \cdots \otimes a_n$.

Here the first map in the composite is induced by the n-fold H-space sum of the suspension map $BU(1) \to \Sigma^t\Omega^t BU(1)$. The second map is (1.1) and the third is projection onto the m-th wedge factor.

This will be proved in section 1.4. We must first recall the decomposition (1.1).

<u>1.3</u>: <u>The Stable Decomposition</u>. The technicalities of [Sn 1] may be rather forbidding so I will give the reader a choice. We need the following basic fact about the S-map of [Sn 1, §3.1].

$$X^n \to F_nC_tX \to \frac{F_mC_tX}{F_{m-1}C_tX} .$$

Basic Fact. Suppose $1 \le m < n$ and $Y \subset X$ is a subspace which does not contain the base-point, $* \in X$. Then the composition of $Y^n \subset X^n$ with the above S-map is equal to the track-group sum of a finite number of maps, each of which factors through a projection of the form $Y^n \to Y^m$.

In §1.4 we prove Theorem 1.2 by applying the Basic Fact when $X = CP^\infty$ and Y is a copy of CP^N. The S-map in question being exhibited below as (1.4.1).

The reader may now skip to §1.4 for the proof of Theorem 1.2. Alternatively for the reader who is interested in examining the details of the S-decomposition of [Sn 1] more closely I will include the details necessary to pass from [Sn 1] to the establishment of the BASIC FACT.

Let I^n denote the unit n-cube and J^n its interior. An (open) little n-cube is a linear embedding, f, of J^n in J^n with parallel axes. Thus $f = f_1 \times \cdots \times f_n$ where $f_i : J \to J$ is a linear function $f_i(t) = (y_i - x_i)t + x_i$ with $0 \le x_i < y_i \le 1$. Let $\mathcal{C}_n(j)$ be the set of j-tuples of little n-cubes whose images are pairwise disjoint. Denote by $\mathcal{D}_n(j)$ the set of j-tuples of little n-cubes without the disjoint image condition. Hence $\mathcal{C}_n(j) \subset \mathcal{D}_n(j)$ $(j > 0)$ and $\mathcal{D}_n(0) = \mathcal{C}_n(0)$ is a point.

Let Y be a space with closed subspace A containing the basepoint, *. Define C_nY and $D_n(Y,A)$ as follows. Form the disjoint union

$$Z_n = \cup_{j\ge 0} \mathcal{D}_n(j) \times Y^j$$

and let $\approx$ be the equivalence relation on Z_n generated by

(i) $$(<c_1,\dots, c_m> \ y_1,\dots, y_{i-1},*,y_{i+1},\dots, y_m)$$
$$\approx (<c_1,\dots, c_{i-1},c_{i+1},\dots, c_m> \ y_1,\dots, y_{i-1},y_{i+1},\dots, y_m)$$

and

(ii) $(<c_1,\dots, c_m> \ y_1,\dots, y_m) \approx (<c_{\sigma(1)},\dots, c_{\sigma(m)}> \ y_{\sigma(1)},\dots, y_{\sigma(m)})$ for $\sigma \in \Sigma_m$, the symmetric group on m letters. Set C_nY $= \{[<c_1,\dots, c_m> \ y_1\dots y_m] \in Z_{n/\approx}$ such that $<c_1,\dots, c_m> \in \mathcal{C}_n(m)\}$ and $D_n(Y,A) = \{[<c_1,\dots, c_m> y_1,\dots, y_m] \in Z_{n/\approx}$ such that if $y_{i_1},\dots, y_{i_k}$ are all the coordinates not in A then $<c_{i_1},\dots, c_{i_k}> \in \mathcal{C}_n(k)\}$.

These spaces may be topologised [Sn 1]. From [Ma 1] we know that, for reasonable $Y, C_t \simeq \Omega^t\Sigma^tY$. The filtration referred to in Theorem 1.2 is the following. F_nC_tY consists of all points $[<c_1,\dots, c_m> y_1,\dots, y_m]$ with $m \le n$.

Now I will describe the ingredients of the construction of (1.1). Firstly suppose given a continuous function $u : BU(1) \to [0,1]$ such that $u^{-1}(0) = *$. Then the following special is a case of a result proved in [Sn 1].

<u>1.3.1</u>: <u>Proposition [Sn 1, Theorem 3.1]</u>. Let $f : F_m C_n BU(1) \to Y$ be a map such that $f(F_{m-1} C_n BU(1)) = *$ and $f(<c_1,\ldots, c_m> x_1,\ldots, x_m) \in A$ if and only if $\min_i u(x_i) \leq \frac{1}{2}$.

Then there is a family of stable maps

$$G_k : F_k C_n BU(1) \to Y/A \qquad (k \geq n)$$

such that $G_k | F_{k-1} C_n BU(1) \simeq G_{k-1}$ and G_m is the composite of f with the collapsing map $Y \to Y/A$.

Henceforth write F_m for $F_m C_n BU(1)$.

Proposition 1.3.1 is applied to $F_m \to F_m/F_{m-1} = Y$ with u and A chosen so that $Y/A \simeq Y$. Then (1.1) is the sum of the stable maps obtained in this manner. Hence I must recall now the proof of Proposition 1.3.1. F_n is constructed from F_{n-1}.

Maps

$$\phi_n : \mathcal{C}_t(n) \to \mathcal{D}_{q_n}(p).$$

$$\psi_n : \mathcal{C}_t(n) \times BU(1)^n \to \mathcal{D}_{\ell_n}(p)$$

$$(p = n!/m!(n-m)!)$$

and

$$\Psi_n : F_n C_t BU(1) \to D_t(F_m/F_{m-1}, A)$$

are constructed. There is a stable map

$$D_t\left(\frac{F_m}{F_{m-1}}, A\right) \to \left(\frac{F_m}{F_{m-1}}\right) \Big/ A$$

called evaluation, eval, and $F_n = \text{eval}\ \Psi_n$. All we need to know about ϕ_n and ψ_n is that $\psi_n(<c_1,\ldots, c_n> x_1)$ is independent of $x_1,\ldots, x_n$ if $u(x_i) = 1$ for $1 \leq i \leq n$. There is a formula for Ψ_n in terms of Ψ_{n-1}, ϕ_n and Ψ_n. Consulting the formula [Sn 1, p. 582] we see that if $u(x_i) = 1$ for $1 \leq i \leq n$ then

$$\Psi_n(<c_1,\ldots, c_n> x_1,\ldots, x_n) = (c, f(z_1),\ldots, f(z_p))$$
$$(p = n!/m!(n-m)!) \tag{1.3.2}$$

Here $c \in \mathcal{C}_t(p)$ is fixed. The element $z_i \in \mathcal{C}_t(m) \times BU(1)^m/\sim$ is of the form $[<d_1,\ldots, d_m> y_1,\ldots, y_m]$ where the d_i's are suitably chose subset of $\{c_1,\ldots, c_n\}$ and the y_i's are the corresponding subset of $\{x_1,\ldots, x_n\}$.

<u>1.4</u>: <u>Proof of Theorem 1.2</u>. For some large N we can find a copy of CP^N in $CP^\infty = BU(1)$ such that $* \notin CP^N$ and such that there exist classes $\alpha_i \in \tilde{H}_*(CP^N;Z)$

mapping to $a_i \in \tilde{H}_*(BU(1);Z)$. Since $* \notin CP^N$ we may assume (by modifying the function $u : BU(1) \to [0,1]$ if necessary) that $CP^N \subset u^{-1}(1)$.

The map $(CP^N)^n \to BU(1)^n \to F_nC_tBU(1)$ sends $(x_1,\ldots, x_n)$ to $[<c_1,\ldots, c_n> x_1,\ldots, x_n]$ for some fixed $<c_1,\ldots, c_n> \in C_t(n)$. We are trying to compute the effect in homology of the stable map

$$(CP^N)^n \to F_n \xrightarrow{\Psi_n} D_t\left(\left(\frac{F_m}{F_{m-1}}\right), A\right) \xrightarrow{\text{eval}} \left(\frac{F_m}{F_{m-1}}\right)\Big/A \simeq \frac{F_m}{F_{m-1}} \tag{1.4.1}$$

From (1.3.2) and the related discussion in section 1.3 $(x_1,\ldots, x_n) \in (CP^N)^n$ goes to $[c,f(z_1),\ldots, f(z_p)]$ in $D_t\left(\left(\frac{F_m}{F_{m-1}}\right), A\right)$.

Composition with evaluation sends the map which sends $(x_1,\ldots, x_n)$ to $[c,f(z_1),\ldots, f(z_p)]$ to the sum (in the sense of track-group addition) of the stable maps

$$g_i : (CP^N)^n \to \left(\frac{F_m}{F_{m-1}}\right)\Big/A \simeq \frac{F_m}{F_{m-1}}$$

given by $g_i(x_1,\ldots, x_n) = f(z_i)$ $(1 \le i \le p)$. Recall $z_i = (<d_1,\ldots, d_m> y_1,\ldots, y_m) \in C_t(m) \times BU(1)^m$ where $\{y_1,\ldots, y_m\}$ is a subset of $\{x_1,\ldots, x_n\}$ and $<d_1,\ldots, d_m>$ is fixed. Therefore g_i factors through the projection $(CP^N)^n \to (CP^N)^m$ which picks out $y_1\ldots y_m$. Hence $(g_i)_*(\alpha_1 \otimes \cdots \otimes \alpha_n) = 0$. The stable map of Theorem 1.2 maps $a_1 \otimes \cdots \otimes a_n$ to $\sum_{i-1}^{p} (g_i)_*(\alpha_1 \otimes \cdots \otimes \alpha_n) = 0$ and the result follows.

§2. AU°(X), ASp°(X) AND COBORDISM

Let $B : S^2 \to BU$ and $B' : S^4 \to BSp$ be generators of $\pi_2(BU)$ and $\pi_4(BSp)$ respectively. The Whitney sum of B with the identity map of BU induces a map $\Sigma^4 BU \to \Sigma^2 BU$. Similarly the sum of B' with the identity induces $\varepsilon' : \Sigma^6 BSP \to \Sigma^2 BSp$. Details are given in section 2.3 below. Using ε and ε' we may define groups (actually rings, see section 3)

$$AU^\circ(X) = \varinjlim_N \{\Sigma^{2N}X,BU\}$$

and

$$ASp^\circ(X) = \varinjlim_N \{\Sigma^{4N}X,BSp\}.$$

Here $\{Y,Z\}$ denotes morphisms of degree zero from the space Y to the space Z in the Adams stable category [Ad 1, Part III]. The limits are taken over the homomorphisms

$$\{\Sigma^{2N}X,BU\} = \{\Sigma^{2N+2}X,\Sigma^2 BU\} \xrightarrow{\varepsilon_\#} \{\Sigma^{2N+2}X,BU\}$$

and

$$\{\Sigma^{4N}X, BSp\} = \{\Sigma^{4N+4}X, \Sigma^4 BSp\} \xrightarrow{\varepsilon'_\#} \{\Sigma^{4N+4}X, BSp\}.$$

Let $c_k \in MU^{2k}(BU)$ and $p_k \in MSp^{4k}(BSp)$ be the Conner-Floyd classes [Ad 1, p. 9]. Define

$$\phi_u : \{\Sigma^{2N}X, BU\} \to \prod_{1\le k} MU^{2k}(\Sigma^{2N}X) \cong \prod_{1\le k} MU^{2k-2N}(X)$$

and

$$\phi_{Sp} : \{\Sigma^{4N}X, BSp\} \to \prod_{1\le k} MSp^{4k}(\Sigma^{4N}X) \cong \prod_{1\le k} MSp^{4k-4N}(X)$$

by the formulae

$$\phi_u(f) = \prod_{1\le k} f^*(c_k),$$

$$\phi_{Sp}(f) = \prod_{1\le k} f^*(p_k).$$

The main results of this section are the following:

<u>2.1</u>: <u>Theorem</u>. ϕ_u induces a well-defined natural homomorphism

$$\phi_u : AU^\circ(X) \to \varinjlim_N \prod_{\ell=-N}^{\infty} MU^{2\ell}(X) = MU^{2*}(X)$$

which is an isomorphism when X is a finite dimensional CW complex.

<u>2.2</u>: <u>Theorem</u>. ϕ_{Sp} induces a well-defined natural homomorphism

$$\phi_{Sp} : ASp^\circ(X) \to \varinjlim_N \prod_{\ell=-N}^{\infty} MSp^{4\ell}(X) = MSp^{4*}(X)$$

which is an isomorphism when X is a finite dimensional CW complex.

<u>2.3</u>: <u>Explanation and Sketch of Proof</u>. Firstly I must define the maps ε and ε'. It will suffice to define ε since ε' is constructed analogously. The Whitney sum of B with 1_{BU} gives

$$B \oplus 1_{BU} : S^2 \times BU \to BU.$$

It is well-known that $\Sigma(S^2 \times BU) = \Sigma S^2 \vee \Sigma BU \vee \Sigma^3 BU$ (for example see [Sn 3]). If we suspend once more the inclusion of $\Sigma^4 BU = \Sigma S^2 \wedge \Sigma BU$ into $\Sigma^2(S^2 \times BU)$ may be represented by the Hopf construction [Hu]

$$\Sigma S^2 \wedge \Sigma BU = \Sigma(S^2 * BU) \xrightarrow{\Sigma H} \Sigma(\Sigma(S^2 \times BU)).$$

Set $\varepsilon = \Sigma^2(B \oplus 1_{BU}) \circ \Sigma H$. There are several ways of including $\Sigma^4 BU$ into $\Sigma^2(S^2 \times BU)$. I have chosen the Hopf construction because it has associativity properties which I will need in section 3 where $AU^\circ(X)$ is discussed in terms of a ring spectrum, AU.

Now let me explain why ϕ_u is well-defined. I wish to establish a commutative diagram of the following form.

$$\begin{array}{ccc} \{\Sigma^{2N}X, BU\} & \xrightarrow{\phi_u} & \prod_{1\le k} MU^{2k-2N}(X) \\ \downarrow{\scriptstyle \varepsilon_{\#}} & & \downarrow{\scriptstyle i} \\ \{\Sigma^{2n+2}X, BU\} & \xrightarrow{\phi_u} & \prod_{1\le k} MU^{2k-2N-2}(X) \end{array} \tag{2.4}$$

Here i is the canonical inclusion. Suppose $f \in \{\Sigma^{2N}X, BU\}$ then the stable map $\varepsilon_{\#}(f)$ is the composite

$$S^2 \wedge (\Sigma^{2N}X) \xrightarrow{\Sigma H} S^2 \times \Sigma^{2N}X \xrightarrow{B \oplus f} BU.$$

$$\begin{aligned} \text{Hence } \varepsilon_{\#}(f)^*(c_k) &= (\Sigma H)^*(B \oplus f)^*(c_k) \\ &= (\Sigma H)^*\Big(\sum_{\ell=0}^{k} B^*(c_\ell) \otimes f^*(c_{k-\ell})\Big) \\ &= B^*(c_1) \otimes f^*(c_{k-1}) \in MU^{2k}(S^2 \wedge \Sigma^{2N}X). \end{aligned}$$

Note that $(\Sigma H)^*(B^*(c_0) \otimes f^*(c_k)) = 0$. However $B^*(c_1)$ is just the suspension class so that $B^*(c_1) \otimes f^*(c_{k-1}) \in MU^{2k}(S^2 \wedge \Sigma^{2N}X)$ corresponds under suspension to $f^*(c_{k-1}) \in MU^{2k-2}(\Sigma^{2N}X)$. This establishes (2.4).

In [Part I, Theorem 4.2] it is shown that there exist stable equivalences

$$\nu_n : BU(n) \to \bigvee_{1\le k\le n} MU(k)$$

such that $\nu_n|BU(n-1) \simeq \nu_{n-1}$. Details are given in section 2.7 below. Hence

$$\{Y, BU\} = \prod_{1\le k} \{Y, MU(k)\} \tag{2.5}$$

and it was shown in [Part I, Theorem 5.1] that if dim $Y \le 4n$ then ϕ_u maps the summand $\prod_{n\le k} \{Y, MU(k)\}$ isomorphically to the summand $\prod_{n\le k} MU^{2k}(Y)$ in $MU^{2*}(Y)$. Setting $Y = \Sigma^{2N}X$ in (2.5) and considering the limit over N it is straightforward to see that Φ_u is surjective. Under the stable map $\varepsilon : \Sigma^2 BU \to BU$ the wedge factor $\Sigma^2 MU(n)$ maps to $\bigvee_{1\le k\le n+1} MU(k)$ and composition with the projection onto $MU(n+1)$ is essentially the canonical map $\Sigma^2 MU(n) \to MU(n+1)$ from the unitary Thom spectrum [Ad 1, p. 135]. That is, up to an automorphism in the S-categary, $\Sigma^2 MU(n) \to MU(n+1)$, is the canonical map. This fact is a consequence of the property [Part I, §4.5(b)] of the stable decomposition maps constructed in [Part I, §4]. Hence if the stable map $\varepsilon : \Sigma^2 BU(n) \to BU(n+1)$ were merely the sum of these pseudo-Thom-spectrum maps

$$\bigvee_{1\le k\le n} \Sigma^2 MU(k) \to \bigvee_{1\le k\le n} MU(k+1)$$

we would be finished. This is because in the limit process by which $AU^\circ(X)$ is defined each element of $\{\Sigma^{2N}X, MU(k)\}$ would eventually be mapped into a summand of the form $\left\{\Sigma^{2M}X, \frac{BU}{BU(t-1)}\right\}$ with $4t \le 2M + \dim X$ on which ϕ_u is injective by [Part I, Theorem 5.1]. Hence to prove Theorem 2.1 we consider the difference, δ_n, between the stable map $\varepsilon : \Sigma^2 BU \to BU$ and the wedge-sum of the pseudo-Thom-spectrum maps mentioned above. In Proposition 2.9 it is shown that $\delta_n : \Sigma^2 BU(n) \to BU(n+1)$ may be chosen to send the double suspension of the 2t-skeleton into the (2t-2)-skeleton. In the limiting process which defines $AU°(X)$ this implies that any class in $\{\Sigma^{2M}X, MU(k)\}$ is mapped eventually into a summand of $\{\Sigma^{2M+2S}X, BU\}$ on which ϕ_u is injective. Therefore Φ_u must be injective.

The proof of Theorem 2.1 is accomplished in a series of results according to the following programme. In Lemma 2.7 the stable equivalence ν_n is computed in homology. This is used in Corollary 2.8 to show that δ_n is zero in homology. Using this δ_n is compressed a little in Proposition 2.9. The proof of Theorem 2.1 is completed in section 2.11.

Section 2.12 contains an outline of the proof of Theorem 2.2 which is entirely analogous to the proof of Theorem 2.1.

2.6. Firstly we need to know what the stable equivalence ν_n of section 2.3 induces in homology. Recall [Ad 1, p. 47] that if $\beta_i \in H_{2i}(BU(1);Z)$ is a generator then $H_*(BU;Z)$ is equal to the polynomial algebra, $Z[\beta_1,\beta_2,\ldots]$. The homology of $BU(n)$ has as a basis those monomials $\beta_{i_1} \otimes \cdots \otimes \beta_{i_t}$ with $t \le n$. Under the collapsing map $BU(n) \to MU(n)$ the homology of $MU(n)$ has as a basis the monomials of the form $\beta_{i_1} \otimes \cdots \otimes \beta_{i_n}$.

2.7: Lemma. In terms of the homology generators of section 2.6 the stable equivalence of section 2.3

$$\nu_n : BU(n) \to \bigvee_{1\le k\le n} MU(k)$$

satisfies $(\nu_n)_*(\beta_{i_1} \otimes \cdots \otimes \beta_{i_t}) = \beta_{i_1} \otimes \cdots \otimes \beta_{i_t}$ $(1 \le t \le n)$.

Proof. Since $\nu_n | BU(n-1) = \nu_{n-1}$ we may proceed by induction on t and therefore assume that $t = n$. The stable map ν_n is equal to a composite of the form

$$\begin{tikzcd}
BU(n) \arrow[r, "\tau(\pi_n)"] & B\Sigma_n \int U(1) \arrow[r] \arrow[dl, "\simeq"] & F_n QBU(1) \\
\bigvee_{1\le k\le n} \frac{F_k QBU(1)}{F_{k-1}QBU(1)} = \bigvee_{1\le k\le n} \frac{B\Sigma_k \int U(1)}{B\Sigma_{k-1}\int U(1)} \arrow[d] & & \\
\bigvee_{1\le k\le n} \frac{BU(k)}{BU(k-1)} . & &
\end{tikzcd}$$

Here $QBU(1) = \Omega^\infty\Sigma^\infty BU(1)$. $\tau(\pi_n)$ is the transfer of the canonical map $\tau_n : B\Sigma_n \int U(1) \to BU(n)$ and the stable equivalence is the subject of [Sn 1, Theorem 3.2]. Details are given in [Part I, section 4]. The class $\beta_{i_1} \otimes \cdots \otimes \beta_{i_n}$ is in the image of the map $H_*(BT^n;Z) \to H_*(BU(n);Z)$ induced by the inclusion of the canonical maximal torus. In [Part I, section 4.6] it is shown that the component of $(\nu_n)_*(\beta_{i_1} \otimes \cdots \otimes \beta_{i_n})$ in $H_*(MU(n);Z)$ is just $\beta_{i_1} \otimes \cdots \otimes \beta_{i_n}$. Therefore we have to show that $(\nu_n)_*(\beta_{i_1} \otimes \cdots \otimes \beta_{i_n})$ has zero component in $H_*(MU(n);Z)$ for $1 \le m < n$.

However there is a commutative diagram of stable maps

$$\begin{tikzcd} BT^n \arrow[r] \arrow[dr] & BU(n) \arrow[d, "\tau(\pi_n)"] \\ & B\Sigma_n \int U(1) \end{tikzcd}$$

in which $BT^n \to B\Sigma_n \int U(1)$ is induced by the canonical inclusion of T^n in the wreath product $\Sigma_n \int U(1)$. This diagram is established by the technique of [Part I, section 2; B-M]. [Part I, Example 2.14] produces a T^n-equivariant vector field, ρ_v, on $\dfrac{U(n)}{\Sigma_n \int U(1)}$, and identifies the singular set of ρ_v as a point. Then [Part I, Proposition 2.2 and section 2.2.1] together with the argument of [Part I, section 2.9] establishes the diagram. The result now follows from Part II, Theorem 2.2 with $t = \infty$.

<u>2.8</u>: <u>Corollary</u>. By means of the stable equivalences ν_n of Lemma 2.7 consider the stable map $\varepsilon : \Sigma^2 BU \to BU$ of section 2.3 as inducing

$$\varepsilon : \bigvee_{1\le k\le n} \Sigma^2 MU(k) \to \bigvee_{1\le k\le n+1} MU(k) \quad \text{for } 1 \le n \le \infty .$$

Let γ_j equal the composite

$$\Sigma^2 MU(j) \to \bigvee_{1\le k\le n} \Sigma^2 MU(k) \xrightarrow{\varepsilon} \bigvee_{1\le k\le n+1} MU(k) \to MU(j+1) .$$

This map is the one discussed in §2.3 above, and up to S-automorphism is equal to the canonical Thom spectrum map. Then the stable map $\delta_n = \varepsilon - \sum_{j=1}^{n} \gamma_j$ induces the zero map in reduced homology and in MU_*.

Here the first and last maps in γ_j are the canonical inclusion and projection respectively.

<u>Proof</u>. We will treat only the homology case, the MU_* case is similar. In terms of the homology generators described in section 2.6

$$(B \oplus 1_{BU})_* : \tilde{H}_*(S^2 \times BU) \to \tilde{H}_*(BU)$$

sends $\sigma \otimes \beta_{i_1} \otimes \cdots \otimes \beta_{i_j}$ to $\beta_1 \otimes \beta_{i_1} \otimes \cdots \otimes \beta_{i_j}$. Here $\sigma \in H_2(S^2)$ is a generator. Hence ε_* does the same. Therefore, by Lemma 2.7, the γ_j are the only components of ε which are non-zero in homology and the result follows at once.

<u>2.9</u>: <u>Proposition</u>. Let $BU(n)_m$ be the m-skeleton of $BU(n)$. Then for any n, $m \geq 1$ there is a stable map

$$\delta'_n : \Sigma^2 BU(n) \to BU(n+1)$$

such that (i) $\delta'_n \simeq \delta_n$, the S-map of Corollary 2,8, and

(ii) δ'_n maps $\Sigma^2 BU(n)_{2m}$ into $BU(n+1)_{2m-2}$.

<u>Proof</u>. Although the map ε of Corollary 2.8 maps $\Sigma^2 BU(n)$ (considered as a wedge of double suspensions of Thom spaces) to $BU(n+1)$ the map $\delta_n = \varepsilon - \sum_{j=1}^{n} \gamma_j$ maps $\Sigma^2 BU(n)$ into the stable summand $BU(n)$ of $BU(n+1)$. $BU(n)$ has only even dimensional cells. By cellular approximation [Sp], we may assume δ_n sends $\Sigma^2 BU(n)_{2m}$ to $BU(n+1)_{2m+2}$. By the Hurewicz isomorphism [Sp]

$$H_{2m+2}(BU(n+1)_{2m+2}, BU(n+1)_{2m}) \simeq \pi_{2m+2}(BU(n+1)_{2m+2}, BU(n+1)_{2m}) .$$

By naturality of the Hurewicz homomorphism and the fact that $(\delta_n)_* = 0$ in homology δ_n on each top cell of $\Sigma^2 BU(n)_{2m}$ may be deformed (relative to the boundary) into $BU(n+1)_{2m}$. Hence δ_n on $\Sigma^2 BU(n)_{2m}$ may be deformed into $BU(n+1)_{2m}$ relative to $\Sigma^2 BU(n)_{2m-2}$. This gives δ''_n on $\Sigma^2 BU(n)_{2m}$ homotopic to δ_n and by homotopy extension δ''_n may be extended to a cellular map, δ''_n, on $\Sigma^2 BU(n)$ which is homotopic to δ_n. Consider $\delta''_n : \Sigma^2 BU(n)_{2m} \to BU(n+1)_{2m}$. It induces zero on homology. Therefore, by the previous argument we may find $\delta'''_n : \Sigma^2 BU(n)_{2m} \to BU(n+1)_{2m}$ such that $\delta'''_n \simeq \delta''_n$ and δ'''_n sends $\Sigma^2 BU(n+1)_{2m-2}$ into $BU(n+1)_{2m-2}$. By homotopy extension we may extend δ'''_n to obtain a cellular stable map on $\Sigma^2 BU(n)$ which is homotopic to δ_n.

Consider the following homotopy commutative diagram of S-maps in which the rows are cofibrations.

$$\begin{tikzcd} \Sigma^2 BU(n)_{2m-2} \arrow[r] \arrow[d, "\delta'''_n"'] & \Sigma^2 BU(n)_{2m} \arrow[r] \arrow[d, "\delta'''_n"'] & \vee_\beta S^{2m+2}_\beta \arrow[d, "\delta^{iv}"] \arrow[dl, "\lambda(n,m)"] \\ BU(n+1)_{2m-2} \arrow[r] & BU(n+1)_{2m} \arrow[r] & \vee_\alpha S^{2m}_\alpha \end{tikzcd} \tag{2.9.1}$$

If, in diagram (2.9.1), $\lambda(n,m)$ is stably trivial then we are finished. Therefore if we are working at odd primes we are done since all stable maps $\vee_\beta S^{2m+2}_\beta \to \vee_\alpha S^{2m}_\alpha$ are of order two [T].

Now we tackle the prime two. This will be done in §2.10 in a series of steps.

§2.10.1. Let us begin by giving names to the spaces we will need. Consider $BT^n = (BS^1)^n$ in the stable category as a wedge sum

$$BT^n = \bigvee_{1\le j\le n} W_j .$$

Here W_j is the wedge sum of all these smash products $BS^1 \wedge \cdots \wedge BS^1$ (j copies) when the product $(BS^1)^n$ is written as sums of smash products of its factors.

We have canonical S-maps

$$\alpha(j,n) : W_j \to BU(n)$$

given by composing the inclusion $W_j \subset BT^n$ with the map induced by the inclusion of the maximal torus into U(n).

2.10.2: Lemma. For $1 \le j \le n$ the S-map

$$\delta_n''' \circ \Sigma^2\alpha(j,n) : \Sigma^2(W_j)_{2m} \to BU(n+1)_{2m}$$

is nullhomotopic.

Proof. From [Part I, Example 2.14] and the techniques of [Part I, §2] we see that the transfer $\tau_u : BU \to QBU(1)$, restricted via $BT^n \to BU(n) \to BU$, is homotopic to the canonical map $BT^n \to B\Sigma_n \int U(1) \to QBU(1)$. Hence, examining the S-map, $\nu_{U(n)}$, which splits BU(n) [Part I, §4] we see that the S-map

$$\Sigma^2 W_j \xrightarrow{\alpha(j,n)} \Sigma^2 BU(n) \xrightarrow{\varepsilon} BU(n+1)$$

(ε as in §2.8 above) is homotopic to the composite

$$\Sigma^2 W_j = S^2 \wedge W_j \subset BS^1 \wedge W_j \subset W_{j+1} \xrightarrow{\alpha(j+1,n+1)} BU(n+1) .$$

But, in the notation of §2.8, this is γ_j restricted to $\Sigma^2 W_j$ and $\delta_n = \varepsilon - \sum_{j=1}^{n} \gamma_j$ so δ_n is trivial when restricted to $\Sigma^2 W_j$. Now carefully following the obstruction argument of §2.9 gives that the deformation $\delta_n''' \circ \alpha(j,n)$ of $\delta_n \circ \alpha(j,n)$ can be deformed to the constant map within the required skeleton.

2.10.3: Lemma. Let $0 \ne \eta \in \pi_1^S(S^\circ)$ and let $\rho_\beta : \Sigma^2 BU(n)_{2m} \to S_\beta^{2m+2}$ be induced by the right-hand map in the top row of (2.9.1). Then there exists j such that Sq^2 detects the composite

$$\Sigma^2(W_j)_{2m} \xrightarrow{\Sigma^2(\alpha(j,n))} \Sigma^2 BU(n)_{2m} \xrightarrow{\rho_\beta} S_\beta^{2m+2} \xrightarrow{\eta} S^{2m+1} .$$

Proof. Sq^2 is non-zero on the mapping cone of $\eta[T]$. Since ρ_β is injective in mod 2 cohomology it follows that Sq^2 is non-zero on the mapping cone of $\eta \circ \rho_\beta$. However the track-group sum of the map $\Sigma^2(\alpha(j,n))$ detects the mod 2 cohomology of $\Sigma^2 BU(n)_{2m}$ so one of the composites $\eta \circ \rho_\beta \circ \Sigma^2(\alpha(j,n))$ $(1 \le j \le n)$ must be detected by Sq^2.

2.10.4: Completion of the Proof of Proposition 2.9. Suppose that for some α_o, β_o there is a non-trivial composition

$$\delta(\alpha_o,\beta_o) : S^{2m+2}_{\beta_o} \subset \underset{\beta}{\vee} S^{2m+2}_{\beta} \xrightarrow{\delta^{iv}} \underset{\alpha}{\vee} S^{2m}_{\alpha} \to S^{2m}_{\alpha_o} .$$

This composition must be stably homotopic to $\eta^2 \in \pi_2^S(S^\circ)$. We will show that $\delta(\alpha_o,\beta_o)$ cannot be non-trivial by use of a well-known argument, due to Adem, [T, p. 84, Example 3] by which one shows $\eta^2 \neq 0$. Choose j as in Lemma 2.10.3 for $\beta = \beta_o$. Consider the composite

$$\Sigma^2(W_j)_{2m} \xrightarrow{A} S^{2m+2}_{\beta_o} \xrightarrow{\eta} S^{2m+1} \xrightarrow{\eta} S^{2m}_{\alpha_o}$$

in which $A = \rho_{\beta_o} \circ \Sigma^2(\alpha(j,n))$. This composite is homotopic to $\delta'''_n \circ \Sigma^2\alpha(j,n)$ which is trivial by Lemma 2.10.2. Hence we may form the space

$$B = (S^{2m}_{\alpha_o} \vee e^{2m+2}_{\eta}) \cup C\,\Sigma^2(W_j)_{2m} .$$

Since Sq^2 is non-trivial in $S^{2m}_{\alpha_o} \cup e^{2m+2}_{\eta}$ and also detects the S-map of Lemma 2.10.3 we see that Sq^2Sq^2 is non-trivial on the integral class generating $H^{2m}(B;Z/2)$. However, this is impossible by the Adem relation $Sq^2Sq^2 = Sq^3\,Sq^1$ because Sq^1 is zero on integral classes.

This completes the proof.

2.11: Proof of Theorem 2.1. In section 2.3 I explained why Φ_U was well-defined.

Also any class in $MU^{2*}(X)$ is contained in

$$\prod_{\ell \ge T-M} MU^{2\ell}(X) = \prod_{k \ge T} MU^{2k}(\Sigma^{2M}X)$$

for some T, M satisfying $4T \ge 2M + \dim X$. By [Part I, Theorem 5.1] ϕ_u restricted to the summand $\left\{\Sigma^{2M}X, \frac{BU}{BU(T-1)}\right\}$ of $\{\Sigma^{2M}X, BU\}$ is an isomorphism onto $\prod_{T \le k} MU^{2k}(\Sigma^{2M}X)$ if $4T \ge 2M + \dim X$. Hence Φ_u is surjective.

Now let $f \in \{X,BU\}$ represent $F \in AU^\circ(X)$ which satisfies $\Phi_u(F) = 0$. For some n,t we may assume that f originates in $\{X,BU(n)_{2t}\} \simeq \bigoplus_{k=1} \{X,MU(k)_{2t}\}$. Suppose $f = \Sigma f_k$ where $f_k \in \{X,MU(k)_{2t}\}$. The computation of Corollary 2.8 shows that each δ_n induces zero in MU*-theory. Here δ_n is the S-map of Corollary 2.8. From this it is easy to see that $\Phi_U(f_k) = 0$ for each k, because $\Phi_U(f_k) \in MU^{2k}(X)$. This is because Φ_U is given by applying ε^* $(= \sum_j \gamma_j^*$ by Corollary 2.8) to the canonical classes in $\oplus\, MU^{2k}(MU(2k))$ and γ_j^* picks out precisely f_j hence we may assume $f = f_n$.

Also, by induction, we shall suppose for all Y that $\Phi_u(f) = 0$ implies $F = 0$ whenever

$$f \in \{Y, MU(m)_{2s}\}$$

satisfies either (i) $s < t$, or

(ii) $\dim Y - 4m < \dim X - 4n$.

Now consider the passage of $f \in \{X, MU(n)_{2t}\}$ in the limit defining $AU^\circ(X)$. The image of f, $\varepsilon_\#(F) \in \{\Sigma^2 X, BU\}$ is equal to

$$((\delta_n)_\#(f), (\gamma_n)_\#(f), 0) \in \{\Sigma^2 X, BU(n)\} \oplus \{\Sigma^2 X, MU(n+1)\} \oplus \left\{\Sigma^2 X, \frac{BU}{BU(n+1)}\right\} .$$

This direct sum expression for $\varepsilon_\#(f)$ uses the fact, mentioned earlier, that δ_n really maps $\Sigma^2 BU(n)$ into $BU(n)$ and not into $BU(n+1)$. By Proposition 2.9 $(\delta_n)_\#(f)$ originates in $\{\Sigma^2 X, BU(n)_{2t-2}\}$ which means it goes by (i) to zero in the direct limit. Also $(\gamma_n)_\#(f)$ goes to zero in the limit by (ii) since $\dim \Sigma^2 X - 4n - 4 < \dim X - 4n$.

This completes the induction step. To start the induction we observe that $\{Y, BU(m)_0\}$ is trivial while Φ_U is injective on $\{Y, MU(m)\}$ when $\dim Y \le 4m$, by [Part I, Theorem 5.1].

2.12: Proof of Theorem 2.2. The symplectic case is entirely analogous to the unitary one. Therefore details will be left to the reader. The obstruction theory analogous to Proposition 2.9 is simpler and the argument permits the deformation of $\delta_n : \Sigma^4 BSp(n)_{4m} \to BSp(n+1)_{4m+4}$ into $BSp(n+1)_{4m-4}$ since the obstructions to such a deformation lie in the 4-stem which is zero rather than the 2-stem as they did in the unitary case.

§3. THE SPECTRA AU AND ASp

In this section two commutative ring spectra, AU and ASp, are defined (section 3.2) and the following result is proved.

3.1: Theorem. There exist commutative ring spectra, AU and ASp, defined in section 3.2 and satisfying the following conditions.[2]

Let $AU^*(X)$ and $ASp^*(X)$ be the generalised cohomology algebras corresponding to these spectra.

Then (a) $AU^\circ(X)$ and $ASp^\circ(X)$ are naturally isomorphic when $\dim X < \infty$ to the groups of Theorems 2.1 and 2.2.

(b) The natural homomorphims $\Phi_u : AU^\circ(X) \to MU^{2*}(X)$ and $\phi_{Sp} : ASp^\circ(X) \to MSp^{4*}(X)$ of Theorems 2.1 and 2.2 are ring homomorphisms which are isomorphisms when $\dim X < \infty$.

For details of spectra the reader is referred to [Ad 1, p. 131 et seq.] and, of course, to [W]. In Proposition 3.6 the homology algebras of these spectra are computed.

3.2: Definition. Let $AU_{2k} = \Sigma^2 BU$ and let $(k \geq 1)$ $\varepsilon : \Sigma^2 AU_{2k} \to AU_{2k+2}$ be the map of section 2.3. Let $\eta : S^4 \to \Sigma^2 BU = AU_4$ be given by $\eta = \Sigma^2 B$ where $B \in \pi_2(BU)$ is as in section 2.3. This data defines the spectrum AU with unit.

Similarly let $ASp_{4k} = \Sigma^4 BSp$ $(k \geq 1)$, let $\varepsilon' : \Sigma^4 ASp_{4k} \to ASp_{4k+4}$ be as in section 2.3 and let $\eta' = \Sigma^4 B' : S^8 \to ASp_8$ where $B' \in \pi_4(BSp)$ is as in section 2.3. This data defines the spectrum Asp with unit η'.

I will now define pairings $AU \wedge AU \to AU$ and $ASp \wedge ASp \to ASp$. It suffices [cf. Ad 1, p. 158; W] to define maps

$$m : AU_{2p} \wedge AU_{2q} \to AU_{2p+2q}$$

and

$$m' : ASp_{4p} \wedge ASp_{4q} \to ASp_{4p+4q}$$

satisfying certain properties which will be stated when they are needed.

The map m is defined as follows. $AU_{2p} \wedge AU_{2q}$ is equal to $S^2 \wedge BU \wedge S^2 \wedge BU$ which is homeomorphic, by switching the first two factors, to $BU \wedge \Sigma^2(S^2 \wedge BU)$. If ΣH is the Hopf map of section 2.3 we have an inclusion $1 \wedge \Sigma H : BU \wedge \Sigma^2(S^2 \wedge BU) \to BU \wedge \Sigma^2(S^2 \times BU)$. Switching factors again we have $BU \wedge \Sigma^2(S^2 \times BU) \simeq \Sigma^2(BU \wedge (S^2 \times BU))$ which includes, by the Hopf map again, into $\Sigma^2(BU \times S^2 \times BU)$. m is defined as the composition of the maps I have just described with the double suspension of $1_{BU} \oplus B \oplus 1_{BU} : BU \times S^2 \times BU \to BU$.

The definition of m' is entirely analogous to that of m.

3.3: Remark. Note that in the definition of m the first copy of S^2 in $S^2 \times BU \times S^2 \times BU$ seems to have a privileged role. However, up to homotopy, we might equally well have used the second S^2 factor since the switching map on $S^2 \wedge S^2$ is homotopic to the identity.

3.4: Proof of Theorem 3.1. I will describe only the proof of the unitary case, leaving the analogous symplectic case to the reader.

Firstly AU is clearly a spectrum and therefore defines cohomology groups [Ad 1, p. 196]

$$AU^j(X) = \varinjlim_k [\Sigma^{2k-j}X, AU_{2k}] = \varinjlim_k [\Sigma^{2k-j}X, \Sigma^2 BU] .$$

If $j = 0$ and $\dim X < \infty$ this limit could equally well be taken over stable homotopy classes so that stabilisation induces a natural isomorphism $\varinjlim_k [\Sigma^{2k}X, \Sigma^2 BU] \xrightarrow{\simeq} \varinjlim_k \{\Sigma^{2k}X, \Sigma^2 BU\}$, the group on the right is $AU^\circ(X)$ as defined in section 2.

In order to show that AU is a commutative ring spectrum with unit I must verify the homotopy commutativity of the following diagrams.

(i) pairing

$$\begin{tikzcd}
\Sigma^2 AU_{2p} \wedge AU_{2q} \arrow[r, "\varepsilon \wedge 1"] \arrow[dd, "\lambda"] & AU_{2p+2} \wedge AU_{2q} \arrow[dr, "m"] & \\
& & AU_{2p+2q+2} \\
\Sigma^2(AU_{2p} \wedge AU_{2q}) \arrow[r, "\Sigma^2 m"] & \Sigma^2 AU_{2p+2q} \arrow[ur, "\varepsilon"] &
\end{tikzcd}$$

Here λ is the canonical homeomorphism.

(ii) associativity

$$\begin{tikzcd}
AU_{2p} \wedge AU_{2q} \wedge AU_{2r} \arrow[r, "1 \wedge m"] \arrow[d, "m \wedge 1"] & AU_{2p} \wedge AU_{2q+2r} \arrow[d, "m"] \\
AU_{2p+2q} \wedge AU_{2r} \arrow[r, "m"] & AU_{2p+2q+2r}
\end{tikzcd}$$

(iii) commutativity

$$\begin{tikzcd}
AU_{2p} \wedge AU_{2q} \arrow[dd, "T"] \arrow[dr, "m"] & \\
& AU_{2p+2q} \\
AU_{2q} \wedge AU_{2p} \arrow[ur, "m"] &
\end{tikzcd}$$

Here T is the switching map.

(iv) unit

$$\begin{tikzcd}
S^2 \wedge S^2 \wedge AU_{2p} \cong S^4 \wedge AU_{2p} \arrow[r, "\eta \wedge 1"] \arrow[d, "1 \wedge \varepsilon"] & AU_4 \wedge AU_{2p} \arrow[d, "m"] \\
S^2 \wedge AU_{2p+2} \arrow[r, "\varepsilon"] & AU_{2p+4}
\end{tikzcd}$$

All these diagrams are easy to verify. One uses the associativity and commutativity of the Whitney sum $BU \times BU \to BU$, the associativity of the Hopf construction and Remark 3.3. Therefore I will omit the details.

By the naturality properties of the product it suffices, in order to complete the proof, to show

$$\Phi_u(F) \otimes \Phi_u(G) = \Phi_u(F \wedge G) \in MU^{2*}(X \wedge Y)$$

where $F \in AU^\circ(X)$ and $G \in AU^\circ(Y)$. Let $\sigma \in MU^2(S^2)$ be the suspension class, so that the suspension isomorphism is multiplication by σ. Let $\Phi_u(\_)_{2t}$ denote the component of Φ_u in $MU^{2t}(\_)$. Suppose that F and G are represented by $f \in [\Sigma^{2k}X, \Sigma^2 BU]$ and $g \in [\Sigma^{2\ell}Y, \Sigma^2 BU]$ respectively. Then $\Phi_u(F)_{2t} \in MU^{2t}(X)$ corresponds to $f^*(\sigma \otimes c_{t+k-1}) \in MU^{2t+2k}(\Sigma^{2k}X)$ while $\Phi_u(G)_{2s}$ corresponds to

$g^*(\sigma \otimes c_{s+\ell-1})$. Similarly $\Phi_u(F \wedge G)_{2v}$ corresponds to the image of $\sigma \otimes c_{v+k+\ell-1}$ in $MU^{2v+2k+2}(\Sigma^{2k}X \wedge \Sigma^{2\ell}Y)$ under the map

$$\begin{array}{ccc} \Sigma^{2k}X \wedge \Sigma^{2\ell}Y & \xrightarrow{f \wedge g} & \Sigma^2 BU \wedge \Sigma^2 BU \\ & \swarrow & \\ \Sigma^2 (BU \times S^2 \times BU) & \xrightarrow{\Sigma^2(1_{BU} \oplus B \oplus 1_{BU})} & \Sigma^2 BU. \end{array}$$

Here the unnamed map is the composition of the Hopf maps described in section 2.2. It is easy to compute that this image is

$$\sum_{p=0}^{v+k+\ell-2} f^*(\sigma \otimes c_{v+k+\ell-2-p}) \otimes g^*(\sigma \otimes c_p)$$

$$= \sum_{j=-\ell+1}^{v+k-2} \Phi_u(F)_{2v-2j} \otimes \Phi_u(G)_{2j} .$$

Now letting $\ell, k \to \infty$ we obtain $\Phi_u(F \wedge G) = \Phi_u(F) \otimes \Phi_u(G)$.

3.5: $H_*(AU;Z)$ and $H_*(ASp;Z)$. Let H_* denote integral singular homology. Then by definition the homology groups of AU and ASp are

$$H_j(AU) = \varinjlim_k H_{j+2k}(AU_{2k})$$

and

$$H_j(ASp) = \varinjlim_k H_{j+4k}(ASp_{4k})$$

respectively.

Let $x \in H^2(CP^\infty)$ and $x' \in H^4(HP^\infty)$ be generators. Define $\beta_j \in H_{2j}(CP^\infty)$ and $\beta_j' \in H_{4j}(HP^\infty)$ by

$$\langle \beta_j', (x')^k \rangle = \langle \beta_j, x^k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} . \end{cases}$$

By means of the canonical map $CP^\infty = BU(1) \to BU$ β_j defines a class $\beta_j \in H_{2j}(BU) \cong H_{2j+2}(AU_2)$ which in turn defines $\beta_j \in H_{2j}(AU)$. Similarly β_j' defines a class $\beta_j' \in H_{4j}(ASp)$.

3.6: Proposition. As algebras

$$H_*(AU) \cong Z[\beta_1, \beta_1^{-1}, \beta_2, \beta_3, \ldots]$$

and

$$H_*(ASp) \cong Z[\beta_1', (\beta_1')^{-1}, \beta_2', \beta_3', \ldots] .$$

Proof. I will prove the unitary result and omit the details of the analogous symplectic case.

There are two products in evidence, one in $H_*(BU)$ and one in $H_*(AU)$. $H_*(BU) \cong Z[\beta_1,\beta_2,\ldots]$, is explained in section 2.6 and in this algebra the product of a and b will be denoted by ab. If $a,b \in H_*(AU)$ their product will be denoted by $a * b$. Identify $H_j(AU_{2k})$ with $H_{j-2}(BU)$ by means of the suspension isomorphism. Consider the map

$$m : AU_{2k} \wedge AU_2 \to AU_{2k+2}$$

defined in section 3.2. If $a \otimes b \in H_{s+2}(AU_{2k}) \otimes H_{t+2}(AU_2) \simeq H_s(BU) \otimes H_t(BU)$ then $m_*(a \otimes b)$ corresponds to $\beta_1 ab \in H_{s+t+2}(BU) \simeq H_{s+t+4}(AU_{2k+2\ell})$. This is because the image of the suspension class under $B : S^2 \to BU$ is β_1. From the diagram (iv) of section 3.4 we see that $\varepsilon_* : H_j(\Sigma^2 AU_{2k}) \to H_j(AU_{2k+2})$ corresponds to multiplication by β_1 on $H_{j-4}(BU) \simeq H_j(\Sigma^2 AU_{2k})$. Hence the limit $H_j(AU)$ is formed from a sequence of injections and is therefore, as an additive group, equal to the union of the images of the $H_{j+2k}(AU_{2k}) \simeq H_{j+2k-2}(BU)$.

In this limit $H_*(AU_2)$ is a subalgebra isomorphic to $Z[\beta_1,\beta_2,\ldots]\ H_*(BU)$. For the product of $a \in H_{s+2}(AU_2)$ and $b \in H_{t+2}(AU_2)$ is $a * b = \beta_1 ab \in H_{s+t+4}(AU_4)$ which is the image of $ab \in H_{s+t+2}(AU_2) \in H_{s+t}(BU)$.

To complete the proof it suffices to show that for $k > 1$ and an element $z = \beta_1^{\varepsilon_1}\beta_2^{\varepsilon_2} \ldots \beta_n^{\varepsilon_n} \in H_s(BU) \simeq H_{s+2}(AU_{2k}) \subset H_{s+2-2k}(AU)$ is in the algebra $Z[\beta_1,\beta_1^{-1},\beta_2,\beta_3,\ldots]$. There are three cases

case (a): If $\varepsilon_1 = 1$ then z is the image under ε_* of $\beta_2^{\varepsilon_2} \ldots \beta_n^{\varepsilon_n} \in H_s(AU_{2k-2}) \simeq H_{s-2}(BU)$.

case (b): If $\varepsilon_1 \geq 2$ then $z = \beta_1 * z'$ where $z' = \beta_1^{\varepsilon_1 - 2}\beta_2^{\varepsilon_2} \ldots \beta_n^{\varepsilon_n} \in H_{s-2}(AU_{2k-2}) \simeq H_{s-4}(BU)$.

case (c): If $\varepsilon_1 = 0$ then $\beta_1 * \beta_1 * \cdots * \beta_1 * z$ ($k-1$ copies of β_1) equals $z'' = \beta_2^{\varepsilon_2} \ldots \beta_n^{\varepsilon_n} \in H_*(AU_2)$ so that $z = (\beta_1^{-1})^{k-1} * z''$.

§4: ADAMS OPERATIONS IN AU-THEORY

In this section and in sections 5-7 several well-known phenomena in cobordism will be described in terms of AU- and ASp-theory. In this section I will show how the Adams operations in K-theory give rise to natural endomorphisms of the graded ring AU*( ). Under the isomorphism, Φ_u, of section 2 these operations will be essentially the Adams operations in cobordism theory [No]. This relationship is made precise in the proof of Corollary 4.3.

Let $KU^\circ(\_;\Lambda)$ denote K-theory with coefficients in a ring with identity Λ. This functor is represented by the H-space $BU\Lambda$. By tradition BUZ is simply written BU. The Adams operations

$$\psi^k : KU^\circ(\_;\Lambda) \to KU^\circ(\_;\Lambda)$$

($k \in \Lambda$) are well-known natural ring homomorphisms [Ad 4; At 2] which are induced by H-maps $\psi^k : BU\Lambda \to BU\Lambda$. The values of k for which ψ^k is defined depend on Λ. For example, if $Z \subset \Lambda$ then ψ^k exists for all $k \in Z$, if $\Lambda = Z/p$ then ψ^k

exists if $(k,p) = 1$ [Mau] and if $\Lambda = \hat{Z}_p$ (the p-adics) then ψ^k exists for all $k \in \hat{Z}_p$ [At - T].

The results of section 3 go through with coefficients in Λ. That is, if we define

$$AU\Lambda^j(X) = \varinjlim_n \{\Sigma^{2N-j}X, BU\Lambda\}$$

by analogy with the definition of $AU^j(X)$. There is a natural ring homomorphism (when $\Lambda = \hat{Z}_p$ or $\Lambda \subset \mathbb{Q}$).

$$\Phi_u : AU\Lambda^\circ(X) \to MU\Lambda^{2*}(X) \tag{4.1}$$

which is an isomorphism if $\dim X < \infty$. This follows from Theorem 3.1 when $X = S^M$ since $AU\Lambda^\circ(S^M) = AU^\circ(S^M) \otimes \Lambda$ and $MU\Lambda^{2*}(S^M) = MU^{2*}(S^M) \otimes \Lambda$. For general A the reader follows by induction on dim X. Details will be left to the reader.

The main result of this section is the following:

<u>4.2</u>: <u>Theorem</u>. Suppose that the Adams operation $\psi^k : BU\Lambda \to BU\Lambda$ is defined and $\frac{1}{k} \in \Lambda$. Then ψ^k induces, in a manner described in section 4.4, a natural graded ring homomorphism

$$\Psi^k : AU\Lambda^*(\_) \to AU\Lambda^*(\_)$$

such that

(a) $\Psi^k\Psi^\ell = \Psi^{k\ell}$

(b) The endomorphism of $MU\Lambda^{2*}(S^{2N})$ given by $\Phi_u \circ \Psi^k \circ \Phi_u^{-1}$ is equal to multiplication by k^{N-t} on $MU\Lambda^{2t}(S^{2N})$ $(t \in Z)$. Here Φ_u is the homomorphism of Theorem 3.1 with coefficients in Λ as in (4.1).

(c) Let $w \in AU\Lambda^\circ(CP^T)$ satisfy $\Phi_u(w) = c_1(y) \in MU\Lambda^2(CP^T)$ where y is the Hopf bundle and $c_s(y)$ is the s-th Conner-Floyd class of y. Then

$$\Phi_u(\Psi^k(w)) = \prod_{1\le j} c_j\left(\frac{y^k}{k}\right) \in MU\Lambda^{2*}(CP^T).$$

In section 5 a general construction is given of natural ring homomorphisms out of $AU^\circ(\_)$ which would suffice to construct Ψ^k. However in this section I will give a self-contained treatment of Ψ^k from a slightly different viewpoint which is better suited to the computations we will need.

The operations Ψ^k will be constructed in section 4.4 and the proof of Theorem 4.2 will be given in section 4.10. Before embarking on the project let us recover the cobordism Adams operations.

<u>4.3</u>: <u>Corollary</u>. Under the condition of Theorem 4.2 there exist natural homomorphisms of graded rings $\Psi^k : MU\Lambda^*(\_) \to MU\Lambda^*(\_)$ such that

(a) $\Psi^k \circ \Psi^\ell = \Psi^{k\ell}$

(b) $\Psi^k : MU\Lambda^{2t}(S^{2N}) \to MU\Lambda^{2t}(S^{2N})$ equals multiplication by k^{N-t}.

(c) If $c_1(y) \in MU\Lambda^2(CP^\infty)$ is the first Conner-Floyd class of the Hopf bundle y then $\Psi^k(c_1(y)) = \frac{1}{k}c_1(y^k) \in MU\Lambda^2(CP^\infty)$.

Proof. Since $AU\Lambda^{n}(\_) \simeq AU\Lambda^{n+2}(\_)$ in a natural manner we may identify these groups and consider $AU\Lambda^{\circ}(\_) \oplus AU\Lambda^{1}(\_)$ as a Z/2-graded, multiplicative cohomology theory. Then Φ_u induces a ring homomorphism $\Phi_u : AU\Lambda^{\circ}(\_) \oplus AU\Lambda^{1}(\_) \to MU\Lambda^{*}(\_)$ which is an isomorphism for finite dimensional spaces. If dim $X < \infty$ and $x \in MU\Lambda^{n}(X)$ define $\Psi^{k}(x) \in MU^{n}(X)$ as the image of x under the composite

$$MU\Lambda^{n}(X) \subset MU\Lambda^{*}(X) \xrightarrow{\Phi_u \circ \Psi^{k} \circ \Phi_u^{-1}} MU\Lambda^{*}(X) \xrightarrow{\pi_n} MU\Lambda^{n}(X)$$

where π_n is the projection onto dimension n. The point here is that $\Phi_u \circ \Psi^{k} \circ \Phi_u^{-1}(x) \in \bigoplus_{s \geq n} MU\Lambda^{s}(X)$, since a natural ring homomorphism satisfying (a) - (c) of Theorem 4.2 cannot decrease dimension, but there may be nonzero components in dimension s with $s > n$. In order to produce a graded homomorphism it is necessary to neglect these components in dimension s with $s > n$. Since $\phi_u \circ \Psi^{k} \circ \Phi_u^{-1}$ is a natural ring homomorphism then so is Ψ^{k}. Ψ^{k} is extended to complexes of arbitrary dimension by a standard limit argument an example of which is sketched in [Ad 1, p. 10]. Parts (a) and (b) are immediate from parts (a) and (b) of Theorem 4.2 and part (c) follows from the equation

$$c_1\left(\frac{y^k}{k}\right) = \frac{1}{k}\, c_1(y^k) \in MU\Lambda^{2}(CP^{\infty}) .$$

4.4: Construction of Ψ^{k}. Since $\frac{1}{k} \in \Lambda$ there exists an H-map $\xi_k = \frac{1}{k}\Psi^{k}$: $BU\Lambda \to BU\Lambda$. On $\pi_2(BU) \simeq \Lambda$, Ψ^{k} is multiplication by k so ξ_k is the identity. Hence we have a homotopy commutative diagram.

$$\begin{array}{ccc} S^2 \times BU\Lambda & \xrightarrow{B \oplus 1_{BU\Lambda}} & BU\Lambda \\ {\scriptstyle 1_{S^2} \times \xi_k}\downarrow & & \downarrow{\scriptstyle \xi_k} \\ S^2 \times BU\Lambda & \xrightarrow[B \oplus 1_{BU\Lambda}]{} & BU\Lambda \end{array} \tag{4.5}$$

The following diagram also homotopy commutes

$$\begin{array}{ccc} \Sigma^4 BU\Lambda & \xrightarrow{\Sigma H} & \Sigma^2(S^2 \times BU\Lambda) \\ {\scriptstyle \Sigma^4\xi_k}\downarrow & & \downarrow{\scriptstyle \Sigma^2(1_{S^2} \times \xi_k)} \\ \Sigma^4 BU\Lambda & \xrightarrow{\Sigma H} & \Sigma^2(S^2 \times BU\Lambda) \end{array} \tag{4.6}$$

where ΣH is the map described in section 2.3. Combining the double suspension of (4.5) with (4.6) we obtain a homotopy commutative diagram.

$$\begin{array}{ccc} \Sigma^4 BU\Lambda = \Sigma^2 AU\Lambda_{2N} & \xrightarrow{\varepsilon} & AU\Lambda_{2N+2} = \Sigma^2 BU \\ {\scriptstyle \Sigma^4\xi_k}\downarrow & & \downarrow{\scriptstyle \Sigma^2\xi_k} \\ \Sigma^2 AU\Lambda_{2N} & \xrightarrow{\varepsilon} & AU\Lambda_{2N+2} \end{array} \tag{4.7}$$

From (4.7) it is clear that composition with $\Sigma^2\xi_k$ induces a natural graded, additive endomorphism of $AU\Lambda^*(\_)$ which will be denoted by Ψ^k.

4.8: The Spectrum for $MU\Lambda^{2*}(\_)$. Now let us discuss $MU\Lambda^{2*}(\_)$ and the map $\Phi_u : AU\Lambda^\circ(\_) \to MU\Lambda^{2*}(\_)$ in terms of a spectrum $EU\Lambda$ such that $EU\Lambda^\circ(\_) = MU^{2*}(\_)$. $EU\Lambda$ will be a wedge of suspensions of the Thom spectrum $MU\Lambda$. The object of the discussion is to provide sufficient information so that we can describe in $AU\Lambda$-theory canonical classes in $MU\Lambda$-theory and its homology.

Define $EU\Lambda_{2k} = \bigvee_{1\le n} MU\Lambda_{2n}$ $(k \ge 1)$ where $MU\Lambda_{2n}$ is the 2n-th space in the $MU\Lambda$-spectrum. Let $\varepsilon : \Sigma^2 EU\Lambda_{2k} \to EU\Lambda_{2k+2}$ be defined as the wedge sum of the Thom spectrum maps $\Sigma^2 MU\Lambda_{2n} \to MU\Lambda_{2n+2}$. Then

$$\begin{aligned} EU\Lambda^\circ(X) &= \varinjlim_M [\Sigma^{2M}X, EU\Lambda_{2M}] \\ &\cong \varinjlim_M \bigoplus_{1\le n} [\Sigma^{2M}X, MU\Lambda_{2n}] \\ &\cong \varinjlim_M \bigoplus_{1-M\le k} MU\Lambda^{2k}(X) \\ &\cong MU\Lambda^{2*}(X). \end{aligned}$$

Inside $EU\Lambda_{2k}$ sits $MU\Lambda_{2k}$ thereby giving a canonical copy of the spectrum $MU\Lambda$ as a subspectrum of $EU\Lambda$. The natural transformation

$$\Phi_u : AU\Lambda^\circ(\_) \to MU\Lambda^{2*}(\_)$$

is induced by a map of spectra $AU\Lambda \to EU\Lambda$.

Now let us describe the canonical element $w \in AU\Lambda^\circ(CP^\infty)$ whose restriction to CP^T is featured in Theorem 4.2(c). There is a canonical map $CP^\infty = BU(1) \to BU \to BU\Lambda$ whose double suspension $w' \in [\Sigma^2 CP^\infty, AU\Lambda_2]$ gives a class $w \in AU\Lambda^\circ(CP^\infty)$. Now the composite

$$[\Sigma^2 CP^\infty, AU\Lambda_2] \to AU\Lambda^\circ(CP^\infty) \xrightarrow{\Phi_u} EU\Lambda^\circ(CP^\infty)$$

sends w' to $\prod_{1\le k} (w')^*(c_k')$ where $c_k' \in MU^{2k+2}(AU\Lambda_2)$ is the image of the k-th Conner-Floyd class, c_k, under the suspension isomorphism. However $(w')^*(c_k')$ is zero for $k > 1$. Hence $\Phi_u(w)$ is represented by the composite

$$\Sigma^2 CP^\infty \xrightarrow{\varepsilon} MU(2) \xrightarrow{\rho} MU\Lambda_4 \subset EU\Lambda_4$$

where $MU(2) = MUZ_4$, ε is the Thom spectrum map and ρ is induced by the coefficient homomorphism $Z \to \Lambda$. This composite represents the same class as that represented by $CP^\infty \simeq MU(1) \xrightarrow{\rho} MU\Lambda_2 \subset EU\Lambda_2$. Hence $\Phi_u(w)$ is equal to the canonical element $c_1(y) \in MU\Lambda^2(CP^\infty)$.

Next we turn to the induced homomorphism $(\Phi_u)_* : H_*(AU\Lambda) \to H_*(EU\Lambda)$. Let $\beta_j \in H_{2j}(CP^\infty)$ denote the element of section 3.5. From [Ad 1, p. 51, Lemma 4.5] we know that if then

$$b_i = c_1(y)_*(\beta_{i+1}) \in H_{2i}(MU\Lambda)$$

$$H_*(MU\Lambda) \simeq [b_1, b_2, \ldots] \qquad (b_0 = 1).$$

Hence

$$H_*(EU\Lambda) \simeq \Lambda(u, u^{-1}, b_1, b_2, \ldots)$$

where $u \in H_{-2}(EU\Lambda)$ is represented by $1 \in H_0(MU\Lambda_2) \subset H_0(EU\Lambda_2)$.

<u>4.9</u>: <u>Lemma</u>. In terms of the generators described in sections 3.5 and 4.8

$$(\Phi_u)_* : H_*(AU\Lambda) \to H_*(EU\Lambda)$$

is given by

$$(\Phi_u)_*(\beta_1^{-1}) = u$$

and

$$(\Phi_u)_*(\beta_j\beta_1^{-1}) = b_{j-1} \qquad (j \geq 1).$$

<u>Proof</u>. From Proposition 3.6 case (c) we see that $\beta_1^{-1} \in H_{-2}(AU\Lambda)$ is represented by $1 \in H_0(BU) \simeq H_2(AU\Lambda_4)$. Since this class is carried by $\Sigma^2CP^\infty \to \Sigma^2BU\Lambda = AU\Lambda_4$ the analysis of w in section 4.8 shows that $(\Phi_u)_*(\beta_1^{-1})$ is represented by

$$\varepsilon_*(1) \in H_2(MU\Lambda_4) \subset H_2(EU\Lambda_4)$$

where $\varepsilon_* : H_0(MU\Lambda_2) \to H_2(MU\Lambda_4)$ is induced by the Thom spectrum map. Therefore $(\Phi_u)_*(\beta_1^{-1})$ is represented by $1 \in H_0(MU\Lambda_2) \subset H_0(EU\Lambda_2)$ and $(\Phi_u)_*(\beta_1^{-1}) = u$.

Also $\beta_j \in H_{2j}(BU\Lambda) \simeq H_{2j+2}(AU\Lambda_4)$ represents the product $\beta_j * \beta_1^{-1}$. A computation similar to the analysis of $(\Phi_u)_*(\beta_1^{-1})$ shows that $(\Phi_u)_*(\beta_j\beta_1^{-1})$ is represented by $(c_1(y))_*(\beta_j) \in H_{2j}(MU\Lambda_2) \subset H_{2j}(EU\Lambda_2)$. Hence $(\Phi_u)_*(\beta_j\beta_1^{-1}) = b_{j-1}$.

<u>4.10</u>: <u>Proof of Theorem 4.2</u>. Firstly we show that Ψ^k is multiplicative. It suffices by naturality to show that if $f \in AU\Lambda^\circ(X)$ and $g \in AU\Lambda^\circ(Y)$ then

$$\Psi^k(fg) = \Psi^k(f)\Psi^k(g) \in AU\Lambda^\circ(X \wedge Y).$$

Recall from section 3.2 that $m : AU\Lambda_{2p} \wedge AU\Lambda_{2q} \to AU\Lambda_{2p+2q}$ is equal to a composition of the form

$$\begin{array}{r} S^2 \wedge BU\Lambda \wedge S^2 \wedge BU\Lambda \to \Sigma^2(BU\Lambda \times S^2 \times BU\Lambda) \\ \Sigma^2(1_{BU\Lambda} \oplus B \oplus 1_{BU\Lambda}) \downarrow \quad \\ \Sigma^2 BU \ . \end{array}$$

Since $\xi_k : BU\Lambda \to BU\Lambda$ is an H-map which induces the identity on $\pi_2(BU\Lambda)$ we have

$$\xi_k \circ (1_{BU\Lambda} \oplus B \oplus 1_{BU\Lambda}) \simeq (1_{BU\Lambda} \oplus B \oplus 1_{BU\Lambda}) \circ (\xi_k \times 1_{S^2} \times \xi_k). \tag{4.11}$$

Since

$$\begin{CD} S^2 \wedge BU\Lambda \wedge S^2 \wedge BU\Lambda @>>> \Sigma^2(BU\Lambda \times S^2 \times BU\Lambda) \\ @V{1_{S^2} \wedge \xi_k \wedge 1_{S^2} \wedge \xi_k}VV @VV{\Sigma^2(\xi_k \times 1_{S^2} \times \xi_k)}V \\ S^2 \wedge BU\Lambda \wedge S^2 \wedge BU\Lambda @>>> \Sigma^2(BU\Lambda \times S^2 \times BU\Lambda) \end{CD} \tag{4.12}$$

is homotopy commutative we may combine the double suspension of (4.11) with (4.12) to yield

$$\Sigma^2\xi_k \circ m \simeq m \circ (\Sigma^2\xi_k \wedge \Sigma^2\xi_k) : AU\Lambda_{2p} \wedge AU\Lambda_{2q} \to AU\Lambda_{2p+2q}$$

which at once implies that Ψ^k is multiplicative.

Part (a) is clear since the same identity is true for Adams operations in K-theory.

On $H_{2\ell}(BU\Lambda)$, Ψ^k_* is multiplication by k^ℓ and $(\xi_k)_*$ is multiplication by $k^{\ell-1}$. Hence on $H_*(AU\Lambda)$ $(\Psi^k)_*(\beta_\ell) = k^{\ell-1}\beta_\ell$. Therefore by Lemma 4.9

$$(\Phi_u \circ \Psi^k \circ \Phi_u^{-1})_*(x) = \begin{cases} k^j b_j & \text{if } x = b_j \\ u & \text{if } x = u. \end{cases}$$

However from [14] we know that $MU^*(S^{2N})$ is torsion free. Therefore the Boardman-Hurewicz homomorphism $h : EU^\circ(S^{2N}) \to H_*(EU)$ is an injection. If $x \in MU^{2t}(S^{2N}) \subset EU^\circ(S^{2N})$ it is represented by an element of $[S^{2N+2M}, MU(M+t)] \subset [S^{2N+2M}, EU_{2M}]$ for some M. Hence $h(x) \in H_{2N}(EU)$ is represented by a class of $H_{2N+2M}(MU(M+t))$. This element must be of the form pu^{-t} where p is a polynomial in $b_1, b_2, \ldots$ of degree $2N - 2t$. Hence $(\Phi_u \circ \Psi^k \circ \Phi_u^{-1})_*(h(x)) = k^{N-t}h(x)$. This proves part (b) when $\Lambda = Z$. Part (b) in general follows from the facts that Ψ^k is in general constructed from Ψ^k with k integral and that $MU\Lambda^{2t}(S^{2N})$ consists of integral classes.

It remains to prove part (c). From the discussion of the canonical element $w \in AU\Lambda^\circ(CP^\infty)$ in section 4.8 and the definition of Φ_u we have at once that

$$\Phi_u(\Psi^k(w)) = \prod_{1 \le k} (\Sigma^2\xi_k \circ w')^*(c_k') \in MU\Lambda^{2*}(\Sigma^2 CP^\infty)$$

where $c_k' \in MU\Lambda^{2K+2}(\Sigma^2 BU)$ and w' are as in section 4.8. However $(\Sigma^2\xi_k w')^*(c_k') \in MU\Lambda^{2k+2}(\Sigma^2 CP^\infty)$ corresponds under suspension to $c_k\left(\frac{y^k}{k}\right) = \left(\frac{\Psi^k(y)}{k}\right)^*(c_k) \in MU\Lambda^{2k}(CP^\infty)$ and the result follows.

§5. IDEMPOTENTS

This section studies idempotents in AU-theory and hence in MU-theory. In Proposition 5.3 a general construction is given which associates to an exponential natural transformation from K-theory to another theory, h*, a natural ring homomorphism from AU-theory to h*. The homomorphisms Φ_u of section 2 and Ψ^k of section 4 may be obtained from this construction (see Examples 5.4.3 and 5.4.3(a)). This construction yields an idempotent $\varepsilon(d) : AU\Lambda^*(\_) \to AU\Lambda^*(\_)$ and an endomorphism $q(d) : AU\Lambda^*(\_) \to AU\Lambda^*(\_)$ for suitable Λ (Theorem 5.1 and Proposition 5.5). $\varepsilon(d)$ induces in MU-theory the idempotent of Adams [Ad 3, p. 107] while q(d) induces the idempotent of Quillen [Ad 1, p. 105].

Some natural endomorphisms of AU-theory may be constructed entirely from K-theory, for example Ψ^k of section 4, others may not. q(d) is an example of the latter kind. I think it is important to remark what can be accomplished in cobordism by use of AU-theory and without prior knowledge of $\pi_*(MU)$ or the use of Adams spectral sequences since analogous arguments may then be possible for ASp-theory. For example the KO*-theory (= KSp^{*-4}-theory) Adams operations will yield Ψ^k operations in ASp-theory. For idempotents the situation is as follows. Adams constructs $\varepsilon(d)$ by use of the Hattori-Stong theorem [Ad-L; Ha; St 2] which in turn has been proved by appealing to the structure of $\pi_*(MU)$ or by use of Adams spectral sequences. The AU-theory construction, which also immediately relates $\varepsilon(d)$ to the Ψ^k's, requires only the K-theory idempotent E_1 of Adams as input. The construction of E_1 requires only easy number theory and easy K-theory [Ad 3, pp. 84-89]--and ingenuity! The formula relating $\varepsilon(d)$ to the Adams operations was originally due to Idar Hansen.

The main result of this section is the following:

<u>5.1</u>: <u>Theorem</u>. Let $d > 1$ be an integer and let $R(d)$ be the ring of fractions a/b such that b contains no prime p with $p \equiv 1(d)$. Then the idempotent of Adams [Ad 3, p. 89]

$$E_1 : BUR(d) \to BUR(d)$$

induces a natural idempotent ring homomorphism

$$\varepsilon(d) : AUR(d)^*(\_) \to AUR(d)^*(\_)$$

such that

(i) if $p \equiv 1(d)$ is a prime then $\varepsilon(d)$ induces

$$\varepsilon(d) : AU\hat{Z}_p^*(\_) \to AU\hat{Z}_p^*(\_)$$

satisfying

$$[\varepsilon(d)(f)]^d = \prod_{j=1}^{d} \Psi^{\alpha_j}(f) \in AU\hat{Z}_p^{\circ}(X)$$

$$(f \in AU\hat{Z}_p^{\circ}(X))$$

(ii) if $\dim X < \infty$ then the composite

$$\begin{tikzcd} \varepsilon : MUR(d)^n(X) \arrow[r] & MUR(d)^*(X) \arrow[dl, "\phi_u \circ \varepsilon(d) \circ \phi_u^{-1}"] \\ MUR(d)^*(X) \arrow[r, "proj"] & MUR(d)^n(X) \end{tikzcd}$$

is equal to the idempotent of Adams [Ad 3, p. 107]. Also with p-adic coefficients as in (i) ε satisfies

$$\varepsilon(f)^d = \prod_{j=1}^{d} \Psi^j(f)$$

$$(f \in MU\hat{Z}_p^{2n}(X)).$$

Here $\alpha_1, \ldots, \alpha_d \in \hat{Z}_p$ are distinct d-th roots of unity and ψ^{α_j}, Ψ^{α_j} denote Adams operations as in section 4.

Proof. I will assume that the reader is familiar with the constructions of $E_1 : BUR(d) \to BUR(d)$ [Ad 3, pp. 84-89].

The existence of $\varepsilon(d)$ is given by Example 5.4.2(b) or by imitation of the construction of ψ^k in section 4. The second viewpoint shows that $\varepsilon(d)$ is idempotent since it is given by composition with $\Sigma^2 E_1 : \Sigma^2 BUR(d) = AUR(d)_{2N} \to AUR(d)_{2N}$ and $E_1 \circ E_1 \simeq E_1$.

It is immediate from the definition of E_1 that

$$(E_1)_* : H_*(BUR(d);Q) \to H_*(BUR(d);Q)$$

satisfies

$$(E_1)_*(x_n) = \begin{cases} x_n & n = kd + 1,\ k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $x_n \in H_{2n}(BUR(d);Q)$ is primitive. Arguing as in section 4.10 we see that

$$\varepsilon(d)_* : H_*(EUR(d);Q) \to H_*(EUR(d);Q)$$

is given by

$$\varepsilon(d)_*(u^s y_n) = \begin{cases} u^s y_n & n = kd,\ k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $y_n \in H_{2n}(EUR(d);Q)$ is a primitive polynomial in $(b_i; i \geq 1)$. This establishes the first half of (ii) since the idempotent of Adams is characterised by this behaviour on $H_*(MUR(d);Q) \subset H_*(EUR(d);Q)$.

Now we relate $\varepsilon(d)$ to the Adams operations. Clearly the formula in part (ii) is implied by that in part (i) and Corollary 4.3. From the formula of Adams [Ad 3, p. 89] we have

$$E_1 = \frac{1}{d} \sum_{j=1}^{d} \frac{1}{\alpha_j} \Psi^{\alpha_j} : BU\hat{Z}_p \to BU\hat{Z}_p . \tag{5.2}$$

Let f be represented by $f \in \{\Sigma^{2M}X, (AU\hat{Z}_p)_{2M}\}$. Then $(\varepsilon(d)(f))^d$ is represented by a composition of the form

$$\begin{tikzcd}
 & \Sigma^{2Md}X \arrow[r, "\text{diag}"] & \Sigma^{2M}X \wedge \cdots \wedge \Sigma^{2M}X \arrow[dll, "f \wedge \cdots \wedge f"] \\
\Sigma^2 BU\hat{Z}_p \wedge \cdots \wedge \Sigma^2 BU\hat{Z}_p \arrow[rr, "g_1"] & & \Sigma^2(BU\hat{Z}_p \times S^2 \times \cdots \times S^2 \times BU\hat{Z}_p) \arrow[dll, "\Sigma^2(E_1 \times 1_{S^2} \times \cdots \times E_1)"] \\
\Sigma^2(BU\hat{Z}_p \times \cdots) \arrow[rr, "g_2"'] & & \Sigma^2 BU\hat{Z}_p = (AU\hat{Z}_p)_{2Md}.
\end{tikzcd}$$

Here "diag" is induced by the diagonal map, g_1 is a composition of Hopf maps as in section 2.3, g_2 is induced by the iterated Whitney sum and there are d copies of X and of $BU\hat{Z}_p$ in each of the displayed products. However by (5.2) we may replace $\Sigma^2(E_1 \times 1_{S^2} \times \cdots)$ in this composition by $\Sigma^2(\xi_{\alpha_1} \times 1_{S^2} \times \cdots \times \xi_{\alpha_d})$, the resulting composition represents $\prod_{j=1}^{d} \Psi^{\alpha_j}(f)$. This completes the proof.

<u>5.3</u>: <u>Proposition</u>. Let $\alpha : KU\Lambda^\circ(\_) \to h^*(\_)$ be a natural transformation into a multiplicative cohomology theory h^* satisfying

(i) For $f,g \in KU\Lambda(X)$

$$\alpha(f + g) = \alpha(f)\alpha(g) \in h^*(X)$$

and (ii) if $B \in KU\Lambda^\circ(S^2)$ is the Bott class then the projection of $\alpha(B)$ into reduced h^*-theory is equal to $\sigma \in h^2(S^2)$ where σ is the image of $1 \in h^\circ(S^\circ)$ under the suspension isomorphism.

Then α induces a natural ring homomorphism $\hat{\alpha} : AU\Lambda^\circ(\_) \to h^*(\_)$. If $F \in AU\Lambda^\circ(X)$ is represented by $f \in \{\Sigma^{2N}X, AU\Lambda_{2N}\}$ then

$$\hat{\alpha}(F) = f^*(\sigma \otimes \alpha) \in h^*(\Sigma^{2N}X) \simeq h^{*-2N}(X)$$

where α is considered as an element of $h^*(BU\Lambda)$.

<u>Proof</u>. Firstly we must check that $\hat{\alpha}$ is well-defined. F is represented by $f \in \{\Sigma^{2N}X, \Sigma^2 BU\Lambda\}$ which is sent to $f^*(\sigma \otimes \alpha) \in h^*(\Sigma^{2N}X) \simeq h^{*-2N}(X)$. However F may be considered as being represented by $\varepsilon_*(f) \in \{\Sigma^{2N+2}X, \Sigma^2 BU\Lambda\}$ where $\varepsilon : \Sigma^2 AU\Lambda_{2N} \to AU\Lambda_{2N+2}$ is described in section 2.3. This second representative is sent to

$$\sigma \otimes f^*(B^*(\alpha) \otimes \alpha) = \sigma \otimes f^*(\sigma \otimes \alpha) \in h^{*+2}(\Sigma^{2N+2}X).$$

Fortunately these two elements are equal when translated to $h^{*-2N}(X)$ and $\hat{\alpha}$ is well-defined.

Now we show $\hat{\alpha}$ is multiplicative. Suppose $f \in \{\Sigma^{2N}X, \Sigma^2 BU\Lambda\}$ and $g \in \{\Sigma^{2M}Y, \Sigma^2 BU\}$ represent $F \in AU\Lambda^\circ(X)$ and $G \in AU\Lambda^\circ(Y)$ respectively. Then fg is represented by

$$\begin{array}{c} \Sigma^{2M+2N}X \wedge Y \simeq \Sigma^{2M}X \wedge \Sigma^{2N}Y \xrightarrow{f \wedge g} \Sigma^2 BU\Lambda \wedge \Sigma^2 BU\Lambda \\ \swarrow \\ \Sigma^2(BU\Lambda \times S^2 \times BU\Lambda) \xrightarrow{\Sigma^2(1_{BU\Lambda} \times B \times 1_{BU\Lambda})} \Sigma^2 BU. \end{array}$$

Hence $\hat{\alpha}(fg) = (f \wedge g)^*(\sigma \otimes \alpha \otimes B^*(\alpha) \otimes \alpha) = f^*(\sigma \otimes \alpha) \otimes g^*(\sigma \otimes \alpha) = \hat{\alpha}(f)\hat{\alpha}(g)$.

<u>5.4.1</u>: <u>Example</u>. Define $\bar{\nu} : KU\Lambda^\circ(X) \to AU\Lambda^\circ(X) = \varinjlim_N \{\Sigma^{2N}X, AU\Lambda_{2M}\}$ by sending $f : X \to BU\Lambda$ to $\Sigma^2 f : \Sigma^2 X \to \Sigma^2 BU\Lambda = AU\Lambda_2$. Clearly $\bar{\nu}(B) \in AU\Lambda^\circ(S^2) \simeq MU\Lambda^{2*}(S^2)$ is the suspension class in $MU\Lambda^2(S^2)$. Set $\nu(f) = 1 + \bar{\nu}(f)$.

Then ν is an exponential map. Suppose $g \in KU\Lambda^{\circ}(Y)$ is represented by $g : Y \to BU\Lambda$ then $\bar{\nu}(f+g)$ is represented by a composite of the form

$$\Sigma^2(X \times Y) \xrightarrow{\Sigma^2(f \times g)} \Sigma^2(BU\Lambda \times BU\Lambda) \xrightarrow{\Sigma^2(1_{BU} \oplus 1_{BU})} \Sigma^2 BU\Lambda = AU\Lambda_2 \qquad \text{(I)}$$

However $\bar{\nu}(f)\bar{\nu}(g)$ is represented by a composite of the form

$$\begin{tikzcd} & \Sigma^2 X \wedge \Sigma^2 Y \arrow[r] & \Sigma^2 BU\Lambda \wedge \Sigma^2 BU\Lambda \arrow[dl, "g'"] & \\ \Sigma^2(BU\Lambda \times S^2 \times BU\Lambda) \arrow[rr, "\Sigma^2(1_{BU\Lambda} \oplus B \oplus 1_{BU\Lambda})"] & & \Sigma^2 BU\Lambda = AU\Lambda_4 . \end{tikzcd} \qquad \text{(II)}$$

Here g' is a Hopf map as described in section 2.3. It is easy to see that if π_X and π_Y are the projections $X \leftarrow X \times Y \to Y$ then (I) represents

$$\bar{\nu}(f)\bar{\nu}(g) + \bar{\nu}(f \circ \pi_X) + \bar{\nu}(g \circ \pi_Y) .$$

Hence

$$\begin{aligned} \nu(f + g) &= 1 + \bar{\nu}(f + g) \\ &= 1 + \bar{\nu}(f) + \bar{\nu}(g) + \bar{\nu}(f)\bar{\nu}(g) \\ &= \nu(f)\nu(g) . \end{aligned}$$

Applying Proposition 5.3 to ν yields the identity map of $AU\Lambda^{\circ}(\_)$.

5.4.2: Example. Applying Proposition 5.3 to $\nu = c_0 + c_1 + c_2 + \cdots \in \prod_{0 \leq k} MU^{2k}(\_)$, the total Conner-Floyd class, yields the homomorphism Φ_u of Theorem 3.1.

In fact we may construct a homomorphism $AU\Lambda^{\circ}(\_) \to h^*(\_)$ in this way for any cohomology theory, h^*, which has a "total Chern class" for complex bundles with coefficients in Λ (cf. [2, p. 55]). For $h^* = KU^*$ this is the Conner-Floyd homomorphism [C-F].

The important property of the Conner-Floyd homomorphism is that it induces an isomorphism

$$MU^*(\_) \otimes_{\pi_* MU} Z \to KU^*(\_)$$

of Z/2-graded rings [C-F] from which incidentally the Hattori-Stong theorem follows as a (quite difficult) corollary by an argument of G. Wolff [Wol].

The methods of AU-theory do give a new proof of the unitary Conner-Floyd theorem. Furthermore, the Conner-Floyd theorem together with AU-theory lead to a rather startling description of BU. This application is given in Part II, section 9. See Remark 9.2.9(b) for the proof of the Conner-Floyd theorem.

Incidentally the real/symplectic Conner-Floyd homomorphism is obtainable

$$MSp^{4*}(\_) \to KO(\_)$$

by a similar method using ASp-theory. Details are left to the reader.

5.4.3: Example. If $\delta : KU\Lambda^{\circ}(\_) \to KU\Lambda^{\circ}(\_)$ is a natural additive homomorphism which is the identity on $KU\Lambda^{\circ}(S^2)$ then Proposition 5.3 may be applied to

$\alpha = \nu \circ \delta$. Here ν is as in Example 5.4.1.

(a) If $\delta = \frac{\psi^k}{k} = \xi_k$ then $\hat{\alpha} = \Psi^k$.

(b) If $\delta = E_1$ the idempotent of Adams [Ad 3, p. 89] in $KUR(d)^\circ(\_)$ then $\alpha = \varepsilon(d)$, the idempotent of Theorem 5.1.

5.4.4: Example. Notice from the splitting principle that a natural exponential map $\alpha : KU\Lambda^\circ(\_) \to MU\Lambda^*(\_)$ determines and is determined by $\alpha(y-1) \in MU\Lambda^*(CP^\infty)$ where y is the Hopf bundle over CP^∞ (cf. [Ad 1, p. 52, Lemma 4.6]).

If $x \in MU^2(CP^\infty)$ is the canonical class them $MU\Lambda^*(CP^\infty) = \pi_*(MU\Lambda)[[x]]$. Following [Ad 1, p. 108] define

$$\operatorname{mog} x = \log x - \frac{1}{d}\sum_{j=1}^{d} \log(\alpha_i x) \qquad (d > 1)$$

where $\alpha_1, \ldots, \alpha_d$ are distinct complex d-th roots of unity and

$$\log x = \sum_{i \geq 0} \frac{[CP^i]}{(i+1)} x^{i+1} \in MUQ^*(CP^\infty).$$

Hence mog x = x + (higher terms) and B*(mog x) equals the suspension class in $MUQ^2(S^2)$. In fact [Ad 1, pp. 108-109]

$$\operatorname{mog} x \in MUZ[\tfrac{1}{d}]^{2*}(CP^\infty) \underset{\sim}{} AUZ[\tfrac{1}{d}]^\circ(CP^\infty).$$

Thus by Proposition 3 mog x determines a natural ring homomorphism

$$q(d) : AUZ[\tfrac{1}{d}]^*(\_) \to AUZ[\tfrac{1}{d}]^*(\_).$$

5.5: Proposition. Let q(d) be the endomorphism of $AUZ[\frac{1}{d}]^*(\_)$ constructed in Example 5.4.4. Suppose dim $X < \infty$.

Then the composite, also called q(d),

$$\begin{tikzcd} & MUZ[\tfrac{1}{d}]^n(X) \subset MUZ[\tfrac{1}{d}]^*(X) \arrow[dl, "\Phi_u \circ q(d) \circ \Phi_u^{-1}"] \\ MUZ[\tfrac{1}{d}]^*(X) \arrow[r, "proj"] & MUZ[\tfrac{1}{d}]^n(X) \end{tikzcd}$$

is the idempotent of Quillen [Ad 1, Theorem 15.1, p. 105].

Proof. Quillen's idempotent is characterised by the fact that it is a ring homomorphism which sends $x \in MUZ[\frac{1}{d}]^2(CP^\infty)$ to mog x. Hence it is equal to the composite

$$MUZ[\tfrac{1}{d}]^n(X) \subset MUZ[\tfrac{1}{d}]^*(X) \xrightarrow{\Phi_u \circ q(d) \circ \Phi_u^{-1}} MUZ[\tfrac{1}{d}]^*(X).$$

§6. THE COMPLEXIFICATION HOMOMORPHISM

The complexification homomorphism $MSp^*(\_) \to MU^*(\_)$ is well-known. By analogy it is reasonable to demand a natural ring homomorphism from $ASp^\circ(X)$ to $AU^\circ(X)$ and to expect that it should be induced by the natural H-map $c : BSp \to BU$.

Unfortunately the obvious map $\{\Sigma^{4n}X,BSp\} \to \{\Sigma^{4n}X,BU\}$, induced by c, is not compatible with the limits whereby $ASp^\circ(X)$ and $AU^\circ(X)$ are formed. However a homomorphism of approximately the right type is available.

Unfortunately the H-map $h : BU \to BSp$ given by symplectification does not induce a homomorphism $AU^*(\_) \to ASp^*(\_)$. This is because AU is constructed using a generator of $\pi_2(BU)$ while ASp uses a generator of $\pi_4(BSp)$ which cannot be expressed in terms of the image of $\pi_2(BU)$ since $\pi_2(BSp) = 0$. Therefore h induces only the trivial map from the AU-spectrum to the ASp-spectrum. This remark will become clearer when the reader has seen below the sort of compatibility conditions that $c : BSp \to BU$ satisfies, which make for an interesting homomorphism from $ASp^*(\_)$ to $AU^*(\_)$.

<u>6.1</u>: <u>Theorem</u>. The complexification H-map $c : BSp \to BU$ induces a natural ring homomorphism

$$c : ASp^\circ(X) \to AU^\circ(X)[(1 - \alpha_{11})^{-1}].$$

Here $\alpha_{11} \in AU^\circ(S^\circ)$ satisfies $\Phi_u(\alpha_{11}) = a_{11}$ where $a_{11} \in MU^2(S^\circ)$ is the coefficient of $x_1 \otimes x_2$ in the formal group law for MU-theory in the notation of [Ad. 1, p. 40].

<u>6.2</u>: <u>Remark</u>. The proof of Theorem 6.1 is straightforward and will be given in section 6.5. One proceeds to analyse how the maps induced by $c : BSp \to BU$ fail to be compatible. Doing this one finds in the diagram

$$\begin{array}{ccc} \{\Sigma^{4k}X,BSp\} & \xrightarrow{\varepsilon'} & \{\Sigma^{4k+4}X,BSp\} \\ \downarrow{\scriptstyle c_\#} & & \downarrow{\scriptstyle c_\#} \\ \{\Sigma^{4k}X,BU\} & \xrightarrow[\varepsilon \circ \varepsilon]{} & \{\Sigma^{4k+4}X,BU\} \end{array}$$

that one route is $(1-\alpha_{11})$ times the other. Hence $(1-\alpha_{11})^{-j}c_\#$, for suitable j, will be compatible.

The localisation is not a serious restriction. For if we were mapping not into $AU^\circ(X) \simeq MU^{2*}(X)$ but into $\prod_{k=-\infty}^{\infty} MU^{2k}(X)$ the element $(1-a_{11})$ would be a unit already.

<u>6.3</u>: <u>Some Elements in $\pi_*^S(BU)$</u>. Here I will use the notation of [Part I, section 6]. By suspension the Bott map $B \in \pi_2(BU)$ yields $x \in \pi_2^S(BU)$. The direct sum on BU induces a product in $\pi_*^S(BU)$ for which $x^2 \in \pi_4^S(BU)$ will denote the square of x. Also the tensor product on BU induces a product on $\pi_*^S(BU)$ for which the square of x is written $x * x \in \pi_4^S(BU)$. In [Part I, Section 6] I showed $\dfrac{\pi_4^S(BU)}{(\text{odd torsion})} \simeq Z \oplus Z$ but the method in fact shows that $\pi_4^S(BU) \simeq Z \oplus Z$ generated by x^2 and $x * x$. The Hurewicz map

$$h : \pi_4^S(BU) \to H_4(BU)$$

satisfies $h(x^2) = \beta_1^2$, $h(x * x) = 2\beta_2$ [Part I, section 6]. $a_{11} \in \pi_2(MU)$ is representable by an S-map $a_{11} : S^4 \to MU(1)$. Using the S-equivalence between BU and $\underset{1\leq k}{\vee} MU(k)$ [Part I] a_{11} may be considered as an element of $\pi_4^S(BU)$. This element is $x * x$.

<u>6.4</u>: <u>Lemma</u>. The generator $B' \in \pi_4(BSp)$, suitably chosen, satisfies

$$c_{\#}(B') = x^2 - x * x \in \pi_4^S(BU).$$

<u>Proof</u>. Using the injectivity of the Hurewicz homomorphism on $\pi_4^S(BU)$ as described in Section 6.3 it suffices to show that the Hurewicz image of $c_{\#}(B')$ is $\beta_1^2 - 2\beta_2 \in H_4(BU)$. This follows from the computations of [Ad 1, pp. 93-98] since x^2 hits β_1^2 and $x * x$ hits $2\beta_2$.

<u>6.5</u>: <u>Proof of Theorem 6.1</u>. Let $A_{4N} = AU_{4N}$ and $\delta = \varepsilon \circ \varepsilon : \Sigma^4 A_{4N} \to A_{4n+4}$. This spectrum will suffice to define AU-theory (cf. section 3). Assigning to $g \in \{\Sigma^{4k}X, ASp_{4k}\}$ the composition $(\Sigma^4 c) \circ g \in \{\Sigma^{4k}X, A_{4k}\}$ defines an additive homomorphism

$$\phi_k : \{\Sigma^{4k}X, ASp_{4k}\} \to \varinjlim_k \{\Sigma^{4k}X, A_{4k}\} = AU^\circ(X).$$

The element g above represents the same element in $ASp^\circ(X)$ as the composite

$$\begin{tikzcd} g' : \Sigma^{4k+4}X \arrow[r, "\Sigma^4 g"] & \Sigma^4(S^4 \wedge BSp) \arrow[dl, "\Sigma H"] & \\ \Sigma^4(S^4 \times BSp) \arrow[r, "\Sigma^4(B' \oplus 1_{BSp})"'] & \Sigma^4 BSp = ASp_{4k+4} . \end{tikzcd} \tag{6.6}$$

Now let us compute $\phi_{k+1}(g')$. By Lemma 6.4 and the fact that c is an H-map we have a commutative diagram of S-maps.

$$\begin{tikzcd} \Sigma^4(S^4 \times BSp) \arrow[r, "\Sigma^4(B' \oplus 1_{BSp})"] \arrow[d, "\Sigma^4(1_{S^4} \times c)"] & \Sigma^4 BSp \arrow[d, "\Sigma^4 c"'] \\ \Sigma^4(S^4 \times BU) \arrow[r, "\Sigma^4((x^2 - x * x) \oplus 1_{BU})"'] & \Sigma^4 BU \end{tikzcd} \tag{6.7}$$

Notice the x^2 in (6.7) is equal to the S-map

$$S^2 \wedge S^2 \to S^2 \times S^2 \xrightarrow{B \oplus B} BU,$$

a composition in which the first map is a Hopf map (cf. section 2.3).

Combining (6.6) and (6.7) we see that $\phi_{k+1}(g')$ is represented by a composite of the form

$$\begin{tikzcd}[column sep=huge] \Sigma^{4k+4}X \arrow[r, "\Sigma^4(\Sigma^4 c\circ g)"] & \Sigma^4(S^4 \wedge BU) \simeq \Sigma^4 A_{4k} \arrow[dl] \\ \Sigma^{4k+4}X \arrow[r, "\Sigma^4((x^2 - x*x)\oplus 1_{BU})"] & \Sigma^4 BU = A_{4K+4}\,. \end{tikzcd}$$

This is clearly the product of $\phi_k(g)$ with the class of $x^2 - x * x \in \{S^4, AU_4\}$ in $AU^\circ(S^\circ)$. The element x^2 represents $1 \in AU^\circ(S^\circ)$. The map $x * x$ is represented by an S-map $x * x \in \pi_4^S(BU(1)) \subset \{S^4, AU_4\}$. Hence $\phi_u(x * x) = c_1(x * x) \in MU^2(S^4) \simeq \pi_2(MU)$. From its Hurewicz image, $2b_1 \in H_2(MU)$, it is clear from section 6.3 that $\phi_u(x * x) = a_{11}$. Thus $\phi_{k+1}(g') = (1 - \alpha_{11})\phi_k(g)$ and the homomorphisms $(k \geq 0)$

$$\tau_{k+1} = (1 - \alpha_{11})^{-k}\phi_{k+1} : \{\Sigma^{4k+4}X, ASp_{4k+4}\} \to AU^\circ(X)[(1 - \alpha_{11})^{-1}]$$

are compatible. They induce an additive homomorphism

$$c : ASp^\circ(X) \to AU^\circ(X)[(1 - \alpha_{11})^{-1}].$$

Next we show that c is multiplicative. Let $f \in \{\Sigma^{4M+4}X, \Sigma^4 BSp\}$ and $g \in \{\Sigma^{4N+4}Y, \Sigma^4 BSp\}$. Then $c(fg)$ is represented by $(1 - \alpha_{11})^{-N-M-1}$ times a composite of the form

$$\begin{tikzcd}[column sep=huge] & \Sigma^{4M+4}X \wedge \Sigma^{4N+4}Y \arrow[r, "f\wedge g"] & \Sigma^4 BSp \wedge \Sigma^4 BSp \arrow[dll] \\ \Sigma^4(BSp \times S^4 \times BSp) \arrow[rr, "\Sigma^4(1_{BSp}\oplus B'\oplus 1_{BSp})"] & & \Sigma^4 BSp \arrow[dl, "\Sigma^4 c"] \\ & \Sigma^4 BU = A_{4N+4M+8} & \end{tikzcd}$$

From (6.7) it is easy to see that this composition is homotopic to $(1 - \alpha_{11})$ times the product of

$$\Sigma^{4M+4}X \xrightarrow{\Sigma^4 c\circ f} \Sigma^4 BU = A_{4M+4}$$

with

$$\Sigma^{4N+4}Y \xrightarrow{\Sigma^4 c\circ g} \Sigma^4 BU = A_{4N+4}\,.$$

Hence $\phi_{M+1}(f)\phi_{N+1}(g) = (1 - \alpha_{11})\phi_{M+N+2}(fg)$ and $c(fg) = c(f)c(g)$.

§7. LANDWEBER-NOVIKOV OPERATIONS AND THE THOM ISOMORPHISM

In this section the Landweber-Novikov operations [Ad 1, p. 12; La] will be described in terms of AU-theory. All the results of this section have symplectic analogues. The statement and method of proof of the symplectic results will be left to the reader. In AU-theory it is most natural to construct the total Landweber-Novikov operation, S. This operation will be the subject of Theorem 7.1. The Thom isomorphism in AU-theory, which appears in the statement of Theorem 7.1, is proved in Proposition 7.2. In Remark 7.4 a brief sketch is given of the manner in which S may be decomposed into the sum of classical Landweber-Novikov operations.

Throughout this section all spaces will be finite dimensional.

7.1: Theorem. For each finitely non-zero sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive integers let $c_\alpha \in MU^{2|\alpha|}(BU)$ denote the α-th Conner-Floyd class [Ad 1, p. 9]. Here $|\alpha| = \sum_i \alpha_i$.

Then

$$C = \sum_\alpha c_\alpha : KU^\circ(X) \to MU^{2*}(X) \cong AU^\circ(X),$$

the "super-total Conner-Floyd class", induces a natural, stable ring homomorphism

$$S : AU^*(X) \to AU^*(X)$$

satisfying the following properties.

(i) If $w \in AU^\circ(CP^N)$ is the canonical element described in section 4.8

$$S(w) = \sum_{i \geq 1} w^i \in AU^\circ(CP^N) \qquad (N \geq 0).$$

(ii) Suppose that E is a complex vector bundle of dimension n over X and consider the following diagram.

$$\begin{array}{ccc} \widetilde{AU}^\circ(Th(E)) & \xrightarrow{S} & \widetilde{AU}^\circ(Th(E)) \\ \lambda \uparrow \cong & & \lambda \uparrow \cong \\ AU^\circ(X) & & AU^\circ(X) \end{array}$$

Then $\lambda^{-1}(S(\lambda(1))) = C(E)$.

Here Th(E) is the Thom space of E and λ is the Thom isomorphism (Proposition 7.2).

7.2: Proposition. (See also Part II, Section 8.) Let $\pi : E \to X$ be a complex vector bundle of dimension n. Let X be compact.

There is a Thom class, $\lambda_E \in \widetilde{AU}^\circ(Th(E))$, such that $\lambda_{E\oplus E'} = \lambda_E \lambda_{E'}$ and

$$\lambda : AU^\circ(X) \to AU^\circ(Th(E))$$

given by $\lambda(x) = \pi^*(x)\lambda_E$ is an isomorphism.

Furthermore if $n = 1$ and $\beta_E : Th(E) \to BU$ is the K-theory Thom class of E then λ_E is represented by $\Sigma^2\beta_E \in [\Sigma^2 Th(E), AU_2]$.

Proof. We know that MU-theory has a Thom class. This (up to multiplication by a unit) will serve as an AU-theory Thom class. The universal Thom class for MU(n) is represented by the S-map inclusion of

$$\Sigma^2 MU(n) \subset \Sigma^2 BU = AU_2$$

given by [Part I, Theorem 4.2]. When $n = 1$ the method of [Part I, section 2], for evaluating restrictions of the transfer, show that the inclusion of $BU(1) \cong MU(1)$ is the canonical map.

In fact, of course, we could show directly that β_E gives a Thom class for line bundles as follows.

By a Mayer-Vietoris argument it suffices to consider the class $E = X \times \mathbb{C}$. In this case β_E is the Bott class B, or rather the composition

$$\mathrm{Th}(X \times \mathbb{C}) = X_+ \wedge S^2 \to S^2 \xrightarrow{B} BU.$$

Hence if $f \in \{\Sigma^{2k}X, BU\}$ represents $F \in AU^\circ(X)$ then $\pi^*(f)\lambda_E$ is represented by

$$\varepsilon_\#(f) \in \{\Sigma^{2k+2}X, BU\} = \{\Sigma^{2k}(\mathrm{Th}(X \times \mathbb{C})), BU\}$$

where $\varepsilon : \Sigma^2 AU_{2k} \to AU_{2k+2}$ is the AU-spectrum map. This clearly gives an isomorphism.

7.3: Proof of Theorem 7.1. The super-total Conner-Floyd class is exponential [Ad 1, p. 9] and $C(w) = \sum_{i\geq 0} w^i$. Therefore we may apply Proposition 5.3 to obtain $\hat{C}$ and set $S = \hat{C}$.

To prove (i) recall that w is represented by $w' = \Sigma^2(y-1) : \Sigma^2 CP^N \to \Sigma^2 BU = AU_2$. Hence $\phi_u(S(w))$ corresponds to $(w')^*(\sigma \otimes C) \in MU^{2*}(\Sigma^2 CP^N)$ where $\sigma \in MU^2(S^2)$ is the suspension class. Therefore by [Ad 1, p. 9] $\Phi_u(S(w)) = \sum_{i\geq 1} \Phi_u(w)^i$. Hence $S(w) = \sum_{i\geq 1} w^i$.

Part (ii) follows from part (i) by the splitting principle which implies that it is sufficient to consider the Hopf line bundle, y, over CP^∞. However in this case the Thom class is $w \in AU^\circ(MU(1))$. Here we have identified CP^∞ and MU(1). The Thom isomorphism is multiplication by w. Therefore

$$\begin{aligned}\lambda^{-1}(S(\lambda(1))) &= (w^{-1}) \sum_{i\geq 1} w^i \\ &= \sum_{i\geq 0} w^i \\ &= C(y).\end{aligned}$$

7.4: Remark. The operation S may be decomposed into the sum of additive operations. Observe that for $\alpha = (\alpha_1, \alpha_2, \ldots)$ we may define

$$t_\alpha : \{\Sigma^{2k}X, BU\} \to MU^{2*}(X)$$

by

$$t_\alpha(f) = f^*(c_\alpha) \in MU^{2|\alpha|}(\Sigma^{2k}X) \cong MU^{2|\alpha|-2k}(X)$$

in such a way that the following diagram commutes

$$\begin{tikzcd} \{\Sigma^{2k}X, BU\} \arrow[rr, "\varepsilon_\#"] \arrow[dr, "t_\alpha"'] & & \{\Sigma^{2k+2}X, BU\} \arrow[dl, "t_\beta"] \\ & MU^{2|\alpha|-2k}(X) & \end{tikzcd} \tag{7.5}$$

Here $\beta = \alpha + (1,0,0,0,\ldots)$. It is not difficult to construct the operations

(S_α) of [Ad 1, p. 12] from diagrams like (7.5) and Theorem 7.1. This task will be left to the reader.

§8. STABLY ALMOST COMPLEX MANIFOLDS

In this section the classical association

$$\left\{\begin{array}{l}\text{bordism classes of stably}\\ \text{almost complex manifolds}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{homotopy of the}\\ \text{MU-spectrum}\end{array}\right\}$$

will be described in terms of AU-theory. For details of unitary bordism and cobordism see [St 1, p. 41].

This association can be made in two ways. The first is the classical geometrical construction of Pontrjagin-Thom and the second uses the cohomological products and the existence of AU-theory Thom classes. Both will be given below and their equivalence established.

The second construction leads to a general cohomological homomorphism which is potentially useful in algebraic geometry to study embeddings which have normal bundles but do not embed algebraically in Euclidean space.

Recall [St 1, Chapter II] that if M^{2n} is a closed stably almost complex manifold then, to a suitable embedding $i : M^{2n} \subset \mathbb{R}^{2n+2k}$, there is associated a Pontrjagin-Thom map

$$P(i) : S^{2n+2k} \to Th(\nu) . \tag{8.1}$$

In (8.1) ν is the normal bundle of i and $Th(\nu)$ is its Thom space. Incidentally, by Atiyah's Duality Theorem [St 1, p. 37] P(i) is S-dual to

$$M \cup (\text{point}) = M^+ \to S^\circ = \{\pm 1\},$$

the map which sends M to (-1) and the disjoint point to (+1).

8.2: Definition. Define $\Lambda_n \in AU^\circ(MU(n))$ to be the element represented by the S-map

$$MU(n) \subset BU$$

given by the stable splitting of [Part I, Theorem 4.2(i)].

The normal bundle ν (mentioned above) is classified by $M^{2n} \to BU(k)$ whose "Thomification" is a map

$$\sigma(\nu) : Th(\nu) \to MU(k) .$$

Using the above notation the first result of this section is as follows:-

8.3: Theorem. Let M^{2n} be a closed stably almost complex manifold. The association

$$\{\_\} : M^{2n} \to \{M^{2n}\} = P(i)^*\sigma(\nu)^*\Lambda_k \in AU^\circ(S^{2n+2k}) \cong AU^\circ(S^\circ)$$

defines a bijection

$$\{\text{unitary bordism classes}\} \longleftrightarrow \pi_0(AU).$$

Proof. It suffices to check that our association, {_}, corresponds with that given by the Pontrjagin-Thom theorem for unitary bordism [St 1, Chapter II] when we identify $AU^\circ(S^\circ)$ with $\bigoplus_s \pi_{2s}(MU)$ by means of Theorem 2.1.

The isomorphism

$$\Phi_u : AU^\circ(S^\circ) = \varinjlim_N \{S^{2N}, BU\} \xrightarrow{\cong} MU^{2*}(S^\circ)$$

is given by taking the induced map in MU-theory of a representative S-map, $S^{2N} \to BU$, and pulling back the total Conner-Floyd class, $\sum_{1\le\ell} c_\ell$. Therefore we must show for the S-map,

$$\Lambda_n : MU(n) \subset BU$$

that $\Lambda_n^*(\sum_{1\le\ell} c_\ell) \in MU^{2*}(MU(n))$ is equal to the canonical Thom class which lies in $MU^{2n}(MU(n))$. To do this it suffices to compute Λ_n^* or equivalently

$$(\Lambda_n)_* : MU_{2*}(MU(n)) \to MU_{2*}(BU).$$

However $MU_{2*}(BU)$ is a polynomial ring over $\pi_{2*}(MU)$. For details see [Ad 1, Part II]. The calculations are entirely analogous to those in integral homology (cf. Part II, §2). In particular there is a natural (algebraic) identification of $MU_{2*}(MU(n))$ with a $\pi_{2*}(MU)$-submodule of $MU_{2*}(BU)$. By the MU_*-homology calculation of [Part II, §2.8] we see that $(\Lambda_n)_*$ is equal to the natural algebraic identification mentioned above. However the dual of the natural identification is well-known (by definition--see [Ad 1, Part III]) to send $\sum_{1\le\ell} c_\ell$ to the required Thom class in $MU^{2n}(MU(n))$.

8.4: Remark. In Part III we will introduce AO-theory. It is to MO*(_) as AU-theory is to $MU^{2*}(\_)$. It is straight-forward to describe unoriented bordism classes of manifolds in terms of AO-theory in a manner similar to that of §8.3. One uses Thom's identification of MO*(_) [Th] together with the results of [Part III, §§2-3]. Details will be left to the interested reader.

8.5: The λ_f Homomorphism of an Embedding, $f : X \to Y$. The following construction will apply to a homology theory with a Thom isomorphism for complex vector bundles.

Let $f : X^{2n} \to Y^{2n+2k}$ by a smooth embedding of stably almost complex manifolds having a stably complex normal bundle, $\nu(f)$. If τ_X and τ_Y are the tangent bundles there is an exact sequence of vector bundles over X

$$0 \to \tau_X \to f^*\tau_Y \to \nu(f) \to 0.$$

There is a Thom class

$$\Lambda(\nu(f)) \subset \widetilde{AU}^\circ(Th(\nu(f))$$

constructed by pulling back $\Lambda_k \in AU^\circ(MU(k))$ by the "Thomification", $Th(\nu(f)) \to MU(k)$, of the classifying map of $\nu(f)$. We may form the homomorphism

$$[\Lambda(\nu(f))\backslash\_] : \widetilde{AU}_\circ(Th(\nu(f))) \to AU_\circ(X) \tag{8.5.1}$$

which is called the AU-theory homology Thom homomorphism. Here $AU_\circ$ is the zero-th homology group associated to the AU-spectrum. If

$$\Delta : Th(\nu(f)) \to Th(\nu(f)) \wedge X$$

is defined by $\Delta(\infty) =$ (base point) and $\Delta(e) = e \wedge x$ for when e belongs to the fibre of $\nu(f)$ over $x \in X$ then

$$[\Lambda(\nu(f))\backslash a] = \Lambda(\nu(f))\backslash \Delta_*(a) \in AU_\circ(X)$$

where $(\_\backslash\_)$ is the slant product described in [Ad 1, Part III].

8.5.2: Lemma. The Thom homomorphism of (8.5.1) is an isomorphism.

Proof. This is a famous isomorphism for MU_{2*}-theory and the classical Thom class. However when $AU_\circ(\_)$ is identified with $MU_{2*}(\_)$ Theorem 8.3 tells us that (up to multiplication by a unit in $\pi_{2*}(MU)$) the Thom class $\Lambda(\nu(f))$ agrees with the classical one.

8.5.3: Definition. There is a also a Kronecker product homomorphism $\widetilde{AU}_\circ(Th(\nu(f))) \to \pi_\circ(AU)$ defined by sending x to $\langle\Lambda(\nu(f)),x\rangle = \Lambda(\nu(f))\backslash x \in \pi_\circ(AU)$ [Ad 1, Part III].

Combining this homomorphism with the inverse of (8.5.1) we obtain a homomorphism

$$\lambda_f : AU_\circ(X) \to \pi_\circ(AU). \tag{8.5.4}$$

8.6: The Relation Between λ_f and the Pontrjagin-Thom Construction. Suppose in §8.5 that $Y = \mathbb{C}^{n+k}$ then the bundle exact sequence becomes

$$0 \to \tau_X \to X \times \mathbb{C}^{n+k} \to \nu(f) \to 0.$$

Each of these bundles has an AU-theory Thom class and a Thom isomorphism. They are related by the following commutative diagram

$$\begin{array}{ccc} \widetilde{AU}_\circ(Th(\tau_X) \wedge Th(\nu(f))) & \cong & \widetilde{AU}_\circ(Th(X \times \mathbb{C}^{n+k})) \\ {\scriptstyle (\Lambda(\tau_X)\backslash\_)}\downarrow\cong & & \downarrow{\scriptstyle\sigma} \\ \widetilde{AU}_\circ(Th(\nu(f))) & \xrightarrow[{[\Lambda(\nu(f))\backslash\_]}]{\cong} & AU_\circ(X) \end{array}$$

in which σ is the suspension isomorphism. There is a fundamental class $[X] \in AU_\circ(X)$ such that

$$\Lambda(\tau_X)\backslash\sigma^{-1}[X] = P(f) \in \widetilde{AU}_\circ(Th(\nu(f)) \tag{8.6.1}$$

where P(f) is the image of the Pontrjagin-Thom map of (8.1) under the Boardman-Hurewicz homomorphism. Formula (8.6.1) follows from the discussion of duality given in [Ad 1, p. 246, et seq.] and Atiyah's Duality Theorem which tells us that an S-duality

$$\mu : S^{2n+2k} \to Th(\nu(f)) \wedge X$$

is given by $\mu(v) = P(f)(v) \wedge \pi(v)$ where $\pi : \nu(f) \to X$ is the projection of the normal bundle. Thus (8.6.1) gives the following formula for $\{X^{2n}\}$ of Theorem 8.3.

$$\begin{aligned}\{X^{2n}\} &= P(f)*\sigma(\nu(f))*\Lambda_k, \quad \text{by definition} \\ &= \langle\Lambda(\nu(f)),\ P(f)\rangle \\ &= \langle\Lambda(\nu(f)),\ [\Lambda(\nu(f))\backslash\_]^{-1}[M]\rangle\ .\end{aligned}$$

8.6.2: Corollary. With the notation of §§8.3-8.6 above

$$\{X^{2n}\} = \lambda_f[M] \in \pi_0(AU).$$

§9. A NEW IDENTITY FOR BU

§9.1. In this section a new construction of BU will be given. It is an application of the previous results of Part II. Also there are applications to the stable homotopy of CP^∞ (Theorem 9.1.2).

I have written this section in terms of spaces (infinite loopspaces) rather than spectra in order to emphasis the familiar space, BU, rather than the more metaphysical spectrum, $\underline{BU}$.

$Z \times BU$ is the classifying space for unitary K-theory. It is an infinite loopspace (henceforth "infinite loop-" will be abbreviated to "Ω^∞-") because of Bott periodicity, which exhibits a homotopy equivalence.

$$Z \times BU \simeq \Omega^2(Z \times BU)$$

where $\Omega^2 Y$ is the second loopspace of Y. A homotopy equivalence of the Bott periodicity type is a rather primitive manner in which to express an Ω^∞-structure, but it is all the more tractible for that reason.

To an Ω^∞-space is associated a spectrum and vice versa. The Bott spectrum for unitary K-theory takes the form

$$\underline{BU}_{2k} = Z \times BU \quad (k \geq 0)$$

with structure maps $\varepsilon_{2k} : \Sigma^2\underline{BU}_{2k} \to \underline{BU}_{2k+2}$ induced by the map $S^2 \times (Z \times BU) \to Z \times BU$ which classifies the tensor product of the reduced Hopf bundle on S^2 with the universal bundle of virtual dimension zero on $Z \times BU$.

For background material on Ω^∞-spaces and spectra the reader is referred to [Ma 1; Ma 2] and [Ad 1] respectively. In particular [Ma 2, Ch. VIII, §2] and [Ad 1, p. 134] deal with K-theory.

Now we introduce another spectrum with evident Bott periodicity. It is constructed in a similar manner to AU of §3.

Let X be an H-space and let $x \in \pi_2(X)$. Form

$$x + 1_X : S^2 \times X \to X$$

the "sum" of x with the identity of X. The Hopf construction applied to this map and suspended once gives a map

$$\varepsilon : \Sigma^4 X \to \Sigma^2 X .$$

There is an associated spectrum, $\underline{X}(x)$, given by

$$\underline{X}(x)_{2k} = \Sigma^2 X \qquad (k \geq 1)$$

and structure maps $\varepsilon : \Sigma^2 \underline{X}(x) \to \underline{X}(x)_{2k+2}$. Denote by P(X) the associated Ω^∞-space, which is defined by

$$P(X) = \varinjlim_n \varinjlim_k \Omega^{2n+2k} \Sigma^{2k} X$$

where the limit is taken over composition with ε. Manifestly $\Omega^2 P(X) = P(X)$ thereby giving an Ω^∞-structure to P(X).

In this section I will prove the following result, which is proved in §9.2.

9.1.1: Theorem. There is an equivalence of Ω^∞-spaces

$$P(\mathbb{C}P^\infty) \xrightarrow{\cong} Z \times BU.$$

Here $\mathbb{C}P^\infty = K(Z,2)$ and x generates $\pi_2(\mathbb{C}P^\infty)$.

Now let $x.\_ : \pi_j^S(\mathbb{C}P^\infty) \to \pi_{j+2}^S(\mathbb{C}P^\infty)$ be the homomorphism of stable homotopy groups induced by "adding" $x \in \pi_2(\mathbb{C}P^\infty)$ by means of the H-space sum. Since

$$\pi_j P(\mathbb{C}P^\infty)) = \varinjlim_k \pi_{j+2k}^S(\mathbb{C}P^\infty),$$

where the limit is taken over $(x.\_)$, we also have the following result which is proved in §9.2.

9.1.2: Theorem. If $y \in \pi_m^S(\mathbb{C}P^\infty)$ is a torsion element then

$$0 = x^k y \in \pi_{m+2k}^S(\mathbb{C}P^\infty)$$

for some k.

Now observe that Theorem 9.1.1 is related to Brauer lifting [Q2; To]. Theorem 9.1.1 expressed $BGL\mathbb{C}(\simeq BU)$ as a functor of $B\mathbb{C}^*(\simeq \mathbb{C}P^\infty)$ where Λ^* denotes the units in a ring Λ. Let p and q be distinct primes and let $(\_)^\wedge_p$ denote p-completion [B-K]. Then, from [Q2], we have

$$BU^\wedge_p \simeq (BGL\overline{\mathbb{F}q}^+)^\wedge_p \tag{9.1.3}$$

where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$ and $(\_)^+$ denotes Quillen's "plus" construction [Wa]. Also

$$(\mathbb{C}P^\infty)^\wedge_p \simeq (B\overline{\mathbb{F}}^*_q)^\wedge_p .$$

Combining these facts with the p-complete version of Theorem 9.1.2 yields an Ω^∞-equivalence of the following form

$$P_\circ((B\overline{\mathbb{F}}^*_q)^\wedge_p) \simeq (BGL\overline{\mathbb{F}}^+_q)^\wedge_p \tag{9.1.4}$$

(where $P_\circ(X)$ denotes the base-point component of P(X)). In (9.1.4) the left-hand space is formed using the element of $\pi_2((B\overline{\mathbb{F}}_q^*)^\wedge_p) \simeq Z^\wedge_p$ which is the image of $x \in \pi_2(\mathbb{C}P^\infty)$. In terms of (9.1.4) the Ω^∞-equivalence of (9.1.3) may be induced from any embedding $\overline{\mathbb{F}}_q^* \subset \mathbb{C}^*$.

Thus theorem 9.1.1 gives a particularly simple viewpoint on the Brauer lifting map of (9.1.3) and of its Ω^∞-space properties, which were first studied in [To] where it is shown that the Brauer lifting is an Ω^∞-map. My methods, as given here, do not give independent proofs of the results of [To]. This could, however, be readily accomplished with the aid of more Ω^∞-technology.

More importantly, I believe, the equivalence of (9.1.4) suggests one might attempt to study the algebraic K-theory of a commutative ring, Λ, by constructing maps

$$BGL\Lambda^+ \to P(B\Lambda^*;y)$$

where $P(B\Lambda^*;y)$ is constructed in a manner analogous to $P(X)$ with $B\Lambda^*$ replacing X and $y \in \pi_1(B\Lambda^*)$ replacing $x \in \pi_2(X)$. However one's expectation of success should not be too high as thefollowing example shows!

<u>9.1.5</u>: <u>Proposition</u>. If $0 \neq \eta \in \pi_1(BZ/2)$ then $P(BZ/2;\eta)$ is contractible.

<u>Proof</u>. Write P for $P(BZ/2;\eta)$. Then $P \simeq \Omega P$ so it suffices to show $\pi_1(P) = 0$. However $\pi_1(P)$ is the direct limit of

$$\cdots \to \pi_n^S(\mathbb{R}P^\infty) \xrightarrow{(\eta\cdot\_)} \pi_{n+1}^S(\mathbb{R}P^\infty) \xrightarrow{(\eta\cdot\_)} \pi_{n+2}^S(\mathbb{R}P^\infty) \to \cdots$$

which is zero because $0 = \eta^3 \in \pi_3^S(\mathbb{R}P^\infty)$ [Li].

Finally I would like to express my gratitude to Dan S. Kahn, who showed enough interest to ask me what $P(\mathbb{C}P^\infty)$ really was. Theorem 9.1.1 solves his problem presented to the problem session of the A.M. Soc. Summer Institute at Stanford in 1976.

§<u>9.2</u>. Theorems 9.1.1 and 9.1.2 will be proved by means of unitary cobordism, MU^{2*} [Ad 1, Part II]. The following description of MU^{2*} is a rephrasing of Theorem 2.1 of Part II.

<u>9.2.1</u>: <u>Proposition</u>. As in §9.1 construct $P(BU^\oplus)$ where $x \in \pi_2(BU^\oplus)$ is a generator and $BU^\oplus$ is BU together with the H-space structure induced from Whitney sum of vector bundles.

Then if Y is a finite dimensional CW complex there is a natural isomorphism (of rings in fact).

$$\phi : [Y, P(BU^\oplus)] \xrightarrow{\simeq} MU^{2*}(Y)$$

<u>9.2.2</u>: <u>Proof of Theorem 9.1.2</u>. Let $y \in \pi_j^S(\mathbb{C}P^\infty)$ be a torsion element. Let $\varepsilon' : \Sigma^4 BU^\oplus \to \Sigma^2 BU^\oplus$ be the structure map of the spectrum associated with $P(BU^\oplus)$ (cf. §9.1). Let $\varepsilon : \Sigma^4\mathbb{C}P^\infty \to \Sigma^2\mathbb{C}P^\infty$ be associated with $P(\mathbb{C}P^\infty)$. If $\det : BU^\oplus \to CP^\infty$ is the H-map induced by the determinant then

$$\varepsilon \circ \Sigma^4(\det) \simeq \Sigma^2(\det) \circ \varepsilon'$$

This is because the diagram

$$\begin{tikzcd} S^2 \times BU^{\oplus} \arrow[r, "x \oplus 1_{BU^{\oplus}}"] \arrow[d, "1 \times \det"'] & BU^{\oplus} \arrow[d, "\det"] \\ S^2 \times \mathbb{C}P^{\infty} \arrow[r, "x + 1_{\mathbb{C}P^{\infty}}"'] & \mathbb{C}P^{\infty} \end{tikzcd}$$

is homotopy commutative. However there is a canonical map $i : \mathbb{C}P^{\infty} \to BU^{\oplus}$ such that $1 \simeq (\det) \circ i$. Since $MU^{2*}(S^j)$ is torsion free [Ad 1, Part II] there exists an integer, k, such that

$$0 = (\varepsilon'_{\#})^k (i_{\#}(y)) \in \pi^S_{j+2k}(BU^{\oplus})$$

Hence
$$\begin{aligned} 0 &= (\det)_{\#}(\varepsilon'_{\#})^k i_{\#}(y) \\ &= (\varepsilon_{\#})^k (\det_{\#} i_{\#}(y) \\ &= (\varepsilon_{\#})^k (y) \\ &= x^k y \in \pi^S_{j+2k}(\mathbb{C}P^{\infty}). \end{aligned}$$

9.2.2. Now we construct an Ω^{∞}-map

$$P(\mathbb{C}P^{\infty}) \to Z \times BU.$$

To do this it suffices to define a stable natural transformation

$$F : \varinjlim_n \{\Sigma^{2n}\_, \mathbb{C}P^{\infty}\} \to KU^{\circ}(\_) \tag{9.2.3}$$

where the limit is taken over composition with $\varepsilon : \Sigma^4 \mathbb{C}P^{\infty} \to \Sigma^2 \mathbb{C}P^{\infty}$ of §9.1. Here "stable" means that F commutes with the periodicity isomorphisms induced by $\Omega^2 P(\mathbb{C}P^{\infty}) \simeq P(\mathbb{C}P^{\infty})$ and $\Omega^2(Z \times BU) \simeq Z \times BU$. Also $\{\_,\_\}$ denotes stable homotopy classes of maps [Ad 1, Part III] and the functors of (9.2.3) are defined on the pointed CW category.

Let $x \in KU^{\circ}(\mathbb{C}P^{\infty}) \simeq Z[[x]]$ be the class of the reduced Hopf bundle. Let $y \in \varinjlim_n \{\Sigma^{2n}X, \mathbb{C}P^{\infty}\}$ be represented by

$$g : \Sigma^{2n+2k}X \to \Sigma^{2k}\mathbb{C}P^{\infty}.$$

Let $\beta \in KU^{\circ}(S^2)$ be the Bott class. Then $\beta^k x \in KU^{\circ}(\Sigma^{2k}\mathbb{C}P^{\infty})$ and we may define

$$F(y) = g^*(\beta^k x) \in KU^{\circ}(\Sigma^{2n+2k}X) \simeq KU^{\circ}(X)$$

where the last isomorphism (Bott periodicity) is the inverse of multiplication by β^{n+k}. Suppose we choose a different representative for y of the form

$$\Sigma^{2n+2k+2\ell}X \xrightarrow{\Sigma^{2\ell}g} \Sigma^{2k+2\ell}\mathbb{C}P^{\infty} \xrightarrow{\varepsilon^m} \Sigma^{2k+2\ell-2m}\mathbb{C}P^{\infty}.$$

Then F(y) will be given by

$$\beta^{-n-k-\ell}\ (\Sigma^{2\ell}g)^*(\varepsilon^*)^m(\beta^{\ell+k-m}x)$$

However $x + 1_{\mathbb{C}P^\infty} : S^2 \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces

$$KU^\circ(\mathbb{C}P^\infty) \to KU^\circ(S^2 \times \mathbb{C}P^\infty)$$

sending x to $\beta \otimes 1 + 1 \otimes x + \beta \otimes x$ so that $\varepsilon^*(x) = \beta x$. Hence the second representative of F(y) becomes

$$\begin{aligned}\beta^{-n-k-\ell}(\Sigma^{2\ell}g)^*\beta^{\ell+k-m}(\varepsilon^*)^m(x) &= \beta^{-n-1-\ell}(\Sigma^{2\ell}g)^*(\beta^\ell(\beta^k x))\\ &= \beta^{-n-k-\ell}\beta^\ell(g^*(\beta^k x))\\ &= \beta^{-n-k}g^*(\beta^k x)\end{aligned}$$

so that F(y) is well-defined. Similarly the periodicity diagram commutes.

$$\begin{array}{ccc} \varinjlim_n \{\Sigma^{2n}X, \mathbb{C}P^\infty\} & \xrightarrow{F} & KU^\circ(X) \\ \downarrow \wr & & \downarrow (\beta \circ \_) \\ \varinjlim_n \{\Sigma^{2n}(\Sigma^2 X), \mathbb{C}P^\infty\} & \xrightarrow{F} & KU^\circ(\Sigma^2 X) \end{array} \qquad (9.2.4)$$

In (9.2.4) the left-hand isomorphism is the obvious (i.e., tautological) one.

Now let $P(BU^\oplus)$ be as in Proposition 9.2.1. We may define an Ω^∞-map $P(BU^\oplus) \to Z \times BU$ in a similar manner. Let $\gamma^i \in KU^\circ(BU^\oplus)$ be the i-th γ-operation [Ad 5] and set

$$\det = \sum_{i \geq 0} \gamma^i \in KU^\circ(BU^\oplus) \cong Z[[\gamma^1, \gamma^2, \ldots]].$$

Define

$$G : \varinjlim_n \{\Sigma^{2n}\_, BU^\oplus\} \to KU^\circ(\_) \qquad (9.2.5)$$

by sending the class of $g : \Sigma^{2n+2k}X \to \Sigma^{2k}BU^\oplus$ to $\beta^{-n-k}g^*(\beta^k \det) \in KU^\circ(X)$. Notice that det is represented by the composite

$$BU^\oplus \xrightarrow{B\det} \mathbb{C}P^\infty \xrightarrow{x} Z \times BU$$

where $x \in KU^\circ(\mathbb{C}P^\infty)$ is the reduced Hopf bundle and B det is induced by the determinant $\det : U(n) \to U(1) = S^1$. Therefore we immediately obtain the following result.

<u>9.2.6</u>: <u>Lemma</u>. The diagram

$$\begin{array}{ccc} \varinjlim_n \{\Sigma^{2n}X, BU^\oplus\} & & \\ (B\det)_\# \downarrow & \overset{G}{\searrow} & KU^\circ(X) \\ \varinjlim_n \{\Sigma^{2n}X, \mathbb{C}P^\infty\} & \overset{F}{\nearrow} & \end{array}$$

commutes.

9.2.7: Proposition. Suppose that X is a finite dimensional CW complex. Then there is a commutative diagram

$$\begin{tikzcd} \varinjlim_n \{\Sigma^{2n}X, BU^{\oplus}\} \simeq [X, P(BU^{\oplus})] \arrow[r, "\simeq"', "\emptyset"] \arrow[dr, "G"'] & MU^{2*}(X) \arrow[d, "H"] \\ & KU^{\circ}(X) \end{tikzcd}$$

in which H is the Conner-Floyd homomorphism [C-F] and $\emptyset$ is the isomorphism of Proposition 9.2.1.

Proof. Recall [Ad 1, Part I] that $KU_{\circ}(BU) = Z[b_1, b_2 \ldots]$ and $KU^{\circ}(BU) = Z[[\gamma^1, \gamma^2, \ldots]]$ where the γ^i are characterised by

$$<z, \gamma^j> = \begin{cases} 1 & \text{if } z = b_1^j \\ 0 & \text{if } z \text{ is any other monomial.} \end{cases}$$

The isomorphism, $\emptyset^{-1}$, is induced by S-maps [Part II, §2]

$$h_n : MU(n) \to BU(n) \to BU$$

using the fact that $MU^{2n}(X) \simeq \{\Sigma^{2t}X, MU(n+t)\}$ if $\dim X < 4n + 2t$. The $\{b_n\}$ have the following property. Let $\pi_n : BU(n) \to BU(n)/BU(n-1) \simeq MU(n)$ be the canonical map. Then

$$(h_n \circ \pi_n)_*(b_{i_1} \ldots b_{i_n}) = b_{i_1} \ldots b_{i_n}$$

where $KU_{\circ}(BU(n))$ is considered to be the subgroup of $KU_{\circ}(BU)$ generated by monomials of weight $\leq n$. Now let $E_n \to BU(n)$ be the universal n-plane bundle and let $\Lambda_n \in KU^{\circ}(MU(n))$ be its Thom class. The restriction of Λ_n to $BU(n)$ is

$$\pi_n^*(\Lambda_n) = \Sigma(-1)^i \lambda^i(E_n) = \gamma^n(E_n - \underline{n}) \in KU^{\circ}(BU(n))$$

where λ^i is the i-th exterior power and $\underline{n}$ is the trivial n-plane. Notice that $\pi_n^* : \tilde{KU}^{\circ}(MU(n)) \to KU^{\circ}(BU(n))$ is injective.

By the above discussion we have

$$\begin{aligned} <b_{i_1} \ldots b_{i_n}, (h_n \circ \pi_n)^*(\sum_{i\geq 0} \gamma^i)> &= <(h_n \circ \pi_n)_*(b_{i_1} \ldots b_{i_n}), \sum_{i\geq 0} \gamma^i> \\ &= <b_{i_1} \ldots b_{i_n}, \sum_{i\geq 0} \gamma^i> \\ &= \begin{cases} 1 & \text{if } b_{i_1} = b_{i_2} = \cdots = b_{i_n} = b_1 \\ 0 & \text{otherwise} \end{cases} \\ &= <b_{i_1} \ldots b_{i_n}, \sum_i \Lambda_i> . \end{aligned}$$

However the Conner-Floyd map is obtained by pulling back the class $\sum_i \Lambda_i \in \bigoplus_i KU^{\circ}(MU(i))$ so that $H_{\circ}\emptyset$ is equal to the homomorphism obtained by pulling back the class $\sum_i \gamma^i \in KU^\circ(BU^\oplus)$. However [Ad 5] $\sum_{i\geq 0} \gamma^i$ is equal to the class $\det \in KU^\circ(BU^\oplus)$ of §9.2.5 by means of which G was defined.

9.2.8: Proof of Theorem 9.1.1. We must show that

$$F : \pi_j(P(\mathbb{C}P^\infty)) \simeq \varinjlim_n \{S^{2n+j}, \mathbb{C}P^\infty\} \to KU^\circ(S^j)$$

is an isomorphism. By a result of J. H. C. Whitehead this will ensure that the base-point components of $P(\mathbb{C}P^\infty)$ and $Z \times BU$ are homotopy equivalent. Since F induces an Ω^∞-map $P(\mathbb{C}P^\infty) \to Z \times BU$ and since both spaces have "Bott periodicity" it follows that the map is a homotopy equivalence.

By the Conner-Floyd theorem [C-F] and §§9.2.6, 9.2.7 we know that F is onto because H is onto in §9.2.7. By Theorem 9.1.2 we know that $\pi_j(P(\mathbb{C}P^\infty))$ is torsion free. Thus we are finished if

$$\operatorname{rank}(\pi_j(P(\mathbb{C}P^\infty))) = \begin{cases} 1 & j \text{ even,} \\ 0 & j \text{ odd} \end{cases}$$

since $\pi_j(BU) = Z$ or 0. However $\pi_{j+2n+2\ell}(\Sigma^{2\ell}\mathbb{C}P^\infty) \otimes \mathbb{Q} \simeq H_{j+2n}(\mathbb{C}P^\infty;\mathbb{Q})$ from which it is easily seen that

$$\varinjlim_n \{\Sigma^{2n}S^j, \mathbb{C}P^\infty\} \otimes \mathbb{Q} = \begin{cases} 1 & \text{if } j \text{ even} \\ 0 & \text{if } j \text{ odd} \end{cases}$$

since $\varepsilon_* : H_j(\Sigma^2\mathbb{C}P^\infty;\mathbb{Q}) \to H_{j+2}(\mathbb{C}P^\infty;\mathbb{Q})$ is an isomorphism.

9.2.9: Remarks. (a) An alternative method to determine $\pi_j(P(\mathbb{C}P^\infty)) \otimes Q$ in §9.2.8 is the following. Write $P(\mathbb{C}P^\infty) = \varinjlim_n \Omega^{2n}Q\mathbb{C}P$ where $QX = \varinjlim_k \Omega^k\Sigma^k X$. The Barratt-Priddy-Quillen theorem provides models for QX [Ba; B-E; Ma 1]. These show that $Q\mathbb{C}P^\infty$ is rationally equivalent to $SP^\infty\mathbb{C}P^\infty$, the infinite symmetric product of $\mathbb{C}P^\infty$. By [D-T] $SP^\infty\mathbb{C}P^\infty$ is rationally equivalent to $\prod_{m\geq 1} K(\mathbb{Q},2m)$ from which it is easy to show the rational type of $P_\circ(\mathbb{C}P^\infty)$ is the same as that of BU, namely $\sum_{m\geq 1} K(\mathbb{Q},2m)$.

(b) Consider the computative diagram

$$\begin{tikzcd}
\widetilde{KU}^\circ(X) \arrow[r] \arrow[d, equal] & \{X,\mathbb{C}P^\infty\} \arrow[r] \arrow[d, equal] & \varinjlim_N \{\Sigma^{2N}X,\mathbb{C}P^\infty\} \arrow[d, "F"] \\
{[X,BU]} \arrow[r, "\tau_U"'] & {[X,Q\mathbb{C}P^\infty]} \arrow[r, "\lambda_U"'] & \widetilde{KU}^\circ(X)
\end{tikzcd}$$

in which $\lambda_U \circ \tau_U = 1$ by Part I, §3.2. This shows F, G and the Conner-Floyd map H (see §9.2.7) are all split epimorphisms. This easily implies the Conner-Floyd theorem as mentioned in §5.4.2.

PART III: UNORIENTED COBORDISM, ALGEBRAIC COBORDISM AND THE X(b)-SPECTRUM

§0. INTRODUCTION

Part III consists of various generalisations of and elaborations upon the material of Parts I and II. In Part II, I constructed spectra, AU and ASp from the H-spaces BU and BSp respectively together with some of their homotopy groups. This process is easy to generalise to the following result. (The reference numbers refer to those used in the body of the text in Part III.)

Theorem 1.1. Let X be a homotopy commutative, homotopy associative H-space. Suppose $b \in \pi_*^S(X)$ is a stable homotopy element.

Then to this data there is associated a (periodic) commutative ring spectrum, X(T), which is described below in section 1.2.

AU and ASp are respectively of the form BU(B) and BSp(B') (see Example 1.4.1).

If $X = BGLR^+$, Quillen's space associated with an arbitrary ring, R, we obtain from Theorem 1.1 the algebraic cobordism of R associated with $b \in \pi_*^S(BGLR)$. The rest of §1 consists of some elementary remarks and computations. Of most interest perhaps are §§1.9, 1.10, and 1.13. The first tells us that many algebraic cobordism spectra associated with a finite field are trivial. The other two will enable us to compute p-adic algebraic cobordism of projective schemes in Part IV.

In Parts I and II we studied $\pi_*^S(BU)$ and MU-theory. We do the same for MO-theory.

From the calculations of [Th] we have:

Theorem 2.1. There is a (4k-2)-equivalence between $\frac{BO(2k)}{BO(2k-2)}$ and a product of $K(Z/2,n)$'s.

As a corollary of Theorem 2.1 and [Part I, Theorem 4.2] we obtain the following result.

Theorem 2.2. There are decompositions $\pi_j^S(BO(2k)) = \pi_j^S(BO(2k-2)) \oplus B_j(k)$ for all j,k. If $j < 4k - 2$ then $B_j(k)$ contains the direct sum of $\beta_j(k)$ copies of Z/2. ($\beta_j(k)$ is defined in section 2.2.)

Take X = BO and T = {generator of $\pi_1(BO)$} in Theorem 1.1 gives a cohomology theory, AO*, which is periodic of period one. In Corollary 1.6 this is shown to be a Z/2-module. The following result shows that this module structure is hardly surprising.

Theorem 3.1. There is a natural ring homomorphism

$$\Phi_0 : AO^\circ(W) \to MO^*(W)$$

which is an isomorphism when dim $W < \infty$. In Theorems 4.1 and 4.2 results for $BO\mathbb{F}_3^+$ are proved which are analogous to Theorems 2.1 and 2.2. The importance of these results lies in the fact that $BO\mathbb{F}_3^+$ is 2-locally the image of J (see the problems at end of Part IV for the connection with $\pi_*^S(S^\circ)$).

In §5 we consider the algebraic cobordism of Z, AZ*, and prove:

Theorem 5.1.1. There is a homomorphism

$$T : AZ^\circ(X) \to MO^*(X)$$

which is onto if dim $X < \infty$ but not generally injective.

§1. THE SPECTRUM X(b)

It is high time that I gave the general construction which has been the motivation for all this work. This construction will generalise Part II, §3. It will be the topic of §§1.1-1.3. Having defined the spectrum, X(b), I will give several examples in §1.4. The rest of the section contains a series of results related to the examples. For instance in §1.9 it is shown that many of the algebraic cobordism theories associated with a finite field are trivial. In §1.10-1.12 we give three basic computational results which will be needed in Part IV.

1.1: Theorem. Let X be a homotopy associative, homotopy commutative H-space. Suppose that $b \in \pi_N^S(X)$. Then to this data there is associated a (periodic) commutative ring spectrum, X(b), which is described in §1.2 below.

1.2: Construction of X(b). Let $b \in \pi_N^S(X)$ be the given stable homotopy element. Find the smallest $t \geq 0$ such that b may be realised as a map

$$b : S^{(t+1)N} \to \Sigma^{tN} X .$$

Then $\Sigma^N b$ will be the unit of X(b). Set $M = (1+t)N$. Now for the spectrum.

Put $$X(b)_{kN} = \Sigma^M X \qquad \text{for } k \geq 1$$

and define structure maps

$$\varepsilon : \Sigma^N X(b)_{kN} \to X(b)_{(k+1)N}$$

by means of the composition

$$\begin{tikzcd}
\Sigma^N(\Sigma^M X) \arrow[r, "H"] & \Sigma^N(S^M \times X) \arrow[r, "\Sigma^N(b \times 1_X)"] & \Sigma^N(\Sigma^{M-N} X \times X) \arrow[dll, "H'"] \\
\Sigma^N(S^{M-N} \times X \times X) \arrow[r, "m"] & \Sigma^N(S^{M-N} \times X) \arrow[r, "n"] & \Sigma^M X .
\end{tikzcd}$$

Here H is a Hopf construction, H' is the composition of $\Sigma^N(\Sigma^{M-N}X \times X) \to \Sigma^N(\Sigma^{M-N}X \wedge X)$ with a Hopf construction, m is induced by the H-space product and n is the canonical quotient map.

The data $(\varepsilon : \Sigma^N X(b)_{kN} \to X(b)_{(k+1)N};\ k \geq 1)$ defines a spectrum in the sense of [Ad 1, Part III]. The product is induced by a pairing

$$X(b)_{kN} \wedge X(b)_{\ell N} \to X(b)_{(k+\ell)N}$$

given by the compositions of the following form.

$$\begin{tikzcd}
 & \Sigma^M X \wedge \Sigma^M X = \Sigma^{M-N}(\Sigma^N(X \wedge S^M \wedge X)) \arrow[dl] & \\
\Sigma^{M-N}(\Sigma^N(X \times S^M \times X)) \arrow[rr, "\Sigma^{M-N}(1_X \oplus b \oplus 1_X)"] & & \Sigma^{M-N}(\Sigma^N(\Sigma^{M-N}X)) \arrow[dl, "\simeq"'] \\
 & \Sigma^N(\Sigma^N(\ldots(\Sigma^N(\Sigma^M X)\ldots) \xrightarrow{\varepsilon^t} \Sigma^M X . &
\end{tikzcd}$$

Here the first map is a Hopf construction, the second is induced by the "product" of $b : S^M \to \Sigma^{M-N}X$ with two copies of 1_X, the third map is the t-fold iterate of ε using up Σ^N's one at a time starting from the right. All the identifications of the type $S^{a+b} \wedge S^c = S^a \wedge S^{b+c}$ use merely associativity of the smash product, no factors are permuted.

1.3: Proof of Theorem 1.1. In the proof of [Part II, Theorem 3.1] replace $\Sigma^2 BU$ and B by $\Sigma^M X$ and b respectively. The proofs are then essentially the same.

1.4: Examples.

1.4.1. We have already met the following examples in Part II, sections 2-3.

(a) $X = BU$, $b = B$;

(b) $X = BSp$ $b = B'$

1.4.2. In section 3 the true identity of the following example will be determined.

Take $X = BO$ and $b = \eta$ where $0 \neq \eta \in \pi_1(BO)$. We will write AO for the X(T) of this example.

1.4.3. Let R be any ring with unit and set $X = BGLR^+$. Details of $BGLR^+$ may be found in [H-S; Wa]. The cohomology theory associated with $BGLR^+(b)$ will be called the algebraic cobordism of R associated with b.

In general these algebraic cobordism groups will be difficult to compute. However the following examples deserve special comment by virtue of the fact that we can say something about the resulting algebraic cobordism theories (see §1.9 and §5).

(a) $R = \mathbb{F}_q$, the field with q elements and $b \in \pi_N(BGL\mathbb{F}_q^+) = K_N(\mathbb{F}_q)$. We will write $A\mathbb{F}_q(b)$ for the resulting spectrum in this example.

(b) $R = Z$, the integers and $b = \eta_1$, the generator of $\pi_1(BGLZ^+) = K_1 Z \cong Z/2$. We will write AZ for the resulting spectrum in this example.

1.4.4. (a) One could replace $BGLR^+$ in §1.4.3 by Karoubi's spaces $BO_\varepsilon R^+$ ($\varepsilon = \pm 1$) [K]. The resulting cohomology theories are probably very subtle invariants of R. (b) Also one could replace $BGLR^+$ by one of its localisations in the sense of [Bou], [B-K] or Su]. For example, p-finite completion X (denoted by $X^\wedge_p$ and called $H^*(\_;Z/p)$ localisation in [Bou]) puts different homotopy elements at our disposal. $BGL\overline{\mathbb{F}}_q^+$ ($\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$) has only non-trivial homotopy groups in odd dimensions. In fact the calculations of [Q] yield

$$\pi_i(BGL\overline{\mathbb{F}}_q^+) = K_i\overline{\mathbb{F}}_q = \begin{cases} 0 & i \text{ even,} \\ \overline{\mathbb{F}}_q^* & i \text{ odd.} \end{cases}$$

Here R* denotes the units in a ring, R. Now let p be a prime not dividing q. From [Bou] or [B-K] we see that the p-finite completion has homotopy groups

$$\pi_i((BGL\overline{\mathbb{F}}_q)^\wedge_p) \cong \begin{cases} 0 & i \text{ odd,} \\ \hat{Z}_p \text{ (the p-adics)} & \text{for } i \text{ even.} \end{cases}$$

Furthermore, the Brauer lifting map of [Q] gives an H-space equivalence

$$(BGL\overline{\mathbb{F}}_q^+)^\wedge_p \simeq BU^\wedge_p .$$

If $b \in \pi_2((BGL\overline{\mathbb{F}}_q^+)^\wedge_p$ corresponds to $1 \in \hat{Z}_p$ we may form the spectrum X(b) where $X = (BGL\overline{\mathbb{F}}_q^+)^\wedge_p$. This spectrum corresponds to the periodic cohomology theory which is $MU\hat{Z}_p^{2*}$ in each even dimension and $MU\hat{Z}_p^{2*+1}$ in each odd dimension. This is seen by identifying the spectrum with $BU^\wedge_p(b)$ and following the proof of Part II, §2.1 with the cells of BU replaced by the p-adic cells of $BU^\wedge_p$. Here $MU\hat{Z}_p$ means MU-theory with $\hat{Z}_p$ coefficients [Ad I, Part III].

1.4.5. (a) $X = CP^\infty(= K(Z,2))$ and $b \in \pi_2(CP^\infty)$, a generator. By Part II, §9.1.1 the resulting spectrum, $CP^\infty(b)$, is periodic, unitary K-theory.

(b) Let $X = B\overline{\mathbb{F}}_q^*$, the classifying space of the units in $\overline{\mathbb{F}}_q$. If p is a prime not dividing q then $X^\wedge_p \simeq (CP^\infty)^\wedge_p$. Hence if $b \in \pi_2(X^\wedge_p) \cong \hat{Z}_p$ corresponds to one then $X^\wedge_p(b)$ is the periodic spectrum associated with $KU\hat{Z}_p$, K-theory with p-adic coefficients. For $X^\wedge_p(b)$ corresponds to the infinite loopspace $P((B\overline{\mathbb{F}}_q^*)^\wedge_p) \simeq P((CP^\infty)^\wedge_p)$. As in Part II, §9 we may construct a map

$$G : P((CP^\infty)^\wedge_p) \to BU^\wedge_p(= BU\hat{Z}_p).$$

By Part II, §9 this is an $H^*(\_;Z/p)$ isomorphism because

$$H^*(P((CP^\infty)^\wedge_p);Z/p) \cong H^*(P(CP^\infty);Z/p)$$

and

$$H^*(BU^\wedge_p;Z/p) \cong H^*(BU;Z/p).$$

Thus G is a homotopy equivalence since both spaces are p-finitely complete (i.e., $H^*(\_;Z/p)$-local).

(c) If $X = BA^*$ for any $\overline{\mathbb{F}}_q$-algebra, A, then we may set b equal to the image of the element $b \in \pi_2((B\overline{\mathbb{F}}_q^*)_p^\wedge)$ of §1.4.5(b) under the map $\pi_2((B\overline{\mathbb{F}}_q^*)_p^\wedge) \to \pi_2(X_p^\wedge)$. The resulting $X_p^\wedge(b)$ spectrum is an analogue of topological K-theory (or rather $KU\hat{Z}_p$) which is natural in the sense that a map $A_1 \to A_2$ of $\overline{\mathbb{F}}_q$-algebras induces a map of ring spectra $(BA_1^*)_p^\wedge(b) \to (BA_2^*)_p^\wedge(b)$.

1.4.6: The Most Important Example.[3] (a) This example generalises §1.4.4(b). If $\mathcal{C}$ is a category with exact sequences [Q3; Q4] we may form Quillen's category $Q\mathcal{C}$ and take the H-space given by the base point component $(\Omega_\circ\text{-})$ of the loop-space of its classifying space $\Omega_\circ BQ\mathcal{C} = X_{\mathcal{C}}$, say. For example, if V is a scheme we may take $\mathcal{C} = \underline{\underline{P}}(V)$ the category of vector bundles over V (= locally free sheaves of $\underline{\underline{O}}_V$-modules of finite rank) equipped with the usual notion (for the Zariski site) of exact sequences. If V = Spec A then $X_{\mathcal{C}} \simeq BGLA^+$ for this example [Q3; Q4], so this example generalises §1.4.4(b). Note that $K_\circ V \times X_{\underline{\underline{P}}(V)} = \Omega BQ\underline{\underline{P}}(V)$ for any scheme V.

One might, of course, replace exactness on the Zariski site for exactness on the étale site. The Q-category construction works equally well for any site on V but we will prefer to pursue variants of $X_{\underline{\underline{P}}(V)}$, introduced above, because for this example we can make some computations, which will be found in Part IV.

(b) Suppose then that $V \to \operatorname{Spec} \overline{\mathbb{F}}_q$ is an $\overline{\mathbb{F}}_q$-scheme. Let p be a prime not dividing q and let $(\_)_p^\wedge$ denote p-finite completion as in §1.4.4(b). We have a map of homotopy commutative, homotopy associative H-spaces

$$(BGL\overline{\mathbb{F}}_q^+)_p^\wedge = \left(X_{\underline{\underline{P}}(\operatorname{Spec}\overline{\mathbb{F}}_q)}\right)_p^\wedge \to \left(X_{\underline{\underline{P}}(V)}\right)_p^\wedge$$

and consequently the element $b \in \pi_2((BGL\overline{\mathbb{F}}_q)_p^\wedge)$ of §1.4.4(b) yields, under the induced homomorphism, $b_V \in \pi_2((X_{\underline{\underline{P}}(V)})_p^\wedge)$. Furthermore b_V is natural with respect to morphisms of schemes over $\operatorname{Spec}\overline{\mathbb{F}}_q$. Forming the spectrum $(X_{\underline{\underline{P}}(V)})_p^\wedge(b_V)$ gives a contravariant functor from $\operatorname{Spec}\overline{\mathbb{F}}_q$ schemes to cohomology theories. This is a "generalised sheaf cohomology theory" in the sense of [Br-G]. We will denote this spectrum by $\underline{A\overline{\mathbb{F}}_{q,V}}$ (notice that it also depends on the once for all choices of b and p). If W is a space we will write $(A\overline{\mathbb{F}}_{q,V})^*(W)$ and $(A\overline{\mathbb{F}}_{q,V})_*(W)$ for the associated cohomology and homology of W.

$\underline{A\overline{\mathbb{F}}_{q,V}}$ is called the p-adic algebraic cobordism spectrum of the scheme V. In Part IV we will identify the homotopy of this spectrum for projective bundle schemes, Severi-Brauer schemes and show how Mayer-Vietoris decompositions of V and devissage and localisation techniques in algebraic K-theory can be used to compute $\pi_*(\underline{A\overline{\mathbb{F}}_{qV}})$.

(c) Example 1.4.6(b) works equally well over the complex field. If $V \to \operatorname{Spec}\mathbb{C}$ is a scheme then, as in §1.4.6(b), we obtain

$$(BGL\mathbb{C}^+)_p^\wedge \to (X_{\underline{\underline{P}}(V)})_p^\wedge .$$

Even though $GL\mathbb{C}$ has the discrete topology (so the left hand space is much bigger than $BU_p^\wedge$) the inclusion of $Z/p^\infty \subset S^1 \subset GL_1\mathbb{C}$ induces $(BZ/p^\infty)_p^\wedge \to (BGL\mathbb{C}^+)_p^\wedge$ and $(BZ/p^\infty)_p^\wedge \simeq (\mathbb{C}P^\infty)_p^\wedge$. Hence we may obtain $b_V \in \pi_2(X_{\underline{\underline{P}}(V)})_p^\wedge$ in a natural manner and construct the spectra $\underline{A\mathbb{C}}_V$ after the manner of §1.$\bar{4}$.6(b).

Now I would like to make some elementary observations concerning the foregoing examples.

1.5: Lemma. If multiplication by $v \in Z$ annihilates the element of $b \in \pi_N^S(X)$ then multiplication by v annihilates $X(b)^j(W)$ for all spaces W and all integers j.

Proof. We use the notation of section 1.2. If $0 : S^N \to X$ is the trivial S-map then the composite S-map

$$\Sigma^N(\Sigma^N X) \xrightarrow{H} \Sigma^N(S^N \times X) \xrightarrow{\Sigma^N(0 \oplus 1_X)} \Sigma^N X$$

is trivial. Suppose that

$$G \in X(b)^j(W) \simeq \varinjlim_k \{\Sigma^{kN-j}W, X\}$$

is represented by the S-map $g : \Sigma^{kN-j}W \to X$. The vg is represented by

$$\begin{aligned} \varepsilon_\#(vg) &= v(\Sigma^N(b \oplus 1_X)) \circ H \circ \Sigma^N g \\ &\simeq \Sigma^N(vb \oplus 1_X) \circ H \circ \Sigma^N g \\ &\simeq \Sigma^N(0 \oplus 1_X) \circ H \circ \Sigma^N g \\ &\simeq 0 \end{aligned}$$

1.6: Corollary. In the notation of 1.4.2/3 both (a) $AO^j(W)$, and (b) $AZ^j(W)$ are annihilated by multiplication by two for all j and W.

(In fact in section 3 we will see that we knew 1.6(a) already!)

1.7: $H_*(AO;Z/2)$ and $H_*(AZ;Z/2)$. The generator of $H_j(RP^\infty;Z/2)$ defines a class in $H_{j+1}(AO_1;Z/2)$ and thence a class

$$u_j \in H_j(AO;Z/2).$$

Since $O(1) = GL_1Z$ there is a commutative diagram of natural maps

$$\begin{array}{ccc} BGL_1Z = RP^\infty & = & BO(1) \\ \downarrow & & \downarrow \\ BGLZ^+ & \xrightarrow{i} & BO \simeq BGL\mathbb{R}\,. \end{array}$$

Therefore the generator of $H_j(RP^\infty;Z/2)$ defines a class

$$v_j \in H_j(AZ;Z/2)$$

satisfying $i_*(v_j) = u_j$.

Computations like those of [Part II, section 3.6] yield the following result.

1.8: Proposition. In the notation of section 1.7

(a) $H_*(AO;Z/2) \simeq Z/2[u_1,u_1^{-1},u_2,\ldots, u_n,\ldots]$;

(b) $H_*(AZ;Z/2) \simeq H_*(BGLZ;Z/2)[v_1^{-1}]$; and

(c) $i : BGLZ^+ \to BO$ induces a homomorphism of spectra $i : AZ \to AO$ such that

$$i_* : H_*(AZ;Z/2) \to H_*(AO;Z/2)$$

is onto. In fact $i_*(v_j) = u_j$.

1.9: Proposition. In examples 1.4.3(a) for all spaces W and integers j

$$\widetilde{A\mathbb{F}_q}(T)^j(W) = 0.$$

Proof. From [Q] we know that any element $b \in K_N(\mathbb{F}_q)$ has zero square. This squaring operation is the one induced by the H-space product on $BGL\mathbb{F}_q^+$ ("Whitney sum"). Hence all iterated products of b must be zero in $\pi_*^S(BGL\mathbb{F}_q)$.

Now arguing as in Lemma 1.5 we see that the 2-fold iterate of the structure map, ε, is stably trivial. Hence the spectrum is trivial.

1.10: Theorem. Suppose, in §1.2, that $X = X_1 \times X_2$ is a product of H-spaces and $b \in \pi_N^S(X)$ is in the summand $\pi_N^S(X_1)$. Then

(a) $X_1(b)$ is a summand in $X(b)$.

(b) In fact $X(b)$ is equal to the spectrum $X_1(b) \wedge (X_2^+)$ where X_2^+ is the union of X_2 with a disjoint point, *.

(c) From (b) we may identify $\pi_*(X(b))$ with $X_1(b)_*(X_2)$, the unreduced $X_1(b)$-homology of X_2. Then the product on $\pi_*(X(b))$ induced by the ring spectrum becomes the produced induced by the H-space structure of X_2.

Proof. (a) This is clear since the $X(b)$ construction is natural for H-maps which preserve b and the maps $X_1 \subset X_1 \times X_2 \to X_1$ fall into this category.

(b) Stably $X_1 \times X_2 \simeq X_1 \vee (X_1 \wedge X_2) \vee X_2$ so for S-homotopy classes we may write

$$\{\_,X_1 \times X_2\} = A_1(\_) \oplus A_2(\_) \oplus A_3(\_) \tag{1.10.1}$$

where
$$A_i(\_) = \{\_,X_i) \qquad (i = 1,2)$$

and
$$A_3(\_) = \{\_,X_1 \wedge X_2\}.$$

The structure map of $(X_1 \times X_2)(b)$ is given by multiplication by b

$$\varepsilon_\# = (b.\_) : \{\_,X_1 \times X_2\} \to \{\Sigma^N\_,X_1 \times X_2\}.$$

We must investigate $(b.\_)$ in terms of the decomposition (1.10.1). Clearly if $x_1 \in A_1(\_) = A_1$ then $b(x_1,0,0) = (bx_1,0,0)$ by (a). Also the Hopf, H, construction has the property that the S-map

$$X_1 \wedge X_2 \xrightarrow{H} X_1 \times X_2 \xrightarrow{\pi} X_1 \wedge X_2$$

is the identity where $\pi(y_1,y_2) = y_1 \wedge y_2$ and H is trivial when composed with either projection $X_1 \times X_2 \to X_i$. From the definition of ε given in §1.2 it follows that

$$\varepsilon_{\#}(0,0,x_3) = (0,0,(\varepsilon \wedge 1_{X_2})(x_3))$$

where $\varepsilon \wedge 1_{X_2}$ is the smash product of 1_{X_2} with the structure map, ε, of $X_1(b)$. Similarly one sees that $\varepsilon_{\#}(0,x_2,0) = (0,0,0)$. Hence when we form the limit over successive compositions with $\varepsilon_{\#}$ we obtain an isomorphism

$$\begin{aligned}&\varinjlim_k (A_1(\Sigma^{kN}\_) \oplus A_2(\Sigma^{kN}\_) \oplus A_3(\Sigma^{kN}\_))\\ &\cong \varinjlim_k A_1(\Sigma^{kN}\_) \oplus \varinjlim_k A_3(\Sigma^{kN}\_)\end{aligned}$$

where the first limit is over composition with ε, the structure map of $X_1(b)$, and the second limit is over composition with $\varepsilon \wedge 1_{X_2}$.

In terms of spectra this means that $(X_1 \times X_2)(b)$ equals the sum of $X_1(b)$ and $X_1(b) \wedge X_2$, which establishes part (b).

(c) We must check the various products of elements represented $a_1, a_1' \in A_1$ and $a_3, a'_3 \in A_3$ in the splitting (1.10.1). By (a) the product a_1a_1' corresponds to the product in $\pi_*(X_1(b))$.

Suppose that a_1 is represented by

$$a_1 : S^{kN+j} \to \Sigma^{kN}X_1 \to \Sigma^{kN}(X_1 \times X_2)$$

and a_3 is represented by

$$a_3 : S^{\ell N+i} \to \Sigma^{\ell N}(X_1 \wedge X_2) \xrightarrow{H} \Sigma^{\ell N}(X_1 \times X_2).$$

The product a_1a_3 is, by definition represented by an S-map of the form

$$\begin{array}{c} S^{kN+j} \wedge S^{\ell N+i} \\ \Big\downarrow a_1 \wedge a_3 \\ \Sigma^{kN}X_1 \wedge \Sigma^{\ell N}(X_1 \wedge X_2) \\ \Big\downarrow H' \\ \Sigma^{(k+\ell)N}(X_1 \times X_1 \times X_2) \\ \Big\downarrow \Sigma^{(k+\ell)N}(m_{X_1} \times 1_{X_2}) \\ \Sigma^{(k+\ell)N}(X_1 \times X_2) \end{array}$$

where H' is a Hopf construction and $m_{X_1} : X_1 \times X_1 \to X_1$ is the H-space product. Projecting to X_1 we see that a_1a_3 has a zero component in the $A_1(\_)$-component of (1.10.1) while the $A_3(\_)$-component represents is precisely the product of a_3 with a_1 under the $\pi_*(X_1(b))$-module structure of $X_1(b)_*(X_2)$.

Similar is the verification that a_3a_3' is the product in $X_1(b)_*(X_2)$ of a_3 and a_3' under the map induced by the product $m_{X_2} : X_2 \times X_2 \to X_2$. Here one uses the fact that $m_{X_1} \times m_{X_2}$ is used in §1.2 to define the product on the spectrum $(X_1 \times X_2)(b)$. Details will be left to the reader.

1.11: Corollary. Suppose that X is a homotopy commutative homotopy associative H-space. Let $b \in \pi_N(X)$ and let $\Delta : X \to X^n$ be the diagonal map. Then $\pi_*(X^n(\Delta_\#(b)))$ is isomorphic to the $\pi_*(X(b))$-algebra $X(b)_*(X^{n-1})$. (Note that b is not a stable homotopy element in this example.)

Proof. Consider the map $\lambda : X^n \to X^n$ given by

$$\lambda(x_1,\ldots, x_n)= (x_1,m_X(x_2, \chi(x_1)),\ldots, m_X(x_n,\chi(x_1)))$$

where m_X is the H-space product and $\chi : X \to X$ is the homotopy inverse. λ is an H-space equivalence and it sends $\Delta^*(b)$ to the image of b under the inclusion of the first factor. Hence $X^n(\Delta_*(b))$ is equivalent to $(X_1 \times X_2)(b)$ as in §1.10 with $X_1 = X$ and $X_2 = X^{n-1}$ and the result follows from §1.10(c).

1.12. Suppose now that $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration of H-spaces and H-maps. Let $b \in \pi_N^S(F)$ and form the spectra $F(b)$ and $E(i_\#(b))$. The map i induces a homomorphism of ring spectra.

$$i : F(b) \to E(i_\#(b)) \tag{1.12.1}$$

In §1.10 we studied this situation for the trivial fibring $E = F \times B$. Since our spectra are constructed using stable homotopy, which does not behave well with respect to fibrations, the best we can hope for is a spectral sequence by means of which to study (1.12.1).

1.13: Theorem. Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ and $b \in \pi_N^S(F)$ be as in §1.12. Then there exists a (strongly convergent) spectral sequence with the following properties:

(i) $E^2_{p,q} = H_p(B, \pi_q(F(b))) \Rightarrow \pi_{p+q}(E(i_\#(b)))$, where the homology is taken with simple coefficients, $(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})$.

(ii) The associated filtration on $\pi_*(E(i_\#(b)))$ has the form

$$0 = F_{-1,s} \subset F_{0,s} \subset F_{1,s} \subset \cdots \subset F_{p,s} \subset \cdots$$

where
$$\bigcup_p F_{p,s} = \pi_s(E(i_\#(b))),$$

and
$$F_{p,s}/F_{p-1,s} \cong E^\infty_{p,s-p} \cong \varinjlim_r E^r_{p,s-p} .$$

Proof. Let B_n be the n-skeleton of B and filter E by $\{\pi^{-1}(B_n)\}$. Applying stable homotopy to this filtration we obtain, by [Sw, Ch. 15], a strongly convergent spectral sequence

$$D^2_{p,q} = H_p(B,\pi^S_q(F)) \Rightarrow \pi^S_{p+q}(E).^4 \tag{1.13.1}$$

By making the multiplication $m: B \times B \to B$ cellular we may assume that the multiplication on E factors as $\{m: E_n \times E_m \to E_{n+m}\}$. Hence, as $F \subset E_0$ we obtain a family of compatible maps

$$m \circ (i \times 1) : F \times E_n \to E_n.$$

Hence multiplication by $i_\#(b)$ on $\pi^S_*(E)$ induces a map of spectral sequences in (1.13.1) which sends $D^2_{p,q}$ to $D^2_{p,q+N}$ by multiplying $\pi^S_q(F)$ by b. Now form the direct limit of the spectral sequence (1.13.1) under successive iterates of the above map. Since $\varinjlim$ is exact we obtain a spectral sequence which evidently satisfies (i). Also, because the filtration associated with (1.13.1) is trivial in negative dimensions the resulting filtration on $\pi_*(E(i_\#(b)))$ will be trivial in negative dimensions. Furthermore, because direct limits and unions commute, $\cup_s F_{p,s}$ equals $\pi_s(E(i_\#(b)))$. Similarly $F_{p,s}/F_{p-1,s} \cong D^\infty_{p,s-p} \cong \varinjlim_r D^r_{p,s-p}$ implies the analogous result for $E^\infty_{p,s-p}$. Hence (ii) is established which means ([C-E] Ch. 15; [Sw] Ch. 15) that the spectral sequence converges strongly.

§2. THE STABLE HOMOTOPY OF BO

The objective of this section is to prove the following result.

<u>2.1</u>: <u>Theorem</u>. For suitable integers, $d(\ell)$, there is a (4k-2)-equivalence between

$$\frac{BO(2k)}{BO(2k-2)} \quad \text{and} \quad \prod_{h=0}^{2k-1} K(Z/2,2k-1+h)^{d(h)} \times \prod_{\ell=0}^{2k} K(Z/2,2k+\ell)^{d(\ell)}.$$

The $d(\ell)$ are defined in section 2.6. $K(Z/2,n)$ is the usual Eilenberg-Maclane space.

The proof of Theorem 2.1 is essentially due to R. Thom [Th, section 6]. However, for completeness, I will give it in full in a series of steps (sections 2.4 - 2.9). As a corollary of [Part I, Theorem 4.2(ii)] and Theorem 2.1 we will obtain the following result. It will be proved in section 2.10.

<u>2.2</u>: <u>Theorem</u>. There is a decomposition of stable homotopy groups

$$\pi^S_j(BO(2k)) \cong \pi^S_j(BO(2k-2)) \oplus B_j(k)$$

for all k,j. If $j < 4k - 2$ then $B_j(k)$ contains the direct sum of $\beta_j(k)$ copies of Z/2 where $\beta_j(k)$ is given by

$$\beta_j(k) = \begin{cases} 0 & \text{if } j < 2k-1 \\ 1 & \text{if } j = 2k-1 \\ d(j-2k+1)+d(j-2k) & \text{if } 2k \le j \le 4k-3 \end{cases}$$

Furthermore the generators of the Z/2's above support the cohomology classes X^h_ω and Y^ℓ_ω defined in section 2.6.

Here d(n) is the integer of Theorem 2.1.

Remark. Of course, one may be able to generate further elements of $\pi_*^S(BO)$ from the "basic" elements which the above result provides by means of the tensor product pairing (cf. Part I, section 6). I do not propose to undertake a detailed calculation of that process here.

For the rest of this section H^* will mean mod 2 singular cohomology. Firstly we must get our notation straight.

2.3: Stiefel-Whitney classes and the ordering. $H^*(BO(1)) \cong Z/2[t]$ where $\deg t = 1$. Hence

$$H^*(BO(1)^{2k}) \cong Z/2[t_1,\dots, t_{2k}]$$

where $\deg t_s = 1$ and t_s belongs to the s-th factor.

Let $h : BO(1)^{2k} \to BO(2k)$ be the natural map then h^* is injective. If $w_v \in H^v(BO(2k))$ is the v-th Stiefel-Whitney class then $h^*(w_v)$ is equal to the v-th elementary symmetric function in $t_1,\dots, t_{2k}$.

$H^*(BO(2k)) \cong Z/2[w_1,\dots, w_{2k}]$ embeds, via h^*, as the algebra of symmetric polynomials.

We may identify $H^*\left(\frac{BO(2k)}{BO(2k-2)}\right)$ with the ideal $\langle w_{2k-1}, w_{2k}\rangle$ in $H^*(BO(2k))$. Hence we may also interpret this ideal as symmetric polynomials in $t_1,\dots, t_{2k}$. Henceforth we will make such identifications without further mention.

If $\underline{t}^{\underline{\varepsilon}} = t_1^{\varepsilon_1} \dots t_{2k}^{\varepsilon_{2k}}$ is a monomial denote by

$$\mathrm{orb}(\underline{t}^{\underline{\varepsilon}}) \in Z/2[t_1,\dots, t_{2k}] \tag{2.3.1}$$

the sum of all distinct translations of $\underline{t}^{\underline{\varepsilon}}$ under the symmetric group, Σ_{2k}. Hence $\mathrm{orb}(\underline{t}^{\underline{\varepsilon}})$ may be interpresented as an element of $H^*(BO(2k))$.

For
$$i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$$
$$j_1 \geq j_2 \geq \cdots \geq j_m \geq 0$$
we write
$$w_{i_1} w_{i_2} \cdots w_{i_n} > w_{j_1} w_{j_2} \cdots w_{j_m} \tag{2.3.2}$$
if and only if for some $s \geq 0$
$$i_1 = j_1,\ i_2 = j_2, \dots,\ i_s = j_s,\ i_{s+1} > j_{s+1} .$$

The monomial ordering is defined by (2.3.2). An expression of the form

$$w_{u_1} \cdots w_{u_s} + (\text{lower monomials})x \tag{2.3.3}$$

will mean a polynomial expression in which $w_{u_1} \cdots w_{u_s}$ is strictly higher in the monomial ordering of (2.3.2) than any monomial in the bracket.

2.4: Lemma. With the conventions of section 2.3 in $H^*\left(\frac{BO(2k)}{BO(2k-2)}\right)$

$$Sq^U(w_{2k}) = w_{u_1} \cdots w_{u_r} w_{2k} + (\text{lower monomials}) w_{2k}$$

and

$$Sq^U(w_{2k-1}) = w_{u_1} \cdots w_{u_r} w_{2k-1} + (\text{lower monomials}) w_{2k-1} \ (\text{mod ideal } \langle w_{2k} \rangle).$$

Here $U = (u_1, \dots, u_r)$ satisfies $\Sigma u_j \le 2k-1$ and U is an admissible sequence of integers [E-S] and Sq^U is the corresponding iteration of Steenrod operations.

Proof. By Wu's formula [M-St; Th, p. 37]

$$Sq^s w_{2k} = w_{2k} w_s$$

while $Sq^s w_{2k-1} = w_{2k-1} w_s + (s-1) w_{s-1} w_{2k}$ so we may start an induction on r.

First consider $Sq^U(w_{2k})$. By induction and the Cartan formula

$$Sq^U(w_{2k}) = \sum_{0 \le a \le u_1} w_{2k} w_a Sq^{u_1 - a}(w_{u_2} \cdots w_{u_r} + (\text{lower monomials}))$$

The term when $a = u_1$ is clearly of the desired form since Sq^0 = Identity. Now if $j < u_2$ then by the Wu formula $Sq^s w_j$ contains w_b's only for $b < 2j < 2u_2$. Since U is admissible $2u_2 \le u_1$ so expanding the expression

$$w_{2k} w_a Sq^{u_1 - a}(w_{u_2} \cdots w_{u_r} + (\text{lower monomials}))$$

with $0 \le a < u_1$ by means of the Cartan formula we see that it contains no terms involving w_j for $j \ge u_1$. Hence this expression is of the form

$$w_{2k}(\text{lower monomials})$$

and the induction is complete.

The same argument establishes the second formula since the ideal $\langle w_{2k} \rangle$ is invariant under the action of the Steenrod algebra.

2.5: Corollary. In $H^j\left(\frac{BO(2k)}{BO(2k-2)}\right)$ with $j \le 4k-2$ the elements $Sq^U w_{2k-1}$ and $Sq^U w_{2k}$ are linearly independent as U ranges through admissible sequences.

Proof. If $\Sigma \lambda_U Sq^U w_{2k} + \Sigma \mu_V Sq^V w_{2k-1} = 0$ consider the highest monomial appearing in the expression $\Sigma \mu_V Sq^V w_{2k-1}$. Reducing the linear relation modulo $\langle w_{2k} \rangle$ and appealing to Proposition 2.4 we see that the highest monomial comes from only one term whose coefficient, μ_V, must therefore be zero. By induction on the monomial ordering all the μ_V are zero. A similar induction shows that all the λ_U are zero also.

2.6: Dyadism. Here we recall R. Thom's terminology concerning dyadic variables, non-dyadic partitions and dyadic ordering of monomials.

First let d(h) be the number of non-dyadic partitions of h. That is, the number of unordered sets of positive integers $(a_1, \dots, a_s)$ such that $\Sigma a_i = h$

and no a_i is of the form $2^m - 1$.

Now let $t_1, \ldots, t_{2k}$ be the variables introduced in section 2.3. If $p(t_1, \ldots, t_{2k})$ is a polynomial we call t_n a <u>dyadic variable</u> of $p(t_1, \ldots)$ if it appears in each monomial with exponent zero or a power of two. If t_n is dyadic in $p(t_1, \ldots)$ it is dyadic in $Sq^r(p(t_1, \ldots))$.

Now we define the <u>dyadic ordering</u> of monomials in $t_1, \ldots, t_{2k}$. Let $\underline{t}^{\varepsilon} = t_1^{\varepsilon_1} \ldots t_{2k}^{\varepsilon_{2k}}$. Set

$$u(\underline{t}^{\varepsilon}) = (\text{number of non-dyadic variables in } \underline{t}^{\varepsilon})$$

and

$$v(\underline{t}^{\varepsilon}) = (\text{the total degree of the non-dyadic variables in } \underline{t}^{\varepsilon}).$$

We say $x = \underline{t}^{\varepsilon} > \underline{t}^{\delta} = y$ if and only if

$$\text{either } u(x) > u(y) \text{ or } u(x) = u(y) \text{ and } v(x) < v(y).$$

For all $h \le 2k - 1$ form

$$X^h_\omega = \mathrm{orb}(t_1^{a_1+1} \ldots t_r^{a_r+1} t_{r+1} \cdots t_{2k-1})$$

where $\omega = (a_1, \ldots, a_r)$ runs through non-dyadic partitions of h. Also for $\ell \le 2k$, form

$$Y^\ell_\omega = \mathrm{orb}(t_1^{b_1+1} \ldots t_s^{b_s+1} t_{s+1} \cdots t_{2k})$$

where $\omega = (b_1, \ldots, b_s)$ runs through non-dyadic partitions of ℓ. Orb (_) was defined in section 2.3.1.

Hence $X^h_\omega \in H^{2k-1+h}\left(\frac{BO(2k)}{BO(2k-2)}\right)$ and $Y^\ell_\omega \in H^{2k+\ell}\left(\frac{BO(2k)}{BO(2k-2)}\right)$.

Finally consider the following set of elements in $H^*\left(\frac{BO(2k)}{BO(2k-2)}\right)$.

$$X^m_\omega,\ Sq^1 X^{m-1}_\omega, \ldots,\ Sq^{U_h} X^h_{\omega_h}, \ldots,\ Sq^U w_{2k-1} \qquad (2.6.1)$$

and

$$Y^n_\omega,\ Sq^1 Y^{n-1}_\omega, \ldots,\ Sq^{V_\ell} Y^\ell, \ldots,\ Sq^V w_{2k}. \qquad (2.6.2)$$

In (2.6.1) m is any integer such that $m \le 2k - 1$, U_h runs through admissible sequences of degree $m - h$ while ω_h runs through the set of d(h) non-dyadic partitions of h. In (2.6.2) n is any integer such that $n \le 2k$, V_ℓ runs through admissible sequences of degree $n - \ell$ while ω_ℓ runs through non-dyadic partitions of ℓ.

<u>2.7</u>: <u>Lemma</u>. Let $\omega = (\alpha_1, \ldots, \alpha_r)$. Let $Sq^I X^h_\omega$ and $Sq^I Y^\ell_\omega$ be elements of (2.6.1) and (2.6.2) respectively. Then, in the dyadic ordering

(i) $\mathrm{orb}(t_1^{\alpha_1+1} \ldots t_r^{\alpha_r+1} Sq^I(t_{r+1} \ldots t_{2k}))$ contains the maximal monomials of $Sq^I Y^\ell_\omega$, and (ii) $\mathrm{orb}(t_1^{\alpha_1+1} \cdots t_r^{\alpha_r+1} Sq^I(t_{r+1} \cdots t_{2k-1}))$ contains the maximal monomials of $Sq^I X_\omega^h$ which are not in the ideal $\langle w_{2k}\rangle$.

Proof. Let us start with (i).

As we remarked in section 2.6 if t_n is dyadic in a polynomial p it is dyadic in $Sq^I(p)$. This is because

$$Sq^r(t_j^m) = \binom{m}{r} t_j^{m+r}.$$

From this observation $Sq^I(t_{r+1} \cdots t_{2k})$ is totally dyadic and hence if x is a monomial in $Sq^I Y_\omega^\ell$ then $u(x) \le r$. There are only two ways to achieve $u(x) = r$. Clearly any monomial in the expression displayed in (i) has u-value equal to r. This term has monomials whose v-value is $r + \ell$. The other way is to obtain monomials from the expansion of $Sq^I Y^\ell$ of the form

$$[Sq^{a_1}(t_1^{a_1+1}) \cdots][Sq^{a_{r+1}}(t_{r+1}) \cdots Sq^{a_{2k}}(t_{2k})]$$

in which $\sum_1^r a_j > 0$ and in which the first bracket contributes a non-dyadic monomial. The v-value for such a monomial is

$$\sum_j a_j + \sum_s (\alpha_s + 1) > r + \ell.$$

Hence the assertion (i) is proved.

The proof of (ii) is similar, all the equations being taken modulo $\langle w_{2k}\rangle$.

2.8: Proposition. The elements of degree j in (2.6.1) and (2.6.2) form a linearly independent subset of

$$H^j\left(\frac{BO(2k)}{BO(2k-2)}\right) \qquad \text{if } j \le 4k - 2.$$

Proof. I will give a proof, using Lemma 2.7, which will show that the elements of (2.6.2) are linearly independent and that the elements of (2.6.1) are linearly independent modulo the ideal $\langle w_{2k}\rangle$. Since the (2.6.2) elements are in $\langle w_{2k}\rangle$ the result follows.

First suppose a relation exists

$$0 = \sum_{\lambda,\omega,\ell} c_\lambda Sq^{I_\lambda} Y_\omega^\ell.$$

There can be no linear dependence relations A = B if the monomials of maximal dyadic order in A are strictly dyadically bigger than all monomials in B. Hence the only possible relations are between $Sq^I Y$'s with the same maximal occurring u-values and v-values. That is, $u = r$ and $v = r + \ell$, by the proof of Lemma 2.7. However for fixed ℓ the maximal terms in $Sq^I Y^\ell$ for different ω are all distinct as ω varies. Hence the only possible relation is of the form

$$0 = \Sigma c_\lambda Sq^{I_\lambda} Y_\omega^\ell$$

for fixed ω, ℓ. But in this expression the dyadically maximal terms, by Lemma 2.7, are orb(z) where

$$z = \sum_\lambda c\, t_1^{\alpha_1+1} \cdots t_r^{\alpha_r+1} (Sq^{I_\lambda}(t_{r+1} \cdots t_{2k}))$$
$$= (t_1^{\alpha_1+1} \cdots) \sum_\lambda c_\lambda Sq^{I_\lambda}(t_{r+1} \cdots t_{2k}).$$

However if orb(z) = 0 then z = 0 but since $(t_1^{\alpha_1+1} \cdots t_r^{\alpha_r+1}) \neq 0$ we obtain a contradiction to the linear independence of the set

$$\{Sq^{I_\lambda}(w_{2k-r-1})\}$$

which is proved as in Lemma 2.4. Hence $c_\lambda = 0$ for all λ.

Now repeat the proof modulo $\langle w_{2k}\rangle$ using Lemma 2.7 (ii) to obtain the linear independence of the $Sq^I X_\omega^h$'s modulo $\langle w_{2k}\rangle$.

2.9: Proof of Theorem 2.1. The product of the cohomology classes X_ω^h and Y_ω^ℓ gives a map from $\frac{BO(2k)}{BO(2k-2)}$ into the product of Eilenberg-Maclane spaces in the statement of Theorem 2.1. In dimensions less than 4k - 2 [Ser] tells us that a basis for the mod 2 cohomology of this product is given by the images of the fundamental classes under admissible Sq^I operations. Hence, to demonstrate an equivalence at the prime 2, we must check that the elements of (2.6.1) and (2.6.2) generate $H^*\left(\frac{BO(2k)}{BO(2k-2)}\right)$ in dimensions $\leq 4k - 2$. Let c(t) equal the number of dyadic partitions of t. Since the only indecomposable Sq^i are those with $i = 2^m$ for some m [E-S, p. 10] c(t) equals the number of admissible Sq^I of degree t. Hence the elements of (2.6.1) and (2.6.2) are a basis for a vector subspace of $H^{t+2k-1}\left(\frac{BO(2k)}{BO(2k-2)}\right)$ which has dimension (if $t \leq 2k - 1$) $\sum_h d(h)c(t-h) + \sum_\ell d(\ell)c(t-1-\ell)$. The first sum is the number of partitions of t since each partition splits uniquely into a dyadic and a non-dyadic part. The second sum is the number of partitions of t - 1. The result is clearly equal to $\dim H^{t+2k-1}\left(\frac{BO(2k)}{BO(2k-2)}\right)$. At other primes both spaces are trivial in these dimensions so the map is a (4k - 2)-equivalence.

2.10: Proof of Theorem 2.2. The decomposition of $\pi_j^S(BO(2k))$ follows from [Part I, Theorem 4.2(ii)]. The fundamental classes of the product of Eilenberg-Maclane spaces of Theorem 2.1 provide the direct sums of Z/2's in $\pi_j\left(\frac{BO(2k)}{BO(2k-2)}\right)$. Since these are detected by cohomology they are non-trivial in $\pi_j^S\left(\frac{BO(2k)}{BO(2k-2)}\right)$. By an argument similar to that of [Part II, Lemma 2.7], which is given in section 3, the S-map

$$BO \to \bigvee_{1\leq k} \frac{BO(2k)}{BO(2k-2)}$$

sends each mod 2 homology class to "itself". Here we interpret homology classes of $H_*\left(\frac{BO(2k)}{BO(2k-2)}\right)$ as elements of $H_*(BO)$ as we did for cohomology in section 2.3. Hence the fundamental classes support the classes Y^{ℓ}_{ω} and X^{h}_{ω} because of the choice of (4k-2)-equivalence given in section 2.9.

§3. UNORIENTED COBORDISM

In this section I wish to identify AO-theory in much the same way as I computed AU- and ASp-theory in [Part II, sections 1/2].

If (X^h_ω) are the elements of (2.6.1) then $U_k = \Pi X^h_\omega$ may be interpreted as an element of $MO^{2k-1}\left(\frac{BO(2k)}{BO(2k-2)}\right)$. Here the classes X^h_ω run through the set exhibited in (2.6.1). Then by means of [Part I, Theorem 4.2] this class may be "lifted" to $U_k \in MO^{2k-1}(BO)$. There is an obvious choice of "lifting" since MO^* is just a sum of copies of $H^*(\_;Z/2)$, namely the one used in the cohomology identifications of section 2.3. When we compute $(\nu_{O(2n)})_*$ in section 3.4 we will see that these two "liftings" are the same. Similarly if (Y^{ℓ}_{ω}) are the elements of (2.6.2) $V_k = \Pi Y^{\ell}_{\omega}$ defines a class (see §3.10)

$$V_k \in MO^{2k}(BO).$$

The main result of this section is the following.

<u>3.1</u>: <u>Theorem</u>. Let $F \in AO^{\circ}(W)$ be represented by $f \in [\Sigma^{N+1}W, \Sigma BO]$. Then

(a) $f^*(\prod_{k\geq 1} U_k + V_k) \in \prod_{k\geq 1} MO^{2k}(\Sigma^N W) \oplus MO^{2k-1}(\Sigma^N W)$ defines an element $\Phi_0(F) \in MO^*(W)$.

(b) $\Phi_0 : AO^{\circ}(W) \to MO^*(W) = \varinjlim_N \prod_{-N\leq k} MO^k(W)$ is a ring homomorphism, and

(c) Φ_0 is an isomorphism when $\dim W < \infty$.

Theorem 3.1 will be proved in section 3.9. The programme of proof is analogous to that sketched in [Part II, section 2.3] in the unitary case. I suggest the reader consult that sketch before becoming embroiled in the technical details which follow. Briefly we wish to take the AO-theory structure map and analyse its square, $\Sigma^2(\Sigma^2 BO) \to \Sigma^2 BO$, in terms of the splitting [Part I, Theorem 4.2]. In terms of this splitting we want to show that this map is stably the sum of maps $\Sigma^2\left(\Sigma^2 \frac{BO(2k)}{BO(2k-2)}\right) \to \Sigma^2 \frac{BO(2k+2)}{BO(2k)}$ $(1 \leq k)$ plus a map which decreases the "BO-skeletal filtration". So we split the map stably into two pieces and apply obstruction theory to compress the "skeletal filtration" of the unwanted piece.

Firstly we must analyse in homology the splitting given by [Part I, Theorem 4.2].

The next result is the real analogue of [Part II, section 1].

<u>3.2</u>: <u>Proposition</u>. Consider the composition of S-maps

$$\begin{tikzcd} BO(1)^{2n} \arrow[r, "k"] & B\Sigma_n \int O(2) \arrow[r] & F_n C_\infty BO(2) \arrow[r, "\alpha_n"] & \bigvee_{t \le n} \dfrac{F_t C_\infty BO(2)}{F_{t-1} C_\infty BO(2)} \arrow[dll] \\ & & \dfrac{F_s C_\infty BO(2)}{F_{s-1} C_\infty BO(2)} & \end{tikzcd}$$

(Here we are using the notation of [Part I, Proposition 3.7 and section 4].) If $s < n$ this composite is the track-group sum of maps $g_1, \ldots, g_{u_{2n}}$ each of which factors through one of the canonical projections from $BO(1)^{2n}$ onto a copy of $BO(1)^{2s}$.

Proof. In the proof of [Part II, Theorem 1.2] given in [Part II, section 1.4] replace BU(1) by BO(2). The proof then shows that

$$BO(2)^n \to B\Sigma_n \int O(2) \to \frac{F_s C_\infty BO(2)}{F_{s-1} C_\infty BO(2)}$$

is the sum of maps which factor through projections $BO(2)^n \to BO(2)^s$. Now restrict to $BO(1)^{2n}$ via the natural map.

3.3: $H_*(BO;Z/2)$. Let $u_j \in H_j(BO;Z/2)$ denote the image of the generator of $H_j(BO(1);Z/2)$ $(j \ge 1)$. Then

$$H_*(BO;Z/2) = Z/2[u_1, u_2, \ldots] \quad \text{(cf. section 1.8).}$$

$H_*(BO(2k);Z/2)$ may be identified with the subspace spanned by monomials of weight $\le 2k$. In this subspace $H_*\left(\frac{BO(2k)}{BO(2k-2)};Z/2\right)$ may be identified with the subspace spanned by monomials of weight $(2k-1)$ or $2k$. With these conventions we have the following result.

3.4: Proposition. Let $\nu_{O(2n)} : BO(2n) \to \bigvee_{1 \le \ell \le n} \frac{BO(2\ell)}{BO(2\ell-2)}$ be the S-equivalence of [Part I, Theorem 4.2]. In the notation of section 3.3

$$(\nu_{O(2n)})_*(u_{i_1} \otimes \cdots \otimes u_{i_p}) = u_{i_1} \otimes \cdots \otimes u_{i_p}$$

for all $1 \le p \le n$.

Proof. Since $\nu_{O(2n)}|BO(2n-2) \simeq \nu_{O(2n-2)}$ we may take $p = 2n-1$ or $2n$. The stable map $\nu_{O(2n)}$ is a composite of the form (in the notation of [Part I, section 4])

$$\begin{array}{c} BO(2n) \xrightarrow{\tau(\pi_n)} B\Sigma_n \int O(2) \to F_n C_\infty(BO(2)) \\ \swarrow \alpha_n \\ \bigvee_{1\le t\le n} \dfrac{F_t C_\infty BO(2)}{F_{t-1} C_\infty BO(2)} = \bigvee_{1\le t\le n} \dfrac{B\Sigma_t \int O(2)}{B\Sigma_{t-1} \int O(2)} \\ \swarrow \vee \pi_t \\ \bigvee_{1\le t\le n} \dfrac{BO(2t)}{BO(2t-2)} \end{array}$$

The class $u_{i_1} \otimes \cdots \otimes u_{i_p} = h_*(x_{i_1} \otimes \cdots \otimes x_{i_p})$ where $0 \neq x_s \in H_s(BO(1);Z/2)$ and h is the natural map (cf. [Part I, Proposition 3.7]). The part of the composite $\nu_{O(2n)}$ which has domain $B\Sigma_n \int O(2)$ and range $\dfrac{B\Sigma_n \int O(2)}{B\Sigma_{n-1} \int O(2)}$ is just the canonical collapse by property (b) of [Part I, section 4.5]. From [Part I, Proposition 3.7]

$$\tau(\pi_n) \circ h = \sum_g I(g)k_g \qquad \text{where } I(g) = \pm 1$$

and k_g is the conjugate of k by the permutation of $BO(1)^{2n}$ induced by g. Hence the part of $\nu_{O(2n)} \circ h$ with range $\frac{BO(2n)}{BO(2n-2)}$ is equal to $\sum_g I(g)h_g$ where $h:BO(1)^{2n} \to BO(2n)$ is the natural map and h_g is the conjugate by g of the composite

$$BO(1)^{2n} \xrightarrow{h} BO(2n) \to \frac{BO(2n)}{BO(2n-2)}$$

in which the second map is the canonical collapse. In mod 2 homology $(h_g)_* = h_*$. Also we have $\sum_g I(g) = 1$. Therefore the proof of [Part I, Proposition 3.6] shows that

$$\begin{aligned}(\Sigma I(g)h_{g*})(x_{i_1} \otimes \cdots \otimes x_{i_p}) &= h_*(x_{i_1} \otimes \cdots \otimes x_{i_p}) \\ &= u_{i_1} \otimes \cdots \otimes u_{i_n}\end{aligned}$$

That deals with the factor $\frac{BO(2n)}{BO(2n-2)}$. However the maps into factors $\frac{BO(2k)}{BO(2k-2)}$ $(k < n)$ induced by $\nu_{O(2n)} \circ h$ are sums of maps which factor through projection maps, by Proposition 3.2. These projections $BO(1)^{2n} \to BO(1)^{2k}$ $(k < n)$ annihilate $x_{i_1} \otimes \cdots \otimes x_{i_p}$ when $p = 2n-1$ or $2n$. Hence the only non-zero component of $(\nu_{O(2n)} \circ h)_*(x_{i_1} \otimes \cdots \otimes x_{i_p})$ is $h_*(x_{i_1} \otimes \cdots \otimes x_{i_p})$ and the proof is complete.

<u>3.5</u>: <u>Corollary</u>. Let $\varepsilon : \Sigma^2 BO \to BO$ be the structure map of the AO-spectrum (Example 1.4.2). By means of the stable decomposition of [Part I, Theorem 4.2]

consider the S-map, $(\Sigma\varepsilon)\circ\varepsilon$, as inducing

$$\varepsilon_n : \bigvee_{1\le t\le n} \Sigma^2 \frac{BO(2t)}{BO(2t-2)} \to \bigvee_{1\le t\le n+1} \frac{BO(2t)}{BO(2t-2)} .$$

Let λ_j equal the composite $(j \le n)$

$$\begin{tikzcd} \Sigma^2 \frac{BO(2j)}{BO(2j-2)} \arrow[r] & \bigvee_{1\le t\le n} \Sigma^2 \frac{BO(2t)}{BO(2t-2)} \arrow[r, "(\Sigma\varepsilon)\circ\varepsilon"] & \bigvee_{1\le t\le n+1} \frac{BO(2t)}{BO(2t-2)} \arrow[dl] \\ & \frac{BO(2j+2)}{BO(2j)} . & \end{tikzcd}$$

Set $\mu_n = \varepsilon_n - \sum_{j=1}^{n} \lambda_j$ then μ_n induces zero in reduced homology with rational coefficients or with Z/q coefficients for any prime q.

Proof. With coefficients in $Z/2$ the result follows at once (cf. [Part II, Corollary 2.8]) from Proposition 3.4 and the fact that

$$(\lambda_j)_*(u_{i_1} \otimes \cdots \otimes u_{i_p}) = \begin{cases} u_1 \otimes u_1 \otimes u_{i_1} \otimes \cdots \otimes u_{i_p} & \text{if } p = j \\ 0 & \text{if } p \ne j \end{cases}$$

With coefficient in Z/q for odd q or with rational coefficients the result is proved by a computation which is essentially the same as the unitary case [Part II, Corollary 2.8].

3.6: Some notation and remarks relating to the proof of Proposition 3.8.

We are now to prove the real analogue of [Part II, Proposition 2.9]. It will be convenient to give names to pieces of the map we wish to deform.

First notice that μ_n has range $\bigvee_{1\le t\le n} \frac{BO(2t)}{BO(2t-2)}$ since the part of ε_n mapping into $\frac{BO(2n+2)}{BO(2n)}$ is λ_n. Define

$$\mu_n(j,\ell) : \Sigma^2 \frac{BO(2j)}{BO(2j-2)} \to \frac{BO(2\ell)}{BO(2\ell-2)}$$

to be that summand of μ_n with this range and domain. For non-triviality we must have $\ell \le j \le n$. For μ_n restricts to μ_s on $\bigvee_{1\le t\le s} \Sigma^2 \frac{BO(2t)}{BO(2t-2)}$ which is stably $\Sigma^2 BO(2s)$ and μ_s maps $\Sigma^2 BO(2s)$ into $BO(2s)$.

Now set $G_j = \frac{BO(2j)}{BO(2j-2)}$. Write $BO(1)^{2n}$ stably as $\bigvee_{j=1}^{n} Y_u$ where Y_j is obtained as follows. $BO(1)^{2n}$ splits as the wedge of q-fold smash products of copies of $BO(1)$ $(1 \le q \le 2n)$. Y_j is the union of all the q-fold smash products where $q = 2j$ or $2j-1$.

3.7: Proposition. In the notation of §3.6 with $\ell \le j \le n$ the following composite S-map is trivial.

$$\Sigma^2 Y_j \subset \Sigma^2(BO(1)^{2n}) \to \Sigma^2 BO(2n) \to \Sigma^2 G_j \xrightarrow{\mu_n(j,\ell)} G_\ell .$$

(Here the nameless maps are the natural ones.)

Proof. Consider the following commutative diagram consisting of S-maps from Part I §4 and from §3.6 above

$$\begin{tikzcd}
\Sigma^2 Y_j \arrow[r, phantom, "\subset"] \arrow[d] & \Sigma^2 BO(1)^{2n} \arrow[r] \arrow[d] & \Sigma^2 BO(2n) \arrow[r, phantom, "\simeq"] \arrow[d, "\varepsilon_n - \lambda_n"'] & \bigvee_{t=1}^{n} G_t \arrow[r] & G_j \arrow[ddl, "\mu_n(j,\ell)"'] \\
Y_{j+1} \arrow[r, phantom, "\overset{i_{j+1}}{\subset}"] & BO(1)^{2n+2} \arrow[r] \arrow[d, "T"'] & BO(2n+2) \arrow[r, phantom, "\simeq"] & \bigvee_{k=1}^{n+1} G_k \arrow[d] & \\
 & \dfrac{F_\ell C_\infty BO(2)}{F_{\ell-1} C_\infty BO(2)} \arrow[rr, "\pi_\ell"] & & G_\ell &
\end{tikzcd}$$

In this diagram T is the S-map of Proposition 3.2 (with n replaced by $n+1$). Hence T is a sum of S-maps, g_i and each g_i factors through a projection of the form $\pi : BO(1)^{2n+2} \to BO(1)^{2\ell}$. The result follows from the fact that $\pi \circ i_{j+1}$ is trivial (i_{j+1} is the inclusion of Y_{j+1} in the diagram).

3.8: Proposition. Let $(G_j)_m$ be the m-skeleton of $G_j = \dfrac{BO(2j)}{BO(2j-2)}$. Then for any $n,m \geq 1$ there is an S-map $\mu_n'(j,\ell) : \Sigma^2 G_j \to G_\ell$ such that

(i) $\mu_n'(j,\ell) \simeq \mu_n(j,\ell)$, the S-map of section 3.6.

(ii) $\mu_n'(j,\ell)$ maps $\Sigma^2(G_j)_m$ to $(G_\ell)_{m-1}$.

Proof. The proof follows the ideas of Part II, §2.9 but the obstruction theory is more subtle. Firstly we observe that all possibly zero obstructions are two primary. This is because $\mu_n(j,\ell)$ is constructed by taking a summand (using the stable splittings of $BO(2k)$) of the S-map $\Sigma^2(BO(2n)) \to BO(2n+2)$ given by the Hopf map on

$$S^1 \times S^1 \times BO(2n) \xrightarrow{\eta \times \eta \times 1} BO(1) \times BO(2n) \to BO(2n+2)$$

and $2\eta = 0 \in \pi_1(BO(1))$ (cf. §1.5). Hence by the mod 2 part of Proposition 3.5 and the argument of Part II, §2.9 we find μ'' homotopic to $\mu_n(j,\ell)$ and fitting into the following diagram on the m-skeleton of G_j.

$$\begin{tikzcd}
\Sigma^2(G_j)_{m-1} \arrow[r] \arrow[d, "\mu''"] & \Sigma^2(G_j)_m \arrow[r] \arrow[d, "\mu''"] & \bigvee_\beta S_\beta^{m+2} \arrow[d, "\mu'''"] \\
(G_\ell)_m \arrow[r] & (G_\ell)_{m+1} \arrow[r] & \bigvee_\alpha S_\alpha^{m+1}
\end{tikzcd} \tag{3.8.1}$$

In (3.8.1) the rows are cofibrations.

We must show that μ''' is trivial. If not then Sq^2 will detect μ''' in $H^*(C(\mu''');Z/2)$ where $C(f)$ means the mapping cone of f. Here "detect" means Sq^2 is non-zero on $H^{m+1}(C(\mu''');Z/2)$. Now consider the S-map $Y_j \to G_j$ introduced in §3.6. It induces

$$g : \frac{\Sigma^2(Y_j)_m}{\Sigma^2(Y_j)_{m-1}} \to \vee_\alpha S_\alpha^{m+1} .$$

Since $H^*(C(\mu''');Z/2) \to H^*(C(g);Z/2)$ is injective Sq^2 would detect g if it detected μ'''. However, by Proposition 3.7, $\mu_n(j,\ell)$ is trivial on $\Sigma^2 Y_j$ if $\ell \le j \le n$ and it is easy to compress the nullhomotopy to give a nullhomotopy of g. Therefor μ''' is trivial.

As in Part II, §2.9 we perform the above argument on consecutive skeleta and then extend to an S-map to get $\mu^{(iv)} \simeq \mu_n(j,\ell)$ fitting into the following diagram.

$$\begin{array}{ccccc}
\Sigma^2(G_j)_{m-1} & \to & \Sigma^2(G_j)_m & \to & \vee_\beta S_\beta^{m+2} \\
\downarrow{\scriptstyle \mu^{(iv)}} & & \downarrow{\scriptstyle \mu^{(iv)}} & & \downarrow{\scriptstyle \mu^{(v)}} \\
(G_\ell)_{m-1} & \to & (G_\ell)_m & \to & \vee_\gamma S_\gamma^m
\end{array} \qquad (3.8.2)$$

In (3.8.2) the rows are cofibrations.

In (3.8.2) we must show that $\mu^{(v)}$ is trivial. I intend to do this by means of a well-known argument due to Adem [T, p. 84, Example 3] by which one shows that $\eta^2 \in \pi_2^S(S^\circ)$ is non-trivial for $0 \ne \eta \in \pi_1^S(S^\circ)$. First consider the composite

$$g' : \Sigma^2(Y_j)_m \to \Sigma^2(G_j)_m \to \vee_\beta S_\beta^{m+2} \xrightarrow{\mu^{(v)}} \vee_\gamma S_\gamma^m .$$

The argument used above to show that the S-map, g, was trivial may be used to show that $g' \simeq 0$. Suppose for β_0 and γ_0 the S-map

$$\mu^{(v)} : S_{\beta_0}^{m+2} \to S_{\gamma_0}^m$$

is non-trivial. Hence it is η^2 where $0 \ne \eta \in \pi_1^S(S^0)$. Thus we have a composition

$$\Sigma^2(Y_j)_m \to S_{\beta_0}^{m+2} \xrightarrow{\eta} S^{m+1} \xrightarrow{\eta} S_{\gamma_0}^m \qquad (3.8.3)$$

which is trivial. However η is non-trivial and detected by Sq^2 in its mapping cone. Also $\Sigma^2(Y_j)_m \to S_{\beta_0}^{m+2} \xrightarrow{\eta} S^{m+1}$ is non-trivial because Sq^2 takes the integral class which is supported by S^{m+1} and maps it non-trivially. Since (3.8.3) is trivial we may attach to the mapping cone $C(\eta)$ the cone on $\Sigma^2(Y_j)_m$. We obtain

$$L = (S^m_{\gamma_0} \cup e^{m+2}) \cup C\Sigma^2(Y_j)_m$$

where $C\Sigma^2(Y_j)_m$ is the cone on $\Sigma^2(Y_j)_m$. Since Sq^2 detects η and the composite of the first two maps in (3.8.3) we see that if $u \in H^m(L;Z/2)$ is the integral class carried by $S^m_{\gamma_0}$ then $0 \neq Sq^2Sq^2u \in H^{m+4}(L;Z/2)$. However $Sq^1u = 0$ since Sq^1 is the Bockstein and annihilates integral classes. Therefore the Adem relation [E-S, p. 2] $Sq^2Sq^2 = Sq^3Sq^1$ shows that $0 = Sq^2Sq^2u$ and we have contradicted the equation $\mu^{(v)} = \eta^2$ so $\mu^{(v)} = 0$. Therefore $\mu^{(iv)}$ is compressible to send $\Sigma^2(G_j)_m$ into $(G_j)_{m-1}$ and $\mu_n'(j,\ell)$ is obtained by extending the compressed map to the whole of Σ^2G_j.

<u>3.9</u>: <u>Proof of Theorem 3.1</u>. If dim $W < \infty$ we have

$$AO^\circ(W) = \varinjlim_N [\Sigma^{N+1}W,\Sigma BO] \simeq \varinjlim_N \{\Sigma^N W, BO\}.$$

Also if dim $W < 4n-2$ then the map induced by $\prod_{k\geq n} (U_k + V_k)$

$$\phi_0 : \left\{W, \frac{BO}{BO(2n-2)}\right\} \to \prod_{\ell\geq 2n-1} MO^\ell(W)$$

is an isomorphism. This is proved like [Part I, Theorem 5.1] using Theorem 2.1 and the identification of MO-theory [T] in terms of mod 2 cohomology (see Remark 3.10 for a sketch of Thom's identification of MO-theory). Hence Φ_0 is a split surjection, by the argument of [Part II, Theorem 2.1].

Φ_0 is, of course, well-defined and since $\prod_{k\geq 1} (U_k + V_k)$ is an exponential map

$$KO^\circ(\_) \to MO^*(\_)$$

Φ_0 is a ring homomorphism. For analogous arguments in the unitary case see [Part II, section 2.3 and 5.3].

It remains to show that Φ_0 is injective. Here the proof is entirely analogous to that of [Part II, Proposition 2.11] using Proposition 3.8 to replace [Part II, Proposition 2.9]. Details are left to the reader.

<u>3.10</u>: <u>Remark</u>. Let H^* denote mod 2 cohomology. If $MO(t)$ is the orthogonal Thom space then $MO(2k-1) = \frac{BO(2k-1)}{BO(2k-2)}$ and we have a natural map $MO(2k-1) \to \frac{BO(2k)}{BO(2k-2)}$. The classes $X^h_\omega \in H^*\left(\frac{BO(2k)}{BO(2k-2)}\right)$ of (2.6.1) pull back to give classes $X^h_\omega \in H^*(MO(2k-1))$. We obtain a map

$$u_k' : \prod_{h,\omega} X^h_\omega : MO(2k-1) \to \prod_h K(Z/2,2k-1+h)^{d(h)}$$

where h,ω run through the set of indices used in (2.6.1) and $d(h)$ is as in §2.6. This map is a $(4k-2)$-equivalence. Similarly there is a collapsing

$\frac{BO(2k)}{BO(2k-2)} \to MO(2k)$ and unique classes $Y^{\ell}_{\omega} \in H^*(MO(2k))$ which pull back to the classes Y^{ℓ}_{ω} of (2.6.2). These maps yield a 4k-equivalence

$$V'_k : \prod_{\ell,\omega} Y^{\ell}_{\omega} : MO(2k) \to \prod_{\ell} K(Z/2, 2k+\ell)^{d(\ell)} .$$

Since $MO^j(W) = \varinjlim_n [\Sigma^{n-j}W, MO(n)]$ the maps U'_k, V'_k (which are compatible with $\varinjlim_n$ here) identify $MO^j(W)$ with products of mod 2 cohomology groups of W when $\dim W < \infty$.

§4. ON THE S-TYPE OF imJ

In this section the S-type of the "image of J" is studied. <u>All spaces will be 2-localised</u> and all homology and cohomology will be taken with coefficients in Z/2. For my purposes the "image of J" will be the space JO(2) of [F-P]. It is defined by an infinite loopspace fibring

$$JO(2) \to BO \xrightarrow{\psi^3 - 1} BSO$$

where ψ^3 is the Adams operation. In [F-P] it is shown that $BO\mathbb{F}_3^+ \simeq JO(2)$. Here $(\_)^+$ is Quillen's construction [Wa] and $O_n\mathbb{F}_3$ is the finite orthogonal group of Part I, §8. Thus JO(2) and $BO\mathbb{F}_3$ have the same (2 local) S-type. In Part I, §8 we stably split $BO\mathbb{F}_3$ into summands of the form $\frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$. By studying these summands we will construct stable homotopy elements in $\pi_*^S(JO(2))$. Since $\pi_*^S(JO(2))$ maps to the stable stem, $\pi_*^S(S^\circ)$, these homotopy elements may prove useful (see the problems at the end of Part IV). The resulting elements in $\pi_*^S(S^\circ)$ almost certainly include Mahawald's highly significant new family [Mah].

<u>4.1</u>: <u>Theorem</u>. There is a 2-local (4n-3)-equivalence between $\frac{BO_{2n}(\mathbb{F}_3)}{BO_{2n-2}(\mathbb{F}_3)}$ and a product of Eilenberg-Maclane spaces

$$\prod_{\varepsilon,\ell,q,J} K(Z/2, 2n+\ell-\varepsilon+J)^{d(\ell)e(2n-q-\varepsilon,J)} .$$

The product is taken over $\varepsilon = 0$ or 1, $q \geq 0$ and $\ell - \varepsilon + J \leq 2n-3$. Also $d(\ell)$ equals the number of partitions $\ell = \Sigma a_i$ of the form $0 < a_i \neq 2^m - 1$ while $e(t,J)$ equals the number of partitions of the form $J + t = \sum_{i=1}^{t} (k_i + 1)$ with $0 \leq k_u \neq k_v$ if $u \neq v$. Furthermore $e(t,J)$ and $d(\ell)$ are defined to be zero if $t < 0$ or $\ell < 0$ respectively and $d(0) = 1 = e(0,0)$.

As an immediate corollary of Theorem 4.1 and Part I, §8 we have the following result.

4.2: Theorem. There is a decomposition of stable homotopy groups (2-localised)

$$\pi_j^S(BO_{2n}\mathbb{F}_3)_{(2)} \cong \pi_j^S(BO_{2n-2}\mathbb{F}_3)_{(2)} \oplus C_j(n)$$

for all j,n. If $j < 4n-2$ then $C_j(n)$ contains the direct sum of $\gamma_j(n)$ copies of Z/2 where $\gamma_j(n)$ is given by

$$\gamma_j(n) = \begin{cases} 0 & \text{if } j < 2n-1 \\ \sum_{\varepsilon,\ell,q,J} d(\ell)e(2n-q-\varepsilon,J) & \text{if} \\ & 2n-1 \leq j \leq 4n-3 \end{cases}$$

Here $j = 2n + \ell - \varepsilon + J$. Also the sum over ε,ℓ,q,J and the functions d(_), e(_,_) are as in Theorem 3.1.

Each Z/2-summand is detected and distinguished by its Hurewicz image in the manner described in §4.14.

4.3: Generators for $H_*(BO\mathbb{F}_3)$. Let us now recall the mod 2 homology of the spaces $BO_{2n}\mathbb{F}_3$. In [F-P] the mod 2 homology of $BO\mathbb{F}_3$ is described. To be precise [F-P] treats an infinite loopspace denoted by $\Gamma_\circ BO\mathbb{F}_3$ which is the zero-component of $\Omega B(\bigcup_{n\geq 0} BO_n\mathbb{F}_3)$. There is a homotopy commutative diagram of maps.

$$\begin{tikzcd} \arrow[r] & BO_n\mathbb{F}_3 \arrow[r] \arrow[d, "(-*[-n])"] & BO_{n+1}\mathbb{F}_3 \arrow[r] \arrow[dl, "(-*[-n-1])"] & \cdots \\ & \Gamma_\circ BO\mathbb{F}_3 & & \end{tikzcd} \tag{4.4}$$

in which (_*[-n]) is the map which sends $BO_n\mathbb{F}_3$ into the n-component by the natural map [F-P] and then translates to the zero-component. (4.4) gives a map

$$BO\mathbb{F}_3 = \varinjlim_n BO_n\mathbb{F}_3 \to \Gamma_\circ BO\mathbb{F}_3 \tag{4.5}$$

which induces an isomorphism in homology. By means of (4.5) we may determine the image of $H_*(BO_{2n}\mathbb{F}_3)$ (mod 2 coefficients) in $H_*(BO\mathbb{F}_3)$. Let T_a and T_b be the copies of Z/2 × Z/2 defined by

$$T_a = \langle A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \rangle \subset O_2\mathbb{F}_3 \qquad \text{and}$$

$$T_b = \langle B_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle = (O_1\mathbb{F}_3)^2 \subset O_2\mathbb{F}_3 .$$

Let $o \neq x_i \in H_i(RP^\infty) = H_i(BZ/2)$ and set $v_{i,j} \in H_{i+j}(BO_2\mathbb{F}_3)$, $v_i \in H_i(BO_1\mathbb{F}_3)$ equal to the images of $x_i \otimes x_j \in H_{i+j}(T_a)$ and $x_i \in H_i(BO_1\mathbb{F}_3)$ respectively. Then set $w_i \in H_i(BO\mathbb{F}_3)$ and $y_i \in H_i(BO\mathbb{F}_3)$ equal to the images of $v_{i,o}$ and v_i respectively. Put $F_n = \operatorname{im}(H_*(BO_{2n}\mathbb{F}_3) \to H_*(BO\mathbb{F}_3))$ then from [F-P, Theorem 3.1 and Proposition 3.11]

4.6: Proposition. F_n is spanned by monomials of the form

$$y_{i_1} y_{i_2} \cdots y_{i_s} w_{j_1} \cdots w_{j_t} \quad \text{with } s + t \le 2n.$$

Here the products are formed in the Pontrjagin ring of $\Gamma_\circ BO\mathbb{F}_3$.

4.7: Proposition. F_n has a basis consisting of the monomials of Proposition 4.6 such that $1 \le j_1 < j_2 < \cdots < j_t$.

Proof. Let $\chi : H_*(BO\mathbb{F}_3) \to H_*(BO\mathbb{F}_3)$ be induced by the H-space inverse on $\Gamma_\circ BO\mathbb{F}_3$. Then if

$$\bar{u}_i = \sum_{a=0}^{i} \chi(w_a)\, y_{i-a} \in H_i(BO\mathbb{F}_3) .$$

[F-P, Theorem 3.2] assures us that there is an isomorphism of algebras

$$H_*(BO\mathbb{F}_3) \cong P(y_1, y_2, \ldots) \otimes E(\bar{u}_1, \bar{u}_2, \ldots).$$

Now $\chi(w_a) = w_a + p(w_1, \ldots, w_{a-1})$ for some polynomial p. Hence $w_i^2 = y_i^2 + q(w_1, \ldots, w_{i-1}, y_1, \ldots, y_{i-1})$ for some polynomial q and the monomials cited above do span F_n. However since

$$\bar{u}_i = w_i + y_i + r(w_1, \ldots, w_{i-1}, y_1, \ldots, y_{i-1}) \tag{4.8}$$

it is easy to show linear independence by induction on the number of and degree of $\bar{u}_i$'s occurring in a monomial.

4.9: Corollary. $H_*\left(\dfrac{BO_{2n}\mathbb{F}_3}{BO_{2n-2}\mathbb{F}_3}\right)$ is isomorphic to the Z/2-vector space with basis consisting of monomials

$$y_{i_1} \cdots y_{i_s} w_{j_1} w_{j_2} \cdots w_{j_t}$$

such that $s + t = 2n - 1$ or $2n$ and $1 \le j_1 < j_2 < \cdots < j_t$. Considered as a subspace of $H_*(BO\mathbb{F}_3)$ the action of the dual Steenrod algebra is induced by

$$(Sq^k)_*(y_{n+k}) = \binom{n}{k} y_n \quad \text{and} \quad (Sq^k)_*(w_{n+k}) = \binom{n}{k} w_n .$$

The diagonal is induced by

$$\Psi(y_i) = \sum_a y_a \otimes y_{i-a} ,$$

$$\Psi(w_i) = \sum_a w_a \otimes w_{i-a} .$$

Proof. By Part I, §8.2.1 $\dfrac{BO_{2n}\mathbb{F}_3}{BO_{2n-2}\mathbb{F}_3}$ is a summand in the S-type of $BO\mathbb{F}_3$ and the inclusion in homology sends a monomial to "itself".

4.10: $H^*\left(\dfrac{BO_{2n}\mathbb{F}_3}{BO_{2n-2}\mathbb{F}_3}\right)$ as an A-module. Let A be the mod 2 Steenrod algebra. We wish to recognise the A-module above and this will be accomplished by comparison with H*(SO) and H*(BO). There is a fibring [F-P]

$$SO \to BO(\mathbb{F}_3)^+ \to BO$$

such that $H_*(BO) = P(\bar{y}_1, \bar{y}_2, \ldots)$ and $H_*(SO) = E(u_1, u_2, \ldots)$ where u_i maps to $\bar{u}_i$ and y_i to $\bar{y}_i$. Let $s_i \in H_i(BO\mathbb{F}_3)$ be the image of the i-th Stiefel-Whitney class. Then s_i is characterised by [cf. Ad 1, p. 49]

$$\langle s_i, y_1^i \rangle = 1 \quad \text{and} \quad \langle s_i, x \rangle = 0$$

for all other monomials $y_i, \ldots w_{j_t}$. We have Wu's formula [M-St, p. 94]

$$Sq^k(s_m) = s_k s_m + \binom{k-m}{1} s_{k-1} s_{m+1} \ldots + \binom{k-m}{1} s_{m+k} \tag{4.11}$$

Now let $S_k(n)$ be the subspace of $H_*\left(\frac{BO_{2n}}{BO_{2n-2}}\right)$ with basis consisting of monomials $y_{i_1} \ldots y_{i_k} w_{j_1} \ldots w_{j_t}$ $(j_1 < j_2 < \cdots; k+t = 2n \text{ or } 2n-1)$. Then $H_*\left(\frac{BO_{2n}}{BO_{2n-2}}\right) = \bigoplus_k S_k(n)$ and the action of the dual Steenrod operations respects this decomposition. In fact $S_k(n)$ together with its $(Sq^\ell)_*$-action is isomorphic to

$$H_*(MO(k)) \otimes W(k)$$

where $W(k) \subset H_*(SO)$ is the vector space spanned by monomials of weight 2n-k or 2n-k-1 in the $\{u_i\}$. To obtain the dual A-module structure on

$$\bigoplus_k S_k(n)^*$$

we must dualise. However although the $\{y_i\}$ have the same diagonal as the $\{\bar{y}_i\}$ the $\{u_i\}$ are primitive while the $\{w_i\}$ are not. The relationship between u_i, w_i and y_i is given in (4.8). Nevertheless the A-module structure is equal "modulo filtration" to that in $H^*(BO) \otimes H^*(SO)$. To make this precise let $\hat{x} \in S_k(n)^*$ denote the dual of a monomial x with respect to the monomial basis described above. The assignment of $\widehat{y_{i_1} \ldots w_{j_t}}$ to $\widehat{\bar{y}_{i_1} \ldots u_{j_t}}$ establishes an additive embedding of $\bigoplus_k S_k(n)^*$ in $H^*(BO) \otimes H^*(SO)$. Note that $\widehat{y_{i_1} \ldots y_{i_\ell}} \otimes \widehat{w_{j_1} \ldots w_{j_t}} = \widehat{y_{i_1} \ldots w_{j_t}}$. Denote by $\bar{S}q^k(\widehat{y_{i_1} \ldots w_{j_t}})$ the element which maps to $Sq^k(y_{i_1} \ldots u_{j_t}) \in H^*(BO) \otimes H^*(SO)$. That is, $\bar{S}q^k$ is a fake Sq^k. Now $\bar{S}q^k = Sq^k$ on elements of the forms $\widehat{y_{i_1} \ldots y_{i_1}}$ and $\hat{w}_j$. Also a simple calculation shows that

$$y^I \widehat{w_{j_1} \ldots w_{j_t}} = \hat{y}^I \hat{W}_{j_i} \ldots \hat{w}_{j_t} + \sum_k y^I \widehat{w^k} \tag{4.12}$$

where $I = (i_1, \ldots)$ and $k = (k_1, \ldots, k_q)$ with $q < t$. By induction on t using equations (4.12) and the relations

$$Sq^R(\widehat{y^I}) = \widehat{(Sq^R)_*(y^I)}$$

$$Sq^R(\hat{w}_j) = \widehat{Sq^R_*(w_j)},\ Sq^\ell(\hat{u}_j) = \widehat{Sq^\ell_*(u_j)}$$

and $\hat{u}_{j_1} \dots \hat{u}_{j_t} = \widehat{u_{j_1} \dots u_{j_t}}$ we obtain the following result.

4.13: Lemma. In the notation of the above discussion

$$Sq^k(\widehat{y^I w_{i_1} \dots w_{i_t}}) - \bar{S}q^k(\widehat{y^I w_{i_1} \dots})$$

is in the subspace spanned by elements of the form $\widehat{y^I w_{k_1} \dots w_{i_q}}$ with $q < t$.

4.14: Proof of Theorem 4.1. Let $z_1, z_2, \dots$ be indeterminates and if z^I is a monomial in $z_1, \dots, z_k$ write $\mathrm{orb}(z^I)$ for the symmetric polynomial which is the sum of the translates of z^I under the action of the symmetric group, Σ_k. If $p(s_1, \dots, s_k)$ is a polynomial in the $s_i \in H^i(BO\mathbb{F}_3)$ of §3.10 write $p(s_1, \dots) = \mathrm{orb}(z^I)$ if the substitution $s_i = \sigma_i(z_1, z_2, \dots)$ (σ_i = i-th elementary symmetric function) makes these expressions equal. With this convention consider the elements of $H^*\left(\dfrac{BO_{2n}\mathbb{F}_3}{BO_{2n-2}\mathbb{F}_3}\right)$

$$\mathrm{orb}(z_1^{a_1+1} z_2^{a_2+1} \dots z_r^{a_r+1} z_{r+1} \dots z_q) \otimes \widehat{w_{j_1} \dots w_{j_t}} \tag{4.15}$$

where $q \le 2n$, $(a_1, \dots, a_r)$ is a partition of $h = \Sigma a_i \le q$ with no a_i of the form $2^m - 1$, $t = 2n - q$ or $2n - q - 1$, $1 \le j_1 < j_2 < \cdots < j_t$ and $h + q + \Sigma j_i \le 4n - 3$.

I claim that in dimensions $\le 4n-3$ the elements $Sq^I(x)$ where I runs through admissible sequences and x runs through (4.15) give a basis for $H^*\left\{\dfrac{BO_{2n}\mathbb{F}_3}{BO_{2n-2}\mathbb{F}_3}\right\}$.

The argument is essentially due to R. Thom and an elaboration of it is given in Part III, §2.

Firstly it suffices to show that the $Sq^I(x)$ are linearly independent. This is because, from the discussion of §4.10, there is an additive isomorphism $S_k(n)^* \cong H^*(MO(k)) \otimes W(k) \subset H^*(MO(k)) \otimes H^*(SO)$ and the counting procedure of Part III, §2.9 shows that $\sum_k \dim S_k(n)^*$ equal the number of $\cdot Sq^I(x)$'s. Now let $a = \mathrm{orb}(z_1^{a_1+1} \dots z_q)$ and $b = \widehat{w_{j_1} \dots w_{j_t}}$ and let $x = a \otimes b$. Then

$$Sq^I(x) = Sq^I(a) \otimes b + \Sigma\, Sq^J(a) \otimes Sq^{J'}(b). \tag{4.16}$$

However by (4.11) $Sq^I(a)$ modulo the ideal generated by $S_{q+1}, S_{q+2}, \dots$ is given by the same formula as $Sq^I(a)$ in $H^*(MO(q))$ when we replace s_i by the i-th

Stiefel-Whitney class. Also, by Lemma 4.13 $Sq^{J'}(b)$ is congruent to $\bar{S}q^{J'}(b)$ modulo the subspace spanned by $\widehat{w_{k_1} \dots w_{k_v}}$ with $v < t$. Define a filtration on $\underset{k}{\oplus} S_k(n)^*$ by considering $\underset{k}{\oplus} S_k(n)^*$ as a subgroup of $H^*(BO) \otimes H^*(SO)$, as in §4.10, and filtring $H^*(BO)$ by the ideals $\langle s_i, s_{i+1}, \dots \rangle$ and filtring $H^*(SO)$ by the dual-weight filtration used in Lemma 4.13. Under this filtration the A-module action on $a \otimes b \in H^*(MO(k)) \otimes W(k) \cong S_k(n)^*$ agrees with that on

$$H^*(MO(k)) \otimes H^*(SO)$$

modulo lower filtration. Under this latter action the $Sq^I(x)$ are known to be linearly independent in dimensions $\leq 4n-3$ by the argument of Part III, §2.

4.17: Remark. In order to illustrate how Theorem 4.2 works in low dimensions I have included below a table of the first few homotopy elements whose existence is asserted. The elements are in $\pi_j^S(BO\mathbb{F}_3)_{(2)}$. The table works as follows. The parameters $j, \ell, \varepsilon, J, q$ and n of Theorem 4.2 are displayed together with a cohomology element in $H^j(BO\mathbb{F}_3; Z/2)$ which pairs non-trivially with the Hurewicz image of the asserted stable homotopy element in π_j^S. The cohomology elements are those of §4.14.

TABLE OF ELEMENTS

j	ℓ	ε	J	q	n	supporting class
1	0	1	0	0	1	$\hat{w}_1$
1	0	1	0	1	1	$\mathrm{orb}(z_1)$
3	0	1	0	3	2	$\mathrm{orb}(z_1z_2z_3)$
3	0	1	0	2	2	$\mathrm{orb}(z_1z_2) \otimes \hat{w}_1$
4	0	0	0	4	2	$\mathrm{orb}(z_1z_2z_3z_4)$
4	0	0	0	3	2	$\mathrm{orb}(z_1z_2z_3) \otimes \hat{w}_1$
4	0	1	1	1	2	$\mathrm{orb}(z_1) \otimes \widehat{w_1w_2}$
4	0	1	1	2	2	$\mathrm{orb}(z_1z_2) \otimes \hat{w}_2$
5	0	0	1	3	2	$\mathrm{orb}(z_1z_2z_3) \otimes \hat{w}_2$
5	0	0	1	2	2	$\mathrm{orb}(z_1z_2) \otimes \widehat{w_1w_2}$
5	2	1	0	3	2	$\mathrm{orb}(z_1^3z_2z_3)$
5	2	1	0	2	2	$\mathrm{orb}(z_1^3z_2) \otimes \hat{w}_1$
5	0	1	2	1	2	$\mathrm{orb}(z_1) \otimes \widehat{w_1w_3}$
5	0	1	2	2	2	$\mathrm{orb}(z_1z_2) \otimes \hat{w}_3$
5	0	1	0	5	3	$\mathrm{orb}(z_1z_2z_3z_4z_5)$
5	0	1	0	4	3	$\mathrm{orb}(z_1z_2z_3z_4) \otimes \hat{w}_1$

§5. ON THE ALGEBRAIC COBORDISM OF Z

The main result of this section is as follows:

5.1.1: Theorem. Let X be a CW complex. Set

$$MO^*(X) = \varinjlim_{N} \prod_{-N \leq k} MO^k(X),$$

the total unoriented cobordism of X. Then there is a natural ring homomorphism

$$T : AZ^\circ(X) \to MO^*(X)$$

such that

(a) T is surjective if $\dim X < \infty$, and

(b) T is not injective if X is the n-sphere for any $n \geq 0$.

Here AZ^* is the algebraic cobordism cohomology theory of Z as defined in §1.4.3(b).

Sketch of Proof. The homomorphism T is induced by a genus in the MO-theory of $BGLZ^+$ induced from Thom's genus on BO by means of the canonical map $BGLZ^+ \to BO$. The spaces in the AZ-spectrum are all equal to $\Sigma^2 BGLZ^+$, the double suspension of $BGLZ^+$. Therefore in order to show T is onto one must construct S-maps from X to $BGLZ^+$. This is accomplished by constructing an S-map from $BO\mathbb{F}_3$ to $BGLZ^+$ and then appealing to the results of Part I, §8 on the decomposition of the 2-local S-type of $BO\mathbb{F}_3$.

To show that T is not an isomorphism we construct an exponential homomorphism

$$\nu_X : [X, BGLZ^+] \to AZ^\circ(X)$$

and prove that when $X = S^3$ there is an element $i \in \pi_3(BGLZ^+) \cong Z/48$ such that $0 \neq \nu_{S^3}(i) \in \operatorname{Ker} T$.

The section is arranged as follows. In §5.2 AZ-theory is recalled and the homomorphism T of Theorem 5.1.1 is defined. Also in §5.2 the real analogue, AO-theory, is recalled. In §5.3 are derived the facts about the homology of the AZ- and AO-spectrum which will be needed. In §5.4 the map ν, from the algebraic K-theory of Z to the algebraic cobordism of Z is defined. It is shown to be non-trivial on $K_i(Z)$ when i = 1, 2 or 3. In §5.5 more elements of $AZ^\circ(X)$ are constructed when $\dim X < \infty$. Finally Theorem 5.1.1 is proved in §5.5.9.

Throughout the section H_* and H^* will denote mod 2 singular homology and cohomology respectively.

§5.2: AZ-theory and the Thom Genus.

5.2.1. Let GL_nA be the general linear group with entries in A. If $GLA = \bigcup_{0 \leq n} GL_nA$ the inclusion $Z \subset \mathbb{R}$ induces a map between classifying spaces $BGLZ \to BGL\mathbb{R} \simeq BO$, where O is the infinite orthogonal group. This map factors as $BGLZ \xrightarrow{j} BGLZ^+ \xrightarrow{r} BO$. Here j is the canonical map associated with Quillen's "plus" construction (cf. [H-S; Wa]). We will require to know that r is a map of H-spaces.

Let $X = BGLZ^+$ or BO and let $0 \neq \eta \in \pi_1(X) \cong Z/2$. The H-space sum of η with 1_X, the identity map of X, gives a map $\eta + 1_X : S^1 \times X \to X$. The suspension of the Hopf construction applied to $\eta + 1_X$ yields $\varepsilon : \Sigma^3 X \to \Sigma^2 X$. Similarly a Hopf construction applied to $\eta + 1_X + \eta + 1_X : S^1 \times X \times S^1 \times X \to X$ yields a map $m : \Sigma^2 X \wedge \Sigma^2 X \to \Sigma^2 X$. In §1.4.3 the spectrum $\underline{AZ}$ is defined by setting

$$AZ_k = \Sigma^2 BGLZ^+ \qquad (k \geq 2)$$

with structure map $\varepsilon : \Sigma AZ_k \to AZ_{k+1}$. The map $m : AZ_k \wedge AZ_\ell \to AZ_{k+\ell}$ makes this spectrum into a commutative, associative ring spectrum with unit $u = \Sigma^2 \eta : S^3 \to \Sigma^2 BGLZ^+ = AZ_3$. In §1.4.2 $\underline{AO}$ is defined to be the spectrum obtained by replacing $BGLZ^+$ by BO in the above construction. The map, r, introduced above induces a map of ring spectra $r : \underline{AZ} \to \underline{AO}$.

The Thom genus is an element (see §3.9)

$$U \in \prod_{0 \leq k} MO^k(BO)$$

satisfying $h^*(U) = U \otimes U$ where $h : BO \times BO \to BO$ is the H-space multiplication associated with Whitney sum. As explained in §3.10 the Thom space, MO(k), is 2k-equivalent to a product of K(Z/2,m)'s. Hence MO-theory is a product of suitable suspensions of H*-theory. Therefore, as explained in §3, the natural class in $MO^k(MO(k))$ can be "lifted" into $MO^k(BO)$. This can be done in such a way as to give the total genus, U. Also if $0 \neq \eta \in \pi_1(BO)$ then $\eta^*(U) = 1 + \sigma$ where $\sigma \in MO^1(S^1)$ is the suspension class. Now we define the homomorphism T of Theorems 1.1. If

$$x \in AZ^\circ(X) = \varinjlim_N [\Sigma^N X, \Sigma^2 BGLZ^+]$$

is represented by $f : \Sigma^n X \to \Sigma^2 BGLZ^+$

define: $T(x) = s_n(f^*(\sigma^2 \otimes r^*U)) \in MO^*(X)$ where

$$s_n : MO^{*+n}(\Sigma^n X) \to MO^*(X)$$

is the suspension isomorphism, $\sigma^2 \in MO^2(S^2)$ is the suspension class, U is the Thom genus and r^*, f^* are the induced maps of r, f. T(x) is independent of the choice of f (cf. Part II, §3.4).

5.2.2: Remark. We may define a ring homomorphism

$$T' : AO^\circ(X) \to MO^*(X)$$

by the above construction in which r^*U is replaced by U. In fact by Part III, §3.1, T' is an isomorphism when dim $X < \infty$. Evidently we have for any X a commutative diagram

$$\begin{tikzcd} AZ^\circ(X) \arrow[rr, "r"] \arrow[dr, "T"'] & & AO^\circ(X) \arrow[dl, "T'"] \\ & MO^*(X) & \end{tikzcd} \qquad (5.2.3)$$

§5.3: $H_*(\underline{AZ})$ and $H_*(BGLZ^+)$.

5.3.1. Since $GL_1Z = O_1$, the first orthogonal group, we obtain a homotopy commutative diagram of natural maps

$$\begin{tikzcd} BGL_1Z = RP^\infty = BO_1 \arrow[d] \arrow[dr] & \\ BGLZ^+ \arrow[r, "r"] & BO \end{tikzcd}$$

If $0 \neq u_j \in H_j(RP^\infty)$ let $v_j \in H_j(BGLZ^+)$ be the image of u_j then $r_*(v_j) = u_j \in H_j(B)$. Let F be the fibre of the map, r, introduced in §5.2.1.

5.3.2: Proposition. There are algebra isomorphisms

(i) $H_*(BO) \cong Z/2[u_1,u_2,\ldots]$

(ii) $H_*(\underline{AO}) \cong H_*(BO)[u_1^{-1}]$

(iii) $H_*(BGLZ^+) \cong H_*(F) \otimes H_*(BO)$

(iv) $H_*(\underline{AZ}) \cong H_*(BGLZ^+)\ [v_1^{-1}]$

Proof. (i) is well-known while (ii) and (iv) are proved in §1.8. To demonstrate (iii) we prove the dual statement by showing that the Serre spectral sequence

$$E_2^{p,q} = H^p(BO) \otimes H^q(F) \Rightarrow H^{p+q}(BGLZ^+)$$

collapses. It is a spectral sequence of Hopf algebras. Since r_* is onto the edge homomorphism, r^*, is injective. Suppose $d_s : E_s^{p,q} \to E_s^{p+s,q-s+1}$ is zero for $s < t$ and for $s = t$, $p + q < n$. Then for $x \in E_t^{p,n-p}$ $d_t(x)$ must be primitive. Thus $d_t(x) = a \otimes 1$ with a primitive. This means that $r^*(a) = 0$. Hence $d_t(x) = 0$ and, by induction, the spectral sequence collapses.

5.3.3: Corollary. $H_3(BGLZ^+) \cong H_3(F) \oplus H_3(BO)$.

Proof. From [L-Sz;Mi 4] we know $\pi_i(F) = 0$ for $i = 0$, 1 and 2 while $\pi_3(F) \cong Z/48$. The result now follows from Propositions 5.2.2(iii) by means of the Hurewicz and universal coefficient theorems.

§5.4: $\nu : K_i(Z) \to AZ^\circ(S^i)$.

5.4.1. If $x \in [X, BGLZ^+]$ then $\Sigma^2 x \in [\Sigma^2 X, AZ_2]$ represents an element $\nu(x) - 1 \in AZ^\circ(X)$. Since multiplication in the spectrum $\underline{AZ}$ is induced by the H-space multiplication on $BGLZ^+$ it follows that $\nu(x+y) = \nu(x)\nu(y)$. In particular when $X = S^i$ we have, by definition, $K_i(Z) = [S^i, BGLZ^+]$ and we obtain an exponential map

$$\nu : K_i(Z) \to AZ^\circ(S^i) \qquad (5.4.2)$$

5.4.3: Proposition. The map ν of (5.4.2) is non-zero on $K_i(Z) \cong Z/2$ when $i = 1$ or 2. When $i = 3$ $K_3(Z)$ is generated by an element y of order 48 [L-Sz] which satisfies

$$0 \neq \nu(y) \in \operatorname{Ker} T \qquad \text{and} \qquad 2\nu(y) = 0.$$

Here T is the homomorphism of Theorem 5.1.1.

Proof. From [Mi 4, Ch. 10] we know that if η generates $K_1(Z)$ then η^2 generates $K_2(Z)$. The Hurewicz images of $\eta \in \pi_1(BO)$ and $\eta^2 \in \pi_2(BO)$ are both non-zero in $H_*(BO)$. Hence, by Proposition 5.3.2 (i) and (ii), the map

$$K_i(Z) \xrightarrow{\nu} AZ^\circ(S^i) \xrightarrow{r} AO^\circ(S^i) \xrightarrow{H} H_i(\underline{AO})$$

is non-zero when $i = 1$ or 2. Here r is as in §5.2.1 and H is the Hurewicz homomorphism.

Since $\pi_3(BO) = 0$ the generator, y, must factor through F, the fibre of r. Since F is 2-connected y is detected in $Z/2 \cong H_3(F)$ and hence in $H_3(BGLZ^+)$ by Corollary 5.3.3. By Proposition 5.3.2 (iii) and (iv) we see that $\nu(y)$ is detected by its Hurewicz image in $H_3(\underline{AZ})$. However $T\nu(y) = T'(r(\nu(y)))$ by (5.2.3) and $r \circ \nu$ factors through

$$r_{\#} : K_i(Z) \to \pi_i(BO) \qquad \text{so} \qquad T\nu(y) = 0 \text{ since } \pi_3(BO) = 0.$$

§5.5: More elements in $AZ^\circ(X)$.

5.5.1. Let $O_n\mathbb{F}_3$ $(1 \le n \le \infty)$ denote the subgroup of $GL_n\mathbb{F}_3$ ($\mathbb{F}_3$ is the three element field) which preserve the form $\sum_{i=1}^{n} X_i^2$. Let $\Sigma_n \int O_2\mathbb{F}_3$ denote the wreath product generated by "diagonal" 2×2 blocks and the symmetric group, Σ_n, which permutes the blocks. Similarly $\Sigma_n \int O_2$ is a subgroup of O_{2n}, the real orthogonal group. Write $O\mathbb{F}_3$ and O for the infinite orthogonal groups $O_\infty\mathbb{F}_3$ and O_∞ respectively.

5.5.2: Proposition. After 2-localisation there exists an S-map $\emptyset : BO\mathbb{F}_3 \to B\Sigma_\infty \int O_2\mathbb{F}_3$ such that, if $\delta_n : BO_2\mathbb{F}_3^n \to BO\mathbb{F}_3$ is the natural map, $(\emptyset \circ \delta_n)_*$ is equal to the homomorphism induced on H_* by the inclusion of $O_2\mathbb{F}_3^n \subset \Sigma_n \int O_2\mathbb{F}_3$.

Proof. This is an elaboration of part of the proof in Part I, §8.2.1.

Let $\emptyset_n : BO_{2n}\mathbb{F}_3 \to B\Sigma_n \int O_2\mathbb{F}_3$ be the S-map given by the transfer associated with the canonical map $i_n : B\Sigma_n \int O_2\mathbb{F}_3 \to BO_{2n}\mathbb{F}_3$. By Part I, §8.1.1 $(\emptyset_n \circ \delta_n')_*$ is the canonical map on H_*, where δ_n' is induced by $O_2\mathbb{F}_3^n \subset \Sigma_n \int O_2\mathbb{F}_3$. Now choose a cofinal family, $(X_\gamma(n))$, of finite subcomplexes of $BO_{2n}\mathbb{F}_3$. Let

$$P_\gamma(n) \subset \{X_\gamma(n), B\Sigma_n \int O_2\mathbb{F}_3\}$$

be the subset of S-maps, f, such that

$$f \circ \delta_n'' : \delta_n''^{-1}(X_\gamma(n)) \to B\Sigma_n \int O_2\mathbb{F}_3$$

induces on H_* the same homomorphism as the canonical map, δ'_n. Here δ''_n : $BO_2\mathbb{F}_3^n \to BO_{2n}\mathbb{F}_3$ is induced by the natural group inclusion. Let $Q_\gamma(n)$ be the image of $P_\gamma(n)$ in $\{X_\gamma(n), B\Sigma_\infty \int O_2\mathbb{F}_3\}$. Now $Q_\gamma(n)$ is finite and non-empty. The inverse limit of compact, non-empty sets is non-empty. So we may choose an element

$$\emptyset \in \varprojlim_{\gamma,n} \{X_\gamma(n), B\Sigma_\infty \int O_2\mathbb{F}_3\} \cong \{BO\mathbb{F}_3, B\Sigma_\infty \int O_2\mathbb{F}_3\} .$$

5.5.3. From [F-P, §3] we know that $BO_{2n}\mathbb{F}_3 \to BO\mathbb{F}_3$ embeds $H_*(BO_{2n}\mathbb{F}_3)$ as a summand in $H_*(BO\mathbb{F}_3)$. The analogous result is true for BO_{2n} and BO. Consequently we may identify

$$H_*\left(\frac{BO_{2n}\mathbb{F}_3}{BO_{2n-2}\mathbb{F}_3}\right) \qquad \text{and} \qquad H_*\left(\frac{BO_{2n}}{BO_{2n-1}}\right)$$

($n \geq 1$) with subgroups of $H_*(BO\mathbb{F}_3)$ and $H_*(BO)$ respectively. Thus we may speak, for example, of a map $BO \to \bigvee_{1\leq k} \frac{BO_{2k}}{BO_{2k-2}}$ sending an H_*-class to "itself"

From Part I, §§4, 8 we have the following result.

5.5.4: Proposition. (i) There exists a 2-local S-equivalence

$$\Psi : BO\mathbb{F}_3 \to \bigvee_{1\leq k} \frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$$

(ii) There exists an S-equivalence

$$\lambda : BO \to \bigvee_{1\leq k} \frac{BO_{2k}}{BO_{2k-2}}$$

(iii) On H_* the induced homomorphisms Ψ_* and λ_* send an element to "itself". in the sense of §5.5.3.

5.5.5. The group $O_2\mathbb{F}_3$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore we we have inclusions $O_2\mathbb{F}_3 \subset GL_2Z \subset GL_2\mathbb{R}$ which induces a diagram

$$\begin{tikzcd} BO_2\mathbb{F}_3 \arrow[rr, "k"] \arrow[dr] & & BGLZ^+ \arrow[dl, "r"] \\ & BO & \end{tikzcd} \tag{5.5.6}$$

Let $QX = \varinjlim_n \Omega^n\Sigma^n X$, this is the free infinite loopspace generated by X. $BGLZ^+$ and BO are infinite loopspaces [Ma 2, Wa]. The diagram (5.5.6) induces a diagram of infinite loop maps

$$\begin{tikzcd} QBO_2\mathbb{F}_3 \arrow[r] \arrow[d, "\lambda_3"'] & QBGL_2Z \arrow[r] \arrow[dl, "\lambda_2"'] & QBO_2 \arrow[dl, "\lambda_1"'] \\ BGLZ^+ \arrow[r, "r"] & BO & \end{tikzcd} \tag{5.5.7}$$

There are structure maps ($n \geq 1$), compatible as n varies

$$i_n : B\Sigma_n \int O_2\mathbb{F}_3 \to QBO_2\mathbb{F}_3$$

and

$$j_n : B\Sigma_n \int O_2 \to QBO_2$$

such that i_1 and j_1 are the natural "suspension" maps. Further details may be found in Part I, §3.5.

5.5.8: Proposition. The composition of maps and S-maps

$$BO_1^{2n} \xrightarrow{\ell} BO_{2n}\mathbb{F}_3 \xrightarrow{\emptyset} B\Sigma_\infty \int O_2\mathbb{F}_3 \xrightarrow{\lambda_3 \circ i_\infty} BGLZ^+ \xrightarrow{r} BO$$

induces the canonical homomorphism on H_*. Here ℓ is induced by the inclusion of the diagonal matrices.

Proof. By definition $r \circ \lambda_3 \circ i_n$ is the Kahn-Priddy transfer, $tr(r \circ i_1 \circ \pi_1)$ [K-P] of (cf. Part I, §3.9; setting $N_k = \Sigma_n \int O_2\mathbb{F}_3$)

$$BN_1 \times BN_{n-1} \xrightarrow{\pi_1} BO_2\mathbb{F}_3 = B\Sigma_1 \int O_2\mathbb{F}_3 \xrightarrow{r \circ k} BO$$

with respect to the covering $BN_1 \times BN_{n-1} \to B\Sigma_n \int O_2\mathbb{F}_3$. Here π_1 is the first projection k, r as in (5.5.6). Now $r \circ k$ $[BO_2\mathbb{F}_3, BO] = \widetilde{KO}(BO_2\mathbb{F}_3)$ representes E - dim E where E is the universal real 2-plane bundle on $BO_2\mathbb{F}_3$. The Kahn-Priddy transfer is additive [K-P, §1.8]. Thus $(r \circ \lambda_3 \circ i_n)_*$ is the Pontrjagin quotient of $tr(E)_*$ by $tr(\dim E)_*$. dim E is a trivial bundle over $BO_2\mathbb{F}_3$. Therefore, by the naturality property of the transfer [K-P], tr(E) factors through the map $B\Sigma_n \int O_2\mathbb{F}_3 \to B\Sigma_n$ induced by $O_2\mathbb{F}_3 \to \{1\}$. Thus $BO_1^{2n} \to B\Sigma_n \int O_2\mathbb{F}_3$ becomes trivial when composed with tr(dim E). However, if $x \in H_*(BO_1^{2n})$ then, by Proposition 5.5.2, $(\emptyset \circ \ell)_*(x)$ equals the image of x under the map induced by the natural map mentioned above. Hence, the previous discussion shows that $tr(\dim E)_*$ is trivial on such elements. Thus,

$$(r \circ \lambda_3 \circ \emptyset \circ \ell)_* = tr(E)_* \circ \ell_* : H_*(BO_1^{2n}) \to H_*(BO).$$

Finally it can be shown as in [Ma 2, Ch. VIII, 1.1] that tr(E) is represented by the canonical map

$$B\Sigma_n \int O_2\mathbb{F}_3 \to BO_{2n} \to BO$$

induced by the inclusions $\Sigma_n \int O_2\mathbb{F}_3 \subset \Sigma_n \int O_2 \subset O_{2n}$.

5.5.9: Proof of Theorem 5.1.1. Let X be a finite dimensional CW complex. Then $MO^*(X)$ is generated by elements of the following form (see §3). $\frac{BO_{2k}}{BO_{2k-2}}$ is (4k-3)-equivalent to a product of $K(Z/2,m)$'s. Thus we may with ease construct homotopy classes, $f : \Sigma^N X \to \frac{BO_{2k}}{BO_{2k-2}}$, when $N + \dim X < 4k - 3$. Composing f with an inverse, λ^{-1}, of the S-map λ of Proposition 5.5.4(ii) gives $x' \in \{\Sigma^N X, BO\}$ and hence $x' \in AO^\circ(X)$. $MO^*(X)$ is generated by the elements, $T'(X)$, constructed in this manner, where T' is as in (5.2.3).

However, in §4 it is shown that $\frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$ is (4k-3)-equivalent to a product of $K(Z/2,m)$'s which contains the (4k-3)-skeleton of $\frac{BO_{2k}}{BO_{2k-2}}$ as the (4k-3)-skeleton of a factor. Hence we may identify $H_j\left(\frac{BO_{2k}}{BO_{2k-2}}\right)$ as a summand of $H_j\left(\frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}\right)$ for $j < 4k - 3$. With this understanding choose $g:\Sigma^N X \to \frac{BO_{2k}\mathbb{F}_3}{BO_{2k-2}\mathbb{F}_3}$ which has the "same" induced map as f on H_*.

Now compose g with $\lambda_3 \circ i_\infty \circ \emptyset \circ \psi^{-1}$, where ψ^{-1} is an inverse to ψ of Proposition 5.5.4(i) and the other maps are as in Proposition 5.5.8. This yields $y \in AZ^\circ(X)$ represented by $y' \in \{\Sigma^N X, BGLZ^+\}$. By Propositions 5.5.4(iii), 5.5.8 and the construction of y we see that $r_\#(y') \in \{\Sigma^N X, BO\}$ induces on H_* the same homomorphism as x. Hence $Ty = T'(r(y)) = T'x$ since a class in $MO^*(X)$ is detected by H_*.

To prove that T is not one-one when $X = S^n$ it suffices, by periodicity, to treat the case $X = S^3$. The element, i, constructed in Proposition 5.4.3 provides a non-zero element in Ker T to complete the proof.

PART IV: ALGEBRAIC COBORDISM AND GEOMETRY

§0. EPILOGUE

The final part of this paper takes up the problem of computing the p-adic algebraic cobordism groups, $\pi_*(\underline{A\overline{\mathbb{F}}_q}, V)$, which were introduced in Part III, §1.4.6 for schemes over Spec $\overline{\mathbb{F}}_q$. In §3 the algebraic cobordism of projective bundles and Severi-Brauer schemes are computed. Also if A is a regular $\overline{\mathbb{F}}_q$-algebra the algebraic cobordism of Spec $A[t,t^{-1}]$ is determined. Here are a few sample results.

__3.3__: __Corollary__. If A is a commutative $\overline{\mathbb{F}}_q$-algebra

$$\pi_*(\underline{A\overline{\mathbb{F}}_q}, \mathbb{P}^r_A) \cong (A\overline{\mathbb{F}}_q, \operatorname{Spec} A)_*(\prod_1^r (BGLA^+)^\wedge_p).$$

__3.4__: __Corollary__. $\pi_i(\underline{A\overline{\mathbb{F}}_q}, \mathbb{P}^r_{\overline{\mathbb{F}}_q}) \cong \begin{cases} 0 & i \text{ odd} \\ MU_{2*}((BU^\wedge_p)^r;\hat{Z}_p) & i \text{ even.} \end{cases}$

__3.10__: __Corollary__. $\pi_i(\underline{A\overline{\mathbb{F}}_q}, \mathbb{A}^1_{\overline{\mathbb{F}}_q} - (\underline{0})) \cong MU_{2*+i}(U^\wedge_p;Z^\wedge_p)$.

Also a spectral sequence is constructed (in §3.12) to analyse the algebraic cobordism in a Mayer-Victoris situation for a regular scheme.

Furthermore to the algebraic cobordism of Spec A there corresponds an analogue of topological K-theory and a surjective homomorphism (§3.15) from cobordism to "K-theory" analogous to the homomorphism of [C-F]. This homomorphism is used in §3.16 to detect elements in the algebraic cobordism of Spec $K[t,t^{-1}]$ $= \mathbb{A}^1_K - (\underline{0})$.

These results and more, together with a discussion of other computational techniques such as devissage and reduction by resolution (§3.17) are in §3. This establishes p-adic algebraic cobordism as a "generalised pre-sheaf cohomology theory" in the sense of [Br-G] which unfortunately is not pseudo-flasque (see §3.13) as algebraic K'-theory is but which nevertheless seems to yield interesting invariants of the geometry. However, in order to attempt to make a case in favour of the use of generalisations of cobordism invariants in algebraic geometry I have included §1 and §2. In §1 a problem about algebraic vector bundles over number fields is examined using unitary K-theory (although cobordism would serve as well) on the étale site. This is intended to emphasize the suitability of generalised cohomology theories (especially K-theory and cobordism) applied to étale homotopy types for treatment of geometrical problems. The reader may then enquire: Why do we need more than classical generalised cohomology of étale homotopy types to get invariants of the geometry? In §2 I have tried to put forward some reasons by examining the generalisation of the unitary Pontrjagin-Thom construction which one obtains from the étale site and by showing in two examples how feeble it can be.

How convincingly §§1, 2 put forward my point of view is a matter of opinion. For better or worse the discussion is elaborated in the introductions to those sections.

In §4 are described the homomorphisms which connect the p-adic algebraic cobordism of an $\overline{\mathbb{F}}_q$-algebra, the topological K-theory of classifying spaces of a subgroup of the group of units and Quillen's K-theory. These homomorphisms are computed in several examples and in these examples the recovery of Quillen K-theory from the other theories is discussed. In §5 are collected a set of problems relating to this paper.

§1. ALGEBRAIC VECTOR BUNDLES OVER NUMBER FIELDS

In this section we use unitary cobordism of the etale site to discuss a question of Atiyah [A-M, p. 2]. This discussion is probably obvious to geometers. However it is brief and the object is to add weight to the idea that one can usefully apply generalised cohomology--particular K-theory (or equally well cobordism theory, see Remark 1.8(iii))--to the étale site of a variety. Having made the point for K-theory and cobordism I will, in the next section, emphasise the limitations--taking the Pontrjagin-Thom construction as my test case--of being contented with applying some classical cobordism theory to a site. Potentially more useful are the p-adic algebraic cobordism theories which are discussed in §3. Now to work.

In [A-M, p. 2] the following question is attributed to M. F. Atiyah. Let L/K be an extension of algebraic number fields with K a (fixed) subfield of the complex numbers, $\mathbb{C}$. Let V be a variety (i.e., an irreducible, separated scheme of finite type) defined over K. Let E be an algebraic vector bundle over $V \otimes_K L$. Let $c : L \to \mathbb{C}$ be an embedding of L/K. Then c will induce a complex vector bundle, E^c, over the topological space $V_{c\ell} = (V \otimes_K \mathbb{C})_{c\ell}$, the associated complex variety with the classical topology. Note that $V_{c\ell}$ has the homotopy type of a finite CW complex.

Let K(X) denote the unitary K-group. K(X) is the set of homotopy classes $[X, Z \times BU]$.

<u>1.1</u>: <u>Problem</u>. How does $[E^c] \in K(V_{c\ell})$, the class of the bundle E^c, depend on $c : L \to \mathbb{C}$?

<u>1.2</u>. If n = dim E we may equivalently study the dependence of $[E^c] - n \in \tilde{K}(V_{c\ell})$, the reduced K-group of $V_{c\ell}$.

In fact let us set $y(E,c) = [E^c] - n \in \tilde{K}(V_{c\ell})$ then the aim of this section will be to discuss the related question:-

<u>1.3</u>: <u>Problem</u>. For $1 \leq i \leq n$ how does $\gamma^i(y(E,c)) \in \tilde{K}(V_{c\ell})$ depend upon $c: L \to \mathbb{C}$? Here γ^i is the i-th γ-operation [At 2].

<u>1.4</u>: <u>Geometrical Comparison of E^{c_1} and E^{c_2}</u>. Let $c_1, c_2 : L \to \mathbb{C}$ be embeddings which agree on K. Also let $\alpha \in \mathrm{Gal}(\mathbb{C}/K)$ be a Galois automorphism of $\mathbb{C}$ such that $\alpha \circ c_1 = c_2$. Form the projective bundle $\mathbb{P}(E) = \mathrm{Proj}\,(SE) \to X$. There is a canonical Hopf line bundle, H, over $\mathbb{P}(E)$ [At 2]. Suppose that $H \otimes_{c_i} \mathbb{C}$ is a very ample line bundle then there is associated to H a morphism [Har, p. 150, §7.1] $f^{c_i} : \mathbb{P}(E^{c_i}) \to \mathbb{P}^N_{\mathbb{C}}$ for some integer N such that

$$(f^{c_i})_{c\ell} : \mathbb{P}(E^{c_i})_{c\ell} \to (\mathbb{P}^N_{\mathbb{C}})_{c\ell} = \mathbb{C}P^N \subset \mathbb{C}P^\infty$$

classifies the topological line bundle

$$(H^{c_i})_{c\ell} \to \mathbb{P}(E^{c_i})_{c\ell}.$$

We have a commutative diagram of morphisms induced by α.

$$\begin{tikzcd}
\mathbb{P}(E^{c_1}) \arrow[rr, "f^{c_1}"] \arrow[d, "\hat{\alpha}"'] \arrow[ddr] & & \mathbb{P}^N_{\mathbb{C}} \arrow[d, "\hat{\alpha}"] \\
\mathbb{P}(E^{c_2}) \arrow[rr, "f^{c_2}"] \arrow[dddr] & & \mathbb{P}^N_{\mathbb{C}} \\
& V \otimes_K \mathbb{C} \arrow[dd, "\hat{\alpha}"'] & \\
& & \\
& V \otimes_K \mathbb{C} &
\end{tikzcd} \tag{1.5}$$

Suppose that $V \otimes_K \mathbb{C}$ is normal. Then every variety in (1.5) is normal. However if X is a normal complex variety then the finite completion, $\hat{X}_{c\ell}$, is (up to homotopy) a functor of the étale site of X. This is Sullivan's domestication of the Artin-Grothendieck comparison theorem [Su, esp. p. 42]. Let BU^ denote the finite completion of BU and write $\tilde{K}(W;\hat{Z}) = [W, BU\hat{}\,]$ for the $\hat{Z}$-K-theory of W. Then $\tilde{K}(X\hat{}_{c\ell};\hat{Z}) \cong \tilde{K}(X_{c\ell};\hat{Z})$ via a natural isomorphism and the above discussion derives from (1.5) the following commutative diagram upon applying $\tilde{K}(-;\hat{Z})$.

$$\begin{tikzcd}
\tilde{K}(\mathbb{P}(E^{c_1})_{c\ell};\hat{Z}) & & K(\mathbb{C}P^N;\hat{Z}) \arrow[ll, "(f^{c_1})^*_{c\ell}"'] \\
\tilde{K}(\mathbb{P}(E^{c_2})_{c\ell};\hat{Z}) \arrow[u, "\hat{\alpha}*"] & & \tilde{K}(\mathbb{C}P^N;\hat{Z}) \arrow[ll, "(f^{c_2})^*_{c\ell}"'] \arrow[u, "\hat{\alpha}*"'] \\
& \tilde{K}((V \otimes_K \mathbb{C})_{c\ell};\hat{Z}) \arrow[uul] & \\
& \tilde{K}((V \otimes_K \mathbb{C})_{c\ell};\hat{Z}) \arrow[u, "\hat{\alpha}*"'] \arrow[uul] &
\end{tikzcd} \tag{1.6}$$

Now for $i = 1,2$, $\tilde{K}(\mathbb{P}(E^{c_i})_{c\ell};\hat{Z}) \oplus \hat{Z} = K(\mathbb{P}(E^{c_i})_{c\ell};\hat{Z})$, by definition,

$$\cong K(V_{c\ell};\hat{Z})[t_i] \Big/ \left(\sum_{r=0}^{n} (-1)^r \gamma^r(y(E,c_i)) t_i^{n-r} \right)$$

where $n = \dim E$ and $\gamma^r(y(E,c_i))$ also denotes the image of this element under the canonical map $j_1 : \tilde{K}(\_) \to \tilde{K}(\_;\hat{Z})$.

<u>1.7</u>: <u>Theorem</u>. In the notation of §1.3/4

(i) $j_1 : \tilde{K}(V_{c\ell}) \to \tilde{K}(V_{c\ell};\hat{Z})$ is injective.

(ii) If $x \in \tilde{K}(\mathbb{C}P^N)$ is the reduced Hopf bundle and $\hat{\alpha}(j_1(x)) = \sum_{s=1}^{N} b_s x^s$ ($b_s \in \hat{Z}$) then $\sum_{r=0}^{n} (-1)^r \hat{\alpha}*(\gamma^r(y(E,c_2)))[\sum_{s=1}^{N} b_s t_1^s]^{n-r}$ is divisible by $\sum_{r=0}^{n} (-1)^r \gamma^r(y(E,c_1)) t_1^{n-r}$ in $K(V_{c\ell};\hat{Z}) = \tilde{K}(V_{c\ell};\hat{Z}) \oplus \hat{Z}$.

<u>Proof</u>. Part (i) is true when $V_{c\ell}$ is replaced by any finite CW complex. This is an easy manipulation of the rationalisation-completion fibre square of BU [Su].

Part (i) guarantees that $\tilde{K}(V_{c\ell})$ equations are faithfully captured in $\tilde{K}(V_{c\ell};\hat{Z})$.

Part (ii) now follows by chasing the diagram (1.6).

<u>1.8</u>: <u>Remark</u>. (i) Theorem 1.7 implies that to answer Problems 1.1, 1.3 we must understand $\hat{\alpha}* \in \mathrm{Aut}(\tilde{K}(V_{c\ell};\hat{Z}))$. This is the difficult part because $\hat{\alpha}$ is not induced by a continuous map. However $\hat{\alpha}*$ is natural for morphisms of normal K-varieties, although this naturality differs from the usual sort of naturality which one has in mind when discussing K-theory operations (see [At 2]).

(ii) A phenomenon related to §1.1 was discovered in [Ser 2]. Let p be a prime and let $Y \subset \mathbb{P}_K^{p-1}$ be the solutions of $\sum_i x_i^p = 0$. Then Y is a Z/p-space by means of cyclic permutation of coordinates. If E is a variety we may form the fibring

$$E^p \to Y \times_{Z/p} E^p \to Y/(Z/p) = X.$$

Choosing p suitably and choosing E to be a suitable elliptic curve Serre [Ser 2] gave two embeddings $d_1, d_2 : K \to \mathbb{C}$ of the number field K such that the fundamental groups of $((Y \times_{Z/p} E^p) \otimes_{d_i} \mathbb{C})_c$ $(i = 1,2)$ were not isomorphic. Hence one would have expected pathological behaviour in §1.1 when V = X. Of course the fundamental groups of Serre's examples are infinite, because their profinite completions are isomorphic. However in [Ab] non-homotopic equivalent, conjugate varieties are given which have finite (hence equal) fundamental groups.

(iii) Equivalently to §1.7 one might study Problems 1.1, 1.3 by studying $\hat{\alpha}^* : MU^{2*}(V_{c\ell};\hat{Z}) \to MU^{2*}(V_{c\ell};\hat{Z})$. For $y(E,c)$ may be obtained as the image of the first Conner-Floyd class of E^c, $c_1(E,c) \in MU^2(V_{c\ell})$, under the Conner-Floyd homomorphism [C-F] and $c_1(E,c)$ is obtained from (1.5) by

$$MU^*(\mathbb{P}(E^c)_{c\ell}) \cong MU^*(V_{c\ell})[t] \left(\sum_{r=0}^{n} (-1)^r c_r(E,c)t^{n-r}\right)$$

where t is the class of $(f \otimes_c \mathbb{C})_{c\ell}$ in MU^2.

(iv) If $H \otimes_{c_1} \mathbb{C} \to \mathbb{P}(E^{c_1})$ is ample then there is a morphism $f^{c_1} : \mathbb{P}(E^{c_1}) \to \mathbb{P}^N_{\mathbb{C}}$ which classifies $(H \otimes_{c_1} \mathbb{C})^n$ for some $n \geq 1$. Using the resulting diagrams analogous to (1.5) and (1.6) relations between $\gamma^j(y(E,c_i))$ $(i = 1,2)$ may be obtained by the method of §§1.3-1.7. Details are left to the interested reader.

§2. THE ANALOGUE OF THE PONTRJAGIN-THOM CONSTRUCTION AND THE ÉTALE SITE

In Part II, §8.5/6 I gave a purely (co-) homological description of a homomorphism, λ_f, associated with an embedding $f : X \to Y$. When $Y = \mathbb{C}^N$ λ_f was seen to capture all the data of the Pontrjagin-Thom construction in unitary cobordism. The construction of λ_f was only MU-cohomological properties of complex vector bundles (especially the Thom isomorphism). The singular cohomology of the etale homotopy type of an algebraic vector bundle (and its "sphere bundle") often behaves in a manner similar to the singular cohomology of a complex vector bundle ([C;Cl;Fr 4]). As we shall see below this permits the definition in reasonable generality, of an analogous homomorphism, λ_f, associated with a smooth algebraic embedding $f : X \to Y$. λ_f is obtained from the MU-theory of the étale homotopy type of X, the normal vector bundle of f and its "normal sphere bundle". As we shall see in a couple of examples this invariant of f has one drawback. Namely its behaviour is so analogous to that of the Pontrjagin-Thom construction for <u>topological</u> geometry that it is insensitive to geometry whose interest is essentially <u>algebraic</u>. The foregoing discussion and that of §1 leads one to conclude, in my opinion, that one should search for a construction of algebraic cobordism invariants in algebraic geometry by other methods than merely applying generalised cohomology theories to étale homotopy types. In §3 we compute with one candidate--the p-adic algebraic cobordism of Part III, §1.4.6.

<u>2.2</u>: <u>Étale Homotopy Types</u>. Let X be a variety over an algebraically closed field, K. Thus X is topological space together with a sheaf of local rings, $\mathcal{O}_X$, which makes X into a reduced, separated scheme of finite type over K [EGA 1, p. 215; Bor p. 21; Sh, p. 263].

A geometric point of a scheme, Y, over any field, F, is a morphism $i_{\bar{y}} : \bar{y} \to Y$ where y Y. This data is a pair (y,ϕ_y) where

$$\phi_y : \mathcal{O}_y \to F_s$$

is an F-algebra homomorphism of the local ring at y into F_s, the separable closure of F. Let $m_{\bar{y}}$ be the kernel of the canonical extension of ϕ_y

$$\bar{\phi}_y : F_s \otimes_F \mathcal{O}_y \to F_s.$$

Then the local ring of the geometric point is the localisation $(F_s \otimes_F \mathcal{O}_y)_{m_{\bar{y}}}$ which will be denoted by $\mathcal{O}_{\bar{y}}$. Let $\hat{\mathcal{O}}_{\bar{y}}$ be the $(m_{\bar{y}})$-adic completion of $\mathcal{O}_{\bar{y}}$.

Now suppose W is a smooth variety [Bor, pp. 65-72; Sh, pp. 72-79] and that $f : Y \to W$ is a morphism. Then f is étale if

$$f^* : \hat{\mathcal{O}}_{\overline{\pi(y)}} \to \hat{\mathcal{O}}_{\bar{y}}$$

is an isomorphism at all geometric points, (y,ϕ_y), of Y [SGA 4, Vol. 270, Expose VIII, p. 343; H-R, p. 84].

Now let us return to the variety X which is henceforth assumed to be smooth. The category of pointed (étale) coverings of X, Cov(X), is described as follows. An object is a family of etale morphisms

$$\mathcal{U} = \{\pi_x : U_x \to X;\ i_x\}$$

indexed by geometric points (x,ϕ_x). Hence $\{\pi_x(U_x)\}$ is a covering of X by Zariski open sets. A morphism between two such objects (families) is defined in the obvious manner to be a collection of morphisms over X which respect geometric points. Form the Čech nerve of the covering $\mathcal{U}$. It is a simplicial set which we will denote by $C_{\mathcal{U}}(X)$ [A-M, p. 96; R-H, p. 86]. If $\pi_\circ$ is the connected component functor [A-M, pp. 111-116] set

$$\pi_{\mathcal{U}}(X) = \pi_\circ(C_{\mathcal{U}}(X)).$$

We may define the notion of one covering, $\mathcal{U}'$, refining another, $\mathcal{U}$. Hence we get an inverse system of simplical sets

$$\{\pi_{\mathcal{U}}(X)\}$$

indexed by the objects of Cov(X). This is a good category for inverse limits (i.e., pseudo-filtering [A-M, p. 148]). The étale homotopy type of X, denoted by X_{et}, is the inverse system of spaces

$$\{|\pi_{\mathcal{U}}(X)|\}$$

where $|\_|$ denotes geometric realisation. Such an inverse system is called a pro-space. X_{et} is defined up to pro-homotopy.

Following [B-K] or [Bou] we may form the p-finite completion (i.e., $H^*(\_;Z/p)$ localisation) of the prospace, X_{et}, to obtain the p-completed étale homotopy type $X^{\wedge}_{et}$. This is a pro-space of p-complete spaces.

2.3: The Analogue of λ_f for a Smooth Algebraic Embedding. Let $f : X \to Y$ be a smooth embedding of K-varieties where K is algebraically closed and char K = q. Let p be a prime different from q and set h^* equal to $MU^*(\_;Z/p)$, mod p unitary cobordism. Let h_* be the associated homology theory.

Consider the normal bundle exact sequence [Sh, p. 275]

$$0 \to \tau_X \to f^*\tau_y \to \nu(f) \to 0$$

which defines the normal bundle $\nu(f) \to X$ of the embedding. The pro-map $(\nu(f) - X)^{\wedge}_{et} \to X^{\wedge}_{et}$ has relative cohomology groups

$$h^*(X^{\wedge}_{et},(\nu(f) - X)^{\wedge}_{et}) \quad \text{and} \quad H^*(X^{\wedge}_{et},(\nu(f) - X)^{\wedge}_{et};Z/p)$$

defined by taking the direct limit of the relative cohomology of maps $p(\mathcal{U},\mathcal{V}):|\pi_{\mathcal{U}}(\nu(f) - X)|^{\wedge} \to |\pi_{\mathcal{V}}(X)|^{\wedge}$ induced by a morphism of pointed étale coverings. Let us abbreviate these groups to $h^*(\nu(f)^{\wedge}_{et})$ and $H^*(\nu(f)^{\wedge}_{et})$. If $\nu(f)$ is n-dimensional there is a Thom isomorphism $H^*(X^{\wedge}_{et};Z/p) \cong H^{*+2n}(\nu(f)^{\wedge}_{et})$. (See [C], [C2] or [Fr4]). This isomorphism is given by multiplying by a Thom class in $H^{2n}(\nu(f)^{\wedge}_{et})$.

Suppose now that there exists a Thom class $\Lambda(\nu(f)) \in h^{2n}(\nu(f)^{\wedge}_{et})$ giving rise to a Thom isomorphism

$$[\Lambda(\nu(f)).\_] : h^*(X^{\wedge}_{et}) \to h^{*+2n}(\nu(f)^{\wedge}_{et}) . \tag{2.3.1}$$

For example this will happen if $H^*(X^{\wedge}_{et})$ is concentrated in even dimensions. To see this consider the Atiyah-Hirzebruch spectral sequence for the mapping cone of the map $p(\mathcal{U},\mathcal{V})$ above. Taking the direct limit of these spectral sequences gives a spectral sequence (cf. Part III, §1.13)

$$E_2^{s,t}(\nu(f)) = H^s(\nu(f)^{\wedge}_{et};\ h^t(\text{point})) \Rightarrow h^{s+t}(\nu(f)^{\wedge}_{et}) .$$

Since $h^t(\text{point}) = 0$ for t odd the spectral sequence is concentrated in even total degree and hence collapses. Thus there exists $\Lambda(\nu(f)) \in h^{2n}(\nu(f)^{\wedge}_{et})$ mapping to the Thom class in $H^{2n}(\nu(f)^{\wedge}_{et})$ under the orientation homomorphism

$h^* \to H^*$. Now multiplication induces an isomorphism of spectral sequences $\{E_r^{s,t}(X) \to E_r^{s+2n,t}(\nu(f))\}$ and hence an isomorphism in (2.3.1). This case will apply to all the examples which we consider.

If we set $h_*(X^\wedge_{et}) = \varprojlim_{\mathcal{U}} h_*(|\pi_{\mathcal{U}}(X)|^\wedge)$ and $h_*(\nu(f)^\wedge_{et}) = \varprojlim_{p(\mathcal{U},\mathcal{V})} h_*(\text{cofibre } p(\mathcal{U},\mathcal{V}))$ then the slant product and Kronecker product induce homomorphisms of the following form (cf. Part II, §8.5).

$$\begin{tikzcd} h_*(\nu(f)^\wedge_{et}) \arrow[r, "{[\Lambda(\nu(f))\backslash -]}"] \arrow[d, "{<\Lambda(\nu(f)),\_>}"'] & h_{*-2n}(X^\wedge_{et}) \arrow[dl] \\ h_{*-2n}(\text{point}) & \end{tikzcd} \tag{2.4}$$

If $[\Lambda(\nu(f))\backslash\_]$ is an isomorphism in (2.4) then we may define λ_f as in the topological case (Part II, §8.6).

Étale homology is not usually defined. This is because $\varprojlim$ is not an exact functor so that the naive definition of $h_*(X^\wedge_{et})$ given here is generally not computable. For the same reason a Thom isomorphism for $h^*(X^\wedge_{et})$ will not in general imply one for h_* in (2.4). However in the cases we will consider λ_f is defined. I am told that the associated Steenrod homology theory to h^* would automatically have homology isomorphism (dual to the h^* Thom isomorphism) which could be used in (2.4) to ensure the definition of a homomorphism similar to λ_f. However, since I am trying to emphasise the short-comings of λ_f I am not concerned here to optimise the establishment of its general theory.

<u>2.5</u>: <u>Two Examples of λ_f</u>. (a) Let $f : \mathbb{P}^2_K \to \mathbb{P}^3_K$ be the natural map. This map is really defined over the integers $\mathbb{P}^2_Z \to \mathbb{P}^3_Z$. Consequently we may compare the h^*- and h_*-behaviour in (2.4) with that in the p-finite completion of the characteristic zero map $f' : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$. This is accomplished by the method of [Fr2] and [Fr5] (where the method is applied in a semi-simplicial context; see also [Fr1] and [Fr3,4]). Accordingly one finds an isomorphism carrying λ_f into $\lambda_{f'} : h_*((\mathbb{P}^2_{\mathbb{C}})^\wedge_{et}) \to h_*(\text{point})$ if either λ_f or $\lambda_{f'}$ is defined. However $\mathbb{P}^2_{\mathbb{C}}$ and $\nu(f')$ are smooth (hence normal) so that the pre-space $(\mathbb{P}^2_{\mathbb{C}})^\wedge_{et}$ consists of spaces $|\pi_{\mathcal{U}}(\mathbb{P}^2_{\mathbb{C}})|^\wedge$ having finite homotopy groups [Su, p. 42]. Hence the inverse system $h_*(|\pi_{\mathcal{U}}(\mathbb{P}^2_{\mathbb{C}})|^\wedge)$ satisfies the Mittag-Leffler condition (cf. [At 1]). Hence we may take the inverse limit of the homology Atiyah-Hirzebruch spectral sequences for $|\pi_{\mathcal{U}}(\mathbb{P}^2_{\mathbb{C}})|^\wedge$ to obtain a spectral sequence, <u>using $h^* = MUZ/p^*$</u>

$$E^2_{s,t} = \varprojlim H_s(|\pi_{\mathcal{U}}(\mathbb{P}^2_{\mathbb{C}})|^\wedge; h_t(\text{point})) \Rightarrow h_{s+t}((\mathbb{P}^2_{\mathbb{C}})^\wedge_{et}) .$$

However $h_t(\text{point})$ is just a finite sum of copies of Z/p so that $E^2_{s,t}$ is the dual vector space to $E_2^{s,-t}(\mathbb{P}^2_{\mathbb{C}})$, the E_2-term of the cohomology spectral sequence introduced in §2.3. Therefore $E^2_{*,*}$ is concentrated in even degrees and the spectral sequence collapses. As the E^2-term is isomorphic to $H_*(\mathbb{C}P^2;h_*(\text{point}))$

we conclude that $h_*((\mathbb{P}^2_{\mathbb{C}})^{\wedge}_{et}) \cong MU_*(\mathbb{C}P^2;Z/p)$ which is a free $\pi_*(\underline{MU};Z/p)$-module on generators $1 = \beta_0,\beta_1,\beta_2$ with deg $\beta_i = 2i$. If $\lambda \in MU^2(\mathbb{C}P^2;Z/p)$ is the first Conner-Floyd class of the Hopf bundle then β_i is defined by

$$<\lambda^k,\beta_j> = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases} .$$

Similarly we may compute $h_*(\nu(f')^{\wedge}_{et}) \cong MU_*(\nu(f);Z/p)$ and conclude that $\lambda_{f'}$ is defined and equal to the analogous homomorphism defined in the topological setting as in Part II, §8.6. Notice that in the topological context $[\Lambda(\nu(f'))\backslash\_]$ is dual to the h*-cohomology Thom isomorphism. Since the Thom space of the topological normal bundle of f' is $\mathbb{P}^3$ it is an easy calculation to show that the $\pi_*(\underline{MU};Z/p)$-module $\lambda_{f'} : MU_*(\mathbb{C}P^2;Z/p) \to \pi_*(\underline{MU};Z/p)$ is given by

$$\lambda_{f'}(\beta_j) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} .$$

(b) Let $f : \mathbb{P}^1_K \times \mathbb{P}^1_K \to \mathbb{P}^3_K$ be given by

$$f([a_0,a_1][b_0,b_1]) = [a_0b_0,a_0b_1,a_1b_0,a_1b_1] .$$

As in (a) this may be compared with the zero characteristic case. One finds

$$h_*((\mathbb{P}^1_I \times \mathbb{P}^1_K)_{et}) \cong MU_*((\mathbb{C}')^2;Z/p)$$

which is isomorphic to the free $\pi_*(MU;Z/p)$-module on generators 1, a_1,a_2,a_3 where deg a_1 = deg a_2 = 2 and deg a_3 = 4. One also finds

$$\lambda_f(a_i) = 0 \quad \text{and} \quad \lambda_f(1) = 1.$$

2.6: Remark. In (a) and (b) the embedding f represents an algebraically significant divisor [Har] but λ_f does not distinguish f from an embedding with trivial normal bundle--because these embeddings (over $\mathbb{C}$) in the topological context are rather uninteresting. Topologically the embeddings which are best detected are into Euclidean space and such embeddings cannot be achieved algebraically.

§3. SOME COMPUTATIONS IN p-ADIC ALGEBRAIC COBORDISM[3]

Recall form Part III, §1.4.6(b) that $\underline{A\overline{\mathbb{F}}}_{q,V}$, the p-adic algebraic cobordism spectrum of the $\overline{\mathbb{F}}_q$-scheme V, gives a functor from schemes over Spec $\overline{\mathbb{F}}_q$ to spectra. Here p is a fixed prime different from q and $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,V})$ is a $\hat{Z}_p$-module because the spectrum is constructed from the p-completion $(X_{\underline{\mathbb{P}}(V)})^{\wedge}_p$. Using results of Quillen [Q3; Q4, G-Q] on the identity of $X_{\underline{P}(V)}$ for certain V we will proceed to make some computations of $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,V})$. All schemes and morphisms will be over Spec $\overline{\mathbb{F}}_q$.

3.1: Projective Bundles. Let PE = Proj(SE), the projective bundle associated to an algebraic vector bundle $E \to V$. Let $f : PE \to V$ be the projection. In [Q4] it is shown that $BQ\underline{\underline{P}}(PE) \simeq \prod_1^r BQ\underline{\underline{P}}(V)$ where f induces inclusion of the first factor and r = dim E. Translation from the first to the i-th factor corresponds to the action of $\underline{\underline{O}}(-i) \in K_0(PE)$ [Q4, §§4,8] on the algebraic K-theory of PE. Hence $X_{\underline{\underline{P}}(PE)} \simeq \prod_1^r X_{\underline{\underline{P}}(V)}$ as H-spaces and similarly for p-completions. From Part III, §1.10 we obtain the following calculation.

3.2: Projective Bundle Theorem. In §3.1 there is an algebra isomorphism

$$\pi_*(\underline{A\overline{\mathbb{F}}}_{q,PE}) \cong (A\overline{\mathbb{F}}_{q,V})_*\left(\prod_1^{r-1} (X_{\underline{\underline{P}}(V)})^{\wedge}_p\right)$$

where the latter group is unreduced homology.

3.3: Corollary. If A is a commutative $\overline{\mathbb{F}}_q$-algebra

$$\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\mathbb{P}^r_A}) \cong (A\overline{\mathbb{F}}_{q,\,\mathrm{Spec}\,A})_*\left(\prod_1^r (BGLA^+)^{\wedge}_p\right).$$

3.4: Corollary.

$$\pi_i(A\overline{\mathbb{F}}_{q,\mathbb{P}^r_{\overline{\mathbb{F}}_q}}) \cong \begin{cases} 0 & (i \text{ odd}) \\ M U\hat{Z}_{p_{2*}}((BU^{\wedge}_p)^r) & (i \text{ even}) \end{cases}.$$

Proof. Set A = $\overline{\mathbb{F}}_q$ in §3.3 and recall from Part III, §1.4 that $(BGL\overline{\mathbb{F}}_q^+)^{\wedge}_p \simeq BU^{\wedge}_p$ in this case. By Part III, §1.4.4 $\underline{A\overline{\mathbb{F}}}_q$ is the $MU\hat{Z}_p^{2*}$-spectrum. Hence the result follows from known results [Ad 1, Part II] from which one sees in fact that when i is even the answer is a very large polynomial ring.

3.5: Proposition. If A is a, not necessarily commutative, $\overline{\mathbb{F}}_q$-algebra there is an algebra isomorphism

$$\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\mathbb{P}^1_A}) \cong (A\overline{\mathbb{F}}_{q,\mathrm{Spec}\,A})((BGLA^+)^{\wedge}_p).$$

Proof. In [Q4, §8.3] it is shown that there are two natural maps $h_0, h_1 : \mathbb{A}^1 \to \mathbb{P}^1_A$ whose "sum" is an equivalence of H-spaces

$$((BGLA^+)^{\wedge}_p)^2 \xrightarrow{\simeq} (X_{\underline{\underline{P}}(\mathbb{P}^1_A)})^{\wedge}_p. \qquad (3.5.1)$$

Under this equivalence the class $b \in \pi_2((BGL\overline{\mathbb{F}}_q)^{\wedge}_p)$ (cf. Part III, §1.4.6(b)) goes to (b,b) in the second homotopy group of the left side of (3.5.1). The result now follows from Part III, §1.11.

3.6: Severi-Brauer Schemes. A morphism $f : V \to S$ of schemes over Spec $\overline{\mathbb{F}}_q$ is a Severi-Brauer scheme of relative dimension r if V is locally isomorphic in the étale topology to $\mathbb{P}^{r-1}_S$. By [Q4, §8.4] there is a vector bundle of rank r, $J \to V$, which restricts to $O(-1)^r$ on each geometric fibre. Let A be the sheaf of (non-commutative) $\underline{\underline{O}}_S$-algebras given by

$$A = f_*(\mathrm{End}_V(J))^{op}$$

where "op" denotes the opposite ring structure. Set A_n equal to the n-fold tensor product $A^{\otimes n}$ on S. A_n is an Azumaya algebra of rank r^{2n}. In [Q4, §8.4] it is shown that when S is quasi-compact there is an H-space equivalence

$$\prod_{n=0}^{r-1} ((BGLA_n)^+)^{\wedge}_p \xrightarrow{\simeq} (X_{\underline{\underline{P}}(V)})^{\wedge}_p . \tag{3.6.1}$$

3.7: Severi-Brauer Scheme Theorem. If S is quasi-compact in §3.6 there is an algebra isomorphism

$$\pi_*(\underline{A\overline{\mathbb{F}}}_{q,V}) \cong (A\overline{\mathbb{F}}_{q;\mathrm{Spec}\, S})_*(\prod_{n=1}^{r-1} (BGLA_n^+)^{\wedge}_p) .$$

Proof. In (3.6.1) the map $j : (BGL\overline{\mathbb{F}}_q^+)^{\wedge}_p \to (X_{\underline{\underline{P}}(V)})^{\wedge}_p$ is the canonical one on the factor corresponding to n = 0, i.e., the map induced by $\overline{\mathbb{F}}_q \to S = A_o$. Let $j = (j_0, j_1, \ldots, j_{r-1})$ in terms of the left side of (3.6.1). Consider the H-space automorphism, k, of the left side of (3.6.1) which is given by "subtracting" from the identity the map sending $(x_0, \ldots, x_{n-1})$ to $(x_0, i_1(x_1), \ldots, i_{r-1}(x_{r-1}))$ where i_j is induced by $S \to A_j$. The composition $k \circ j$ is homotopic to $(j_o, *, *, \ldots, *)$, the canonical map into the first factor. Consequently the spectrum $\underline{A\overline{\mathbb{F}}}_{q,V}$ is isomorphic to one to which we may apply Part III, §1.10 from which the computation follows.

3.8: Example. If V is a complete, non-singular curve of genus zero over $K = H^\circ(V;\underline{\underline{O}}_V)$ and having no rational point. Then [Q4, §8.4] V is a Severi-Brauer scheme over K of relative dimension one and J is the unique indecomposable vector bundle over V of rank 2 and degree -2. In this example

$$\pi_*(\underline{A\overline{\mathbb{F}}}_{q,V}) \cong (A\overline{\mathbb{F}}_{q,\mathrm{Spec}\, K})_*((BGLA^+)^{\wedge}_p) . \tag{3.8.1}$$

Next we have a localisation result.

3.9: Theorem. If A is a regular $\overline{\mathbb{F}}_q$-algebra then there are isomorphisms:

(i) $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\, A[t]}) \xrightarrow{\cong} \pi_*(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\, A})$ (of algebras) induced by $A \to A[t]$.

(ii) $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\, A[t,t^{-1}]}) \cong (A\overline{\mathbb{F}}_{q,\ \mathrm{Spec}\, A})_*((BQ\underline{\underline{P}}(\mathrm{Spec}\ A))^{\wedge}_p)$, (of $\pi_*(\underline{A\overline{\mathbb{F}}}_{q\ \mathrm{Spec}\, A})$-modules) where $BQ\underline{\underline{P}}(\_)$ is the classifying space introduced in Part III, §1.4.6.

Proof. Part (i) follows from the equivalence (the fundamental theorem for regular rings) $BGLA^+ \simeq BGLA[t]^+$ established in [Q4, §6]. Also from [Q4, §6] one obtains a homotopy equivalence of spaces

$$(BGLA[t,t^{-1}])^{\wedge}_p \simeq (BGLA^+)^{\wedge}_p \times (BQ\underline{\underline{P}}(\mathrm{Spec}\ A))^{\wedge}_p .$$

This is not an H-space equivalence so that the argument in Part III, §1.10 yields only a $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\, A})$-module isomorphism.

3.10: Corollary.

$$\pi_i(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\, \overline{\mathbb{F}}_q[t,\ t^{-1}]}) \cong MU_{2*+i}(U^{\wedge}_p;\hat{Z}_p)$$

where U is the infinite special unitary group.

Proof. Since $(BGL\overline{\mathbb{F}}_q^+)^\wedge_p \simeq BU^\wedge_p$ we have $\Omega_o BQ\underline{\underline{P}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)^\wedge_p \simeq BU^\wedge_p$. Also $BQ\underline{\underline{P}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)$ is a connected space. Hence, taking classifying spaces of both sides will yield the universal covers (see e.g., [Ma 1]). That is, the universal cover of $BQ\underline{\underline{P}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)^\wedge_p$ is $B\Omega BQ\underline{\underline{P}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)^\wedge_p = BBU^\wedge_p = SU^\wedge_p$. Since $\pi_1(BQ\underline{\underline{P}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)^\wedge_p)$ $= (K_o\overline{\mathbb{F}}_q)^\wedge_p \cong \hat{Z}_p$ it is now clear that $BQ\underline{\underline{P}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)^\wedge_p \simeq (S^1)^\wedge_p \times SU^\wedge_p \simeq U^\wedge_p$. The result now follows by combining the above discussion with Part III, §1.4.4(b).

3.11: Remark. (i) The right hand side of §3.10 is additively isomorphic to the i-dimensional part of $\pi_{2*}(MU;\hat{Z}_p) \otimes_{\hat{Z}_p} E(v_1,v_3,\ldots)$ where $E(v_1,v_3,\ldots)$ is an exterior algebra on odd dimensional generators. This follows easily by a spectral sequence argument (cf. [Ad 1, Part II]. Recall that these spectra all have periodic homotopy and that the result depends only on i mod 2.

(ii) Localisation in general. If R is an $\overline{\mathbb{F}}_q$-algebra and $S \subset R$ is a multiplicative set of central non-zero divisors there is a fibration [Q5, p. 233], $K_oR \times (X_{\underline{\underline{P}}(\mathrm{Spec}\ R)}) \to K_oR_S \times (X_{\underline{\underline{P}}(\mathrm{Spec}\ R_S)}) \to (BQH)$ (3.11.1)

where H is the category of finitely generated R-modules, M, of projective dimension ≤ 1 and $M_S = 0$. However (3.11.1) is not necessarily a fibration of H-spaces so that we cannot construct (as in Part III, §1.11) a spectral sequence to compute $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\ (R_S)})$ from $H_*((BQH)^\wedge_p;\ \pi_*(\underline{A\overline{\mathbb{F}}}_{q,\ \mathrm{Spec}\ R}))$.

However if (3.11.1) is split then we may compute as in §3.10.

3.12: Mayer-Vietoris Theorem. Let $V = U_1 \cup U_2$ where U_i are Zariski opens and suppose V is regular ([Q4, §7.1]). Suppose also that $K_oV \to K_oU_1 \oplus K_oU_2$ is injective. Then there is a strongly convergent spectral sequence

$$E^2_{*,*} = H_*((X_{\underline{\underline{P}}(U_1 \cap U_2)})^\wedge_p;\ \pi_*(\underline{A\overline{\mathbb{F}}}_{q,V})) \Rightarrow (A\overline{\mathbb{F}}_{q,\ U_1})_*((X_{\underline{P}(U_2)})^\wedge_p)\ .$$

(Note that the abutment is symmetrical in U_1 and U_2.)

Proof. In [Br-G, §3] it is shown that algebraic K-theory is pseudo-flasque. This means that $K_oV \times X_{\underline{\underline{P}}(V)} \to \prod_{i=1}^{2} K_oU_i \times X_{\underline{\underline{P}}(U_i)} \to K_o(U_1 \cap U_2) \times X_{\underline{\underline{P}}(U_1 \cap U_2)}$ is a fibration. Since $K_oV \to K_oU_1 \oplus KU_2$ is injective the fibration property is preserved by taking base-point components--and also by completion. Hence

$$(X_{\underline{\underline{P}}(V)})^\wedge_p \to (X_{\underline{\underline{P}}(U_1)})^\wedge_p \times (X_{\underline{\underline{P}}(U_2)})^\wedge_p \to (X_{\underline{\underline{P}}(U_1 \cap U_2)})^\wedge_p$$

is a fibration. We obtain a spetral sequence from Part III, §1.11. However the argument used in §3.7 shows that the "diagonal" $(BGL\overline{\mathbb{F}}_q^+)^\wedge_p \to \prod_{i=1}^{2} (X_{\underline{\underline{P}}(U_i)})^\wedge_p$ is conjugate by an H-space equivalence to a map into the first factor (or into the second). Hence the abutment of the spectral sequence may be identified by Part III, §1.10.

3.12.1: Remark. The result of [Br-G, §3] referred to above makes no reference to the regularity condition yet claims the pseudo-flasque condition for the K-groups rather than for Quillen's K'-groups. In [Q4, §7.3] the Mayer-Vietoris fibration is only proved for the spaces corresponding to K'-groups. The regularity assumption ensures that K' and K coincide [Q4, §7].

3.13: p-adic Algebraic Cobordism is not Pseudo-Flasque. In [Br-G] the notion of pseudo-flasqueness is defined for a generalised presheaf cohomology, like $\{\underline{A\overline{\mathbb{F}}}_{q,V};\ V \to \operatorname{Spec}\overline{\mathbb{F}}_q\}$. If $V = U_1 \cup U_2$ is the union of two Zariski opens then being pseudo-flasque would imply an exact homotopy sequence of the following form.

$$\cdots \to \pi_{i+1}(\underline{A\overline{\mathbb{F}}}_{q,U_1 \cap U_2}) \to \pi_i(\underline{A\overline{\mathbb{F}}}_{q,V}) \to \bigoplus_{j=1}^{2} \pi_i(\underline{A\overline{\mathbb{F}}}_{q,U_i}) \cdots$$

However if we set $V = \mathbb{P}^1_{\overline{\mathbb{F}}_q}$ and $U_1 = U_2 = \mathbb{A}^1_{\overline{\mathbb{F}}_q}$ this would yield an exact sequence, by §§3.4, 3.9 and 3.10, of the form

$$0 \to (MU\hat{Z}_p)_{2*+1}(U_p^\wedge) \to (MU\hat{Z}_p)_{2*}(BU_p^\wedge) \longrightarrow \bigoplus_1^2 \pi_{2*}(MU\hat{Z}_p) \to (MU\hat{Z}_p)_{2*}(U_p^\wedge) \to 0.$$

All the groups are free $\pi_{2*}(MU\hat{Z}_p)$-modules so applying $(\_\otimes_{\pi_{2*}(MU\hat{Z}_p)} \hat{Z}_p)$ yields an exact sequence

$$0 \to H_{odd}(U_p^\wedge) \to H_{even}(BU_p^\wedge) \to \hat{Z}_p \oplus \hat{Z}_p \to H_{even}(U_p^\wedge) \to 0$$

which is impossible since the $\hat{Z}_p$-rank of the right hand group is countably infinite.

3.14: Units, the Analogue of p-adic Topological K-theory and the Conner-Floyd Homomorphism. Suppose that A is a commutative $\overline{\mathbb{F}}_q$-algebra whose units are denoted by A^*. We have an inclusion $\overline{\mathbb{F}}_q^* \to A^*$ which induces $(B\overline{\mathbb{F}}_q^*)_p^\wedge \to (BA^*)_p^\wedge$. As in Part III, §1.4.5(b) we use this map to form a spectrum $(BA^*)_p^\wedge(b)$ where b is the image of $b \in \pi_2((B\overline{\mathbb{F}}_q^*)_p^\wedge)$. The determinant induces an H-map (preserving $b \in \pi_2$)

$$\mathrm{Det} : (X_{\underline{\underline{P}}(\operatorname{Spec} A)})_p^\wedge \simeq (BGLA^+)_p^\wedge \to (BA^*)_p^\wedge$$

which is split as a map of spaces. Consequently on homotopy and stable homotopy Det is surjective. Since

$$\pi_j(\underline{A\overline{\mathbb{F}}}_{q,\operatorname{Spec} A}) = \varinjlim_n \pi^S_{j+2n}((X_{\underline{\underline{P}}(\operatorname{Spec} A)})_p^\wedge)$$

and

$$\pi_j((BA^*)_p^\wedge(b)) = \varinjlim_n \pi^S_{j+2n}((BA^*)_p^\wedge)$$

we obtain the following result--in view of Part III, §1.4.5(a),(b) and Part II, §9.1.1 this result is a generalisation of the Conner-Floyd theorem [C-F].

3.15: Generalised Conner-Floyd Theorem. In §3.14 Det induces a surjective ring homomorphism

$$\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\operatorname{Spec} A}) \to \pi_*((BA^*)_p^\wedge(b)) .$$

3.16: Application of Generalised Conner-Floyd Theorem. Let K be an $\overline{\mathbb{F}}_q$-algebra without divisors of zero. Then for all $i \geq 0$, $\pi_i(\underline{A\overline{\mathbb{F}}}_{q,\, \mathrm{Spec}\, K[t,t^{-1}]})$ has $\hat{Z}_p$ as a quotient.

Proof. Let L be the quotient field of K. Since $L[t,t^{-1}]^* = \{xt^m \mid c \in L^*,\ m \in Z\}$ we have $K[t,t^{-1}]^* \cong K^* \times Z$ and $(BK[t,t^{-1}]^*)^{\wedge}_p \simeq (BK^*)^{\wedge}_p \times (S^1)^{\wedge}_p$. Also K^* retracts onto $\overline{\mathbb{F}}^*_q$, because if $\overline{L}$ is the separable closure of L then $\overline{L}^*$ has $\overline{\mathbb{F}}^*_q$ as a direct factor. Hence $(BK^*)^{\wedge}_p \simeq (B\overline{\mathbb{F}}^*_q)^{\wedge}_p \times Y$ for some Y. Arguing as in §3.9(ii) and Part III, §1.10 we have a homomorphism of

$$\pi_*((B\overline{\mathbb{F}}^*_q)^{\wedge}_p(b)) \cong \pi_*(\underline{KU\hat{Z}_p})\text{-modules}$$

$$\pi_i((BK[t,t^{-1}]^*)^{\wedge}_p(b)) \cong KU_i(Y \times (S^1)^{\wedge}_p;\ \hat{Z}_p).$$

The identification with the $\underline{KU\hat{Z}_p}$-spectrum comes from Part II, §1.4.5(c). Hence by §3.15,

$$\mathrm{Det} : \pi_i(\underline{A\overline{\mathbb{F}}}_{q,\, \mathrm{Spec}\, K[t,t^{-1}]}) \to KU_i(Y \times (S^1)^{\wedge}_p;\ \hat{Z}_p)$$

is onto and the result follows since the latter (unreduced) $KU\hat{Z}_p$-group has $\hat{Z}_p$ as a quotient for all $i \geq 0$.

3.17: Remark. From the foregoing examples it should be clear how to use Part II, §§1.4, 1.10, 1.11 to pass from a K-theory computational result to a result about p-adic algebraic cobordism. Consequently the techniques of dévissage [Q4, §5] and reduction by resolution [Q4, §4] have their applications to computations in p-adic algebraic cobordism. However these applications are to specific examples and will not be examined here. In further papers I intend to examine some examples of the process of converting algebraic geometry phenomena into algebra through $A\overline{\mathbb{F}}_q$-theory.

§4. UNITS AND THE p-ADIC COBORDISM OF $\overline{\mathbb{F}}_q$-ALGEBRAS AND THEIR QUILLEN K-THEORY

Through this section A will be a commutative $\overline{\mathbb{F}}_q$-algebra and p will be a prime different from the characteristic of $\overline{\mathbb{F}}_q$. Also $X^{\wedge}_p$ will denote the p-completion of X, as in §3.

If P(A) is the category of finitely generated projective A-modules (so that $P(A) = \underline{P}(\mathrm{Spec}\, A)$ of Part III, §1.4.6), then

$$\Omega BQP(A) \simeq K_oA \times BGLA^+ \quad [G\text{-}Q].$$

I wish now to relate the p-adic algebraic cobordism, $\pi_*(\cdot\, \overline{\mathbb{F}}_{q,\, \mathrm{Spec}\, A})$, of §3 to the $K_iA = \pi_{i+1}(BQP(A) \cong \pi_i(BGLA^+)$ $(i > 0)$.

Firstly we have the short exact sequence of [A-M, p. 183]

$$0 \to \mathrm{Ext}(Z/p^\infty, K_iA) \to \pi_i((BGLA^+)^{\wedge}_p) \to \mathrm{Hom}(Z/p^\infty, K_{i-1}A) \to 0 \qquad (4.1)$$

where the left hand term may be identified with the p-adic completion of K_iA. Hence I propose to concentrate for the moment on $\pi_*((BGLA^+)^{\wedge}_p)$. This is a graded ring with the addition induced by direct sum of matrices and with the product induced by tensor product [Sn 7, p. 71]. If $b \in \pi_2((BGL\overline{\mathbb{F}}_q^+)^\wedge_p)$ is as in Part III, §1.4.4 then $\pi_*((BGL\overline{\mathbb{F}}_q^+)^\wedge_p) \cong \hat{Z}_p[b]$ because of the equivalence $(BGL\overline{\mathbb{F}}_q^+)^\wedge_p \simeq BU^\wedge_p$ $(= BU\hat{Z}_p)$. Thus

$$\pi_*((BGL\overline{\mathbb{F}}_q^+)^\wedge_p)[b^{-1}] = \varinjlim_n \pi_{*+2n}((BGL\overline{\mathbb{F}}_q^+)^\wedge_p) \cong \hat{Z}_p[b,b^{-1}]$$

where the direct limit is taken over successive multiplication by b. If $b \in \pi_2((BGLA^+)^\wedge_p)$ also denotes the image of b we may hope to recover information about K_iA by means of (4.1) and by studying

$$\varinjlim_n \pi_{*+2n}((BGLA^+)^\wedge_p) \cong \pi_*((BGLA^+)^\wedge_p)[b^{-1}].$$

Henceforth we will consider this localisation as a Z/2-graded ring and denote it by K_*A. Explicitly for i = 0 or 1 $K_iA = \varinjlim_n \pi_{i+2n}((BGLA^+)^\wedge_p)$, the limit being taken over iterated products with b.

<u>4.2</u>: <u>Theorem</u>. Let $G_A = A^*/\overline{\mathbb{F}}_q^*$ where$(_)^*$ denotes units.

(a) Then there is an epimorphism of Z/2-graded rings

$$\lambda_A : \pi_*(\underline{A\overline{\mathbb{F}}}_{q,\,\mathrm{Spec}\,A}) \to KU\hat{Z}_{p_*}((BG_A)^\wedge_p) .$$

(b) There is a homomorphism of Z/2-graded rings

$$\gamma_A : KU\hat{Z}_{p_*}((BG_A)^\wedge_p) \to K_*A .$$

Both λ_A and γ_A are natural with respect to $\overline{\mathbb{F}}_q$-algebra homomorphisms.

<u>Proof</u>. (a) The determinant induces a (split surjective) map of H-spaces

$$(BGLA^+)^\wedge_p \to (BA^*)^\wedge_p .$$

This map preserves the element $b \in \pi_2$ of Part III, §1.4.4. Hence a ring homomorphism is induced[5]

$$\begin{array}{rcl} \pi_*(A\overline{\mathbb{F}}_{q,\,\mathrm{Spec}\,A}) & = & \varinjlim_n \pi^S_{*+2n}((BGLA^+)^\wedge_p) \\ & & \downarrow \lambda'_A \\ & & \varinjlim_n \pi^S_{*+2n}((BA^*)^\wedge_p) \ . \end{array}$$

Being a direct limit of epimorphisms λ'_A is onto also. However $A^* \cong \overline{\mathbb{F}}_q^* \times G_A$, for we may map A to K, an algebraically closed field of finite characteristic, in such a way that $\overline{\mathbb{F}}_q \subset A$ injects. Then K* is the product of a divisible torsion group, $\overline{\mathbb{F}}_q^*$, and a divisible torsion free group so that projecting to $\overline{\mathbb{F}}_q^*$ splits A*. Therefore

$$(BA^*)^\wedge_p \simeq (B\overline{\mathbb{F}}_q^*)^\wedge_p \times (BG_A)^\wedge_p \simeq (\mathbb{C}P^\infty)^\wedge_p \times (BG_A)^\wedge_p .$$

Combining Part III, §1.4.5(b) and §1.10 identifies the limit which is the range of λ'_A as $KU\hat{Z}_{p_*}((BG_A)^\wedge_p)$ thereby constructing λ_A as required.

(b) Since $(BGLA^+)^\wedge_p$ is an infinite loopspace [Ma 2] $[\_,(BGLA^+)^\wedge_p]$ (and its localisation by inverting b) is the zero-th group of a cohomology functor. However if $m : (BA^*)^\wedge_p \times (BA^*)^\wedge_p \to (BA^*)^\wedge_p$ is the H-space product and if

$$x \in [(BA^*)^\wedge_p,\ (BGLA^+)^\wedge_p]$$

is the canonical class, then

$$m^*(x) = x \otimes 1 + 1 \otimes x + x \otimes x\ .$$

Also x restricted via $b \in \pi_2((BA^*)^\wedge_p)$ is just $b \in \pi_2((BGLA^+)^\wedge_p)$. Therefore composition with x induces

$$x_\# : [\_,(BA^*)^\wedge_p] \to \bigoplus_{n=-\infty}^{\infty} [\Sigma^n(\_),\ (BGLA^+)^\wedge_p][b^{-1}] = h^*(\_),\ \text{say}\ .$$

Also $x_\#$ is an exponential to which we may apply the version of Part II, §5.3 in which $KU\wedge(\_)$ is replaced by $[\_,(BA^*)^\wedge_p]$ to obtain

$$\hat{x} : \varinjlim_n \{\Sigma^{2n}(\_),(BA^*)^\wedge_p\} \to h^*(\_)\ .$$

Applying this map in the case of a sphere and identifying (as in (a))

$$\varinjlim_n \pi^S_{*+2n}((BA^*)^\wedge_p) \cong KU\hat{Z}_{p_*}((BG_A)^\wedge_p)$$

we obtain γ_A.

4.3: Examples

4.3.1. Let $A = \overline{\mathbb{F}}_q[t]$ then $K_iA \cong K_i(\overline{\mathbb{F}}_q) = \begin{cases} \hat{Z}_p & i \text{ even,} \\ 0 & i \text{ odd} \end{cases}$ by [Q4, Theorem 8]. However $A^* = \overline{\mathbb{F}}_q^*$ so that G_A is trivial and γ_A takes the form

$$\pi_*(KU\hat{Z}_p) \xrightarrow{\cong} K_*(\overline{\mathbb{F}}_q[t]) \cong K_*(\overline{\mathbb{F}}_q)\ .$$

which is an isomorphism by Part III, §§1.4.4/5.

4.3.2. Let $A = \overline{\mathbb{F}}_q[t]/(t^n)$, then $K_i(A) \cong K_i(\overline{\mathbb{F}}_q)$ in fact. However G_A in this case is isomorphic to the kernel of the augmentation homomorphism $A^* \to \overline{\mathbb{F}}_q^*$, which is a q-group. Hence $(BG_A)^\wedge_p \simeq *$ and γ_A is an isomorphism as in the previous example, because K_*A and $K_*\overline{\mathbb{F}}_q$ differ only in q-torsion.

The details are left to the interested reader.

In both 4.3.1 and 4.3.2 λ_A is just the classical Conner-Floyd homomorphism $\pi_*(MU\hat{Z}_p) \to \pi_*(KU\hat{Z}_p)$, by the p-adic version of Part II, §9.2.7.

4.3.3. Let $A = \overline{\mathbb{F}}_q[t,t^{-1}]$. In this case [Q4, Theorem 8] $K_i(A) \cong K_i(\overline{\mathbb{F}}_q) \oplus K_{i-1}(\overline{\mathbb{F}}_q) \simeq \overline{\mathbb{F}}_q^*$ $(i > 0)$, and so $K_i(A) \simeq \hat{Z}_p$ for each i. The generator of $K_i(A)$ with i even is the image of a generator of $K_i(\overline{\mathbb{F}}_q)$ under the inclusion of $\overline{\mathbb{F}}_q$ into A. Loday has shown that the isomorphism $K_{2i}(A) \xrightarrow{\cong} K_{2i-1}(\overline{\mathbb{F}}_q)$ is given by cup product with $t \in A^*$ [G-Q, p. 237]. Hence the generator of $K_i(A)$ with i odd is represented by the generator of an odd dimensional homotopy group of $(BGLA^+)^\wedge_p$. However $A^* \cong \overline{\mathbb{F}}^*_q \times Z$ where Z is generated by t. Hence

$$\pi_1((BA^*)^\wedge_p) = \pi_1(K(Z;1)^\wedge_p) = \hat{Z}_p$$

and $K(Z,1)^\wedge_p = (S^1)^\wedge_p$ is the classifying space of G_A. Also the split inclusion $(BA^*)^\wedge_p \to (BGLA^+)^\wedge_p$ induces an isomorphism on fundamental groups so that $K_1(A)$ is generated by $\pi_1((BG_A)^\wedge_p) = \pi_1((S^1)^\wedge_p)$. The same conclusions apply to the generation of $KU\hat{Z}_{p_*}(BG_A) = KU\hat{Z}_{p_*}((S^1)^\wedge_p)$ so from Example 4.3.1 we deduce that γ_A is again an isomorphism.

In this case, from §§3.9/3.10, we see that λ_A takes the form of a composition

$$MU\hat{Z}_{p_*}(U^\wedge_p) \to KU\hat{Z}_{p_*}(U^\wedge_p) \to KU\hat{Z}_p((S^1)^\wedge_p)$$

in which the first map is the Conner-Floyd map and the second is induced by the determinant $U \to S^1$.

<u>4.3.4</u>. Let π be a finite abelian group of order prime to q then $\mathbb{F}_q[\pi]$, the group ring of π, is a product of Galois fields. This is seen by writing $\pi = \pi_1 \times \pi_2 \times \cdots \times \pi_s$, a product of cyclic groups of prime power order so that $\mathbb{F}_q[\pi] \cong \mathbb{F}_q[\pi_1] \underset{\mathbb{F}_q}{\otimes} \mathbb{F}_q[\pi_2] \otimes \cdots$ and remarking that $\mathbb{F}_q[t]/(t^{u^m} - 1)$ is a product of Galois extensions of $\mathbb{F}_q$ when u is a prime not equal to the characteristic. Hence $\overline{\mathbb{F}}_q[\pi] = \prod_{i=1}^{N} A_i$ where A_i is a copy of $\overline{\mathbb{F}}_q$.

$$\text{Let } A = \overline{\mathbb{F}}_q[\pi] \text{ then } K_iA = \begin{cases} \bigoplus_1^N \hat{Z}_p & , \text{ i even} \\ 0 & , \text{ i odd.} \end{cases}$$

In this case G_A is the $(N-1)$-fold product of $\overline{\mathbb{F}}_q$ and γ_A takes the form (as $(BA_i^*)^\wedge_p \simeq (\mathbb{C}P^\infty)^\wedge_p$)

$$KU\hat{Z}_{p_*}\left(\prod_1^{N-1} (\mathbb{C}P^\infty)^\wedge_p\right) \to K_*A .$$

This is onto because the generators of $K_o(A) = \bigoplus_1^N \hat{Z}_p$ are represented by the generators of the $K_o(A_i)$ which originate in $\pi_2((BA_i^*)^\wedge_p) \cong \pi_2((\mathbb{C}P^\infty)^\wedge_p)$. However each of these elements factors through $(KU\hat{Z}_p)_o\left(\prod_1^{N-1} (CP^\infty)^\wedge_p\right)$.

From §1.10 and §3 it is easy to see that λ_A in this example takes the form

$$MU\hat{Z}_{p_*}\left(\prod_1^{N-1} BU^\wedge_p\right) \to KU\hat{Z}_{p_*}\left(\prod_1^{N-1} BU^\wedge_p\right) \to KU\hat{Z}_{p_*}\left(\prod_1^{N-1} (\mathbb{C}P^\infty)^\wedge_p\right)$$

in which the first map is the Conner-Floyd homomorphism and the second is induced by the determinant $BU \to \mathbb{C}P^\infty$.

4.3.5: Speculative Example. Let $A = \overline{\mathbb{F}}_q[x,y]/(f)$ where $f(x,y) = x^3 - x - y^2$. The K-theory of this ring is, I believe, unknown. However it is related to the K-theory of the elliptic curve (q odd)

$$E = \{[x,y,z] \in \mathbb{P}^2_{\overline{\mathbb{F}}_q} : zy^2 = x^3 - xz^2\}.$$

If we excise from E the closed point $Z = (0,1,0)$ at infinity there is an exact sequence [Q4, §7.3.2]

$$\cdots \to K_{a+1}(E - Z) \to K_a(\overline{\mathbb{F}}_q) \to K_a(E) \to \cdots .$$

Also $E - Z = \operatorname{Spec} A$ so that $K_*(E - Z) = K_*(A)$.

Since $K_n(A)$ is unknown what about $\mathcal{K}_*(A)$? The previous examples suggest that

$$\gamma_A : KU\hat{Z}_{p_*}((BG_A)^\wedge_p) \to \mathcal{K}_*(A)$$

is an isomorphism. For it is elementary to show that $\underline{0} = G_A$ so that γ_A takes the form $\gamma_A : \pi_*(KU\hat{Z}_p) \to \mathcal{K}_*A$. γ_A is clearly injective (by considering the map $\overline{\mathbb{F}}_q[x,y]/(f) \to \overline{\mathbb{F}}_q$ given by sending y and x to zero). Hence the preceding examples suggest γ_A is an isomorphism. Theorem 4.5 and Examples 4.6.1/4 then suggest that

$$K_i(\mathbb{F}_q) \to K_i(\mathbb{F}_q[x,y]/(f))$$

is an isomorphism on torsion of order prime to q.

Incidentally attempts that I have made to compute the K-theory of the affine cuspidal cubic ($A = \overline{\mathbb{F}}_q[x,y]/(x^3 - y^2)$) suggest that in this case K_*A and $K_*\overline{\mathbb{F}}_q$ are isomorphic away from q-torsion and that γ_A is an isomorphism.

4.4: Adams Operations and the Frobenius Map.[6] Suppose now that $A = B \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ where B is a commutative $\mathbb{F}_q$-algebra. Write $V = (BA^*)^\wedge_p$ and $W = (BGLA^+)^\wedge_p$. We will now discuss how the Frobenius map $\phi_q : A \to A$ given by $\phi_q(b \otimes \lambda) = b \otimes \lambda^q$ induces an operation on the cohomology theory of which the zero-th group is $\varinjlim_n [\Sigma^{2n}(\_),W]$ and hence on the homotopy groups $\mathcal{K}_iA$. However the operation will be unstable in the sense that it does not commute with the evident isomorphism $\mathcal{K}_iA \cong \mathcal{K}_{i+2}A$. Therefore we must remember for $i \in \mathbb{Z}$ that

$$\mathcal{K}_iA \cong \varinjlim_n [S^{2n+i},W].$$

The homomorphism ϕ_q induces, via $M \to \phi_q^* M$ a natural transformation on the category of finitely generated projective A-modules. This induces an H-map $\phi_q : W \to W$. When $A = \overline{\mathbb{F}}_q$ it is known [Q2] that $W \simeq BU\hat{Z}_p$ and ϕ_q coincides with the Adams operation ψ^q so that $\phi_{q\#}(b) = qb \in \pi_2(W)$ for any $A = B \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}}_q$. Also if M is an A-module of the form $M' \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}}_q$ for some B-module, M', then $\phi_q^*(M) = M$. Hence we have a homotopy commutative diagram of natural maps and completion maps as follows.

$$\begin{tikzcd} BGLB^+ \arrow[rr, "i"] \arrow[d, "i"'] & & BGLA^+ \arrow[d, "\pi"] \\ BGLA^+ \arrow[r, "\pi"'] & W \arrow[r, "\phi_q"] & W \end{tikzcd} \tag{4.4.1}$$

Consequently the natural homomorphism induced by $\pi \circ i$

$$\rho_B : [\_, BGLB^+] \to \underset{n}{\varinjlim}\, [\Sigma^{2n}(\_), W]$$

satisfies

$$\phi_q \circ \rho_B = \rho_B \,. \tag{4.4.2}$$

There is a commutative diagram

$$\begin{tikzcd} {[X,W]} \arrow[r, "\phi_q"] \arrow[d] & {[X,W]} \arrow[d] \\ {[\Sigma^2 X, W]} \arrow[r, "(\frac{1}{q}\phi_q)"'] & {[\Sigma^2 X, W]} \end{tikzcd} \tag{4.4.3}$$

in which the vertical maps are those from the direct system--that is, multiplication by b. This follows from the fact that ϕ_q is a ring homomorphism so that

$$b(\phi(x)) = \frac{1}{q}(qb)\phi_q(x) = \frac{1}{q}\phi_q(bx) \ .$$

Here $\frac{1}{q} : W \to W$ is a homotopy inverse to the equivalence given by multiplication by q. From (4.4.2) we obtain a ring homomorphism

$$\Phi_q : \underset{n}{\varinjlim}\, [\Sigma^{2n}(\_), W] \to \underset{n}{\varinjlim}\, [\Sigma^{2n}(\_), W] \tag{4.4.4}$$

by composition with the maps $\left\{\frac{1}{q^n}\phi_q\right\}$. The verification is analogous to that of part II, §6.5.

Now consider the limit $\underset{n}{\varinjlim}\, \{\Sigma^{2n}(\_), V\}$ where, as in Theorem 4.2(a) the limit is the one associated with the spectrum of Part III, §1.4.5(c). The Frobenius on A* induces

$$\hat{\phi}_q : V \to V$$

and the following diagram commutes.

$$\begin{array}{ccc} \{\_,V\} & \xrightarrow{\hat{\phi}_q} & \{\_,V\} \\ {\scriptstyle \varepsilon_\#}\downarrow & & \downarrow{\scriptstyle \varepsilon_\#} \\ \{\Sigma^2(\_),V\} & \xrightarrow[(\frac{1}{q}\hat{\phi}_q)]{} & \{\Sigma^2(\_),V\}. \end{array} \qquad (4.4.5)$$

Here $\frac{1}{q}$ is formed using the track addition and the diagram commutes because $\varepsilon_\#$ is multiplication by b and $\hat{\phi}_q$ is a multiplicative homomorphism. We obtain a ring homomorphism analogous to (4.4.4)

$$\hat{\Phi}_q : \varinjlim_n \{\Sigma^{2n}(\_),V\} \to \varinjlim_n \{\Sigma^{2n}(\_),V\}. \qquad (4.4.6)$$

Also for $y \in \{\Sigma^{2n}X,V\}$ and $x : V \to W$ as in Theorem 4.2(b) we have

$$(\frac{1}{q^n}\hat{\phi}_q(y))^*(x) = (\frac{1}{q^n}\hat{\phi}_q)(y^*(x)) \in [\Sigma^{2n}X,W]. \qquad (4.4.7)$$

When $n = 0$ this follows from the definition of ϕ_q and $\hat{\phi}_q$ and when $n \geq 1$ we need only note that the addition in $[\Sigma^{2n}X,W]$ can be defined by track addition or by the H-space sum on W so that dividing by q^n means the same on both sides of the above equation.

The groups $\varinjlim_n \{\Sigma^{2n}(\_),V\}$ and $\varinjlim_n [\Sigma^{2n}(\_),W]$ are the zero-th groups of cohomology theories and $\hat{\Phi}_q$-(Identity), Φ_q-(Identity) extend to natural transformations of cohomology theories. Therefore we may form the "fibre" cohomology theories. That is, there exist infinite loopspaces $F\hat{\Phi}_q$ and $F\Phi_q$ together with homotopy exact sequences related by γ_A as follows.

$$\begin{array}{ccccccc} \cdots \to \pi_m(F\hat{\Phi}_q) & \to & \varinjlim_n \pi^S_{2n+m}(V) & \xrightarrow{\hat{\Phi}_q - 1} & \varinjlim_n \pi^S_{2n+m}(V) & \to & \cdots \\ \downarrow & & \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_A} & & \\ \cdots \to \pi_m(F\Phi_q) & \to & \varinjlim_n \pi_{2n+m}(W) & \xrightarrow{\Phi_q - 1} & \varinjlim_n \pi_{2n+m}(W) & \to & \cdots \end{array}$$

Here γ_A is as in §4.2 and the diagram commutes by virtue of (4.4.7).

We are now ready to summarise the above discussion.

<u>4.5</u>: <u>Theorem</u>.[6] (a) In the notation of §§4.4.1/7 there is a commutative diagram with exact rows in which broken-line triangle commutes for $m \geq 1$.

$$
\begin{tikzcd}
K_m(B) \arrow[d, "\rho'_B"'] \arrow[dr, dashed, "\rho_B"] & & & \\
\cdots \to \pi_m(F\Phi_q) \arrow[r] & \varinjlim_n \pi_{2n+m}(W) \arrow[r, "\Phi_q - 1"] & \varinjlim_n \pi_{2n+m}(W) \arrow[r] & \pi_{2m-1}(F\Phi_q) \cdots \\
\cdots \to \pi_m(F\hat{\Phi}_q) \arrow[r] \arrow[u] & \varinjlim_n \pi^S_{2n+m}(V) \arrow[r, "\hat{\Phi}_q - 1"] \arrow[u, "\gamma_A"'] \arrow[d, "\cong"] & \varinjlim_n \pi^S_{2n+m}(V) \arrow[r] \arrow[u, "\gamma_A"'] & \pi_{2m-1}(F\hat{\Phi}_q) \cdots \\
& (KU\hat{Z}_p)_m((BG_A)^\wedge_p) & &
\end{tikzcd}
$$

(b) If B is an augmented $\mathbb{F}_q$-algebra then $\hat{\phi}_q : A^* \to A^* \cong \overline{\mathbb{F}}^*_q \times G_A$ induces a homomorphism $\hat{\phi}_q : G_A \to G_A$. In this case $\hat{\Phi}_q$ on $(KU\hat{Z}_p)_m((BG_A)^\wedge_p)$ is equal to the composition of the Adams operation, ψ^q, with $(B\hat{\phi}_q)_*$.

Proof. Part (a) was established during the discussion. For example, the commutative triangle of broken arrows results from (4.4.2).

For part (b) we observe that $\hat{\phi}_q$ may be identified with the product of the Frobenius on $\overline{\mathbb{F}}^*_q$ and $\hat{\phi}_q$ on G_A. Hence $\hat{\Phi}_q$ will be, by Theorem 1.10, the composition of the map induced by $B\hat{\phi}_q : BG_A \to BG_A$ with the natural transformation of

$$(KU\hat{Z}_p)_o(\_) \cong \varinjlim_n \{\Sigma^{2n}(\_),\ (B\overline{\mathbb{F}}^*_q)^\wedge_p\}$$

induced by $\hat{\phi}_p : \overline{\mathbb{F}}^*_q \to \overline{\mathbb{F}}^*_q$. On $(KU\hat{Z}_p)_o(S^2) \cong \hat{Z}_p$ this map is multiplication by q since the generator of $\pi^S_2((B\overline{\mathbb{F}}^*_q)^\wedge_p)$ is the image of b. However this class generates the ring $\pi_*(KU\hat{Z}_p)$ so that $\Phi_q = \psi^q$ on $\pi_*(KU\hat{Z}_p)$ which is sufficient to ensure that the ring homomorphisms Φ_q and ψ^q coincide as natural transformations.

4.6: Examples.

4.6.1. When $B = \mathbb{F}_q$ it is the substance of [Q] that $\rho'_{\mathbb{F}_q}$ is an isomorphism on p-torsion and of Part III, §1.4.5(c) that $\gamma_{\overline{\mathbb{F}}_q}$ is an isomorphism.

4.6.2. Let $B = \mathbb{F}_q[t]$ or $\mathbb{F}_q[t]/(t^n)$. From §§4.3.1/2, 4.6.1 we see that ρ'_B is an isomorphism on p-torsion.

4.6.3. Let $B = \mathbb{F}_q[t,t^{-1}]$ then $G_A \cong Z$ and $\hat{\phi}_q$ on G_A is trivial. Hence applying Theorem 4.5(b) and §4.3.3 we obtain a diagram ($\varepsilon = 0$ or 1) with exact rows.

$$
\begin{tikzcd}
& & & K_{2m-\varepsilon}(\mathbb{F}_q]t,t^{-1}]) \arrow[d, "\rho'_B"'] & \\
0 \arrow[r] & \varinjlim_n \pi_{2n+2m-\varepsilon}(W) \cong \hat{Z}_p \arrow[r, "(q^m-1)"] & \hat{Z}_p \arrow[r] & \pi_{2m-\varepsilon}(F\Phi_q) \arrow[r] & 0
\end{tikzcd}
$$

From the results of [Q] it is not difficult to show that ρ_B' is an ismorphism on p-torsion in this example also.

4.6.4. Let π be a finite abelian group of order prime to q and let $B = \mathbb{F}_q[\pi]$. Then B is a product of fields $\mathbb{F}_{q^d}$ and ρ_B' is again an isomorphism on p-torsion. To see this it suffices to consider the case $B = \mathbb{F}_{q^d}$ because both $K_i(\_)$ and the top row of the diagram of Theorem 4.5(a) are additive for finite products. Also, since γ_A is onto in this example (by §4.3.4), we may compute Φ_q from $\hat{\Phi}_q$ which is given in terms of Adams operations. However there is a ring isomorphism

$$\mu : \mathbb{F}_{q^d} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \xrightarrow{\sim} \bigoplus_0^{d-1} \overline{\mathbb{F}}_q$$

with j-th component of $\mu(a \otimes b)$ given by $a^{q^j} b$ $(0 \le j \le d-1)$. Hence on $\bigoplus_1^d \overline{\mathbb{F}}_q$ the Frobenius takes the form

$$\phi_q(x_0, \ldots, x_{d-1}) = (x_{d-1}^q, x_0^q, \ldots, x_{d-2}^q).$$

Therefore

$$\Phi_q - 1 : \varinjlim_n \pi_{2n+2m}(W) \to \varinjlim_n \pi_{2n+2m}(W) \cong \bigoplus_0^{d-1} \hat{Z}_p$$

is given by

$$(\Phi_q - 1)(y_0, \ldots, y_{d-1}) = (q^m x_{d-1} - x_0,\ q^m x_0 - x_1, \ldots) .$$

which is a monomorphism with cokernel isomorphism to $\hat{Z}_p/(q^m - 1)\hat{Z}_p$. Also $0 = \varinjlim_n \pi_{2n+2m-1}(W)$ in this case and again from the results of [Q] one sees that ρ_B' is an isomorphism on p-torsion.

4.7: Remark. The object of examining the homomorphisms of Theorems 4.2 and 4.5 in the foregoing examples is to substantiate a conjecture that for induced $\overline{\mathbb{F}}_q$-algebras we may recover the torsion of the Quillen K-groups from the p-adic algebraic cobordism. See §5, Problems 11 and 12.

§5. PROBLEMS

Stable Homotopy Groups

Problem 1. Part I, §§5-6, Part III, §2 and [K-Sn] elements were constructed $\pi_*^S(BG)$ for G = U, O or Sp. There exist several K-theory pairings induced by the K-theory product. For example,

$$BO \wedge O \to O$$
$$BSp \wedge O \to Sp$$
$$BO \wedge Sp \to O$$
$$BSp \wedge Sp \to O$$
$$\text{and} \quad BU \wedge U \to U.$$

Detect the elements in $\pi_*^S(O)$, $\pi_*^S(Sp)$ and $\pi_*^S(U)$ which are obtained by pairing my elements with known elements. For example detect the pairing of an Arf invariant one element in $\pi^S_{2^r-2}(0)$ with some of the newly constructed elements in $\pi^S_*(BO)$.

<u>Problem 2</u>. In Part III, §4 some elements of order two were constructed in $\pi^S_*(BO\mathbb{F}_3)_{(2)} \cong \pi^S_*(JO(2))_{(2)}$. However the solution of the Adams conjecture [Q2; Su] implied that there is a split injection (2-localised) $JO(2) \to SG$. Since $SG_{(2)}$ is an infinite loopspace its stable homotopy maps onto its homotopy be means of the Dyer-Lashof map $d : QSG \to SG$ [Ma 1]. Hence we obtain a homomorphism ($j \geq 1$)

$$\pi^S_j(BO\mathbb{F}_3)_{(2)} \to \pi^S_j(SG)_{(2)} \xrightarrow{d_\#} \pi_j(SG)_{(2)} \cong {}_2\pi^S_j(S^\circ) .$$

Detect in the stable stems the images of some of the elements constructed in Part III, §4. It is possible that the Hurewicz homomorphism

$$\pi_j(SG_{(2)}) \to H_j(SG;Z/2)$$

may detect some of these elements. (Originally Part III, §4 was motivated by the hope that the Arf invariant elements might arise this way--as far as I know that is still a possibility.)

<u>K-Theory</u>

<u>Problem 3</u>. In Part II, §9 a model for unitary K-theory was given $KU(X) \cong \varinjlim_n \{\Sigma^{2n}X, \mathbb{C}P^\infty\}$. Consider the following S-map

$$\varepsilon_0 : \Sigma^8 \mathbb{H}P_\infty \xrightarrow{f \wedge g} BO \wedge BSp \xrightarrow{\otimes} BSp \xrightarrow{\pi} \mathbb{H}P^\infty$$

in which $f \in \pi_8(BO)$ is the Bott class, $\otimes$ is the KO-product and π is the collapsing map onto the MSp(1) summand in the stable splitting of BSp of Part I.

Taking the limit over successive compositions with ε_0 we can form $\varinjlim_n \{\Sigma^{8n}X, \mathbb{H}P^\infty\}$. This group maps to KO(X) (cf. Part I, §9). Is this a description of KO(X) when $\dim X < \infty$?

<u>The Spectrum X(T)</u>

<u>Problem 4</u>. The homology mod 2 of $BO\mathbb{F}_3^+ \simeq JO(2)$ looks like $H_*(SO) \otimes H_*(BO)$. This suggests if we replace $\mathbb{R}$ by $\mathbb{F}_3$ in Part III, §§1.4.2 and 3.1 we will obtain $MO^*(\_\wedge SO)$ instead of MO^* for the cohomology theory associated with $BO\mathbb{F}_3^+(\eta)$. Prove this using the spectral sequence of Part III, §1.11 applied to the fibring $JO(2) \to BO \to BSO$. Can this be used to solve Problem 6 (cf. Part III, §5)?

<u>Problem 5</u>. Let $x \in \pi^S_j(BGL\Lambda^+)$ then we have a map $\nu : K_i(\Lambda) \to \pi_i(\underline{BGL\Lambda^+(x)})$ given by setting $\nu(f) - 1$ equal to the element represented by the stable homotopy class of $f : S^i \to BGL\Lambda^+$ (cf. Part III, §5.4.1). Also $\nu(f+g) = \nu(f)\nu(g)$. In [Sn 6] or [Sn 7] we find $K_3Z/4 \cong Z/6$. If x is taken as the generator of $K_1Z/4 \cong Z/2$ and $BGLZ/4^+(x)$ is denoted by $AZ/4$ is $K_3Z/4 \to \pi_3(AZ/4)$ non-trivial?

Problem 6. Since $2x = 0$ in Problem 5 then $\pi_i(AZ/4)$ is a $Z/2$-vector space. In Part III, §5 we found an epimorphism $\pi_o(AZ) \to MO^*(S^o)$. Does this epimorphism fact through reduction mod 4 $\pi_o(AZ) \to \pi_o(AZ/4)$?

Algebraic Vector Bundles Over Number Fields

Problem 7. In Part IV, §1 we saw how Atyiyah's problem concerning algebraic vector bundles reduces to the study of Galois group actions of $\alpha \in \mathrm{Gal}(\mathbb{C}/K)$.

$$\hat{\alpha}_V : \tilde{K}(V_{c\ell};\hat{Z}) \to \tilde{K}(V_{c\ell};\hat{Z}).$$

$\hat{\alpha}_V$ is a natural ring homomorphism as V varies over K-varieties. Can this naturality be used to determine the possible $\hat{\alpha}_V$?

p-adic Algebraic Cobordism

Problem 8. In the notation of Part IV, §3 we have an exponential homomorphism

$$\nu : K_i(V) \to \pi_i(\underline{A\overline{\mathbb{F}}}_{q,V})$$

natural in the $\overline{\mathbb{F}}_q$-scheme, V. This is constructed in a similar manner to ν in Part III, §5.4.1 as in Problem 5 using $X_{\underline{\underline{P}}(V)} \to (X_{\underline{\underline{P}}(V)})^\wedge_p$.

For $i \leq 2$ many K_iV are known and for higher K-groups we have the results of [Q4]. Analyse ν in some of these cases.

Problem 9. In Part II, §§5-7 we saw how unitary K-theory operations induced cobordism operations. K_iA is a λ-ring and therefore has operations. If A is an $\overline{\mathbb{F}}_q$-algebra these operations will induce operations on $\pi_*(\underline{A\overline{\mathbb{F}}}_{q,\,\mathrm{Spec}\,A})$. What can be said about them in the examples of Part IV, §3?

Classical Cobordism

Problem 10. In Part II, §8 we saw one (very homotopy theoretic) method of describing the Pontrjagan-Thom construction in AU-theory. Give a direct geometrical construction in terms of AU-theory.

Quillen K-theory

Problem 11. In Part IV, §4.5 is

$$\rho_B' : K_i(B) \to \pi_i(F\Phi_q)$$

an isomorphism on p-torsion? For example, what about Example 4.3.6?

Problem 12. In Part IV, §4.2 is the homomorphism

$$\gamma_A : KU\hat{Z}_{p^*}((BG_A)^\wedge_p) \to \varinjlim_n \pi^S_{2n+*}((BGLA^+)^\wedge_p)$$

surjective for an $\overline{\mathbb{F}}_q$-algebra, A? For instance, what about the test case, Example 4.3.6?

In Problems 11 and 12 it may be possible that we should restrict to regular A. For example $A = \overline{\mathbb{F}}_q[x,y]/(x^3 - y^2)$--the coordinate ring of the affine cusp--has $A^* \cong \overline{\mathbb{F}}^*_q$ but $K_oA \cong Z \oplus \overline{\mathbb{F}}_q \not\cong K_o\overline{\mathbb{F}}_q$. Notice however that this example is not conclusive since the K-groups differ by q-torsion only. In fact that is the only difference I have found in singular affine curves over $\overline{\mathbb{F}}_q$ and it is a difference which vanishes upon p-completion ($p \neq q$). See, for example, the singular ring in Example 4.3.2.

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FOOTNOTES

1. F. Cohen has pointed out to me that the maps $\frac{F_n C_\infty BG_1}{F_{n-1} C_\infty BG_1} \to \frac{B\Sigma_n \int G_1}{B\Sigma_{n-1} \int G_1}$ is <u>not</u> a homeomorphism. In fact the range is not simply connected. This means that ν_{G_n} is still defined and yields a stable equivalence, by the proof of Part I, Theorem 4.2. However μ_{G_n} is not defined. This is simple to rectify. In [C-M-T,2] is given a stable splitting of classifying spaces

$$B\Sigma_m \int G_1 \simeq \bigvee_{1 \le k \le n} \frac{B\Sigma_k \int G_1}{B\Sigma_{k-1} \int G_1} .$$

The resulting (stable) inclusions of $\frac{B\Sigma_k \int G_1}{B\Sigma_{k-1} \int G_1}$ into $B\Sigma_m \int G_1$ may be used in the definition (p. 20) of μ_{G_m} to replace β_n the proof then proceeds with only minor changes.

With reference to the stable splittings of [C-M-T,1] and its applications to my results see also footnote 2.

2. The proof of Theorem 3.1 given here involves three ingredients. Firstly we have stably to split BU(n) and BSp(n) in a controlled manner, which uses the transfer map as developed in Part I and the stable splittings of QX developed in [Sn 1]. In addition we need the obstruction theory arguments of Part II §§2.9, 2.10. However, since my first proofs (circa 1975/6) approaches to the transfer [Fe, Fe 2] and to stable splittings [C^2-M-T; C-M-T 1 & 2; K-Sa; Ma 3] have been developed which are more technological. For example the stable maps of loopspaces, which are used in [Sn 1], become much easier to handle when one uses the right model for the loopspaces. This is done in [Sn 8].

If one uses the combinatorial geometry of the S-maps of [C-M-T 2] together with the double coset formula of [Fe; Fe 2] one can show--without any obstruction theory--that there are equivalences of ring spectra

$$AU \simeq \bigvee_{1 \le k} MU(k) \quad \text{and}$$

$$ASp \simeq \bigvee_{1 \le k} MSp(k).$$

This, of course, gives a very short proof of Part II, Theorems 2.1 and 3.1. This proof will appear in [Sn 8] together with a simpler proof of Part II, Theorems 9.1.1/9.1.2.

3. This remark applies to all the examples in which p-completion is used, namely Part III, §§1.4.4-1.4.6.

It is not true that p-completion commutes with suspension so that in general if $b \in \pi_2(X)$ then $X_p^\wedge(b)$ will not be $X(b)$ with $\hat{Z}_p$-coefficients. Although in the simple examples of Part III, §§1.4.4, 1.4.5 it seems to make little difference. However, in order to ensure that the p-adic algebraic cobordism theories are cohomology theories whose associated infinite loopspaces are p-complete (this is used for example in the sketch proof in §1.4.4) we must p-complete them again. This means forming $P(X_p^\wedge(b))$, in the notation of Part II, §9.1, and then completing again to obtain the infinite loopspace $P(X_p^\wedge(b))_p^\wedge$. Hence p-adic algebraic cobordism shall mean the cohomology theory associated to the latter infinite loopspace. This convention applies in particular to the computations of Part IV, §3.

4. In order to use unreduced homology the spectral sequence $[D^2_{p,q}]$ must be taken with respect to unreduced stable homotopy. If F^+ and E^+ are the disjoint unions of F and E with a point, , then the spectral sequence should read

$$D^2_{p,q} = H_p(B;\pi_q^S(F^+)) \Rightarrow \pi^S_{p+q}(E^+).$$

In Part III, §1.13 set $x = i_\#(b) \in \pi_N^S(E)$. We may make E^+ into an H-space without unit by defining $m_+ : E^+ \times E^+ \to E^+$ to be $m : E \times E \to E$ on $E \times E$ and to send all other points to ∞. Let i denote an inclusion map and let S(_) denote the unreduced suspension. We have a homotopy commutative diagram

$$\begin{tikzcd}
E*E \arrow[r, "\mathrm{Hopf}"] \arrow[d, "i\times i"'] & S(E\times E) \arrow[r, "Sm"] \arrow[d, "S(i\times i)"] & SE \arrow[d, "Si"] \arrow[dr, "i"] & \\
(E^+)*(E^+) \arrow[r, "\mathrm{Hopf}"] & S(E^+\times E^+) \arrow[r, "Sm_+"] & S(E^+) \simeq & SE\vee S^1
\end{tikzcd}$$

The above diagram ensures that the localisation of $\pi_*^S(E^+) \cong \pi_*^S(E) \oplus \pi_*^S(S^0)$ by inverting x (using m_+ to induce multiplication by x) is isomorphic to $\pi_*^S(E)[1/x] = \pi_*(E(x))$ as defined in Part III, §1.2.

This argument justifies the disappearance of the F^+ and E^+ in the spectral sequence in the statement of Theorem 1.13.

5. The argument given in Part IV, §4.2 blithely ignores Footnote #3. For this I apologise--and I hope it does not confuse the reader. The problem is as follows.

In (a) the construction given on p. 134 yields $\lambda_A\colon P((X_{\underline{\underline{P}}(\mathrm{Spec}\,A)})^{\wedge}_p) \to P((BA^*)^{\wedge}_p)$ where $P(-)$ is as in Part II, §9.1. However in Footnote #3 we observed that p-adic algebraic cobordism should really be given by the homotopy groups of the p-completion of the domain space. Thus we must p-complete λ_A. The proof of §4.2(a) then applies to show $\pi_*(P((BA^*)^{\wedge}_p)^{\wedge}_p) \cong KU\hat{Z}_p((BG_A)^{\wedge}_p)$.

In (b) a similar remark must be added to the argument. Namely that, because $P((BA^*)^{\wedge}_p)$ has been replaced by $P((BA^*)^{\wedge}_p)^{\wedge}_p$ we should redefine K_*A as follows. Form the space $\varinjlim_n \Omega^{2n}(BGLA^+)^{\wedge}_p = K_A$ where the limit is taken over iterated products with $b \in \pi_2$ and set $K_iA = \pi_i((K_A)^{\wedge}_p)$ instead of $\pi_i(K_A)$.

Actually in the example of Part IV, §4.3 this alteration leaves K_*A unchanged.

Incidentally since writing this paper I have come to realise (influenced considerably by conversations with C. Soulé) that the "p-complete" approach to these algebraic cobordism spectra raises many difficulties because of the fact that p-completion and stable homotopy type do not commute. This may be overcome by considering algebraic cobordism spectra with coefficients in Z/p^m ($m \geq 1$). In this setting the results of Part IV, Theorem 4.2 become very nice. For example if $p \neq 2$ and $A = \mathbb{F}_q$ with p dividing $(q-1)$ the mod p analogue of $\gamma_{\mathbb{F}_q}$ is an isomorphism.

This and related results will appear elsewhere.

6. Part IV, §§4.4/4.5 require amendments similar to those detailed in Footnote 5. Details are left to the reader. However the changes do not affect Examples 4.6.1-4.6.4.

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