Arizona Winter School 2017: Adic Spaces

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1 An introduction to adic spaces

This year's AWS topic is *perfectoid spaces*, a difficult topic to treat in one week if there ever was one. But given the interest in the topic, and the huge amount of important work awaiting young mathematicians who want to work on this field, it is certainly a worthy effort. The lecture notes here are meant to be a motivated introduction to adic spaces, perfectoid spaces and diamonds, for the reader who knows some algebraic geometry.[1].

1.1 What is a "space"?

Consider the different kinds of geometric "spaces" you know about. First you learned about topological spaces. Then came various sorts of manifolds, which are topological spaces which locally look like a *model space* (an open subset of $\mathbf{R}^n$). Then you learned that manifolds could carry different structures (differentiable, smooth, complex, ...). You could express these structures in terms of the transition functions between charts on your manifold. But this is a little awkward, thinking of everything in terms of charts. Later you learned a more efficient definition: a manifold with one of these structures is a *ringed space* $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, such that locally on X the pair $(X, \mathcal{O}_X)$ is isomorphic to one of the model spaces, together with its sheaf of (differentiable, smooth, complex) functions. An advantage of this point of view is that it becomes simple to define a morphism $f\colon X \to Y$ between such objects: it is a continuous map of topological spaces together with a homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (in other words, *functions pull back*).

This formulation of spaces in terms of pairs $(X, \mathcal{O}_X)$ was good preparation for learning about schemes, the modern language of algebraic geometry. This time the model spaces are affine schemes, which are spectra of rings. For a ring A, the topological space $\operatorname{Spec} A$ may have initially seemed strange—in particular it is not generally Hausdorff. But then you learn some advantages of working

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with schemes. For instance, an integral scheme X has a generic point η. It is enormously useful to take an object associated with X (a morphism to X, an $\mathcal{O}_X$-module, an étale sheaf on X, ...) and pass to its generic fiber, which is associated with the function field of X. Usually if some property is true on the generic fiber, then it is also true "generically" on X (that is, on a dense open subset). Number theorists use this language all the time in the setting of $\operatorname{Spec} \mathbf{Z}$: if a property holds over the generic point $\operatorname{Spec} \mathbf{Q}$, then it holds at almost all special points $\operatorname{Spec} \mathbf{F}_p$.

The language of *formal schemes* is useful for studying what happens in an infinitesimal neighborhood of a closed subset of a scheme. Thus formal schemes often arise in deformation theory. This time, the model spaces are formal spectra $\operatorname{Spf} A$, where A is an admissible topological ring. (Examples of such A include $\mathbf{Z}_p$ and $\mathbf{Z}[\![T]\!]$.) The notation Spf stands for "formal spectrum", and refers to the collection of open prime ideals of A. This can be given the structure of a topological space X, which is equipped with a sheaf $\mathcal{O}_X$ of topological rings.

In the theory of *complex-analytic spaces*, the model space is the vanishing locus of a collection of holomorphic functions on an open subset of $\mathbf{C}^n$. Thus it is like the theory of complex manifolds, except that some singularities are allowed. The theory of complex-analytic spaces has many nice interactions with the theory of schemes. If X is a finite-type scheme over $\operatorname{Spec} \mathbf{C}$, then there is a complex-analytic space X^{an}, the *analytification* of X, which is universal for the property of admitting a morphism of ringed spaces $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$. Conversely, if $\mathcal{X}$ is a complex-analytic space admitting a closed immerson into projective space, then $\mathcal{X}$ is the analytification of a projective complex variety X, and then X and $\mathcal{X}$ have equivalent categories of coherent sheaves, and the equivalence respects cohomology groups (Serre's *GAGA* theorem). In this situation there are *comparison isomorphisms* between the étale cohomology groups of X and $\mathcal{X}$. There are further relations known as *uniformizations*; most well-known of these is the phenomenon that if E is an elliptic curve over $\operatorname{Spec} \mathbf{C}$, then there exists a lattice $L \subset \mathbf{C}$ such that $E^{\mathrm{an}} \cong \mathbf{C}/L$ as complex-analytic spaces.

1.2 Rigid-analytic spaces

Let us turn our attention from archimedean fields ($\mathbf{R}$ and $\mathbf{C}$) to non-archimedean fields ($\mathbf{Q}_p$, $\mathbf{C}_p$, $k(\!(t)\!)$ for any field k). Both are kinds of complete metric fields, so it is natural to expect a good theory of manifolds or analytic spaces for a non-archimedean field K. Which ringed spaces $(X, \mathcal{O}_X)$ should serve as our model spaces? The naïve answer is that (to define a manifold) X should be an open subset of K^n, and $\mathcal{O}_X$ should be its sheaf of continuous K-valued functions. The problem with this approach is that X is totally-disconnected, which makes it too easy to glue functions together. This problem will ruin an attempt to emulate the complex theory: if $X = \mathbf{P}^1$ defined this way, then $H^0(X, \mathcal{O}_X) \neq K$ (violating GAGA) and $H^0_{\text{ét}}(X, A) \neq A$ (violating the comparison isomorphism).

Nonetheless, Tate observed that some elliptic curves over K (those with multiplicative reduction) admit an explicit uniformization by $K^\times$, which hints that there should be a good theory of analytic varieties. Tate's uniformization involved power series which converged on certain sorts of domains in $K^\times$. Tate's theory of *rigid-analytic spaces* is a language which satisfies most of the desiderata of an analytic space, including GAGA and the comparison isomorphisms. A brief summary of the theory: we define the *Tate algebra* $K\langle T_1, \ldots, T_n\rangle$ to be the K-algebra of power series in $K[\![T_1, \ldots, T_n]\!]$ whose coefficients tend to zero. (Alternately, this is the completion of the polynomial ring $K[T_1, \ldots, T_n]$ with respect to the "Gauss norm".) The Tate algebra has various nice properties: it is Noetherian, all ideals are closed, and there is a bijection between the maximal spectrum $\operatorname{Spm} K\langle T_1, \ldots, T_n\rangle$ and the closed unit disc in $\overline{K}^n$, modulo the action of $\operatorname{Gal}(\overline{K}/K)$. An *affinoid K-algebra* is a quotient of a Tate algebra.

The model spaces in the theory of rigid-analytic spaces are $\operatorname{Spm} A$, where A is an affinoid K-algebra, and Spm means the set of maximal ideals. But the topology Tate puts on $\operatorname{Spm} A$ is not the one coming from $\overline{K}^n$, and in fact is not a topology at all, but rather a *Grothendieck topology*, with a collection of "admissible opens" and a notion of "admissible open covering". With this topology, $\operatorname{Spm} A$ carries a sheaf of rigid-analytic functions, whose global sections recover A. Then a rigid-analytic space over K is a pair $(X, \mathcal{O}_X)$, where X is a set carrying a Grothendieck topology and $\mathcal{O}_X$ is a sheaf of K-algebras, which is locally isomorphic to a model space $\operatorname{Spm} A$.

Despite this quirk about Grothendieck topologies, the theory of rigid-analytic spaces has had spectacular successes as a non-archimedean analogue to complex-analytic spaces: there is a rigid-analytic GAGA theorem, comparison theorems, fascinating theorems about uniformization of curves and of Shimura varieties, new moduli spaces which are local analogues of Shimura varieties (implicated in the proof of the local Langlands correspondence for GL_n over a p-adic field).

1.3 A motivation for adic spaces

Despite these successes, the theory of rigid-analytic spaces has a few shortcomings, which are addressed by the more general theory of *adic spaces*. One is the problem with topologies, illustrated in the following examples:

Example 1.3.1. Let $X = \operatorname{Spm} K\langle T\rangle$ be the rigid-analytic closed unit disk, and let Y be the disjoint union of the open unit disc U with the circle $S = \operatorname{Spm} K\langle T, T^{-1}\rangle$. There is an open immersion $Y \to X$, which is a bijection on the level of points. But it is not an isomorphism, because the two spaces have different Grothendieck topologies. (The trouble is that $\{U, S\}$ is not an admissible cover of X, because U is not a finite union of affinoid subdomains.)

Another example: let $X = \operatorname{Spm} K\langle T\rangle$, let α be an element of the completion of $\overline{K}$ which is transcendental over K, and let $Y \subset X$ be the union of all affinoid subdomains U which do not "contain" α, in the sense that α does not satisfy the collection of inequalities among power series which define U. Then the open immersion $Y \to X$ is once again a bijection on points. Indeed, a point $x \in X$ is a Galois orbit of roots of an irreducible polynomial $f(T) \in K[T]$. Since $f(\alpha) \neq 0$, we have $|f(\alpha)| > |t|$ for some nonzero $t \in K$, and then x belongs to the rational subdomain defined by $|f| \leq |t|$, hence it belongs to Y. However, $Y \to X$ cannot be an isomorphism: the collection of affinoid subdomains U used to define Y does not admit a finite subcover of Y, whereas (since X is affinoid), any admissible cover of X by affinoid subdomains admits a finite subcover.

In both examples there was an open immersion $Y \to X$ which is a bijection on points but which is not an isomorphism. This suggests that there are certain hidden "points" in X which Y is missing. In fact in the world of adic spaces, Y is simply the complement in X of a single point.

Another shortcoming, if we may be so greedy as to point it out, is that rigid-analytic spaces are too narrowly tailored to the class of K-affinoid algebras studied by Tate. Whereas the category of adic spaces encompasses the categories of rigid-analytic spaces, formal schemes, and even ordinary schemes. This allows to pass between these categories very easily. For instance, if X is a formal scheme over $\operatorname{Spf} \mathbf{Z}_p$ (satisfying certain finiteness assumptions), then there should be a corresponding rigid space X^{rig}, its *rigid generic fiber*. This was worked out by Berthelot [Ber91], but is rather subtle: if $X = \operatorname{Spf} \mathbf{Z}_p[\![T]\!]$, then X^{rig} is the rigid-analytic open unit disc, which is not even affinoid. Whereas in the adic world, there is a formal unit disc fibered over a two-point space $\operatorname{Spa} \mathbf{Z}_p$, and its generic fiber is simply the open subset lying over the generic point $\operatorname{Spa} \mathbf{Q}_p$.

1.4 Huber rings

The model spaces in the theory of adic spaces are associated to certain topological rings A. In light of our desiderata, A should be allowed to be $\mathbf{Z}_p[\![T]\!]$, or $\mathbf{Q}_p\langle T\rangle$, or even any ring whatsoever with its discrete topology. In the first and third case, the topology of A is generated by a finitely-generated ideal. In the second case, the topology of $\mathbf{Q}_p\langle T\rangle$ certainly isn't generated by p (since this is invertible in A), but rather there is an open subring $\mathbf{Z}_p\langle T\rangle$ whose topology *is* generated by p.

Definition 1.4.1. A *Huber ring*[2] is a topological ring A containing an open subring A_0 carrying the linear topology induced by a finitely generated ideal $I \subset A_0$. The ring A_0 and the ideal I are called a *ring of definition* and an *ideal of definition*, respectively. (The data of A_0 and I are part of the data of a Huber ring; only their existence is.)

A Huber ring A is *Tate* if it contains a topologically nilpotent unit. Such an element is called a *pseudo-uniformizer*.

Example 1.4.2.

1. Any ring A can be given the discrete topology; then A is a Huber ring with $A_0 = A$ and $I = 0$.

2. Let K be a nonarchimedean field: this means a topological field which is complete with respect to a nontrivial nonarchimedean real-valued metric

[2] called an f-adic ring by Huber [Hub94].

$|\,|$. Since $|\,|$ is nontrivial, K contains an element ϖ with $0 < |\varpi| < 1$, which is then a pseudo-uniformizer of K. Then K is a Huber ring, $K^\circ = \{|x| \leq 1\}$ is a ring of definition, and (ϖ) is an ideal of definition.

3. Continuing with the previous example, we have the Tate K-algebra $A = K\langle T_1, \dots, T_n\rangle$. This is a Tate Huber ring. The subring $K^\circ\langle T_1, \dots, T_n\rangle$ is a ring of definition, and (ϖ) is an ideal of definition.

4. Let R be any ring with its discrete topology. Then the power series ring $A = R[\![T_1, \dots, T_n]\!]$ is a Huber ring which is not Tate. Then A itself is a ring of definition, and $(T_1, \dots, T_n)$ is an ideal of definition.

5. Similarly, if K is a nonarchimedean field with pseudouniformizer ϖ, then $A = K^\circ[\![T_1, \dots, T_n]\!]$ is a Huber ring. Then A itself is a ring of definition, and $(\varpi, T_1, \dots, T_n)$ is an ideal of definition.

6. Let K be a nonarchimedean field which is perfect of characteristic p. The ring of Witt vectors $A = W(K^\circ)$ is a Huber ring, A itself is a ring of definition, and $(p, [\varpi])$ is an ideal of definition.

7. Let $A = \mathbf{Q}_p[\![T]\!]$. It is tempting to say that A is a Huber ring, with a ring of definition $A_0 = \mathbf{Z}_p[\![T]\!]$ and an ideal of definition (p, T). But in fact one cannot put a topology on A which makes this work. Indeed, in such a topology $T^n \to 0$, and since multiplication by p^{-1} is continuous, $p^{-1}T^n \to 0$ as well. But this sequence never enters A_0, and therefore $A_0 \subset A$ is not open. (It is fine to say that $\mathbf{Q}_p[\![T]\!]$ is a Huber ring with ring of definition $\mathbf{Q}_p[\![T]\!]$ and ideal of definition (T), but then you are artificially suppressing the topology of $\mathbf{Q}_p$, so that the sequence p^n does not approach 0.) There is a similar obstruction to $\mathbf{Z}_p[\![T]\!][1/p]$ being a Huber ring.

We need a few more basic definitions.

Definition 1.4.3. A subset S of a topological ring A is *bounded* if for all open neighborhoods U of 0, there exists an open neighborhood V of 0 such that $VS \subset U$. An element $f \in A$ is *power-bounded* if $\{f^n\} \subset A$ is bounded. Let A° be the subset of power-bounded elements. If A is linearly topologized (for instance if A is Huber) then $A^\circ \subset A$ is a subring.

A Huber ring A is *uniform* if $A^\circ \subset A$ is bounded.

All of the Huber rings in Example 1.4.2 are uniform. A non-uniform Huber ring is $A = \mathbf{Q}_p[T]/T^2$, because $A^\circ = \mathbf{Z}_p + \mathbf{Q}_p T$ is unbounded.

Remark 1.4.4. In a uniform Huber ring A, the power-bounded subring $A_0 \subset A$ serves as a ring of definition. Complete uniform Huber rings which are Tate are especially convenient because they are Banach rings. Indeed, suppose A is a uniform Tate Huber ring, and let $\varpi \in A$ be a pseudo-uniformizer. Then the topology on A is induced from the norm

$$|a| = 2^{\inf\{n\colon\, \varpi^n a \in A^\circ\}}.$$

1.5 Continuous valuations

The idea now is to associate to a Huber ring A a ringed space $\operatorname{Spa} A = (X, \mathcal{O}_X)$, which will serve as the model space for the theory of adic spaces. The points of X are quite interesting: they correspond to continuous valuations on the ring A.

Recall that an *ordered abelian group* is an abelian group Γ endowed with a translation-invariant total order $\leq$. These will be written multiplicatively. Examples include $\mathbf{R}_{>0}$ and any subgroup thereof. Another example is $\Gamma = \mathbf{R}_{>0} \times \mathbf{R}_{>0}$ under its *lexicographical ordering*: $(a, b) \leq (c, d)$ means that either $a < c$ or else $a = c$ and $b \leq d$. A feature of this Γ is that it contains $\mathbf{R}_{>0}$ (embedded along the first coordinate) together with, for each $a \in \mathbf{R}_{>0}$, elements (such as $(a, 1/2)$, respectively $(a, 2)$) which are between a and every real number less than (respectively, greater than) a. This concept easily generalizes to finite products $\mathbf{R}^n_{>0}$, or even infinite products of $\mathbf{R}_{>0}$ indexed by an ordinal.

Definition 1.5.1. For an ordered abelian group Γ, a subgroup $\Gamma' \subset \Gamma$ is *convex* if any element of Γ lying between two elements of Γ' itself lies in Γ'.

It is a nice exercise to show that if $\Gamma', \Gamma'' \subset \Gamma$ are two convex subgroups then either $\Gamma' \subset \Gamma''$ or $\Gamma'' \subset \Gamma'$. Therefore the set of nontrivial convex subgroups forms a totally ordered set with respect to inclusion. The cardinality of this set is called the *rank* of Γ. Thus the rank of $\mathbf{R}^n_{>0}$ is n.

The condition for Γ to be rank 1 is equivalent to the following archimedean property: given $a, b \in \Gamma$ with $a > 1$, then there exists $n \in \mathbf{Z}$ with $b < a^n$. We remark that a rank 1 ordered abelian group can always be embedded into $\mathbf{R}_{>0}$.

Definition 1.5.2. Let A be a topological ring. A *continuous valuation* on A is a map

$$\left|\cdot\right| : A \to \Gamma \cup \{0\},$$

where Γ is a totally ordered abelian group, and $\Gamma \cup \{0\}$ is the ordered monoid with least element 0. It is required that

- $|ab| = |a|\,|b|$,
- $|a + b| \leq \max(|a|, |b|)$,
- $|1| = 1$,
- $|0| = 0$,
- (Continuity) For all $\gamma \in \Gamma$, $\left\{ a \in A \,\middle|\, |a| < \gamma \right\}$ is open in A.

Two continuous valuations $|\ | : A \to \Gamma \cup \{0\}$ and $|\ |' : A \to \Gamma' \cup \{0\}$ are *equivalent* if for all $a, b \in A$ we have $|a| \geq |b|$ if and only if $|a|' \geq |b|'$. In that case, after replacing Γ by the subgroup generated by the image of A, and similarly for Γ', there exists an isomorphism $\iota \colon \Gamma \cong \Gamma'$ such that $\iota(|a|) = |a|'$ for all $a \in A$.

Note that the kernel of $|\cdot|$ is a prime ideal of A which only depends on the equivalence class of $|\cdot|$.

Definition 1.5.3. Let $\mathrm{Cont}(A)$ denote the set of equivalence classes of continuous valuations of A. For an element $x \in \mathrm{Cont}(A)$, we use the notation $f \mapsto |f(x)|$ to denote a continuous valuation representing x. We give $\mathrm{Cont}(A)$ the topology generated by subsets of the form $\left\{ x \;\middle|\; |f(x)| \leq |g(x)| \neq 0 \right\}$, with $f, g \in A$. For $x \in \mathrm{Cont}(A)$, the rank of x is the rank of the ordered abelian group generated by the image of a continuous valuation representing x.

Some remarks on the topology of $\mathrm{Cont}(A)$: Note that sets of the form $\{|g(x)| \neq 0\}$ are open, as are sets of the form $\{|f(x)| \leq 1\}$. This blends features of the Zariski topology on schemes and topology on rigid spaces. Furthermore, $\mathrm{Cont}(A)$ is quasi-compact, just as the spectrum of a ring is quasi-compact.

When A is a Huber ring, the set $\mathrm{Cont}(A)$ is a good candidate for the model space we want to build. For instance if A is a discrete ring, then $\mathrm{Cont}(A)$ contains one point x for each prime $\mathfrak{p} \in \operatorname{Spec} A$, namely the valuation pulled back from the trivial valuation on the residue field of $\mathfrak{p}$. The set $\mathrm{Cont}(\mathbf{Q}_p)$ is a single point, namely the equivalence class of the usual p-adic valuation on $\mathbf{Q}_p$.

Now consider $\mathrm{Cont}(\mathbf{Q}_p\langle T \rangle)$, which is our hypothetical "adic closed unit disc". For each maximal ideal $\mathfrak{m} \in \operatorname{Spm} \mathbf{Q}_p\langle T \rangle$, we do get a point in $\mathrm{Cont}(A)$ by pulling back the valuation on the nonarchimedean field $\mathbf{Q}_p\langle T \rangle/\mathfrak{m}$ (this is a finite extension of $\mathbf{Q}_p$). Thus there is a map $\operatorname{Spm} \mathbf{Q}_p\langle T \rangle \to \operatorname{Cont} \mathbf{Q}_p\langle T \rangle$. But the latter set contains many more points. For instance, we can let $\alpha \in \mathbf{C}_p$ be a transcendental element with $|\alpha| \leq 1$, and define a continuous valuation on $\mathbf{Q}_p\langle T \rangle$ by $f \mapsto |f(\alpha)|$. This is going to address one of the problems in classical rigid geometry brought up in Example 1.3.1.

Addressing the other problem brought up in that example, we can also define an element $x^- \in \operatorname{Cont} \mathbf{Q}_p\langle T \rangle$ as follows: let $\Gamma = \mathbf{R}_{>0} \times \gamma^{\mathbf{Z}}$, where the order is determined by the relations $a < \gamma < 1$ for all real $a < 1$. (If you like, Γ can be embedded as a subgroup of $\mathbf{R}_{>0} \times \mathbf{R}_{>0}$ by $a\gamma^n \mapsto (a, 1/2^n)$.) Now define x^- by

$$\sum_{n=0}^{\infty} a_n T^n \mapsto \sup_{n \geq 0} |a_n| \, \gamma^n.$$

Thus x^- "thinks" that T is infinitesimally smaller than one: we have $|T(x^-)| = \gamma < 1$, but $|T(x^-)| > |a|$ for all $a \in \mathbf{Q}_p$ with $|a| < 1$. The point x^- prevents us from disconnecting $\operatorname{Cont} \mathbf{Q}_p\langle T \rangle$ by the disjoint open sets $\cup_{n \geq 1} \{|T^n(x)| < |p|\}$ and $\{|T(x)| = 1\}$, because neither of these contains x^-!

However, this example suggests that we have more points in $\operatorname{Cont} \mathbf{Q}_p\langle T \rangle$ than we bargained for. There is also a point x^+ with the same definition, except that γ is now infinitesimally greater than 1. Morally, whatever the closed adic disc is, it should not contain any points which think that T is greater than 1, and so we need to modify our model spaces a little.

1.6 Integral subrings

Definition 1.6.1. Let A be a Huber ring. A subring $A^+ \subset A$ is a *ring of integral elements* if it is open and integrally closed and $A^+ \subset A^\circ$. A *Huber pair*[3] is a pair (A, A^+), where A is Huber and $A^+ \subset A$ is a ring of integral elements. Given a Huber pair, we let $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ be the subset (with its induced topology) of continuous valuations x for which $|f(x)| \leq 1$ for all $f \in A^+$. We will sometimes write $\mathrm{Spa}\, A$ for $\mathrm{Spa}(A, A^\circ)$.

We remark that $\mathrm{Spa}(A, A^+)$ is always quasi-compact.

Thus the closed adic disc should be $\mathrm{Spa}(A, A^+)$, where $A = \mathbf{Q}_p\langle T\rangle$ and $A^+ = A^\circ = \mathbf{Z}_p\langle T\rangle$. But one could also define an integral subring

$$A^{++} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in A^+ \;\middle|\; |a_n| < 1 \text{ for all } n \geq 1 \right\}.$$

We have $A^{++} \subset A^+$, and so $\mathrm{Spa}(A, A^+) \subset \mathrm{Spa}(A, A^{++})$. In fact the complement of $\mathrm{Spa}(A, A^+)$ in $\mathrm{Spa}(A, A^{++})$ is the single point x^+ from our discussion above. Furthermore, if we embed $\mathrm{Spa}(A, A^+)$ into an adic closed disc of larger radius, then it will be an *open* subset of the larger disc, and its closure will be $\mathrm{Spa}(A, A^{++})$.

1.7 The classification of points in the adic unit disc

Suppose C is a nonarchimedean field which is algebraically closed, and suppose that $\alpha \mapsto |\alpha|$ is an absolute value inducing the topology on C. We review here the classification of points in $X = \mathrm{Spa}(C\langle T\rangle, C^\circ\langle T\rangle)$ as in [Sch12]. The points of X are divided into five types; we warn that this division into five types breaks down for other adic spaces. Generally, one may work with adic spaces without consciously knowing what each point looks like.

- Points of Type 1 correspond to elements $\alpha \in C$ with $|\alpha| \leq 1$. The corresponding continuous valuation is $f \mapsto |f(\alpha)|$.

- Points of Type 2 and 3, also called Gauss points, correspond to closed discs $D = D(\alpha, r)$. Here $\alpha \in C$ has $|\alpha| \leq 1$, $0 < r \leq 1$ is a real number, and $D = \left\{ \beta \in C \;\middle|\; |\alpha - \beta| \leq r \right\}$. The corresponding valuation is

  $$f \mapsto \sup_{\beta \in D} |f(\beta)|.$$

  Explicitly, if we expand f as a series in $T - \alpha$, say $f(T) = \sum_{n=0}^{\infty} a_n (T-\alpha)^n$, then this works out to be $\sup_n |a_n|\, r^n$.

  If r belongs to $|C|$, then the point is of Type 2; otherwise it is of Type 3.

[3] Called an *affinoid algebra* in [Hub94].

- Points of Type 4 appear because of the strange phenomenon that C may not be *spherically complete*. That is, there may be a descending sequence of closed discs $D_1 \supset D_2 \supset \cdots$ with empty intersection. (For instance, this occurs when $C = \mathbf{C}_p$.) The corresponding continuous valuation is $f \mapsto \inf_i \sup_{\beta \in D_i} |f(\beta)|$.
- Points of Type 5 have rank 2. For each $\alpha \in C$ with $|\alpha| \leq 1$, each $0 < r \leq 1$, and each sign $\pm$ (excluding the positive sign if $r = 1$), we let $\Gamma = \mathbf{R}_{>0} \times \gamma^{\mathbf{Z}}$ be the ordered abelian group generated by $\mathbf{R}_{>0}$ and an element γ which is infinitesimally less than or greater than r, depending on the sign. The corresponding continuous valuation is

$$\sum_{n=0}^{\infty} a_n (T-\alpha)^n \mapsto \sup_n |a_n| \gamma^n.$$

If C has value group $\mathbf{R}_{>0}$, then there are no points of Type 3. If C is spherically complete, then there are no points of Type 4 either: every descending sequence of closed discs has an intersection which is either itself a closed disc or a single point.

The only non-closed points in X are the Type 2 points, which correspond to discs D: the closure of such a point contains all Type 5 labeled with a triple $(\alpha, r, \pm)$, where $D = D(\alpha, r)$.

1.8 The structure presheaf, and the definition of an adic space

In the construction of affine schemes, one starts with a ring A, defines the topological space $X = \operatorname{Spec} A$, and then defines the structure sheaf $\mathcal{O}_X$ this way: there is a basis of open sets of the form $U_f = \left\{x \middle| f(x) \neq 0\right\}$ for $f \in A$, and one puts $\mathcal{O}_X(U_f) = A[1/f]$; it is easy enough to check that there is a unique sheaf of rings $\mathcal{O}_X$ with this property. (Here we use the notational convention that if x corresponds to a prime ideal $\mathfrak{p} \subset A$, then $f(x)$ is the image of x in the residue field of $\mathfrak{p}$.) The idea behind this definition is that U_f should be an affine scheme in its own right, namely $\operatorname{Spec} A[1/f]$. The key observation here is that $\operatorname{Spec} A[1/f] \to \operatorname{Spec} A$ is an open immersion with image U_f, and is universal for this property in the sense that for any A-algebra B, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ factors through U_f if and only if $A \to B$ factors through $A[1/f]$.

It is somewhat more subtle to define $\mathcal{O}_X$ for $X = \operatorname{Spa}(A, A^+)$, where (A, A^+) is a Huber pair. We single out a class of open sets called rational subsets.

Definition 1.8.1. Let $s_1, \ldots, s_n \in A$ and let $T_1, \ldots, T_n \subset A$ be finite subsets such that $T_i A \subset A$ is open for all i. We define a subset

$$U\left(\left\{\frac{T_i}{s_i}\right\}\right) = U\left(\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right) = \left\{x \in X \,\middle|\, |t_i(x)| \leq |s_i(x)| \neq 0, \text{ for all } t_i \in T_i\right\}.$$

This is open because it is an intersection of a finite collection of the sort of opens which generate the topology on X. Subsets of this form are called *rational subsets*.

Note that a finite intersection of rational subsets is again rational, just by concatenating the data that define the individual rational subsets.

The following theorem shows that rational subsets are themselves adic spectra.

Theorem 1.8.2 ([Hub94, Proposition 1.3]). *Let $U \subset \operatorname{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the map $\operatorname{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \operatorname{Spa}(A, A^+)$ factors over U, and is final among such maps. Moreover, this map is a homeomorphism onto U. In particular, U is quasi-compact.*

Definition 1.8.3. Define a presheaf $\mathcal{O}_X$ of topological rings on $\operatorname{Spa}(A, A^+)$: If $U \subset X$ is rational, $\mathcal{O}_X(U)$ is as in the theorem. On a general open $W \subset X$, we define

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U).$$

We defines $\mathcal{O}_X^+$ analogously. If $\mathcal{O}_X$ is a sheaf, we call (A, A^+) a *sheafy* Huber pair.

Proposition 1.8.4. *For all $U \subset X = \operatorname{Spa}(A, A^+)$,*

$$\mathcal{O}_X^+(U) = \left\{ f \in \mathcal{O}_X(U) \;\middle|\; |f(x)| \leq 1, \text{ for all } x \in U \right\}.$$

In particular, $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is. If (A, A^+) is complete, then $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X^+(X) = A^+$.

Let (A, A^+) be a sheafy Huber pair, and let $X = \operatorname{Spa} A$. Then $(X, \mathcal{O}_X)$ is a locally ringed topological space, and $\mathcal{O}_X$ is a sheaf of topological rings. The locally ringed space $(X, \mathcal{O}_X)$ comes equipped with some extra data: for each $x \in \operatorname{Spa} A$, we have a continuous valuation $|\cdot|_x$ on the local ring $\mathcal{O}_{X,x}$. (Note that $\mathcal{O}_X^+$ can be recovered from the data of these valuations, by Proposition 1.8.4.)

We can now define the category of adic spaces.

Definition 1.8.5. An *adic space* is a triple $(X, \mathcal{O}_X, \{|\cdot|_x\}_{x \in X})$, where $(X, \mathcal{O}_X)$ is locally ringed topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings, and for each $x \in X$, $|\cdot|_x$ is a continuous valuation on $\mathcal{O}_{X,x}$. We require that locally on X, this is the triple associated to $\operatorname{Spa}(A, A^+)$, where (A, A^+) is a sheafy Huber pair. A morphism between adic spaces is a morphism between locally ringed topological spaces, which is compatible with the topology on $\mathcal{O}_X$ and with the given valuations $|\cdot|_{x \in X}$, in the evident manner.

Of course one wants some criteria for determining whether a given Huber pair is sheafy.

Theorem 1.8.6 ([Hub94]). *A Huber pair (A, A^+) is sheafy in the following situations.*

1. *The ring A is discrete. Thus, there is a functor from schemes to adic spaces, which sends* $\operatorname{Spec} A$ *to* $\operatorname{Spa}(A, A)$*.*

2. *The ring A is finitely generated (as an algebra) over a noetherian ring of definition. Thus, there is a functor from noetherian formal schemes to adic spaces, which sends* $\operatorname{Spf} A$ *to* $\operatorname{Spa}(A, A)$*.*

3. *The ring A is Tate and* strongly noetherian*, which means that the rings*

$$A\langle X_1, \dots, X_n\rangle = \left\{ \sum_{\underline{i}=(i_1,\dots,i_n)\geq 0} a_{\underline{i}} T^{\underline{i}} \;\middle|\; a_{\underline{i}} \in A,\ a_{\underline{i}} \to 0 \right\}$$

are noetherian for all $n \geq 0$. Thus there is a functor from rigid spaces over a nonarchimedean field K to adic spaces over $\operatorname{Spa} K$*, which sends* $\operatorname{Spm} A$ *to* $\operatorname{Spa}(A, A^\circ)$ *for an affinoid K-algebra A.*

Example 1.8.7 (The adic closed disc over $\mathbf{Q}_p$)**.** Let $A = \mathbf{Q}_p\langle T\rangle$, and let $A^+ = A^\circ = \mathbf{Z}_p\langle T\rangle$. Then $\operatorname{Spa}(A, A^+)$ is the adic closed disc over $\mathbf{Q}_p$.

Example 1.8.8 (The adic open disc over $\mathbf{Q}_p$)**.** Let $A = \mathbf{Z}_p[\![T]\!]$. Since A is its own ring of definition and is noetherian, (A, A) is sheafy and $\operatorname{Spa}(A, A)$ is an adic space. We have a morphism $\operatorname{Spa}(A, A) \to \operatorname{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$. The latter is a two-point space, with generic point $\eta = \operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. The generic fiber of $\operatorname{Spa}(A, A)$ is $\operatorname{Spa}(A, A)_\eta$, the preimage of η. It is worthwhile to study this space in detail.

Let $x \in \operatorname{Spa}(A, A)_\eta$. We have $|p(x)| \neq 0$. We also know that since p and T are topologically nilpotent in A, we have $|T(x)|^n \to 0$ as $n \to \infty$. Therefore, there exists an $n \geq 0$ with $|T^n(x)| \leq |p(x)|$. This means that x lies in the rational subset $U(T^n/p)$. From this we see that the increasing sequence of rational subsets $U(T^n/p)$ covers $\operatorname{Spa}(A, A)_\eta$. Since this covering has no finite subcovering, we can conclude that $\operatorname{Spa}(A, A)_\eta$ is not quasi-compact.

Example 1.8.9 (The adic affine line over $\mathbf{Q}_p$)**.** Let D be the adic closed disc over $\mathbf{Q}_p$. We let $\mathbf{A}^1_{\mathbf{Q}_p} = \varinjlim D$, where the colimit is taken over the transition map $T \mapsto pT$. Put another way, $\mathbf{A}^1_{\mathbf{Q}_p}$ is the ascending union of closed discs of unbounded radius. Then $\mathbf{A}^1_{\mathbf{Q}_p}$ is not quasi-compact. As we remarked earlier, the closure of the unit disc $D \subset \mathbf{A}^1_{\mathbf{Q}_p}$ is $\operatorname{Spa}(A, A^{++})$ for a strict subring $A^{++} \subset A^\circ$.

Example 1.8.10 (The projective line over $\mathbf{Q}_p$)**.** Let D be the adic closed disc over $\mathbf{Q}_p$. The projective line $\mathbf{P}^1_{\mathbf{Q}_p}$ is obtained by gluing together two copies of D along the map $T \mapsto T^{-1}$ on the "circle" $\{|T| = 1\}$. Then $\mathbf{P}^1_{\mathbf{Q}_p}$ contains $\mathbf{A}^1_{\mathbf{Q}_p}$ as an open subspace; the complement is a single point.

1.9 Partially proper adic spaces

Given an adic space X, one can consider its *functor of points*: whenever (R, R^+) is a complete sheafy Huber pair, we define $X(R, R^+)$ to be the set of morphisms from $\mathrm{Spa}(R, R^+)$ to X. We also have the relative version of this functor: If X is fibered over a base space S, then we may consider the relative functor of points on the category of morphisms $\mathrm{Spa}(R, R^+) \to S$, which sends such an object to the set of S-morphisms $\mathrm{Spa}(R, R^+) \to X$. Since every adic space is covered by affinoid spaces, an adic space is determined by its functor of points.

Let us compute the functor of points for the examples in the previous section.

Example 1.9.1. Let (R, R^+) be a complete sheafy Huber pair over $(\mathbf{Q}_p, \mathbf{Z}_p)$.

1. Let D be the closed unit disc over $\mathbf{Q}_p$. Then

$$D(R, R^+) = \mathrm{Hom}(\mathbf{Z}_p\langle T\rangle, R^+) \cong R^+$$

(via $f \mapsto f(T)$). (The Hom here and below is in the category of topological $\mathbf{Z}_p$-algebras.)

2. Let D° be the open unit disc over $\mathbf{Q}_p$. Then

$$D^\circ(R, R^+) = \mathrm{Hom}(\mathbf{Z}_p[\![T]\!], R^+) \cong R^{\circ\circ}$$

is the set of topologically nilpotent elements of R, again via $f \mapsto f(T)$. Now, *a priori* the image is $R^{\circ\circ} \cap R^+$. However, the fact that R^+ is open and integrally closed means that if $a \in R^{\circ\circ}$, then $a^n \in R^+$ for n large enough, and thus $a \in R^+$. Thus, $R^{\circ\circ} \subset R^+$.

3. Let $\mathbf{A}^1_{\mathbf{Q}_p}$ be the adic affine line over $\mathbf{Q}_p$. Then

$$\mathbf{A}^1_{\mathbf{Q}_p}(R, R^+) = R.$$

If $\overline{D}$ is the closure of D in $\mathbf{A}^1_{\mathbf{Q}_p}$, then

$$\overline{D}(R, R^+) = \mathrm{Hom}(A^{++}, R^+) = \left\{ a \in R \;\middle|\; pa^n \in R^\circ \text{ for all } n \geq 1 \right\}.$$

Again, *a priori* the condition on a is that $pa^n \in R^+$ for all $n \geq 1$. But if $pa^n \in R^\circ$ for all $n \geq 1$, then also $(pa^n)^2 = p(pa^{2n}) \in pR^\circ \subset R^+$, so $pa^n \in R^+$ as well.

4. Let $\mathbf{P}^1_{\mathbf{Q}_p}$ be the adic projective line over $\mathbf{Q}_p$. Then $\mathbf{P}^1_{\mathbf{Q}_p}(R, R^+)$ is the set of projective rank 1 quotients of R^2.

Definition 1.9.2. Let X be an adic space. We say X is *partially proper* if it is quasi-separated[4] and if for every sheafy Huber pair (R, R^+) and every morphism

[4] A topological space is quasi-separated if the intersection of any two quasi-compact open subsets of X is again quasi-compact. If (A, A^+) is a Huber pair, then $\mathrm{Spa}(A, A^+)$ is quasi-separated.

$\mathrm{Spa}(R, R^\circ) \to X$, there exists a unique morphism $\mathrm{Spa}(R, R^+) \to X$ making the diagram commute:

$$\begin{tikzcd} \mathrm{Spa}(R,R^\circ) \arrow[r] \arrow[d] & \mathrm{Spa}(R,R^+) \arrow[dl] \\ X. & \end{tikzcd}$$

Thus if X is partially proper, $X(R, R^+) = X(R, R^\circ)$ only depends on R. Finally, X is *proper* if it is quasi-compact and partially proper.

There is a relative definition of partial properness for a morphism $X \to S$, which we leave to the reader to work out. Note that the definition of partial properness is similar to the valuative criteria for properness and separatedness for schemes. There is also a definition of properness involving universally closed morphisms, cf. [Hub96].

Intuitively, a space is partially proper when it has no boundary. In the examples above, D°, $\overline{D}$, $\mathbf{A}^1_{\mathbf{Q}_p}$ and $\mathbf{P}^1_{\mathbf{Q}_p}$ are partially proper, but of these only $\overline{D}$ and $\mathbf{P}^1_{\mathbf{Q}_p}$ are proper. Then D is not partially proper, as its functor of points really depends on R^+.

2 Perfectoid fields

We are now going to take a sudden change of direction to talk about perfectoid fields. The idea is that perfectoid fields are the one-point perfectoid spaces, so they are rather a prerequisite to study perfectoid spaces in general. Besides, perfectoid fields have an interesting history, even if the name and formal definition did not appear until [Sch12] and [KL15].

A class of perfectoid fields plays a crucial role in Tate's study of p-divisible groups [Tat67]. Let K be the fraction field of a mixed-characteristic discrete valuation ring with perfect residue field of characteristic p (*e.g.*, a finite extension of $\mathbf{Q}_p$). Tate considered a tower of Galois extensions K_n/K satisfying the conditions (a) $\mathrm{Gal}(K_n/K) \cong (\mathbf{Z}/p^n\mathbf{Z})^h$ for some $h \geq 1$ and (b) K_n/K is totally ramified. (For Tate, such a tower came by adjoining the torsion in a p-divisible group.) Let $K_\infty = \cup_n K_n$ and let $\widehat{K}_\infty$ be its completion.

Let C be the completion of an algebraic closure of K. Tate proved some basic facts about the cohomology of C as a $\mathrm{Gal}(\overline{K}/K)$-module, using K_∞ as an intermediary. (The ultimate goal was to prove a p-adic Hodge decomposition for p-divisible groups and abelian varieties.) Along the way he proved a curious fact: if L/K_∞ is a finite extension, then the ideal of K_∞° generated by traces of elements of L° contains the maximal ideal $\mathfrak{m}_{K_\infty}$ of K_∞°. (Thus it is either $\mathfrak{m}_{K_\infty}$ or else it is all of K_∞°.) Now, if L were instead a finite extension of K, then this ideal of traces is related to the different ideal of L/K, and measures the ramification: the bigger the ideal, the less ramified L/K is. Tate's result is that any finite extension of K_∞ is *almost unramified*, or put another way, the corresponding extension of K_∞° is *almost étale*.

The next work along these lines comes from Fontaine and Wintenberger [FW79]. They considered a more general infinite algebraic extension K_∞/K which is highly ramified, in the technical sense that $G_K^u G_{K_\infty} \subset G_K$ is open for all $u \geq -1$, where G_K^u is a higher ramification group. Such extensions are called *arithmetically profinite* (APF). For instance, if K_∞/K is a totally ramified Galois extension with $\mathrm{Gal}(K_\infty/K)$ a p-adic Lie group, then K_∞/K is APF. To such an extension, Fontaine and Wintenberger attached a nonarchimedean field X, the *field of norms*, whose multiplicative monoid is the inverse limit $\varprojlim K_n$, where the transition maps in the limit are norms. The field X has characteristic p; in fact it is a Laurent series field over the residue field of K. Rather surprisingly, we have an isomorphism of Galois groups $\mathrm{Gal}(\overline{X}/X) \cong \mathrm{Gal}(\overline{K}/K_\infty)$. This isomorphism is fundamental to the classification of p-adic Galois representations via (ϕ, Γ)-modules (see [Ked15] for a discussion of these) and the proof of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ [Col10].

The themes of almost étale extensions and passage to characteristic p are the hallmarks of perfectoid fields, which we now define.

Definition 2.0.1. A nonarchimedean field K of residue characteristic p is a *perfectoid field* if (a) its value group is nondiscrete, and (b) the pth power Frobenius map on K°/p is surjective.

Example 2.0.2.

1. The basic examples of perfectoid fields are the completions of $\mathbf{Q}_p(\mu_{p^\infty})$ and $\mathbf{Q}_p(p^{1/p^\infty})$. The completion of any strictly APF extension is perfectoid.

2. One source of APF extensions (and therefore perfectoid fields) comes from *p-divisible formal group laws.* Let E be a local field of characteristic 0 with residue characteristic p and uniformizer π. Recall that a 1-dimensional formal group law over $\mathcal{O}_E$ is a power series $\mathcal{F}(X, Y) = X + Y + \text{higher order terms in } \mathcal{O}_E[\![X, Y]\!]$ which satisfies the axioms of an abelian group. Iterating $\mathcal{F}$ on itself p times produces a power series $[p]_{\mathcal{F}}(T)$. If $[p]_{\mathcal{F}}(T)$ modulo π is nonzero, then $\mathcal{F}$ is p-divisible; in that case $[p]_{\mathcal{F}}(T) \mod \pi = g(T^{p^h})$ for some power series g and some maximal h, called the height of $\mathcal{F}$. The set of roots $\mathcal{F}[p^n]$ of $[p^n]_{\mathcal{F}}$ is isomorphic to $(\mathbf{Z}/p^n\mathbf{Z})^h$. Let $E_\infty = E(\mathcal{F}[p^\infty])$ be the field obtained by adjoining all p-power torsion points to E. The extension E_∞/E is APF, and therefore the completion of E_∞ is perfectoid.

3. If a nonarchimedean field has characteristic p, then it is perfectoid if and only if it is perfect. A basic example is $k(\!(t^{1/p^\infty})\!)$, where $k/\mathbf{F}_p$ is a perfect field: this is defined to be the completion of the perfection of $k(\!(t)\!)$. This example is rather fundamental: if K is a perfectoid field of characteristic p and residue field k, then K contains $k(\!(t^{1/p^\infty})\!)$, where t is any element of K with $0 < |t| < 1$.

2.1 Tilting

Let K be a perfectoid field with absolute value $|\ |$. We let $K^\circ = \{|x| \leq 1\}$ be its ring of integers.

We define

$$K^\flat = \varprojlim K,$$

where the transition map is $x \mapsto x^p$. Thus, elements of $K^\flat$ are sequences $(a_0, a_1, \dots)$ of elements of K with $a_n^p = a_{n-1}$ for all $n \geq 1$. (If K has characteristic p, then trivially $K^\flat \cong K$; this operation is only interesting in characteristic 0.) *A priori* $K^\flat$ is a topological multiplicative monoid. We define an addition law on $K^\flat$ by the rule $(a_n) + (b_n) = (c_n)$, where

$$c_n = \lim_{m\to\infty} (a_{m+n} + b_{m+n})^{p^m}. \tag{2.1.1}$$

It it easy to check that the limit exists (here we use the fact that K is complete). It can be verified directly that $K^\flat$ is a field, but the easiest route is to pass to the quotient K°/p. The reduction map $K^\circ \to K^\circ/p$ induces a map of topological multiplicative monoids

$$\varprojlim_{x\mapsto x^p} K^\circ \to \varprojlim_{x\mapsto x^p} K^\circ/p.$$

Now one observes that this map is an isomorphism; the inverse sends a sequence $(a_n \bmod p)$ to (b_n), where

$$b_n = \lim_{m\to\infty} a_{m+n}^{p^m}.$$

(The limit does not depend on the choice of lift of a_n.) Therefore $\varprojlim K^\circ$ inherits the structure of a ring, with addition law as in (2.1.1); its fraction field is $K^\flat$. Let $f \mapsto f^\sharp$ denote the projection map $K^\flat \to K$ which sends (a_n) to a_0. We define an absolute value on $K^\flat$ by $|f| = \left|f^\sharp\right|$. One checks that this is a nontrivial nonarchimedean absolute value inducing the topology on $K^\flat$, and that $K^\flat$ is complete with respect to it. Finally, the very definition of $K^\flat$ shows that it is perfect of characteristic p. Therefore $K^\flat$ is a perfectoid field of characteristic p; it is called the *tilt* of K.

The perfectoid field $K^\flat$ contains a pseudo-uniformizer ϖ with $|\varpi| = |p|$. An important observation is that $K^{\flat\circ} \cong \varprojlim_{x\mapsto x^p} K^\circ/p$, and that

$$K^{\flat\circ}/\varpi \cong K^\circ/p.$$

Example 2.1.1.

1. Let $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$. Then $K^\flat$ contains the element $t = (p, p^{1/p}, \dots)$ with $|t| = |p|$. Thus t is a pseudo-uniformizer of $K^\flat$, and since $K^\flat$ is perfectoid, $K^\flat$ contains $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$ (as remarked in Example 2.0.2). In fact $K^\flat = \mathbf{F}_p(\!(t^{1/p^\infty})\!)$. To see this, observe that $K^\circ/p = \mathbf{Z}_p[p^{1/p^\infty}]/p \cong \mathbf{F}_p[t^{1/p^\infty}]/t$, and apply $\varprojlim$ along $x \mapsto x^p$ to both sides.

2. If $K = \mathbf{Q}_p(\mu_{p^\infty})^\wedge$, then $K^\flat$ (considered as the fraction field of $\varprojlim K^\circ/p$) contains the element $t = (1-\zeta_p, 1-\zeta_{p^2}, \dots)$, and then once again $K^\flat = \mathbf{F}_p(\!(t^{1/p^\infty})\!)$. In fact if K is the completion of any APF extension of a p-adic field (see Example 2.0.2), then $K^\flat \cong k(\!(t^{1/p^\infty})\!)$, where k is the residue field of K.

2.2 The tilting equivalence for perfectoid fields

For a perfectoid field K of characteristic 0, the structures of K and $K^\flat$ seem quite different: of course their characteristics are different, and even though there is a multiplicative map $K^\flat \to K$ ($f \mapsto f^\sharp$), this is far from being surjective in general. Nonetheless we will encounter a family of theorems known as *tilting equivalences* which relate the arithmetic of a perfectoid object and its tilt. The most basic tilting equivalence concerns the Galois groups of perfectoid fields.

Theorem 2.2.1. *Let K be a perfectoid field of characteristic 0. Then for any finite extension L/K (necessarily separable), L is also a perfectoid field, and $L^\flat/K^\flat$ is a finite extension of the same degree as L/K. The categories of finite extensions of K and $K^\flat$ are equivalent, via $L \mapsto L^\flat$. Consequently there is an isomorphism* $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K}^\flat/K^\flat)$.

Example 2.2.2. Theorem 2.2.1 allows us to describe the tilt of the perfectoid field $\mathbf{C}_p = \overline{\mathbf{Q}}_p^\wedge$. Since $\mathbf{C}_p$ is the completion of the algebraic closure of the perfectoid field $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, $\mathbf{C}_p^\flat$ is the completion of the algebraic closure of $K^\flat \cong \mathbf{F}_p(\!(t^{1/p^\infty})\!)$.

There is an explicit inverse to $L \mapsto L^\flat$ which merits discussion. Since we want to move from characteristic p to characteristic 0, it is not surprising that Witt vectors appear. Recall that for a perfect ring R of characteristic p, we have the ring of Witt vectors $W(R)$, which is characterized by the following properties: $W(R)$ is p-adically complete and p-torsion free, and $W(R)/pW(R) \cong R$. This is a ring which is separated and complete for the p-adic topology; there is a surjective morphism $W(R) \to R$ which admits a multiplicative (not additive) section $R \to W(R)$, written $x \mapsto [x]$. The ring $W(R)$ has the following universal property: For a p-adically complete, p-torsion free ring S and a map of multiplicative monoids $R \to S$ for which the composition $R \to S \to S/p$ is a ring homomorphism, there exists a unique continuous ring homomorphism $W(R) \to S$ such that the diagram

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ W(R) \arrow[r] & S \end{tikzcd}$$

commutes. Elements of $W(R)$ may be written uniquely as formal power series $[x_0] + [x_1]p + [x_n]p^2 + \dots$.

In the context of Theorem 2.2.1, we have the perfect ring $K^{\flat\circ}$, the p-adically complete p-torsion free ring K°, and the ring homomorphism $K^{\flat\circ} \to K^\circ/p$, which factors through a map of multiplicative monoids $K^{\flat\circ} \to K^\circ$, namely $f \mapsto f^\sharp$. Therefore by the universal property of Witt vectors, there exists a unique continuous ring homomorphism $\theta\colon W(K^{\flat\circ}) \to K^\circ$ satisfying $\theta([f]) = f^\sharp$. Since p is invertible in K, the map θ extends to a homomorphism of $\mathbf{Q}_p$-algebras $W(K^{\flat\circ})[1/p] \to K$, which we continue to call θ.

Lemma 2.2.3. *The homomorphism $\theta\colon W(K^{\flat\circ})[1/p] \to K$ is surjective. Its kernel is a principal ideal, generated by an element of the form $[\varpi] + \alpha p$, where $\varpi \in K^\flat$ is a pseudo-uniformizer and $\alpha \in W(K^{\flat\circ})$ is a unit.*

We can now describe the inverse to the tilting functor $L \mapsto L^\flat$ in Theorem 2.2.1. Suppose that $M/K^\flat$ is a finite extension. Then M° is perfect, and $W(M^\circ)$ is an algebra over $W(K^{\flat\circ})$. We put

$$M^\sharp = W(M^\circ) \otimes_{W(K^{\flat\circ}),\theta} K.$$

Then $M^\sharp$ is a perfectoid field, and there is a multiplicative map $M \to M^\sharp$ given by $f \mapsto f^\sharp = [f] \otimes 1$. There is an isomorphism $M \cong M^{\sharp\flat}$ given by $f \mapsto (f^\sharp, (f^{1/p})^\sharp, \dots)$.

2.3 Untilts of a perfectoid field of characteristic p

Let K be a perfectoid field of characteristic p. Does there always exists a characteristic 0 perfectoid field whose tilt is K, and if so, can one describe the set of such "untilts"? Certainly an untilt is not unique in general: In Example 2.1.1 we saw that there at least two distinct perfectoid fields whose tilts are isomorphic to $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$.

Definition 2.3.1. An *untilt* of K is a pair $(K^\sharp, \iota)$, where $K^\sharp$ is a perfectoid field and $\iota\colon K \xrightarrow{\sim} K^{\sharp\flat}$ is an isomorphism.

We remark that our definition includes K as an untilt of itself, since after all $K^\flat = K$.

Given an untilt $(K^\sharp, \iota)$, the multiplicative map $K^\circ \xrightarrow{\iota} K^{\sharp\flat\circ} \xrightarrow{\sharp} K^{\sharp\circ}$ induces a surjective ring homomorphism

$$\begin{aligned} \theta_{K^\sharp}\colon W(K^\circ) &\to K^{\sharp\circ} \\ \sum_{n=0}^\infty [f_n]p^n &\mapsto \sum_{n=0}^\infty f_n^\sharp p^n. \end{aligned}$$

Then $\ker \theta_{K^\sharp}$ is an ideal which is *primitive of degree 1*: this means that I is generated by an element of the form $\sum_{n\geq 0}[f_n]p^n$, where f_0 is topologically nilpotent and $f_1 \in K^\circ$ is a unit.

Theorem 2.3.2. *The map $I \mapsto (W(K^\circ)/I)[1/p]$ is a bijection between the set of primitive ideals of $W(K^\circ)$ of degree 1, and the set of isomorphism classes of untilts of K.*

Note that $I = (p)$ is the unique ideal which produces the trivial untilt K.

Theorem 2.3.2 suggests that untilts of K of characteristic 0 are parametrized by some kind of geometric object which is related to $W(K^\circ)$. An approximation to this object might be $\operatorname{MaxSpec} W(K^\circ)[1/p[\varpi]]$, where ϖ is a pseudo-uniformizer of K. After all, every characteristic 0 untilt $K^\sharp$ of K induces a surjective ring homomorphism $\theta_{K^\sharp}\colon W(K^\circ)[1/p] \to K^\sharp$ for which $\theta_{K^\sharp}([\varpi]) = \varpi^\sharp$ is a pseudo-uniformizer of $K^\sharp$ (and is therefore nonzero); thus $\ker \theta_{K^\sharp}$ determines a maximal ideal of $W(K^\circ)[1/p[\varpi]]$. However, $\operatorname{MaxSpec} W(K^\circ)[1/p[\varpi]]$ isn't a rigid-analytic space, as $W(K^\circ)[1/p[\varpi]]$ isn't an affinoid algebra.

The approach of Fargues and Fontaine requires looking at $W(K^\circ)$ as a ring equipped with its $([\varpi], p)$-adic topology. (This is called the *weak topology* in [FF11].) This makes $W(K^\circ)$ into a Huber ring (with itself as ring of definition), and so we may make the following definition.

Definition 2.3.3 (The adic Fargues–Fontaine curves $\mathcal{Y}_K$ and $\mathcal{X}_K$)**.** Let

$$\mathcal{Y}_K = \operatorname{Spa} W(K^\circ) \backslash \{|p[\varpi]| = 0\},$$

where ϖ is a pseudo-uniformizer of K. The Frobenius automorphism on K° induces a properly discontinuous automorphism $\phi\colon \mathcal{Y}_K \to \mathcal{Y}_K$; we let

$$\mathcal{X}_K = \mathcal{Y}_K/\phi^{\mathbf{Z}}.$$

We claim that $\mathcal{Y}_K$ is covered by rational subsets of the form

$$U\left(\frac{\{p, [\varpi^a]\}}{[\varpi^a]}, \frac{\{p, [\varpi^b]\}}{p}\right) = \left\{\left|[\varpi^b]\right| \le |p| \le |[\varpi^a]|\right\} \subset \operatorname{Spa} W(K^\circ)$$

as a and b (with $a \le b$) range through $\mathbf{Z}[1/p]_{>0}$. Indeed, suppose that $x \in \operatorname{Spa} W(K^\circ)$ satisfies $|p[\varpi](x)| \ne 0$. Since $[\varpi]$ is topologically nilpotent and $|p(x)| \ne 0$, there exists a $b > 0$ with $\left|[\varpi]^b(x)\right| \le |p(x)|$. Similarly, there exists an $a > 0$ with $|p(x)| \le |[\varpi^a](x)|$.

For an interval $I = [a, b] \subset (0, \infty)$ with endpoints lying in $\mathbf{Z}[1/p]_{>0}$, let $\mathcal{Y}_{K,I}$ be the rational subset defined above, and let $B_{K,I} = H^0(\mathcal{Y}_{K,I}, \mathcal{O}_{\mathcal{Y}_K})$. Finally, let

$$B_K = H^0(\mathcal{Y}_K, \mathcal{O}_{\mathcal{Y}_K}) = \varprojlim_I B_{K,I}.$$

These rings can be defined in terms of a family of norms on the ring $W(K^\circ)[1/p[\varpi]]$. For $r > 0$, let

$$\left|\sum_{n \in \mathbf{Z}} [x_n] p^n\right|_r = \max\left\{p^{-n} |x_n|^r\right\}.$$

For the interval $I = [a, b]$, the ring $B_{K,I}$ is the completion of $W(K^\circ)[1/p[\varpi]]$ with respect to the norm $\max\{|\ |_a, |\ |_b\}$, and B_K is the Fréchet completion of $W(K^\circ)[1/p[\varpi]]$ with respect to the family of norms $|\ |_r$.

Theorem 2.3.4 ([Ked16])**.** *The Huber ring $B_{K,I}$ is strongly noetherian. Thus $\mathcal{Y}_K$ and $\mathcal{X}_K$ are adic spaces.*

Theorem 2.3.5 ([FF11], Corollary 2.5.4)**.** *Suppose that $K = C$ is algebraically closed. There is a bijection between the set of isomorphism classes of closed maximal ideals of B_C and the set of isomorphism classes of characteristic 0 untilts of C, given by $I \mapsto B_C/I$.*

This means that there is an embedding of the set of isomorphism classes of characteristic 0 untilts of C into the set of closed points of $\mathcal{Y}_C$ (although this is far from being surjective). For a characteristic 0 untilt $C^\sharp$ with corresponding ideal I, the homomorphism $\theta_{C^\sharp}\colon W(C^\circ) \to C^{\sharp\circ}$ extends to a surjection $\theta_{C^\sharp}\colon B_C \to C^\sharp$ with kernel I.

2.4 Explicit parametrization of untilts by a formal $\mathbf{Q}_p$-vector space

Theorems 2.3.2 and 2.3.5 do not give particularly explicit parametrizations for the set of untilts of a perfectoid field K. The problem is that, even though it is easy to exhibit elements of $W(K^\circ)$ which generate primitive ideals of degree 1, it is not easy to decide whether two such elements generate the same ideal.

We offer now a different perspective. Assume that $K = C$ is an algebraically closed perfectoid field of characteristic p; we want to classify untilts of C. Suppose that $(C^\sharp, \iota\colon C \to C^{\sharp\flat})$ is an untilt of C of characteristic 0. By Theorem 2.2.1, the field $C^\sharp$ is also algebraically closed. Therefore it contains a compatible system of primitive pth power roots of unity: $1, \zeta_p, \zeta_{p^2}, \dots$. Let $\varepsilon = \iota^{-1}(1, \zeta_p, \zeta_{p^2}, \dots) \in C$. The idea is that the element $\varepsilon \in C$ is an invariant of the untilt $C^\sharp$. Now, this element isn't quite well-defined, because there is an ambiguity in the choice of system of roots of unity.

Before resolving this ambiguity, we introduce some notation. Let $H = \widehat{\mathbf{G}}_m$ be the formal multiplicative group over $\mathbf{Z}_p$. This is the completion of $\mathbf{G}_{m,\mathbf{Z}_p}$ along the origin of $\mathbf{G}_{m,\mathbf{F}_p}$. It is perhaps easiest to think of H as a functor from complete adic $\mathbf{Z}_p$-algebras to $\mathbf{Z}_p$-modules, which sends R to the abelian group $1 + R^{\circ\circ}$ under multiplication. This group gets its $\mathbf{Z}_p$-module structure this way: for $a \in \mathbf{Z}_p$, the action of a sends x to x^a (defined using power series). The underlying formal scheme of H is isomorphic to $\operatorname{Spf} \mathbf{Z}_p[\![T]\!]$. We also define the *universal cover*

$$\widetilde{H} = \varprojlim_{x \mapsto x^p} H,$$

so that if R is an adic $\mathbf{Z}_p$-algebra, $\widetilde{H}(R)$ is the $\mathbf{Q}_p$*-vector space* $\varprojlim_{x\mapsto x^p}(1+R^{\circ\circ})$. There is a reduction map

$$\widetilde{H}(R) \to \widetilde{H}(R/p), \tag{2.4.1}$$

which one checks is an isomorphism, rather along the lines of the proof that $K^{\flat\circ} \cong \varprojlim_{x\mapsto x^p} K^\circ/p$ for a perfectoid field K. Consequently,

$$\widetilde{H}(R) \cong \widetilde{H}(R/p) \cong \varprojlim_{x\mapsto x^p} R^{\circ\circ}/p \cong \varprojlim_{x\mapsto x^p} R^{\circ\circ},$$

so that $\widetilde{H}$ is representable by the formal scheme $\operatorname{Spf} \mathbf{Z}_p[\![T^{1/p^\infty}]\!]$. Thus $\widetilde{H}$ is a $\mathbf{Q}_p$-vector space object in the category of formal schemes, which is to say, a *formal*

$\mathbf{Q}_p$*-vector space*. Whenever K is a perfectoid field, $\widetilde{H}(K^\circ) \cong \widetilde{H}(K^{\flat\circ}) \cong H(K^{\flat\circ})$ (the last isomorphism holds because $K^\flat$ is perfect).

Given a characteristic 0 untilt $C^\sharp$ of C, we obtain a nonzero element $\varepsilon \in \widetilde{H}(C^\circ)$ defined as the image of $(1, \zeta_p, \zeta_{p^2}, \dots)$ under $\widetilde{H}(C^{\sharp\circ}) \cong \widetilde{H}(C^\circ)$. This element is well-defined up to translation by an element of $\mathbf{Z}_p^\times$. Note that $\theta_{C^\sharp}([\varepsilon^{1/p^n}]) = \zeta_{p^n}$ for all $n \geq 0$; therefore the element

$$\xi = \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = [1] + [\varepsilon^{1/p}] + \dots + [\varepsilon^{(p-1)/p}] \tag{2.4.2}$$

lies in the kernel of $\theta_{C^\sharp}$. One checks that the ideal (ξ) is primitive of degree 1, and therefore $C^\sharp$ corresponds to the ideal (ξ) under the bijection in Theorem 2.3.2.

On the other hand, we could start with a nonzero element $\varepsilon \in \widetilde{H}(C^\circ)$, form ξ as above, and from this construct the untilt $C^\sharp = W(C^\circ)[1/p]/(\xi)$. In fact:

Theorem 2.4.1 (see [FF11, Proposition 3.4] and [FF11, Remarque 3.6])**.** *The map $C^\sharp \mapsto \varepsilon$ gives a bijection between equivalence classes (respectively, Frobenius-equivalence classes) of characteristic 0 untilts of C and $(\widetilde{H}(C^\circ)\setminus\{0\})/\mathbf{Z}_p^\times$ (respectively, $(\widetilde{H}(C^\circ)\setminus\{0\})/\mathbf{Q}_p^\times$).*

The $\mathbf{Q}_p$-vector space $\widetilde{H}(C^\circ)$ is rather interesting. On the one hand it is huge: it certainly has uncountable dimension. To get a handle on it, we first choose a characteristic 0 untilt $C^\sharp$ of C, so that $\widetilde{H}(C^\circ) \cong \widetilde{H}(C^{\sharp\circ})$. We have a *logarithm map* $\log\colon H(C^{\sharp\circ}) \to C^\sharp$, defined by the usual formula

$$\log x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}.$$

The logarithm map is a $\mathbf{Z}_p$-module homomorphism, which sits in an exact sequence

$$0 \to \mu_{p^\infty}(C^\sharp) \to H(C^{\sharp\circ}) \to C^\sharp \to 0, \tag{2.4.3}$$

where $\mu_{p^\infty}(C^\sharp) = H[p^\infty](C^{\sharp\circ})$ is the group of pth power roots of 1 in $C^\sharp$. Let us check that the logarithm map is surjective. If $x \in C^\sharp$, there exists an n large enough so that $p^n x$ is in the region of convergence of the exponential map; then $z = \exp(p^n x) \in H(C^{\sharp\circ})$ satisfies $\log(z) = p^n x$, so that $\log(z^{1/p^n}) = x$ for any p^nth root z^{1/p^n} of z in $C^{\sharp\circ}$.

Taking inverse limits along multiplication by p in (2.4.3) gives an exact sequence of $\mathbf{Q}_p$-vector spaces:

$$0 \to VH(C^\sharp) \to \widetilde{H}(C^{\sharp\circ}) \to C^\sharp \to 0, \tag{2.4.4}$$

where $VH = \varprojlim_p H[p^\infty](C^\sharp)$; note that VH is a $\mathbf{Q}_p$-vector space of dimension 1, spanned by a compatible system of primitive pth power roots of 1.

The exact sequence in (2.4.4) sheds some light onto the structure of the $\mathbf{Q}_p$-vector space $\widetilde{H}(C^\circ)$. Once a characteristic 0 untilt $C^\sharp$ is chosen, together with a system of pth power roots of 1 in $C^\sharp$, there is a "presentation" of $\widetilde{H}(C^\circ)$ as an extension of $C^\sharp$ by $\mathbf{Q}_p$.

2.5 The schematic Fargues-Fontaine curve

We give here another interpretation of the exact sequence (2.4.4). Given an element $\varepsilon \in \widetilde{H}(C^{\sharp\circ})$, we define its logarithm

$$t = \log[\varepsilon] = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n} \in B_C = H^0(\mathcal{Y}_C, \mathcal{O}_{\mathcal{Y}_C}). \tag{2.5.1}$$

One has to check here that the sum converges in the Fréchet topology on B_C, but this is just a matter of checking that $\left|[\varepsilon]-1\right|_r < 1$ for all $0 < r < \infty$. Then formally we have

$$\phi(t) = \log \phi([\varepsilon]) = \log[\varepsilon^p] = p \log[\varepsilon] = pt,$$

and so t lies in the $\mathbf{Q}_p$-vector space $B_C^{\phi=p}$ consisting of elements that exhibit this behavior. The element t also has the property that $\theta_{C^\sharp}(t) = 0$, since $\theta_{C^\sharp}([\varepsilon]) = 1$.

In general we can take any element $\alpha \in \widetilde{H}(C^\circ)$ and produce $\log[\alpha] \in B_C^{\phi=p}$. We have the following commutative diagram, in which the first row is (2.4.4):

$$\begin{tikzcd} 0 \arrow[r] & VH(C^\sharp) \arrow[r] \arrow[d] & \widetilde{H}(C^{\sharp\circ}) \arrow[r] \arrow[d, "\log[\cdot]"'] & C^\sharp \arrow[r] \arrow[d, "="'] & 0 \\ 0 \arrow[r] & \mathbf{Q}_p t \arrow[r] & B_C^{\phi=p} \arrow[r, "\theta_{C^\sharp}"'] & C^\sharp \arrow[r] & 0. \end{tikzcd}$$

Theorem 2.5.1 ([FF11])**.** *The map $\varepsilon \mapsto \log[\varepsilon]$ defines an isomorphism of $\mathbf{Q}_p$-vector spaces $\widetilde{H}(C^\circ) \cong B_C^{\phi=p}$. Furthermore, for each $t \in B_C^{\phi=p} \setminus \{0\}$, there is a unique Frobenius-equivalence class of characteristic 0 untilts $C^\sharp$ such that $\theta_{C^\sharp}(t) = 0$. Therefore there is a bijection between the set of Frobenius-equivalence classes of characteristic 0 untilts of $C^\sharp$ and the set $(B_C^{\phi=p} \setminus \{0\})/\mathbf{Q}_p^\times$.*

Recall that $\mathcal{Y}_C$ is the adic space which is (informally) supposed to parametrize equivalence classes of characteristic 0 untilts of C, and $\mathcal{X}_C = \mathcal{Y}_C/\phi^{\mathbf{Z}}$ parametrizes Frobenius-equivalence classes of such untilts. A key insight of [FF11] is that $\mathcal{X}_C$ resembles a proper smooth analytic curve, and so should be the analytification of an algebraic curve, just as the Tate curve $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$ is the analytification of an elliptic curve over a p-adic field K. In this context, the usual thing to do is to find an ample line bundle $\mathcal{L}$ on $\mathcal{X}_C$, and then define

$$X_C = \operatorname{Proj} \bigoplus_{n \geq 0} H^0(\mathcal{X}_C, \mathcal{L}^{\otimes n}).$$

In the case of $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$, the line bundle is $\mathcal{O}(P)$, where P is the origin of $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$; the graded ring in the above construction is $K[x,y,z]/f(x,y,z)$, where f is a cubic whose coefficients depend on q according to the usual formulas.

For the Fargues-Fontaine curve, the requisite line bundle $\mathcal{L}$ on $\mathcal{X}_C$ should pull back to a line bundle on $\mathcal{Y}_C$ which is ϕ-equivariant. And so we define a free line bundle $\mathcal{O}_{\mathcal{Y}_C}e$, with the ϕ-equivariance defined by $\phi(e) = p^{-1}e$. This $\mathcal{O}_{\mathcal{Y}_C}e$ descends to a line bundle on $\mathcal{X}_C$, which we call $\mathcal{O}_{\mathcal{X}_C}(1)$. For $n \in \mathbf{Z}$ we define $\mathcal{O}_{\mathcal{X}_C}(n) = \mathcal{O}_{\mathcal{X}_C}^{\otimes n}$ (with the usual convention regarding negative n).

The algebraic Fargues-Fontaine curve is defined by declaring $\mathcal{O}_{\mathcal{X}_C}(1)$ to be very ample. Note that

$$H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(n)) \cong H^0(\mathcal{Y}_C, \mathcal{O}_{\mathcal{Y}_C}e^{\otimes n})^{\phi=1} \cong B_C^{\phi=p^n}$$

Definition 2.5.2 (The schematic Fargues-Fontaine curve)**.** Define $X_C = \operatorname{Proj} P$, where

$$P = \bigoplus_{d \geq 0} P_d, \text{ where } P_d = B_C^{\phi=p^d}.$$

Theorem 2.5.3.

1. *The "ring of constants" $H^0(X_C, \mathcal{O}_{X_C}) = P_0 = B_C^{\phi=1}$ is exactly $\mathbf{Q}_p$.*
2. *The graded ring P is factorial: the irreducible homogenous elements are exactly the nonzero elements of P_1, and for every $d \geq 1$, a nonzero element of P_d admits a factorization into irreducibles in P_1, unique up to units.*
3. *As a result, X_C is an integral Noetherian scheme of dimension 1, which admits a cover by spectra of Dedekind rings (in fact PIDs).*

In these respects X_C resembles nothing so much as the projective line $\mathbf{P}^1_C = \operatorname{Proj} C[S,T]$, where $C[S,T]$ is graded by total degree. But unlike $\mathbf{P}^1_C$, the scheme X_C is not finitely generated over any field.

Since X_C is an integral Noetherian scheme of dimension 1, it is the union of its generic point together with its set of closed points $|X_C|$. In light of Theorem 2.5.3, it is easy to describe the closed points: they correspond to nonzero homogenous prime ideals of P (other than the irrelevant ideal); since every homogenous element of P factors as a product of elements of P_1, every such ideal is generated by a nonzero element of P_1. Since $P^\times = \mathbf{Q}_p$, we find that $|X_C|$ is in bijection with $(P_1 \backslash \{0\})/\mathbf{Q}_p^\times$. Summing up our investigations of untilts of $C^\sharp$ gives the following theorem.

Theorem 2.5.4. *Let C be an algebraically closed perfectoid field of characteristic p. The following sets are in bijection:*

- *Frobenius-equivalence classes of characteristic 0 untilts of C,*
- $(\widetilde{H}(C^\circ) \backslash \{0\})/\mathbf{Q}_p^\times$,
- *closed points of the scheme X_C.*

2.6 Universal covers of other p-divisible groups

What are the $\mathbf{Q}_p$-vector spaces $P_d = B_C^{\phi=p^d}$ for $d \geq 2$? It is easy enough to exhibit elements of P_d; for $x \in C^{\circ\circ}$ the element

$$\sum_{n\in\mathbf{Z}} \frac{[x^{p^n}]}{p^{dn}}$$

belongs to P_d. However, it is probably not the case that all elements of P_d admit such a presentation, nor is it clear that such a presentation is unique.

The situation is better for the $\mathbf{Q}_p$-vector space $B_C^{\phi^h=p}$, where $h \geq 1$. As in the case $h = 1$, this is isomorphic to the universal cover of a p-divisible formal group. Let $H_{1/h}/\mathbf{Z}_p$ be the 1-dimensional formal group whose logarithm is

$$\log_{H_{1/h}}(T) = \sum_{n=1}^{\infty} T^{p^{hn}}/p^n.$$

This means that the underlying formal scheme of $H_{1/h}$ is $\operatorname{Spf} \mathbf{Z}_p[\![T]\!]$, and the addition law $+_{H_{1/h}}$ is determined by the relation

$$\log_{H_{1/h}}(X +_{H_{1/h}} Y) = \log_{H_{1/h}}(X) + \log_{H_{1/h}}(Y)$$

as power series in $\mathbf{Q}_p[\![X, Y]\!]$. Then $H_{1/h} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ has height h; in fact $[p]_{H_{1/h}}(T) \equiv T^{p^h} \pmod p$ (See [Haz12] for proofs of these assertions. The formal group $H_{1/h}$ is an example of a *p-typical* formal group.) We remark that if $\mathbf{Q}_{p^h}/\mathbf{Q}_p$ is the unramified extension of degree h, and if $\mathbf{Z}_{p^h}$ is the ring of integers in $\mathbf{Q}_{p^h}$, then $H_{1/h} \otimes_{\mathbf{Z}_p} \mathbf{Z}_{p^h}$ admits endomorphisms by $\mathbf{Z}_{p^h}$. In fact $H_{1/h} \otimes_{\mathbf{Z}_p} \mathbf{Z}_{p^h}$ is a Lubin-Tate formal $\mathbf{Z}_{p^h}$-module in the sense of [LT65].

Let

$$\widetilde{H}_{1/h} = \varprojlim_{x \mapsto [p]_{H_{1/h}}(x)} H_{1/h},$$

a priori as a functor from adic $\mathbf{Z}_p$-algebras to $\mathbf{Q}_p$-vector spaces. Just as with H_1, one uses the congruence between $[p]_{H_{1/h}}$ and a power of Frobenius to show that for any adic $\mathbf{Z}_p$-algebra R, we have isomorphisms

$$\widetilde{H}_{1/h}(R) \cong \widetilde{H}_{1/h}(R/p) \cong \varprojlim_{x \mapsto x^p} R^{\circ\circ}.$$

Applied to $R = W(C^\circ)$, the first isomorphism has inverse

$$\begin{array}{rcl} \widetilde{H}_{1/h}(C^\circ) & \to & \widetilde{H}_{1/h}(W(C^\circ)) \\ (x_n) & \mapsto & (y_n), \end{array}$$

where

$$y_n = \lim_{m\to\infty} p^m [x_{m+n}].$$

This isomorphism respects the action of Frobenius ϕ on either side, and therefore the identity $\phi^h = p$ holds in $\operatorname{End} \widetilde{H}_{1/h}(W(C^\circ))$, since it holds in $\operatorname{End} \widetilde{H}_{1/h}(C^\circ)$. Given an element $(x_n) \in \widetilde{H}_{1/h}(W(C^\circ))$, its logarithm $\log_{\widetilde{H}_{1/h}}(x_0)$ lies in $B_C^{\phi^h=p}$.

Theorem 2.6.1 ([FF11, Proposition 3.4.5])**.** *The map* $(x_n) \mapsto \log_{\widetilde{H}_{1/h}}([x_0])$ *gives an isomorphism* $\widetilde{H}_{1/h}(C^\circ) \xrightarrow{\sim} B_C^{\phi^h=p}$.

We can be quite explicit about this isomorphism. There is a commutative diagram

$$\begin{tikzcd}
\widetilde{H}_{1/h}(C^\circ) \arrow[r, "\sim"] \arrow[d, "\cong"'] & \widetilde{H}_{1/h}(W(C^\circ)) \arrow[r, "(x_n)\mapsto\log_{H_{1/h}}(x_0)"] & B_C^{\phi^h=p} \\
C^{\circ\circ} \arrow[urr] & &
\end{tikzcd}$$

in which all maps are isomorphisms; the diagonal map is

$$x \mapsto \lim_{m\to\infty} p^m \log_{H_{1/h}}[x^{1/p^m}] = \sum_{n\in\mathbf{Z}} \frac{[x^{p^{hn}}]}{p^n}.$$

Note that the latter expression visibly lies in $B_C^{\phi^h=p}$.

Theorem 2.6.1 generalizes to p-divisible groups of arbitrary height $h \geq 1$ and dimension $d \geq 0$, whenever $0 \leq d/h \leq 1$. The universal cover of such a formal group parametrizes $B_C^{\phi^h=p^d}$.

2.7 Interpretation in terms of vector bundles on X

A major theorem in [FF11] is the classification of vector bundles on the Fargues-Fontaine curve X. This classification is in terms of isocrystals.

Definition 2.7.1. Let k be a perfect field of characteristic $p > 0$, and let $K = W(k)[1/p]$. Let $\phi\colon K \to K$ be the Frobenius automorphism. An *isocrystal* over k is a finite-dimensional K-vector space N together with an isomorphism $\phi_N\colon \phi^*N \to N$.

These form an abelian tensor category. When k is algebraically closed, the category of isocrystals over k is well understood. It is a semisimple category, with one irreducible object $N = N_{d/h}$ for each pair (d,h), where $d \in \mathbf{Z}$ and $h \geq 1$ are relatively prime. The underlying K-vector space of N has basis $e, \phi_N(e), \dots, \phi_N^{h-1}(e)$, and $\phi_N^h(e) = p^d e$. Morphisms between the simple objects go as follows: There are no nonzero morphisms between distinct $N_{d/h}$s, and the endomorphism algebra of $N_{d/h}$ is a central division algebra over K of rank h^2, with invariant $d/h \in \mathbf{Q}/\mathbf{Z}$.

Given an isocrystal N over k, and an algebraically closed perfectoid field C of characteristic p with residue field k, we can define the graded P-module

$$\widetilde{N} = \bigoplus_{d\geq 0} (B_C \otimes_{W(k)} N)^{\phi=p^d}.$$

Let $\mathcal{E}(N)$ be the corresponding $\mathcal{O}_X$-module. Then $\mathcal{E}(N)$ is a vector bundle of rank $\dim N$. For a relatively prime pair (d,h) with $d \geq 0$ and $h \geq 1$, we let $\mathcal{O}_{X_C}(d/h) = \mathcal{E}(N_{-d/h})$. Then $H^0(X_C, \mathcal{O}_X(d/h)) \cong B_C^{\phi^h=p^d}$.

Theorem 2.7.2 ([FF11]). *Let C be an algebraically closed perfectoid field of characteristic $p > 0$. Every vector bundle on X_C is isomorphic to $\mathcal{E}(N)$ for an isocrystal N, which is unique up to isomorphism.*

It must be emphasized that the functor $N \mapsto \mathcal{E}(N)$ is far from being an equivalence of categories, as it is not full. Each nonzero element of $B^{\phi=p}$ gives a morphism $\mathcal{O}_X \to \mathcal{O}_X(1)$ which does not arise from a map of isocrystals. However if $N = N_{d/h}$ as above, then $\operatorname{End} N \to \operatorname{End} \mathcal{E}(N)$ is an isomorphism.

In the last subsection we saw that if $0 \leq d/h \leq 1$, then there is a p-divisible group $\overline{H} = \overline{H}_{d/h}/\overline{\mathbf{F}}_p$ of height h and dimension d, and a natural isomorphism $\widetilde{\overline{H}}(C^\circ) \cong H^0(X_C, \mathcal{O}_{X_C}(d/h))$. Let $C^\sharp$ be a characteristic 0 untilt of C, and let H be a lift of $\overline{H}$ to $C^{\sharp\circ}$. (The question of the existence of such lifts is addressed in [Mes72, Chapter IV]. As a special case, p-divisible groups can always be lifted from C°/p to C°.) Then there is an exact sequence of $\mathbf{Z}_p$-modules

$$0 \to H[p^\infty](C^{\sharp\circ}) \to H(C^{\sharp\circ}) \stackrel{\log_H}{\to} \operatorname{Lie} H \otimes_{C^{\sharp\circ}} C^\sharp \to 0,$$

Taking an inverse limit along multiplication by p (this is right-exact because $H[p^\infty](C^{\sharp\circ})$ is p-divisible) gives an exact sequence of $\mathbf{Q}_p$-vector spaces

$$0 \to VH(C^{\sharp\circ}) \to \widetilde{H}(C^{\sharp\circ}) \stackrel{\log_H}{\to} \operatorname{Lie} H \otimes_{C^{\sharp\circ}} C^\sharp \to 0. \tag{2.7.1}$$

Note that the middle term, which is naturally isomorphic to $\overline{H}(C^\circ)$, does not depend on the lift H. Also note that this exact sequence presents a very large $\mathbf{Q}_p$-vector space as an extension of a finite-dimensional $C^\sharp$-vector space by a finite-dimensional $\mathbf{Q}_p$-vector space; this is an instance of the theory of *Banach–Colmez spaces*, which we will investigate systematically in the last lecture.

Let $x \in |X_C|$ be the closed point corresponding to the Frobenius equivalence class of $C^\sharp$ under Theorem 2.5.4. The exact sequence in (2.7.1) can be reinterpreted as the global sections of the following exact sequence of $\mathcal{O}_{X_C}$-modules:

$$0 \to \mathcal{O}_{X_C} \otimes_{\mathbf{Q}_p} VH \to \mathcal{O}_{X_C}(d/h) \to i_* \operatorname{Lie} H \otimes C^\sharp \to 0,$$

where i is the inclusion of $x = \operatorname{Spec} C^\sharp$ into X_C.

We mention in passing that [FF11] deduces the following theorem from Theorem 2.7.2:

Theorem 2.7.3. *The curve X_C is geometrically simply connected over $\mathbf{Q}_p$. That is, any finite étale cover of X_C is isomorphic to $X_C \times_{\operatorname{Spec} \mathbf{Q}_p} \operatorname{Spec} A$ for an étale $\mathbf{Q}_p$-algebra A. Thus, the étale fundamental group of the scheme X_C is $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.*

There are versions of Theorems 2.7.2 and 2.7.3 for the adic curve $\mathcal{X}_C$, owing to the equivalence of categories between coherent sheaves on X_C and $\mathcal{X}_C$ [Far15].

3 Perfectoid spaces and diamonds

3.1 Definitions

Definition 3.1.1. Let A be a Huber ring. A Huber ring A is *perfectoid* if the following conditions hold:

1. A is Tate, meaning it contains a pseudo-uniformizer (a topologically nilpotent unit),
2. A is uniform, meaning that $A^\circ \subset A$ is bounded,
3. A contains a pseudo-uniformizer ϖ such that $\varpi^p | p$ in A°, and such that the pth power map $A^\circ/\varpi \to A^\circ/\varpi^p$ is an isomorphism.

Remark 3.1.2. In the definition above it is always possible to choose a pseudo-uniformizer ϖ which contains a compatible system of pth power roots.

Theorem 3.1.3. *Let (A, A^+) be a Huber pair, with A perfectoid. Then (A, A^+) is sheafy, so that $X = \operatorname{Spa}(A, A^+)$ is an adic space. Furthermore, $\mathcal{O}_X(U)$ is a perfectoid ring for every rational subset $U \subset X$.*

Theorem 3.1.3 shows that adic spaces $\operatorname{Spa}(R, R^+)$ with R perfectoid can serve as model spaces for the category of perfectoid spaces:

Definition 3.1.4. A *perfectoid space* is an adic space that may be covered by affinoids of the form $\operatorname{Spa}(A, A^+)$, where A is perfectoid.

Example 3.1.5.

- If K is a perfectoid field and $K^+ \subset K$ is a ring of integral elements, then $\operatorname{Spa}(K, K^+)$ is a perfectoid space.
- (The perfectoid closed disc.) Let K be a perfectoid field. Let $A = K\langle T^{1/p^\infty}\rangle$; this is the completion of the polynomial algebra $K[T^{1/p^\infty}]$. Then A is a perfectoid ring, and $\operatorname{Spa}(A, A^\circ)$ is a perfectoid space.
- (The perfectoid open disc.) This time let $A = K^\circ[\![T^{1/p^\infty}]\!]$, the completion of $K^\circ[T^{1/p^\infty}]$ with respect to the (ϖ, T)-adic topology (here ϖ is a pseudo-uniformizer of K). Then A is not a perfectoid ring, because it is not Tate. It is not even clear that (A, A) is sheafy (although this is probably true). Nonetheless, the generic fiber of $\operatorname{Spa} A$ over $\operatorname{Spa} K^\circ$ is perfectoid: it is covered by the affinoids $\operatorname{Spa}(A_n, A_n^\circ)$, where $A_n = K\langle (T/\varpi^{1/p^n})^{1/p^\infty}\rangle$.
- Let k be a perfect field of characteristic p with its discrete topology. Let $A = k[\![T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]\!]$; this is defined as the $(T_1, \dots, T_n)$-adic completion of $k[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$. Then A is not a perfectoid ring (it is not Tate), but the analytic locus in $\operatorname{Spa} A$ is perfectoid. This is the complement in $\operatorname{Spa} A$ of the single non-analytic point satisfying $|T_i| = 0$ for $i = 1, \dots, n$. Note that if $n > 1$, this perfectoid space does not live over any particular perfectoid field.

- (Some totally disconnected perfectoid spaces.) Let K be a perfectoid field and let S be a profinite set. Let $A = \mathrm{Cont}(S, K)$ be the ring of continuous maps $S \to K$. Give A the structure of a Banach K-algebra under the sup norm; we have $A^\circ = \mathrm{Cont}(S, K^\circ)$. Then $\mathrm{Spa}(A, A^\circ)$ is a perfectoid space whose underlying topological space is S. The construction globalizes to the case that S is only locally profinite. If K is understood, we write $\underline{S}$ for the resulting perfectoid space.

The tilting operation we discussed in 2.1 extends to perfectoid spaces. For a perfectoid ring A with pseudo-uniformizer ϖ as in Remark 3.1.2, we define its tilt by

$$A^\flat = \left(\varprojlim_{x \mapsto x^p} A^\circ/\varpi \right) [1/\varpi^\flat],$$

where $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots)$. Then $A^\flat$ is a perfectoid ring of characteristic p.

We gather here some results from [Sch12] (which assumes a fixed perfectoid field of scalars, but the proofs carry over in general).

Theorem 3.1.6. *Let A be a perfectoid ring.*

1. *There is a homeomorphism of topological monoids:*

$$A^\flat \cong \varprojlim_{x \mapsto x^p} A.$$

If $f \in A^\flat$ corresponds to the sequence (f_n) with $f_n \in A$, define $f^\sharp = f_0$.

2. *There is a bijection $A^+ \mapsto A^{\flat +} = \varprojlim_{x \mapsto x^p} A^+/\varpi$ between rings of integral elements of A and $A^\flat$.*

3. *Given a ring of integral elements $A^+ \subset A$, there is a homeomorphism*

$$\begin{array}{ccc} \mathrm{Spa}(A, A^+) & \xrightarrow{\sim} & \mathrm{Spa}(A^\flat, A^{\flat +}) \\ x & \mapsto & x^\flat \end{array}$$

where $x^\flat$ is defined by $\left|f(x^\flat)\right| = \left|f^\sharp(x)\right|$ for $f \in A^\flat$. This homeomorphism identifies rational subsets on either side.

4. *The categories of perfectoid algebras over A and $A^\flat$ are equivalent, via $B \mapsto B^\flat$.*

5. *Let B be a finite étale A-algebra, so that B becomes a topological ring. Then B is also perfectoid. The categories of finite étale algebras over A and $A^\flat$ are equivalent, via $B \mapsto B^\flat$.*

One way to construct perfectoid spaces comes from universal covers of p-divisible groups, which we discussed in (2.6). Let k be a perfect field of characteristic p, and let H be a p-divisible group over k. We have the universal cover $\widetilde{H} = \varprojlim_p H$, which we may consider as a functor from k-algebras to $\mathbf{Q}_p$-vector spaces. For now let us assume that H is connected, so that H is representable by $\operatorname{Spf} k[\![T_1, \ldots, T_d]\!]$, where $d = \dim H$; then $\widetilde{H}$ is representable by a formal scheme $\operatorname{Spf} k[\![T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]\!]$. (This follows from two facts: multplication by p in H factors through Frobenius, and a sufficiently high power of Frobenius on H factors through multiplication by p.)

Let $\widetilde{H}^{\mathrm{ad}}$ be the corresponding adic space. Then $\widetilde{H}^{\mathrm{ad}}$ isn't quite a perfectoid space (it isn't analytic), but the punctured version $\widetilde{H}^{\mathrm{ad}} \backslash \{0\}$ is a perfectoid space, as in Example 3.1.5. If we want to create a perfectoid space version of $\widetilde{H}$ without puncturing it, we can introduce a separate perfectoid field K/k, and define $\widetilde{H}_K$ as the adic generic fiber of $\widetilde{H} \times_{\operatorname{Spec} k} \operatorname{Spf} K^\circ$. Then $\widetilde{H}_K$ is a $\mathbf{Q}_p$-vector space object in the category of perfectoid spaces over K.

A similar object exists in characteristic 0. Suppose now that K is a perfectoid field of characteristic 0 whose residue field contains k. Then the ring homomorphism $K^\circ/p \to k$ admits a canonical section, namely $k \to K^{\flat\circ} \to K^{\flat\circ}/p \cong K^\circ/p$. We may define $\widetilde{H}_K$ as the perfectoid space over K whose tilt is $\widetilde{H}_{K^\flat}/K^\flat$. Then if G is any lift of $H \otimes_k K^\circ/p$ to K°, then we have the following functorial interpretation of $\widetilde{H}_K$: it is the sheafification of the functor $R \mapsto \widetilde{G}(R^\circ)$ on perfectoid K-algebras R. Note that this does not depend on the choice of lift G.

In fact, the requirement that H be formal is just a red herring; there is a functor $H \mapsto \widetilde{H}_K$ from the whole category of p-divisible groups over k to the category of perfectoid spaces with $\mathbf{Q}_p$-vector space structure. For instance if $H = \mathbf{Q}_p/\mathbf{Z}_p$ is the constant p-divisible group, then $\widetilde{H} = \underline{\mathbf{Q}_p}$ is the constant $\mathbf{Q}_p$-vector space.

Finally, if we allow K to be any nonarchimedean field with residue field containing k, then $\widetilde{H}_K$ will be a *pre-perfectoid space*, meaning that it becomes perfectoid after extending scalars from K to any perfectoid field.

3.2 Untilts of perfectoid spaces in characteristic p, and a motivation for diamonds

Let X be a perfectoid space lying over $\operatorname{Spa} \mathbf{F}_p$. As we did with perfectoid fields, we can investigate the set of equivalence classes of untilts of X. What we would like is a *moduli space* M lying over $\operatorname{Spa} \mathbf{F}_p$, for which there is a natural bijection between the following sets:

- Morphisms $X \to M$, and
- Equivalence classes of characteristic 0 untilts $X^\sharp \to \operatorname{Spa} \mathbf{Q}_p$.

This object M will ultimately be called $\operatorname{Spd} \mathbf{Q}_p$, where the "d" stands for *diamond*; it lives in a category of diamonds, which contains the category of perfectoid spaces as a full subcategory.

In the special case $X = \operatorname{Spa} C$ for a perfectoid field C of characteristic p, Theorem 2.5.4 gave the following parametrizations:

1. Equivalence classes of untilts correspond to primitive ideals $I \subset W(C^\circ)$ of degree 1, via $C^\sharp \mapsto \ker \theta_{C^\sharp}$.

2. Frobenius-equivalence classes of characteristic 0 untilts correspond to closed points on the Fargues-Fontaine curve X constructed from C as in (2.5); the inverse map sends a point to its residue field.

3. Frobenius-equivalence classes of characteristic 0 untilts correspond to elements of the quotient $(\widetilde{H}(C^\circ)\backslash\{0\})/\mathbf{Q}_p^\times$, where $\widetilde{H}$ is the universal cover of the formal multiplicative group as in (2.4).

The parametrization described in (1) relativizes quite easily. Suppose R is a perfectoid $\mathbf{F}_p$-algebra with pseudo-uniformizer ϖ. Then we have the Witt ring $W(R^\circ)$, equipped with its $(p,[\varpi])$-adic topology. A *primitive ideal of degree 1* in $W(R^\circ)$ is a principal ideal generated by an element of the form $\xi = \sum_{n=0}^\infty [x_n]p^n$, where $x_0 \in R$ is topologically nilpotent and $x_1 \in R^\circ$ is a unit.

Theorem 3.2.1 ([Fon13])**.** *Ideals $I \subset W(R^\circ)$ which are primitive of degree 1 are in bijection with isomorphism classes of untilts of R, via $I \mapsto (W(R^\circ)/I)[1/p]$.*

As in the case with perfectoid fields, however, this does not give us much in the way of defining the object $\operatorname{Spd} \mathbf{Q}_p$; it is not easy to tell whether two such ideals are the same, given their generators.

We turn now to (2). It is easy to define a relative Fargues–Fontaine curve: given a perfectoid ring $R/\mathbf{F}_p$, first define the relative adic curve

$$\mathcal{Y}_R = \operatorname{Spa} W(R^\circ)\backslash\{|p[\varpi]| = 0\}$$

and the ring $B_R = H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R})$. Then B_R has an action of Frobenius ϕ, and we define the *relative schematic Fargues-Fontaine curve* as

$$X_R = \operatorname{Proj} \bigoplus_{d \geq 0} B_R^{\phi = p^d}.$$

However, when R is not a field, we cannot expect X_R to have any nice properties (*e.g.* it may not be Noetherian). Nor should we expect that closed points of X_R parametrize Frobenius-equivalence classes of characteristic 0 untilts; after all, the residue field of such a point is a field, whereas an untilt $R^\sharp$ very well may not be.

Perhaps (3) has more promise. In the case that $R = C$ is an algebraically closed field of characteristic p, Theorem 2.5.4 says that isomorphism classes of characteristic 0 untilts of C are in bijection with $(\widetilde{H}(C^\circ)\backslash\{0\})/\mathbf{Z}_p^\times$, where H is the formal multiplicative group over $\mathbf{F}_p$. Recall the construction: if $C^\sharp$ is a characteristic 0 untilt, we choose a compatible system $(1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive pth power roots of 1 in $C^\sharp$, which determines a nonzero element of $\widetilde{H}(C^{\sharp\circ}) \cong \widetilde{H}(C^\circ)$, well-defined up to multiplication by an element of $\mathbf{Z}_p^\times$.

Let $\mathbf{Q}_p^{\text{cycl}}$ be the completion of $\mathbf{Q}_p(\mu_{p^\infty})$. Then $\operatorname{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \cong \mathbf{Z}_p^\times$ acts continuously on $\mathbf{Q}_p^{\text{cycl}}$. Finally, let $\widetilde{H}^{\text{ad}}\backslash\{0\}$ be the punctured adic space attached to the formal scheme $\widetilde{H}$.

Lemma 3.2.2. *There is an isomorphism* $\widetilde{H}^{\mathrm{ad}} \backslash \{0\} \cong \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$ *which is* $\mathbf{Z}_p^\times$*-equivariant.*

Proof. Since $\widetilde{H} \cong \operatorname{Spf} \mathbf{F}_p[\![t^{1/p^\infty}]\!]$, we have $\widetilde{H}^{\mathrm{ad}} \backslash \{0\} \cong \operatorname{Spa} \mathbf{F}_p(\!(t^{1/p^\infty})\!)$. we have already identified the latter with $\mathbf{Q}_p^{\mathrm{cycl},\flat}$ in Example 2.1.1, so one only needs to check that the $\mathbf{Z}_p^\times$-action is preserved. □

(There is a generalization of this lemma to Lubin–Tate extensions of any local field [Wei17, Proposition 3.5.3].)

Therefore, we can restate our parametrization of untilts of C as follows:

$$\{\text{Char. 0 untilts of } C\} \cong \operatorname{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^{\mathrm{cycl},\flat}, C)/\mathbf{Z}_p^\times = \operatorname{Hom}(\operatorname{Spa} C, \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times. \tag{3.2.1}$$

We could also have derived this directly: if $C^\sharp$ is a characteristic 0 untilt of C, then there exists an embedding $\mathbf{Q}_p^{\mathrm{cycl}} \hookrightarrow C^\sharp$ which is well-defined up to the action of $\mathbf{Z}_p^\times$; tilting this gives $\mathbf{Q}_p^{\mathrm{cycl},\flat} \hookrightarrow C$.

The bijections in (3.2.1) suggest that $\operatorname{Spd} \mathbf{Q}_p$ should be the quotient

$$\text{“}(\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times.\text{”}$$

But this quotient doesn't exist in the category of adic spaces. The subfield of $\mathbf{Q}_p^{\mathrm{cycl},\flat}$ fixed by $\mathbf{Z}_p^\times$ is just $\mathbf{F}_p$.

We would like to formulate a generalization of (3.2.1) for general perfectoid rings $R/\mathbf{F}_p$. We begin with the case that $R = K$ is a perfectoid field which is not algebraically closed. Let $K^\sharp/\mathbf{Q}_p$ be an untilt. Then $K^\sharp$ might not contain all pth power roots of unity. For each $n \geq 1$, the field $K_n^\sharp := K^\sharp(\mu_{p^n})$ is a perfectoid field, whose tilt K_n is a finite Galois extension of K. Let $K_\infty^\sharp$ be the completion of $\cup_{n\geq 1} K_n^\sharp$; then $K_\infty^\sharp$ is perfectoid. Let $K_\infty = K_\infty^{\sharp\flat}$. Let $G = \operatorname{Gal}(K^\sharp(\mu_{p^\infty})/K^\sharp)$; then G acts continuously on K_∞. If we choose a compatible sequence of pth power roots of 1 in $K_\infty^\sharp$, we obtain a nonzero element $\varepsilon \in \widetilde{H}(K_\infty^{\sharp\circ}) \cong \widetilde{H}(K_\infty^{\sharp\flat\circ}) = \widetilde{H}(K_\infty^\circ)$. Since G acts on ε through the cyclotomic character, the class of ε in $\widetilde{H}(K_\infty^\circ)/\mathbf{Z}_p^\times$ is G-invariant.

Thus, given an untilt $K^\sharp/\mathbf{Q}_p$, there exists a perfectoid field K_∞/K, equal to the completion of a Galois extension with group G, together with a class $\varepsilon \in \operatorname{Hom}(\operatorname{Spa} K_\infty, \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times$ which is G-invariant. Conversely, if we are given such data, the class ε determines a characteristic 0 untilt $K_\infty^\sharp$ of K_∞ together with a continuous action of G; then $K^\sharp := (K_\infty^\sharp)^G$ is a characteristic 0 untilt of K.

It may happen that two data of the form (K_∞, ε) give rise to the same untilt. The proper way to sort this out is in the language of sheaves on the *pro-étale site*, in which $\operatorname{Spa} K_\infty \to \operatorname{Spa} K$ is considered a covering.

3.3 The pro-étale topology

The extension of fields K_∞/K appearing in the previous section was the completion of the union of a tower of finite separable (that is, étale) extensions of K. Such an extension K_∞/K is said to be *pro-étale*. The definition works in families as follows.

Definition 3.3.1. A morphism $f\colon X \to Y$ of perfectoid spaces is *pro-étale* if locally on X it is of the form $\operatorname{Spa}(A_\infty, A_\infty^+) \to \operatorname{Spa}(A, A^+)$, where A and A_∞ are perfectoid rings, and

$$(A_\infty, A_\infty^+) = \left[\varinjlim (A_i, A_i^+)\right]^\wedge$$

is a filtered colimit of pairs (A_i, A_i^+), such that $\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A, A^+)$ is étale.

(The notion of an étale morphism between analytic affinoid adic spaces appears in [Sch12, Definition 7.1].)

Example 3.3.2. Let K be a perfectoid field, and let S be a profinite set; we have the perfectoid space $\underline{S}$ as in Example 3.1.5. Then $\underline{S} \to \operatorname{Spa} K$ is pro-étale. If $K = C$ is algebraically closed and $X \to \operatorname{Spa} C$ is pro-étale, then $X = \underline{S}$ for a locally profinite set S.

Example 3.3.3. Somewhat counterintuitively, the inclusion of a Zariski-closed subset is pro-étale. For instance, let K be a perfectoid field, let ϖ be a pseudo-uniformizer of K, and let $Y = \operatorname{Spa} K\langle T^{1/p^\infty}\rangle$. For $n = 1, 2, \dots$, let $Y_n \subset Y$ be the rational subset $\{|T| \leq |\varpi|^n\}$. Then "evaluation at 0" induces an isomorphism $\left[\varinjlim \mathcal{O}_Y(Y_n)\right]^\wedge \to K$, so that the inclusion-at-0 map $\operatorname{Spa} K \to Y$ is pro-étale.

Definition 3.3.4. Consider the category Pfd of perfectoid spaces of characteristic p. We endow this with the structure of a site by declaring that a collection of morphisms $\{f_i\colon X_i \to X\}$ is a covering (a *pro-étale cover*) if the f_i are pro-étale, and if for all quasi-compact open $U \subset X$, there exists a finite subset $I_U \subset I$, and a quasi-compact open $U_i \subset X_i$ for $i \in I_U$, such that $U = \cup_{i \in I_U} f_i(U_i)$.

If K is eitehr a discrete perfect field (such as $\overline{\mathbf{F}}_p$) or a perfectoid field of characteristic p, we write Pfd_K for the category of perfectoid spaces lying over $\operatorname{Spa} K$, endowed with the topology obtained by restriction from Pfd.

Remark 3.3.5. The finiteness condition in Definition 3.3.4 excludes certain "pointwise" morphisms from being pro-étale covers. For instance if Y is the perfectoid unit disc, we can consider the inclusion $f_x\colon \operatorname{Spa}(K_x, K_x^+) \to Y$ for each point $x \in |Y|$; this is pro-étale by similar reasoning as in Example 3.3.3, but we don't want $\{f_x\}_{x \in |Y|}$ to be a pro-étale covering.

Remark 3.3.6. The notions of a pro-étale morphism of schemes and of a pro-étale site appear in [BS15], where they were used to define a pro-étale fundamental group of a scheme, and also to give the "morally correct" definition of the ℓ-adic cohomology group $H^i(X, \mathbf{Q}_\ell)$ for a scheme X.

It now makes sense to talk about a sheaf on Pfd: this is a presheaf on Pfd (that is, a contravariant set-valued functor) which satisfies the sheaf axioms with respect to the pro-étale topology. If X is a perfectoid space of characteristic p, we have the representable presheaf h_X defined by $h_X(Y) = \operatorname{Hom}(Y, X)$.

Proposition 3.3.7 ([SW, Proposition 8.2.7]). *The presheaf h_X is a sheaf.*

If $\mathcal{F}$ is a sheaf on Pfd, and if X is an object of Pfd, then a morphism $h_X \to \mathcal{F}$ is the same thing as a section of $\mathcal{F}(X)$. Note that the functor $X \mapsto h_X$ exhibits Pfd as a full subcategory of the category of sheaves on Pfd.

Definition 3.3.8.

1. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves on Pfd is *pro-étale* if for all perfectoid spaces X and maps $h_X \to \mathcal{G}$, the pullback $h_X \times_{\mathcal{G}} \mathcal{F}$ is representable by a perfectoid space Y, and the morphism $Y \to X$ (corresponding to $h_Y = h_X \times_{\mathcal{G}} \mathcal{F} \to h_X$) is pro-étale.

2. Let $\mathcal{F}$ be a sheaf on Pfd. A *pro-étale equivalence relation* is a monomorphism of sheaves $\mathcal{R} \hookrightarrow \mathcal{F} \times \mathcal{F}$, such that each projection $\mathcal{R} \to \mathcal{F}$ is pro-étale, and such that for all objects S of Pfd, the image of the map $\mathcal{R}(S) \to \mathcal{F}(S) \times \mathcal{F}(S)$ is an equivalence relation on $\mathcal{F}(S)$.

3. A *diamond* is a sheaf $\mathcal{F}$ on Pfd which is the quotient of a perfectoid space by a pro-étale equivalence relation. That is, there exists a perfectoid space X and a pro-étale equivalence relation $\mathcal{R} \to h_X \times h_X$ such that
$$\mathcal{R} \rightrightarrows h_X \to \mathcal{F}$$
is a coequalizer diagram in the category of sheaves on Pfd.

4. If X is a perfectoid space (of whatever characteristic), let $X^\diamond$ be the representable sheaf $h_{X^\flat}$; this is a diamond. In the case $X = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid, we also write $\mathrm{Spd}(A, A^+)$ for $X^\diamond$.

5. A diamond X is *partially proper* if it satisfies the criterion appearing in Definition 1.9.2: for a perfectoid Huber pair (R, R^+), we have $X(R, R^\circ) \xrightarrow{\sim} X(R, R^+)$ only depends on R. If X is partially proper we write $X(R) = X(R, R^\circ)$.

Remark 3.3.9. The definition of diamonds given above is meant to mimic the notion of an *algebraic space*, which is the quotient of a scheme by an étale equivalence relation. The category of algebraic spaces is a mild generalization of the category of schemes. Some algebraic spaces arise as quotients of schemes by finite groups. Suppose that X is a scheme and G is a finite group acting on X. Assume that the action is *free* in the sense that for all nontrivial $g \in G$ and all $x \in X$ fixed by g, the action of g on the residue field of x is nontrivial. Then the quotient X/G is an algebraic space [Sta14, Tag 02Z2]; it is the quotient of X by the étale equivalence relation $G \times X \to X \times X$, $(g, x) \mapsto (x, g(x))$. (The freeness condition is necessary for this morphism to be a monomorphism.) Algebraic spaces are not to be confused with the larger category of algebraic stacks, which include stacky quotients $[X/G]$ for arbitrary actions of G on X.

3.4 The diamond $\operatorname{Spd} \mathbf{Q}_p$

Recall that we seek an object like "$(\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^{\times}$" which parametrizes characteristic 0 untilts of a perfectoid space of characteristic p. Now that we have the category of diamonds, we may make the following *ad hoc* definition.

Definition 3.4.1. We define $\operatorname{Spd} \mathbf{Q}_p = (\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\underline{\mathbf{Z}}_p^{\times}$. That is, $\operatorname{Spd} \mathbf{Q}_p$ is the coequalizer of

$$\underline{\mathbf{Z}}_p^{\times} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \rightrightarrows \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}, \tag{3.4.1}$$

where one map is the projection and the other is the action.

Thus $\operatorname{Spd} \mathbf{Q}_p$ is the sheafification of the presheaf on Pfd which assigns to an object S the set $\operatorname{Hom}(S, \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^{\times}$.

Lemma 3.4.2. $\operatorname{Spd} \mathbf{Q}_p$ *is a partially proper diamond.*

Proof. Each of the maps $\underline{\mathbf{Z}}_p^{\times} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$ is pro-étale (see Example 3.3.2). One must show that $\underline{\mathbf{Z}}_p^{\times} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$ is a monomorphism, which ultimately boils down to the fact that the map $\mathbf{Z}_p^{\times} \to \operatorname{Aut} \mathbf{Q}_p^{\mathrm{cycl},\flat}$ is injective. From there it is formal that (3.4.1) is a pro-étale equivalence relation, and thus that $\operatorname{Spd} \mathbf{Q}_p$ is a diamond. The partial properness of $\operatorname{Spd} \mathbf{Q}_p$ follows from that of $\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$. $\square$

If S is an object of Pfd, then to give an element of $(\operatorname{Spd} \mathbf{Q}_p)(S)$ is to give a pro-étale cover $\widetilde{S} \to S$ and an element of the set $\operatorname{Hom}(\widetilde{S}, \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^{\times}$ which comes equipped with a descent datum along $\widetilde{S} \to S$. In the case $S = \operatorname{Spa} K$ for a perfectoid field $K/\mathbf{F}_p$, one way to do this would be to give a perfectoid field $\widetilde{K}/K$, equal to the completion of a Galois extension of K with group G, and an element of $\operatorname{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^{\mathrm{cycl},\flat}, \widetilde{K})/\mathbf{Z}_p^{\times}$ which is G-invariant. We have already seen that such data gives an untilt of K. More generally, we have the following theorem.

Theorem 3.4.3 ([SW, Theorem 3.4.5])**.** *Let X be a perfectoid space of characteristic p. Then the set of isomorphism classes of untilts of X to characteristic 0 is naturally in bijection with* $(\operatorname{Spd} \mathbf{Q}_p)(X)$*. In other words, there is an equivalence of categories between perfectoid spaces over $\mathbf{Q}_p$, and the category of perfectoid spaces X of characteristic p together with a "structure morphism" $X^{\diamondsuit} \to \operatorname{Spd} \mathbf{Q}_p$.*

3.5 The functor $X \mapsto X^{\diamondsuit}$

The construction of $\operatorname{Spd} \mathbf{Q}_p$ from $\operatorname{Spa} \mathbf{Q}_p$ is a special case of a general phenomenon.

Definition 3.5.1. Let X be an analytic adic space on which p is topologically nilpotent (that is, X is fibered over $\operatorname{Spa} \mathbf{Z}_p$). Let $X^{\diamondsuit}$ be the functor on Pfd which sends an object S to the set of isomorphism classes of pairs $(S^{\sharp} \to X, \iota)$, where $S^{\sharp}$ is a perfectoid space and $\iota \colon S^{\sharp\flat} \xrightarrow{\sim} S$ is an isomorphism.

Thus $X^\diamondsuit$ classifies "untilts to X". If X itself is a perfectoid space and S is a test object in Perf, then the tilting equivalence in Theorem 3.1.6(3) shows that morphisms $S \mapsto X^\flat$ are in bijection with untilts $S^\sharp \to X$. Thus $X^\diamondsuit$ agrees with the notation introduced in Definition 3.3.8, namely $X^\diamondsuit = h_{X^\flat}$. Finally, Theorem 3.4.3 shows that $\operatorname{Spd} \mathbf{Q}_p = (\operatorname{Spa} \mathbf{Q}_p)^\diamondsuit$.

If $X = \operatorname{Spa}(R, R^+)$ is affinoid, we may write $\operatorname{Spd}(R, R^+)$ (or just $\operatorname{Spd} R$, if $R^+ = R^\circ$) to mean $X^\diamondsuit$.

Theorem 3.5.2 ([SW, Theorem 10.1.3])**.** *The functor $X^\diamondsuit$ is a diamond.*

The idea behind this, which appears in [Fal02] and [Col02], is that if $X = \operatorname{Spa} R$ for a Tate Huber $\mathbf{Z}_p$-algebra R, then there exists a tower of finite étale R-algebras R_i, such that $\widetilde{R} = [\varinjlim R_i]^\wedge$ is a perfectoid ring. Let $\widetilde{X} = \operatorname{Spa} \widetilde{R}$; then

$$\widetilde{X}^\diamondsuit \times_{X^\diamondsuit} \widetilde{X}^\diamondsuit \rightrightarrows \widetilde{X}^\diamondsuit \to X^\diamondsuit$$

presents $X^\diamondsuit$ as a quotient of a perfectoid space by a pro-étale equivalence relation.

Example 3.5.3. Let K be a perfectoid field of characteristic 0, and let $R = K\langle T^{\pm 1}\rangle$. Then $\widetilde{R} = K\langle T^{\pm 1/p^\infty}\rangle$ is pro-étale over R. If K contains all pth power roots of 1, then $\widetilde{R}/R$ is even a $\mathbf{Z}_p$-torsor.

Example 3.5.4. Let K be a perfectoid field of characteristic 0, and let ϖ be a pseudo-uniformizer which divides p in K°. Let $R = K\langle T \rangle$. This time, adjoining pth roots of T produces ramification at the origin in $\operatorname{Spa} R$ (and everywhere if K has characteristic p!), so that $K\langle T^{1/p}\rangle$ will not be finite étale over R. Instead one can adjoin a root of an Artin–Schreier polynomial, such as $U^p - \varpi U = T$, to produce a finite étale R-algebra R_1 for which T is a pth power in R_1°/ϖ. Iteration of this process produces the desired $\widetilde{R}$.

Thus we have a well-defined functor $X \to X^\diamondsuit$ from analytic adic spaces over $\operatorname{Spa} \mathbf{Z}_p$ to diamonds. One might wonder whether this functor is fully faithful, which would allow us to view analytic adic spaces over $\mathbf{Z}_p$ as a subcategory of the category of diamonnds. This cannot be true as stated, since $\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl}}$ and $\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$ are non-isomorphic adic spaces, while $(\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}})^\diamondsuit \cong (\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl},\flat})^\diamondsuit$. But if we fix a nonarchimedean scalar field K, we may instead consider the functor $X \mapsto X^\diamondsuit$ from analytic adic spaces over $\operatorname{Spa} K$ to diamonds over $\operatorname{Spd} K$. This also fails to be fully faithful, as shown by the following example.

Example 3.5.5. Let X be the cuspidal cubic $y^2 = x^3$, considered as an adic space over $\mathbf{Q}_p$. Let $X' \to X$ be the usual desingularization of X. That is, X' is the affine line in one variable t, and $X' \to X$ is $t \mapsto (t^2, t^3)$. We claim that $(X')^\diamondsuit \to X^\diamondsuit$ is an isomorphism. This is equivalent to the claim that $X'(R) \to X(R)$ is a bijection for every perfectoid $\mathbf{Q}_p$-algebra R. We leave injectivity as an exercise to the reader (hint: R is reduced). Surjectivity is a little subtle; we refer to the reader to [KL, Theorem 3.7.4] for details.

A ring is R *seminormal* $t \mapsto (t^2, t^3)$ is a bijection from R onto the set of pairs $(x, y) \in R^2$ satisfying $y^2 = x^3$. A rigid-analytic space X over a nonarchimedean field K is seminormal if locally it is $\mathrm{Spa}(A, A^+)$, where A is a seminormal ring. The following theorem states that Example 3.5.5 is essentially the only obstruction to $X \mapsto X^\diamondsuit$ being fully faithful.

Theorem 3.5.6 ([SW, Proposition 10.2.4])**.** *For a nonarchimedean field K of characteristic 0, the functor $X \mapsto X^\diamondsuit$ from seminormal rigid-analytic spaces over K onto diamonds over* $\mathrm{Spd}\, K$ *is fully faithful.*

3.6 A diamond version of the Fargues-Fontaine curve

Let C be an algebraically closed perfectoid field of characteristic $p > 0$, and let ϖ be a pseudo-uniformizer of C. Recall the adic space

$$\mathcal{Y}_C = \mathrm{Spa}\, W(C^\circ) \backslash \{|p[\varpi]| = 0\}\,.$$

Since $\mathcal{Y}_C$ is a analytic adic space over $\mathrm{Spa}\, \mathbf{Q}_p$, Theorem 3.5.2 indicates that $\mathcal{Y}_C^\diamondsuit$ makes sense and is a diamond.

Proposition 3.6.1 (The diamond formula)**.** $\mathcal{Y}_C^\diamondsuit \cong \mathrm{Spd}\, C \times \mathrm{Spd}\, \mathbf{Q}_p$

Proof. (Sketch.) The isomorphism says that for a perfectoid ring R in chararacteristic p, the following categories are equivalent:

1. Pairs consisting of an untilt $R^\sharp/\mathbf{Q}_p$ of R and a continuous homomorphism $C \to R$, and
2. Pairs consisting of an untilt $R^\sharp$ of R and a morphism $\mathrm{Spa}\, R^\sharp \to \mathcal{Y}_C$ (whose existence means that $R^\sharp/\mathbf{Q}_p$).

(Both sides are partially proper, so there is no need to discuss rings of integral elements.) We now describe the equivalence assuming an untilt $R^\sharp/\mathbf{Q}_p$: A continuous homomorphism $C \to R$ induces a homomorphism $\theta_C \colon W(C^\circ) \to R^\sharp$, in which the images of p and $[\varpi]$ are invertible; then θ_C induces a morphism $\mathrm{Spa}\, R^\sharp \to \mathcal{Y}_C$. Conversely if $\mathrm{Spa}\, R^\sharp \to \mathcal{Y}_C$ is given, we get a homomorphism $W(C^\circ) \to R^{\sharp\circ}$, in which the images of p and $[\varpi]$ are invertible in $R^\sharp$. This induces $C^\circ \to R^{\sharp\circ}/p$. Take the inverse limit under Frobenius to get $C^\circ \to R^\circ$, and then invert ϖ to get $C \to R$. □

As for the adic Fargues–Fontaine curve $\mathcal{X}_C$, we have the diamond formula

$$\mathcal{X}_C^\diamondsuit \cong (\mathrm{Spd}\, C \times \mathrm{Spd}\, \mathbf{Q}_p)/(\phi \times \mathrm{id}),$$

where ϕ is the Frobenius automorphism of C. Recall from Theorem 2.7.3 (or rather the adic version of this theorem) that the étale fundamental group of $\mathcal{X}_C$ is $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. The notion of an étale morphism exists for diamonds, and for an analytic adic space Y, there is an equivalence of sites $Y_{\text{ét}} \xrightarrow{\sim} Y_{\text{ét}}^\diamondsuit$, via the diamond functor. Therefore:

$$\pi_1((\mathrm{Spa}\, C \times \mathrm{Spd}\, \mathbf{Q}_p)/(\phi \times \mathrm{id})) \cong \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p). \tag{3.6.1}$$

Scholze observed that this formula resembles a theorem of Drinfeld [Dri80]. Suppose U and V are two algebraic curves (not necessarily projective) over a common algebraically closed field of characteristic p. The Künneth formula $\pi_1(U \times V) \cong \pi_1(U) \times \pi_1(V)$ fails (the left side is much larger), but it can be salvaged by means of a "partial Frobenius". There is a group $\pi_1((U \times V)/(\phi \times \mathrm{id}))$ classifying finite étale covers of $U \times V$ equipped with an automorphism lying over $\phi \times \mathrm{id}$. Drinfeld's theorem is that $\pi_1((U \times V)/(\phi \times \mathrm{id})) \cong \pi_1(U) \times \pi_1(V)$. The goal of [Dri80] (and its successor [Laf02]) was to establish the Langlands correspondence for GL_n over a function field, using moduli spaces of *shtukas*. Scholze's goal as laid out in [SW] is to define a moduli space of *mixed-characteristic local shtukas* to establish a local Langlands correspondence for p-adic groups.

There are yet other versions of the diamond formula. Let H/C° be the multiplicative formal group. We have seen that the universal cover $\widetilde{H}$ is a formal $\mathbf{Q}_p$-vector space, whose adic generic fiber $\widetilde{H}_C$ is a $\mathbf{Q}_p$-vector space object in the category of perfectoid spaces. The underlying perfectoid space of $\widetilde{H}_C$ is the perfectoid open unit disc. Let $\widetilde{H}_C^* = \widetilde{H}_C \setminus \{0\}$. Then $\widetilde{H}_C^*$ admits an action of $\mathbf{Q}_p^\times$.

Proposition 3.6.2 ([Wei17])**.** *There is an isomorphism of diamonds*

$$\widetilde{H}_C^{*\diamondsuit}/\mathbf{Q}_p^\times \cong (\operatorname{Spd} C \times \operatorname{Spd} \mathbf{Q}_p)/(\mathrm{id} \times \phi).$$

The étale fundamental group of $\widetilde{H}_C^{*\diamondsuit}/\mathbf{Q}_p^\times$ *is isomorphic to* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

Proof. By Lemma 3.2.2, $\widetilde{H}_C^*$ is isomorphic to $\operatorname{Spa} C \times \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$, where the $\mathbf{Z}_p^\times$ action on $\widetilde{H}_C^*$ becomes the Galois action on $\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$. Therefore $\widetilde{H}_C^{*\diamondsuit}/\mathbf{Z}_p^\times$ is isomorphic to $\operatorname{Spd} C \times (\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl},\flat}/\mathbf{Z}_p^\times) \cong \operatorname{Spd} C \times \operatorname{Spd} \mathbf{Q}_p$. One can also check that the action of p on $\widetilde{H}_C^*$ corresponds to the action of Frobenius on $\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$, which gives the claimed isomorphism.

The statement about the étale fundamental group looks like (3.6.1), but the partial Frobenius is on the wrong side. No matter: the composition of two partial Frobenii is the absolute Frobenius, which is an equivalence on the étale site of any diamond. □

Remark 3.6.3. There is a generalization of the above proposition which concerns a finite extension $E/\mathbf{Q}_p$. One has to replace H with the Lubin-Tate formal $\mathcal{O}_E$-module H_E. Then the diamond $Z_E = \widetilde{H}_{E,C}^{*\diamondsuit}/E^\times$ classifies untilts of a perfectoid C-algebra to a perfectoid E-algebra, up to Frobenius.

Remark 3.6.4. The diamond $(\operatorname{Spd} C \times \operatorname{Spd} \mathbf{Q}_p)/(\mathrm{id} \times \phi)$ is called the *mirror curve* by Fargues, who identifies it as the moduli space of divisors of degree 1 on X_C.

4 Banach-Colmez spaces

4.1 Definition and first examples

So far we have considered objects belonging to a progression of categories: rigid spaces over a nonarchimedean field of residue characteristic p, analytic adic spaces over $\operatorname{Spa} \mathbf{Z}_p$, perfectoid spaces, and finally diamonds, which (in the limited sense of Theorem 3.5.6) generalize all three. This lecture will introduce some examples of diamonds which carry the structure of $\mathbf{Q}_p$-vector spaces. Throughout, we fix an algebraically closed perfectoid field $C/\mathbf{Q}_p$ with residue field k.

Example 4.1.1. The following are two examples of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C.

1. If V is a finite-dimensional $\mathbf{Q}_p$-vector space, we have the constant sheaf $\underline{V}$. If (R, R^+) is a perfectoid Huber pair over (C, C°), then $\underline{V}(R, R^+)$ is the $\mathbf{Q}_p$-vector space of continuous maps $|\operatorname{Spa}(R, R^+)| \to V$. If $\operatorname{Spa}(R, R^+)$ is connected, then $\underline{V}(R, R^+) = V$.

2. The additive group $\mathbf{G}_\mathrm{a}$ may be considered as a sheaf of $\mathbf{Q}_p$-vector spaces on Pfd_C, by $\mathbf{G}_\mathrm{a}(R, R^+) = R$. For a finite-dimensional C-vector space W, the sheaf $W \otimes_C \mathbf{G}_\mathrm{a}$ is $(R, R^+) \mapsto W \otimes_C R$.

Both sorts of examples are diamonds arising from analytic adic spaces over C.

Definition 4.1.2. The category of *Banach-Colmez spaces* over C is the smallest abelian subcategory of the category of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C which contains the objects $\underline{V}$ and $W \otimes_{\mathbf{Q}_p} \mathbf{G}_\mathrm{a}$ from Example 4.1.1 and which is closed under extensions.

An equivalent category was introduced by Colmez [Col02] without reference to perfectoid spaces; the definition above appears in [Bra], where it shown that the two definitions are equivalent. The "Banach" half of the name refers to Colmez' definition, in which the objects are functors taking values in the category of $\mathbf{Q}_p$-Banach spaces.

Example 4.1.3. Let H_0 be a p-divisible group over k, and let H be a lift of $H_0 \otimes_k C^\circ/p$ to C°. We have seen that the universal cover $\widetilde{H} = \varprojlim_p H$ does not depend on the choice of lift H, and that the generic fiber $\widetilde{H}_C$ is a $\mathbf{Q}_p$-vector space object in the category of perfectoid spaces over C. The logarithm map on H induces an exact sequence of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C, as in (2.7.1):

$$0 \to \underline{VH} \to \widetilde{H}_C \to \operatorname{Lie} H \otimes_{C^\circ} \mathbf{G}_\mathrm{a} \to 0.$$

One has to check exactness on the right. This is a matter of showing that, for any perfectoid C-algebra R and any $v \in \operatorname{Lie} H \otimes_{C^\circ} R$, that there exists a pro-étale R'/R and a sequence $(x_0, x_1, \cdots) \in \widetilde{H}(R^\circ)$ with $\log_H(x_0) = v$. After replacing v with $p^n v$ for $n \gg 0$, we may assume that $\exp_H(v)$ converges to

$x_0 \in H(R^\circ)$. The pro-étale extension R' is then obtained by adjoining all pth power division points of x_0 to R.

Since $\widetilde{H}_C$ is an extension of $(\operatorname{Lie} H)[1/p] \otimes_C \mathbf{G}_{\mathrm{a}}$ by $\underline{VH}$, it is a Banach-Colmez space, which happens to be representable by a perfectoid space.

In (2.7) we saw a connection between $\widetilde{H}_C$ and vector bundes on the Fargues-Fontaine curve $X_{C^\flat}$. Let $D(H)$ be the (contravariant) Dieudonné module of H, so that $D(H)$ is a free finite-rank $W(k)$-module equipped with actions of Frobenius and Verschiebung. Then $N := \operatorname{Hom}_{W(k)}(D(H), W(k)[1/p])$ is an isocrystal, all of whose slopes lie in the range $[0,1]$. Let $\mathcal{E}(N)$ be the associated vector bundle. For a perfectoid C-algebra R, we have the relative Fargues–Fontaine curve $X_{R^\flat}$ constructed in (3.2), which lies over $X_{C^\flat}$.

Proposition 4.1.4. *Let R be a perfectoid C-algebra. There is an isomorphism $\widetilde{H}(R^\circ) \xrightarrow{\sim} H^0(X_{R^\flat}, \mathcal{E}(N))$.*

Proof. (Sketch.) The left-hand side is $\widetilde{H}(R^\circ) \cong \widetilde{H}(R^{\flat\circ})$ and right-hand side is $(B_{R^\flat} \otimes_{W(k)} N)^{\phi=1}$. (Thus both sides only depend on the tilt R°.) $R^{\flat\circ}$ is a perfect ring; by [SW13, Theorem 4.1.4], the covariant crystalline Dieudonné module functor on p-divisible groups over $R^{\flat\circ}$ up to isogeny is fully faithful. Applied to morphisms $(\mathbf{Q}_p/\mathbf{Z}_p)_{R^{\flat\circ}} \to H_{R^{\flat\circ}}$, that result gives an isomorphism $\widetilde{H}(R^{\flat\circ}) \xrightarrow{\sim} (B_{\mathrm{cris}}(R^{\flat\circ}) \otimes_{W(k)} N)^{\phi=1}$, where $B_{\mathrm{cris}}(R^{\flat\circ})$ is the crystalline period ring. A hint as to why $(B_{R^\flat} \otimes_{W(k)} N)^{\phi=1} \cong (B_{\mathrm{cris}}(R^{\flat\circ}) \otimes_{W(k)} N)^{\phi=1}$ is [FF11, Corollaire 1.10.13], although strictly speaking that result only applies to the case that R is a field. $\square$

Example 4.1.5. Let $C'/\mathbf{Q}_p$ be an untilt of $C^\flat$, not necessarily equal to C itself. We define a sheaf $\mathbf{G}'_{\mathrm{a}}$ on Pfd_C by sending a perfectoid C-algebra R to the untilt of $R^\flat$ over C'. That is, $\mathbf{G}'_{\mathrm{a}}(R) = W(R^\flat) \otimes_{W(C^{\flat\circ})} C'$. We claim that $\mathbf{G}'_{\mathrm{a}}$ is a Banach-Colmez space. To see this, let H be the formal multiplicative group over C°. Theorem 2.5.1 produces two nonzero elements $t, t' \in \widetilde{H}(C^{\flat\circ}) \cong \widetilde{H}(C^\circ)$, well-defined up to multiplication by $\mathbf{Q}_p^\times$, corresponding to the untilts C and C' of $C^\flat$, respectively. There are now two intersecting exact sequences of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & \underline{\mathbf{Q}}_p t' \arrow[d] & & \\ 0 \arrow[r] & \underline{\mathbf{Q}}_p t \arrow[r] & \widetilde{H} \arrow[r] \arrow[d] & \mathbf{G}_{\mathrm{a}} \arrow[r] & 0. \\ & & \mathbf{G}'_{\mathrm{a}} \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

Thus $\mathbf{G}'_{\mathrm{a}}$ is the quotient of a Banach–Colmez space, and so must be one itself.

4.2 Banach-Colmez spaces of slope > 1

Now suppose N is a general isocrystal over k, which doesn't necessarily arise from a p-divisible group. We may consider the functor $H^0(\mathcal{E}(N))$ on Pfd_C, which sends a perfectoid C-algebra R to the $\mathbf{Q}_p$-vector space $H^0(X_{R^\flat}, \mathcal{E}(N))$. It suffices to consider the case $\mathcal{E}(N) = \mathcal{O}_X(\lambda)$ for $\lambda \in \mathbf{Q}$, because a general $\mathcal{E}(N)$ is isomorphic to a direct sum of these. If $\lambda < 0$ then $H^0(X, \mathcal{O}_X(\lambda)) = 0$, and if $\lambda \in [0,1]$, then Proposition 4.1.4 shows that $H^0(\mathcal{O}_X(\lambda)) \cong \widetilde{H}_\lambda$ is an absolute perfectoid space.

What if $\lambda > 1$? For instance, if $\lambda = 2$, then $H^0(X_{R^\flat}, \mathcal{O}_X(2)) = B_{R^\flat}^{\phi=p^2}$. For brevity's sake, let $B^{\phi=p^2} = H^0(\mathcal{O}_X(2))$. Let C' be an untilt of $C^\flat$ which is not Frobenius-equivalent to C. As in Example 4.1.5, the two untilts C and C' correspond to $\mathbf{Q}_p$-linearly independent elements $t, t' \in B_{C^\flat}^{\phi=p} \cong H^0(X_{C^\flat}, \mathcal{O}_X(1))$.

Proposition 4.2.1. *There is an exact sequence of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C:*

$$0 \to \underline{\mathbf{Q}}_p \to B^{\phi=p} \times B^{\phi=p} \longrightarrow B^{\phi=p^2} \longrightarrow 0$$
$$t \longmapsto (t, t')$$
$$(x, x') \longmapsto xt' - x't$$

Proof. First we check that the map $B^{\phi=p} \times B^{\phi=p} \to B^{\phi=p^2}$ is surjective. Let R be a perfectoid C-algebra, and let $s \in B_{R^\flat}^{\phi=p^2}$. Let R' be the untilt of $R^\flat$ over C'. We have two ring homomorphisms $\theta\colon B_{R^\flat} \to R$ and $\theta'\colon B_{R^\flat} \to R'$, such that $\ker \theta \cap B_{R^\flat}^{\phi=p} = \mathbf{Q}_p t$ and $\ker \theta' \cap B_{R^\flat}^{\phi=p} = \mathbf{Q}_p t'$. Thus $\theta(t')$ and $\theta'(t)$ are both nonzero. We have the elements $\theta(t')^{-1}\theta(s) \in R$ and $\theta'(t)^{-1}\theta'(s) \in R'$. After replacing R with a pro-étale extension, we can find elements $x, x' \in B_{R^\flat}^{\phi=p}$ such that $\theta(x) = \theta(t')^{-1}\theta(s)$ and $\theta'(x') = -\theta'(t)^{-1}\theta'(s)$. Then the element

$$\alpha = xt' - x't - s \in B_{R'}^{\phi=p^2}$$

has the property that $\theta(\alpha) = 0$ and $\theta'(\alpha) = 0$. This implies that $\alpha = att'$ for some $a \in H^0(X_{R^\flat}, \mathcal{O}_X) = \underline{\mathbf{Q}}_p(R)$; this shows that $s = (x - at)t' - x't$ lies in the image of $B^{\phi=p} \times B^{\phi=p}$ as required. $\square$

As a corollary, we find that $B^{\phi=p^2}$ is a Banach-Colmez space, and also a diamond. Indeed, Proposition 4.2.1 gives a presentation of $B^{\phi=p^2}$ as a quotient of a perfectoid space by a pro-étale equivalence relation. More generally, if N is an isocrystal over k, then $H^0(\mathcal{E}(N))$ is a Banach-Colmez space and a diamond.

4.3 The de Rham period ring

We begin with a definition from p-adic Hodge theory.

Definition 4.3.1. Let R be a perfectoid ring. The *de Rham period ring* $B_{\mathrm{dR}}^+(R)$ is the completion of $W(R^{\flat\circ})[1/p]$ with respect to the kernel of $\theta_R \colon W(R^{\flat\circ})[1/p] \to R$.

If $R = C$ is an algebraically closed perfectoid field, then $B_{\mathrm{dR}}^+(C)$ is a discrete valuation ring with uniformizer ξ_C, residue field C and fraction field $B_{\mathrm{dR}}(C)$. These objects were constructed by Fontaine. They appear in the context of p-adic p-adic Galois representations, particularly in the comparison isomorphism linking étale and de Rham cohomology of a variety over a p-adic field [Fal89]. They also appear in the study of the Fargues-Fontaine curve: the untilt C of $C^\flat$ determines a closed point $\infty \in X_{C^\flat}$, and $B_{\mathrm{dR}}^+(C)$ is the completed local ring $\widehat{\mathcal{O}}_{X_{C^\flat},\infty}$.

Definition 4.3.2. For $n \geq 1$, let $B_{\mathrm{dR}}^+/\operatorname{Fil}^n$ be the sheaf on Pfd_C which assigns to (R, R^+) the $\mathbf{Q}_p$-vector space $B_{\mathrm{dR}}^+(R)/(\ker\theta_R)^n$.

Theorem 4.3.3. *$B_{\mathrm{dR}}^+/\operatorname{Fil}^n$ is a Banach–Colmez space and a diamond.*

Note that $B_{\mathrm{dR}}^+/\operatorname{Fil}^1 = \mathbf{G}_{\mathrm{a}}$, because for a perfectoid C-algebra R, we have $B_{\mathrm{dR}}(R)^+/(\ker\theta_R) = R$.

Proof. We sketch the proof for $B_{\mathrm{dR}}/\operatorname{Fil}^2$; the general case works by induction. Consider the complex of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C:

$$0 \to \operatorname{Fil}^1/\operatorname{Fil}^2 \to B_{\mathrm{dR}}^+/\operatorname{Fil}^2 \to B_{\mathrm{dR}}^+/\operatorname{Fil}^1 \to 0. \tag{4.3.1}$$

We already observed that $B_{\mathrm{dR}}^+/\operatorname{Fil}^1 = \mathbf{G}_{\mathrm{a}}$. As for $\operatorname{Fil}^1/\operatorname{Fil}^2$, we claim that it is $\mathbf{G}_{\mathrm{a}}(1) = \mathbf{G}_{\mathrm{a}} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1)$. We construct an isomorphism $\operatorname{Fil}^1/\operatorname{Fil}^2 \to \mathbf{G}_{\mathrm{a}}$ over $\operatorname{Spd} C$. Form the element t as in (2.5.1), and consider it as an element of $B_{\mathrm{dR}}^+(C)$. Then t generates the kernel of $B_{\mathrm{dR}}^+(R) \to R$ for any perfectoid

C-algebra R. This shows that $\mathrm{Fil}^1/\mathrm{Fil}^2 \cong \mathbf{G}_{\mathrm{a}}$, and therefore that $B_{\mathrm{dR}}^+/\mathrm{Fil}^2$ is a Banach-Colmez space.

Given a perfectoid C-algebra R, a section of $\mathrm{Fil}^1/\mathrm{Fil}^2$ over R consists of a pro-étale cover $\operatorname{Spa}\widetilde{R} \to \operatorname{Spa} R$ and an element $\alpha \in tB_{\mathrm{dR}}(\widetilde{R})^+$, together with a descent datum through $\operatorname{Spa}\widetilde{R} \to \operatorname{Spa} R$ for the image of α modulo t^2. Our morphism sends this section to $\theta_{\widetilde{R}}(\alpha/t)$, which (because of the descent datum) lies in R. (We leave it to the reader to construct the morphism in the opposite direction.) Note that $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{cycl}}/\mathbf{Q}_p)$ acts on t via the cyclotomic character; this is what we need to descend the morphism through $\mathbf{Q}_p^{\mathrm{cycl}}/\mathbf{Q}_p$.

Now we claim that the complex in (4.3.1) locally splits. Let H be the formal multiplicative group over C, and let $\widetilde{H}_C$ be the generic fiber of its universal cover. Then $\widetilde{H}_C$ is a perfectoid space, and the logarithm map $\widetilde{H}_C \to \mathbf{G}_{\mathrm{a}}$ is a pro-étale cover. Define a morphism $\widetilde{H}_C \to B_{\mathrm{dR}}^+$ by $(x_0, x_1, \dots) \mapsto \log[(x_i)]$. Then the following diagram commutes:

$$\begin{tikzcd}
 & & \widetilde{H}_C \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \mathbf{G}_{\mathrm{a}} \arrow[r] & B_{\mathrm{dR}}^+/\mathrm{Fil}^2 \arrow[r] & \mathbf{G}_{\mathrm{a}} \arrow[r] & 0.
\end{tikzcd}$$

We can now give a presentation of $B_{\mathrm{dR}}^+/\mathrm{Fil}^2$: it is the quotient of $\mathbf{G}_{\mathrm{a}} \times \widetilde{H}_C$ by the pro-étale equivalence relation of "having the same image in $B_{\mathrm{dR}}^+/\mathrm{Fil}^2$". $\square$

In general, $B_{\mathrm{dR}}^+/\mathrm{Fil}^i$ is a Banach-Colmez space admitting an i-step filtration, where the quotients are isomorphic to $\mathbf{G}_{\mathrm{a}}$.

As with the Banach-Colmez spaces of the previous section, $B_{\mathrm{dR}}^+/\mathrm{Fil}^n$ is the space of global sections of a (Zariski) sheaf on the Fargues-Fontaine curve $X_{C^\flat}$. The untilt C of $C^\flat$ determines a closed point $\infty \in X_{C^\flat}$. The completion of $X_{C^\flat}$ at ∞ is $\operatorname{Spec} B_{\mathrm{dR}}^+(C)$. Let $i_\infty\colon \operatorname{Spec} B_{\mathrm{dR}}^+(C) \to X_{C^\flat}$ be the corresponding morphism. Then $\mathcal{F} := i_\infty(B_{\mathrm{dR}}^+/(\ker\theta_C)^n)$ is a coherent sheaf on $X_{C^\flat}$ supported at ∞. Proposition 4.3.3 says that $H^0(\mathcal{F})$ (meaning the sheaf $R \mapsto H^0(X_{R^\flat}, \mathcal{F})$) is a Banach-Colmez space and a diamond.

Our examples show a strong connection between Banach–Colmez spaces and coherent sheaves on the Fargues–Fontaine curve. Indeed, for any coherent sheaf $\mathcal{F}$ on $X_{C^\flat}$, the sheaf $R \mapsto H^0(X_{R^\flat}, \mathcal{F})$ is a Banach-Colmez space. The complete story is a theorem of Le Bras [Bra], which gives an equivalence between the category $\mathcal{BC}(C)$ of Banach–Colmez spaces relative to C and the core of a certain t-structure on the derived category of coherent sheaves on $X_{C^\flat}$. Every object of $\mathcal{BC}(C)$ is isomorphic to $H^0(\mathcal{F}^+) \oplus H^1(\mathcal{F}^-)$, where $\mathcal{F}^+$ and $\mathcal{F}^-$ are coherent $\mathcal{O}_{X_{C^\flat}}$-modules. (An example of type $H^1(\mathcal{F}^-)$ is discussed in Project 5.5.) As a corollary, $\mathcal{BC}(C)$ only depends on the tilt $C^\flat$. Finally, every Banach-Colmez space is a diamond.

4.4 A survey of the diamond landscape

In these lectures we have introduced a hierarchy of nonarchimedean analytic spaces: rigid spaces, adic spaces, perfectoid spaces, and diamonds. We have highlighted the role of $\mathbf{Q}_p$-vector space objects in each category. In the last two sections, we studied $\mathbf{Q}_p$-vector space diamonds arising as global sections of sheaves on the Fargues-Fontaine curve (vector bundles and torsion sheaves, respectively).

Since we presented these objects without much context, you have a right to wonder about motivation. Why do we care that certain sheaves on Pfd are diamonds? And why are these particular objects so important?

Fundamentals of diamond geometry. The device of étale cohomology allows us to apply our intuitions about algebraic topology to schemes. To wit, if X is a scheme, there is a notion of an étale site $X_{\text{ét}}$, whose objects are étale morphisms over X; these can be used to define the ℓ-adic cohomology groups $H^i(X_{\text{ét}}, \mathbf{Q}_\ell)$. If in addition X is a smooth projective variety over an algebraically closed field k, and ℓ is invertible in k, then the $H^i(X_{\text{ét}}, \mathbf{Q}_\ell)$ have some nice properties: they are zero outside of the range $i = 0, 1, \dots, 2\dim X$, they satisfy Poincaré duality, there is a Lefschetz fixed-point formula one can apply to endomorphisms of X, and so on.

Underpinning these properties is a framework of results concerning different kinds of morphisms (finite type, étale, proper, smooth, etc.) and their effects on étale sheaves. For instance, we have a notion of a smooth morphism of schemes $f\colon X \to Y$, which is meant to mimic the same notion for manifolds, and which can be checked using a Jacobian criterion. The Poincaré duality theorem mentioned above is a special case of a relative version: there is an isomorphism $f^!\mathcal{F} \cong f^*\mathcal{F}[2d](d)$, valid whenever f is a smooth morphism of relative dimension d, and $\mathcal{F}$ is an étale sheaf of $(\mathbf{Z}/n\mathbf{Z})$-modules on Y, where n is invertible on Y.

Many of these fundamentals are carried over into the world of rigid and adic spaces in [Hub96]. Huber defines the important classes of morphisms of adic spaces (finite type, étale, proper, smooth, etc.), and proves theorems (base change theorems, Poincaré duality) about how they interact with étale cohomology.

Perfectoid spaces seem at first glance to be immune to this sort of treatment. For instance, let K be a perfectoid field, and let $\widetilde{D} = \operatorname{Spa} K\langle T^{1/p^\infty}\rangle$ be the perfectoid closed disc from Example 3.1.5. The ring $K\langle T^{1/p^\infty}\rangle$ isn't finitely generated over K, nor is it even topologically finitely generated, so already we run into problems if we wish to think of $\widetilde{D}$ as being "finite type" over K. The situation seems even worse if one tries to define smooth morphisms of perfectoid spaces using a Jacobian criterion. (If f belongs to $A = K\langle T^{1/p^\infty}\rangle$, then "$df/dT$", naïvely defined, may fail to lie in A, so this is certainly not the right way to proceed.)

Nonetheless, Scholze [Sch17] has defined a notion of *cohomological smoothness* for a morphism of diamonds (relative to a prime ℓ distinct from the residue characteristic), which essentially says that relative Poincaré duality holds. In this sense, $\widetilde{D} \to \operatorname{Spa} K$ is cohomologically smooth, as are the Banach–Colmez spaces $\widetilde{H}$ and $B_{\mathrm{dR}}^+/\operatorname{Fil}^n$ (over the base $\operatorname{Spd} C$). Recent work of Fargues-Scholze [FS] even gives a Jacobian criterion of sorts to determine whether a morphism of diamonds is cohomologically smooth.

Moduli spaces of mixed-characteristic local shtukas. Let $C/\mathbf{Q}_p$ be an algebraically closed perfectoid field with residue field k, and let H_0 be a p-divisible group over k. Recall from the discussion in (2.7) that we have an exact sequence of $\mathbf{Q}_p$-vector spaces

$$0 \to VH(C^\circ) \to \widetilde{H}(C^\circ) \to \operatorname{Lie} H \otimes C \to 0,$$

which can be interpreted as an exact sequence of $\mathcal{O}_{X_{C^\flat}}$-modules:

$$0 \to \mathcal{O}_{X_{C^\flat}} \otimes_{\mathbf{Q}_p} VH \to \mathcal{E}(H_0) \to i_* \operatorname{Lie} H \otimes C \to 0,$$

where $\mathcal{E}(H_0)$ is the vector bundle corresponding to (the isocrystal corresponding to) H_0, and i is the morphism $\operatorname{Spec} B_{\mathrm{dR}}^+(C) \to X_{C^\flat}$.

Define a (partially proper) sheaf $\mathcal{M}_{H_0}$ on Pfd_C as follows. For a perfectoid C-algebra R, we define $\mathcal{M}_{H_0}(R)$ to be the set of injective morphisms $s\colon \mathcal{O}_{X_{R^\flat}}^h \to \mathcal{E}(H_0)$ of $\mathcal{O}_{X_{R^\flat}}$-modules, whose cokernel is a sheaf of the form i_*W, where W is a projective R-module. (We have used the same letter i to denote the morphism $\operatorname{Spec} B_{\mathrm{dR}}^+(R) \to X_{R^\flat}$.)

Results in [SW13] show that $\mathcal{M}_{H_0}$ is a perfectoid space, and that it is isomorphic to the moduli space of deformations H of H_0 together with a $\mathbf{Q}_p$-basis for VH. The space $\mathcal{M}_{H_0}$ admits commuting actions of the groups $J = \operatorname{Aut}^0 H_0$ (automorphisms up to isogeny; this acts on $\mathcal{E}(H_0)$) and $\mathrm{GL}_h(\mathbf{Q}_p)$ (which acts on $\mathcal{O}_{X_{R^\flat}}^h$). The cohomology groups $H_c^i(\mathcal{M}_{H_0,\mathbf{C}_p}, \overline{\mathbf{Q}}_\ell)$ admit an action of $\mathrm{GL}_h(\mathbf{Q}_p) \times J \times W_{\mathbf{Q}_p}$, where $W_{\mathbf{Q}_p}$ is the Weil group. In the case that H_0 is basic, the *Kottwitz conjectures* predict that these cohomology groups realize Langlands functoriality. In the special case that H_0 is connected of dimension 1, the space $\mathcal{M}_{\overline{H}}$ is called a Lubin-Tate space (at infinite level). The Kottwitz conjectures are known to be true in for Lubin-Tate space [HT01].

The introduction of diamonds allows us to generalize the situation considerably. Fix an integer $h \geq 1$ and an isocrystal b of rank h. Write $\mathcal{E}_b$ for the corresponding vector bunde on X_C. Fix an h-tuple of integers $\mu = (a_1, a_2, \dots, a_h)$ with $a_1 \geq \cdots \geq a_h \geq 0$. Such a μ determines a class of modules over a discrete valuation ring (A, M), namely those of the form $\bigoplus_{i=1}^h A/M^{a_i}$. The set of such μ forms a partially ordered set.

Definition 4.4.1 (The space of infinite-level local shtukas with one leg [SW])**.** Let $\mathcal{M}_{b,\mu}$ be the (partially proper) functor on Pfd_C which assigns to R the set of exact sequences

$$0 \to \mathcal{O}_{X_{R^\flat}}^h \to \mathcal{E}_{b,R^\flat} \to i_*W \to 0,$$

where W is a $B_{\mathrm{dR}}^+(R)$-module quotient of $i^*\mathcal{E}_{b,R^\flat}$ which (at every geometric point of $\operatorname{Spa} R$) is of type $\leq \mu$.

One refers to the exact sequence above as a *modification* of $\mathcal{E}_b$ of type $\leq \mu$ which produces the trivial vector bundle.

When b is an isocrystal with slopes in $[0,1]$ and μ is *minuscule* (meaning $a_i \leq 1$ for all i), we recover the moduli space $\mathcal{M}_{\overline{H}_0}$ as above, so long as a certain compatibility is satisfied between b and μ.

The name "shtuka" recalls Drinfeld's constructions for a smooth projective curve over a finite field [Dri80]. Drinfeld defined a space of rank 2 shtukas and studied the cohomology of this space, and in doing so proved the Langlands conjectures for GL_2 over a function field. This was generalized to GL_n by L. Lafforgue [Laf02]. (There is a strong but highly non-obvious analogy between the two sorts of shtukas.)

Theorem 4.4.2 ([SW])**.** *The sheaf $\mathcal{M}_{b,\mu}$ is a diamond.*

The idea is that $\mathcal{M}_{b,\mu}$ admits a pro-étale morphism to the space of possible Ws, which is a kind of flag variety; one wants to show that this latter space is a diamond. For this it helps to know that $B^+_{\mathrm{dR}}/\operatorname{Fil}^i$ is a diamond, which is Theorem 4.3.3. (More details are supplied by the lecture notes of Kedlaya in this series.)

It therefore makes sense to consider the étale cohomology of the $\mathcal{M}_{b,\mu}$, and to pose generalizations of the Kottwitz conjecture for it. The construction of the $\mathcal{M}_{b,\mu}$ answers a question of Rapoport–Viehmann about the existence of "local Shimura varieties" [RV14].

A geometric Langlands program for p-adic fields. Let X be a smooth projective curve over a finite field k, with function field K. The set

$$\prod_{x\in|X|} \mathrm{GL}_n(K_x^\circ)\backslash \mathrm{GL}_n(\mathbf{A}_K)/\mathrm{GL}_n(K)$$

has two interpretations: (1) it classifies the set of isomorphism classes of rank n vector bundles on X, and (2) functions on this set are automorphic forms on K of level 1. Now, automorphic forms on K of level 1 which are Hecke eigenforms are supposed to correspond to n-dimensional Galois representations of K which are unramified everywhere, which is to say, rank n local systems on X.

The idea behind *geometric Langlands* is to geometrize the above statement, along the lines of the function-sheaf correspondence of Grothendieck. The set $\prod_{x\in|X|} \mathrm{GL}_n(K_x^\circ)\backslash \mathrm{GL}_n(\mathbf{A}_K)/\mathrm{GL}_n(K)$ is the set of k-points of the stack Bun_n which classifies vector bundles of rank n. Instead of considering functions on this set, we consider $\overline{\mathbf{Q}}_\ell$-sheaves on Bun_n.

The Hecke operators from the usual theory get geometrized as well. The stack Bun_n admits Hecke correspondences indexed by n-tuples $\mu = (a_1, \dots, a_n)$, with $a_1 \geq \cdots \geq a_n$. For each such μ, there is a diagram of stacks

$$\begin{tikzcd} & \mathrm{Hecke}_\mu \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times X. \end{tikzcd}$$

Here Hecke_μ classifies pairs of rank n vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$, together with a modification of $\mathcal{E}_2$ at a point $P \in X$ which produces $\mathcal{E}_1$; the morphisms h_1 and h_2 take such a datum to $\mathcal{E}_1$ and $(\mathcal{E}_2, P)$, respectively. The Hecke operator $\mathcal{H}_\mu$ inputs a sheaf on Bun_n and outputs a sheaf on $\mathrm{Bun}_n \times X$. In the case that μ is minuscule (meaning all a_i are 0 or 1), then $\mathcal{H}_\mu(\mathcal{F})) = (h_2)_! h_1^* \mathcal{F}$.

Theorem 4.4.3 ([FGV02]). *For every irreducible and everywhere unramified ℓ-adic representation $\phi\colon \mathrm{Gal}(K^s/K) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$, there exists a nonzero perverse sheaf $\mathcal{F}_\phi$ on Bun_n, which is a Hecke eigensheaf with respect to ϕ in the following sense: for all μ, $\mathcal{H}_\mu(\mathcal{F}) \cong \mathcal{F} \boxtimes (r_\mu \circ \phi)$, where r_μ is the algebraic representation of GL_n with highest weight μ.*

There is a marvelous suite of conjectures due to Fargues [Far] which replaces X with the Fargues–Fontaine curve in the above discussion. In this context we define the stack Bun_n as the sheaf on Pfd which assigns to a perfectoid $\mathbf{F}_p$-algebra R the groupoid of rank n vector bundles on X_R.

Theorem 4.4.4 ([FS]). *The sheaf* Bun_n *is a* smooth Artin stack *in the category of perfectoid spaces: it admits a smooth surjective morphism from a smooth diamond.*

As before, the stack Bun_n admits Hecke correspondences. For each μ, there is a corresponding Hecke operator H_μ which inputs a sheaf on Bun_n and outputs a sheaf on $\mathrm{Bun}_n \times \mathrm{Spd}\, \mathbf{Q}_p$. Part of Fargues' conjecture is the following.

Conjecture 4.4.5. *Let $\phi\colon W_{\mathbf{Q}_p} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ be an irreducible ℓ-adic representation. There exists a nonzero perverse sheaf $\mathcal{F}_\phi$ on Bun_n such that for all μ we have $H_\mu(\mathcal{F}_\phi) \cong \mathcal{F}_\phi \otimes (r_\mu \circ \phi)$.*

There is a connection between the Hecke operators H_μ and spaces of shtukas $\mathcal{M}_{b,\mu}$, and in fact the full statement Fargues' conjecture implies the generalized Kottwitz conjecture for $\mathcal{M}_{b,\mu}$ in the case that b is basic.

5 Projects

5.1 Basic examples of adic spaces

1. Classify points in $\mathrm{Spa}\, \mathbf{Q}_p\langle T\rangle$; describe the set-theoretic fibers of $\mathrm{Spa}\, \mathbf{C}_p\langle T\rangle \to \mathrm{Spa}\, \mathbf{Q}_p\langle T\rangle$.
2. Classify points in $\mathrm{Spa}\, W(C^\circ)$, where C is an algebraically closed perfectoid field of characteristic $p > 0$.

5.2 Perfectoid fields

1. Let $K = \mathbf{Q}_2(2^{1/2^\infty})^\wedge$. Identify $K^\flat$ with $\mathbf{F}_2(\!(t^{1/2^\infty})\!)$, where t corresponds to the sequence $(2, 2^{1/2}, \dots)$. Let $L = K(\sqrt{-1})$, so that L/K has degree 2. Thus L is perfectoid. Identify $L^\flat$ as a separable extension of $\mathbf{F}_2(\!(t^{1/p^\infty})\!)$. Repeat for all other quadratic field extensions of K.

2. Let K be a perfectoid field with residue field k. Show that $K^\flat \cong k(\!(t^{1/p^\infty})\!)$ if and only if the following criterion holds: K admits no proper perfectoid subfields with the same residue field and value group.

5.3 Some commutative algebra

1. Let K be a perfectoid field. Describe the group of units in $K\langle T^{1/p^\infty}\rangle$.

2. Let C be an algebraically closed perfectoid field of characteric p, and let $f \in C\langle T^{1/p^\infty}\rangle$ be a non-unit. Let $D = \{|x| \leq 1\} \subset C$. Does there always exist $\alpha \in D$ with $f(\alpha) = 0$? Is the set of zeros of f finite? Profinite? Which subsets of D are zero sets of such f?

3. Is there a generalization of the preceding exercise in characteristic 0?

4. Continuing this theme, let C be an algebraically closed perfectoid field of characteristic p, and let $f_1, \dots, f_m \in A = C\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle$ be elements which do not generate the unit ideal. Does there exist a common zero of the f_i in D^n? (This is something like a perfectoid Nullstellensatz statement.)

5.4 Closed subsets of adic spaces

For a scheme X, a closed subset $T \subset X$ is (rather by definition) Zariski closed: it is the zero locus of an ideal sheaf in $\mathcal{O}_X$. There is a scheme, the *reduced induced subscheme* Z, and a closed immersion $Z \to X$ whose set-theoretic image is T. This property is universal: for a reduced scheme Y, a morphism $f \colon Y \to X$ has $f(Y) \subset T$ (set-theoretically) if and only if f factors as $Y \to Z \to X$.

It is quite different with adic spaces. One difference is that closed subsets are not necessarily Zariski-closed.

1. Consider $\mathbf{Q}_p$ as a closed subset of the underlying topological space of $\mathbf{A}^1$, considered as an adic space over $\mathbf{Q}_p$. Show that $\mathbf{Q}_p$ is not Zariski closed.

2. Nonetheless, show that there exists a reduced adic space Z and a morphism $Z \to \mathbf{A}^1$, which is a monomorphism and has image $\mathbf{Q}_p$, and which satisfies a universal property.

Let $H/\overline{\mathbf{F}}_p$ be a formal p-divisible group of height 2 and dimension 1. Its universal cover $\widetilde{H}$ lifts to a formal $\mathbf{Q}_p$-vector space over $\breve{\mathbf{Z}}_p = W(\overline{\mathbf{F}}_p)$; let $\widetilde{H}_{\breve{\mathbf{Q}}_p}$ be its generic fiber. Then $\widetilde{H}_{\breve{\mathbf{Q}}_p}$ is a preperfectoid space. Let $M(H)$ be the Dieudonné module of H; this is a free $\breve{\mathbf{Z}}_p$-module of rank 2. There is a quasi-logarithm map of adic spaces

$$\mathrm{qlog}_H \colon \widetilde{H}_{\breve{\mathbf{Q}}_p} \to M(H) \otimes_{\breve{\mathbf{Z}}_p} \mathbf{G}_{\mathrm{a}} \cong \mathbf{G}_{\mathrm{a}}^2$$

which respects the $\mathbf{Q}_p$-vector space structure on either side. We describe it as a natural transformation between functors from $\mathrm{Pfd}_{\breve{\mathbf{Q}}_p}$ to $\mathbf{Q}_p$-vector spaces. Let

R be a perfectoid $\breve{\mathbf{Q}}_p$-algebra. We have an isomorphism $\widetilde{H}_{\breve{\mathbf{Q}}_p}(R) = \widetilde{H}(R^\circ) \cong (B(R^\flat) \otimes_{\breve{\mathbf{Z}}_p} M(H))^{\phi=1}$. Then $\mathrm{qlog}_H(R)$ is the composition of this map with $\theta_R \otimes 1\colon B(R^\flat) \otimes_{\breve{\mathbf{Z}}_p} M(H) \to R \otimes_{\breve{\mathbf{Z}}_p} M(H)$.

3. Prove that qlog is a monomorphism.

4. Let Z be the image of qlog_H, considered as a subset of the underlying topological space of $\mathbf{G}_{\mathrm{a}}^2$. Show that Z is closed and generalizing.

5. Show that the residue fields of nonzero points of Z are never finite extensions of $\breve{\mathbf{Q}}_p$. That is, the image of qlog_H contains no "classical points" other than the origin.

6. Show that if Y is a perfectoid space over $\mathrm{Spa}\,\breve{\mathbf{Q}}_p$ and $f\colon Y \to \mathbf{G}_{\mathrm{a}}^2$ has set-theoretic image contained in Z, then f factors through qlog_H.

Thus we have a closed subset of the adic space $\mathbf{G}_{\mathrm{a}}^2$ which (considered as a subfunctor on the category of perfectoid spaces) is representable by a preperfectoid space. In fact, it is a theorem of Scholze [Sch17] that any closed generalizing subset of a diamond, when considered as a subfunctor on the category of perfectoid spaces, is itself a diamond.

5.5 Computations with Banach-Colmez spaces

Recall our discussion of Banach-Colmez spaces, which are sheaves of $\mathbf{Q}_p$-algebras on the category of perfectoid spaces. There are two projects here. The first has to do with some "ineffective" Banach-Colmez spaces. Fix an algebraically closed perfectoid fied C of characteristic 0.

1. We begin with the space $H^1(\mathcal{O}_X(-1))$, which inputs a perfectoid C-algebra R and outputs the $\mathbf{Q}_p$-vector space $H^1(X_R, \mathcal{O}_X(-1))$. Show that there is an isomorphism of sheaves of $\mathbf{Q}_p$-vector spaces on Pfd_C:
$$H^1(\mathcal{O}_X(-1)) \cong \mathbf{G}_{\mathrm{a}}/\underline{\mathbf{Q}}_p.$$

2. The sheaf $H^1(\mathcal{O}_X(-1))$ parametrizes extension classes
$$0 \to \mathcal{O}_X(-1) \to \mathcal{E} \to \mathcal{O}_X \to 0,$$
or (after twisting by $\mathcal{O}_X(1)$) extension classes
$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X(1) \to 0.$$
Show that if this latter extension is nonsplit, then there exists an isomorphism $\mathcal{E} \cong \mathcal{O}_X(1/2)$. Recall that global sections of $\mathcal{O}_X(1/2)$ are representable by a formal scheme $\widetilde{H}$, where $H/\bar{\mathbf{F}}_p$ is a formal p-divisible group

of dimension 1 and height 2. Let us abbreviate $\widetilde{H}_C^* = \widetilde{H}_C \setminus \{0\}$; this is a perfectoid space over C. Show that there is an isomorphism

$$H^1(\mathcal{O}_X(-1)) \setminus \{0\} \cong (\widetilde{H}_C^* \times \underline{\mathbf{Q}}_p(1)^*)/D^\times,$$

where $D = \operatorname{Aut}^0 H$ is the nonsplit quaternion algebra over $\mathbf{Q}_p$, where $\mathbf{Q}_p(1)^* = \mathbf{Q}_p(1) \setminus \{0\}$, and where $D^\times$ acts on $\mathbf{Q}_p(1)^*$ through the reduced norm map.

3. Let $\Omega = \mathbf{G}_a \setminus \underline{Q}_p$. Combining the previous two exercises gives an isomorphism $\Omega/\mathbf{Q}_p \cong (\widetilde{H}_C^* \times \underline{\mathbf{Q}}_p(1)^*)/D^\times$. This isomorphism means there is a diamond M carrying an action of $\mathbf{Q}_p \times D^\times$, whose quotient by $D^\times$ is Ω, and whose quotient by $\mathbf{Q}_p$ is $\widetilde{H}_C^* \times \mathbf{Q}_p(1)^*$. Show that M (with this action) is isomorphic to the Lubin-Tate tower for $\mathrm{GL}_2(\mathbf{Q}_p)$.

4. Is there a similar story for $H^1(X, \mathcal{O}_X(\lambda))$ for other negative values of $\lambda \in \mathbf{Q}$?

The other project is due to David Hansen. Let $M \to \operatorname{Spd} C$ be the infinite-level Lubin–Tate tower for $\mathrm{GL}_2(\mathbf{Q}_p)$. Then M can be interpreted as the space of "mixed-characteristic shtukas" of a certain type. To wit, M is the sheafification of the presheaf which assigns to a perfectoid C-algebra R, the set of exact sequences of the form

$$0 \to \mathcal{O}_{X_{R^\flat}}^2 \to \mathcal{O}_{X_{R^\flat}}(1/2) \to i_* W \to 0,$$

where $i\colon \operatorname{Spec} B_{\mathrm{dR}}(R) \to X_{R^\flat}$ is the usual morphism, and W is a rank 1 projective quotient of $i^* \mathcal{O}_{X_{R^\flat}}(1/2)$. Then M admits an action of the product group $\mathrm{GL}_2(\mathbf{Q}_p) \times D^\times$, where $D = \operatorname{Aut}_{\mathcal{O}_X} \mathcal{O}_X(1/2)$ is the nonsplit quaternion algebra over $\mathbf{Q}_p$.

Here is a different space of shtukas, which we'll call N: it is the sheafification of the presheaf which assigns to a perfectoid C-algebra R the set of exact sequences of the form

$$0 \to \mathcal{O}_{X_{R^\flat}}^2 \to \mathcal{O}_{X_{R^\flat}}(1)^2 \to i_* V \to 0,$$

where this time V is a projective $B_{\mathrm{dR}}^+(R^\flat)/(\ker \theta_R)^2$-module of rank 1. Then N admits an action of $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p)$.

5. Show that N is a perfectoid space.

6. Show that, in the category of diamonds admitting an action of $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p)$, the sheaf N is isomorphic to the quotient $(M \times M)/D^\times$, where the action of $D^\times$ is the diagonal one.

7. Are there other isomorphisms of these type, for different spaces of shtukas?

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