Japan J. Math.
Vol. 3, No. 1, 1977

# Ring Spectra with Coefficients in $V(1)$ and $V(2)$, I

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(Received April 30, 1976)

For a (reduced) cohomology theory $h$ the mod $q$ cohomology theory $h(\ ;Z_q)$ is defined by $h^*(X;Z_q)=h^{*+2}(X\wedge M_q)$ where $M_q$ is a co-Moore space of type $(Z_q,2)$. By the representability theorem any (multiplicative) cohomology theory $h$ is represented by a certain (ring) spectrum $E$. $\Sigma^{-1}M_q$ is a Moore spectrum of type $Z_q$, so we put $V_q(0)=\Sigma^{-1}M_q$. Since $V_q(0)$ is self dual, $E\wedge V_q(0)$ is a represented spectrum of $h(\ ;Z_q)$ so that $h^*(X;Z_q)\cong\{X,E\wedge V_q(0)\}_{-*}$. In [1] Araki-Toda discussed the multiplicative structure in mod $q$ cohomology theories. In other words they investigated several conditions on a ring spectrum $E$ under which $E\wedge V_q(0)$ is a nice ring spectrum.

Let $p$ be a fixed prime. A spectrum $V(n)$ is defined to be a finite $CW$-spectrum having $H^*(V(n);Z_p)\cong E(Q_0,Q_1,\cdots,Q_n)$ as a module over the mod $p$ Steenrod algebra where $Q_i$ are Milnor elements. For example, we can take as $V(0)$ a Moore spectrum of type $Z_p$, i.e., $V(0)=V_p(0)$, and the existence of $V(n)$ is assured for $n=1$, $p\geqq 3$, for $n=2$, $p\geqq 5$ and for $n=3$, $p\geqq 7$. Making use of Adams spectral sequence Toda [4] computed the homotopy groups of $V(1)$ and $V(2)$ up to some range, and he then determined the structure of the algebra $\{V(1),V(1)\}_*$ in [5].

Let $E$ be a ring spectrum equipped with a multiplication $\mu$ and a unit $\iota$. The purpose of the present work is to give conditions on $E$ under which $E\wedge V(1)$ and $E\wedge V(2)$ are nice ring spectra (Theorem 4.2), by means of Toda's computations. In § 1 we restate several results of Araki-Toda [1], mainly existence theorems of admissible multiplications for $E\wedge V(0)$, but they are presented here in terms of the stable homotopy category of $CW$-spectra. If $p\geqq 3$, then $V(0)$ becomes a ring spectrum which admits a unique multiplication $\psi$. In § 2 we first give a condition under which $E\wedge V(1)$ has a multiplication whose restriction to $E\wedge V(0)$ is $(\mu\wedge\psi)(1\wedge T\wedge 1)$ where $T$ denotes the map switching two factors. We next study a condition for the commutativity of $E\wedge V(1)$. In particular, when $p\geqq 5$ $V(1)$ is a ring spectrum whose multiplication $\psi_{1,1}$ is a unique extension of $\psi$. In § 3 we give a condition under which $E\wedge V(2)$ has a multiplication whose restriction to $E\wedge V(1)$ is induced by $\mu$ and $\psi_{1,1}$, and then discuss the commutativity of $E\wedge V(2)$.

In § 4 we show that in the $p\geqq 3$ cases $BP\wedge V(n)$ are associative and commutative ring spectra for the Brown-Peterson spectrum $BP$ (Theorem 4.7), although it seems difficult to investigate the associativity of $E\wedge V(1)$ and $E\wedge V(2)$ for a general $E$. We can construct a certain $CW$-spectrum $P(n)$ using the Baas-Sullivan technique of defining bordism theories with singularities (see [2]). Since $BP\wedge V(n)$ is isomorphic to $P(n+1)$, the above result means that $P(n+1)$ is an associative and commutative ring spectrum if $p\geqq 3$ and $V(n)$ exists (Theorem 4.10). In appendix we show that $P(n)$ is always a ring spectrum even if $V(n)$ does not exist, and in addition that it is commutative for $p\geqq 3$.

In this note we shall work in the stable homotopy category of $CW$-spectra.

The authors wish to thank Professor S. Oka for his kind advices.

## § 1. Admissible multiplications of $E\wedge V(0)$

**1.1.** Let us fix a prime $p$ and denote by $V(0)$ the Moore spectrum of type $Z_p$, so we have a cofibering

$$\Sigma^0 \xrightarrow{p} \Sigma^0 \xrightarrow{i} V(0) \xrightarrow{\pi} \Sigma^1.$$

A $CW$-spectrum $X$ is called a *$Z_p$-spectrum* if the identity $1_X: X\to X$ has order $p$. Thus a $Z_p$-spectrum $X$ is equipped with two maps

$$\psi_X: X\wedge V(0)\longrightarrow X \quad \text{and} \quad \phi_X: \Sigma^1 X\longrightarrow X\wedge V(0)$$

satisfying the equalities

$$\begin{gathered}\psi_X\cdot\phi_X=0,\\ \psi_X(1\wedge i)=(1\wedge\pi)\phi_X=1_X \quad \text{and} \quad (1\wedge i)\psi_X+\phi_X(1\wedge\pi)=1_{X\wedge V(0)}.\end{gathered} \tag{1.1}$$

Remark that $\psi_X$ and $\phi_X$ are uniquely determined when $\{\Sigma^1 X, X\}=0$.

It is well known that

$$\begin{aligned} &p\cdot 1_{V(0)}=0 && \textit{if } p \textit{ is odd, but}\\ &p\cdot 1_{V(0)}=i\cdot\eta\cdot\pi\neq 0 && \textit{if } p=2,\end{aligned} \tag{1.2}$$

where $\eta: \Sigma^1\to\Sigma^0$ is the Hopf map [1, Theorem 1.1]. This means that $V(0)$ is a $Z_p$-spectrum for an odd $p$, but not so for $p=2$. Let $N$ denote the mapping cone of the composition $i\cdot\eta: \Sigma^1\to V(0)$. By Verdier's lemma (see [3]) we then have a cofibering

$$\Sigma^1 \xrightarrow{j_0} V(0)\wedge V(0) \xrightarrow{k_0} N \xrightarrow{p\cdot\pi_N} \Sigma^2$$

making the diagram below commutative

$$
\begin{array}{ccccccc}
 & & & & \Sigma^1 & = & \Sigma^1 \\
 & & & & \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle i} \\
V(0) & \xrightarrow{p} & V(0) & \xrightarrow{1\wedge i} & V(0)\wedge V(0) & \xrightarrow{1\wedge\pi} & \Sigma^1 V(0) \\
\downarrow{\scriptstyle \pi} & & \| & & \downarrow{\scriptstyle k_0} & & \downarrow{\scriptstyle \pi} \\
\Sigma^1 & \xrightarrow[i\eta]{} & V(0) & \xrightarrow[i_N]{} & N & \xrightarrow[\pi_N]{} & \Sigma^2
\end{array}
$$

in which the right-lower square commutes up to the sign $-1$.

For any $Z_p$-spectrum $Y$ $(1\wedge\pi)^*: \{\Sigma^1 X, Y\}\to\{X\wedge V(0), Y\}$ is monic. Hence

LEMMA 1.1. *$X\wedge V(0)$ is a $Z_p$-spectrum if and only if $1_X\wedge i\cdot\eta: \Sigma^1 X\to X\wedge V(0)$ is trivial.*

We say that a map $\gamma: X\wedge V(0)\wedge V(0)\to X\wedge V(0)$ is a *pre multiplication of* $X\wedge V(0)$ if it satisfies $\gamma(1\wedge 1\wedge i)=\gamma(1\wedge i\wedge 1)=1$. Assume that $X\wedge V(0)$ is a $Z_p$-spectrum, so we have a left inverse $\gamma_N: X\wedge N\to X\wedge V(0)$ of $1\wedge i_N$. Making use of this left inverse we define a map

$$\gamma_0: X\wedge V(0)\wedge V(0)\longrightarrow X\wedge V(0)$$

as the composition $\gamma_0=\gamma_N(1\wedge k_0)$.

LEMMA 1.2. *The map $\gamma_0$ is a pre multiplication of $X\wedge V(0)$.*

PROOF. The difference $i\wedge 1-1\wedge i$ belongs to $\pi^*\{\Sigma^1, V(0)\wedge V(0)\}=\pi^* j_{0*}\{\Sigma^1, \Sigma^1\}$ as $\{\Sigma^1, N\}=0$. So we get immediately

$$\gamma_N(1\wedge k_0)(1\wedge i\wedge 1)=\gamma_N(1\wedge k_0)(1\wedge 1\wedge i)=\gamma_N(1\wedge i_N)=1.$$

The above result means that

(1.3) *$X\wedge V(0)$ is a $Z_p$-spectrum if and only if it has a pre multiplication.*

For two pre multiplications $\gamma$, $\gamma'$ of $X\wedge V(0)$ we can choose a map $b: X\wedge\Sigma^1 V(0)\to X\wedge V(0)$ such that $\gamma-\gamma'=b(1\wedge 1\wedge\pi)$. Obviously $b(1\wedge i)(1\wedge\pi)=0$ and hence $b(1\wedge i)=0$ because $p\{\Sigma^1 X, X\wedge V(0)\}=0$. Consequently there exists a unique map

$$B(\gamma,\gamma'): \Sigma^2 X\longrightarrow X\wedge V(0) \tag{1.4}$$

so that $\gamma-\gamma'=B(\gamma,\gamma')(1\wedge\pi\wedge\pi)$. $B(\gamma,\gamma')$ measures the difference of pre multiplications $\gamma$ and $\gamma'$.

Let $E$ be a ring spectrum, i.e., it has given maps $\mu: E\wedge E\to E$ and $\iota: \Sigma^0\to E$ such that $\mu(1\wedge\iota)=\mu(\iota\wedge 1)=1$. Every pre multiplication $\gamma$ of $E\wedge V(0)$

gives rise to a map

$$\mu_\gamma: E\wedge V(0)\wedge E\wedge V(0)\longrightarrow E\wedge V(0)$$

defined by the composition $\mu_\gamma=\gamma(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)$ where $T$ is the map switching factors. This map satisfies the property

$$(\Lambda_1)\qquad \mu_\gamma(1\wedge T\wedge 1)(1\wedge 1\wedge 1\wedge i)=\mu_\gamma(1\wedge T\wedge 1)(1\wedge 1\wedge i\wedge 1)=\mu\wedge 1.$$

Therefore this gives $E\wedge V(0)$ the structure of a ring spectrum having $\iota\wedge i$ as the unit. On the other hand, each multiplication $\tilde{\mu}$ of $E\wedge V(0)$ satisfying $(\Lambda_1)$ yields a pre multiplication $\gamma_{\tilde{\mu}}$ by putting $\gamma_{\tilde{\mu}}=\tilde{\mu}(1\wedge 1\wedge\iota\wedge 1)$. This correspondence is a left inverse of the previous $\gamma\to\mu_\gamma$.

PROPOSITION 1.3. *Let $E$ be a ring spectrum. The following conditions are equivalent:*

i) *$E\wedge V(0)$ is a $Z_p$-spectrum,*

ii) *$E\wedge V(0)$ has a multiplication satisfying $(\Lambda_1)$, and*

iii) *$E\wedge V(0)$ is a ring spectrum with the unit $\iota\wedge i$.*

PROOF. The above observations show the implications i)→ii)→iii), and iii)→i) is immediate.

By the same argument as (1.4) we obtain a unique map

$$B(\tilde{\mu},\tilde{\mu}'): \Sigma^2E\wedge E\longrightarrow E\wedge V(0) \tag{1.5}$$

so that $\tilde{\mu}-\tilde{\mu}'=B(\tilde{\mu},\tilde{\mu}')(1\wedge\pi\wedge 1\wedge\pi)$ for two multiplications $\tilde{\mu},\tilde{\mu}'$ of $E\wedge V(0)$ satisfying $(\Lambda_1)$.

LEMMA 1.4. *If a multiplication $\tilde{\mu}$ of $E\wedge V(0)$ satisfies the property $(\Lambda_1)$, then there exists a unique map $\tilde{\gamma}_N; E\wedge E\wedge N\to E\wedge V(0)$ such that $\tilde{\mu}(1\wedge T\wedge 1)=\tilde{\gamma}_N(1\wedge 1\wedge k_0)$.*

PROOF. Take a left inverse $\gamma_N$ of $1\wedge i_N$ and fix our multiplication $\mu_0=\mu_{\gamma_0}$ associated with the pre multiplication $\gamma_0=\gamma_N(1\wedge k_0)$. Since

$$\begin{aligned}&\tilde{\mu}(1\wedge T\wedge 1)(1\wedge 1\wedge j_0)\\&\quad=\mu_0(1\wedge T\wedge 1)(1\wedge 1\wedge j_0)+B(\tilde{\mu},\mu_0)(1\wedge 1\wedge\pi\wedge\pi)(1\wedge 1\wedge j_0)\\&\quad=0,\end{aligned}$$

we can find a required map which is unique.

A similar discussion to the above shows that

(1.6) *every pre multiplication $\gamma$ of $X\wedge V(0)$ admits a factorization $\gamma=\gamma_N(1\wedge k_0)$.*

**1.2.** We put $\rho=\eta$ in the $p=2$ case and $\rho=0$ in the other cases and denote by $P$ its mapping cone. There exists a cofibering

$$\Sigma^0 \xrightarrow{p\cdot i_P} P \xrightarrow{j_N} N \xrightarrow{k_N} \Sigma^1$$

so that the diagram below is commutative

$$\begin{array}{ccccccc}
\Sigma^1 & \xrightarrow{\rho} & \Sigma^0 & \xrightarrow{i_P} & P & \xrightarrow{\pi_P} & \Sigma^2 \\
\| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j_N} & & \| \\
\Sigma^1 & \xrightarrow[i\eta]{} & V(0) & \xrightarrow[i_N]{} & N & \xrightarrow[\pi_N]{} & \Sigma^2. \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle k_N} & & \\
 & & \Sigma^1 & = & \Sigma^1 & &
\end{array}$$

Take a map $k: N\to\Sigma^1$ such that $(1\wedge\pi)(1+T)=i\cdot k\cdot k_0$ as $\pi_*(1\wedge\pi)(1+T)=0$ and $k_0^*: \{N, \Sigma^1\}\to\{V(0)\wedge V(0), \Sigma^1\}$ is epic. Setting $k\cdot j_N=a\eta\cdot\pi_P$, $a\in Z_2$, where $a=0$ in the $p=2$ case, the map $k$ is expressed as a sum $k=a\eta\cdot\pi_N+bk_N$, $b\in Z$. Therefore $(1\wedge\pi)(1+T)=bi\cdot k_N\cdot k_0$. Applying $(1\wedge i)^*$ on both sides we get $b\equiv 1 \bmod p$. Thus

(1.7) $$(1\wedge\pi)(1+T)=i\cdot k_N\cdot k_0.$$

Let $D$ be the Moore spectrum of type $Z_{p^2}$ and $j: \Sigma^0\to D$ be the canonical inclusion. Then we have a cofibering

$$\Sigma^{-1}V(0) \xrightarrow{\delta} V(0) \xrightarrow{i_D} D \xrightarrow{\pi_D} V(0)$$

so that $p\cdot j=i_D\cdot i$ and $\pi_D\cdot j=i$, corresponding to the short exact sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$. Put $\rho_0=j\cdot\eta$ in the $p=2$ case and $\rho_0=0$ in the other cases. Denoting by $Q$ its mapping cone there exists a commutative diagram

$$\begin{array}{ccccccc}
\Sigma^1 & \xrightarrow{\rho_0} & D & \xrightarrow{i_Q} & Q & \xrightarrow{\pi_Q} & \Sigma^2 \\
\| & & \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle j'_N} & & \| \\
\Sigma^1 & \xrightarrow[i\eta]{} & V(0) & \xrightarrow[i_N]{} & N & \xrightarrow[\pi_N]{} & \Sigma^2 \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle k'_N} & & \\
 & & \Sigma^1 V(0) & = & \Sigma^1 V(0) & &
\end{array}$$

consisting of four cofiberings. The above $k'_N$ coincides with the composition $i\cdot k_N$ as $k'_N-i\cdot k_N$ belongs to $\pi_N^*\{\Sigma^1, V(0)\}=0$.

Let $E$ be a ring spectrum such that $1\wedge\rho_0: \Sigma^1E\to E\wedge D$ is trivial. For any left inverse $\gamma_Q: E\wedge Q\to E\wedge D$ of $1\wedge i_Q$ we now construct a left inverse $\gamma_N: E\wedge N\to E\wedge V(0)$ of $1\wedge i_N$ which is compatible with it. Considering the diagram

$$(1.8)\qquad \begin{array}{ccccccc} E\wedge V(0) & \xrightarrow{1\wedge i_Q i_D} & E\wedge Q & \xrightarrow{1\wedge j'_N} & E\wedge N & \xrightarrow{1\wedge k'_N} & E\wedge\Sigma^1 V(0) \\ \| & & \downarrow{\scriptstyle \gamma_Q} & & & & \| \\ E\wedge V(0) & \xrightarrow[1\wedge i_D]{} & E\wedge D & \xrightarrow[1\wedge\pi_D]{} & E\wedge V(0) & \xrightarrow[1\wedge\delta]{} & E\wedge\Sigma^1 V(0) \end{array}$$

with two cofiberings, we get a map $\gamma': E\wedge N\to E\wedge V(0)$ which makes the entire diagram commute. Five lemma shows that $\gamma'(1\wedge i_N)$ is a homotopy equivalence. So we put $\gamma_N=\{\gamma'(1\wedge i_N)\}^{-1}\cdot\gamma'$, which makes the above diagram commutative again and satisfies $\gamma_N(1\wedge i_N)=1$.

Consider the multiplication $\mu_0$ of $E\wedge V(0)$ associated with the pre multiplication $\gamma_0=\gamma_N(1\wedge k_0)$. This satisfies the property

$$(\Lambda_2)'\qquad (1\wedge\delta)\mu_0=(1\wedge 1\wedge\pi)(1+1\wedge T)(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)$$

because of (1.7). In other words the equality

$$(\Lambda_2)\qquad (1\wedge\delta)\mu_0=\mu_0(1\wedge\delta\wedge 1\wedge 1)+\mu_0(1\wedge 1\wedge 1\wedge\delta)$$

holds. Thus $1\wedge\delta$ behaves as a derivation.

PROPOSITION 1.5. *Let $E$ be a ring spectrum. In the $p=2$ case $E\wedge V(0)$ has a multiplication satisfying $(\Lambda_1)$ and $(\Lambda_2)$ if and only if $1\wedge j\cdot\eta:\Sigma^1 E\to E\wedge D$ is trivial. In the other cases $E\wedge V(0)$ has always a multiplication satisfying $(\Lambda_1)$ and $(\Lambda_2)$.*

PROOF. Our multiplication $\mu_0$ constructed suitably as the previous satisfies the properties $(\Lambda_1)$ and $(\Lambda_2)$. We next assume that there exists a multiplication $\tilde\mu$ of $E\wedge V(0)$ with the two properties when $p=2$. Lemma 1.4 says that $\tilde\mu$ has a factorization $\tilde\mu=\tilde\gamma_N(1\wedge 1\wedge k_0)(1\wedge T\wedge 1)$. By use of (1.7) the equality $(\Lambda_2)'$ yields

$$\begin{aligned}(1\wedge\delta)\tilde\gamma_N(1\wedge 1\wedge k_0)&=(1\wedge 1\wedge\pi)(1+1\wedge T)(\mu\wedge 1\wedge 1)\\ &=(1\wedge i)(1\wedge k_N)(1\wedge k_0)(\mu\wedge 1\wedge 1).\end{aligned}$$

This then implies that $(1\wedge\delta)\tilde\gamma_N=(1\wedge i)(1\wedge k_N)(\mu\wedge 1)$ as $2\{\Sigma^1 E\wedge E, E\wedge V(0)\}=0$. Putting $\gamma_N=\tilde\gamma_N(\iota\wedge 1\wedge 1)$, it is a left inverse of $1\wedge i_N$ which has $(1\wedge\delta)\gamma_N=1\wedge k'_N$. By the same argument as (1.8) we can find a left inverse $\gamma_Q$ of $1\wedge i_Q$ such that $(1\wedge\pi_D)\gamma_Q=\gamma_N(1\wedge j'_N)$. Hence $1\wedge j\cdot\eta:\Sigma^1 E\to E\wedge Q$ is trivial.

**1.3.** Let $E$ be a ring spectrum and $f: A\to B$ be a map which induces the trivial $1\wedge f: E\wedge A\to E\wedge B$. Denote by $C$ the mapping cone of the map $f$, so we have a cofibering

$$A\xrightarrow{f}B\xrightarrow{i_C}C\xrightarrow{\pi_C}\Sigma^1 A.$$

For any $\xi: \Sigma^1 A \to E\wedge C$ with $(1\wedge\pi_C)\xi=\iota\wedge 1$ we define a left inverse of $1\wedge i_C$

$$\gamma_\xi: E\wedge C \longrightarrow E\wedge B$$

by the formula $(1\wedge i_C)\gamma_\xi=1-(\mu\wedge 1)(1\wedge\xi)(1\wedge\pi_C)$. As is easily checked, the correspondence $\xi\to\gamma_\xi$ has a left inverse and hence it is injective.

LEMMA 1.6. *Let $\xi: \Sigma^1 A\to E\wedge C$ be a map such that $(1\wedge\pi_C)\xi=\iota\wedge 1$.*

i) *If $E$ is associative, then the relation $\gamma_\xi(\mu\wedge 1)=(\mu\wedge 1)(1\wedge\gamma_\xi)$ holds.*

ii) *If $E$ is associative and commutative, then the relation $\gamma_\xi(\mu\wedge 1)(1\wedge T)$ $=(\mu\wedge 1)(1\wedge T)(\gamma_\xi\wedge 1)$ holds.*

PROOF. Under our assumptions a routine computation shows that

$$(1\wedge i_C)\gamma_\xi(\mu\wedge 1)=(\mu\wedge 1)(1\wedge 1\wedge i_C)(1\wedge\gamma_\xi)$$

and

$$(1\wedge i_C)\gamma_\xi(\mu\wedge 1)(1\wedge T)=(\mu\wedge 1)(1\wedge T)(1\wedge i_C\wedge 1)(\gamma_\xi\wedge 1).$$

Let $E$ be an associative ring spectrum such that $E\wedge V(0)$ is a $Z_p$-spectrum. Take a map $\xi: \Sigma^2\to E\wedge N$ satisfying $(1\wedge\pi_N)\xi=\iota$ and consider the left inverse $\gamma_\xi$ of $1\wedge i_N$ induced by the map $\xi$. This gives us a pre multiplication $\gamma_0$ by putting $\gamma_0=\gamma_\xi(1\wedge k_0)$. Note that there exists a map $\xi_0: \Sigma^2\to N$ satisfying $\pi_N\cdot\xi_0=1$ whenever $p$ is odd. By means of Lemma 1.6 we see that the above $\gamma_0$ is compatible with the multiplication $\mu$ of $E$ in the sense that

$$(A_3)' \qquad \begin{aligned}&\gamma_0(\mu\wedge 1\wedge 1)=(\mu\wedge 1)(1\wedge\gamma_0),\\ &\gamma_0(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)(1\wedge 1\wedge T)=(\mu\wedge 1)(1\wedge T)(\gamma_0\wedge 1)\end{aligned}$$

when $E$ is commutative or $\xi=(\iota\wedge 1)\xi_0$. The property $(A_3)'$ implies that the multiplication $\mu_0$ induced by the left inverse $\gamma_\xi$ is *quasi associative*, i.e.,

$$(A_3) \qquad \begin{aligned}&\mu_0(\mu\wedge 1\wedge 1\wedge 1)=(\mu\wedge 1)(1\wedge\mu_0),\\ &\mu_0(\mu\wedge 1\wedge 1\wedge 1)(1\wedge T\wedge 1\wedge 1)=\mu_0(1\wedge 1\wedge\mu\wedge 1),\end{aligned}$$

and

$$\mu_0(1\wedge 1\wedge\mu\wedge 1)(1\wedge 1\wedge 1\wedge T)=(\mu\wedge 1)(1\wedge T)(\mu_0\wedge 1).$$

A multiplication of $E\wedge V(0)$ is said to be *admissible* if it satisfies the properties $(A_1)$, $(A_2)$ and $(A_3)$ (see [1]).

PROPOSITION 1.7. *Let $E$ be an associative ring spectrum. In the $p=2$ case $E\wedge V(0)$ has an admissible multiplication if $1\wedge j\cdot\eta: \Sigma^1 E\to E\wedge D$ is trivial and $E$ is commutative. In the other cases admissible multiplications of $E\wedge V(0)$ exist always.* (Cf., [1, Theorem 5.9]).

PROOF. For any $\xi': \Sigma^2 \to E\wedge Q$ satisfying $(1\wedge\pi_Q)\xi'=\iota\wedge 1$ we put $\xi=(1\wedge j'_N)\xi'$. This determines the left inverse $\gamma_\xi$ of $1\wedge i_N$, which satisfies $\gamma_\xi(1\wedge j'_N)=(1\wedge\pi_D)\gamma_{\xi'}$ and $(1\wedge\delta)\gamma_\xi=(1\wedge i)(1\wedge k_N)$. When $p$ is odd we can take the composition $(\iota\wedge 1)\xi'_0$ as $\xi'$ where $\xi'_0: \Sigma^2\to Q$ satisfies $\pi_Q\cdot\xi'_0=1$. Therefore our multiplication $\mu_0=\gamma_\xi(1\wedge k_0)(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)$ is admissible.

REMARK. Araki-Toda [1, Corollary 3.9] showed that admissible multiplications $\tilde\mu$, $\tilde\mu'$ of $E\wedge V(0)$ coincide if and only if

$$B(\tilde\mu, \tilde\mu')(\iota\wedge\iota)=0\in\{\Sigma^2, E\wedge V(0)\}.$$

**1.4.** Taking as a ring spectrum $E$ the sphere spectrum $S$, Proposition 1.3 implies that $V(0)$ is a ring spectrum with the unit $i$ whenever $p$ is odd. Its multiplication

$$\psi: V(0)\wedge V(0)\longrightarrow V(0)$$

is unique as $\{\Sigma^1 V(0), V(0)\}=0$. $V(0)$ is commutative when $p\geqq 3$ and associative when $p\geqq 5$. However it is not associative in the $p=3$ case [4, Lemma 6.2]. Thus

$$(1.9)\qquad \begin{array}{ll} \psi\cdot T=\psi,\ \psi(\psi\wedge 1)=\psi(1\wedge\psi)+i\cdot\alpha_1(\pi\wedge\pi\wedge\pi) & \text{when } p=3,\\ \psi\cdot T=\psi,\ \psi(\psi\wedge 1)=\psi(1\wedge\psi) & \text{when } p\geqq 5,\end{array}$$

where $\alpha_1: \Sigma^3\to\Sigma^0$ is the generator of the 3-primary part (see **2.1**).

We next discuss the commutativity of $E\wedge V(0)$ for $p=2$. When $p=2$, choose maps $\bar\eta: \Sigma^1 V(0)\to\Sigma^0$ and $\tilde\eta: \Sigma^2\to V(0)$ such that $\bar\eta\cdot i=\eta$ and $\pi\cdot\tilde\eta=\eta$, then put $\eta_1=i\cdot\bar\eta$ and $\eta_2=\tilde\eta\cdot\pi$. Since $\{\Sigma^1 V(0), V(0)\}$ is generated by two $\eta_1$ and $\eta_2$, a routine computation shows that

$$\{V(0)\wedge V(0), V(0)\}\cong Z_2+Z_2+Z_2$$

with generators $\eta_1(1\wedge\pi)$, $\eta_2(1\wedge\pi)$ and $i\cdot\eta\cdot k_N\cdot k_0$.

Put

$$k_0(T-1)=ai_N\eta_1(1\wedge\pi)+bi_N\eta_2(1\wedge\pi),\qquad a, b\in Z$$

as $\pi_N k_0(T-1)=(\pi\wedge\pi)(1-T)=0$. We use the relation $(1\wedge i)(\eta_1+\eta_2)(1\wedge\pi)=2\cdot 1_{V(0)\vee V(0)}$ to rewrite

$$k_0(T-1)=(a-b)i_N\eta_1(1\wedge\pi)+2bk_0=(b-a)i_N\eta_2(1\wedge\pi)+2ak_0.$$

We here assume $a\equiv b \bmod 2$, and set $T=(2b+1)+cj_0k_Nk_0$ for some $c\in Z_4$. Then $c\equiv 1 \bmod 2$, because $\delta=(1\wedge\pi)T(1\wedge i)=c(1\wedge\pi)j_0k_Nk_0(1\wedge i)=c\delta$. By the above setting we have

$$T^2=1+2j_0k_Nk_0$$

which implies that $2j_0k_Nk_0=(1\wedge i)_*i\cdot\eta\cdot k_Nk_0=0$. This is a contradiction. Therefore

(1.10) $\quad k_0(T-1)\equiv i_N\eta_1(1\wedge\pi)\equiv i_N\eta_2(1\wedge\pi) \mod 2\{V(0)\wedge V(0), N\}$

(cf., [1, Theorem 7.4]).

PROPOSITION 1.8. *Let $E$ be a commutative ring spectrum such that $E\wedge V(0)$ is a $Z_2$-spectrum. Then the following conditions are equivalent:*

i) *$E\wedge V(0)$ has at least one commutative multiplication satisfying $(\Lambda_1)$,*

ii) *$1\wedge\eta_1(1\wedge\pi)=1\wedge\eta_2(1\wedge\pi): E\wedge V(0)\wedge V(0)\to E\wedge V(0)$ is trivial,*

iii) *$1\wedge\bar\eta: E\wedge\Sigma^1V(0)\to E$ is trivial, and*

iv) *$1\wedge\tilde\eta: \Sigma^2E\to E\wedge V(0)$ is trivial.*

PROOF. Since $E\wedge V(0)$ is a $Z_2$-spectrum, $1\wedge\eta_1(1\wedge\pi)=1\wedge\eta_2(1\wedge\pi)$ and the conditions ii), iii) and iv) are equivalent.

i)→ii): Let $\tilde\mu$ be a commutative multiplication satisfying $(\Lambda_1)$. By virtue of Lemma 1.4 we obtain a decomposition $\tilde\mu=\tilde\gamma_N(1\wedge1\wedge k_0)(1\wedge T\wedge1)$. By the commutativity of $\tilde\mu$ we have

$$\begin{aligned}\tilde\mu(\iota\wedge1\wedge\iota\wedge1)&=\tilde\mu(1\wedge T\wedge1)(T\wedge T)(1\wedge T\wedge1)(\iota\wedge1\wedge\iota\wedge1)\\&=\tilde\gamma_N(1\wedge1\wedge k_0)(1\wedge1\wedge T)(\iota\wedge\iota\wedge1\wedge1)\\&=\tilde\mu(\iota\wedge1\wedge\iota\wedge1)+(\mu\wedge1)(1\wedge1\wedge\eta_1(1\wedge\pi))(\iota\wedge\iota\wedge1\wedge1)\end{aligned}$$

which implies that $\iota\wedge\eta_1(1\wedge\pi)=0$, and hence $1\wedge\eta_1(1\wedge\pi)=0$.

ii)→i): For any left inverse $\gamma_N$ of $1\wedge i_N$ we see

$$\gamma_N(1\wedge k_0)(1\wedge T)=\gamma_N(1\wedge k_0)+(1\wedge\eta_1)(1\wedge1\wedge\pi)$$

by (1.10). Therefore the pre multiplication $\gamma_0=\gamma_N(1\wedge k_0)$ is commutative, and so our multiplication $\mu_0$ associated with the above $\gamma_0$ is commutative as $E$ is commutative.

**1.5.** In order to discuss the associativity of $E\wedge V(0)$ when $p=2$ we require the following lemmas.

LEMMA 1.9. *In the $p=2$ case there exists a map $p_0: N\to P\wedge V(0)$ so that $p_0i_N=i_P\wedge1$, $(\pi_P\wedge1)p_0=i\cdot\pi_N$, $p_0j_N=1\wedge i$ and $(1\wedge\pi)p_0=i_Pk_N$.*

PROOF. First, consider the diagram

$$\begin{array}{ccccccc}
\Sigma^1 & \xrightarrow{i\eta} & V(0) & \xrightarrow{i_N} & N & \xrightarrow{\pi_N} & \Sigma^2 \\
\downarrow{\scriptstyle i} & & \| & & & & \downarrow{\scriptstyle i} \\
\Sigma^1 V(0) & \xrightarrow[\eta\wedge 1]{} & V(0) & \xrightarrow[i_P\wedge 1]{} & P\wedge V(0) & \xrightarrow[\pi_P\wedge 1]{} & \Sigma^2 V(0)
\end{array}$$

with two cofiberings. We then have a map $p_0'\colon N\to P\wedge V(0)$ which makes the entire diagram commute. In the commutative diagram

$$\begin{array}{ccccc}
\{N, V(0)\} & \xrightarrow{(i_P\wedge 1)_*} & \{N, P\wedge V(0)\} & \xrightarrow{(\pi_P\wedge 1)_*} & \{N, \Sigma^2 V(0)\} \\
\downarrow{\scriptstyle j_N^*} & & \downarrow{\scriptstyle j_N^*} & & \downarrow{\scriptstyle j_N^*} \\
\{P, V(0)\} & \xrightarrow[(i_P\wedge 1)_*]{} & \{P, P\wedge V(0)\} & \xrightarrow[(\pi_P\wedge 1)_*]{} & \{P, \Sigma^2 V(0)\}
\end{array}$$

with two exact rows, the left $j_N^*$ is epic. As $(\pi_P\wedge 1)_*(p_0'j_N)=(\pi_P\wedge 1)_*(1\wedge i)$ we get a map $q\colon N\to V(0)$ such that $(i_P\wedge 1)q\cdot j_N=1\wedge i-p_0'j_N$. Put $p_0=p_0'+(i_P\wedge 1)q$, it is clear that $(\pi_P\wedge 1)p_0=i\cdot\pi_N$ and $p_0j_N=1\wedge i$. Further $p_0i_N=i_P\wedge 1$ because $q\cdot i_N\in 2\{V(0), V(0)\}$ and $2(i_P\wedge 1)=(i_P\wedge 1)i\cdot\eta\cdot\pi=0$. On the other hand, we may set $(1\wedge\pi)p_0=ai_Pk_N$, $a\in Z$, as $j_N^*(1\wedge\pi)p_0=0$. We apply $i_N^*$ on both sides to get $a\equiv 1 \bmod 2$. Hence $(1\wedge\pi)p_0=i_Pk_N$.

LEMMA 1.10. *There exist maps* $\kappa_0$, $\kappa_0'\colon N\wedge V(0)\to P\wedge V(0)$ *such that* $\kappa_0(j_N\wedge 1)=\kappa_0'(j_N\wedge 1)=1$, $\kappa_0(i_N\wedge 1)=\kappa_0'(i_N\wedge 1)T=p_0k_0$ *and* $\kappa_0(1\wedge i)=\kappa_0'(1\wedge i)=p_0$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc}
\{\Sigma^2 V(0), P\wedge V(0)\} & \xrightarrow{(\pi_N\wedge 1)^*} & \{N\wedge V(0), P\wedge V(0)\} \\
\| & & \downarrow{\scriptstyle (j_N\wedge 1)^*} \\
\{\Sigma^2 V(0), P\wedge V(0)\} & \xrightarrow[(\pi_P\wedge 1)^*]{} & \{P\wedge V(0), P\wedge V(0)\}
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{(i_N\wedge 1)^*} & \{V(0)\wedge V(0), P\wedge V(0)\} & \xrightarrow{(i\eta\wedge 1)^*} & \{\Sigma^1 V(0), P\wedge V(0)\} \\
& \downarrow{\scriptstyle (i\wedge 1)^*} & & \| \\
\xrightarrow[(i_P\wedge 1)^*]{} & \{V(0), P\wedge V(0)\} & \xrightarrow[(\eta\wedge 1)^*]{} & \{\Sigma^1 V(0), P\wedge V(0)\}
\end{array}$$

consisting of two exact rows. Since $p_0k_0(i\wedge 1)=p_0k_0(1\wedge i)=p_0i_N=i_P\wedge 1$, both $p_0k_0$ and $p_0k_0T$ are contained in the $(i_N\wedge 1)^*$-image. By chasing the above diagram we get immediately maps $\kappa_0$ and $\kappa_0'$ which satisfy the first two equalities. The last equality composed $j_N$ from the right is valid. Since $j_N^*\colon\{N, P\wedge V(0)\}\to\{P, P\wedge V(0)\}$ is monic, the last is satisfied.

Let us consider the short exact sequence

$$0 \longrightarrow \{\Sigma^1 V(0), P\wedge V(0)\} \xrightarrow{(k_N\wedge 1)^*} \{N\wedge V(0), P\wedge V(0)\}$$
$$\xrightarrow{(j_N\wedge 1)^*} \{P\wedge V(0), P\wedge V(0)\} \longrightarrow 0.$$

This sequence is split as $\kappa_0(j_N\wedge 1)=1$. The first group is generated by $(i_P\wedge 1)\eta_1$ of order 2, and the last is generated by $1_{P\wedge V(0)}$ of order 4 and $\tilde{\zeta}\cdot\pi_P\wedge 1$ of order 2 where $\tilde{\zeta}: \Sigma^2\to P$ is defined by $\pi_P\tilde{\zeta}=2\cdot 1_{\Sigma^2}$ (see [1, Theorem 8.3]). Hence we see

$$\{N\wedge V(0), P\wedge V(0)\}\cong Z_4+Z_2+Z_2$$

with generators $\kappa_0$, $\tilde{\zeta}\cdot\pi_N\wedge 1$ and $(i_P\wedge 1)\eta_1(k_N\wedge 1)$.

For the Hopf map $\nu: \Sigma^3\to\Sigma^0$ we may put

$$(i_P\wedge i)\nu(\pi_N\wedge\pi)=a\kappa_0+b\tilde{\zeta}\cdot\pi_N\wedge 1+c(i_P\wedge 1)\eta_1(k_N\wedge 1)$$

with $a\in Z_4$ and $b, c\in Z_2$. Applying $(j_N\wedge 1)^*$ on both sides we get

$$(i_P\wedge i)\nu(\pi_P\wedge\pi)=a+b\tilde{\zeta}\cdot\pi_P\wedge 1.$$

Recall the relation $2\cdot 1_{P\wedge V(0)}=(i_P\wedge i)\nu(\pi_P\wedge\pi)$ obtained in [1, Theorem 8.3]. This implies that $a=2$ and $b=0$. Similarly, applying $(i_N\wedge 1)^*$ we get

$$0=2p_0k_0+c(i_P\wedge 1)\eta_1(\pi\wedge 1).$$

Since $2p_0k_0=p_0k_0(1\wedge i)(1\wedge\eta)(1\wedge\pi)=(i_P\wedge 1)(1\wedge\eta)(1\wedge\pi)=0$, we find $c=0$. Thus the relation

(1.11) $$(i_P\wedge i)\nu(\pi_N\wedge\pi)=2\kappa_0$$

holds.

We here compare with the composition maps $\kappa_0(k_0\wedge 1)$ and $\kappa_0'(k_0\wedge 1)(1\wedge T)(T\wedge 1)$. Making use of the above results we have

(1.12)
$$\kappa_0(k_0\wedge 1)(1\wedge i\wedge 1)=\kappa_0(k_0\wedge 1)(i\wedge 1\wedge 1)=\kappa_0(i_N\wedge 1)T=p_0k_0,$$
$$\kappa_0(k_0\wedge 1)(1\wedge 1\wedge i)=\kappa_0'(k_0\wedge 1)(1\wedge T)(T\wedge 1)(i\wedge 1\wedge 1)=p_0k_0$$
and
$$\kappa_0'(k_0\wedge 1)(1\wedge T)(T\wedge 1)(1\wedge 1\wedge i)=\kappa_0'(k_0\wedge 1)(1\wedge T)(T\wedge 1)(1\wedge i\wedge 1)$$
$$=\kappa_0'(i_N\wedge 1)T=p_0k_0.$$

LEMMA 1.11. $\kappa_0(k_0\wedge 1)\equiv\kappa_0'(k_0\wedge 1)(1\wedge T)(T\wedge 1) \bmod 2\{V(0)\wedge V(0)\wedge V(0), P\wedge V(0)\}$.

PROOF. Using the exact sequences

$$\{\Sigma^1, V(0)\} \xrightarrow{\eta_*} \{\Sigma^2, V(0)\} \xrightarrow{(i_P \wedge 1)_*} \{\Sigma^2, P \wedge V(0)\} \longrightarrow 0$$

$$\{V(0), V(0)\} \xrightarrow{\eta_*} \{\Sigma^1 V(0), V(0)\} \xrightarrow{(i_P \wedge 1)_*} \{\Sigma^1 V(0), P \wedge V(0)\} \longrightarrow 0$$

we see that $\{\Sigma^2, P \wedge V(0)\}$ and $\{\Sigma^1 V(0), P \wedge V(0)\}$ are $Z_2$-modules which have one generator $(i_P \wedge 1)\tilde{\eta}$ and $(i_P \wedge 1)\eta_1$ respectively. Therefore $\pi^* : \{\Sigma^2, P \wedge V(0)\} \to \{\Sigma^1 V(0), P \wedge V(0)\}$ and $(1 \wedge \pi)^* : \{\Sigma^1 V(0), P \wedge V(0)\} \to \{V(0) \wedge V(0), P \wedge V(0)\}$ are monic. Hence (1.12) implies that

$$\kappa_0(k_0 \wedge 1) - \kappa_0'(k_0 \wedge 1)(1 \wedge T)(T \wedge 1) \in (\pi \wedge \pi \wedge \pi)^* \{\Sigma^3, P \wedge V(0)\}.$$

Observe that $(i_P \wedge 1)_* : \{\Sigma^3, V(0)\} \to \{\Sigma^3, P \wedge V(0)\}$ is epic, then we have the equality that $\kappa_0(k_0 \wedge 1) - \kappa_0'(k_0 \wedge 1)(1 \wedge T)(T \wedge 1) = a(i_P \wedge i)\nu(\pi \wedge \pi \wedge \pi)$ for some $a \in Z_2$. The result is now immediate from (1.11).

Let $E$ be a ring spectrum such that $1 \wedge \eta : \Sigma^1 E \to E$ is trivial. Take a map $\xi'' : \Sigma^2 \to E \wedge P$ with $(1 \wedge \pi_P)\xi'' = \iota \wedge 1$ and $\xi = (1 \wedge j_N)\xi''$. Between the left inverses $\gamma_\xi$ and $\gamma_{\xi''}$ induced by the maps $\xi$ and $\xi''$ we have the relation

$$\gamma_\xi = (\gamma_{\xi''} \wedge 1)(1 \wedge p_0)$$

because the $(1 \wedge i_P \wedge 1)_*$-images of both sides coincide.

We say that a pre multiplication $\gamma$ is *associative* if it satisfies the relation $\gamma(\gamma \wedge 1) = \gamma(T \wedge 1)(1 \wedge \gamma)(T \wedge 1 \wedge 1)$.

LEMMA 1.12. *The pre multiplication $\gamma_0 = \gamma_\xi(1 \wedge k_0)$ is associative when $p = 2$.*

PROOF. By definition $(1 \wedge i_P)\gamma_{\xi''}(\mu \wedge 1)(1 \wedge \xi'') = 0$, and hence $\gamma_{\xi''}(\mu \wedge 1)(1 \wedge \xi'') = 0$. Using Lemma 1.10 and this result we have

$$\begin{aligned} &\gamma_\xi(1 \wedge k_0)(\gamma_\xi \wedge 1) \\ &\quad = (\gamma_{\xi''} \wedge 1)(1 \wedge \kappa_0)(1 \wedge i_N \wedge 1)(\gamma_\xi \wedge 1) \\ &\quad = (\gamma_{\xi''} \wedge 1)(1 \wedge \kappa_0)(1 - (\mu \wedge 1 \wedge 1)(1 \wedge 1 \wedge j_N \wedge 1)(1 \wedge \xi'' \wedge 1)(1 \wedge \pi_N \wedge 1)) \\ &\quad = (\gamma_{\xi''} \wedge 1)(1 \wedge \kappa_0), \end{aligned}$$

and similarly

$$\gamma_\xi(1 \wedge k_0 T)(\gamma_\xi \wedge 1) = (\gamma_{\xi''} \wedge 1)(1 \wedge \kappa_0').$$

The above equalities yield

$$\gamma_0(\gamma_0 \wedge 1) = (\gamma_{\xi''} \wedge 1)(1 \wedge \kappa_0)(1 \wedge k_0 \wedge 1),$$

and

$$\gamma_0(T\wedge 1)(1\wedge\gamma_0)(T\wedge 1\wedge 1)=\gamma_0(1\wedge T)(\gamma_0\wedge 1)(1\wedge 1\wedge T)(1\wedge T\wedge 1)$$
$$=(\gamma_{\xi''}\wedge 1)(1\wedge\kappa_0')(1\wedge k_0\wedge 1)(1\wedge 1\wedge T)(1\wedge T\wedge 1).$$

Making use of Lemma 1.11 we obtain that

$$1\wedge\kappa_0(\kappa_0\wedge 1)=1\wedge\kappa_0'(k_0\wedge 1)(1\wedge T)(T\wedge 1),$$

which implies

$$\gamma_0(\gamma_0\wedge 1)=\gamma_0(T\wedge 1)(1\wedge\gamma_0)(T\wedge 1\wedge 1).$$

Let $\mu_\gamma$ be a multiplication of $E\wedge V(0)$ associated with a pre multiplication $\gamma$. If $\gamma$ is compatible with $\mu$, i.e., if it satisfies $(\Lambda_3)'$, then a routine computation shows

$$\mu_\gamma(\mu_\gamma\wedge 1\wedge 1)$$
$$=\gamma(\gamma\wedge 1)(\mu(\mu\wedge 1)\wedge 1\wedge 1\wedge 1)(1\wedge 1\wedge T\wedge 1\wedge 1)(1\wedge T\wedge T\wedge 1)$$
$$\mu_\gamma(1\wedge 1\wedge\mu_\gamma)$$
$$=\gamma(T\wedge 1)(1\wedge\gamma)(T\wedge 1)(\mu(1\wedge\mu)\wedge 1\wedge 1\wedge 1)(1\wedge 1\wedge T\wedge 1\wedge 1)(1\wedge T\wedge T\wedge 1).$$

Hence we see that

(1.13) *$\mu_\gamma$ is associative if $\mu$ and $\gamma$ are associative and if $\gamma$ is compatible with $\mu$.*

By means of (1.9) and Lemma 1.12 with (1.13) we obtain

PROPOSITION 1.13. *Let $E$ be an associative ring spectrum. Assume that $E$ is commutative and $1\wedge\eta:\Sigma^1E\to E$ is trivial if $p=2$ and that $1\wedge i\cdot\alpha_1:\Sigma^3E\to E\wedge V(0)$ is trivial if $p=3$. Then there exists an associative admissible multiplication of $E\wedge V(0)$.*

## § 2. Multiplications of $E\wedge V(1)$

**2.1.** For any $Z_p$-spectra $X, Y$ a map $f:\Sigma^kX\to Y$ is called a *$Z_p$-map* if it satisfies $f\cdot\psi_X=\psi_Y(f\wedge 1)$ and $(f\wedge 1)\phi_X=(-1)^k\phi_Y\cdot f$. Let $C$ denote the mapping cone of a $Z_p$-map $f:\Sigma^kX\to Y$, so we have a cofibering

$$\Sigma^kX\xrightarrow{f}Y\xrightarrow{i_C}C\xrightarrow{\pi_C}\Sigma^{k+1}X.$$

By a similar discussion to (1.8) we find a map $\psi_C: C\wedge V(0)\to C$ such that $\psi_C(1\wedge i)=1_C$. Thus $C$ is a $Z_p$-spectrum if $f:\Sigma^kX\to Y$ is a $Z_p$-map. For any $Z_p$-spectra $X$ and $Y$ Toda [5] introduced an operation

$$\theta:\{\Sigma^k X, Y\}\longrightarrow\{\Sigma^{k+1}X, Y\}$$

by the formula $\theta(f)=\psi_Y(f\wedge 1)\phi_X$. This operation has the properties

(2.1) i) *$\theta$ is derivative*, i.e., $\theta(g\cdot f)=g\cdot\theta(f)+(-1)^{\deg(f)}\theta(g)\cdot f$,
ii) *$f$ is a $Z_p$-map if and only if $\theta(f)=0$.*

LEMMA 2.1 ([5, Lemma 2.3]). *Let $X$ and $Y$ be $Z_p$-spectra and $C$ be the mapping cone of a map $f:\Sigma^k X\to Y$. Then $C$ is a $Z_p$-spectrum if $\theta(f)=0$. The converse is valid under the assumption that $\{Y,\Sigma^k X\}=\{\Sigma^1 X, X\}=\{\Sigma^1 Y, Y\}=0$.*

PROOF. The above observations show the first half. On the other hand, we get

$$\begin{aligned}(1\wedge i)_*(i_C\theta(f)\pi_C)&=(i_C\wedge 1)(1-\phi_Y(1\wedge\pi))(f\wedge 1)\phi_X\pi_C\\&=(-1)^{k+1}(i_C\wedge 1)\phi_Y f\cdot\pi_C=0.\end{aligned}$$

Hence $i_C\theta(f)\pi_C=0$ when $p\{C,C\}=0$. The latter half is now immediate.

In the following we always assume that a fixed prime $p$ is odd. $V(0)$ is a $Z_p$-spectrum, so that it has unique maps

$$\psi: V(0)\wedge V(0)\longrightarrow V(0),\qquad \phi:\Sigma^1 V(0)\longrightarrow V(0)\wedge V(0)$$

which satisfy (1.1) and moreover which are commutative, i.e., $\psi\cdot T=\psi$ and $T\cdot\phi=-\phi$. So we note that

(2.2) $$\psi(1\wedge i)=\psi(i\wedge 1)=1,\qquad (1\wedge\pi)\phi=-(\pi\wedge 1)\phi=1.$$

A $Z_p$-spectrum $X$ is said to be *associative* if $\psi_X(\psi_X\wedge 1)-\psi_X(1\wedge\psi)=0$ and $(\phi_X\wedge 1)\phi_X+(1\wedge\phi)\phi_X=0$. There exists uniquely a map $\alpha_X:\Sigma^2 X\to X$ so that

$$\psi_X(\psi_X\wedge 1)-\psi_X(1\wedge\psi)=\alpha_X(1\wedge\pi\wedge\pi)$$

and

$$(\phi_X\wedge 1)\phi_X+(1\wedge\phi)\phi_X=(1\wedge i\wedge i)\alpha_X$$

when $\{\Sigma^1 X, X\}=0$ (see [5, Proposition 2.1]). In particular $X$ is associative if $\{\Sigma^1 X, X\}=\{\Sigma^2 X, X\}=0$.

As an analogy of $\theta$ Toda [5] defined another operation

$$\lambda=\lambda_X:\{\Sigma^k V(0), V(0)\}\longrightarrow\{\Sigma^{k+1}X, X\}$$

by the formula $\lambda(h)=\psi_X(1\wedge h)\phi_X$ for each $Z_p$-spectrum $X$. From the commutativities of $\psi$ and $\phi$ we obtain

$$\lambda_{V(0)}(h)=-\theta(h)$$

for every $h: \Sigma^k V(0)\to V(0)$.

Recall the spectrum $V(n)$ whose ordinary cohomology is a certain exterior algebra over the mod $p$ Steenrod algebra. For $n=1$, $p\geqq 3$, for $n=2$, $p\geqq 5$ and for $n=3$, $p\geqq 7$ spectra $V(n)$ were constructed in [4]. However $V(1)$ for $p=2$ and $V(2)$ for $p=3$ do not exist [4, Theorem 1.2]. Consider the following cofiberings

$$\Sigma^q V(0)\xrightarrow{\alpha} V(0)\xrightarrow{i_1} V(1)\xrightarrow{\pi_1}\Sigma^{q+1}V(0), \qquad p\geqq 3$$
$$\Sigma^{pq+q} V(1)\xrightarrow{\beta} V(1)\xrightarrow{i_2} V(2)\xrightarrow{\pi_2}\Sigma^{pq+q+1}V(1), \qquad p\geqq 5$$

where we set $q=2(p-1)$. When $p=3$ a map $[\beta i_1]: \Sigma^{16}V(0)\to V(1)$ exists even though $\beta$ does not exist.

We use the notations

$$i_0=i_1\cdot i: \Sigma^0\longrightarrow V(1), \qquad \pi_0=\pi\cdot\pi_1: V(1)\longrightarrow \Sigma^{q+2}$$
$$\delta_1=i_1\cdot\pi_1: \Sigma^{-q-1}V(1)\longrightarrow V(1) \quad \text{and} \quad \delta_0=i_0\cdot\pi_0: \Sigma^{-q-2}V(1)\longrightarrow V(1)$$

and put

$$\alpha'=\alpha_1\wedge 1: \Sigma^{q-1}V(1)\longrightarrow V(1) \quad \text{and} \quad \beta'=\beta_1\wedge 1: \Sigma^{pq-2}V(1)\longrightarrow V(1)$$

for the elements $\alpha_1=\pi\cdot\alpha\cdot i\in\pi_{q-1}(S)$ and $\beta_1=\pi_0\cdot\beta\cdot i_0\in\pi_{pq-2}(S)$. Then we obtain maps

$$\alpha'': \Sigma^{q-2}V(1)\longrightarrow V(1) \quad \text{and} \quad \beta'': \Sigma^{pq+2q-3}V(1)\longrightarrow V(1)$$

such that $\alpha''\cdot i_1=\alpha'\cdot i_1\cdot\delta$ and $\beta''\cdot i_1=\alpha''\cdot\beta\cdot i_1\cdot\delta$ [5, Lemmas 3.1 and 3.5].

Notice that $V(1)$ and $V(2)$ are $Z_p$-spectra. Making use of the Adams spectral sequence Toda computed the homotopy groups of $V(1)$ and $V(2)$ (see [4, Theorem 5.2 and Corollary 5.4] and [5, Theorem 3.2 and Proposition 6.9]):

(2.3) i) $\pi_*(V(1))\cong P(\beta,\beta')\otimes\{1,\alpha',\delta_1\beta,\alpha''\beta,\delta_0\beta^2,\delta_0\beta^2\alpha'\}\otimes\{i_0\}$

for degree $<p^2q-3$ when $p\geqq 5$ and for degree $<31$ when $p=3$,

ii) $\pi_*(V(2))\cong\{i_2\}\otimes P(\beta')\otimes\{1,\alpha',\delta_1\beta,\alpha''\beta,\delta_0\beta^2,\delta_0\beta^2\alpha'\}\otimes\{i_0\}$

for degree $<p^2q-3$ when $p\geqq 5$.

By applying the operations $\theta$ and $\lambda$ Toda [5, Theorems 3.6 and 6.11] determined an additive basis of the algebra $\{V(1), V(1)\}_*$ up to some range:

$$\text{(2.4)}\quad \begin{aligned}\{V(1),V(1)\}_*\cong & P(\beta,\beta')\otimes\{1,\alpha',\delta_1\beta,\alpha''\beta,\delta_0\beta^2,\delta_0\beta^2\alpha'\}\otimes E(\delta_0)\\ &+P(\beta,\beta')\otimes\{\delta_1,\alpha'',\delta_1\beta\delta_1,\delta_0\beta,\alpha''\beta\delta_1,\beta'',\delta_0\beta^2\delta_1,\delta_0\beta^2\alpha''\}\end{aligned}$$

for degree $<(p^2-1)q-5$ when $p\geqq 5$ and for degree $<14$ when $p=3$.

The $p=3$ case is quite different from the other cases. Besides the previous examples we have that the products $\alpha''\cdot\alpha''$ and $\alpha'\cdot\alpha''=\alpha''\cdot\alpha'$ are not trivial for $p=3$. Thus the relations

(2.5) $$\alpha''\cdot\alpha''=\beta'\cdot\delta_0 \quad\text{and}\quad \alpha'\cdot\alpha''=\alpha''\cdot\alpha'=\beta'\cdot\delta_1$$

hold [5, Theorem 6.2]. Further we see [5, Theorem 6.4] that

(2.6) $$\theta([\beta i_1])=\alpha''[\beta i_1]\delta \qquad \text{for } p=3.$$

**2.2.** As $\{\Sigma^1 V(1), V(1)\}=0$ the $Z_p$-spectrum $V(1)$ has unique maps

$$\psi_1: V(1)\wedge V(0)\longrightarrow V(1), \qquad \phi_1: \Sigma^1 V(1)\longrightarrow V(1)\wedge V(0)$$

satisfying (1.1). As is easily checked, $\psi_1$ and $\phi_1$ are compatible with $\psi$ and $\phi$ respectively in the sense that the relations

(2.7) $$\begin{aligned}\psi_1(i_1\wedge 1)&=i_1\psi, & \pi_1\psi_1&=\psi(\pi_1\wedge 1),\\ \phi_1 i_1&=(i_1\wedge 1)\phi \quad\text{and} & (\pi_1\wedge 1)\phi_1&=-\phi\cdot\pi_1\end{aligned}$$

hold.

By means of Lemma 2.1 we see that $\alpha: \Sigma^q V(0)\to V(0)$ is a $Z_p$-map, i.e.,

(2.8) $$\psi(\alpha\wedge 1)=\alpha\cdot\psi=\psi(1\wedge\alpha), \qquad (\alpha\wedge 1)\phi=\phi\cdot\alpha=(1\wedge\alpha)\phi.$$

Whenever $p\geqq 5$ $V(1)$ is associative, but it is not so in the $p=3$ case. Thus we have

(2.9) $$\begin{aligned}\psi_1(\psi_1\wedge 1)-\psi_1(1\wedge\psi)&=\alpha''(1\wedge\pi\wedge\pi),\\ (\phi_1\wedge 1)\phi_1+(1\wedge\phi)\phi_1&=(1\wedge i\wedge i)\alpha''\end{aligned}$$

when $p=3$ [5, Lemma 6.5].

We here give a decomposition of the smash product $1\wedge\alpha: \Sigma^q V(1)\wedge V(0)\to V(1)\wedge V(0)$. By virtue of (2.4) we have

$$\begin{aligned}&\{\Sigma^{q-1}V(1), V(1)\}\cong Z_p && \text{with a generator } \alpha',\\ &\{\Sigma^{pq-q-4}V(1), V(1)\}\cong Z_p && \text{with a generator } \beta'\cdot\delta_0,\\ &\{\Sigma^{pq-q-3}V(1), V(1)\}\cong Z_p+Z_p && \text{with generators } \delta_1\cdot\beta\cdot\delta_0,\ \beta'\cdot\delta_1.\end{aligned}$$

So we may set

$$\begin{aligned}1\wedge\alpha=\phi_1\cdot\alpha'\cdot\psi_1&+w\phi_1\cdot\beta'\delta_0(1\wedge\pi)\\ &+(1\wedge i)(x\delta_1\beta\delta_0+y\beta'\delta_1)(1\wedge\pi)+z(1\wedge i)\beta'\delta_0\cdot\psi_1\end{aligned}$$

where $w, x, y, z\in Z_p$. The following result was implicitly given in Toda [5].

LEMMA 2.2. *$1\wedge\alpha=\phi_1\cdot\alpha'\cdot\psi_1-\phi_1\cdot\beta'\delta_0(1\wedge\pi)+(1\wedge i)\beta'\delta_1(1\wedge\pi)-(1\wedge i)\beta'\delta_0\cdot\psi_1$ when $p=3$, but $1\wedge\alpha=\phi_1\cdot\alpha'\cdot\psi_1$ when $p\geqq 5$.*

PROOF. The latter half is clear by the dimensional reason. We prove only the $p=3$ case. We first use (2.2) and (2.8) to verify

$$\theta(\alpha\cdot\delta)=\psi(\alpha\wedge 1)(i\wedge 1)(\pi\wedge 1)\phi=-\alpha.$$

By use of (2.5) and (2.9) we compute

$$\begin{aligned}\lambda_{V(1)}(\alpha)&=\psi_1(1\wedge\alpha)\phi_1\\&=-\psi_1(1\wedge\psi)(1\wedge\alpha\cdot\delta\wedge 1)(1\wedge\phi)\phi_1\\&=\psi_1(1\wedge\psi T)(1\wedge\alpha\cdot\delta\wedge 1)(1\wedge T\phi)\phi_1\\&=(\psi_1(\psi_1\wedge 1)-\alpha''(1\wedge\pi\wedge\pi))(1\wedge 1\wedge\alpha\cdot\delta)((1\wedge i\wedge i)\alpha''-(\phi_1\wedge 1)\phi_1)\\&=\alpha''(1\wedge\pi)(1\wedge\pi\wedge 1)(1\wedge 1\wedge\alpha)(1\wedge 1\wedge i)\phi_1\\&=\alpha''(1\wedge\pi)(1\wedge\alpha)(1\wedge i)=\beta'\delta_1.\end{aligned}$$

This implies $x=0$ and $y=1$. Next, by (2.8) and (2.9) we get

$$\begin{aligned}&\psi_1(\psi_1\wedge 1)(1\wedge\alpha\wedge 1)(1\wedge i\wedge 1)\phi_1\\&\quad=(\psi_1(1\wedge\psi)+\alpha''(1\wedge\pi\wedge\pi))(1\wedge\alpha\wedge 1)(1\wedge i\wedge 1)\phi_1\\&\quad=\psi_1(1\wedge\alpha)\phi_1+\alpha''\alpha'=-\beta'\delta_1,\end{aligned}$$

and similarly

$$\psi_1(1\wedge\pi\wedge 1)(1\wedge\alpha\wedge 1)(\phi_1\wedge 1)\phi_1=-\beta'\delta_1.$$

On the other hand, by (2.2) and (2.7) we see

$$\theta(\delta_0)=\psi_1(i_1\wedge 1)(i\wedge 1)(\pi\wedge 1)(\pi_1\wedge 1)\phi_1=-i_1\psi(i\wedge 1)(\pi\wedge 1)\phi\cdot\pi_1=\delta_1.$$

Consequently it follows that $z=w=-1$.

Since $\delta\cdot\psi=1\wedge\pi+\pi\wedge 1$ we have

COROLLARY 2.3. *$\psi_1(1\wedge\alpha)=\beta'\cdot i_1(1\wedge\pi-\pi\wedge 1)(\pi_1\wedge 1)$ when $p=3$, but $\psi_1(1\wedge\alpha)=0$ when $p\geqq 5$.*

**2.3.** A map $\gamma: X\wedge V(1)\wedge V(1)\to X\wedge V(1)$ is said to be a *pre multiplication of $X\wedge V(1)$* if $\gamma(1\wedge 1\wedge i_1)=\gamma(1\wedge i_1\wedge 1)(1\wedge T)=1\wedge\psi_1$. We here construct a pre multiplication of $X\wedge V(1)$ under a suitable assumption on $X$. Let $V$ be the mapping cone of $\psi_1(1\wedge\alpha)$. Then there exists a map $v: V(1)\wedge V(1)\to V$ which makes the diagram below commutative

$$\begin{array}{ccccccc}
\Sigma^q V(1)\wedge V(0) & \xrightarrow{1\wedge\alpha} & V(1)\wedge V(0) & \xrightarrow{1\wedge i_1} & V(1)\wedge V(1) & \xrightarrow{1\wedge\pi_1} & \Sigma^{q+1}V(1)\wedge V(0) \\
\| & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle v} & & \| \\
\Sigma^q V(1)\wedge V(0) & \xrightarrow[\psi_1(1\wedge\alpha)]{} & V(1) & \xrightarrow[i_V]{} & V & \xrightarrow[\pi_V]{} & \Sigma^{q+1}V(1)\wedge V(0).
\end{array}$$

We put $\rho_1=\beta'\cdot i_1(1\wedge\pi-\pi\wedge 1)$ in the $p=3$ case and $\rho_1=0$ in the other cases, and denote by $R$ its mapping cone. We then have a commutative diagram

$$\begin{array}{ccccccc}
 & & & & \Sigma^{q+1}V(0)\wedge V(0) & = & \Sigma^{q+1}V(0)\wedge V(0) \\
 & & & & \downarrow{\scriptstyle j_V} & & \downarrow{\scriptstyle i_1\wedge 1} \\
\Sigma^q V(1)\wedge V(0) & \xrightarrow{\psi_1(1\wedge\alpha)} & V(1) & \xrightarrow{i_V} & V & \xrightarrow{\pi_V} & \Sigma^{q+1}V(1)\wedge V(0) \\
\downarrow{\scriptstyle \pi_1\wedge 1} & & \| & & \downarrow{\scriptstyle k_V} & & \downarrow{\scriptstyle \pi_1\wedge 1} \\
\Sigma^{2q+1}V(0)\wedge V(0) & \xrightarrow[\rho_1]{} & V(1) & \xrightarrow[i_R]{} & R & \xrightarrow[\pi_R]{} & \Sigma^{2q+2}V(0)\wedge V(0)
\end{array}$$

involving four cofiberings in which the right-lower square commutes up to the sign $-1$.

Assume that $1\wedge\rho_1: X\wedge\Sigma^{2q+1}V(0)\wedge V(0)\to X\wedge V(1)$ is trivial. Each left inverse $\gamma_R: X\wedge R\to X\wedge V(1)$ of $1\wedge i_R$ gives rise to a map

$$\gamma_1: X\wedge V(1)\wedge V(1)\longrightarrow X\wedge V(1)$$

defined by the composition $\gamma_1=\gamma_R(1\wedge k_V)(1\wedge v)$.

LEMMA 2.4. *The map $\gamma_1$ is a pre multiplication of $X\wedge V(1)$.*

PROOF. Obviously $\gamma_R(1\wedge k_V)(1\wedge v)(1\wedge 1\wedge i_1)=1\wedge\psi_1$. Since $\pi_{V*}(v(i_1\wedge 1))=\pi_{V*}(j_V(1\wedge\pi_1))$ we set

$$v(i_1\wedge 1)=j_V(1\wedge\pi_1)+a i_V\psi_1 T, \qquad a\in Z_p.$$

We apply $(i\wedge i_0)^*$ on both sides to get that $i_V i_0=a i_V i_0$ which implies $a=1$. Thus $v(i_1\wedge 1)=j_V(1\wedge\pi_1)+i_V\psi_1 T$. Hence we see

$$\gamma_R(1\wedge k_V)(1\wedge v)(1\wedge i_1\wedge 1)(1\wedge T)=\gamma_R(1\wedge k_V)(1\wedge i_V)(1\wedge\psi_1)=1\wedge\psi_1.$$

Let $E$ be a ring spectrum equipped with a multiplication $\mu$ and a unit $\iota$. For any pre multiplication $\gamma$ of $E\wedge V(1)$ we define a map

$$\mu_\gamma: E\wedge V(1)\wedge E\wedge V(1)\longrightarrow E\wedge V(1)$$

as the composition $\gamma(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)$. This map satisfies the property

$$(A_1)_1 \qquad \begin{aligned}&\mu_\gamma(1\wedge T\wedge 1)(1\wedge 1\wedge 1\wedge i_1)\\ &\quad=\mu_\gamma(1\wedge T\wedge 1)(1\wedge 1\wedge i_1\wedge 1)(1\wedge 1\wedge T)=\mu\wedge\psi_1.\end{aligned}$$

Every map $\tilde{\mu}$ with $(A_1)_1$ gives $E\wedge V(1)$ the structure of a ring spectrum having $\iota\wedge i_0$ as the unit. As a consequence we obtain

PROPOSITION 2.5. *Let $E$ be a ring spectrum and assume that*

$$1\wedge\beta'\cdot i_1(1\wedge\pi-\pi\wedge 1): E\wedge\Sigma^{2q+1}V(0)\wedge V(0)\longrightarrow E\wedge V(1)$$

*is trivial if $p=3$. Then $E\wedge V(1)$ is a ring spectrum which has a multiplication satisfying the property $(A_1)_1$.*

**2.4.** Take the sphere spectrum $S$ as the ring spectrum $E$ in Proposition 2.5 when $p\geqq 5$. Then $V(1)$ becomes a ring spectrum equipped with the unit $i_0$. Its multiplication

$$\psi_{1,1}: V(1)\wedge V(1)\longrightarrow V(1)$$

is unique and it is associative and commutative because

$$(1\wedge i_0)^*: \{V(1)\wedge V(1), V(1)\}\longrightarrow\{V(1), V(1)\}$$

and $$(1\wedge 1\wedge i_0)^*: \{V(1)\wedge V(1)\wedge V(1), V(1)\}\longrightarrow\{V(1)\wedge V(1), V(1)\}$$

are isomorphic. Thus $\psi_{1,1}$ satisfies the equalities

(2.10) $$\psi_{1,1}T=\psi_{1,1} \quad\text{and}\quad \psi_{1,1}(\psi_{1,1}\wedge 1)=\psi_{1,1}(1\wedge\psi_{1,1}) \qquad \text{when } p\geqq 5.$$

We here study the commutativity of $E\wedge V(1)$ in the $p=3$ case. Denoting by $M$ the mapping cone of $\beta'\cdot i_1: \Sigma^{2q+2}V(0)\to V(1)$ when $p=3$, then we have a commutative (up to sign) diagram

$$\begin{array}{ccccccc}
\Sigma^q V(1)\wedge V(0) & \xrightarrow{1\wedge\alpha} & V(1)\wedge V(0) & \xrightarrow{1\wedge i_1} & V(1)\wedge V(1) & \xrightarrow{1\wedge\pi_1} & \Sigma^{q+1}V(1)\wedge V(0)\\
\downarrow{\scriptstyle \pi_1\wedge 1} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle k_v v} & & \downarrow{\scriptstyle \pi_1\wedge 1}\\
\Sigma^{2q+1}V(0)\wedge V(0) & \xrightarrow{\rho_1} & V(1) & \xrightarrow{i_R} & R & \xrightarrow{\pi_R} & \Sigma^{2q+2}V(0)\wedge V(0)\\
\downarrow{\scriptstyle 1\wedge\pi-\pi\wedge 1} & & \| & & \downarrow{\scriptstyle k_R} & & \downarrow{\scriptstyle 1\wedge\pi-\pi\wedge 1}\\
\Sigma^{2q+2}V(0) & \xrightarrow[\beta' i_1]{} & V(1) & \xrightarrow[i_M]{} & M & \xrightarrow[\pi_M]{} & \Sigma^{2q+3}V(0)
\end{array}$$

consisting of three cofiberings. In the exact sequence

$$\{\Sigma^1 V(1)\wedge V(0), V(1)\}\xrightarrow{(1\wedge\alpha)^*}\{\Sigma^{q+1}V(1)\wedge V(0), V(1)\}\xrightarrow{(1\wedge\pi_1)^*}\{V(1)\wedge V(1), V(1)\}$$
$$\xrightarrow{(1\wedge i_1)^*}\{V(1)\wedge V(0), V(1)\}\xrightarrow{(1\wedge\alpha)^*}\{\Sigma^q V(1)\wedge V(0), V(1)\},$$

$(1\wedge\pi_1)^*$ is epic as $(1\wedge\alpha)^*\psi_1\neq 0$. Therefore $\{V(1)\wedge V(1), V(1)\}$ is spanned by $(1\wedge\pi_1)^*(\delta_1\beta\delta_1(1\wedge\pi))$, $(1\wedge\pi_1)^*(\delta_1\beta\delta_0\psi_1)$ and $(1\wedge\pi_1)^*(\beta'\delta_1\psi_1)$. But $(1\wedge\alpha)^*$

$(\alpha''(1\wedge\pi))=\alpha''(\alpha'\psi_1-\beta'\delta_0(1\wedge\pi))=\beta'\delta_1\psi_1$ because of Lemma 2.2 and (2.5). Hence we have

$$\{V(1)\wedge V(1), V(1)\}\cong Z_3+Z_3$$

with generators $\delta_1\beta\cdot i_1(1\wedge\pi)(\pi_1\wedge\pi_1)$ and $\delta_1\beta\cdot i_1\delta\cdot\psi(\pi_1\wedge\pi_1)$.

Setting $k_1=k_R\cdot k_V\cdot v$, $\pi_{M*}(k_1(T-1))=(1\wedge\pi-\pi\wedge 1)(\pi_1\wedge\pi_1)(T-1)=0$. So we put

$$k_1(T-1)=ai_M\delta_1\beta\cdot i_1(1\wedge\pi)(\pi_1\wedge\pi_1)+bi_M\delta_1\beta\cdot i_1\delta\cdot\psi(\pi_1\wedge\pi_1)$$

where $a, b\in Z_3$. Applying $T$ from the right we get

$$k_1(1-T)=-ai_M\delta_1\beta\cdot i_1(\pi\wedge 1)(\pi_1\wedge\pi_1)-bi_M\delta_1\beta\cdot i_1\delta\cdot\psi(\pi_1\wedge\pi_1).$$

We substract the first equality from the latter to obtain

$$k_1(T-1)=(b-a)i_M\delta_1\beta\cdot i_1\delta\cdot\psi(\pi_1\wedge\pi_1).$$

Thus

$$k_1T=k_1+c(k_1)i_M\delta_1\beta\cdot i_1\delta\cdot\psi(\pi_1\wedge\pi_1),\qquad c(k_1)\in Z_3. \tag{2.11}$$

PROPOSITION 2.6. *Let $E$ be a commutative ring spectrum and assume that $1\wedge\beta'\cdot i_1: E\wedge\Sigma^{2q+2}V(0)\to E\wedge V(1)$ is trivial if $p=3$. Then there exists a commutative multiplication of $E\wedge V(1)$ satisfying the property $(A_1)_1$.*

PROOF. We may assume $p=3$. Take a left inverse $\gamma_M: E\wedge M\to E\wedge V(1)$ of $1\wedge i_M$ and put $\gamma_1'=\gamma_M(1\wedge k_1)-c(k_1)\delta_1\beta\cdot i_1\delta\cdot\psi(\pi_1\wedge\pi_1)$, $c(k_1)\in Z_3$. Making use of Lemma 2.4 and (2.9) we see that $\gamma_1'$ is a pre multiplication of $E\wedge V(1)$ such that $\gamma_1'(1\wedge T)=\gamma_1'$. Therefore our multiplication $\mu_1$ associated with the above $\gamma_1'$ is commutative.

## § 3. Multiplications of $E\wedge V(2)$

**3.1.** In this section we assume $p\geqq 5$, so $V(2)$ exists. The $Z_p$-spectrum $V(2)$ has unique maps

$$\psi_2: V(2)\wedge V(0)\longrightarrow V(2),\qquad \phi_2: \Sigma^1V(2)\longrightarrow V(2)\wedge V(0)$$

satisfying (1.1) as $\{\Sigma^1V(2), V(2)\}=0$. Note that $V(2)$ is associative. As is easily seen, $\psi_2$ and $\phi_2$ are compatible with $\psi_1$ and $\phi_1$ respectively, thus

$$\begin{aligned}&\psi_2(i_2\wedge 1)=i_2\psi_1,\qquad \pi_2\psi_2=\psi_1(\pi_2\wedge 1),\\ &\phi_2i_2=(i_2\wedge 1)\phi_1\quad\text{and}\quad(\pi_2\wedge 1)\phi_2=-\phi_1\pi_2.\end{aligned}\tag{3.1}$$

Recall that $V(1)$ has a unique multiplication

$$\psi_{1,1}: V(1)\wedge V(1) \longrightarrow V(1)$$

which is associative and commutative whenever $p \geqq 5$. Of course this is an extension of $\psi_1$, i.e.,

$$\psi_{1,1}(1\wedge i_1)=\psi_1 \quad \text{and} \quad \psi_{1,1}(i_1\wedge 1)=\psi_1 T.$$

Note that $\beta: \Sigma^{pq+q}V(1)\to V(1)$ is an attaching map of the $Z_p$-spectrum $V(2)$. Lemma 2.1 shows that it is a $Z_p$-map, i.e., $\psi_1(\beta\wedge 1)=\beta\cdot\psi_1$. The equalities

$$\psi_{1,1}(\beta\wedge 1)=\beta\cdot\psi_{1,1}=\psi_{1,1}(1\wedge\beta) \tag{3.2}$$

hold because the aboves composed $1\wedge i_1$ or $i_1\wedge 1$ from the right are valid. Hence there exists a map

$$\psi_{2,1}: V(2)\wedge V(1) \longrightarrow V(2)$$

making the diagram below commutative

$$\begin{array}{ccccccc}
\Sigma^{pq+q}V(1)\wedge V(1) & \xrightarrow{\beta\wedge 1} & V(1)\wedge V(1) & \xrightarrow{i_2\wedge 1} & V(2)\wedge V(1) & \xrightarrow{\pi_2\wedge 1} & \Sigma^{pq+q+1}V(1)\wedge V(1) \\
\downarrow{\scriptstyle\psi_{1,1}} & & \downarrow{\scriptstyle\psi_{1,1}} & & \downarrow{\scriptstyle\psi_{2,1}} & & \downarrow{\scriptstyle\psi_{1,1}} \\
\Sigma^{pq+q}V(1) & \xrightarrow[\beta]{} & V(1) & \xrightarrow[i_2]{} & V(2) & \xrightarrow[\pi_2]{} & \Sigma^{pq+q+1}V(1).
\end{array}$$

$\psi_{2,1}$ becomes an extension of $\psi_2$, i.e., $\psi_{2,1}(1\wedge i_1)=\psi_2$. A routine computation shows that $\psi_{2,1}$ is associative in the sense that

$$\psi_{2,1}(1\wedge\psi_{1,1})=\psi_{2,1}(\psi_{2,1}\wedge 1) \qquad \text{when } p\geqq 7. \tag{3.3}$$

But the authors don't know whether $\psi_{2,1}$ is so or not in the $p=5$ case, although the equality

$$\psi_{2,1}(1\wedge\psi_1)=\psi_2(\psi_{2,1}\wedge 1)$$

holds in general.

We now consider the composition $\psi_{2,1}(1\wedge\beta): \Sigma^{pq+q}V(2)\wedge V(1)\to V(2)$. Since $\psi_{2,1}(1\wedge\beta)(i_2\wedge 1)=i_2\psi_{1,1}(1\wedge\beta)=i_2\beta\cdot\psi_{1,1}=0$ by (3.1) and (3.2) there exists a map

$$\rho_2: \Sigma^{2pq+2q+1}V(1)\wedge V(1) \longrightarrow V(2)$$

such that $\psi_{2,1}(1\wedge\beta)=\rho_2(\pi_2\wedge 1)$.

LEMMA 3.1. *$\rho_2=x(\rho_2)i_2\delta_1\beta\cdot\beta'^2 i_1\psi(\pi_1\wedge\pi_1)$, $x(\rho_2)\in Z_5$, if $p=5$ and $\rho_2=0$ if $p\geqq 7$.*

PROOF. Consider the following diagram

$$
\begin{array}{ccc}
 & \{\Sigma^{2pq+3q+2}V(1)\wedge V(0), V(2)\} & \xrightarrow{(1\wedge\pi_1)^*} \{\Sigma^{2pq+2q+1}V(1)\wedge V(1), V(2)\} \\
 & \big\downarrow (1\wedge i)^* & \\
\{\Sigma^{2pq+4q+3}V(0), V(2)\} & \xrightarrow[\pi_1^*]{} \{\Sigma^{2pq+3q+2}V(1), V(2)\} & \\
\big\downarrow i^* & & \\
\{\Sigma^{2pq+4q+3}, V(2)\}. & &
\end{array}
$$

By use of (2.3) ii) we see directly that all maps in the above are isomorphic, and also that $\pi_{2pq+4q+3}(V(2))$ is spanned by one generator $i_2\delta_1\beta\cdot\beta'^2 i_0$ in the $p=5$ case, but it is zero in the other cases. Therefore

$$
\{\Sigma^{2pq+2q+1}V(1)\wedge V(1), V(2)\} \cong \begin{cases} Z_5 & \text{when } p=5 \\ 0 & \text{when } p\geqq 7, \end{cases}
$$

where the former has a generator $i_2\delta_1\beta\cdot\beta'^2\delta_1\psi_1(1\wedge\pi_1)$. The result is now immediate.

**3.2.** Denote by $W$ and $U$ the mapping cones of $\psi_{2,1}(1\wedge\beta)$ and $\rho_2$ respectively. Then we have commutative diagrams

$$
\begin{array}{ccccccc}
\Sigma^{pq+q}V(2)\wedge V(1) & \xrightarrow{1\wedge\beta} & V(2)\wedge V(1) & \xrightarrow{1\wedge i_2} & V(2)\wedge V(2) & \xrightarrow{1\wedge\pi_2} & \Sigma^{pq+q+1}V(2)\wedge V(1) \\
\| & & \big\downarrow \psi_{2,1} & & \big\downarrow w & & \| \\
\Sigma^{pq+q}V(2)\wedge V(1) & \xrightarrow[\psi_{2,1}(1\wedge\beta)]{} & V(2) & \xrightarrow[i_W]{} & W & \xrightarrow[\pi_W]{} & \Sigma^{pq+q+1}V(2)\wedge V(1)
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & & \Sigma^{pq+q+1}V(1)\wedge V(1) & = & \Sigma^{pq+q+1}V(1)\wedge V(1) \\
 & & & & \big\downarrow j_W & & \big\downarrow i_2\wedge 1 \\
\Sigma^{pq+q}V(2)\wedge V(1) & \xrightarrow{\psi_{2,1}(1\wedge\beta)} & V(2) & \xrightarrow{i_W} & W & \xrightarrow{\pi_W} & \Sigma^{pq+q+1}V(2)\wedge V(1) \\
\big\downarrow \pi_2\wedge 1 & & \| & & \big\downarrow k_W & & \big\downarrow \pi_2\wedge 1 \\
\Sigma^{2pq+2q+1}V(1)\wedge V(1) & \xrightarrow[\rho_2]{} & V(2) & \xrightarrow[i_U]{} & U & \xrightarrow[\pi_U]{} & \Sigma^{2pq+2q+2}V(1)\wedge V(1),
\end{array}
$$

where the right-lower square commutes up to the sign $-1$.

As the $V(1)$ case a map $\gamma: X\wedge V(2)\wedge V(2)\to X\wedge V(2)$ is said to be a *pre multiplication of* $X\wedge V(2)$ if $\gamma(1\wedge 1\wedge i_2)=\gamma(1\wedge i_2\wedge 1)(1\wedge T)=1\wedge\psi_{2,1}$. Assume that $1\wedge\rho_2: X\wedge\Sigma^{2pq+2q+1}V(1)\wedge V(1)\to X\wedge V(2)$ is trivial. For any left inverse $\gamma_U: X\wedge U\to X\wedge V(2)$ of $1\wedge i_U$ we define a map

$$
\gamma_2: X\wedge V(2)\wedge V(2)\longrightarrow X\wedge V(2)
$$

by putting $\gamma_2=\gamma_U(1\wedge k_W)(1\wedge w)$.

LEMMA 3.2. *The map $\gamma_2$ is a pre multiplication of $X\wedge V(2)$.*

PROOF. Clearly $\gamma_U(1\wedge k_W)(1\wedge w)(1\wedge 1\wedge i_2)=1\wedge\psi_{2,1}$. $\{V(1)\wedge V(2), V(2)\}$ is generated by $\psi_{2,1}T$ because $(i_0\wedge 1)^*:\{V(1)\wedge V(2), V(2)\}\to\{V(2), V(2)\}$ is isomorphic. We set

$$w(i_2\wedge 1)=j_W(1\wedge\pi_2)+ai_W\psi_{2,1}T, \qquad a\in Z_p$$

as $\pi_{W*}(w(i_2\wedge 1))=\pi_{W*}(j_W(1\wedge\pi_2))$. The above equality yields that $w(i_2i_0\wedge i_2i_0)=i_Wi_2i_0=ai_Wi_2i_0$ which implies a$=1$. Therefore

$$\gamma_U(1\wedge k_W)(1\wedge w)(1\wedge i_2\wedge 1)=\gamma_U(1\wedge k_W)(1\wedge i_W)(1\wedge\psi_{2,1}T)=1\wedge\psi_{2,1}T.$$

For a ring spectrum $E$ every pre multiplication $\gamma$ of $E\wedge V(2)$ gives us a map

$$\mu_\gamma: E\wedge V(2)\wedge E\wedge V(2)\longrightarrow E\wedge V(2)$$

defined by the composition $\mu_\gamma=\gamma(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)$. As is easily seen,

$$(A_1)_2 \qquad \begin{aligned}&\mu_\gamma(1\wedge T\wedge 1)(1\wedge 1\wedge 1\wedge i_2)\\ &\quad=\mu_\gamma(1\wedge T\wedge 1)(1\wedge 1\wedge i_2\wedge 1)(1\wedge 1\wedge T)=\mu\wedge\psi_{2,1}.\end{aligned}$$

The above observation shows

PROPOSITION 3.3. *Let $E$ be a ring spectrum and assume that $1\wedge i_2\delta_1\beta\beta'^2 i_1\psi(\pi_1\wedge\pi_1): E\wedge\Sigma^{2pq+2q+1}V(1)\wedge V(1)\to E\wedge V(2)$ is trivial if $p=5$. Then $E\wedge V(2)$ is a ring spectrum equipped with a multiplication satisfying $(A_1)_2$.*

**3.3.** According to Proposition 3.3, $V(2)$ is a ring spectrum having $i_2i_0$ as the unit when $p\geqq 7$. As is easily checked, its multiplication

$$\psi_{2,2}: V(2)\wedge V(2)\longrightarrow V(2)$$

is unique and it is associative and commutative. Thus

$$(3.4) \qquad \psi_{2,2}T=\psi_{2,2} \quad\text{and}\quad \psi_{2,2}(\psi_{2,2}\wedge 1)=\psi_{2,2}(1\wedge\psi_{2,2}) \qquad \text{when } p\geqq 7.$$

We next discuss the commutativity of $E\wedge V(2)$ in the $p=5$ case. Put $\rho_2'=x(\rho_2)i_2\delta_1\beta\cdot\beta'^2 i_1$ when $p=5$, i.e., $\rho_2=\rho_2'\psi(\pi_1\wedge\pi_1)$, and denote by $L$ its mapping cone. Then we have a commutative (up to sign) diagram

$$\begin{array}{ccccccc}
\Sigma^{pq+q}V(2)\wedge V(1) & \xrightarrow{1\wedge\beta} & V(2)\wedge V(1) & \xrightarrow{1\wedge i_2} & V(2)\wedge V(2) & \xrightarrow{1\wedge\pi_2} & \Sigma^{pq+q+1}V(2)\wedge V(1)\\
\downarrow{\scriptstyle \pi_2\wedge 1} & & \downarrow{\scriptstyle \psi_{2,1}} & & \downarrow{\scriptstyle k_W w} & & \downarrow{\scriptstyle \pi_2\wedge 1}\\
\Sigma^{2pq+2q+1}V(1)\wedge V(1) & \xrightarrow[\rho_2]{} & V(2) & \xrightarrow[i_U]{} & U & \xrightarrow[\pi_U]{} & \Sigma^{2pq+2q+2}V(1)\wedge V(1)\\
\downarrow{\scriptstyle \psi(\pi_1\wedge\pi_1)} & & \| & & \downarrow{\scriptstyle k_U} & & \downarrow{\scriptstyle \psi(\pi_1\wedge\pi_1)}\\
\Sigma^{2pq+4q+3}V(0) & \xrightarrow[\rho_2']{} & V(2) & \xrightarrow[i_L]{} & L & \xrightarrow[\pi_L]{} & \Sigma^{2pq+4q+4}V(0)
\end{array}$$

with three cofiberings.

Setting $k_2=k_U\cdot k_W\cdot w$, $k_2(T-1)$ belongs to $i_{L*}\{V(2)\wedge V(2), V(2)\}$ as $\pi_{L*}(k_2(T-1))=-\psi(\pi_1\wedge\pi_1)(\pi_2\wedge\pi_2)(T-1)=0$. In order to compute the group $\{V(2)\wedge V(2), V(2)\}$ we use the exact sequence

$$\{\Sigma^{pq+q+1}V(2)\wedge V(1), V(2)\}\xrightarrow{(1\wedge\pi_2)^*}\{V(2)\wedge V(2), V(2)\}$$
$$\xrightarrow{(1\wedge i_2)^*}\{V(2)\wedge V(1), V(2)\}\xrightarrow{(1\wedge\beta)^*}\{\Sigma^{pq+q}V(2)\wedge V(1), V(2)\}.$$

A routine computation shows that $\{\Sigma^{pq+q+1}V(2)\wedge V(1), V(2)\}=0$ and $\{V(2)\wedge V(1), V(2)\}$ is generated by $\psi_{2,1}$. If $\psi_{2,1}(1\wedge\beta)\neq 0$, then $\{V(2)\wedge V(2), V(2)\}=0$ which implies $k_2T=k_2\in\{V(2)\wedge V(2), L\}$.

PROPOSITION 3.4. *Let $E$ be a commutative ring spectrum and assume that $1\wedge i_2\delta_1\beta\cdot\beta'^2 i_1: E\wedge\Sigma^{2pq+4q+3}V(0)\to E\wedge V(2)$ is trivial if $p=5$. Then there exists a commutative multiplication of $E\wedge V(2)$ which satisfies the property $(A_1)_2$.*

PROOF. If $\psi_{2,1}(1\wedge\beta)=0$ for $p=5$, then $\rho_2=0$. So we have a multiplication $\psi_{2,2}: V(2)\wedge V(2)\to V(2)$ even if $p=5$. Since $(1\wedge i_2 i_0)^*: \{V(2)\wedge V(2), V(2)\}\to\{V(2), V(2)\}$ is always monic, $\psi_{2,2}$ is commutative. So we may assume that $\psi_{2,1}(1\wedge\beta)\neq 0$ for $p=5$. Any left inverse $\gamma_L: E\wedge L\to E\wedge V(2)$ of $1\wedge i_L$ gives rise to a pre multiplication $\gamma_2$ of $E\wedge V(2)$ defined by the composition $\gamma_L(1\wedge k_2)$, which is commutative. Consequently the multiplication of $E\wedge V(2)$ associated with the above $\gamma_2$ is commutative.

## §4. Brown-Peterson spectrum *BP*

**4.1.** Let $E$ be a ring spectrum equipped with a multiplication $\mu$ and a unit $\iota$. For any map $f: A\to B$ the smash $1\wedge f: E\wedge A\to E\wedge B$ is rewritten as the composition $(\mu\wedge 1)(1\wedge\iota\wedge 1)(1\wedge f)$. So we have

(4.1) $1\wedge f: E\wedge A\to E\wedge B$ *is trivial if* $\{A, E\wedge B\}=0$.

Recall that $\pi_n(S)$ is a finite group for each $n\geqq 1$.

LEMMA 4.1. *Let $f\in\pi_n(S)$, $n\geqq 1$, be a $p$-torsion element. If $\pi_n(E)$ is $p$-torsion free, then $1\wedge f: \Sigma^n E\to E$ is trivial.*

As a summary of Propositions 1.7, 1.13, 2.5, 2.6, 3.3 and 3.4 and (1.9), (2.10) and (3.4) we obtain

THEOREM 4.2. *Let $E$ be an associative and commutative ring spectrum.*

i) *The $p=2$ case: $E\wedge V(0)$ is an associative ring spectrum if $\pi_1(E)$ is 2-torsion free.*

ii) *The $p=3$ case*: *$E \wedge V(0)$ is an associative and commutative ring spectrum if $\pi_3(E)$ is 3-torsion free, and $E \wedge V(1)$ is a commutative ring spectrum if $\pi_{pq-2}(E)$ is 3-torsion free.*

iii) *The $p=5$ case*: *$E \wedge V(1)$ is always associative and commutative ring spectrum, and $E \wedge V(2)$ is a commutative ring spectrum if $\pi_{2pq-4}(E)$ is 5-torsion free.*

iv) *The $p \geqq 7$ case*: *$E \wedge V(1)$ and $E \wedge V(2)$ are always associative and commutative ring spectra.*

Let $E$ be an associative and commutative ring spectrum such as $\pi_*(E)$ is torsion free. For example, as candidates of $E$ we have the $BU$-spectrum $K$, the unitary Thom spectrum $MU$, the Brown-Peterson spectrum $BP$ and so on. Since the above $E$ satisfies all assumptions stated in Theorem 4.2,

(4.2) *$E \wedge V(0)$, $E \wedge V(1)$ and $E \wedge V(2)$ are all ring spectra, and moreover the last two are commutative.*

**4.2.** Fix a prime $p$ and denote by $BP$ the Brown-Peterson spectrum at the prime $p$. This ring spectrum has a coefficient ring $BP_*(=\pi_*(BP)) \cong Z_{(p)}[v_1, \cdots, v_n, \cdots]$ where the degree of $v_n$ is $2(p^n-1)$. There is an equivalent characterization of the $V(n)$ spectra in terms of the $BP$ homology. Thus we may define the spectrum $V(n)$ by specifying the structure of its $BP$-homology as a $BP_*$-module (see [3]):

$$BP_*(V(n)) \cong BP_*/(p, v_1, \cdots, v_n).$$

If $V(n)$ exists and if we can find a map $\omega_n : \Sigma^{2(p^{n+1}-1)}V(n) \to V(n)$ for which $\omega_{n*} : BP_{*-2(p^{n+1}-1)}(V(n)) \to BP_*(V(n))$ is the multiplication by $v_{n+1}$, then $V(n+1)$ is constructed as the mapping cone of $\omega_n$, so

$$\Sigma^{2(p^{n+1}-1)}V(n) \xrightarrow{\omega_n} V(n) \xrightarrow{i_n} V(n+1) \xrightarrow{\pi_n} \Sigma^{2p^{n+1}-1}V(n) \tag{4.3}$$

is a cofibering.

Note that $\pi_*(BP \wedge V(n)) \cong Z_p[v_{n+1}, \cdots]$, $n \geqq 0$. This shows that the canonical inclusion $j_n : \Sigma^0 \to V(n)$ induces isomorphisms

$$\{V(n), BP \wedge V(n)\} \longrightarrow \{\Sigma^0, BP \wedge V(n)\} \qquad \text{when } p \geqq 2,$$
$$\{V(n) \wedge V(n), BP \wedge V(n)\} \longrightarrow \{V(n), BP \wedge V(n)\} \qquad \text{when } p \geqq 3,$$

and

$$\{V(n) \wedge V(n) \wedge V(n), BP \wedge V(n)\} \longrightarrow \{V(n) \wedge V(n), BP \wedge V(n)\} \qquad \text{when } p \geqq 5,$$

because $V(n)$ is $2(p^{n+1}-1)/(p-1)-(n+1)$ dimensional. If $p$ is odd, then there exists a unique map

$$(4.4)_n \qquad q_n : V(n)\wedge V(n) \longrightarrow BP\wedge V(n)$$

whose restriction onto $\Sigma^0$ is the canonical inclusion $\iota\wedge j_n$.

Clearly we have

LEMMA 4.3. *The map $q_n$ satisfies the equalities $q_n(j_n\wedge 1)=q_n(1\wedge j_n)=\iota\wedge 1$ and $q_nT=q_n$.*

It follows immediately that the map $q_n$ has the relation

$$(A_a)_n \qquad (\mu\wedge 1)(1\wedge q_n)(q_n\wedge 1)=(\mu\wedge 1)(1\wedge q_n)(T\wedge 1)(1\wedge q_n)$$

whenever $p\geqq 5$.

We now assume $p=3$, so $V(1)$ exists only. We shall next show that the map $q_1$ satisfies the property $(A_a)_1$, too. By the sparseness of $\pi_*(BP\wedge V(1))$ we get that the sequence

$$0\longrightarrow\{\Sigma^{3q+3}V(0)\wedge V(0)\wedge V(0), BP\wedge V(1)\}$$
$$\xrightarrow{(\pi_1\wedge\pi_1\wedge\pi_1)^*}\{V(1)\wedge V(1)\wedge V(1), BP\wedge V(1)\}\xrightarrow{(i_0\wedge i_0\wedge i_0)^*}\{\Sigma^0, BP\wedge V(1)\}$$

is exact, and

$$(\pi_1\wedge\pi_1\wedge\pi_1)^*: \{\Sigma^{3q+4}V(0)\wedge V(0)\wedge V(0), BP\wedge V(1)\}$$
$$\longrightarrow\{\Sigma^1 V(1)\wedge V(1)\wedge V(1), BP\wedge V(1)\}$$

is isomorphic. Since $\{\Sigma^{16}V(0), BP\wedge V(1)\}$ is spanned by one generator $(\iota\wedge 1)[\beta i_1]$, we have

$$\{\Sigma^{3q+3}V(0)\wedge V(0)\wedge V(0), BP\wedge V(1)\}\cong Z_3+Z_3+Z_3$$

with generators

$$(\iota\wedge 1)[\beta i_1]\psi(\pi\wedge 1\wedge 1),\ (\iota\wedge 1)[\beta i_1]\psi(1\wedge\pi\wedge 1)\quad\text{and}\quad(\iota\wedge 1)[\beta i_1]\psi(1\wedge 1\wedge\pi),$$

and

$$\{\Sigma^{3q+4}V(0)\wedge V(0)\wedge V(0), BP\wedge V(1)\}\cong Z_3$$

with a generator $(\iota\wedge 1)[\beta i_1]\psi(\psi\wedge 1)$.

For the map $q_1: V(1)\wedge V(1)\rightarrow BP\wedge V(1)$ of $(4.4)_1$ we put

$$\nu_1=(\mu\wedge 1)(1\wedge q_1)(T\wedge 1)(1\wedge q_1): V(1)\wedge V(1)\wedge V(1)\longrightarrow BP\wedge V(1).$$

This satisfies the equality

$$(4.5) \qquad \nu_1(1\wedge T)=\nu_1.$$

LEMMA 4.4.

$$\nu_1(T\wedge 1)=\nu_1+a(\iota\wedge 1)[\beta i_1]\psi(\pi\wedge 1\wedge 1+1\wedge\pi\wedge 1+1\wedge 1\wedge\pi)(\pi_1\wedge\pi_1\wedge\pi_1)$$

*where* $a\in Z_3$.

PROOF. Set

$$\nu_1(T\wedge 1)=\nu_1+(\iota\wedge 1)[\beta i_1]\psi(a_1\pi\wedge 1\wedge 1+a_2 1\wedge\pi\wedge 1+a_3 1\wedge 1\wedge\pi)(\pi_1\wedge\pi_1\wedge\pi_1),$$

$a_1, a_2, a_3\in Z_3$ as $(i_0\wedge i_0\wedge i_0)^*(\nu_1(T\wedge 1-1))=0$. Composing $1\wedge T$ from the right we get

$$\begin{aligned}&\nu_1(T\wedge 1)(1\wedge T)\\&\quad=\nu_1-(\iota\wedge 1)[\beta i_1]\psi(a_1\pi\wedge 1\wedge 1+a_2 1\wedge 1\wedge\pi+a_3 1\wedge\pi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1).\end{aligned}$$

We apply $(T\wedge 1)^*$ on two equalities to obtain

$$\begin{aligned}&\nu_1=\nu_1(T\wedge 1)-(\iota\wedge 1)[\beta i_1]\psi(a_1 1\wedge\pi\wedge 1+a_2\pi\wedge 1\wedge 1+a_3 1\wedge 1\wedge\pi)(\pi_1\wedge\pi_1\wedge\pi_1),\\&\nu_1(T\wedge 1)(1\wedge T)\\&\quad=\nu_1(T\wedge 1)+(\iota\wedge 1)[\beta i_1]\psi(a_1 1\wedge\pi\wedge 1+a_2 1\wedge 1\wedge\pi+a_3\pi\wedge 1\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1).\end{aligned}$$

The former implies $a_1=a_2$, and the latter does $a_1=a_3$ and $a_2=a_3$. Thus $a_1=a_2=a_3$.

Recall that $V(1)$ is a $Z_p$-spectrum equipped with unique structure maps $\psi_1$ and $\phi_1$. For any $CW$-spectrum $X$ we may regard $X\wedge V(1)$ as a $Z_p$-spectrum whose structure maps are $1\wedge\psi_1$ and $1\wedge\phi_1$. Abbreviating

$$\begin{aligned}A=[\beta i_1]\psi(\pi\wedge 1\wedge 1+1\wedge\pi\wedge 1+1\wedge 1\wedge\pi)(\pi_1\wedge\pi_1\wedge\pi_1):&\\ V(1)\wedge V(1)\wedge V(1)\longrightarrow V(1),&\end{aligned}$$

we operate the derivation $\theta$ on it.

LEMMA 4.5. $\theta(A)=[\beta i_1]\psi(\psi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1)$.

PROOF. Making use of (1.9), (2.6) and (2.7) we compute

$$\begin{aligned}\theta(A)&=\psi_1([\beta i_1]\wedge 1)(\psi\wedge 1)(\pi\wedge 1\wedge 1\wedge 1+1\wedge\pi\wedge 1\wedge 1+1\wedge 1\wedge\pi\wedge 1)\\&\qquad\qquad(\pi_1\wedge\pi_1\wedge\pi_1\wedge 1)(1\wedge\phi_1)\\&=\psi_1([\beta i_1]\wedge 1)((1\wedge i)\psi+\phi(1\wedge\pi))(\psi\wedge 1)((1\wedge\phi)(\pi\wedge 1\wedge 1+1\wedge\pi\wedge 1)+1)\\&\qquad\qquad(\pi_1\wedge\pi_1\wedge\pi_1)\\&=[\beta i_1](\psi(1\wedge\psi)+i\alpha_1(\pi\wedge\pi\wedge\pi))(1\wedge\phi)(\pi\wedge 1\wedge 1+1\wedge\pi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1)\\&\quad+[\beta i_1]\psi(\psi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1)\\&\quad+\theta[\beta i_1]\psi(\pi\wedge 1\wedge 1+1\wedge\pi\wedge 1+1\wedge 1\wedge\pi)(\pi_1\wedge\pi_1\wedge\pi_1)\end{aligned}$$

$$=[\beta i_1]\psi(\psi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1)$$
$$+\alpha''[\beta i_1](\pi\wedge 1+1\wedge\pi)(\pi\wedge 1\wedge 1+1\wedge\pi\wedge 1+1\wedge 1\wedge\pi)(\pi_1\wedge\pi_1\wedge\pi_1)$$
$$=[\beta i_1]\psi(\psi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1).$$

PROPOSITION 4.6. *The map* $q_n: V(n)\wedge V(n)\to BP\wedge V(n)$ *satisfies the equality* $(\mu\wedge 1)(1\wedge q_n)(q_n\wedge 1)=(\mu\wedge 1)(1\wedge q_n)(T\wedge 1)(1\wedge q_n)$.

PROOF. The $(p, n)=(3,1)$ case: By Lemmas 4.4 and 4.5 we obtain

$$\theta(\nu_1(T\wedge 1-1))=a(\iota\wedge 1)[\beta i_1]\psi(\psi\wedge 1)(\pi_1\wedge\pi_1\wedge\pi_1),\qquad a\in Z_3.$$

On the other hand, it is clear that

$$\theta(\nu_1)=(1\wedge\psi_1)(\mu\wedge 1\wedge 1)(1\wedge q_1\wedge 1)(T\wedge 1\wedge 1)(1\wedge q_1\wedge 1)(1\wedge 1\wedge\phi_1)=0,$$

and

$$\theta(\nu_1(T\wedge 1))=\theta(\nu_1)(T\wedge 1)=0$$

because $\theta(q_1)$ belongs to $\{\Sigma^1 V(1)\wedge V(1), BP\wedge V(1)\}=0$. Consequently we have $a=0$, so $\nu_1(T\wedge 1)=\nu_1$. We use this relation and (4.5) to compute

$$(\mu\wedge 1)(1\wedge q_1)(q_1\wedge 1)=(\mu\wedge 1)(1\wedge q_1)(1\wedge T)(q_1\wedge 1)(1\wedge T)(1\wedge T)$$
$$=\nu_1(T\wedge 1)(1\wedge T)=\nu_1.$$

The other cases have already been done.

**4.3.** When $p\geqq 3$, we consider the map

$$\gamma_n: BP\wedge V(n)\wedge V(n)\longrightarrow BP\wedge V(n)$$

given by the composition $(\mu\wedge 1)(1\wedge q_n)$. A routine computation shows that

$(A_3)'_n$
$$\gamma_n(\mu\wedge 1\wedge 1)=(\mu\wedge 1)(1\wedge\gamma_n)$$
$$\gamma_n(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)(1\wedge 1\wedge T)=(\mu\wedge 1)(1\wedge T)(\gamma_n\wedge 1)$$

as $\mu$ is associative and commutative. Moreover Lemma 4.3 and Proposition 4.6 imply that $\gamma_n$ satisfies the relations

(4.6)
$$\gamma_n(1\wedge j_n\wedge 1)=\gamma_n(1\wedge 1\wedge j_n)=1,\ \gamma_n(1\wedge T)=\gamma_n\quad\text{and}$$
$$\gamma_n(\gamma_n\wedge 1)=\gamma_n(T\wedge 1)(1\wedge\gamma_n)(T\wedge 1\wedge 1).$$

As before we define a multiplication

$$\mu_n: BP\wedge V(n)\wedge BP\wedge V(n)\longrightarrow BP\wedge V(n)$$

to be the composition $\gamma_n(\mu\wedge 1\wedge 1)(1\wedge T\wedge 1)$. By use of (1.13) and (4.6) we obtain

THEOREM 4.7. *If $p\geqq 3$, then $BP\wedge V(n)$ is a ring spectrum equipped with the unit $\iota\wedge j_n$ which is associative and commutative.*

**4.4.** Let $E$ be an associative $BP$-module spectrum whose coefficient module $\pi_*(E)$ is finitely presented as a $BP_*$-module, further $Y$ be a finite $CW$-spectrum and $W$ be a connective $CW$-spectrum such that $HZ_{(p)*}(W)$ is $Z_{(p)}$-free. Since $BP^*(W)$ is $BP_*$-flat, the pairing $BP\wedge E\to E$ gives us an isomorphism

$$BP^*(W)\underset{BP^*}{\otimes}E^*(Y)\longrightarrow E^*(W\wedge Y). \tag{4.7}$$

Assume that $E^*(\ )$ is a $BP_*/(p, v_1, \cdots, v_n)$-module. The generator $v_n$ yields a homomorphism $v_n^*: BP^*(BP)\to BP^{*-2(p^n-1)}$ whose image is contained in the prime ideal $(p, v_1, \cdots, v_n)$ (see [2, Lemma 1.7]). Hence $v_n^*\otimes 1$: $BP^*(BP)\otimes_{BP^*}E^*(Y)\to BP^{*-2(p^n-1)}\otimes_{BP^*}E^*(Y)$ is trivial. Making use of (4.7) the triviality of $v_n^*\otimes 1$ implies that

$$(v_n\wedge 1)^*: E^*(BP\wedge Y)\longrightarrow E^{*-2(p^n-1)}(Y) \tag{4.8}$$

is trivial for any finite $Y$.

Using the Baas-Sullivan theory of manifolds with singularities we can construct $BP$-module spectra $P(n)$ with coefficient modules $P(n)_*(=\pi_*(P(n)))$ $\cong BP_*/(p, v_1, \cdots, v_{n-1})$ (see [2]). In particular

$$P(0)=BP \quad\text{and}\quad P(1)=BP\wedge V(0).$$

$P(n+1)$ is related to $P(n)$ by a cofibering of $BP$-module spectra

$$\Sigma^{2(p^n-1)}P(n)\xrightarrow{\cdot v_n}P(n)\xrightarrow{g_n}P(n+1)\xrightarrow{h_n}\Sigma^{2p^n-1}P(n) \tag{4.9}$$

where $\cdot v_n$ is given by the composition $m_n(v_n\wedge 1): \Sigma^{2(p^n-1)}P(n)\to BP\wedge P(n)\to P(n)$.

Since $E^*(P(n)\wedge X)$ is always Hausdorff for $n\geqq 1$, (4.8) is true for $P(n)\wedge X$. Hence we have

LEMMA 4.8 ([2, Lemma 2.8]). *Let $E$ be an associative $BP$-module spectrum whose coefficient module $\pi_*(E)$ is a finitely presented $BP_*$-module. If $E^*(\ )$ is a $P(n+1)_*$-module, then the cofibering* (4.9) *induces a short exact sequence*

$$0\longrightarrow E^{*-2p^n+1}(X\wedge P(n))\xrightarrow{(1\wedge h_n)^*}E^*(X\wedge P(n+1))\xrightarrow{(1\wedge g_n)^*}E^*(X\wedge P(n))\longrightarrow 0$$

*for any $X$.*

PROPOSITION 4.9. *$BP\wedge V(n)$ is homotopy equivalent to $P(n+1)$.*

PROOF. Beginning with $BP\wedge V(0)=P(1)$ the proof is inductively proceeded. We now assume that there exists a homotopy equivalence $\tau_n: P(n) \to BP\wedge V(n-1)$ which induces the identity in homotopy groups. Note that $BP\wedge V(n)^*(\ )$ becomes a $P(n+1)_*$-module because $BP\wedge V(n)$ is a ring spectrum. In virtue of Lemma 4.8 we can choose a map

$$\tau_{n+1}: P(n+1) \longrightarrow BP\wedge V(n)$$

such that $\tau_{n+1}g_n=(1\wedge i_{n-1})\tau_n$. Since the map $\tau_{n+1}$ induces the identity in homotopy groups, it is a homotopy equivalence.

Theorem 4.7 combined with Proposition 4.9 shows

THEOREM 4.10. *Assume* $p\geqq 3$. *If* $V(n)$ *exists, then* $P(n+1)$ *is an associative and commutative ring spectrum.*

## Appendix

Recall that $P(n)$ is an (associative) $BP$-module spectrum. Thus there exists a pairing $m_n: BP\wedge P(n)\to P(n)$ which satisfies $m_n(\iota\wedge 1)=1$. Denote by $\varepsilon_n: BP\to P(n)$ the composition $g_{n-1}\cdots g_0$.

LEMMA A.1. *There exist multiplications* $\phi_n: P(n)\wedge P(n)\to P(n)$ *such that* $\phi_n(\varepsilon_n\wedge 1)=m_n$, $\phi_n(1\wedge\varepsilon_n)=m_nT$ *and* $\phi_{n+1}(g_n\wedge g_n)=g_n\phi_n$.

PROOF. Assume inductively that there exists a multiplication $\phi_n$ such that $\phi_n(\varepsilon_n\wedge 1)=m_n$ and $\phi_n(1\wedge\varepsilon_n)=m_nT$. We consider the commutative diagram

$$\begin{array}{ccccc}
0\longrightarrow P(n+1)^*(P(n)\wedge P(n)) & \longrightarrow & P(n+1)^*(P(n)\wedge P(n+1)) & \longrightarrow & P(n+1)^*(P(n)\wedge P(n))\longrightarrow 0\\
\downarrow & & \downarrow & & \downarrow\\
0\longrightarrow P(n+1)^*(BP\wedge P(n)) & \longrightarrow & P(n+1)^*(BP\wedge P(n+1)) & \longrightarrow & P(n+1)^*(BP\wedge P(n))\longrightarrow 0,
\end{array}$$

where two rows are induced by the cofibering (4.9) and all vertical arrows are done by the map $\varepsilon_n$. By Lemma 4.8 two rows are exact and all vertical arrows are epic. Note that $g_n$ is a $BP$-module map, i.e., $m_{n+1}(1\wedge g_n)=g_nm_n$. By chasing the above diagram we can choose a map

$$\psi_{n+1}: P(n)\wedge P(n+1)\longrightarrow P(n+1)$$

so that $\psi_{n+1}(1\wedge g_n)=g_n\phi_n$ and $\psi_{n+1}(\varepsilon_n\wedge 1)=m_{n+1}$. We again consider the commutative diagram

$$\begin{array}{ccccc}
0\longrightarrow & P(n+1)^*(P(n)\wedge P(n+1)) & \longrightarrow & P(n+1)^*(P(n+1)\wedge P(n+1)) & \\
& \downarrow & & \downarrow & \\
0\longrightarrow & P(n+1)^*(P(n)\wedge BP) & \longrightarrow & P(n+1)^*(P(n+1)\wedge BP) &
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & P(n+1)^*(P(n)\wedge P(n+1)) & \longrightarrow 0 \\
& \downarrow & \\
\longrightarrow & P(n+1)^*(P(n)\wedge BP) & \longrightarrow 0
\end{array}$$

which consists of two exact rows induced by the cofibering (4.9) and of three vertical arrows induced by $\varepsilon_{n+1}$. By a similar diagram chasing to the above we get a map

$$\phi_{n+1}: P(n+1)\wedge P(n+1)\longrightarrow P(n+1)$$

such that $\phi_{n+1}(g_n\wedge 1)=\psi_{n+1}$ and $\phi_{n+1}(1\wedge\varepsilon_{n+1})=m_{n+1}T$. Clearly $\phi_{n+1}$ has the properties as required.

LEMMA A.2. *If $p\geqq 3$, then we can take as $\phi_n$'s in the above lemma commutative ones.*

PROOF. Assuming that a multiplication $\phi_n$ is commutative we shall construct a commutative one $\phi_{n+1}$ which satisfies the properties stated in Lemma A.1. We use the commutative diagram

$$\begin{array}{ccccc}
& 0 & & 0 & \\
& \downarrow & & \downarrow & \\
0\longrightarrow & P(n+1)^*(P(n)\wedge P(n)) & \longrightarrow & P(n+1)^*(P(n+1)\wedge P(n)) & \\
& \downarrow & & \downarrow & \\
0\longrightarrow & P(n+1)^*(P(n)\wedge P(n+1)) & \longrightarrow & P(n+1)^*(P(n+1)\wedge P(n+1)) & \\
& \downarrow & & \downarrow & \\
0\longrightarrow & P(n+1)^*(P(n)\wedge P(n)) & \longrightarrow & P(n+1)^*(P(n+1)\wedge P(n)) & \\
& \downarrow & & \downarrow & \\
& 0 & & 0 &
\end{array}$$

$$\begin{array}{ccc}
& 0 & \\
& \downarrow & \\
\longrightarrow & P(n+1)^*(P(n)\wedge P(n)) & \longrightarrow 0 \\
& \downarrow & \\
\longrightarrow & P(n+1)^*(P(n)\wedge P(n+1)) & \longrightarrow 0 \\
& \downarrow & \\
\longrightarrow & P(n+1)^*(P(n)\wedge P(n)) & \longrightarrow 0 \\
& \downarrow & \\
& 0 &
\end{array}$$

where all rows and columns are induced by the cofibering (4.9) and they are exact. First, choose a map $\phi'_{n+1}: P(n+1)\wedge P(n+1)\rightarrow P(n+1)$ so that

$$\phi'_{n+1}(\varepsilon_{n+1}\wedge 1)=m_{n+1},\ \phi'_{n+1}(1\wedge\varepsilon_{n+1})=m_{n+1}T \quad\text{and}\quad \phi'_{n+1}(g_n\wedge g_n)=g_n\phi_n.$$

Then we may assume that $\phi'_{n+1}(1\wedge g_n)=\phi'_{n+1}(g_n\wedge 1)T$. So there exists a unique map $w: \Sigma^{4p^n-2}P(n)\wedge P(n)\to P(n+1)$ such that

$$\phi'_{n+1}T=\phi'_{n+1}+w(h_n\wedge h_n).$$

We compose $T$ from the right to obtain

$$\phi'_{n+1}=\phi'_{n+1}T-wT(h_n\wedge h_n).$$

So we find $w=wT$. Putting

$$\phi_{n+1}=\phi'_{n+1}+w/2(h_n\wedge h_n),$$

it becomes commutative, and moreover it has the properties as required.

Consequently we obtain

PROPOSITION A.3. *$P(n)$ is a ring spectrum equipped with $\varepsilon_n\iota$ as unit, and $g_n: P(n+1)\to P(n)$ is a map of ring spectra. Besides $P(n)$ is commutative in the $p\geqq 3$ case.*

REMARK. If $3n<2(p-1)$, then $g_{n-1}$ yields an isomorphism

$$P(n)^*(P(n)\wedge P(n)\wedge P(n))\longrightarrow P(n)^*(P(n-1)\wedge P(n-1)\wedge P(n-1))$$

(cf., [2, Remark 2.14]). In this case $P(n)$ is an associative and commutative ring spectrum.

## References


[1] S. Araki and H. Toda: Multiplicative structures in mod $q$ cohomology theories, I and II, Osaka J. Math., **2** (1965), 71–115 and **3** (1966), 81–120.

[2] D. C. Johnson and W. S. Wilson: BP operations and Morava's extraordinary $K$-theories, Math. Z., **144** (1975), 55–75.

[3] E. Thomas and R. Zahler: Generalized higher order cohomology operations and stable homotopy groups of spheres, Advances in Math. **20** (1976), 287–328.

[4] H. Toda: On spectra realizing exterior parts of the Steenrod algebra, Topology, **10** (1971), 53–65.

[5] H. Toda: Algebra of stable homotopy of $Z_p$-spaces and applications, J. Math. Kyoto Univ., **11** (1971), 197–251.



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