A plan of a positive solution of the Snaith Conjecture (2009) on the Kervaire invariants

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Аннотация

The Kervaire Invariant 1 Problem until recently was an open problem in algebraic topology. For setting and applications, see [A1]. Hill, Hopkins, and Ravenel solved this problem for all dimensions except $n = 126$. In dimension $n = 126$, the problem has not been solved and has the status of a hypothesis by V.P. Snaith (2009). We consider an alternative (with respect to the approach in [H-H-R]) geometric to the Kervaire Invariant 1 Problem and prove the Snaith Conjecture.

1 Self-intersections of generic immerisions and the Kervaire invariant; the problem statement

Let us consider a smooth generic immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$, $n = 2^\ell - 2$, $\ell > 1$ of the codimension 1. Denote by $g : N^{n-2} \looparrowright \mathbb{R}^n$ the immersion of self-intersection manifold of f.

Let us recall a definition of the cobordism group $Imm^{sf}(n-k,k)$, a particular case $k = 1$ is better known: $Imm^{sf}(n-1,1)$. The cobordism group is defined as equivalent classes of triples up to the standard cobordism relation, equipped with a disjoint union operation;

- $f : M^{n-k} \looparrowright \mathbb{R}^n$ is a codimension k immersion;

• $\Xi : \nu(f) \cong k\kappa_M^*(\gamma)$ is a bundle map, which is invertible;
• $\kappa_M \in H^1(M^{n-k};\mathbb{Z}/2)$ is a prescribed cohomology class, which is a mapping $M^{n-k} \to \mathbb{RP}^\infty = K(\mathbb{Z}/2,1)$.

By $\nu(f)$ is denoted the normal bundle of the immersion f, by $\kappa_M^*(\gamma)$ is denoted the pull-back of the universal line bundle γ over $\mathbb{RP}^\infty$ by the mapping κ_M, by $k\kappa_M^*(\gamma)$ is denoted the Whitney sum of the k copies of the line bundles, below for short we will write $k\kappa_M$. The isomorphism Ξ is called a skew-framing of the immersion f.

The Kervaire invariant is an invariant of cobordism classes, which is homomorphism

$$\Theta^{sf} : Imm^{sf}(n-k,k) \to \mathbb{Z}/2. \tag{1}$$

Let us recall the homomorphism (1) in the case $k=1$.

The normal bundle ν_g of the immersion $g : N^{n-2} \looparrowright \mathbb{R}^n$ is a 2-dimensional bundle over N^{n-2}, which is equipped by a $\mathbf{D}$–framing Ξ, where $\mathbf{D}$ is the dihedral group of the order 8. The classifying map of this bundle (and also the corresponding characteristic class) is denoted by $\eta_N : N^{n-2} \to K(\mathbf{D},1)$. The triple (g,η_N,Ξ) represents an element in the cobordism group $Imm^{\mathbf{D}}(n-2,2)$. The correspondence $(f,\kappa,\Psi) \mapsto (g,\eta_N,\Xi)$ defines a homomorphism

$$\delta^{\mathbf{D}} : Imm^{sf}(n-1,1) \to Imm^{\mathbf{D}}(n-2,2). \tag{2}$$

Definition 1. The Kervaire invariant of an immersion f is defined by the formula:

$$\Theta_{sf}(f) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle. \tag{3}$$

It is not difficult to prove that the formula (3) determines a homomorphism

$$\Theta^{\mathbf{D}} : Imm^{\mathbf{D}}(n-2,2) \to \mathbb{Z}/2. \tag{4}$$

The homorphism (4) is called the Kervaire invariant of a **D**–framed immersion. The composition of the homomorphisms (2), (4) determines the homomorphism (1).

The group $Imm^{\mathbf{D}}(n-2,2)$ admits a standard generalization to the cobordism group $Imm^{\mathbf{D}}(n-2k,2k)$ with a parameter $k \ge 1$. We will need the case $k = 7$.

Main Result

In the case $n = 2^{\ell} - 2$, $\ell \ge 7$ the homomorphism $\Theta_{sf}(f)$ (3) is trivial. In the case $n \ge 8$ this result is proved in [H-H-R]. For $n = 7$ this is a new result, conjectured (2009) in [S].

2 First step in proof

In the present and the next sections the cobordism groups $Imm^{sf}(n-k,k)$, $Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k)$ will be used. In the case the first argument in the bracket is strongly positive, the cobordism group is finite.

The dihedral group $\mathbb{Z}/2^{[2]} = \mathbf{D}$ is defined by its corepresentation

$$\{a,b \,|\, b^4 = a^2 = e, [a,b] = b^2\}.$$

This group is represented by rotations a asubgroup in the group $O(2)$ of orthogonal transformation of the standard plane. Elements transforms the base $\{\mathbf{e}_1, \mathbf{e}_2\}$ on the plane $Lin(\mathbf{e}_1, \mathbf{e}_2)$ to itself, a non-ordered pair of coordinate lines on the plane are keeped by transformations. The element b is represented by the rotation of the plane by the angle $\frac{\pi}{2}$. The element a is represented by the reflection of the plane relative to the straight line $l_1 = Lin(\mathbf{e}_1 + \mathbf{e}_2)$ parallel to the vector $\mathbf{e}_1 + \mathbf{e}_2$.

Let us consider a subgroup $\mathbf{I}_{a \times \dot{a}} = \mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[2]}$ in the dihedral group, which is generated by the elements $\{a, b^2a\}$. This is an elementary 2-group of the rank 2. Transformations of this group keep each line l_1, l_2 with the base vectors $\mathbf{f}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{f}_2 = \mathbf{e}_1 - \mathbf{e}_2$ correspondingly. The cohomology group $H^1(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2)$ contains two generators $\kappa_a, \kappa_{\dot{a}}$.

Let us define the cohomology classes

$$\kappa_a \in H^1(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2)), \quad \kappa_{\dot{a}} \in H^1(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2)). \tag{5}$$

Denote by $p_a : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a$ a projection, the kernel of p_a consists the symmetry transformation with respect to the bisector of the second coordinate angle and the identity.

Denote $\kappa_a = p_a^*(t_a)$, where $e \neq t_a \in H^1(K(\mathbf{I}_a, 1); \mathbb{Z}/2) \simeq \mathbb{Z}/2$. Let us denote by $p_{\dot{a}} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \dot{\mathbf{I}}_a$ the projection, the kernel of $p_{\dot{a}}$ consists of the symmetry with respect to the bisector of the first coordinate angle and the identity.

Let us denote $\kappa_{\dot{a}} = p_{\dot{a}}^*(t_{\dot{a}})$, where $e \neq t_{\dot{a}} \in H^1(K(\dot{\mathbf{I}}_a, 1); \mathbb{Z}/2) \cong \mathbb{Z}/2$.

A standardized immersion with the dihedral framing

Let us consider a $\mathbf{D}$-framed immersion (g, Ψ, η_N) of codimension $2k$. Let us assume that the image of g contains in a regular neighbourhood $U(\mathbb{RP}^2)$ of the embedding $\mathbb{RP}^2 \subset \mathbb{R}^n$. The following mapping $\pi \circ \eta_N : N^{n-2k} \to K(\mathbf{D}, 1) \to K(\mathbb{Z}/2, 1)$ is well-defined, where $K(\mathbf{D}, 1) \to K(\mathbb{Z}/2, 1)$ is the epimorphism with the kernel $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$. It is required that this mapping coincides to the composition $i \circ PROJ \circ \eta_N$, where $PROJ : U(\mathbb{RP}^2) \to \mathbb{RP}^2$ is the projection of the neighbourhood onto its central line, $i : \mathbb{RP}^2 \subset K(\mathbb{Z}/2, 1)$ is the standard inclusion, which transforms the fundamental class into the generator.

Below we will clarify a small constant d and denote an integer $\delta = \frac{n}{4} + d \approx \frac{n}{4}$. The integer $\delta \in \mathbb{N}$ is called the co defect of a standardization.

C1 Let us consider the submanifold $N_{sing}^{n-2k-2} \subset N^{n-2k}$, $N_{sing} = PROJ^{-1}(\mathbb{RP}^0), \mathbb{RP}^0 \subset \mathbb{RP}^2$. Require that the mapping

η_N, restricted to N_{sing}, be skipped trough the skeleton of $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$ the dimension $n-2k-\delta$ (δ is the codimension of the considered skeleton inside the skeleton of the dimension $n-2k = dim(N_{sing})$).

Geometrically, this means that the submanifold N_{sing} is a defect of a reduction of the structured mapping of the normal bundle with a control of the subgroup $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ over $\mathbb{RP}^2$ to a mapping with a control over $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

In the case $k \to 0+$ structured mapping on the defect manifold is skipped through a polyhedron of the dimension $\frac{3n}{4}$, in the case $2k \to \frac{n}{2}$ through a polyhedron of the dimension $\frac{n}{4}$.

The complement $N \setminus N_{sing}$ to the defect we get an open manifold, for which the classifying normal bundle mapping is given by $\eta_N : N \setminus N_{sing} \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \rtimes \mathbb{RP}^1$.

Definition 2. Let us say that a $\mathbf{D}$-framed immersion (g, Ψ, η_N) of the codimension $2k$ is standardized with a codefect $\delta \approx \frac{n}{4}$, if the condition $C1$ is satisfied.

Definition 3. Assume $[(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k)$, $f : M^{n-k} \looparrowright \mathbb{R}^n$, $\kappa_M \in H^1(M^{n-k}; \mathbb{Z}/2)$, Ξ is a skew-framing. Let us say that the pair (M^{n-k}, κ_M) admits a compression of an order q, if the mapping $\kappa_M : M^{n-k} \to \mathbb{RP}^\infty$ is represented (up to homotopy) by the following composition: $\kappa = I \circ \kappa'_M : M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^\infty$. Let us say that the element $[(f, \Xi, \kappa_M)]$ admits a compression of an order q, if in its cobordism class exists a triple $(f', \Xi', \kappa_{M'})$, which admits a compression of the order q.

Теорема 4. *1. Assume $n = 2^\ell - 2$, $\ell \geq 7$, $m_\sigma = 14$. An arbitrary class of $\mathbf{D}$-framed cobordism from the image of the homomorphism (2) (for $k > 1$ from the image of the left homomorphism on the bottom line of the diagram (3)) is represented by a standardized dihedral immersion (g, η_N, Ψ) with $d = 2$.* [1]

[1] The constant d cannot be large, in the prove the inequality $d < \frac{3q}{4} - \frac{3}{2}$, which relates

2. Assume that an element $x \in Imm^{sf}(n-\frac{n-m_\sigma}{16}, \frac{n-m_\sigma}{16})$ *admits a compression of the order* $q = \frac{m_\sigma}{2}+1$ *(in particular, for* $q=8$, $k = \frac{n-m_\sigma}{16} = 7$, $m_\sigma = 14$*). By this additional assumption one may get a standardization of* x *with the following additional condition:*

– a submanifold $N^{m_\sigma}_{a \times \dot{a}} \subset N^{n-2k}$*,*

$$N^{m_\sigma}_{a\times\dot{a}} = \eta_N^{-1}(K(n-2k-m_\sigma) \subset K(\mathbf{D},1))$$

admits a structured subgroup $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ *of the normal bundle (equivalently, the mapping* $\pi \circ g : N^{n-2k} \looparrowright U(\mathbb{R}\mathrm{P}^2) \to \mathbb{R}\mathrm{P}^2$*, restricted on the submanifold* $N^{m_\sigma}_{a\times\dot{a}}$*, is homotopic to the constant mapping).*

To prove the theorem a preliminary construction is required.

Let

$$d^{(2)} : \mathbb{R}\mathrm{P}^{n-k'} \times \mathbb{R}\mathrm{P}^{n-k'} \setminus \mathbb{R}\mathrm{P}^{n-k'}_{diag} \to \mathbb{R}^n \times \mathbb{R}^n \qquad (6)$$

be an arbitrary $(T_{\mathbb{R}\mathrm{P}^{n-k'}\times\mathbb{R}\mathrm{P}^{n-k'}}, T_{\mathbb{R}^n\times\mathbb{R}^n})$–equivariant mapping, which is transversal along the diagonal $\mathbb{R}^n_{diag} \subset \mathbb{R}^n \times \mathbb{R}^n$. The diagonal in the pre-image is mapped into the diagonal of the image, by this reason the equivariant mapping $d^{(2)}$ is defined on the open manifold outside of the diagonal (in the pre-image).

(To Condition 1 in Theorem 4 the case $k'=k$ is required, for Condition 2 the codimension k' has to be defined by the formula: $k' = k+q+1$ (parameters k,q correspond to denotations of Theorem 4).

Let us re-denote $(d^{(2)})^{-1}(\mathbb{R}^n_{diag})/T_{\mathbb{R}\mathrm{P}^{n-k'}\times\mathbb{R}\mathrm{P}^{n-k'}}$ by $\mathbb{N}_{circ} = \mathbb{N}(d^{(2)})_\circ$ for short, this polyhedron is called a polyhedron of (formal) self-intersection of the equivariant mapping $d^{(2)}$.

The polyhedron $\mathbb{N}_\circ$ is an open polyhedron, this polyhedron admits a compactification, which is denoted by $\mathbb{N}$ with a boundary

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d with an order q of the compression is required. Based on Theorem [?] by Kee Yuen Lam and Duane Randall (2006), for $n=126$ it is sufficient to get a proof of Main Result in the case $q=10$.

$\partial\mathbb{N}$. The boundary consists of all critical (formal critical) points of the mapping $d^{(2)}$. Let us denote by $\mathbb{N}_\circ$ an open polyhedron $\mathbb{N}\setminus\partial\mathbb{N}$, by $U(\partial\mathbb{N})_\circ$ a thin regular neighbourhood of the diagonal $\partial\mathbb{N}$.

The polyhedron $\mathbb{N}_\circ$ is equipped by the mapping $d^{(2)}$, which admits the following lift:

$$\eta_{circ;Ab} : \mathbb{N}_\circ \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1).$$

On the polyhedron $U(\partial\mathbb{N})_\circ$ the mapping $\eta_{circ;Ab}$ gets the values into the following subcomplex: $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} 2\mathbb{Z}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ (the projection on $K(\mathbb{Z}, 1)$ is the analogue of the projection of the Moebius band onto its central line).

Definition 5. Let us call that a formal (equivariant) mapping $d^{(2)}$, given by (6), is holonomic, if this mapping is the formal extension of a mapping

$$d : \mathbb{RP}^{n-k'} \to \mathbb{R}^n. \qquad (7)$$

Definition 6. Assume a formal (equivariant) mapping (6) is holonomic. Let us say $d^{(2)}$ admits an abelian structure, if the following two conditions are satisfied.

– 1. On the open polyhedron $\mathbb{N}_\circ$ the following mapping is well-defined:

$$\eta_{\circ;Ab} : \mathbb{N}_\circ \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1), \qquad (8)$$

which is a lift of the structured mapping

$$\eta_\circ : \mathbb{N}_\circ \to K(\mathbb{Z}/2^{[2]}, 1).$$

–2. Let us consider the Moebius band M^2 and represent the Eilenberg-MacLane space $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ as a skew-product $K(\mathbf{I}_{a\times\dot{a}}, 1)\tilde{\times}M^2 \to M^2$; the restriction of the fibration over the boundary circle $S^1 = \partial(M^2)$ is identified with the subspace $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} 2\mathbb{Z}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$. Let us include the space $K(\mathbf{I}_{a\times\dot{a}}, 1)\tilde{\times}M^2$ into $K(\mathbf{I}_{a\times\dot{a}}, 1)\tilde{\times}\mathbb{RP}^2$ by a gluing of the trivial bundle over the boundary S^1 by the trivial bundle over a small disk with a central point $\mathbf{x}_\infty \in \mathbb{RP}^2$. The resulting space is denoted by $K(\mathbf{I}_{a\times\dot{a}},1) \rtimes \mathbb{RP}^2$. The following mapping:

$$\eta_{\circ;Ab} : (\mathbb{N}_\circ, U(\partial \mathbb{N}_\circ)) \to (K(\mathbf{I}_{a\times\dot{a}},1) \rtimes \mathbb{RP}^2, K(\mathbf{I}_{a\times\dot{a}},1) \times \mathbf{x}_\infty), \quad (9)$$

is well-defined, where the inverse image by $\eta_{\circ;Ab}$ of the fibre $K(\mathbf{I}_{a\times\dot{a}},1) \times \mathbf{y}_\infty \subset K(\mathbf{I}_{a\times\dot{a}},1) \rtimes \mathbb{RP}^2$, ($\mathbf{y}_\infty \approx \mathbf{x}_\infty$ over a closed point is a subpolyhedron of the dimension less (or, equals) to $\frac{3}{4}(n-k') = \frac{3}{4}\dim(\mathbb{N}_\circ)$ (up to a small constant d).

The following lemma is proved in [A-P2].

Лемма 7. *Small Lemma*
For

$$n - k' \equiv 1 \pmod 2, \quad n \equiv 0 \pmod 2 \quad (10)$$

there exist a holonomic formal mapping $d^{(2)}$, which admits an Abelian structure, Definition 6.

Proof of Theorem 4

Theorem 4 follows from Lemma 4 and a theorem by Kee Yuen Lam and Duane Randall. □

3 Local coefficients and homology groups

Let us define the group $(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \rtimes_{\chi^{[2]}} \mathbb{Z}$ and the epimorphism $(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[2]}$. Consider the automorphism

$$\chi^{[2]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a \times \dot{\mathbf{I}}_a \quad (11)$$

of the exterior conjugation of the subgroup $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ by the element $ba \in \mathbf{D}$, this element is represented by the reflection of the plane with respect to the line $Lin(\mathbf{e}_1)$. Let us define the authomorphism (denotations are not changed)

$$\chi^{[2]} : \mathbb{Z}/2^{[2]} \to \mathbb{Z}/2^{[2]}, \qquad (12)$$

by permutations of the base vectors. It is not difficult to chack, that the inclusion $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[2]}$ commutes with automorphisms (11), (12) in the image and the preimage.

Define the group

$$\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}. \qquad (13)$$

Let us consider the quotient of the group $\mathbf{I}_{a\times\dot{a}} * \mathbb{Z}$ (the free product of the group $\mathbf{I}_{a\times\dot{a}}$ and $\mathbb{Z}$) by the relation $zxz^{-1} = \chi^{[2]}(x)$, where $z \in \mathbb{Z}$ is the generator, $x \in \mathbf{I}_{a\times\dot{a}}$ is an arbitrary element.

This group is a particular example of a semi-direct product $A \rtimes_\phi B$, $A = \mathbf{I}_{a\times\dot{a}}$, $B = \mathbb{Z}$, by a homomorphism $\phi : B \to Aut(A)$; the set $A \times B$ is equipped with a binary operation $(a_1, b_1) * (a_2, b_2) \mapsto (a_1\phi_{b_1}(b_2), b_1b_2)$. Let us define the group (13) by this construction for $A = \mathbf{I}_{a\times\dot{a}}$, $B = \mathbb{Z}$, $\phi = \chi^{[2]}$.

The classifying space $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ is a skew-product over the circle S^1 with $K(\mathbf{I}_{a\times\dot{a}}, 1)$, where the shift mapping in the cyclic covering $K(\mathbf{I}_{a\times\dot{a}}, 1) \to K(\mathbf{I}_{a\times\dot{a}}, 1)$ over $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ is induced by the automorphism $\chi^{[2]}$. The projection onto the circle is denoted by

$$p_{a\times\dot{a}} : K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \to S^1. \qquad (14)$$

Take a marked point $pt_{S^1} \in S^1$ and define the subspace

$$K(\mathbf{I}_{a\times\dot{a}}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \qquad (15)$$

as the inverse image of the marked point pt_{S^1} by the mapping (14).

A description of the standard base of the group $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z})$ is sufficiently complicated and is not required. The group $H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z})$ is described using the Kunneth formula:

$$\begin{array}{c} 0 \to \bigoplus_{i_1+i_2=i} H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z}) \longrightarrow \\ \longrightarrow H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}) \longrightarrow \\ \longrightarrow \bigoplus_{i_1+i_2=i-1} Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z}), H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z}) \to 0. \end{array} \tag{16}$$

The standard base of the group $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1))$ contains the following elements:

$$x \otimes y/(x \otimes y) - (y \otimes x),$$

where $x \in H_j(K(\mathbf{I}_a,1))$, $y \in H_{i-j}(K(\dot{\mathbf{I}}_a,1))$ ($\mathbb{Z}/2$–coefficients is the formulas are omitted).

In particular, for odd i the group $H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z})$ contains elements, which are defined by the fundamental classes of the following submanifolds: $\mathbb{R}\mathrm{P}^i \times pt \subset \mathbb{R}\mathrm{P}^i \times \mathbb{R}\mathrm{P}^i \subset K(\mathbf{I}_a,1) \times K(\dot{\mathbf{I}}_a,1) = K(\mathbf{I}_{a\times\dot{a}},1)$, $pt \times \mathbb{R}\mathrm{P}^i \subset \mathbb{R}\mathrm{P}^i \times \mathbb{R}\mathrm{P}^i \subset K(\mathbf{I}_a,1) \times K(\dot{\mathbf{I}}_a,1) = K(\mathbf{I}_{a\times\dot{a}},1)$. Let us denote the corresponding elements as following:

$$t_{a,i} \in H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}), t_{\dot{a},i} \in H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}). \tag{17}$$

The following analogues of the homology groups $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1))$, $H_i(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z})$ with local coefficients [2] is defined, the groups are denoted by

$$H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]), \tag{18}$$

$$H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2]). \tag{19}$$

The following epimorphism

$$p_{a\times\dot{a}} : \mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z} \tag{20}$$

[2]more simple homology groups $H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z})$ with $\mathbb{Z}$-local coefficient system also can be defined, in this case on the last step of the construction difficulties with triple (non-commutative) local coefficient system arise.

is well-defined by the formula: $x * y \mapsto y$, $x \in \mathbf{I}_{a\times\dot{a}}$, $y \in \mathbb{Z}$. The following homomorphism

$$\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2, \qquad (21)$$

is well-defined by the formula $p_{a\times\dot{a}} \pmod 2$.

Let us define the group (19). Let us consider the group ring $\mathbb{Z}[\mathbb{Z}/2] = \{a + bt\}$, $a, b \in \mathbb{Z}$, $t \in \mathbb{Z}/2$. The generator $t \in \mathbb{Z}[\mathbb{Z}/2]$ is represented by the involution

$$\chi^{[2]} : K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1),$$

the restriction of the involution on the subspace $K(\mathbf{I}_{a\times\dot{a}}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ is the reflection, which is induced by the automorphism $\mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a \times \dot{\mathbf{I}}_a$, which permutes factors. Because all non-trivial homology classes of the space $K(\mathbf{I}_{a\times\dot{a}}, 1)$ are of the order 2, transformation of signs is not required. Nevertheless, even-dimensional $n - 2k$-simplexes are transformed in the case of odd k by the opposition of the orientation, in the case of even k the orientation is preserved.

Let us consider the local system of the coefficient ρ_t : $\mathbb{Z}/2[\mathbb{Z}/2] \to Aut(K(\mathbf{I}_{a\times\dot{a}}, 1) \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1))$, using this local system a chain $(a + bt)\sigma$ with the support on a simplex $\sigma \subset K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ is transformed into a chain $(at + b)\chi^{[2]}(\sigma)$.

The group (19) is well-defined. The group (18) is defined analogously. A complete calculation of the groups (18), (19) is not required.

Let us define the following subgroup:

$$D_i^{loc}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}[\mathbb{Z}/2]) \subset H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]) \qquad (22)$$

by the formula: $D_i^{loc}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}[\mathbb{Z}/2]) = \mathrm{Im}(H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]))$, where the homomorphism

$$H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2])$$

is induced by the inclusion of the subgroup.

From the definition, there exist a natural epimorphism:

$$H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}[\mathbb{Z}/2]) \to D_i^{loc}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}[\mathbb{Z}/2]). \qquad (23)$$

The following natural homomorphism $H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z})) \otimes \mathbb{Z}[\mathbb{Z}/2] \to H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}[\mathbb{Z}/2])$ is well-defined, this homomorphism is an isomorphism by the universal coefficients formula. This calculation is following from the isomorphism $\mathbb{Z}[\mathbb{Z}/2] \equiv \mathbb{Z} \oplus \mathbb{Z}$ an the additivity of the derived functor $Tor^{\mathbb{Z}}$. Analogously, $H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}/2)) \otimes \mathbb{Z}/2[\mathbb{Z}/2] \equiv H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}/2[\mathbb{Z}/2])$.

The subgroup (22) is generated by various elements $X+Yt$, $X,Y \in H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z})$. The following equivalent relation determines the equality of two elements: $X \equiv \chi_*^{[2]}(X)t$, where the authomorphism

$$\chi_*^{[2]} : H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}) \to H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}) \qquad (24)$$

is induced by the authomorphism (11). The authomorphism (11) induces also the authomorphism

$$\chi_*^{[2]} : H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}[\mathbb{Z}/2]) \to H_i(K((\mathbf{I}_{a\times\dot{a}});\mathbb{Z}[\mathbb{Z}/2]). \qquad (25)$$

A subgroup

$$D_i^{loc}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2[\mathbb{Z}/2]) \subset H_i^{loc}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]) \quad (26)$$

is defined analogously as (22). A description of this subgroup is more easy, because this group is generated by elements $X+tY$, $X = x\otimes y, Y = x'\otimes y'$, wherein $\chi_*^{[2]}(x\otimes y) = y \otimes x$.

Let us define the homomorphism

$$\Delta^{[2]} : H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}) \qquad (27)$$

by the formula: $\Delta^{[2]}(X+Yt) = X + \chi_*^{[2]}(Y)$. Let us prove that the homomorphism (27) admits a factorisomorphism:

$$\Delta^{[2]} : D_i^{loc}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{a\times\dot{a}},1);\mathbb{Z}), \qquad (28)$$

which is denoted the same. This fact follows from the fact that the kernel of the homomorphism (23) is generated by elements $X-\chi^{[2]}_*(X)t$, which is clear by geometrical reason.

Analogously, the homomorphism with modulo 2 coefficients is well-defined.

$$\Delta^{[2]}: D^{loc}_i(\mathbf{I}_{a\times \dot{a}};\mathbb{Z}/2[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{a\times \dot{a}},1)). \qquad (29)$$

The composition of the homomorphisms

$$H_i(K(\mathbf{I}_{a\times \dot{a}},1);\mathbb{Z}) \to H_i(K(\mathbf{I}_{a\times \dot{a}},1);\mathbb{Z}[\mathbb{Z}/2]) \to \qquad (30)$$

$$D^{loc}_i(\mathbf{I}_{a\times \dot{a}};\mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{a\times \dot{a}},1);\mathbb{Z}),$$

where the first homomorphism is the natural inclusion, the middle is (23), and the last in the composition is defined by the formula (28), is the identity homomorphism.

Let us define the forgetful homomorphism:

$$forg_*: H^{loc}_i(K(\mathbf{I}_{a\times \dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{a\times \dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}) (31)$$

This homomorphism is induced by the omitting the local coefficient system and is analogous to the homomorphism $\Delta^{[2]}$.

In the case $i=2s$ the basis elements $y \in D^{loc}_{i-1}(\mathbf{I}_{a\times \dot{a}};\mathbb{Z}[\mathbb{Z}/2])$, with the condition $\chi^{[2]}_*(y)=y$, are the following:

1. $y=r$, $r=t_{a,2s-1}+t_{\dot{a},2s-1}$, where $t_{a,2s-1},t_{\dot{a},2s-1} \in H_{2s-1}(K(\mathbf{I}_{a\times \dot{a}},1);\mathbb{Z})$ are defined by the formula (17).

2. $y=z(i_1,i_2)$, where $z(i_1,i_2)=tor(r_{a,i_1},r_{\dot{a},i_2})+tor(r_{a,i_2},r_{\dot{a},i_1})$, $i_1,i_2 \equiv 1 \pmod 2$, $i_1+i_2=2s-2$, $tor(r_{a,i_1},r_{\dot{a},i_2}) \in Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z}),H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z}))$. The elements $\{z(i_1,i_2)\}$ belong to the kernel of the homomorphism:

$$D^{loc}_{i-1}(\mathbf{I}_{a\times \dot{a}};\mathbb{Z}[\mathbb{Z}/2]) \to D^{loc}_{i-1}(\mathbf{I}_{a\times \dot{a}};\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (32)$$

which is defined as the modulo 2 reduction of the coefficients: $\mathbb{Z}\to\mathbb{Z}/2$.

The elements $R, \quad Z(i_1,i_2) \in H^{lok}_{2s}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$ are defined as the direct product of the corresponding $2s-1$–cycle $f : C_{2s-1} \to K(\mathbf{I}_{a\times\dot{a}},1)$ on the circle. The mapping of the cycle into $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1)$ is defined by the composition of the cartesian product mapping $f \times id : C_{2s-1} \times S^1 \to K(\mathbf{I}_{a\times\dot{a}},1) \times S^1$ with the standard 2-sheeted covering $K(\mathbf{I}_{a\times\dot{a}},1) \times S^1 \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1)$.

Лемма 8. *The group* $H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$ *is isomorphic to the direct sum of the subgroup* $D^{loc}_{2s}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}[\mathbb{Z}/2])$ *and the subgroup, which is generated by elements* $R, \quad \{Z(i_1,i_2)\}$. *The elements from the subgroup*

$$\bigoplus_{i_1+i_2=2s} H_{i_1}(K(\mathbf{I}_a,1);\mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{I}}_a,1);\mathbb{Z}) \subset D^{loc}_{2s}(\mathbf{I}_a \times \dot{\mathbf{I}}_a;\mathbb{Z}[\mathbb{Z}/2]),$$

and the element R *generate the image* $\mathrm{Im}(A)$ *of the following homomorphism:*

$$A : H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{b}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2]) \to \tag{33}$$
$$H^{loc}_{2s}(K(\mathbf{I}_{a\times\dot{b}} \rtimes_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]),$$

which is the modulo 2 reduction of the coefficient system.

Let us concider the diagonal subgroup: $i_{\mathbf{I}_d,\mathbf{I}_{a\times\dot{a}}} : \mathbf{I}_d \subset \mathbf{I}_a \times \dot{\mathbf{I}}_a = \mathbf{I}_{a\times\dot{a}}$. This subgroup coincides to the kernel of the homomorphism

$$\omega^{[2]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbb{Z}/2, \tag{34}$$

which is defined by the formula $(x \times y) \mapsto xy$.

Let us define the homomorphism

$$\Phi^{[2]} : \mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[2]} \tag{35}$$

by the formula: $\Phi^{[2]}(z) = ab$, $z \in \mathbb{Z}$ is the generator (the element ab is represented by the inverson of the first basis vector $\mathbb{Z}^{[2]} \subset O(2)$),

$$\Phi^{[2]}|_{\mathbf{I}_{a\times\dot{a}}\times\{0\}} : \mathbf{I}_{a\times\dot{a}} \subset \mathbf{D}$$

is the standard inclusion;

$$\Phi^{[2]}|_{z^{-1}\mathbf{I}_a \times \dot{\mathbf{I}}_a z} : \mathbf{I}_{a\times\dot{a}} \subset \mathbf{D}$$

is the conjugated inclusion, by the composition with the exterior automorphism in the subgroup $\mathbf{I}_{a\times\dot{a}} \subset \mathbb{Z}/2^{[2]}$.

Let us define

$$(\Phi^{[2]})^*(\tau_{[2]}) = \tau_{a\times\dot{a}},$$

where $\tau_{a\times\dot{a}} \in H^2(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1))$, $\tau_{[2]} \in H^2(K(\mathbb{Z}^{[2]}, 1))$.

4 The fundamental class of the canonical covering over the self-intersection D-framed standardized immersion without a defect

Let us consider a $\mathbb{Z}/2^{[2]}$–framed immersion (g, Ψ, η_N), $g : N^{n-2k} \looparrowright \mathbb{R}^n$, and assume that this immersion is standardized.

The mapping η_N admits a reduction, the image of this mapping belongs to the total bundle space over $\mathbb{R}\mathrm{P}^2$ with the fibre $K(\mathbf{I}_{a\times\dot{a}}, 1)$. Assume that the manifold N^{n-2k} is connected. Assume that a marked point $PROJ^{-1}(\mathbf{x}_\infty) \in N^{n-2k}$ is fixed. Assume that the immersion g translates a marked point into a small neighbourhood of the central point $\mathbf{x}_\infty \in \mathbb{R}\mathrm{P}^2$, over this point the defect is well-defined.

The image of the fundamental class $\eta_{N,*}([N^{n-2k}])$ belongs to $H_{n-2k}(\mathbf{D}; \mathbb{Z}/2)$. The standardization condition of implies an existence of a natural lift of this homology class into the group $H_{n-2k}(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}; \mathbb{Z})$, because the condition C1 implies that the submanifold $N_{sing} \subset N^{n-2k}$, which is a fibre over the point $\mathbf{x}_\infty \in \mathbb{R}\mathrm{P}^2$ determines the trivial fundamental class in $H_{n-2k-2}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z})$. If N^{n-2k} is a self-intersection of the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$, then the compression condition of of the class $\kappa \in H^1(M; \mathbb{Z}/2)$ in the codimension q, for $q \geq 2$, implies the existence of a lift of the class $\eta_{N,*}([N^{n-2k}])$ into $H_{n-2k}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z})$. Moreover, the homology

Euler class $PD(e_*(\nu(g)))$ of the normal bundle also is lifted into $H_{n-4k}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z})$.

Let us consider the self-intersection manifold L^{n-4k} of the immersion g and its canonical double covering $\bar{L}^{n-4k}$, which is immersed into N^{n-2k}. Let us assume (this assumption gives no restriction in the proof) that the defect N_{sing} is empty and, therefore, the characteristic mapping η_N has the target space $K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$.

Let us investigate the Hurewicz image of the fundamental class $[\bar{L}^{n-4k}]$. Define a local coefficient system and prove that the image of the fundamental class $[\bar{L}^{n-4k}, pt]$ by η_N determines an element

$$\eta_*([\bar{L}^{n-4k}, pt]) \in H^{loc}_{n-2k}(K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]). \qquad (36)$$

in the homology group with this system.

Let us consider a skeleton of the space $K(\mathbf{I}_{a\times\dot{a}}, 1)$, which is realized as the structured group $\mathbf{I}_{a\times\dot{a}} \subset O(2+2)$ of a bundle over the corresponding Grasmann manifold $Gr_{\mathbf{I}_{a\times\dot{a}}}(2+2, n)$ of non-oriented $2+2=4$-plans in n–space. Denote this Grasmann manifold by

$$KK(\mathbf{I}_{a\times\dot{a}}) \subset K(\mathbf{I}_{a\times\dot{a}}, 1). \qquad (37)$$

On the space (37) a free involution

$$\chi^{[2]} : KK(\mathbf{I}_{a\times\dot{a}}) \to KK(\mathbf{I}_{a\times\dot{a}}), \qquad (38)$$

acts, this involution corresponds to the automorphism (11) and represents by a permutation of planes of fibres (by a block-antidiagonal matrix).

Let us define a family of spaces (37) with the prescribed symmetry group, are parametrized over the base $\mathbb{RP}^2$, the generator of $\mathbb{RP}^2$ transforms fibres by the involution 37. This is the required classifying space, the target space of the mapping η_N. This space is a subspace (a skeleton) in the following subspace, which is a skew-product: $K(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{RP}^2 \subset K(\mathbf{D}, 1)$.

To define the required space

$$KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{R}\mathrm{P}^2) \qquad (39)$$

we start by the torus of the involution (38), which is denoted by

$$KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}). \qquad (40)$$

Then define the space (41) by a quotient of the direct product $KK(\mathbf{I}_{a\times\dot{a}}) \times S^2$ with respect to the involution $\chi^{[2]} \times (-1)$, where $-1 : S^2 \to S^2$–is the antipodal involution. The following inclusion is well-defined:

$$KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}) \subset KK(\mathbf{I}_{a\times\dot{a}}) \rtimes \mathbb{R}\mathrm{P}^2. \qquad (41)$$

The involution (38) induces the S^1-fibrewise involution

$$\chi^{[2]} : KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}) \to KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}), \qquad (42)$$

which is extended to the involution

$$\chi^{[2]} : KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{R}\mathrm{P}^2) \to KK(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{R}\mathrm{P}^2) \qquad (43)$$

on $KK(\mathbf{I}_{a\times\dot{a}}) \rtimes \mathbb{R}\mathrm{P}^2$, all this extended involutions denote the same.

The universal 4-dimensional $\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}/2$–bundle over $KK(\mathbf{I}_{a\times\dot{a}}, 1) \rtimes \mathbb{R}\mathrm{P}^2$ is well-defined, the restriction of the bundle on the subspace (37) is well-defined. Let us denote this universal bundle by $\eta_{a\times\dot{a}}$.

Let us define $\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}$–reduction of the structured mapping of $\bar{L}^{n-4k}$ for the standardized immersion g and the framing $\Psi_{\bar{L}} : \nu_g|_{\bar{L}} \equiv k\eta^*_{a\times\dot{a}}(\tau_{a\times\dot{a}})$. By the construction, the normal bundle ν_g of the immersion g over $\bar{L}^{n-4k}$ is the Whitney sum of k isomorphic 2-dimensional blocks. The normal bundle over $\bar{L}^{n-4k}$ inside N^{n-2k} is isomorphic is the Whitney sum of k isomorphic 2-block, analogously. Therefore the normal bundle over $\bar{L}^{n-4k}$ is the Whitney sum of k $2+2$ blocks, each first term in a block is orthogonal to $g(N^{n-2k})$, the second term is parallel to the tangent space of $g(N^{n-2k})$. Each term are decomposed into line bundles, which are classified by the classes $\kappa_a, \kappa_{\dot{a}}$ correspondingly, the decomposition of the 2-block into line subbundles is globally defined, but the denotation $\kappa_a, \kappa_{\dot{a}}$ are well-defined up to the permutation.

The two 2-terms in a 4-block are permuted along a path $l \subset L^{n-4k}$, $\bar{L}^{n-4k} \to L^{n-4k}$, which corresponds to a permutation of leafs of the self-intersection manifold. Such a path represents also an odd element in $H_1(S^1)$ by the mapping (14).

Let us consider an arbitrary cell α of a regular cell-decomposition of the manifold N^{n-2k}. Assume that a prescribed path ϕ_α, which connects central points of a cell α with a marked point $pt \in N^{n-2k}$. The restriction of the fibre bundle $\nu_{a \times \dot{a}}$ on α is classified by the mapping $\eta_{a \times \dot{a}}(\alpha, \phi_\alpha) : \alpha \to KK(\mathbf{I}_{a \times \dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1)$ into the space (41).

A change of the path ϕ_α with a fixed boundary points in a conjugacy class with respect to the subgroup $H_1(S^1)$ determines an alternative mapping $\eta_{a \times \dot{a}}(\alpha, \phi_\alpha)$, the two mappings are related by the composition with the involution (42). In a cell complex of the space (40) with $\mathbb{Z}[\mathbb{Z}/2]$–local coefficients this changing of a path for a chain on αcorresponds to the multiplication of the coefficient of the chain by the variable t and, simultaneously, by the changing of the classified map to the Grasmann manifold by the involution (42). Therefore, the element (36) is well-defined.

Let us apply this to the canonical covering $\bar{L}^{n-4k}$ globally, $\bar{L}^{n-4k} = \nu_g \oplus T(\nu_g)$. On the g-normal 2-block of $\nu(\bar{L})$, this block is parallel to the normal bundle of ν_g, define this reduction as above; on the second 2-block, this block is parallel to the tangent bundle $T(\nu_g)|_{N^{n-2k}}$, define the $\chi^{[2]}$-conjugated reduction. In the case a path to the marked point is changed into the conjugated path, the both terms are transformed by $\chi^{[2]}$ simultaneously. Therefore, the local coefficient system on $\bar{L}$ is well-defined.

Let us consider the image of the element (36) by the projection onto $D^{loc}_{n-4k}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2[\mathbb{Z}/2])$, this projection algebraically is determined in Lemma 8.

In the case $dim(N)=n-8k$, we get $dim(N_{atimes\dot{a}})=m_\sigma=n-16k$ (see Theorem 4); when $n=126$, $k=7$ implies $m_\sigma=14$.

Let us prove that the image of the fundamental class of the canonical covering $\bar{L}$, $dim(\bar{L})=m_\sigma$, determines an element

$$\eta_*(\bar{L}^{m_\sigma}_{a\times\dot{a}}) \in D^{loc}_{m_\sigma}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (44)$$

which is transformed to the element

$$\eta_*(N^{m_\sigma}_{a\times\dot{a}}) \in H_{m_\sigma}(K(\mathbf{I}_{a\times\dot{a}},1)) \qquad (45)$$

by the homomorphism (29).

The manifold N^{n-8k} is assumed connected. Let us consider a decomposition of the self-intersection manifold into components:

$$\bar{L}^{n-16k}_{a\times\dot{a}} \equiv \cup_i \bar{L}^{n-16k}_{i,a\times\dot{a}} \subset N^{n-8k}. \qquad (46)$$

For each connected component of (46) let us take a marked point $pt_i \in N^{n-16k}_{i,a\times\dot{a}}$. Take a path ρ_i on N^{n-2k} from the point pt_i into pt. For each i an isomorphism of the fibres of $\kappa_a \oplus \kappa_{\dot{a}}$ over pt and over pt_i is defined along the corresponding path.

Therefore, the mapping

$$\eta_{\bar{L}_{i,a\times\dot{a}}}(\rho_i) : \bar{L}^{n-16k}_{i,a\times\dot{a}} \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1) \qquad (47)$$

is well-defined. An immersed manifold $\bar{L}^{n-16k}_{a\times\dot{a}}$ is framed by $\Psi_{a\times\dot{a}} \oplus \bar{\Psi}_{a\times\dot{a}}$. The framing $\Psi_{a\times\dot{a}} \oplus \bar{\Psi}_{a\times\dot{a}}$ over each component $\bar{L}^{n-16k}_{i,a\times\dot{a}}$ is totally determined by a coordinate system into the fibre over the marked point.

Alternatively, a coordinate system in the fibre over pt_i can be changed along the path, which corresponds to the element $s(i) \in \mathbb{Z}/2$ in the residue class of the subgroup $\mathbf{I}_{a\times\dot{a}} \subset \mathbb{Z}/2^{[2]}$. For the element in the nontrivial resedue class the transformation is given by the element $ba \in \mathbf{D}$. Let us consider an element

$\eta_{i,*}(\rho_i)(\bar{L}^{n-16k}_{i,a\times\dot{a}}) \in H_{m_\sigma}(\mathbf{I}_{a\times\dot{a}})$, which is denoted by $x_i(\rho_i)$. Let us define the element

$$X_i \in D_{m_\sigma}(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]) \qquad (48)$$

equals to $x_i(\rho_i) + 0t$, in the case the framing $\Psi_{a\times\dot{a}}$ on $\bar{L}^{n-16k}_{i,a\times\dot{a}}$ corresponds with the framing, which is defined by an extension of the framing $\Psi_{a\times\dot{a}}$ along ρ_i; and define $Xi_i = 0 + \chi^{[2]}_*(x_i)t$, in the case the considered framing are not agree. The element (44) is well-defined.

By the construction, the element (44) is not depend of a choice of the path ρ_i.

Лемма 9. *–1. The element* (44) *has the image by the homomorphism* (29)*, such as the decomposition of this image with respect to the standard base of the group* $H_{m_\sigma}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$ *contains not more then the only monomial* $t_{a,i} \otimes t_{\dot{a},i}$*, see.* (17)*,* $i = \frac{m_\sigma}{2} = \frac{n-16k}{2}$*. The coeffisient of this monomial coinsids with the Kervaire invariant, which is calculated for a* $\mathbb{Z}/2^{[2]}$*-framed immersion* (g, Ψ, η_N)*.*

–2. The element (36) *belongs to the subgroup* (26)*,* $i = n - 2k$*.*

Proof of Lemma 9

Let us proof Statement 1 for the case $m_\sigma = 14$. Let us consider the manifold $N^{14}_{a\times\dot{a}}$, which we re-denote in the proof by N^{14} for sort. This manifold is equipped with the mapping $\eta : N^{14} \to K(\mathbf{I}_{a\times\dot{a}}, 1)$. Let us consider all the collection of characteristic $\mathbb{Z}/2$-numbers for the mapping η, which is induced from the universal classes.

Let us consider the manifold $\bar{L}^{n-16k}$ and its fundamental class (44). By arguments from Herbert theorem the following two classes $-[N^{14}_{a\times\dot{a}}]$ (45) and (44) coincides in the group $D^{loc}_{m_\sigma}(\mathbf{I}_{a\times\dot{a}}; \mathbb{Z}/2[\mathbb{Z}/2])$. Analogously, the projection of the opposite class (but modulo 2 coefficients the opposition is not required) the classes (45) coincides to (44) in $H_{m_\sigma}(K(\mathbf{I}_{a\times\dot{a}},1))$.

Let us estimate the fundamental class $[N^{14}]$ and prove that this class is pure (see definition 14) below. Because N^{14} is oriented, among characteristic numbers: $\kappa_a\kappa_{\dot{a}}^{13}$, $\kappa_a^3\kappa_{\dot{a}}^{11}$, $\kappa_a^5\kappa_{\dot{a}}^9$, $\kappa_a^7\kappa_{\dot{a}}^7$, $\kappa_a^9\kappa_{\dot{a}}^5$, $\kappa_a^3\kappa_{\dot{a}}^{11}$, $\kappa_a\kappa_{\dot{a}}^{13}$. the only number $\kappa_a^7\kappa_{\dot{a}}^7$ could be nontrivial.

It is sufficiently to note, that the characteristic number $\langle\kappa_a^7\kappa_{\dot{a}}^7;[N^{14}]\rangle$ coincides with the characteristic number in the lemma. Statement 1 is proved.

Statement 2 is an refrmulation of Statement 2 of Theorem 4. Lemma 9 is proved. $\square$

Definition 10. Let (g,η_N,Ψ) be the standardized $\mathbf{D}$-framed immersion in the codimension $2k$.

Let us say that this standardized immersion is negligible, if its fundamental class $\pmod 2$

$$\eta_*(N_{a\times\dot{a}}^{n-2k}) \in D_{n-2k}^{loc}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2[\mathbb{Z}/2]) \qquad (49)$$

is trivial.

Let us say that this standardized immersion is pure, if its fundamental class $\pmod 2$ satisfies the condition of Lemma 9, i.e. the Hurewicz image of the fundamental class in the group $\in D_{n-2k}^{loc}(\mathbf{I}_{a\times\dot{a}};\mathbb{Z}/2[\mathbb{Z}/2])$ contains no monomials $t_{a,\frac{n-2k}{2}+i}\otimes t_{\dot{a},\frac{n-2k}{2}-i}$, $i\in\{\pm1;\pm2;\pm3;\pm4;\pm5;\pm6;\pm7\}$, but, probably, contains the monomial $t_{a,\frac{n-2k}{2}}\otimes t_{\dot{a},\frac{n-2k}{2}}$.

5 $\mathbf{H}_{a\times\dot{a}}$–structure on $\mathbb{Z}/2^{[3]}$–framed immersion $\mathbf{J}_b\times\dot{\mathbf{J}}_b$–structure on $\mathbb{Z}/2^{[4]}$–framed immersion

The group $\mathbf{I}_b$ is defined as the cyclic subgroup of the order 4 in the dihedral group: $\mathbf{I}_b\subset\mathbb{Z}/2^{[2]}$.

Let us define an analogous subgroup

$$i_{\mathbf{J}_b \times \dot{\mathbf{J}}_b} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbb{Z}/2^{[4]}, \qquad (50)$$

which is isomorphic to the Cartesian product of the two cyclic groups of the order 4.

The group $\mathbb{Z}/2^{[4]}$ (the monodromy group fot the 3-uple iterated self-intersection of a skew-framed immersion) is defined using the base $(\mathbf{e}_1, \dots, \mathbf{e}_8)$ in the Euclidean space $\mathbb{R}^8$. Let us denote the generators of the subgroup $\mathbf{J}_b \times \dot{\mathbf{J}}_b$ by b, $\dot{b}$ correspondingly. Let us describe transformations in $\mathbb{Z}/2^{[4]}$, which corresponds to each generator.

Let us consider an orthogonal base $\{\mathbf{f}_1, \dots, \mathbf{f}_8\}$, which is determined by the formulas: $\mathbf{f}_{2i-1} = \frac{\mathbf{e}_{2i-1}+\mathbf{e}_{2i}}{\sqrt{2}}$, $\mathbf{f}_{2i} = \frac{\mathbf{e}_{2i-1}-\mathbf{e}_{2i}}{\sqrt{2}}$, $i = 1, \dots, 4$.

The transformation group $\mathbf{J}_b \times \dot{\mathbf{J}}_b$ have invariant pairwise orthogonal $(2,2,2,2)$-dimensional subspaces, which we denote by $\mathbb{R}^2_{b,+}$, $\mathbb{R}^2_{b,-}$, $\mathbb{R}^2_{\dot{b},+}$, $\mathbb{R}^2_{\dot{b},-}$ correspondingly.

The subspace $\mathbb{R}^2_{b,+}$ is generated by linear combinations of pairs of vectors: $Lin(\mathbf{f}_1 + \mathbf{f}_5, \mathbf{f}_3 + \mathbf{f}_7)$. The subspace $\mathbb{R}^2_{b,-}$ is generated by linear combinations $Lin(\mathbf{f}_1 - \mathbf{f}_5, \mathbf{f}_3 - \mathbf{f}_7)$. The subspace $\mathbb{R}^2_{\dot{b},+}$ is generated by linear combinations $Lin(\mathbf{f}_2 + \mathbf{f}_4, \mathbf{f}_6 + \mathbf{f}_8)$. The subspace $\mathbb{R}^2_{\dot{b},-}$ is generated by linear combinations $Lin(\mathbf{f}_2 - \mathbf{f}_4, \mathbf{f}_6 - \mathbf{f}_8)$.

It is convenient to pass to a new basis:

$$\frac{\mathbf{f}_1 + \mathbf{f}_5}{\sqrt{2}} = \mathbf{h}_{1,+}, \frac{\mathbf{f}_1 - \mathbf{f}_5}{\sqrt{2}} = \mathbf{h}_{1,-}, \frac{\mathbf{f}_3 + \mathbf{f}_7}{\sqrt{2}} = \mathbf{h}_{2,+}, \frac{\mathbf{f}_3 - \mathbf{f}_7}{\sqrt{2}} = \mathbf{h}_{2,-}, \qquad (51)$$

$$\frac{\mathbf{f}_2 + \mathbf{f}_4}{\sqrt{2}} = \dot{\mathbf{h}}_{1,+}, \frac{\mathbf{f}_2 - \mathbf{f}_4}{\sqrt{2}} = \dot{\mathbf{h}}_{1,-}, \frac{\mathbf{f}_6 + \mathbf{f}_8}{\sqrt{2}} = \dot{\mathbf{h}}_{2,+}, \frac{\mathbf{f}_6 - \mathbf{f}_8}{\sqrt{2}} = \dot{\mathbf{h}}_{2,-}. \qquad (52)$$

In the denotations above linear combinations $Lin(\mathbf{h}_{1,+}, \mathbf{h}_{2,+})$, $Lin(\mathbf{h}_{1,-}, \mathbf{h}_{2,-})$ determines linear subspaces $\mathbb{R}^2_{b,+}$, $\mathbb{R}^2_{b,-}$ with prescribed basis correspondingly.

Pairs of vectors $(\dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+})$, $(\dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$ determines basis in the the subspaces $Lin(\dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+})$, $Lin(\dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$, which is denoted by $\mathbb{R}^2_{\dot{b},+}$, $\mathbb{R}^2_{\dot{b},-}$ correspondingly. Linear combinations $Lin(\dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+})$, $Lin(\dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$ determines basis in the subspaces $\mathbb{R}^2_{\dot{b},+}$, $\mathbb{R}^2_{\dot{b},-}$ correspondingly.

The generator b is represented by the rotation on the angle $\frac{\pi}{2}$ in each plane $\mathbb{R}^2_{b,+}$, $\mathbb{R}^2_{b,-}$ and by the central symmetry in the planes $\mathbb{R}^2_{\dot{b},-}$, (this symmetry is commuted with the transformation on the element $\dot{b}$, which will be described below). The generator $\dot{b}$ is represented by the rotation on the angle $\frac{\pi}{2}$ in each plane $\mathbb{R}^2_{\dot{b},+}$ $\mathbb{R}^2_{\dot{b},-}$ and by the central symmetry in the plane $\mathbb{R}^2_{b,-}$, which is commuted with the b transformation, described above. The subgroup (50) is defined.

Let us denote the subgroup $i_{\mathbf{H}_{a\times\dot{a}},\mathbf{J}_b\times\dot{\mathbf{J}}_b} : \mathbf{H}_{a\times\dot{a}} \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$, which is the product of the diagonal subgroup, which we denote by $\mathbf{I}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$ with the elementary subgroup $\mathbb{Z}/2$ of the second factor, which is denoted by $\dot{\mathbf{J}}_d \subset \dot{\mathbf{J}}_b$. The subgroup $\mathbf{H}_{a\times\dot{a}}$ coincides with the preimage of the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/4$ by the homomorphism

$$\omega^{[4]} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \to \mathbb{Z}/4, \qquad (53)$$

which is defined by the formula $(x \times y) \mapsto xy$.

Define the subgroup $i_{\mathbf{I}_{a\times\dot{a}},\mathbf{H}_{a\times\dot{a}}} : \mathbf{I}_{a\times\dot{a}} \subset \mathbf{H}_{a\times\dot{a}}$ as the kernel of the epimorphism

$$\omega^{[3]} : \mathbf{H}_{a\times\dot{a}} \to \mathbb{Z}/2, \qquad (54)$$

which is defined by the formula: $(x \times y) \mapsto x$ using generators x, y of the group $\mathbf{J}_b \times \dot{\mathbf{J}}_b$.

Let us consider the diagonal subgroup $\mathbb{Z}/2^{[3]} \subset \mathbb{Z}/2^{[3]} \times \mathbb{Z}/2^{[3]} \subset \mathbb{Z}/2^{[4]}$, which is generated by the invariant transformations in the direct sum of the subspaces $diag(\mathbb{R}^2_{b,+}, \mathbb{R}^2_{\dot{b},+})$, $diag(\mathbb{R}^2_{b,-}, \mathbb{R}^2_{\dot{b},-})$. This group is a subgroup of transformations of the base vectors, which are defined by the formulas: $\mathbf{h}_{1,+} + \dot{\mathbf{h}}_{1,+}, \mathbf{h}_{2,+} + \dot{\mathbf{h}}_{2,+}, \mathbf{h}_{1,-} + \dot{\mathbf{h}}_{2,-}, \mathbf{h}_{2,-} + \dot{\mathbf{h}}_{2,-}$. This collection of the vectors determines the standard base in the space $diag(\mathbb{R}^2_{b,+}, \mathbb{R}^2_{\dot{b},+}) \oplus diag(\mathbb{R}^2_{b,-}, \mathbb{R}^2_{\dot{b},-})$. The complement is defined by the formula: $antidiag(\mathbb{R}^2_{b,+}, \mathbb{R}^2_{\dot{b},+}) \oplus antidiag(\mathbb{R}^2_{b,-}, \mathbb{R}^2_{\dot{b},-})$. In this space the standard base is analogously defined.

Let us define a subgroup $\mathbb{Z}/2^{[2]} \subset \mathbb{Z}/2^{[3]}$ as the subgroup, which is generated by transformations of unite vectors, which are parallel to the vectors $\mathbf{h}_{1,+} + \dot{\mathbf{h}}_{1,+} + \mathbf{h}_{2,+} + \dot{\mathbf{h}}_{2,+}$, $\mathbf{h}_{1,-} + \dot{\mathbf{h}}_{2,-} + \mathbf{h}_{2,-} + \dot{\mathbf{h}}_{2,-}$. It is easy to see, that the first vector in the collection is parallel to the vector $\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_5 + \mathbf{e}_7$. The second vetor is parallel to the vector $\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6$.

The inclusion $i_{\mathbf{H}_{a\times\dot{a}}} : \mathbf{H}_{a\times\dot{a}} \subset \mathbb{Z}/2^{[3]}$, which is corresponded to the inclusion (50) is well defined, such that the following diagram is commutative:

$$\begin{array}{rcc}
\mathbf{I}_{a\times\dot{a}} & \stackrel{i_{a\times\dot{a}}}{\longrightarrow} & \mathbb{Z}/2^{[2]} \\
i_{a\times\dot{a},\mathbf{H}_{a\times\dot{a}}} \downarrow & & i^{[3]} \downarrow \\
\mathbf{H}_{a\times\dot{a}} & \stackrel{i_{\mathbf{H}_{a\times\dot{a}}}}{\longrightarrow} & \mathbb{Z}/2^{[3]} \\
i_{\mathbf{H}_{a\times\dot{a}},\mathbf{J}_b\times\dot{\mathbf{J}}_b} \downarrow & & i^{[4]} \downarrow \\
\mathbf{J}_b \times \dot{\mathbf{J}}_b & \stackrel{i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array} \qquad (55)$$

Let us define automorphisms of the order 2:

$$\chi^{[3]} : \mathbf{H}_{a\times\dot{a}} \to \mathbf{H}_{a\times\dot{a}}, \qquad (56)$$

$$\chi^{[4]} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \to \mathbf{J}_b \times \dot{\mathbf{J}}_b, \qquad (57)$$

and

$$\chi^{[3]} : \mathbb{Z}/2^{[3]} \to \mathbb{Z}/2^{[3]}, \qquad (58)$$

$$\chi^{[4]} : \mathbb{Z}/2^{[4]} \to \mathbb{Z}/2^{[4]}, \qquad (59)$$

which are marked with a loss of the strictness.

Define the automorphism (58) by the permutation of the base vectors in each direct factor $diag(\mathbb{R}^2_{b,+}, \mathbb{R}^2_{\dot{b},+}) \oplus diag(\mathbb{R}^2_{b,-}, \mathbb{R}^2_{\dot{b},-})$ with inexes b and $\dot{b}$. Define the automorphism (58) as the automorphism, which restricted on the diagonal subgroup $diag(\mathbf{J}_b, \dot{\mathbf{J}}_b) = \mathbf{I}_b \subset \mathbf{H}_{a\times\dot{a}}$is the identity, and the restriction on the subgroup $\mathbf{I}_{a\times\dot{a}} \subset \mathbf{H}_{a\times\dot{a}}$ coincides with the automorphism $\chi^{[2]}$. Evidently, the definition is correct.

Define the automorphism (59) in the standard basis in the subspaces $diag(\mathbb{R}^2_{b,+}, \mathbb{R}^2_{\dot{b},+}) \oplus diag(\mathbb{R}^2_{b,-}, \mathbb{R}^2_{\dot{b},-})$, $antidiag(\mathbb{R}^2_{b,+}, \mathbb{R}^2_{\dot{b},+}) \oplus antidiag(\mathbb{R}^2_{b,-}, \mathbb{R}^2_{\dot{b},-})$ as above.

The following triple of $\mathbb{Z}$-extensions of the group $\mathbf{J}_b \times \dot{\mathbf{J}}_b$, defined below, is required.

The group $\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}$ was defined above by the formula (13). Analogously, the groups

$$\mathbf{H}_{a\times\dot{a}} \rtimes_{\chi^{[3]}} \mathbb{Z}, \tag{60}$$

$$(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z}, \tag{61}$$

are defined as semi-direct products of the corresponding groups, equipped by automorphisms, with the group $\mathbb{Z}$.

The classifying space $K(\mathbf{H}_{a\times\dot{a}} \rtimes_{\chi^{[3]}} \mathbb{Z}, 1)$ is a skew-product of the circle S^1 with the space $K(\mathbf{H}_{a\times\dot{a}}, 1)$, moreover, the mapping $K(\mathbf{H}_{a\times\dot{a}}, 1) \to K(\mathbf{H}_{a\times\dot{a}}, 1)$, which corresponds to a shift of the cyclic covering over $K(\mathbf{H}_{a\times\dot{a}} \rtimes_{\chi^{[3]}} \mathbb{Z}, 1)$, is defined by the involution, which is induced by the automorphism $\chi^{[3]}$. The definition of the group (61) is totally analogous.

Let us define another two $\mathbb{Z}$-extensions (Laurent extensions), which is conjugated to each other by the automorphism $\chi^{[4]}$. Let us denote this pair of extensions on the subgroup $\mathbf{J}_b \times \dot{\mathbf{J}}_b$, on this subgroup the extentions are commuted. Let us denote the extensions by

$$\rtimes_{\mu^{(4)}_b}, \qquad \rtimes_{\mu^{(4)}_{\dot{b}}}.$$

The Loran extension on the factor $\mathbf{J}_b$ is determined by the authomorphism $\omega_b^{(4)} : \mathbf{J}_b \to \mathbf{J}_b$, which inverse the generator. The Laurent expansion on the factor $\dot{\mathbf{J}}_b$ is determined by the automorphism $\mu_{\dot{b}}^{(4)} : \dot{\mathbf{J}}_b \to \dot{\mathbf{J}}_b$, which inverse the generator. We have defined the double Loran extension group, which is denoted by

$$\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\omega^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z}.$$

Let us extend the extension constructed above to the extension

$$\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[4]}} \mathbb{Z}. \tag{62}$$

The Loran extensions (62), and (61) (see below) are naturally represented into $\mathbb{Z}/2^{[4]}$.

Analogously, let us define a $\mathbb{Z}$-extension (a Laurent expansion), which is denoted by $\omega_b^{(3)}$. Let us firstly define this pair of extensions on the subgroup $\mathbf{H}_{a\times\dot{a}}$. This Laurent extension is defined using an automorphism $\mu_b^{(3)} : \mathbf{H}_{a\times\dot{a}} \mapsto \mathbf{H}_{a\times\dot{a}}$, which inverses the generator in the subgroup $\mathbf{I}_b \subset \mathbf{H}_{a\times\dot{a}}$. This Laurent extension is denoted by $\mathbf{H}_{a\times\dot{a}} \rtimes_{\mu_a^{(3)}} \mathbb{Z}$. Let us extend this extension to the extension

$$\mathbf{H}_{a\times\dot{a}} \rtimes_{\mu_a^{(3)}} \mathbb{Z} \rtimes_{\chi^{[3]}} \mathbb{Z}. \tag{63}$$

The extension (63) contains a sub extension (60), the both groups are represented into $\mathbb{Z}/2^{[3]}$ in an agreed way, moreover, this representation agrees to the representation (62), as a subrepresentation on a subgroup of the index 2 and extends the representation (70), which is denoted below.

Corresponding $\mathbb{Z}/2$–reductions of extensions (62), (63) denote by

$$\mathbf{H}_{a\times\dot{a}} \rtimes_{\mu_a^{(3)}} \mathbb{Z}/2 \rtimes_{\chi^{[3]}} \mathbb{Z}, \tag{64}$$

$$\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z}/2 \times \mathbb{Z}/2 \rtimes_{\chi^{[4]}} \mathbb{Z}. \tag{65}$$

The $\mathbb{Z}/2$–reductions are required, because integer extensions (62) is not sufficient, a parametrization of the structured group $\mathbf{J}_b \times \dot{\mathbf{J}}_b$ is not over S^1 (over the Moebius band), but also over the projective plane $\mathbb{RP}^2$. For the reduction (65) this (minimal) parametrization is assumed.

The following $\mathbb{Z}/4$-line bundle

$$\beta_{b\times\dot{b}}. \tag{66}$$

with the hermitian conjugation (in fact, a $\mathbf{D}$-bundle) over the classifying space of the group (62), by means of the authomorphism $\mu^{(4)}_{b\times\dot{b}}$. Below this bundle determines a local coefficients system, with inversions of cyclic factors.

On the subgroup (61) the bundle (66) is defined as the inverce image of the cannonical $\mathbb{C}$-bundle over $K(\mathbb{Z}/4,1)$, using an extended homomorphism (53). Over each extension $\rtimes_{\mu^{(4)}_{b}}$, $\rtimes_{\mu^{(4)}_{\dot{b}}}$ a fibre of $\beta_{b\times\dot{b}}$ is transformed by the complex conjugation (separately for the each factor) along the generator of the Laurent extension; generators of the factors $\mathbf{J}_b$, $\dot{\mathbf{J}}_b$ transforms a fibre by the rotation on the angle $\frac{\pi}{2}$.

As the result, the commutation of the generator $b \in \mathbf{J}_b$ of the subgroup (62) with the corresponding generator of the Laurent extension $\rtimes_{\mu^{(4)}_{\dot{b}}}$, the action of the generator on the fibre is changed into the opposite, and this b-part of the double Laurent extension commutes with the generator $\dot{b} \in \dot{\mathbf{J}}_b$. The same effect for the generator of $\rtimes_{\mu^{(4)}_{b}}$ with $\dot{b}$-part is.

Over the subgroup (65), which is extended using $\rtimes_{\chi^{[4]}}$, the bundle $\beta_{b\times\dot{b}}$ is well-defined and is denoted the same. The Laurent generator, associated with $\rtimes_{\chi^{[4]}}$, permutes the Laurent generators for $\rtimes_{\mu^{(4)}_{b}}$, $\rtimes_{\mu^{(4)}_{\dot{b}}}$ as well as permutes the generators of the factors $\mathbf{J}_b \times \dot{\mathbf{J}}_b$.

Let us consider the homology group $H_i(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z}, 1))$. In particular, for an odd i the second group contains the fundamental classes of the manifolds

$$S^i/\mathbf{i} \times pt \subset S^i/\mathbf{i} \times S^i/\mathbf{i} \subset K(\mathbf{J}_b, 1) \times K(\dot{\mathbf{J}}_b, 1),$$

$$pt \times S^i/\mathbf{i} \subset S^i/\mathbf{i} \times S^i/\mathbf{i} \subset K(\mathbf{J}_b, 1) \times K(\dot{\mathbf{J}}_b, 1),$$

which are denoted by

$$t_{b,i}, \in H_i(K(\mathbf{J}_b, 1)); \quad t_{\dot{b},i'} \in H_{i'}(K(\dot{\mathbf{J}}_b, 1)). \tag{67}$$

Let us define the homology groups

$$H_i^{loc}(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z}, 1)).$$

The local $\mathbb{Z} \times \mathbb{Z}$-coefficients system $\mu^{(4)}_{b\times\dot{b}}$ is agree with the bundle (66); when the generators of the local system acts (along paths), the generators $t_{b,i}$, $t_{\dot{b},i'}$ for $i, i' \equiv 1 \pmod 4$ change singes; for $i, i' \equiv 3 \pmod 4$ preserve singes.

Then let us define the homology groups with the 3-uple local coefficient system with over the module $\mathbb{Z}[\mathbb{Z}/2]$, using the automorphism $\chi^{[4]}$, which changes the factors of the system (as in (19)).

The $\mathbb{Z}[\mathbb{Z}/2]$-homology groups with 3-uple system are defined by the formulas analogously to (18), (19), (22); for exemple:

$$\begin{gathered} D_i^{loc}(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z}; \mathbb{Z}[\mathbb{Z}/2]) \subset \\ H_i^{loc}(K((\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z}) \rtimes_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]). \end{gathered} \tag{68}$$

Analogous groups are defined with $\mathbb{Z}/2[\mathbb{Z}/2]$-coefficient.

Analogously to (28), (29) the homomorphism

$$\Delta^{[4]} : D_i^{loc}(\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z}; \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{J}_b \times \dot{\mathbf{J}}_b, 1)), \tag{69}$$

is well-defined, this homomorphism is an isomorphism.

The following lemma is analogous to Lemma 8.

Лемма 11. *–1. The subgroup* $H_{2s}(K((\mathbf{J}_b\times\dot{\mathbf{J}}_b)\rtimes_{\chi^{[4]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$ *and the group* $H_{2s}(K((\mathbf{J}_b\times\dot{\mathbf{J}}_b)\rtimes_{\chi^{[4]};\mu^{[4]}_{a\times\dot{a}}}\mathbb{Z}^3,1);\mathbb{Z}[\mathbb{Z}/2])$ *contain a direct term* $D_{2s}(\mathbf{J}_b\times\dot{\mathbf{J}}_b;\mathbb{Z}[\mathbb{Z}/2])$.

–2. Basis in the subgroups $\bigoplus_{i_1+i_2=2s}H_{i_1}(K(\mathbf{J}_b,1);\mathbb{Z})\otimes H_{i_2}(K(\dot{\mathbf{J}}_b,1);\mathbb{Z})\subset D^{loc}_{2s}(\mathbf{J}_b\times\dot{\mathbf{J}}_b;\mathbb{Z}[\mathbb{Z}/2])$ *generate a basis in the group* $\mathrm{Im}(B)\cap D^{loc}_{2s}(\mathbf{J}_{b\times\dot{b}};\mathbb{Z}/2[Z/2])$*, where*

$$B: H^{loc}_{2s}(K((\mathbf{J}_b\times\dot{\mathbf{J}}_b)\rtimes_{\chi^{[4]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])\to$$
$$H^{loc}_{2s}(K((\mathbf{J}_b\times\dot{\mathbf{J}}_b)\rtimes_{\chi^{[4]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$$

is the modulo 2 reduction homomorphism.

The representation $\Phi^{[2]}$, defined by the formula (35), is generalized into the following representation:

$$\Phi^{[4]}:(\mathbf{J}_b\times\dot{\mathbf{J}}_b\rtimes_{\mu^{(4)}_{b\times\dot{b}}}\mathbb{Z}\times\mathbb{Z})\rtimes_{\chi^{[4]}}\mathbb{Z}\to\mathbb{Z}/2^{[4]}, \qquad (70)$$

where the generator of the factor $\mathbb{Z}$ is represented in $\mathbb{Z}/2^{[4]}$ by the rensformation $\chi^{[4]}$, which is defined by the formula (59).

The automorphisms $\chi^{[i]}$, $i=2,3,4$ in the images and pre-images of the diagram (55) are agree with respect to horizontal arrows. Therefore the following diagram is well-defined:

$$\begin{array}{ccc}
\mathbf{I}_{a\times\dot{a}}\rtimes_{\chi^{[2]}}\mathbb{Z} & \stackrel{\Phi^{[2]}}{\longrightarrow} & \mathbb{Z}/2^{[2]}\\
i_{a\times\dot{a},\mathbf{H}_{a\times\dot{a}}}\downarrow & & i^{[3]}\downarrow\\
\mathbf{H}_{a\times\dot{a}}\rtimes_{\chi^{[3]}}\mathbb{Z} & \stackrel{\Phi^{[3]}}{\longrightarrow} & \mathbb{Z}/2^{[3]}\\
i_{\mathbf{H}_{a\times\dot{a}},\mathbf{J}_b\times\dot{\mathbf{J}}_b}\downarrow & & i^{[4]}\downarrow\\
(\mathbf{J}_b\times\dot{\mathbf{J}}_b)\rtimes_{\chi^{[4]}}\mathbb{Z} & \stackrel{\Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array} \qquad (71)$$

The two bottom arrows of the diagram are included into the corresponding diagrams of $\mathbb{Z}$- and $\mathbb{Z}/2$- extensions, which are constructed by the formulas (62), (65). This diagram is not written-down.

Standardized $\mathbf{J}_b \times \dot{\mathbf{J}}_b$-immersions

Let us formulate notions of standardized (and pre-standartized) $\mathbf{J}_b \times \dot{\mathbf{J}}_b$-immersion.

Let us consider a $\mathbb{Z}/2^{[4]}$-framed immersion (g, Ψ, η_N) of the codimension $8k$. Assume that the image of the immersion g belongs to a regular neighbourhood of an embedding $I : \mathbb{RP}^2_b \times \mathbb{RP}^2_{\dot{b}} \rtimes S^1 \subset \mathbb{R}^n$. In this formula the third factor S^1 is not a direct, the action (an involution) of the generator $[S^1]$ on $\mathbb{RP}^2_b \times \mathbb{RP}^2_{\dot{b}}$ is given by the permutation of the factors. Denote by $U \subset \mathbb{R}^n$ a regular thin neighbourhood of the embedding I.

Let us consider a $\mathbb{Z}/2^{[4]}$–framed immersion $g : N^{n-8k} \looparrowright U \subset \mathbb{R}^n$, for which te following condition (Y) of a control of the structured group of the normal bundle is defined. The immersion g admits a reduction of a general structured group to the subgroup (65). Additionally, the projection $\pi \circ g : N^{n-8k} \to \mathbb{RP}^2_b \times \mathbb{RP}^2_{\dot{b}} \rtimes S^1$ of this immersion onto the central manifold of U is agreed with the projection of the structured group $\mathbb{Z}/2 \times \mathbb{Z}/2 \rtimes \mathbb{Z}$ onto the factors of the extension.

Formally, weaker but, in fact, an equivalent condition (Y1) is following: the image of the immersion g can be outside of U, but a mapping $\phi : N^{n-8k} \to \mathbb{RP}^2_b \rtimes \mathbb{RP}^2_{\dot{b}} \rtimes S^1$, which is agree with the reduction of the structured group of the normal bundle of the immersion g into the subgroup (65) is fixed.

Additional, assume that the obstruction to the mapping onto the polyhedron $S^1_b \times S^1_{\dot{b}} \rtimes S^1 \subset \mathbb{RP}^2_b \times \mathbb{RP}^2_{\dot{b}} \rtimes S^1$, this obstruction is a codimension 2 submanifold in N^{n-8k} have to be compressed onto a skeleton of the dimension $\approx \frac{n}{4}$ in the corresponding classifying space (a codefect of the reduction the group structure (65) to (62) equals to $\approx \frac{n}{4}$, see Definition 2).

Definition 12. Let us say that a $\mathbb{Z}/2^{[4]}$-framed immersion (g,Ψ,η_N) of the codimension $8k$ is standardized if its codefect equals to $\delta \approx \frac{n}{4}$, and the condition (Y) is satisfied.

Let us say that $\mathbb{Z}/2^{[4]}$-framed immersion (g,Ψ,η_N) of the codimension $8k$ is pre-standardized if its codefect equals to $\delta \approx \frac{n}{4}$, and the condition (Y1) is satisfied. The conditions (Y1), (Y) are equivalent (formally, (Y) implies (Y1)), the definitions are equivalent.

Standardized $\mathbb{Z}/2^{[4]}$-framed immersions with a given codefect generates a cobordism group, this group is naturally mapped into $Imm^{\mathbb{Z}/4^{[4]}}(n-8k,8k)$ when a standardization structure is omitted.

The following definition is analogous to Definition 14. In this definition a projection of the Hurewicz image of a fundamental class onto a corresponding subgroup is defined using Lemma 11.

Definition 13. Let (g,η_N,Ψ) be a pre-standardized $\mathbb{Z}/2^{[4]}$-framed immersion in the codimension $8k$.

Let us say that this pre-standardized immersion is pure, if the Hurewicz image of a fundamental class in $\in D^{loc}_{n-8k}(\mathbf{J}_b \times \dot{\mathbf{J}}_b;\mathbb{Z}/2[\mathbb{Z}/2])$ contains not monomial $t_{b,\frac{n-8k}{2}+i} \otimes t_{\dot{b},\frac{n-8k}{2}-i}$, $i \in \{\pm1;\pm2;\pm3;\pm4;\pm5;\pm6;\pm7\}$, but, probably, contains the only non-trivial monom $t_{b,\frac{n-8k}{2}} \otimes t_{\dot{b},\frac{n-8k}{2}}$.

Definition 14. Let (g,η_N,Ψ) be a $\mathbb{Z}/2^{[4]}$-framed immersion, for which a structured group $\mathbb{Z}/2^{[4]}$ of the normal bundle is reduced to the group $\mathbb{Z}/2^{[3]} \times \mathbb{Z}/2^{[3]} \rtimes \mathbb{Z}$, where the generator $\mathbb{Z}$ acts on $\mathbb{Z}/2^{[3]} \times \mathbb{Z}/2^{[3]}$ by the permutation of the factors.

Let us say that such an immersion is negligible immersion, if the image of its fundamental mod 2 class in the group

$$\eta_*(N^{n-8k}) \in D^{loc}_{n-8k}(\mathbf{I}_a \times \dot{\mathbf{I}}_a;\mathbb{Z}/2[\mathbb{Z}/2]) \qquad (72)$$

by the transfer homomorphism onto 4-sheeted covering with the tower (71) equals to zero.

Теорема 15. *Let* (g, Ψ, η_N) *be a standardized* $\mathbf{D}$*-framed immersion in the codimension* $2k$ *(see Definition 2),* $k \geq 7$*. Let* $(g_1, \Psi_1, \eta_{N_1})$ *be the* $\mathbb{Z}/2^{[3]}$*-framed immersion of self-intersection points of* g*;* $(g_2, \Psi_2, \zeta_{N_2})$ *be the* $\mathbb{Z}/2^{[4]}$*-framed immersion of iterated self-intersection points of* g*. Then there exist a formal deformation of the immersion* $g^{[4]}$ *(the double equivariant extension of* g*), for which the iterated self-intersection manifold is regular cobordant to a disjoin union of pre-standardized pure* $\mathbb{Z}/2^{[4]}$*-immersion and neglected* $\mathbb{Z}/2^{[4]}$*-immersion.*

6 Proof of Theorem 15

Let us reformulate Definition 6 and Lemma 7, replacing the subgroup $\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbf{D}$ into the subgroup $\mathbf{I}_b \subset \mathbf{D}$.

Definition 16. Assume an equivariant (formal) mapping

$$d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \mathbb{RP}^{n-k}_{diag} \to \mathbb{R}^n \times \mathbb{R}^n, \qquad (73)$$

which, generally speaking, is not a holonomic. Let us say that $d^{(2)}$ admits a cyclic structure, if the following condition is satisfied. The polyhedron of the (formal) self-intersection of $d^{(2)}$ is divided into two components: a closed component N_b and a component with a boundary $\mathbb{N}_{a\times\dot{a},\circ}$, moreower, the following conditions of a reduction of the structured mapping are satisfied.

– 1. On an open polyhedra $\mathbb{N}_{a\times\dot{a},\circ}$ the structured mapping admits a reduction:

$$\eta_{a\times\dot{a}} : \mathbb{N}_\circ \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1), \qquad (74)$$

which is a double covering (a transfer) mapping into a classified space of a central $\mathbb{Z}/4$-extension. In particular, the fundamental class of the polyhedron $\mathbb{N}_\circ$ has only trivial characteristic numbers modulo 2 in the image of the mapping (74) (see Definition 14).

–2. The restriction of the structured mapping on the polyhedron $\mathbb{N}_b$ admits a reduction

$$\eta_{b,\mathbb{R}\mathrm{P}^2} : \mathbb{N}_b \to K(\mathbf{I}_b, 1) \rtimes_{\chi^{[2]}} \mathbb{R}\mathrm{P}^2. \tag{75}$$

Moreover, there exist a polyhedron $\mathbb{N}_{reg} \subset \mathbb{N}_b$ (this polyhedron is an obstruction to a reduction of the structured mapping (8) to a mapping $\eta_{a\times\dot{a}} : \mathbb{N}_\circ \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1))$, the dimension $\dim(\mathbb{N}_{reg})$ equals (up to a small constant d) $\frac{3\dim(\mathbb{N}_\circ)}{4} = \frac{3(n-k)}{4}$, and which is a preimage of a marked point on $\mathbf{x}_\infty \in \mathbb{R}\mathrm{P}^2$ of the composition $\mathbb{N}_b \to K(\mathbf{I}_{a\times\dot{a}} \rtimes_{\chi^{[2]}} \mathbb{Z}, 1) \to \mathbb{R}\mathrm{P}^2$.

Let us recall the following lemma [A-P2].

Лемма 17. *Big Lemma*

Assuming

$$n - k' \equiv 1 \pmod 2, \quad n \equiv 0 \pmod 2, k' \geq 2 \tag{76}$$

there exist a formal mapping $d^{(2)}$, which admits a cyclic structure in the cence of Definition 6. The formal mapping $d^{(2)}$ is a result of a non-holonomic (a formal vertical) small deformation of the formal extension of a special mapping $d : \mathbb{R}\mathrm{P}^{n-k'} \to S^{n-k'} \subset \mathbb{R}^n$ with the image on the sphere $S^{n-k'}$ with the assumption $k' \geq 2$.

The mapping in the lemma $d : \mathbb{R}\mathrm{P}^{n-k'} \to S^{n-k'}$ has to be generalized, using two-stages tower (80) of ramified coverings.

Let us start a proof of Theorem 15 with the following construction. Let us recall, that a positive integer $m_\sigma = 14$. Denote by $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ the Cartesian product o standard lens space lence $\pmod 4$, namely,

$$ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} = S^{n-\frac{n-m_\sigma}{8}+1}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{8}+1}/\mathbf{i}. \tag{77}$$

Evidently, $\dim(ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}) = \frac{7}{4}(n+m_\sigma)+2 > n$.

On the space $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ a free involution $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \to ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ acts by the formula: $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}(x\times y) = (y\times x)$.

Let us define a subpolyhedron (a manifold with singularities) $X_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \subset ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$. Let us consider the following family $\{X_j, \quad j=0,1,\dots,j_{max}\}$, $j_{max}\equiv 0 \pmod 2$, of submanifolds $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$:

$$X_0 = S^{n-\frac{n-m_\sigma}{8}+1}/\mathbf{i}\times S^1/\mathbf{i},$$

$$X_1 = S^{n-\frac{n-m_\sigma}{8}-1}/\mathbf{i}\times S^3/\mathbf{i}, \quad \dots$$

$$X_j = S^{n-\frac{n-m_\sigma}{8}+1-2j}/\mathbf{i}\times S^{2j+1}/\mathbf{i},$$

$$X_{j_{max}} = S^1/\mathbf{i}\times S^{n-\frac{n-m_\sigma}{8}+1}/\mathbf{i},$$

where

$$j_{max} = \frac{7n+m_\sigma}{16} = 2^{n-1}, \quad m_\sigma = 14. \tag{78}$$

The dimension of each manifold in this family equals to $n-\frac{n-m_\sigma}{8}+2$ and the codimension in $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ equals to $n-\frac{n-m_\sigma}{8}$. Let us define an embedding

$$X_j \subset ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$$

by a Cartesian product of te two standard inclusions Let us denote by $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \to ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ the involution, which permutes coordinates. Evidently, we get: $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}(X_j) = X_{j_{max}-j}$.

A polyhedron $X_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} = \bigcup_{j=0}^{j_{max}} X_j \subset ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ is well-defined. This polyhedron is invariant with respect to the involution $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$. The polyhedron $X_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ can be considered as a stratified manifolds with strata of the codimension 2. The restriction of the involution $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ on the polyhedron $X_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ denote by $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$.

Let us written-down a sequence of the index 2 subgroup from diagram (55):

$$\mathbf{I}_a\times\dot{\mathbf{I}}_a \longrightarrow \mathbf{H}_{a\times\dot{a}} \longrightarrow \mathbf{J}_b\times\dot{\mathbf{J}}_b. \tag{79}$$

Let us define the following tower of double coverings, which is associated with the sequence (79):

$$ZZ_{a\times\dot a} \longrightarrow ZZ_{\mathbf{H}_a\times\dot a} \longrightarrow ZZ_{\mathbf{J}_b\times\dot{\mathbf{J}}_b}. \qquad (80)$$

The bottom space of the tower (80) coincides to a skeleton of the Eilenberg-MacLane space $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \subset K(\mathbf{J}_a,1)\times K(\dot{\mathbf{J}}_a,1)$. The tower of double coverings

$$K(\mathbf{I}_a,1)\times K(\dot{\mathbf{I}}_a,1) \to K(\mathbf{H}_{a\times\dot a},1) \to K(\mathbf{J}_b,1)\times K(\dot{\mathbf{J}}_b,1),$$

which is associated with the sequence (79) is well-defined. This tower determines the tower (80) by means of the inclusion $ZZ_{\mathbf{J}_b\times\dot{\mathbf{J}}_b} \subset K(\mathbf{J}_b,1)\times K(\dot{\mathbf{J}}_b,1)$.

Let us define the following tower of double coverings:

$$X_{a\times\dot a} \longrightarrow X_{\mathbf{H}_{b\times\dot b}} \longrightarrow X_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}. \qquad (81)$$

The bottom space of the tower (81) is a subspace of the bottom space of the tower (80) by means of an inclusion посредством включения $X_{\mathbf{J}_b\times\dot{\mathbf{J}}_b} \subset ZZ_{\mathbf{J}_b\times\dot{\mathbf{J}}_b}$. The tower (81) determines as the restriction of the tower (80) on this subspace.

Let us describe a polyhedron $X_{a\times\dot a} \subset ZZ_{a\times\dot a}$ explicitly. Let us define a family $\{X'_0, X'_1, \dots, X'_{j_{max}}\}$ of standard submanifolds in the manifold $ZZ_{a\times\dot a} = \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1}\times\mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1}$ by the following formulas:

$$X'_0 = \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1}\times\mathbb{RP}^1 \dots \qquad (82)$$
$$X'_j = \mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1-2j}\times\mathbb{RP}^{2j+1} \dots$$
$$X'_{j_{max}} = \mathbb{RP}^1\times\mathbb{RP}^{n-\frac{n-m_\sigma}{8}+1}.$$

In this formulas the integer index j_{max} is defined by the formula (78). The polyhedron $X_{a\times\dot a} \subset ZZ_{a\times\dot a}$ is defined as the union of standard submanifolds in this family. The polyhedron $X_{\mathbf{H}_{a\times\dot a}} \subset ZZ_{\mathbf{H}_{a\times\dot a}}$ a factorspace of the double covering, which corresponds to the tower of the groups.

The spaces $X_{\mathbf{H}_{a\times\dot{a}}}$, $X_{a\times\dot{a}}$ admit free involutions, which are pullbacks of the involution $\chi_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ by the projection on the bottom space of the tower.

The cylinder of the involution $\chi_{a\times\dot{a}}$ is well-defined, (correspondingly, of the involution $\chi_{\mathbf{H}_{a\times\dot{a}}}$), which is denoted by $X_{a\times\dot{a}} \rtimes_\chi S^1$ (correspondingly, by $X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi S^1$). The each space is embedded into the corresponding fibred space over $\mathbb{R}\mathrm{P}^2$:

$$X_{a\times\dot{a}} \rtimes_\chi S^1 \subset X_{a\times\dot{a}} \rtimes_\chi \mathbb{R}\mathrm{P}^2,$$

$$X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi S^1 \subset X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{R}\mathrm{P}^2.$$

Then let us define a polyhedron $J_{b\times\dot{b}}$, which is a base of a ramified covering $X_{a\times\dot{a}} \to J_{b\times\dot{b}}$.

Then let us extend the ramified covering over the bottom space of the tower to the ramified covering: $X_{a\times\dot{a}} \rtimes_\chi \mathbb{R}\mathrm{P}^2 \to J_{b\times\dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2$, and the ramified covering over the middle space of the tower of the ramified covering $X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{R}\mathrm{P}^2 \to J_{b\times\dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2$.

Let us define a polyhedron (a manifold with singularities) $J_{b\times\dot{b}}$. For an arbitrary $j = 0, 1, \dots, j_{max}$, where j_{max} is defined by the formula (78), let us define the polihedron $J_j = S^{n-\frac{n-m_\sigma}{8}-2j+1} \times S^{2j+1}$ (the Cartesian product). Spheres (components of this Cartesian product) $S^{n-\frac{n-m_\sigma}{8}-2j+1}$, S^{2j+1} are redenoted by $J_{j,1}$, $J_{j,2}$ correspondingly. Using this denotations, we get:

$$J_j = J_{j,1} \times J_{j,2}.$$

The standard inclusion $i_j : J_{j,1} \times J_{j,2} \subset S^{\frac{n-m_\sigma}{8}+1} \times S^{\frac{n-m_\sigma}{8}+1}$ is well-defined, each factor is included into the target sphere as the standard subsphere. The union $\bigcup_{j=0}^{j_{max}} Im(i_j)$ of images of this embeddings ar denoted by

$$J_{b\times\dot{b}} \subset S^{\frac{n-m_\sigma}{8}+1} \times S^{\frac{n-m_\sigma}{8}+1}. \qquad (83)$$

The polyhedron $J_{b\times\dot{b}}$ is constructed.

Let us define a ramified covering

$$\varphi_{a\times\dot{a}} : X_{a\times\dot{a}} \to J_{b\times\dot{b}}. \tag{84}$$

The covering (84) is defined as the union of the Cartesian products of the ramified coverings, which was constructed in Lemma 17.

The covering (84) is factorized into the following ramified covering:

$$\varphi_{\mathbf{H}_{a\times\dot{a}}} : X_{\mathbf{H}_{a\times\dot{a}}} \to J_{b\times\dot{b}}. \tag{85}$$

Because $X_{a\times\dot{a}} \to X_{\mathbf{H}_{a\times\dot{a}}} \to J_{b\times\dot{b}}$ is a double covering, the number of sheets of the covering 85 is greather by the factor 2^r, where r is the denominator of the ramification.

The polyhedron $J_{b\times\dot{b}}$ is equipped by the involution χ, which is defined analogously to the involutions $\chi_{a\times\dot{a}}$, $\chi_{\mathbf{H}_{a\times\dot{a}}}$. The cylinder of the involution is well-defined, let us denote this cylinder by $J_{b\times\dot{b}} \rtimes_\chi S^1$. The inclusion $J_{b\times\dot{b}} \rtimes_\chi S^1 \subset J_{b\times\dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2$. is well-defined.

The ramified covering (84) commutes with the involutions $\chi_{a\times\dot{a}}$, $\chi_{\mathbf{H}_{a\times\dot{a}}}$ in the origine and the target. Therefore the ramified covering

$$c_X : X_{a\times\dot{a}} \rtimes_\chi \mathbb{R}\mathrm{P}^2 \to J_{b\times\dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2, \tag{86}$$

which is factorized into the ramified covering

$$c_Y : X_{\mathbf{H}_{a\times\dot{a}}} \rtimes_\chi \mathbb{R}\mathrm{P}^2 \to J_{b\times\dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2. \tag{87}$$

is well-defined.

Лемма 18. *There exist an inclusion*

$$i : J_{b\times\dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2 \times D^8 \subset \mathbb{R}^n, \tag{88}$$

where D^8 is the standard 8-dimensional disk (of a small radius).

Proof of Lemma 18

Let us put $m_\sigma = 14$, $k = 7$, $n = 126$. The polyhedron $J_{b \times \dot{b}}$ is embedded into the sphere S^{n-12}, which is the unite sphere (the target of the standard projection) of the trivial $n-11$-dimensional fiber ε^{n-11} over $emb : \mathbb{R}\mathrm{P}^2 \subset \mathbb{R}^n$. The normal bundle of the embedding emb is the Whitney sum: $\nu(emb) = \kappa \oplus \varepsilon^{n-11} \oplus \varepsilon^8$. Lemma 18 is proved. □

Лемма 19. *There exists a formal mapping, which satisfied conditions of Definition 16.*

Proof of Lemma 19

Let us use Lemma 18. Let us consider the ramified covering (87), then take an embedding of the base into $\mathbb{R}^n$ and get a vertical lift of the ramified covering along the subbundle ε^8 of the normal bundle of the embedding by the formula: $\varepsilon^8 = \varepsilon^4 \oplus \kappa^4 = \mathbb{R}^4 \tilde{\times} \mathbb{R}^4$.

Let us write-down into each fibre the space $S^1 \times S^1 \cup S^1 \times S^1$ equivalently. By Lemma 17 a formal vertical χ-invariant lift of the iterated coverings is well-defined. Because the considered vertical lift has to be a χ-equivariant, this lift is well-defined over $J_{b \times \dot{b}} \times S^2$, the each lift in the family is defined by Lemma 17 to the both coordinates of the Cartesian product over each elementary block of the polyhedron.

We get a vertical (formal) lift over $J_{b \times \dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^1$, which extend to a (formal) vertical lift over the polyhedron $J_{b \times \dot{b}} \rtimes_\chi \mathbb{R}\mathrm{P}^2$.

As the result of this (formal) deformation the polyhedron of self-intersection is divided into two subpolyhedra, for the subpolyhedra properties 1,2 follows from the corresponding properties of the (formal) deformation by Lemma 17. □

Proof of Theorem 15

The proof is analogous to Theorem 4, but re required deformation is not holonomic (a formal) using Lemma 19. Deformation contains two steps, which corresponds to coverings (85), (84).

Constructions for both steps are analogous, the bottom step is a little simpler. Theorem 15 is proved. □

7 $\mathbf{Q} \times \mathbb{Z}/4$–structure (quaternionic-cyclic structure) on self-intersection manifold of a standardazed $\mathbb{Z}/2^{[4]}$–framed immersion

Let us recall the definition of the quaternionic subgroup $\mathbf{Q} \subset \mathbb{Z}/2^{[3]}$, which contains the subgroup $\mathbf{J}_b \subset \mathbf{Q}$.

Let us define the following subgroups:

$$i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b,\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbf{Q} \times \mathbb{Z}/4, \tag{89}$$

$$i_{\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{Q} \times \mathbb{Z}/4 \subset \mathbb{Z}/2^{[5]}, \tag{90}$$

$$i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b\times\mathbb{Z}/2} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2 \subset \mathbb{Z}/2^{[5]}. \tag{91}$$

Define the subgroup (89). Define the epimorphism $\mathbf{J}_b \times \dot{\mathbf{J}}_b \to \mathbb{Z}/4$ by the formula $(x \times y) \mapsto xy$. The kernel of this epimorphism coincides with the antydiagonal subgroup $\dot{\mathbf{I}}_b = antidiag(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$, and this epimorphism admits a section, the kernel is a direct factor (the subgroup $\mathbf{J}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$). This kernel is mapped onto the group $\mathbb{Z}/4$ by the formula: $(x \times x^{-1}) \mapsto x$. The subgroup (89) is well-defined.

Let us define subgroups (90), (91). Consider the basis $(\mathbf{h}_{1,+}, \mathbf{h}_{2,+}, \mathbf{h}_{1,-}, \mathbf{h}_{2,-}, \dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+}, \dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$ of the space $\mathbb{R}^8$, which is defined by the formulas (51), (52).

Let us define an analogous basis of the space $\mathbb{R}^{16}$. This basis contains 16 vectors, the basis vectors are divided into two subset $16 = 8 + 8$.

$$\mathbf{h}_{1,*,**}, \mathbf{h}_{2,*,**}, \tag{92}$$

$$\dot{\mathbf{h}}_{1,*,**}, \dot{\mathbf{h}}_{2,*,**}; \tag{93}$$

where the symbols $*$, $**$ takes values $+, -$ independently.

Let us define the subgroup (90). The representation $i_{\mathbf{Q}\times\mathbb{Z}/4}$ is given such that the generator $\mathbf{j}$ of the quaternionic factor $\mathbf{Q} \subset \mathbf{Q} \times \mathbb{Z}/4$ acts in each 4-dimensional subspace from the following list

$$diag(Lin(\mathbf{h}_{1,*,**}, \mathbf{h}_{2,*,**}, \mathbf{h}_{1,*,-**}, \mathbf{h}_{2,*,-**}), \tag{94}$$
$$Lin(\dot{\mathbf{h}}_{1,*,**}, \dot{\mathbf{h}}_{2,*,**}, \dot{\mathbf{h}}_{1,*,-**}, \dot{\mathbf{h}}_{2,*,-**})),$$

$$diag(Lin(\mathbf{h}_{1,-*,**}, \mathbf{h}_{2,-*,**}, \mathbf{h}_{1,-*,-**}, \mathbf{h}_{2,-*,-**}), \tag{95}$$
$$Lin(\dot{\mathbf{h}}_{1,-*,**}, \dot{\mathbf{h}}_{2,-*,**}, \dot{\mathbf{h}}_{1,-*,-**}, \dot{\mathbf{h}}_{2,-*,-**})),$$

$$antidiag(Lin(\mathbf{h}_{1,*,**}, \mathbf{h}_{2,*,**}, \mathbf{h}_{1,*,-**}, \mathbf{h}_{2,*,-**}), \tag{96}$$
$$Lin(\dot{\mathbf{h}}_{1,*,**}, \dot{\mathbf{h}}_{2,*,**}, \dot{\mathbf{h}}_{1,*,-**}, \dot{\mathbf{h}}_{2,*,-**})),$$

$$antidiag(Lin(\mathbf{h}_{1,-*,**}, \mathbf{h}_{2,-*,**}, \mathbf{h}_{1,-*,-**}, \mathbf{h}_{2,-*,-**}), \tag{97}$$

$$Lin(\dot{\mathbf{h}}_{1,-*,**}, \dot{\mathbf{h}}_{2,-*,**}, \dot{\mathbf{h}}_{1,-*,-**}, \dot{\mathbf{h}}_{2,-*,-**}))$$

by the standard matrix, which is defined in the standard basis of the corresponding space.

Each of 4-space, described above, corresponds to one of the pair of spaces $\mathbb{R}^2_{b,*}$, or, to one of the space $\mathbb{R}^2_{\dot{b},*}$, which isdefined below the formulas (51), (52).

The generator $\mathbf{i} \in \mathbf{Q}$ acts in the direct sum of the two exemplars of the corresponding space as the generator of the group $\mathbf{J}_b$ by the corresponding matrix. The generator of the factor $\mathbb{Z}/4 \subset \mathbf{Q} \times \mathbb{Z}/4$ acts of the direct sum of the two exemplairs of the corresponding space as the generator of the group $antidiag(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b$. The representation (90) is well-defined.

Let us define the representation (91) as following. The factor $\mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$ is represented in each 4-dimensional subspace (94)-(97) by the formula (50), which is applied separately to standard basis of each spaces. The factor $\mathbb{Z}/2 \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$ is represented

–in 8-dimensional subspace, which is defined as the direct sum of the subspaces (94), (96) by the identity transformation.

–in 8-dimensional subspace, which is defined as the direct sum of the subspaces (95), (97) by the central symmetry.

The representation (91) is well-defined.

On the group $\mathbf{Q} \times \mathbb{Z}/4$ define theauthomorphism $\chi^{[5]}$ of the order 4. This authomorphism on the subgroup (89) is defined as the restriction of the authomorphism $\chi^{[4]}$. The extension of $\chi^{[4]}$ from the subgroup $\chi^{[5]}$ to the group is defined by the simplest way: the automorphism $\chi^{[5]}$ keeps the generator$\mathbf{j}$. It is easy to see that the automorphism with such property exists and uniquely.

Consider the projection

$$p_{\mathbf{Q}} : \mathbf{Q} \times \mathbb{Z}/4 \to \mathbf{Q} \qquad (98)$$

on the first factor. The kernel of the homomorphism $p_{\mathbf{Q}}$ coincides with the antydiagonal subgroup $\dot{\mathbf{I}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \subset \mathbf{Q} \times \mathbb{Z}/4$.

Evidently, the following equality is satisfied:

$$p_{\mathbf{Q}} \circ \chi^{[5]} = p_{\mathbf{Q}}. \qquad (99)$$

Analogously, on the group $\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$ define the authomorphim $\chi^{[5]}$ of the order 2 (this new authomorphism denete the same) Define the projection

$$p_{\mathbb{Z}/4\times\mathbb{Z}/2} : \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2, \qquad (100)$$

the kernel of this projection coincids with the diagonal subgroup $\dot{\mathbf{I}}_b \subset \mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2$. Obviously, the following formula is satisfied: $\chi^{[5]} \circ p_{\mathbb{Z}/4\times\mathbb{Z}/2} = p_{\mathbb{Z}/4\times\mathbb{Z}/2}$.

This allos to define analogously with (13), (60), (61) the groups

$$(\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z}, \qquad (101)$$

$$(\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z}, \qquad (102)$$

as semi-direct products of the corresponding groups with automorphisms with the group $\mathbb{Z}$.

Let us define the epimorphism:

$$\omega^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbf{Q}, \qquad (103)$$

the restriction of this epimorphism on the subgroup (89) coincides with the epimorphism (98). For this definition use the formula (99) and define $z \in Ker(p_{\mathbf{Q}})$, where $z \in \mathbb{Z}$ is the generator.

Evidently, the epimorphism

$$\omega^{[5]} : (\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/4 \times \mathbb{Z}/2, \qquad (104)$$

analogously is well-defined, denote this automorphism as the automorphism (103), the same.

In the group $\mathbb{Z}/2^{[5]}$ let us define the involution, which is denoted by $\chi^{[5]}$ as on the resolution group. In the stanard basis of the spaces

(94)-(97) the authomorphism $\chi^{[5]}$ is defined by the same formulas as $\chi^{[4]}$, the each considered space is a proper space for $\chi^{[5]}$. This definition implies that $\chi^{[5]}$ is commuted with the representations (90), (91).

Moreower, the following homomorphisms

$$\Phi^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}, \qquad (105)$$

$$\Phi^{[5]} : (\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}, \qquad (106)$$

are analogously to (70) well-defined, they are included into the following commutative dyagrames (107), (108) of the homomorphisms, which are analogous to the datagram (71).

$$\begin{array}{ccc}
(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]}\times\Phi^{[4]})}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\
i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b,\mathbf{Q}\times\mathbb{Z}/4} \downarrow & & i_{[5]} \downarrow \\
(\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]},
\end{array} \qquad (107)$$

In this diagram the left vertical homomorphism

$$i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b,\mathbf{Q}\times\mathbb{Z}/4} : (\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z} \to (\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\chi^{[5]}} \mathbb{Z}$$

is induced by the homomorphism (89), the right vertical homomorphism

$$i_{[5]} : \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \subset \mathbb{Z}/2^{[5]}.$$

is the inclusion of the subgroup of the index 2.

The following dyagrame

$$\begin{array}{ccc}
(\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]}\times\Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\
i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b,\mathbf{J}_b\times\dot{\mathbf{J}}_b\times\mathbb{Z}/2} \downarrow & & i_{[5]} \downarrow \\
(\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]},
\end{array} \qquad (108)$$

in which the left vertical homomorphism

$$i_{\mathbf{J}_b\times\dot{\mathbf{J}}_b,\mathbf{J}_b\times\dot{\mathbf{J}}_b\times\mathbb{Z}/2} : (\mathbf{J}_b \times \dot{\mathbf{J}}_b) \rtimes_{\chi^{[4]}} \mathbb{Z} \to (\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\chi^{[5]}} \mathbb{Z}$$

is an inclusion, is well-defined.

Лемма 20. *The homomorphism (105) is extended to the homomorphism*

$$(\mathbf{Q} \times \mathbb{Z}/4) \rtimes_{\mu^{(5)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[5]}} \mathbb{Z} \mapsto \mathbb{Z}/2^{[5]} \qquad (109)$$

from the subgroup $\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[4]}} \mathbb{Z}$ *of the index 2. to the group.*

Proof of Lemma 20

Let us construct the extension

$$(\mathbf{Q} \times \mathbf{Q}) \rtimes_{\mu^{(5)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[5]}} \mathbb{Z},$$

then let us pass to the required subgroup of the index 2. The authomorphism $\mu^{(5)}_{b\times\dot{b}}$ is induced from the authomorphism $\mu_{\mathbf{Q}} : \mathbf{Q} \to \mathbf{Q}$ of the factors: $\mu_{\mathbf{Q}}(\mathbf{i}) = -\mathbf{i}$, $\mu_{\mathbf{Q}}(\mathbf{j}) = -\mathbf{j}$, $\mu_{\mathbf{Q}}(\mathbf{k}) = \mathbf{k}$. $\square$

7.1 An additional remark, which is required to check properties 1,2 in Theorem 21

Let us investigate as the automorphism $\mu_{\mathbf{Q}}$, described above, is defined on the 3-dimensional quaternionic lance space $S^3/\mathbf{Q}$ (the quaternion groups acks on S^3 on the right). The authomorphism $\mu_{\mathbf{Q}}$ is given by the right transformation of S^3, which is given by the formula above of the authomorphism. The authomorphism $\mu_{\mathbf{Q}}$ commutes with the transformation by the unite base quaternions (a calculation of the commutator of $\mu_{\mathbf{Q}}$ with the unite quaternion $\mathbf{k}$ is required). Therefore on the factorspace $S^3/\mathbf{Q}$ the automorphism is well-defined

A trivialization of the tangent space of the lence $S^3/\mathbf{Q}$ is given by the left multiplication on the unite quaternions $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, is changed by the automorphism correspondingly. Therefore the image of the tangent framing by the differential $D(\mu_{\mathbf{Q}})$:

$T(S^3/\mathbf{Q}) \to T(S^3/\mathbf{Q})$ is fibre-preserved isotopic to the identity. This isotopy of tangent spaces is a family of rotations of fibres on the angle π in planes orthogonal to the quaternion $\mathbf{k}$.

The homomorphism (106) is extened to the homomorphism of the Laurent extension from the subgroup $\mathbf{J}_b \times \dot{\mathbf{J}}_b \rtimes_{\mu^{(4)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[4]}} \mathbb{Z}$ of the index 2 to the all group:

$$(\mathbf{J}_b \times \dot{\mathbf{J}}_b \times \mathbb{Z}/2) \rtimes_{\mu^{(5)}_{b\times\dot{b}}} \mathbb{Z} \times \mathbb{Z} \rtimes_{\chi^{[5]}} \mathbb{Z} \mapsto \mathbb{Z}/2^{[5]}. \qquad (110)$$

Definition 21. Let us say that $\mathbb{Z}/2^{[4]}$-framed immersion (g, Ψ, η_N) of the codimension $8k$, whic is stanardized in the sence of Definition 12, is an immersion with $\mathbf{Q} \times \mathbb{Z}/4$-structure (with a quaternion-cyclic structure), if g is self-intersects along a $\mathbb{Z}/2^{[5]}$-framed immersion (h, Ξ, ζ_L) with the self-intersection manifold L of the dimension $n - 16k$ (for $k = 7$, $n = 126$ we have $n - 16k = 14$). additionally, the self-intersection manifold is divided into components: $L = L_{\mathbf{Q}\times\mathbb{Z}/4} \cup L_{negl}$ and the following conditions are satisfied:

1. the Hurewicz image of the fundamental class $[L_{\mathbf{Q}\times\mathbb{Z}/4}]$ belongs to the sum of images of the Laurent extension (62) by the homomorphisms (109) and (110).

2. The component L_{negl} is negligible; the Hurewicz image of the class $[\bar{L}_{negl}]$ belongs to the image of the extesion of (62).

Лемма 22. *An arbitrary $\mathbb{Z}/2^{[4]}$-framed immersion (g, Ψ, η_N) of the codimension $8k$, which is a standardized in the sence of Definition 12, admits in its regular cobordism class a a standardized representative with $\mathbf{Q} \times \mathbb{Z}/4$-structure in the sence on Definition 21.*

Proof of Lemma 22

A proof is analogous to Theoem 15. A quaternionic analoug of constructions in Lemma 18, which is based on the Massey embedding $S^3/\mathbf{Q} \subset \mathbb{R}^4$ and Remark 7.1 are needed. Local coefficients systemis constructed fot the Laurent extension (105), but not for an extension over $\mathbb{RP}^2 \times \mathbb{RP}^2$, because a standardized immersion (g, Ψ, η_N) has a defect of the bicyclic structure of high codimension, this reduces the Hurewicz image. □

From the following theorem the Snaith Conjecture is deduced. The proof is a transformation of the result of the paper [A-P1] to the more complicated case, described above.

Propositions 1,2 from Definition 21, formulated above, which are required in the statement are satisfied. A generalization of (77) and below on the quaternionic case is used.

Теорема 23. *Let (g, Ψ, η_N) be a $\mathbb{Z}/2^{[4]}$–framed standardized (in the sense of Definition 12), which is an immersion with $\mathbf{Q} \times \mathbb{Z}/4$-structure (in the sense of Definition 21). Assume that (g, Ψ, η_N) is a pure (in the sense of Definition 13). Then (g, Ψ, η_N) is negligible.*

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