

Geometric approach towards stable homotopy groups of spheres. The Steenrod-Hopf invariant

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Аннотация

In this paper a geometric approach toward stable homotopy groups of spheres, based on the Pontrjagin-Thom [P] construction is proposed. From this approach a new proof of Hopf Invariant One Theorem by J.F.Adams [A1] for all dimensions except 15, 31, 63, 127 is obtained.

It is proved that for $n > 127$ in the stable homotopy group of spheres Π_n there is no elements with Hopf invariant one. The new proof is based on geometric topology methods. The Pontrjagin-Thom Theorem (in the form proposed by R.Wells [W]) about the representation of stable homotopy groups of the real projective infinite-dimensional space (this groups is mapped onto 2-components of stable homotopy groups of spheres by the Khan-Priddy Theorem [A2]) by cobordism classes of immersions of codimension 1 of closed manifolds (generally speaking, non-orientable) is considered. The Hopf Invariant is expressed as a characteristic number of the dihedral group for the self-intersection manifold of an immersed codimension 1 manifold that represents the given element in the stable homotopy group. In the new proof the Geometric Control Principle (by M.Gromov)[Gr] for immersions in a given regular homotopy classes based on Smale-Hirsch Immersion Theorem [H] is required.

Let $f : M^{n-1} \looparrowright \mathbb{R}^n$, $n = 2^l - 1$, be a smooth immersion of codimension 1.

The characteristic number

$$\langle w_1^{n-1}(M); [M^{n-1}] \rangle = h(f)$$

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is called the stable Hopf invariant or the Steenrod-Hopf invariant. This characteristic number depends only on the immersed manifold M^{n-1} itself. The relationship with the definition of the Steenrod-Hopf invariant in algebraic topology is considered in [E],[K],[L].

Theorem (by J.F.Adams), [A]

$$\text{For } l \geq 4, h(f) = 0.$$

Skew-framed immersions

Let $f : M^{n-k} \looparrowright \mathbb{R}^n$ be an immersion of codimension k . Let $\kappa : E(\kappa) \rightarrow M^{n-k}$ be a line bundle over M^{n-k} and let $\Xi : k\kappa \rightarrow \nu(f)$ be an isomorphism of the normal bundle of the immersion f with the Whitney sum of k copies of the line bundle κ .

We shall call the triple (f, κ, Ξ) a skew-framed immersion with characteristic class $\kappa \in H^1(M^{n-k}; \mathbb{Z}/2)$. (If n is odd then $w_1(\kappa)$ is the orientation class of M^{n-k} , see [A-E] for more details).

The Steenrod-Hopf invariants for skew-framed immersions

The characteristic class $\langle w_1(\kappa)^{n-k}; [M^{n-k}] \rangle = h(f, \kappa, \Xi)$ is called the Steenrod-Hopf invariant of the skew-framed immersion (f, κ, Ξ) .

The Main Theorem 1

Let $(f : M^{n-k} \looparrowright \mathbb{R}^n, \kappa, \Xi)$ be a skew-framed immersion, $n = 2^l - 1$, $\dim(M) = n - k = \frac{n+1}{2} + 7$. Then for $n \geq 255$ (i.e. for $l \geq 8$)

$$h(f, \kappa, \Xi) = \langle w_1(\kappa)^{\dim(M)}; [M] \rangle = 0 \pmod{2}.$$

Corollary

Adams' Theorem for $n \geq 255$.

Let \mathbf{D}_4 be the dihedral group of order 8,

$$\mathbf{D}_4 = \{a, b | a^4 = b^2 = e, [a, b] = a^2\}.$$

This is the group of symmetries of the two coordinate axes in the plane. Let

$$\mathbf{I}_a = \{e, a, a^2, a^3\}, \mathbf{I}_b = \{e, b, a^2, a^2b\}, \mathbf{I}_c = \{e, ab, a^2, a^3b\}$$

be the subgroups in \mathbf{D}_4 of index 2. The cyclic $\mathbb{Z}/4$ -subgroup \mathbf{I}_a is generated by rotation of the plane through the angle $\frac{\pi}{2}$. The $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ -subgroup \mathbf{I}_b (\mathbf{I}_c) is generated by symmetries (or reflections) with respect to the bisectors of the coordinate axes. The $\mathbb{Z}/2$ -subgroup $\mathbf{I}_b \cap \mathbf{I}_c = \{e, a^2\}$ is generated by the central symmetry.

Let $(f : M^{n-k} \looparrowright \mathbb{R}^n, \kappa, \Xi)$ be a skew-framed generic immersion. Its double points manifold N^{n-2k} is immersed into \mathbb{R}^n , $g : N^{n-2k} \looparrowright \mathbb{R}^n$, and the normal bundle $\nu(g)$ admits a canonical decomposition $\Psi : \nu(g) = k\eta^*$, where η^* is a two-dimensional bundle over N^{n-2k} with \mathbf{D}_4 -structure. The bundle η^* is a pull-back of the universal bundle $E(\mathbf{D}_4) \rightarrow K(\mathbf{D}_4, 1)$, via the classifying map $\eta : N^{n-2k} \rightarrow K(\mathbf{D}_4, 1)$.

Let us consider the canonical 2-fold covering $\bar{N}^{n-2k} \rightarrow N^{n-2k}$ over the double point manifold of the immersion g . This covering corresponds to the subgroup $\mathbf{I}_c \subset \mathbf{D}_4$, $\mathbf{I}_c = \{e, a^2, ab, a^3b\}$.

Let $\bar{\kappa} \in H^1(\bar{N}^{n-2k}; \mathbb{Z}/2)$ be the cohomology class corresponding to the epimorphism $\mathbf{I}_c \rightarrow \mathbf{I}_d$ with the image $\mathbf{I}_d = \{e, a^2 \simeq ab\} = \mathbb{Z}/2$ (the kernel is generated by the element a^3b). By the definition $\bar{\kappa} = i^*(\kappa)$, $i : \bar{N}^{n-2k} \looparrowright M^{n-k}$ is the canonical immersion of the double point covering. Let us define the following characteristic number

$$\bar{h}(g, \eta, \Psi) = \langle \bar{\kappa}^{n-2k}; [\bar{N}^{n-2k}] \rangle.$$

Lemma 2

$$h(f, \kappa, \Xi) = \bar{h}(g, \eta, \Psi).$$

Proof of Lemma 2

Immediate from Herbert's Theorem. (Concerning Herbert's Theorem, see e.g. [E-G].)

Definition (Cyclic structure for skew-framed immersions)

Let (f, κ, Ξ) be a skew-framed immersion, N^{n-2k} be the (odd-dimensional) double self-intersection point manifold of f . A mapping

$$\mu : N^{n-2k} \rightarrow K(\mathbf{I}_a, 1)$$

($\mathbf{I}_a = \{e, a, a^2, a^3\}$) is called a *cyclic structure* for f if

$$\langle \mu^*(t); [N^{n-2k}] \rangle = h(f),$$

where $t \in H^{n-2k}(K(\mathbf{I}_a, 1); \mathbb{Z}/2)$ is the generator.

The following lemma is proved by an explicit calculation.

The Main Lemma 3 (jointly with P.J.Eccles (1998))

Let $n - k = \frac{n+1}{2} + 7$ ($n - 2k = 15$), $n \geq 31$, $\mu : N^{n-2k} \rightarrow K(\mathbf{D}_4, 1)$ be a cyclic structure for f . Then $h(f, \kappa, \Xi) = 0 \pmod{2}$.

Lemma 4. Cyclic Structure for skew-framed immersions

For $n \geq 255$, an arbitrary skew-framed immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$, $n - k = \frac{n+1}{2} + 7$ is regularly homotopic to an immersion with a cyclic structure.

Proof

The proof is a corollary of Lemma 5 and Proposition 6.

Definition (Cyclic Structure for generic mappings of the standard projective space)

Let $g : \mathbb{R}P^{n-k} \rightarrow \mathbb{R}^n$ be a generic mapping, $n \geq 3k$, $n = 2^l - 1$ with double point manifold N^{n-2k} and critical points $(n-2k-1)$ -dimensional submanifold $(\partial N)^{n-2k-1} \subset \mathbb{R}P^{n-k}$.

Let $\eta : (N^{n-2k}, \partial N) \rightarrow (K(\mathbf{D}_4, 1), K(\mathbf{I}_b, 1))$ be the structured mapping corresponding to g . This structured mapping is defined analogously with the case of skew-framed immersions. Obviously, the restriction of the mapping η to the boundary of the double points manifold, i.e. to the critical points submanifold, has the target $K(\mathbf{I}_b, 1) \subset K(\mathbf{D}_4, 1)$. The standard inclusion $\mathbf{I}_d \subset \mathbf{I}_a$ as the subgroup of the index 2 is well-defined. We shall call a mapping

$$\mu : (N^{n-2k}, \partial N^{n-2k-1}) \rightarrow (K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1))$$

a cyclic structure for g , if the following conditions hold:

- (*) the homological condition:

$$\langle \mu^*(t); [N^{n-2k}, \partial N^{n-2k-1}] \rangle = 1 \pmod{2},$$

where $t \in H^{n-2k}(K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1); \mathbb{Z}/2)$ is the generator in the cokernel of the homomorphism $H^{n-2k}(K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1); \mathbb{Z}/2)$ (remark: this relative characteristic number is well-defined, by explicit calculations $\mu_*([\partial N^{n-2k-1}]) \in H_{n-2k-1}(K(\mathbf{I}_d, 1; \mathbb{Z}/2))$ is trivial);

- (**) the boundary condition :

$$p_c \circ \eta|_{\partial N} : \partial N^{n-2k-1} \rightarrow K(\mathbf{I}_b, 1) \rightarrow K(\mathbf{I}_d, 1),$$

$\mathbf{I}_b = \{e, b, a^2, ba^2\}$, where $p_c : K(\mathbf{I}_b, 1) \rightarrow K(\mathbf{I}_d, 1)$ is the standard projection with the image $\mathbf{I}_d = \{e, a^2 \simeq ba^2\}$, coincides with $\mu|_{\partial N^{n-2k-1}}$.

Lemma 5. Geometric Control

Let

$$(f, \kappa, \Xi), \quad f : M^{n-k} \looparrowright \mathbb{R}^n$$

be a skew-framed immersion. Let us assume (see Proposition 6 below) that there exists a generic mapping

$$g : \mathbb{R}P^{n-k} \rightarrow \mathbb{R}^n, \quad n \geq 3k,$$

with a cyclic structure

$$\mu : (N^{n-2k}, \partial N^{n-2k-1}) \rightarrow (K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1)).$$

Then there exists a skew-framed immersion (f', κ, Ξ') in the regular homotopy class of f with a cyclic structure.

The idea of the proof of Lemma 5.

Take the mapping

$$g \circ \kappa : M^{n-k} \rightarrow \mathbb{R}P^{n-k} \rightarrow \mathbb{R}^n.$$

By [Gr], 1.2.2, in the regular homotopy class of f there exists a generic immersion

$$f' : M^{n-k} \looparrowright \mathbb{R}^n$$

defined as a C^0 -close generic regular perturbation (arbitrary small) of a (singular) mapping

$$g \circ \kappa.$$

A construction by S.A.Melikhov (2004)

Let us denote by J the join of $(2^{l-4} + 1) = r$ copies of the standard $\mathbb{Z}/4$ - lens space S^7/i , $\dim(J) = 2^{l-1} + 7 = n - k$. There is a PL -embedding $i_J : J \subset \mathbb{R}^n$ for $l \geq 8$.

Let $p' : S^{n-k} \rightarrow J$ be the join of r copies of the standard cover $S^7 \rightarrow S^7/i$, $\hat{p} : S^{n-k}/i \rightarrow J$ be the quotient mapping of p' , $p : \mathbb{R}P^{n-k} \rightarrow J$ be the composition of the standard projection $\pi : \mathbb{R}P^{n-k} \rightarrow S^{n-k}/i$ with \hat{p} . The composition $i_J \circ \hat{p} : S^{n-k}/i \rightarrow J \rightarrow \mathbb{R}^n$ is well-defined. Let $\hat{g} : S^{n-k}/i \rightarrow \mathbb{R}^n$ be an ε -small generic alternation of the mapping $i_J \circ \hat{p}$, $d : \mathbb{R}P^{n-k} \rightarrow \mathbb{R}^n$ be defined by a ε_1 -small generic alternation, $\varepsilon_1 \ll \varepsilon$, of the composition $\hat{g} \circ \pi$.

Proposition 6¹

The Melikhov map $d : \mathbb{R}P^{n-k} \rightarrow \mathbb{R}P^n$ is equipped with a cyclic structure.

The rest of the paper concerns the proof of this result. This Proposition with Lemma 5 implies Lemma 4 and Lemma 4 with the Main Lemma 3 implies Lemma 2 and the main Theorem 1.

The beginning of the proof of the Proposition 6

Let Γ_0 be the *deleted product* of the standard projective space $\mathbb{R}P^{n-k}$

$$\Gamma_0 = (\mathbb{R}P^{n-k} \times \mathbb{R}P^{n-k} \setminus \Delta_{diag})/T',$$

where the quotient is determined with respect to the free involution

$$\begin{aligned} T' : \mathbb{R}P^{n-k} \times \mathbb{R}P^{n-k} \setminus \Delta_{diag} &\rightarrow \mathbb{R}P^{n-k} \times \mathbb{R}P^{n-k} \setminus \Delta_{diag} \\ T'(x, y) &= (y, x). \end{aligned}$$

The classifying map

$$\eta_{\Gamma_0} : \Gamma_0 \rightarrow K(\mathbf{D}_4, 1)$$

is well-defined. (Note that $\pi_1(\Gamma_0) = \mathbf{D}_4$ and the involution T' corresponds to the subgroup $\mathbf{I}_c \subset \mathbf{D}_4$.) Let $\Delta_{antidiag} \subset \Gamma_0$ be a subspace, called the antidiagonal, defined as

$$\Delta_{antidiag} = \{(x, y) \in (\mathbb{R}P^{n-k} \times \mathbb{R}P^{n-k}/T') \mid T(x) = y\},$$

where

$$T : \mathbb{R}P^{n-k} \rightarrow \mathbb{R}P^{n-k}$$

is the standard involution on the covering

$$\mathbb{R}P^{n-k} \rightarrow S^{n-k}/i.$$

Let us denote by Γ the space

$$\Gamma_0 \setminus (U(\Delta_{antidiag}) \cup U(\Delta_{diag})),$$

where $U(\Delta_{antidiag})$ is a small regular neighborhoods of $\Delta_{antidiag}$, $U(\Delta_{diag})$ is a small regular neighborhood of the end of Γ_0 near the deleted diagonal Δ_{diag} .

¹The author was developed this proof following conversations with Prof. O.Saeki and Dr. R.R.Sadykov (2006)

(The radius of the regular neighborhoods depends on a constant ε of an approximation in the Melikhov construction.)

The space Γ is a manifold with boundary. The involution

$$T : \mathbb{R}P^{n-k} \rightarrow \mathbb{R}P^{n-k}$$

induces the free involution

$$T_\Gamma : \Gamma \rightarrow \Gamma.$$

A polyhedron

$$\Sigma_0 = \{[(x, y)] \in \Gamma_0, p(x) = p(y)\}, \quad \Sigma_0 \subset \Gamma_0$$

of double points of the mapping

$$p : \mathbb{R}P^{n-k} \rightarrow J$$

is called *the singular set* or *the singular polyhedron*.

The mapping

$$\eta_{\Sigma_0} : \Sigma_0 \rightarrow K(\mathbf{D}_4, 1)$$

is well-defined as the restriction of the mapping

$$\eta_{\Gamma_0}|_{\Sigma_0}.$$

The subpolyhedron

$$\Sigma_0 \subset \Gamma_0$$

decomposes as

$$\Sigma_0 = \Sigma_{antidiag} \cup K, \quad K \subset \Gamma,$$

where

$$\Sigma_{antidiag} = \Sigma_0 \cap U(\Delta_{antidiag}).$$

The restriction $\eta_{\Gamma_0}|_K$ will be denoted by $\eta_K : K \rightarrow K(\mathbf{D}_4, 1)$.

Boundary conditions of η_K

The diagonal and antidiagonal boundary components of K will be denoted by

$$Q_{diag} = K \cap \partial U(\Delta_{diag}), \quad Q_{antidiag} = K \cap \partial U(\Delta_{antidiag}).$$

The restriction $\eta_K|_{Q_{antidiag}} : Q_{antidiag} \rightarrow K(\mathbf{D}_4, 1)$ is decomposes as

$$i_a \circ \eta_{antidiag} : Q_{antidiag} \rightarrow K(\mathbf{I}_a, 1) \subset K(\mathbf{D}_4, 1).$$

The restriction $\eta_K|_{Q_{diag}} : Q_{diag} \rightarrow K(\mathbf{D}_4, 1)$ is decomposed as

$$i_b \circ \eta_{diag} : Q_{diag} \rightarrow K(\mathbf{I}_b, 1) \subset K(\mathbf{D}_4, 1).$$

The resolution space RK .

Let us construct the space RK called *the resolution space* for K . This space is included into the diagram

$$K(\mathbf{I}_a, 1) \xleftarrow{\phi} RK \xrightarrow{pr} K.$$

Let us denote $pr^{-1}(Q_{diag})$ by RQ_{diag} , $pr^{-1}(Q_{antidiag})$ by $RQ_{antidiag}$. The boundary conditions on $Q_{antidiag}$ are:

$$\begin{array}{ccc} RQ_{antidiag} & \xrightarrow{pr} & Q_{antidiag} \\ \phi \searrow & & \swarrow \eta_{antidiag} \\ & K(\mathbf{I}_a, 1). & \end{array}$$

The boundary conditions on Q_{diag} are:

$$\begin{array}{ccc} RQ_{diag} & \xrightarrow{pr} & Q_{diag} \\ \phi \downarrow & & \downarrow \eta_{diag} \\ K(\mathbf{I}_d, 1) & \xleftarrow{pb} & K(\mathbf{I}_b, 1). \end{array}$$

The diagrams above are included into the following diagram:

$$\begin{array}{ccc} K(\mathbf{I}_a, 1) & & \\ \uparrow \phi & & \swarrow \\ RK & \longleftarrow & RQ_{diag} \cup RQ_{antidiag} \\ \downarrow & & \downarrow \\ K & \supset & Q_{diag} \cup Q_{antidiag} \\ \downarrow \eta & & \swarrow \\ K(\mathbf{D}_4, 1). & & \end{array}$$

Let us consider Melikhov's mapping $d : \mathbb{RP}^{n-k} \rightarrow \mathbb{R}^n$ (this is a small generic alternation of the composition $i \circ p \circ \pi : \mathbb{RP}^{n-k} \rightarrow S^{n-k}/i \rightarrow J \subset \mathbb{R}^n$). Let N^{n-2k} be the double point manifold (with boundary) of d , the embedding $N^{n-2k} \subset \Gamma_0$ is well-defined. The manifold N^{n-2k} is decomposed into two manifolds (with boundary) along the common component of the boundary

$$\begin{aligned} N^{n-2k} &= N_{antidiag} \cup N_d, \\ N_{antidiag} &= N^{n-2k} \cap U(\Delta_{antidiag}), \\ N_d &= N^{n-2k} \cap \Gamma. \end{aligned}$$

Lemma 7

There exists a mapping $res : N_d \rightarrow RK$ called the *resolution mapping* that induces a mapping $\mu : N_d \rightarrow K(\mathbf{I}_a, 1)$ included into the following diagram:

$$\begin{array}{ccc}
 K(\mathbf{I}_a, 1) & = & K(\mathbf{I}_a, 1) \\
 \uparrow \phi & & \uparrow \mu \\
 RK & \xleftarrow{res} & N_d \quad \supset \quad W_{diag} \cup W_{antidiag} \\
 \downarrow \eta & & \downarrow \eta \\
 K(\mathbf{D}_4, 1) & = & K(\mathbf{D}_4, 1),
 \end{array}$$

with boundary conditions on $W_{antidiag}$:

$$\mu|_{W_{antidiag} \subset N_d} = i_a \circ \eta_{antidiag} : W_{antidiag} \longrightarrow K(\mathbf{I}_a, 1) \xrightarrow{i_a} K(\mathbf{D}_4, 1),$$

and with boundary conditions on W_{diag} :

$$\mu = i_a \circ p_b \circ \eta_{diag} : W_{diag} \rightarrow K(\mathbf{I}_b, 1) \rightarrow K(\mathbf{I}_d, 1) \rightarrow K(\mathbf{I}_a, 1).$$

Lemma 8

The mapping

$$\mu_a = \eta|_{N_{antidiag}} \cup \mu : N^{n-2k} = N_{antidiag} \cup N_d \rightarrow K(\mathbf{I}_a, 1)$$

determines a cyclic structure for d .

Proof of Lemma 8

We have to prove the equality * in the Definition of the Cyclic Structure for generic mappings.

Let us consider the free involution $T_\Gamma : \Gamma \rightarrow \Gamma$ and the quotient Γ/T_Γ . The fundamental group $\pi_1(\Gamma/T_\Gamma)$, denoted by \mathbb{E} , is a quadratic extension of \mathbf{D}_4 by means of an element $c \in \mathbb{E} \setminus \mathbf{D}_4$, $c^2 = a^2$. The element c of order 4 is commutes with all elements in the subgroup $\mathbf{D}_4 \subset \mathbb{E}$. The following diagram is well-defined:

$$\begin{array}{ccccccc}
N_d & \longrightarrow & RK & \longrightarrow & K & \subset & \Gamma \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N_d/T & \longrightarrow & RK/T & \longrightarrow & K/T & \subset & G/T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(\mathbf{I}_a, 1) & = & K(\mathbf{I}_a, 1) & & K(\mathbb{E}, 1) & = & K(\mathbb{E}, 1).
\end{array}$$

The composition $RK \rightarrow RK/T \rightarrow K(\mathbf{I}_a, 1)$ coincides with ϕ (The $c \in \pi_1(RK/T) \setminus \pi_1(RK)$ commutes with all elements in the subgroup $\pi_1(RK) \subset \pi_1(RK/T)$ of the index 2 and the mapping $RK/T \rightarrow K(\mathbf{I}_a, 1)$ is well-defined. The image of the element c is the generator of the cyclic group \mathbf{I}_a .) The composition of the left vertical arrows in the diagram $N_d \rightarrow N_d/T_{N_d} \rightarrow K(\mathbf{I}_a, 1)$ coincides with the mapping $\mu : N_d \rightarrow K(\mathbf{I}_a, 1)$. The pair (N^{n-2k}, μ_a) is cobordant to a pair (N'^{n-2k}, μ'_a) , where $N'^{n-2k} = N'_{cycl}{}^{n-2k} \cup N_d^{n-2k}$, and $N'_{cycl}{}^{n-2k}$ is a closed manifold. The manifold (with boundary) $N'_{cycl}{}^{n-2k}$ is the double covering over the oriented manifold (with boundary) $N'_{cycl}{}^{n-2k}/T_{N_d}$. The base of the cover represents a cycle in $H_{n-2k}(K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1); \mathbb{Z})$. Therefore the relative cycle $\mu'_{a,*}([N_d'^{n-2k}, \partial N']) \in H_{n-2k}(K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1))$ is trivial and

$$\langle \mu_a^*(t); [N^{n-2k}, \partial N^{n-2k-1}] \rangle = \langle \mu_a^*(t); [N'_{cycl}{}^{n-2k}] \rangle.$$

The last characteristic number coincides with

$$\langle \kappa^{n-k}; [\bar{N}^{n-k}] \rangle = 1.$$

Lemma 8 is proved.

A natural stratification of the polyhedron K . Proof of Lemma 7

Let J be the join of lens spaces $(S^7/i)_j$, $j = 1 \dots, r$. The space J admits a natural stratification defined by the collection of subjoins $J(k_1, \dots, k_s)$ generated by lenses with numbers $0 < k_1 < \dots < k_s < r$.

The preimage

$$p^{-1}(J(k_1, \dots, k_s)) \subset \mathbb{R}P^{n-k},$$

$p : \mathbb{R}P^{n-k} \rightarrow J$, is denoted by $R(k_1, \dots, k_s)$.

A point

$$x \in R(k_1, \dots, k_s) \subset \mathbb{R}P^{n-k}$$

is determined by the collection of coordinates

$$(x_{k_1}, \dots, x_{k_s}, \lambda)$$

(up to the antipodal transformation of the first s coordinates), where $x_{k_j} \in S_j^7$, and λ is a barycentric coordinate on the standard $(s-1)$ -simplex.

The polyhedron K admits a natural stratification

$$K(k_1, \dots, k_s), \quad 1 \leq s \leq r,$$

correspondingly to the stratification of J . The maximal stratum $K(1, \dots, r)$ is represented by the disjoint union of connected components of different types.

Let a point (x_1, x_2) belongs to $K(1, \dots, r)$. Let $(x_{1,1}, x_{2,1}, \dots, x_{1,r}, x_{2,r}, \lambda)$ be the collection of the coordinates of the point. The first $2r$ terms of this collection is r ordered pairs of points on the standard sphere S^7 . The collection is defined up to the permutation of the coordinates in the pair and up to the independent antipodal transformation of first point or second point in each pair. The equivalence class of a collection of the coordinates of a point (x_1, x_2) contains 8 collections.

Types of components of the maximal stratum

Let $x \in K(1, \dots, r)$ be a point with the prescribed pair of collections of spherical coordinates $(x_{1,i}, x_{2,i})$. The following possibilities are: the coordinates in the i -th pair

(1) coincide, or (2) are antipodal, or (3) are related by means of the generator of the $\mathbb{Z}/4$ -cyclic cover.

This determines a sequence of r complex numbers $v_i \in \{1, -1, +i, -i\}$, $i = 1, \dots, r$, with respect to (1),(2), or (3). We will call such a sequence the characteristic. For an arbitrary point in a prescribed component $K(1, \dots, r)$ the characteristic is well-defined up to the multiplication of each term by -1 and this characteristic does not depend on a point on the component. We shall say that the prescribed component of the maximal stratum is of the \mathbf{I}_a -type (\mathbf{I}_b -type) if the corresponded characteristic contains only $\{+i, -i\}$ ($\{+1, -1\}$); the component is of the \mathbf{I}_d -type, if the characteristic contains at least 3 different values. It is easy to prove that the restriction of the canonical mapping $\eta : \Gamma \rightarrow K(\mathbf{D}_4, 1)$ on a stratum of the $\mathbf{I}_a, \mathbf{I}_b$ or \mathbf{I}_d -type admits (up to homotopy equivalence of the mappings) a reduction with the target in the subspace $K(\mathbf{I}_a, 1), K(\mathbf{I}_b, 1), K(\mathbf{I}_d, 1)$ of the space $K(\mathbf{D}_4, 1)$ correspondingly. This reduction (for strata of the \mathbf{I}_a and \mathbf{I}_b -types) is well-defined up to the composition with mapping of the corresponding classified space given by the conjugation automorphism $\mathbf{D}_4 \rightarrow \mathbf{D}_4, x \rightarrow (ba)x(ba)^{-1}, x \in \mathbf{D}_4, ba \in \mathbf{I}_c$.

The resolution space RK

Let us denote by $K_1 \subset K$ the disjoint union of all singular strata of the length 1, by K_0 the disjoint union of maximal strata, and by $K_{reg} \subset K$ the subpolyhedron defined as $K_{reg} = K_0 \cup K_1$. The component of the

boundary $K_{reg} \cap Q_{antidiag}$ is denoted by $Q_{reg,antidiag}$ and the component of the boundary $K_{reg} \cap Q_{diag}$ is denoted by $Q_{reg,diag}$. Note that $Q_{reg,antidiag}$ ($Q_{reg,antidiag}$) contains only points for which no more than two numbers in the characteristic are different from $+i$ ($+1$). Components of the space K_0 are divided into 3 classes: diagonal, antidiagonal and generic class. A component of the diagonal (antidiagonal) class intersects with the diagonal (the antidiagonal) by a maximal subcomponent of the boundary. A regular stratum of the considered type contains only points for which the only number in the characteristic is different from $+i$ ($+1$).

Let us denote by \bar{K}^1 the 2-sheeted covering space over K^1 with respect to the inclusion $\mathbf{I}_c \subset \mathbf{D}_4$. This covering coincides with the canonical double covering over the polyhedron of self-intersection points.

The space RK is defined from the following diagram:

$$\bar{K}^1 \rightarrow K^1 \subset K \supset K_{reg}.$$

The space K_{reg} is defined by the gluing by means of the collection of 2-sheeted coverings over a regular neighborhood of each component of the singular stratum of the length 1 with respect to the mapping

$$U(\bar{K}_1) \setminus \bar{K}_1 \longrightarrow (K_{reg} \setminus K_1).$$

The cyclic mapping $\phi : RK \rightarrow K(\mathbf{I}_a, 1)$

The union of all components in the given class is denoted by $K_{0,diag}$, $K_{0,antidiag}$, $K_{0,int}$ correspondingly. The restriction of the mapping $\eta : K \rightarrow K(\mathbf{D}_4, 1)$ to $K_{0,diag} \subset K$ ($K_{0,antidiag} \subset K$) is given by the composition

$$\begin{aligned} K_{0,diag} &\rightarrow K(\mathbf{I}_b, 1) \subset K(\mathbf{D}_4, 1) \\ (K_{0,antidiag} &\rightarrow K(\mathbf{I}_a, 1) \subset K(\mathbf{D}_4, 1)). \end{aligned}$$

Because of the prescribed boundary condition, the structured reduction for a subcomponent of $K_{0,diag}$, $K_{0,antidiag}$ is canonical. The structured reduction for a component of $K_{0,int}$ is non-canonical.

The mapping

$$\phi_0 : K_0 \rightarrow K(\mathbf{I}_a, 1)$$

extends to a mapping

$$\phi : RK \rightarrow K(\mathbf{I}_a, 1).$$

The composition

$$\bar{K}_1 \rightarrow K_1 \xrightarrow{\eta} K(\mathbf{D}_4, 1)$$

admits a natural reduction with the target

$$K(\mathbf{I}_c, 1) \subset K(\mathbf{D}_4, 1).$$

Let $K_{0,\alpha}, K_{0,\beta} \subset K_0$ be two prescribed components of the same type \mathbf{I}_b (or \mathbf{I}_a) with a common boundary stratum $K_{1,\gamma} \subset K_1$. A cyclic mapping

$$\phi_{0,*} = \pi_d \circ \eta_* : K_{0,*} \rightarrow K(\mathbf{I}_b, 1) \rightarrow K(\mathbf{I}_d, 1)$$

$$(\eta_* : K_{0,*} \rightarrow K(\mathbf{I}_a, 1)), \quad * \in \{\alpha, \beta\},$$

where $K_{0,*} \rightarrow K(\mathbf{I}_b, 1)$ ($K_{0,*} \rightarrow K(\mathbf{I}_a, 1)$) is well-defined up to a composition with the mapping

$$K(\mathbf{I}_b, 1) \rightarrow K(\mathbf{I}_b, 1) \quad (K(\mathbf{I}_a, 1) \rightarrow K(\mathbf{I}_a, 1)).$$

The last mapping is induced by the automorphism

$$\mathbf{D}_4 \rightarrow \mathbf{D}_4, \quad x \rightarrow (ba)x(ba)^{-1},$$

$$x \in \mathbf{D}_4, \quad ba \in \mathbf{I}_c.$$

The transfer with respect to the inclusion $\mathbf{I}_c \subset \mathbf{D}_4$ determines a unique mapping

$$\eta_*^! : \bar{K}_{0,*} \rightarrow K(\mathbf{I}_d, 1).$$

This proves that the extension ϕ with the prescribed boundary conditions exists.

The lift $res : N_d \rightarrow RK$

Let us consider a generic PL -homotopy

$$F(\tau) : S^{n-k}/i \rightarrow \mathbb{R}^n, \quad \tau \in [0; 1]$$

with the boundary conditions

$$F(0) = i \circ \hat{p} : S^{n-k}/i \rightarrow J \subset \mathbb{R}^n.$$

For a given $\tau \in (0; 1]$ the double points of $F(\tau)$ is denoted by $\hat{N}(\tau)$. This manifold with boundary is a submanifold of a quotient of the space $\Gamma/T_\Gamma \times \{\tau\}$. The polyhedron $\cup_\tau \hat{N}(\tau)$, $\tau \in (0, \varepsilon]$ (ε is sufficiently small) is denoted by $\hat{N}_{(0;\varepsilon]}$.

Because the mapping F is a PL -mapping, the bottom boundary of $\hat{N}_{(0;\varepsilon]}$, denoted by \hat{N}_0 , is a 15-dimensional subpolyhedron in a quotient of the space

$\Gamma/T_\Gamma \times \{0\}$. The polyhedron $\hat{N}_{(0;\varepsilon]}$ is the base of the 4-sheeted cover $N_{(0;\varepsilon]} \rightarrow \hat{N}_{(0;\varepsilon]}$, where $N_{(0;\varepsilon]}$ is the set of self-intersection points of the composition (after a small alteration)

$$F(\tau) \circ \pi : \mathbb{RP}^{n-k} \rightarrow S^{n-k}/i \rightarrow \mathbb{R}^n.$$

Because of the general position arguments the following condition holds:

(1) the polyhedron N_0 does not intersect (if ε is small enough) with singular strata in Γ of the length ≥ 2 (of codimension ≥ 16).

(2) the polyhedron N_0 is in general position with respect to the stratum of length 1; in particular, the restriction of $F(\tau)|_{p^{-1}(J^1)}$, $J^1 \subset J$, $\tau \in (0, \varepsilon]$ to the singular stratum of length 1 is an embedding.

A resolution mapping

$$res : N_d(\varepsilon) \rightarrow RK$$

with the prescribed boundary conditions is well-defined from (1),(2). Note that $diam(U_{antidiag}), diam(U_{diag})$ has to be less then the distance between N_0 and $K_2 \subset K$.

Proposition 6 is proved.

Discussion

Conjecture 1

There exists a 7-dimensional manifold K^7 with a normal \mathbf{D}_4 -framing Ξ_K in codimension $2^l - 8$, $l \geq 4$, such that the pair (K^7, Ξ_K) has the Steenrod-Hopf invariant 1.

Remark

An arbitrary cyclic \mathbf{I}_a -framed manifold (N^7, Ξ_N) in codimension $2^l - 8$ has the trivial Steenrod-Hopf invariant. The conjectured \mathbf{D}_4 -framed manifold (K^7, Ξ_K) cannot be realized as a double-point manifold for a skew-framed immersion $f : M^{2^{l-1}+3} \looparrowright \mathbb{R}^{2^l-1}$.

Conjecture 2

The Main Theorem holds for $n \geq 31$, i.e. for an arbitrary skew-framed immersion $f : M^{2^{l-1}+7} \looparrowright \mathbb{R}^{2^l-1}$, $l \geq 5$ the Steenrod-Hopf invariant is trivial.

Remark

A proof of the Conjecture 2 could be obtained by means of a straightforward generalization of the Melikhov mapping. The join of the standard mappings

$$\mathbb{R}P^7 \rightarrow S^7/i \subset \mathbb{R}^{14}$$

is replaced by the join of several copies of the standard mapping

$$\mathbb{R}P^3 \rightarrow Q^3 \subset \mathbb{R}^4,$$

where Q^3 is the quotient of the 3-sphere by the quaternions group of the order 8 (a homogeneous space) standard space,

$$\mathbb{R}P^3 \rightarrow Q^3$$

is the standard 4-sheeted cover,

$$Q^3 \subset \mathbb{R}^4$$

is the Massey embedding. This embedding is explicitly described in [M], Example 4.

This generalized construction determines the mapping

$$\mathbb{R}P^{4k-1} \rightarrow \mathbb{R}^{5k-1}$$

(below the metastable range) with a cyclic structure. The case

$$k = 6, \quad 4k - 1 = 23 = 2^4 + 7,$$

$$5k - 1 = 29 < 31 = 2^5 - 1$$

is required for a generalization of the The Main Theorem 1.

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Список литературы

- [A1] J.F.Adams, *On the non-existence of Elements of Hopf-Invariant One*, Ann Math., Ser.2, Vol 72 (1960) 20-104.
- [A2] J.F.Adams, *The Khan-Priddy Theorem*, Proc. Cambridge Philos. Soc. 73 (1973) 45-55.
- [A-E] P.M.Akhmet'ev and P.J.Eccles, *The relationship between framed bordism and skew-framed bordism*, Bull. London Math. Soc., vol. 39, 4 (2007) 473-481.
- [E] P.J.Eccles, *Codimension One Immersions and the Kervaire Invariant One Problem*, Math. Proc. Cambridge Phil. Soc. vol 90 (1981) 483- 493.
- [E-G] P.J.Eccles and M. Grant *Bordism groups of immersions and classes represented by self-intersections*, Algebraic and Geometric Topology 7 (2007) 1081–1097.
- [Gr] M.Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Springer-Verlag, Berlin 1986.
- [H] M.W.Hirsch, *Immersions of manifolds*, Trans.Amer.Math.Soc., 93, (1960) 242-276.
- [K] U.Koschorke, *Multiple points of immersions and the Kahn-Priddy theorem*, Math.Z., 169 (1979) 223-236.
- [L] J.Lannes, *Sur les immersions de Boy*, Lecture Notes in Mathematics, 1051, Springer-Verlag (1984) 263-270.
- [M] S.A.Melikhov, *Sphere eversions and the realization of mappings*, Труды Математического Института им. В.А.Стеклова, т.247, 159-181. Геометрическая топология и теория множеств, сборник статей к 100-летию Л.В.Келдыш, English transl.: Geom. Topol. i Teor. Mnozhh., Proc. Steklov Inst. Math. 2004, no. 4 (247), 143–163
- [P] L.S.Pontryagin, *Smooth manifolds and their applications to homotopy theory*, 2-nd ed., Nauka, Moscow (1976), English transl.: Amer. Math. Soc. Transl. (2) 11 (1959) 1-114; reprinted (1964).
- [W] R.Wells, *Cobordism groups of immersions*, Topology 5 (1966) 281- 294.