The v_1-periodic part of the Adams spectral sequence at an odd prime

by

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Abstract

We tell the story of the stable homotopy groups of spheres for odd primes at chromatic height 1 through the lens of the Adams spectral sequence. We find the "dancers to a discordant system."

We calculate a Bockstein spectral sequence which converges to the 1-line of the chromatic spectral sequence for the odd primary Adams E_2-page. Furthermore, we calculate the associated algebraic Novikov spectral sequence converging to the 1-line of the BP chromatic spectral sequence. This result is also viewed as the calculation of a direct limit of localized modified Adams spectral sequences converging to the homotopy of the v_1-periodic sphere spectrum.

As a consequence of this work, we obtain a thorough understanding of a collection of q_0-towers on the Adams E_2-page and we obtain information about the differentials between these towers. Moreover, above a line of slope $1/(p^2-p-1)$ we can completely describe the E_2 and E_3-pages of the mod p Adams spectral sequence, which accounts for almost all the spectral sequence in this range.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics

Acknowledgments

Without the support of my mother and my advisor, Haynes, I have no doubt that this thesis would have ceased to exist.

There are many things I would like to thank my mother for. Most relevant is the time she dragged me to Oxford. I had decided, at sixteen years of age, that I was not interested in going to Oxbridge for undergraduate study but she knew better. Upon visiting Oxford, I experienced for the first time the wonder of being able to speak to others who love maths as much as I do. My time there was mathematically fulfilling and the friends I made, I hope, will be lifelong. Secondly, it was her who encouraged me to apply to MIT for grad school. There's no other way to put it, I was terrified of moving abroad and away from the friends I had made. I would come to be the happiest I could ever have been at MIT. Cambridge is a beautiful place to live and the energy of the faculty and students at MIT is untouched by many institutes. Her support during my first year away, during the struggle of qualifying exams, from over $3,000$ miles away, and throughout the rest of my life is never forgotten.

Haynes picked up the pieces many times during my first year at MIT. His emotional support and kindness in those moments are the reasons I chose him to be my advisor. He has always been a pleasure to talk with and I am particularly appreciative of how he adapted to my requirements, always giving me the level of detail he knows I need, while holding back enough so that our conversations remain exciting. His mathematical influence is evident throughout this thesis. In particular, theorem 1.4.4 was his conjecture and the results of this thesis build on his work in [10] and [11]. It has been a pleasure to collaborate with him in subsequent work [2]. On the other hand, it is wonderful to have an advisor that I consider a friend and who I can talk to about things other than math. I will always remember him giving me strict orders to go out and buy a guitar amp when he could tell I was suffering without. Thank you, Haynes.

There are many friends to thank for their support during my time at MIT. I am grateful to Michael, Dana, Jiayong, and Saul, particularly for their support during

my first year at MIT. I am grateful to Rosa, Stuart, Pat and Nate for putting up with me as a roommate. Thank you, Nisa. I have been a better person since knowing you. Thank you, Alex, for letting me talk your ear off about permanent cycles for months and months, and for being the best friend one could hope for.

The weeks I spent collaborating with Will as he coded up spectral sequence charts were some of my most enjoyable as a mathematician. Before Will's work, no-one had seen a trigraded spectral sequence plotted in 3D with rotation capability, or a 70 term cocycle representative for e_0. His programs made for a particularly memorable thesis defense and will be useful for topologists for a long time, I am sure. Thank you, Will.

There are three courses I feel very lucky to have been a part of during my time at MIT. They were taught by Haynes, Mark Behrens and Emily Riehl. Haynes' course on the Adams spectral sequence was the birthplace for this thesis.

Mark taught the best introductory algebraic topology course that you can imagine. It was inspiring for my development as a topologist and a teacher. I wish to thank him for the advice he gave me during the microteaching workshop, the conversations we have had about topology and for the energy he brings to everything he is involved in. I hope we will work together more in the future.

Emily made sure that I finally learned some categorical homotopy theory. Each one of her classes was like watching Usain Bolt run the 100m over and over again for an hour. They were incredible. I thank her for her rigour, her energy and for showing me that abstract nonsense done right is beautiful. Although, the final version of this thesis contains less categorical homotopy theory than the draft, her course gave me the tools I needed to prove proposition 8.1.9.

Finally, I wish to thank Jessica Barton for her support, John Wilson who made it possible for me to take my GRE exams, and Yan Zhang who helped me plot pictures of my spectral sequences, which inspired the proof of proposition 5.4.5.

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Chapter 1

Introduction

1.1 The stable homotopy groups of spheres

Algebraic topologists are interested in the class of spaces which can be built from spheres. For this reason, when one tries to understand the continuous maps between two spaces up to homotopy, it is natural to restrict attention to the maps between spheres first. The groups of interest

$$\pi_{n+k}(S^k) = \{\text{homotopy classes of maps } S^{n+k} \longrightarrow S^k\}$$

are called the *homotopy groups of spheres*.

Topologists soon realized that it is easier to work in a stable setting. Instead, one asks about the *stable homotopy groups of spheres* or, equivalently, the homotopy groups of the *sphere spectrum*

$$\pi_n(S^0) = \text{colim}_k\ \pi_{n+k}(S^k).$$

Calculating all of these groups is an impossible task but one can ask for partial information. In particular, one can try to understand the global structure of these groups by proving the existence of recurring patterns. These patterns are clearly visible in spectral sequence charts for calculating $\pi_*(S^0)$ and this thesis came about

because of the author's desire to understand the mystery behind these powerful dots and lines, which others in the field appeared so in awe of. It tells the story of the stable homotopy groups of spheres for odd primes at chromatic height 1, through the lens of the Adams spectral sequence.

1.2 Calculational tools in homotopy theory

The *Adams spectral sequence (ASS)* and the *Adams-Novikov spectral sequence (ANSS)* are useful tools for homotopy theorists. Theoretically, they enable a calculation of the stable homotopy groups but they have broader utility than this. Much of contemporary homotopy theory has been inspired by analyzing the structure of these spectral sequences.

The ASS has E_2-page given by the *cohomology of the dual Steenrod algebra* $H^*(A)$ and it converges p-adically to $\pi_*(S^0)$. The ANSS has as its E_2-page the cohomology of the Hopf algebroid BP_*BP given to us by the p-typical factor of *complex cobordism* and it converges p-locally to $\pi_*(S^0)$.

The ANSS has the advantage that elements constructed using *non-nilpotent self maps* occur in low filtration. This means that the classes they represent are less likely to be hit by differentials in the spectral sequence and so proving such elements are nontrivial in homotopy often comes down to an algebraic calculation of the E_2-page. The ASS has the advantage that such elements have higher filtration and, therefore, less indeterminacy in the spectral sequence. For this reason, among others, arguing with both spectral sequences is fruitful.

$$\begin{array}{ccc} H^*(P;Q) & \xRightarrow{\text{CESS}} & H^*(A) \\ \Big\Downarrow{\scriptstyle \text{alg.NSS}} & & \Big\Downarrow{\scriptstyle \text{ASS}} \\ H^*(BP_*BP) & \xRightarrow{\text{ANSS}} & \pi_*(S^0) \end{array} \tag{1.2.1}$$

The relationship between the two spectral sequences is strengthened by the existence of an algebra $H^*(P;Q)$, which serves as the E_2-page for two spectral sequences:

the *Cartan-Eilenberg spectral sequence (CESS)* which converges to $H^*(A)$, and the *algebraic Novikov spectral sequence (alg.NSS)* converging to $H^*(BP_*BP)$. We will say more about the algebra $H^*(P;Q)$ shortly. For now it will be a black box and we will give the relevant definitions in the next section.

Continuing our comparison of the two spectral sequences for calculating $\pi_*(S^0)$, we note that the ASS has the advantage that its E_2-page can be calculated, in a range, efficiently with the aid of a computer. The algebra required to calculate the E_2-page of the ANSS is more difficult. For this reason, the *chromatic spectral sequence (v-CSS)* was developed in [12] to calculate the 1 and 2-line.

$$\begin{array}{ccc} \bigoplus_{n\geq 0} H^*(P; q_n^{-1}Q/(q_0^\infty, \ldots, q_{n-1}^\infty)) & \xRightarrow{q\text{-CSS}} & H^*(P;Q) \\ \Big\Downarrow{\scriptstyle \text{alg.NSS}} & & \Big\Downarrow{\scriptstyle \text{alg.NSS}} \\ \bigoplus_{n\geq 0} H^*(BP_*BP; v_n^{-1}BP_*/(p^\infty, \ldots, v_{n-1}^\infty)) & \xRightarrow{v\text{-CSS}} & H^*(BP_*BP) \end{array}$$

In [10, §5], Miller sets up a *chromatic spectral sequence* for computing $H^*(P;Q)$. To distinguish this spectral sequence from the more frequently used chromatic spectral sequence of [12], we call it the *q-CSS*. At odd primes, Miller [10, §4] shows that the E_2-page of the ASS can be identified with $H^*(P;Q)$ and so he compares the q-CSS and the v-CSS to explain some differences between the Adams and Adams-Novikov E_2-terms. He also observes that it is almost trivial to calculate the 1-line in the BP case ([12, §4]), but notes that it is more difficult to calculate the 1-*line* of the q-CSS. The main result of this thesis is a calculation of the 1-line of the q-CSS, that is, of

$$H^*(P; q_1^{-1}Q/q_0^\infty).$$

The most interesting application of this work is a calculation of the ASS, at odd primes, above a line of slope $1/(p^2-p-1)$. We note that as the prime tends to infinity, the fraction of the ASS described tends to 1. As a consequence of this work, we are able to describe, for the first time, differentials of arbitrarily long length in the ASS.

1.3 Some BP_*BP-comodules and the corresponding P-comodules

Our main result is the calculation of a *Bockstein spectral sequence* converging to $H^*(P; q_1^{-1}Q/q_0^\infty)$, the 1-line of the chromatic spectral sequence for $H^*(P;Q)$. First, we recall how P, Q and related P-comodules are defined. They come from mimicking constructions used in the chromatic spectral sequence for $H^*(BP_*BP)$ and so we also recall some relevant BP_*BP-comodules. p is an odd prime throughout this thesis.

Recall that the coefficient ring of the *Brown-Peterson spectrum* BP is a polynomial algebra $\mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$ on the Hazewinkel generators.

$$p \in BP_* \text{ and } v_1^{p^{n-1}} \in BP_*/p^n$$

are BP_*BP-comodule primitives and so we have BP_*BP-comodules $v_1^{-1}BP_*/p$,

$$\begin{aligned} BP_*/p^\infty &= \operatorname{colim}(\ldots \longrightarrow BP_*/p^n \xrightarrow{p} BP_*/p^{n+1} \longrightarrow \ldots), \text{ and} \\ v_1^{-1}BP_*/p^\infty &= \operatorname{colim}(\ldots \longrightarrow (v_1^{p^{n-1}})^{-1}BP_*/p^n \xrightarrow{p} (v_1^{p^n})^{-1}BP_*/p^{n+1} \longrightarrow \ldots). \end{aligned}$$

By filtering the BP *cobar construction* by powers of the kernel of the augmentation $BP_* \longrightarrow \mathbb{F}_p$ we obtain the algebraic Novikov spectral sequence

$$H^*(P;Q) \implies H^*(BP_*BP).$$

$P = \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \ldots]$ is the polynomial sub Hopf algebra of the dual Steenrod algebra A and

$$Q = \operatorname{gr}^* BP_* = \mathbb{F}_p[q_0, q_1, q_2, \ldots]$$

is the associated graded of BP_*; q_n denotes the class of v_n. Similarly to above, we have P-comodules $q_1^{-1}Q/q_0$, Q/q_0^∞ and $q_1^{-1}Q/q_0^\infty$ and there are appropriate algebraic Novikov spectral sequences (the first three vertical spectral sequences in figure 1-2).

1.4 Main results

We have a Bockstein spectral sequence, the q_1^{-1}-*Bockstein spectral sequence (q_1^{-1}-BSS)* coming from q_0-multiplication:

$$\left[H^*(P; q_1^{-1}Q/q_0)\left[q_0\right]\right]/q_0^\infty \implies H^*(P; q_1^{-1}Q/q_0^\infty).$$

Our main theorem is the complete calculation of this spectral sequence, and this, as we shall describe, tells us a lot about the Adams E_2-page.

The key input for the calculation is a result of Miller, which we recall presently.

Theorem 1.4.1 (Miller, [10, 3.6])**.**

$$H^*(P; q_1^{-1}Q/q_0) = \mathbb{F}_p[q_1^{\pm 1}] \otimes E[h_{n,0} : n \geq 1] \otimes \mathbb{F}_p[b_{n,0} : n \geq 1].$$

Here $h_{n,0}$ and $b_{n,0}$ are elements which can be written down explicitly, though their formulae are not important for the current discussion. To state the main theorem in a clear way we change these exterior and polynomial generators by units.

Notation 1.4.2. For $n \geq 1$, let $p^{[n]} = \frac{p^n-1}{p-1}$, $\epsilon_n = q_1^{-p^{[n]}} h_{n,0}$, and $\rho_n = q_1^{1-p^{[n+1]}} b_{n,0}$.

We have $H^*(P; q_1^{-1}Q/q_0) = \mathbb{F}_p[q_1^{\pm 1}] \otimes E[\epsilon_n : n \geq 1] \otimes \mathbb{F}_p[\rho_n : n \geq 1]$.

We introduce some convenient notation for differentials in the q_1^{-1}-BSS.

Notation 1.4.3. Suppose $x, y \in H^*(P; q_1^{-1}Q/q_0)$. We write $d_r x = y$ to mean that for all $v \in \mathbb{Z}$, $q_0^v x$ and $q_0^{v+r} y$ survive until the E_r-page and that $d_r q_0^v x = q_0^{v+r} y$. In this case, notice that $q_0^v x$ is a permanent cycle for $v \geq -r$.

$H^*(P; q_1^{-1}Q/q_0)$ is an algebra and with the notation just introduced differentials are derivations, i.e. from differentials $d_r x = y$ and $d_r x' = y'$ we deduce that $d_r(xx') = yx' + (-1)^{|x|} xy'$.

Using $\doteq$ to denote equality up to multiplication by an element in $\mathbb{F}_p^\times$, we are now ready to state the main theorem.

Theorem 1.4.4. *In the q_1^{-1}-BSS we have two families of differentials. For $n \geq 1$,*

1. $d_{p^{[n]}} q_1^{kp^{n-1}} \doteq q_1^{kp^{n-1}} \epsilon_n$, *whenever* $k \in \mathbb{Z} - p\mathbb{Z}$;

2. $d_{p^n-1} q_1^{kp^n} \epsilon_n \doteq q_1^{kp^n} \rho_n$, *whenever* $k \in \mathbb{Z}$.

Together with the fact that $d_r 1 = 0$ for $r \geq 1$, these two families of differentials determine the q_1^{-1}-BSS.

We describe the significance of this theorem in terms of the Adams spectral sequence E_2-page. To do so, we need to recall how the 1-line of the chromatic spectral sequence manifests itself in $H^*(A)$. In the following zig-zag, L is the natural localization map, ∂ is the boundary map coming from the short exact sequence of P-comodules $0 \longrightarrow Q \longrightarrow q_0^{-1}Q \longrightarrow Q/q_0^\infty \longrightarrow 0$, and the isomorphism $H^*(P;Q) \cong H^*(A)$ is the one given by Miller in [10, §4].

$$H^*(P; q_1^{-1}Q/q_0^\infty) \xleftarrow{L} H^*(P; Q/q_0^\infty) \xrightarrow{\partial} H^*(P;Q) \xleftarrow{\cong} H^*(A) \quad (1.4.5)$$

If an element of $H^*(P; q_1^{-1}Q/q_0^\infty)$ is a permanent cycle in the q-CSS, then we can lift it under L and map via ∂ (and the isomorphism) to $H^*(A)$. If there is no lift of an element of $H^*(P; q_1^{-1}Q/q_0^\infty)$ under L then it must support a nontrivial chromatic differential.

We now turn to figure 1-1. Recall that q_0 is the class detecting multiplication by p in the ASS. Figure 1-1 displays selected "q_0-towers" in the ASS at the prime 3; most of these are visible in the charts of Nassau [14]. In the range displayed, we see that there are "principal towers" in topological degrees which are one less than a multiple of $2p-2$ and "side towers" in topological degrees which are two less than a multiple of $p(2p-2)$. Under the zig-zag of (1.4.5) (lifting uniquely under ∂ and applying L) we obtain q_0-towers in $H^*(P; q_1^{-1}Q/q_0^\infty)$. The principal towers are sent to q_0-towers which correspond to differentials in the first family of 1.4.4. The side towers are sent to q_0-towers which correspond to differentials in the second family of 1.4.4. In the ASS, in the range plotted, there are as many differentials as possible between each

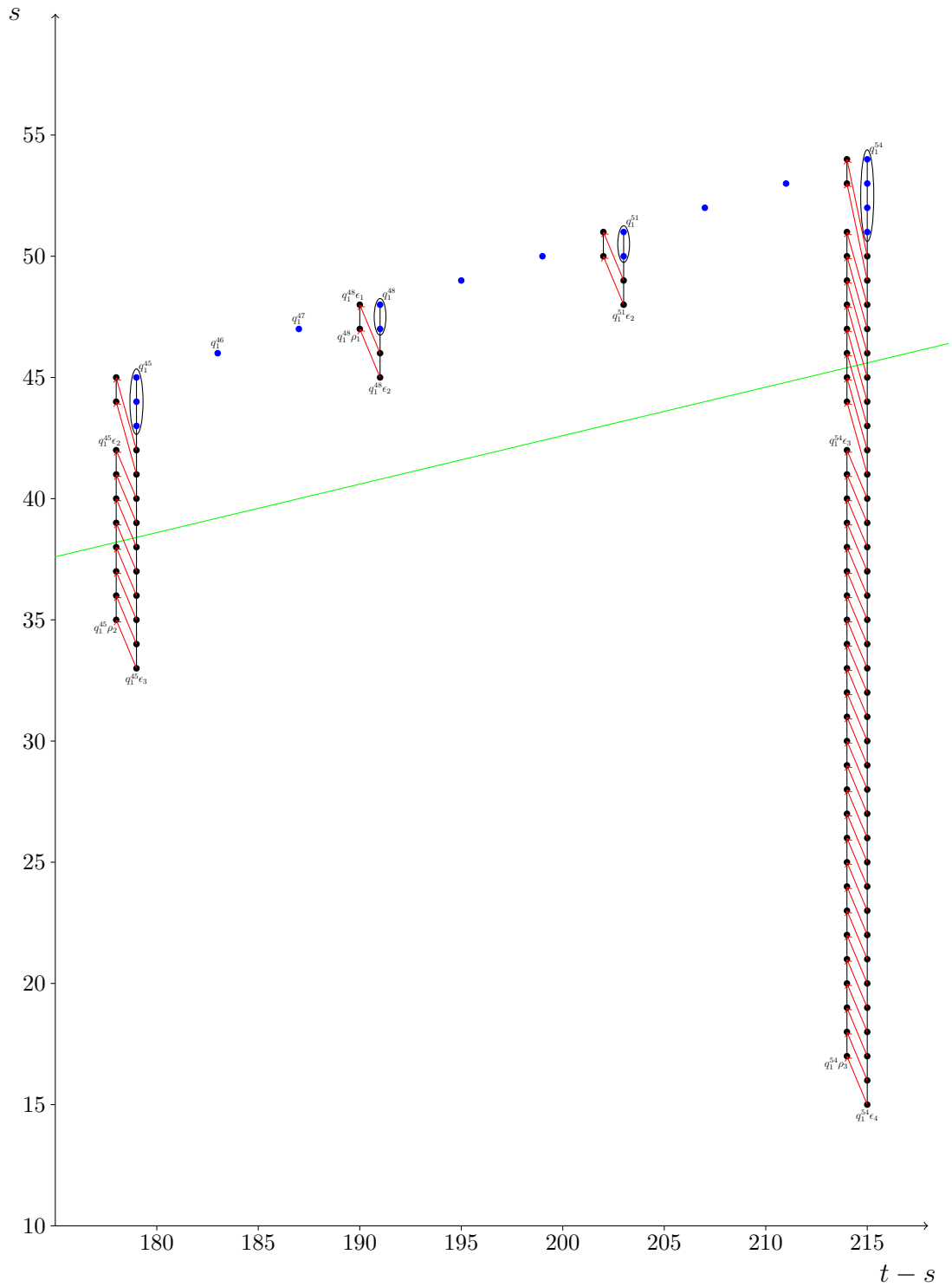


Figure 1-1: The relevant part of $H^{s,t}(A)$ when $p = 3$, in the range $175 < t - s < 218$, with a line of slope $1/(p^2 - p - 1) = 1/5$ drawn. Vertical black lines indicate multiplication by q_0. The top and bottom of selected q_0-towers are labelled by the source and target, respectively, of the corresponding Bockstein differential. Red arrows indicate Adams differentials up to higher Cartan-Eilenberg filtration.

principal tower and its side towers. Some permanent cycles are left at the top of each principal tower. They detect v_1-periodic elements in the given dimension.

Almost all of what we have described about figure 1-1 is true in general.

In each positive dimension D which is one less than a multiple of $2p-2$ there is a "principal tower." As long as $N = (D+1)/(2p-2)$ is not a power of p, the principal tower maps under the zig-zag (1.4.5) to the q_0-tower corresponding to the Bockstein differential on q_1^N. If $N = (D+1)/(2p-2)$ is a power of p, so that $D = p^n(2p-2)-1$ where $n \geq 0$, the principal tower has length p^n and it starts on the 1-line at $h_{1,n}$. This is a statement about the existence of chromatic differentials: for $n \geq 1$, there are chromatic differentials on the q_0-tower corresponding to the Bockstein differential on $q_1^{p^n}$.

In each positive dimension D which is two less than a multiple of $p(2p-2)$ there are "side towers." If p^n is the highest power of p dividing $N = (D+2)/(2p-2)$, then there are n side towers. In most cases, the j^{th} side tower (we order from higher Adams filtration to lower Adams filtration) maps under the zig-zag (1.4.5) to the q_0-tower corresponding to the Bockstein differential on $q_1^N \epsilon_j$. However, if $N = (D+2)/(2p-2)$ is a power of p so that $D = p^n(2p-2)-2$ where $n \geq 1$, the n^{th} side tower has length $p^n - p^{[n]}$ and it starts on the 2-line at $b_{1,n-1}$; for $n \geq 2$, there are chromatic differentials on the q_0-tower corresponding to the Bockstein differential on $q_1^{p^n}\epsilon_n$.

To make the assertions above we have to calculate some differentials in a Bockstein spectral sequence for $H^*(P;Q)$. We omit stating the relevant result here.

We have not described all the elements in $H^*(P; q_1^{-1}Q/q_0^\infty)$. The remaining elements line up in a convenient way but to be more precise we must talk about the *localized algebraic Novikov spectral sequence (loc.alg.NSS)*

$$H^*(P; q_1^{-1}Q/q_0^\infty) \implies H^*(BP_*BP; v_1^{-1}BP_*/p^\infty).$$

This is also important if we are to address the Adams differentials between principal towers and their side towers.

Theorem 1.4.4 allows us to understand the associated graded of the E_2-page of the

loc.alg.NSS with respect to the Bockstein filtration. Since the Bockstein filtration is respected by $d_2^{\text{loc.alg.NSS}} : H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t) \longrightarrow H^{s+1,u}(P; [q_1^{-1}Q/q_0^\infty]^{t+1})$ we have a *filtration spectral sequence (q_0-FILT)*

$$E_0(q_0\text{-FILT}) = E_\infty(q_1^{-1}\text{-BSS}) \implies E_3(\text{loc.alg.NSS}).$$

Theorem 1.4.4 enables us to write down some obvious permanent cycles in the q_1^{-1}-BSS. The next theorem tells us that they are the only elements which appear on the E_1-page of the q_0-FILT.

Theorem 1.4.6. *$E_1(q_0$-FILT$)$ has an $\mathbb{F}_p$-basis given by the following elements.*

$$\left\{q_0^v : v < 0\right\} \cup \left\{q_0^v q_1^{kp^{n-1}} : \ n \geq 1,\ k \in \mathbb{Z} - p\mathbb{Z},\ -p^{[n]} \leq v < 0\right\}$$
$$\cup \left\{q_0^v q_1^{kp^n} \epsilon_n : \ n \geq 1,\ k \in \mathbb{Z},\ 1 - p^n \leq v < 0\right\}$$

This theorem tells us that the d_2 differentials in the loc.alg.NSS which do not increase Bockstein filtration kill all the q_0-towers except those corresponding to the differentials of theorem 1.4.4. This is precisely what we meant when we said that "the remaining elements line up in a convenient way." Once theorem 1.4.6 is proved, the calculation of the remainder of the loc.alg.NSS is straightforward because one knows $H^*(BP_*BP; v_1^{-1}BP_*/p^\infty)$ by [12, §4].

We now turn to the Adams differentials between principal towers and their side towers, which is the motivation for drawing figure 1-2. In [11], Miller uses the square analogous to (1.2.1) for the mod p Moore spectrum to deduce Adams differentials (up to higher Cartan-Eilenberg filtration) from algebraic Novikov differentials. The algebraic Novikov spectral sequence he calculates is precisely the one labelled as the v_1-alg.NSS in figure 1-2 and this is the key input to proving theorem 1.4.6. We can use the same techniques to deduce Adams differentials for the sphere from differentials in the alg.NSS. We make this statement precise (see also, [2, §8]).

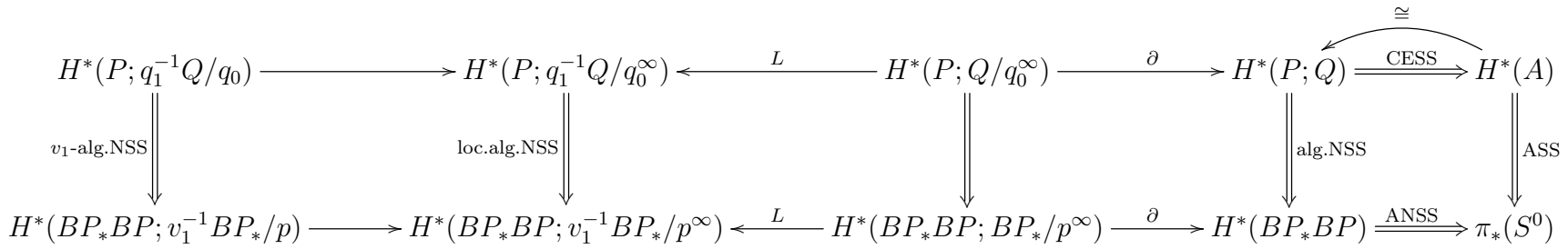


Figure 1-2: *Obtaining information about the Adams spectral sequence from the Miller's v_1-algebraic Novikov spectral sequence.* Having calculated the q_1^{-1}-BSS, Miller's calculation of the v_1-alg.NSS allows us to calculate the loc.alg.NSS. Above a line of slope $1/(p^2-p-1)$ the E_2-page of the loc.alg.NSS is isomorphic to the E_2-page of the alg.NSS. Thus, our localized algebraic Novikov differentials allow us to deduce unlocalized ones, which can, in turn, be used to deduce Adams d_2 differentials up to higher Cartan-Eilenberg filtration.

Theorem 1.4.7 (Miller, [11, 6.1])**.** *Suppose $x \in H^{s,u}(P;Q^t)$. Use the identification $H^*(A) = H^*(P;Q)$ to view x as lying in $H^{s+t,u+t}(A)$. Then we have*

$$d_2^{ASS}x \in \bigoplus_{i\geq 0}^{t+1} H^{s+i+1,u+i}(P;Q^{t-i+1}) \subset H^{s+t+2,u+t+1}(A),$$

where the zero-th coordinate is $d_2^{alg.NSS}x \in H^{s+1,u}(P;Q^{t+1})$.

Moreover, the map $\partial : H^*(P;Q/q_0^\infty) \longrightarrow H^*(P;Q)$ is an isomorphism away from low topological degrees, since $H^*(P;q_0^{-1}Q) = \mathbb{F}_p[q_0^{\pm 1}]$ and we have the following result concerning the localization map L.

Proposition 1.4.8. *The localization map*

$$L : H^{s,u}(P;[Q/q_0^\infty]^t) \longrightarrow H^{s,u}(P;[q_1^{-1}Q/q_0^\infty]^t)$$

is an isomorphism if $(u+t) < p(p-1)(s+t) - 2$. In particular, the localization map is an isomorphism above a line of slope $1/(p^2-p-1)$ when we plot elements in the $(u-s, s+t)$-plane, the plane that corresponds to the usual way of drawing the Adams spectral sequence.

The upshot of all of this is that as long as we are above a particular line of slope $1/(p^2-p-1)$, the d_2 differentials in the loc.alg.NSS can be transferred to d_2 differentials in the unlocalized spectral sequence (the alg.NSS), and using theorem 1.4.7 we obtain d_2 differentials in the Adams spectral sequence. In fact, we can do even better. Proposition 1.4.8 states the isomorphism range which one proves when one chooses to use the bigrading $(\sigma, \lambda) = (s+t, u+t)$. We can also prove a version which makes full use of the trigrading (s,t,u) and this allows one to obtain more information. In particular, it allows one to show that the bottom of a principal tower in the Adams spectral sequence always supports d_2 differentials which map to the last side tower.

To complete the story we discuss the higher Adams differentials between principal towers and their side towers. Looking at figure 1-1 one would hope to prove that if a

principal tower has n side towers, then the j^{th} side tower is the target for nontrivial d_{n-j+2} differentials. We have just addressed the case when $j = n$ and one finds that in the loc.alg.NSS everything goes as expected. The issue is that theorem 1.4.7 does not exist for higher differentials. For instance, $d_2^{\text{alg.NSS}}x = 0$, simply says that $d_2^{\text{ASS}}x$ has higher Cartan-Eilenberg filtration. In this case $d_3^{\text{alg.NSS}}x$ lives in the wrong trigrading to give any more information about $d_2^{\text{ASS}}x$. Instead, we set up and calculate a spectral sequence which converges to the homotopy of the *v_1-periodic sphere spectrum*

$$v_1^{-1}S/p^\infty = \text{hocolim}(\ldots \longrightarrow (v_1^{p^{n-1}})^{-1}S/p^n \xrightarrow{p} (v_1^{p^n})^{-1}S/p^{n+1} \longrightarrow \ldots).$$

This is the *localized Adams spectral sequence for the v_1-periodic sphere (LASS-∞)*

$$H^*(P; q_1^{-1}Q/q_0^\infty) \implies \pi_*(v_1^{-1}S/p^\infty).$$

This spectral sequence behaves as one would like with respect to differentials between principal towers and their side towers (i.e. in the same way as the loc.alg.NSS) and moreover, the zig-zag of (1.4.5) consists of maps of spectral sequences, which enables a comparison with the Adams spectral sequence. It is this calculation that allows us to describe differentials of arbitrarily long length in the ASS. They come from differentials between primary towers and side towers. We find such differentials in the LASS-∞, sufficiently far above the line of slope $1/(p^2-p-1)$, and transfer them across to the ASS.

In order to set up the LASS-∞ we prove an odd primary analog of a result of Davis and Mahowald, which appears in [6]. This is of interest in its own right and we state it below.

In [1] Adams shows that there is a CW spectrum B with one cell in each positive dimension congruent to 0 or -1 modulo $q = 2p-2$ such that $B \simeq (\Sigma^\infty B\Sigma_p)_{(p)}$. Denote the skeletal filtration by a superscript in square brackets. We use the following notation.

Notation 1.4.9. For $1 \leq n \leq m$ let $B_n^m = B^{[mq]}/B^{[(n-1)q]}$.

The following theorem allows a very particular construction of a v_1 self-map for S/p^{n+1}.

Theorem 1.4.10. *The element $q_0^{p^n-n-1}h_{1,n} \in H^{p^n-n,p^n(q+1)-n-1}(A)$ is a permanent cycle in the Adams spectral sequence represented by a map*

$$\alpha : S^{p^nq-1} \xrightarrow{i} B^{p^n}_{p^n-n} \xrightarrow{f} B^{p^n-1}_{p^n-n-1} \longrightarrow \ldots \longrightarrow B^{n+2}_2 \xrightarrow{f} B^{n+1}_1 \xrightarrow{t} S^0.$$

Here, i comes from the fact that the top cell of $B^{[p^nq-1]}/B^{[(p^n-n-1)q-1]}$ splits off, t is obtained from the transfer map $B_1^\infty \longrightarrow S^0$, and each f is got by factoring a multiplication-by-p map.

Moreover, there is an element $\tilde{\alpha} \in \pi_{p^nq}(S/p^{n+1})$ whose image in $BP_{p^nq}(S/p)$ is $v_1^{p^n}$, and whose desuspension maps to α under

$$\pi_{p^nq-1}(\Sigma^{-1}S/p^{n+1}) \longrightarrow \pi_{p^nq-1}(S^0).$$

1.5 Outline of thesis

Chapter 2 is an expository chapter on spectral sequences. A correspondence approach is presented, terminology is defined, and we say what it means for a spectral sequence to converge. In chapter 3 we introduce all the Bockstein spectral sequences that we use and prove their important properties, namely, that differentials in the Q-BSS and the q_0^∞-BSS coincide, and that the differentials in the q_1^{-1}-BSS are derivations.

Chapter 4 contains our first important result. After finding some vanishing lines we examine the range in which the localization map $H^*(P; Q/q_0^\infty) \to H^*(P; q_1^{-1}Q/q_0^\infty)$ is an isomorphism. We do this from a trigraded and a bigraded perspective.

Chapter 5 contains our main results. We calculate the q_1^{-1}-BSS and find some differentials in the Q-BSS. We address the family of differentials corresponding to the principal towers using an explicit argument with cocycles. The family of differentials corresponding to the side towers is obtained using a Kudo transgression theorem. A combinatorial argument gives the E_∞-page of the q_1^{-1}-BSS.

Chapter 6 contains the calculation of the localized algebraic Novikov spectral sequence. The key ingredients for the calculation are the combinatorics used to describe the E_∞-page of the q_1^{-1}-BSS and Miller's calculation of the v_1-algebraic Novikov spectral sequence.

In chapter 7 we construct representatives for some permanent cycles in the Adams spectral sequence using the geometry of stunted projective spaces and the transfer map.

In chapter 8 we set up the localized Adams spectral sequence for the v_1-periodic sphere (LASS-∞), calculate it, and demonstrate the consequences the calculation has for the Adams spectral sequence for the sphere. Along the way we construct a *modified Adams spectral sequence for the mod p^n Moore spectrum* and the *Prüfer sphere*. We lift the permanent cycles of the previous chapter to permanent cycles in these spectral sequences and we complete the proof of the last theorem stated in the introduction.

In the appendices we construct various maps of spectral sequences and check the convergence of our spectral sequences.

Chapter 2

Spectral sequence terminology

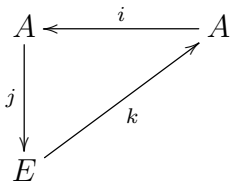
Spectral sequences are used in abundance throughout this thesis. Graduate students in topology often live in fear of spectral sequences and so we take this opportunity to give a presentation of spectral sequences, which, we hope, shows that they are not all that bad. We also fix the terminology which is used throughout the rest of the thesis.

All of this chapter is expository. Everything we say is surely documented in [3].

2.1 A correspondence approach

The reader is probably familiar with the notion of an exact couple which is one of the most common ways in which a spectral sequence arises.

Definition 2.1.1. An *exact couple* consists of abelian groups A and E together with homomorphisms i, j and k such that the following triangle is exact.

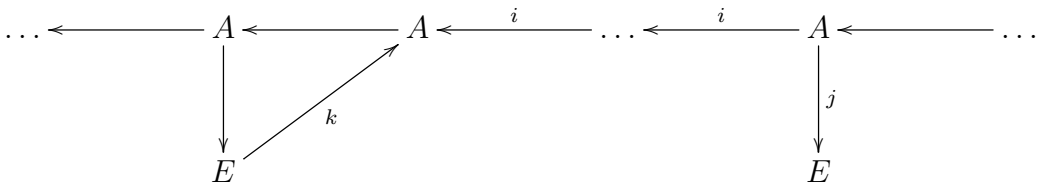


Given an exact couple, one can form the associated *derived exact couple*. Iterating this process gives rise to a spectral sequence. Experience has led the author to conclude that, although this inductive definition is slick, it disguises some of the

important features that spectral sequences have and which are familiar to those who work with them on a daily basis. Various properties become buried in the induction and the author feels that first time users should not have to struggle for long periods of time to discover these properties however rewarding that process might be.

An alternative approach exploits correspondences. A *correspondence* $f : G_1 \longrightarrow G_2$ is a subgroup $f \subset G_1 \times G_2$. The images of f under the projection maps are the *domain* $\mathrm{dom}(f)$ and the *image* $\mathrm{im}(f)$ of the correspondence. We can also define the *kernel* of a correspondence $\ker(f) \subset \mathrm{dom}(f)$.

We will find that the picture becomes clearer, especially once gradings are introduced, when we spread out the exact couple:

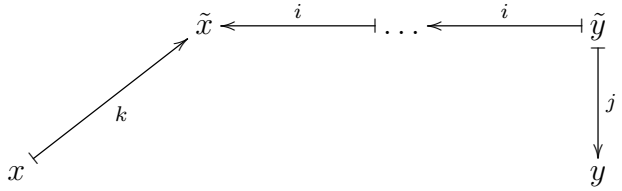


Let $\pi : E \times A \times A \times E \longrightarrow E \times E$ be the projection map. Then we make the following definitions.

Definition 2.1.2. For each $r \geq 1$ let

$$\widetilde{d_r} = \{(x, \tilde{x}, \tilde{y}, y) \in E \times A \times A \times E : kx = \tilde{x} = i^{r-1}\tilde{y} \text{ and } j\tilde{y} = y\}$$

and $d_r = \pi(\widetilde{d_r})$. Let $d_0 = E \times 0 \subset E \times E$.



Since i, j, k and π are homomorphisms of abelian groups $\widetilde{d_r}$ and d_r are subgroups of $E \times A \times A \times E$ and $E \times E$, respectively. In particular, $d_r : E \longrightarrow E$ is a correspondence. We note that d_0 is the zero homomorphism and that $d_1 = jk$.

Notation 2.1.3. We write $d_r x = y$ if $(x, y) \in d_r$.

We have the following useful observations.

Lemma 2.1.4.

1. *For $r \geq 1$, $d_r x$ is defined if and only if $d_{r-1} x = 0$, i.e.*

$$(x, 0) \in d_{r-1} \iff \exists y : (x, y) \in d_r.$$

2. *For $r \geq 1$, $d_r 0 = y$ if and only if there exists an x with $d_{r-1} x = y$, i.e.*

$$(0, y) \in d_r \iff \exists x : (x, y) \in d_{r-1}.$$

We note that the first part of the lemma says that $\mathrm{dom}(d_r) = \ker(d_{r-1})$ for $r \geq 1$. The second part of the lemma has the following corollary.

Corollary 2.1.5. *For $r \geq 1$, the following conditions are equivalent:*

1. *$d_r x = y$ and $d_r x = y'$;*

2. *$d_r x = y$ and there exists an x' with $d_{r-1} x' = y' - y$.*

It is also immediate from the definitions that the following lemma holds.

Lemma 2.1.6. *Suppose $r \geq 1$ and that $d_r x = y$. Then $d_s y = 0$ for any $s \geq 1$.*

Spectral sequences consist of pages.

Definition 2.1.7. For $r \geq 1$, let $E_r = \ker d_{r-1} / \operatorname{im} d_{r-1}$. This is the r^{th} *page* of the spectral sequence.

One is often taught that a spectral sequence begins with an E_1 or E_2-page and that one obtains successive pages by calculating differentials and taking homology. We relate our correspondence approach to this one presently.

We have a surjection $E_r \longrightarrow \ker d_{r-1}/\bigcup_s \operatorname{im} d_s$, an injection $\bigcap_s \ker d_s/\operatorname{im} d_{r-1} \longrightarrow E_r$, and the preceding lemmas show that d_r defines a homomorphism allowing us to form the following composite which, for now, we call δ_r.

$$E_r \longrightarrow \ker d_{r-1}/\bigcup_s \operatorname{im} d_s \longrightarrow \bigcap_s \ker d_s/\operatorname{im} d_{r-1} \longrightarrow E_r.$$

We have an identification of the E_{r+1}-page as the homology of the E_r-page with respect to the *differential* δ_r. We will blur the distinction between the correspondence d_r and the differential δ_r, calling them both d_r.

We note that the E_1 page is E. Our Bockstein spectral sequences have convenient descriptions from the E_1-page and so we use the correspondence approach. Consequently, all our differentials will be written in terms of elements on the E_1-page. Our topological spectral sequences have better descriptions from the E_2-page. The correspondence approach also allows us to write all our formulae in terms of elements of the E_2-pages.

Here is some terminology that we will use freely throughout this thesis.

Definition 2.1.8. Suppose $d_r x = y$. Then x is said to *survive* to the E_r-page and *support* a d_r differential. y is said to be the *target* of a d_r differential, to be *hit* by a d_r differential, and to be a *boundary*. If, in addition, $y \notin \operatorname{im} d_{r-1}$, then the differential is said to be *nontrivial* and x is said to *support a nontrivial differential*.

Definition 2.1.9. Elements of $\bigcap_s \ker d_s$ are called *permanent cycles*.

We write E_∞ for $\bigcap_s \ker d_s/\bigcup_s \operatorname{im} d_s$, permanent cycles modulo boundaries, the E_∞-page of the spectral sequence.

Note that lemma 2.1.6 says that targets of differentials are permanent cycles or, said another way, elements that are hit by a differential survive to all pages of the spectral sequence. In particular, note that we use the word *hit*, not *kill*.

2.2 Convergence

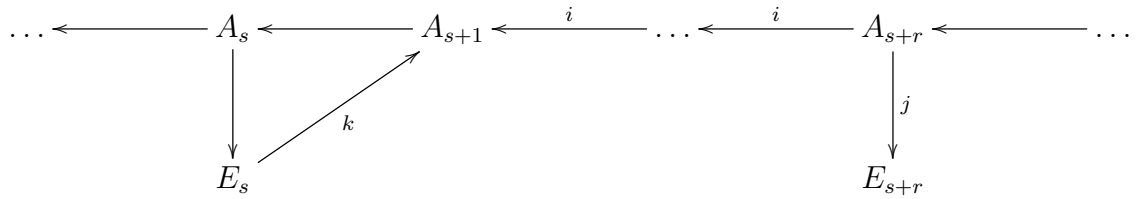
The purpose of a spectral sequence is to give a procedure to calculate an abelian group of interest M. This procedure can be viewed as having three steps, which we outline below, but first, we give some terminology.

Definition 2.2.1. A *filtration* of an abelian group M is a sequence of subgroups

$$M \supset \ldots \supset F^{s-1}M \supset F^sM \supset F^{s+1}M \supset \ldots \supset 0, \quad s \in \mathbb{Z}.$$

The *associated graded* abelian group corresponding to this filtration is the graded abelian group $\bigoplus_{s\in\mathbb{Z}} F^sM/F^{s+1}M$.

The E_∞-page of a spectral sequence should tell us about the associated graded of an abelian group M we are trying to calculate. In particular, the E_∞-page should be $\mathbb{Z}$-graded, so we consider the story described in the previous section, with the added assumption that A and E have a $\mathbb{Z}$-grading s, that $i : A_{s+1} \longrightarrow A_s$, $j : A_s \longrightarrow E_s$ and $k : E_s \longrightarrow A_{s+1}$. We can redraw the exact couple as follows.



We see that d_r has degree r and so E_∞ becomes $\mathbb{Z}$-graded. In all the cases we consider in the thesis the abelian group M we are trying to calculate will be either the limit or colimit of the directed system $\{A_s\}_{s\in\mathbb{Z}}$.

We now describe the way in which a spectral sequence can be used to calculate an abelian group M.

1. Define a filtration of M and an identification $E^s_\infty = F^sM/F^{s+1}M$ between the E_∞-page of the spectral sequence and the associated graded of M.

2. Resolve *extension problems*. Depending on circumstances this will give us either F^sM for each s or M/F^sM for each s.

3. Recover M. Depending on circumstances this will either be via an isomorphism $M \longrightarrow \lim_s M/F^sM$ or an isomorphism $\mathrm{colim}_s F^sM \longrightarrow M$.

In all the cases we consider, M will be graded and the filtration will respect this grading. Thus, the associated graded will be bigraded. Correspondingly, the exact couple will be bigraded. There are three cases which arise for us. We highlight how each affects the procedure above.

1. Each case is determined by the way in which the filtration behaves.

   (a) $F^0M = M$ and $\bigcap F^sM = 0$.

   (b) $F^0M = 0$ and $\bigcup F^sM = M$.

   (c) $\bigcup F^sM = M$ and keeping track of the additional gradings the identification in the first part of the procedure becomes $E_\infty^{s,t} = F^sM_{t-s}/F^{s+1}M_{t-s}$; moreover, for each u there exists an s such that $F^sM_u = 0$.

2. The way in which we would go about resolving extension problems varies according to which case we are in.

   (a) $M/F^0M = 0$ so suppose that we know M/F^sM where $s \geq 0$. The first part of the procedure gives us $F^sM/F^{s+1}M$ and so resolving an extension problem gives $M/F^{s+1}M$. By induction, we know M/F^sM for all s.

   (b) $F^0M = 0$ so suppose that we know $F^{s+1}M$ where $s < 0$. The first part of the procedure gives us $F^sM/F^{s+1}M$ and so resolving an extension problem gives F^sM. By induction, we know F^sM for all s.

   (c) This is similar to (2b). Fixing u, there exists an s_0 with $F^{s_0}M_u = 0$. Suppose that we know $F^{s+1}M_u$ where $s < s_0$. The first part of the procedure gives us $F^sM_u/F^{s+1}M_u$ and so resolving an extension problem gives F^sM_u. By induction, we know F^sM_u for all s. We can now vary u.

3. In case (a) we need an isomorphism $M \longrightarrow \lim_s M/F^sM$. In cases (b) and (c) we have an isomorphism $\mathrm{colim}_s F^sM \longrightarrow M$.

When we say that our spectral sequences converge we ignore whether or not we can resolve the extension problems. This is paralleled by the fact that, when making such a statement, we ignore whether or not it is possible to calculate the differentials in the spectral sequence. The point is, that theoretically, both of these issues can be overcome even if it is extremely difficult to do so in practice. We conclude that the important statements for convergence are given in stages (1) and (3) of our procedure and we make the requisite definition.

Definition 2.2.2. Suppose given a graded abelian group M and a spectral sequence E^*_*. Suppose that M is filtered, that we have an identification $E^s_\infty = F^sM/F^{s+1}M$ and that one of the following conditions holds.

1. $F^0M = M$, $\bigcap F^sM = 0$ and the natural map $M \to \lim_s M/F^sM$ is an isomorphism.
2. $F^0M = 0$ and $\bigcup F^sM = M$.
3. $\bigcup F^sM = M$, if we keep track of the additional gradings then we have $E^{s,t}_\infty = F^sM_{t-s}/F^{s+1}M_{t-s}$, and for each u there exists an s such that $F^sM_u = 0$.

Then the spectral sequence is said to *converge* and we write $E^s_1 \overset{s}{\Longrightarrow} M$ or $E^s_2 \overset{s}{\Longrightarrow} M$ depending on which page of the spectral sequence has the more concise description.

It would appear that the notation $E^s_1 \overset{s}{\Longrightarrow} M$ is over the top since s appears twice, but once other gradings are recorded it is the s above the " $\Longrightarrow$ " that indicates the filtration degree.

Suppose that $E^s_1 \overset{s}{\Longrightarrow} M$ (or equivalently $E^s_2 \overset{s}{\Longrightarrow} M$). We have some terminology to describe the relationship between permanent cycles and elements of M.

Definition 2.2.3. Suppose that x is a permanent cycle defined in E^s_r (usually $r = 1$ or $r = 2$) and $z \in F^sM$. Then we say that x *detects* z or that z *represents* x, to mean that the image of x in E^s_∞ and the image of z in $F^sM/F^{s+1}M$ correspond under the given identification $E^s_\infty = F^sM/F^{s+1}M$.

Suppose that $x \in E^s_r$ detects $z \in F^sM$. Notice that x is a boundary if and only if $z \in F^{s+1}M$.

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Chapter 3

Bockstein spectral sequences

In this chapter, we set up all the Bockstein spectral sequences used in this thesis and prove the properties that we require of them.

3.1 The Hopf algebra P and some P-comodules

Throughout this thesis p is an odd prime.

Definition 3.1.1. Let P denote the polynomial algebra over $\mathbb{F}_p$ on the Milnor generators $\{\xi_n : n \geq 1\}$ where $|\xi_n| = 2p^n - 2$. P is a Hopf algebra when equipped with the Milnor diagonal

$$P \longrightarrow P \otimes P, \quad \xi_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i, \quad (\xi_0 = 1).$$

Definition 3.1.2. Let Q denote the polynomial algebra over $\mathbb{F}_p$ on the generators $\{q_n : n \geq 0\}$ where $|q_n| = 2p^n - 2$. Q is an algebra in P-comodules when equipped with the coaction map

$$Q \longrightarrow P \otimes Q, \quad q_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes q_i.$$

We write Q^t for the sub-P-comodule consisting of monomials of length t.

Note that the multiplication on Q is commutative, which is the same as graded

commutative since everything lives in even degrees. We shall see later that t is the *Novikov weight*. Miller [10] also refers to t as the *Cartan degree*.

q_0 is a P-comodule primitive and so Q/q_0 and $q_0^{-1}Q$ are P-comodules.

Definition 3.1.3. Define Q/q_0^∞ by the following short exact sequence of P-comodules.

$$0 \longrightarrow Q \longrightarrow q_0^{-1}Q \longrightarrow Q/q_0^\infty \longrightarrow 0$$

Q/q_0^∞ is a Q-module in P-comodules.

We find that $q_1 \in Q/q_0$ is a comodule primitive so we may define $q_1^{-1}Q/q_0$ which is an algebra in P-comodules. We may also define $q_1^{-1}Q/q_0^\infty$, a Q-module in P-comodules, but this requires a more sophisticated construction, which we now outline.

Definition 3.1.4. For $n \geq 1$, M_n is the sub-P-comodule of Q/q_0^∞ defined by the following short exact sequence of P-comodules. M_n is a Q-module in P-comodules.

$$0 \longrightarrow Q \longrightarrow Q\langle q_0^{-n}\rangle \longrightarrow M_n \longrightarrow 0.$$

Lemma 3.1.5. *$q_1^{p^{n-1}} : M_n \longrightarrow M_n$ is a homomorphism of Q-modules in P-comodules.*

Definition 3.1.6. For each $k \geq 0$ let $M_n(k) = M_n$. $q_1^{-1}M_n$ is defined to be the colimit of the following diagram.

$$M_n(0) \xrightarrow{q_1^{p^{n-1}}} M_n(1) \xrightarrow{q_1^{p^{n-1}}} M_n(2) \xrightarrow{q_1^{p^{n-1}}} M_n(3) \xrightarrow{q_1^{p^{n-1}}} \dots$$

Definition 3.1.7. We have homomorphisms $q_1^{-1}M_n \longrightarrow q_1^{-1}M_{n+1}$ induced by the inclusions $M_n \longrightarrow M_{n+1}$. $q_1^{-1}Q/q_0^\infty$ is defined to be the colimit of following diagram.

$$q_1^{-1}M_1 \longrightarrow q_1^{-1}M_2 \longrightarrow q_1^{-1}M_3 \longrightarrow q_1^{-1}M_4 \longrightarrow \dots$$

Notation 3.1.8. If $\mathfrak{Q}$ is a P-comodule then we write $\Omega^*(P;\mathfrak{Q})$ for the *cobar construction* on P with coefficients in $\mathfrak{Q}$. In particular, we have

$$\Omega^s(P;\mathfrak{Q}) = \overline{P}^{\otimes s} \otimes \mathfrak{Q}$$

where $P = \mathbb{F}_p \oplus \overline{P}$ as $\mathbb{F}_p$-modules. We write $[p_1|\dots|p_s]q$ for $p_1 \otimes \dots \otimes p_s \otimes q$. We set $\Omega^* P = \Omega^*(P; \mathbb{F}_p)$.

We recall (see [10, pg. 75]) that the differentials are given by an alternating sum making use of the diagonal and coaction maps. We also recall that if $\mathfrak{Q}$ is an algebra in P-comodules then $\Omega^*(P; \mathfrak{Q})$ is a DG-$\mathbb{F}_p$-algebra; if $\mathfrak{Q}'$ is a $\mathfrak{Q}$-module in P-comodules then $\Omega^*(P; \mathfrak{Q}')$ is a DG-$\Omega^*(P; \mathfrak{Q})$-module.

Definition 3.1.9. If $\mathfrak{Q}$ is a P-comodule then $H^*(P; \mathfrak{Q})$ is the cohomology of $\Omega^*(P; \mathfrak{Q})$.

We remark that in our setting $H^*(P; \mathfrak{Q})$ will always have three gradings. There is the cohomological grading s. P and its comodules are graded and so we have an internal degree u. The Novikov weight t on Q persists to Q/q_0, $q_0^{-1}Q$, Q/q_0^∞, $q_1^{-1}Q/q_0$, and $q_1^{-1}Q/q_0^\infty$.

Later on, we will use an algebraic Novikov spectral sequence. From this point of view, right P-comodules are more natural (see [2], for instance). However, Miller's paper [10] is such a strong source of guidance for this work that we choose to use left P-comodules as he does there.

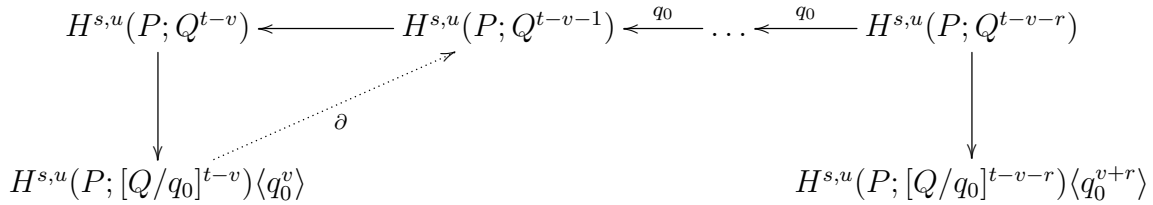
3.2 The Q-Bockstein spectral sequence (Q-BSS)

Applying $H^*(P; -)$ to the short exact sequence of P-comodules

$$0 \longrightarrow Q \xrightarrow{q_0} Q \longrightarrow Q/q_0 \longrightarrow 0$$

gives a long exact sequence. We also have a trivial long exact sequence consisting of the zero group every three terms and $H^*(P; Q)$ elsewhere. Intertwining these long exact sequences gives an exact couple, the nontrivial part, of which, looks as follows

$(v \geq 0)$.



Here ∂ raises the degree of s by one relative to what is indicated and the powers of q_0 are used to distinguish copies of $H^*(P;Q/q_0)$ from one another.

Definition 3.2.1. The spectral sequence arising from this exact couple is called the *Q-Bockstein spectral sequence (Q-BSS)*. It has E_1-page given by

$$E_1^{s,t,u,v}(Q\text{-BSS}) = \begin{cases} H^{s,u}(P;[Q/q_0]^{t-v})\langle q_0^v\rangle & \text{if } v \geq 0 \\ 0 & \text{if } v < 0 \end{cases}$$

and d_r has degree $(1,0,0,r)$.

The spectral sequence converges to $H^*(P;Q)$ and the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{s,t,u,v}(Q\text{-BSS}) = F^v H^{s,u}(P;Q^t)/F^{v+1}H^{s,u}(P;Q^t)$$

where $F^vH^*(P;Q) = \mathrm{im}(q_0^v : H^*(P;Q) \longrightarrow H^*(P;Q))$ for $v \geq 0$. The identification is given by lifting an element of $F^vH^*(P;Q)$ to the v^{th} copy of $H^*(P;Q)$ and mapping this lift down to $H^*(P;Q/q_0)\langle q_0^v\rangle$ to give a permanent cycle.

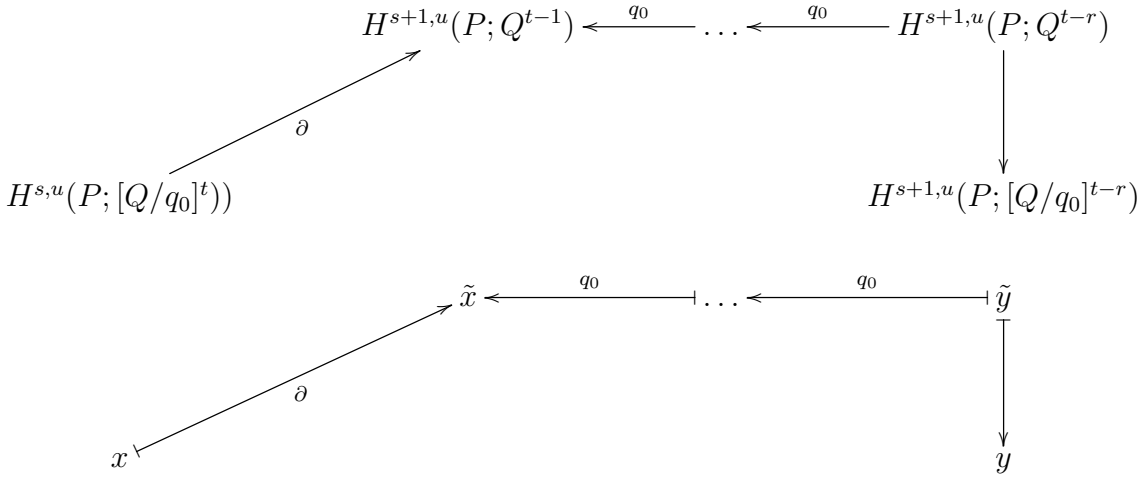
Remark 3.2.2. One can describe the E_1-page of the Q-BSS more concisely as the algebra $H^*(P;Q/q_0)[q_0]$. The first three gradings (s,t,u) are obtained from the gradings on the elements of $H^*(P;Q/q_0)$ and q_0; the adjoined polynomial generator q_0 has v-grading 1, whereas elements of $H^*(P;Q/q_0)$ have v-grading 0.

Notation 3.2.3. Suppose $x, y \in H^*(P;Q/q_0)$. We write $d_rx = y$ to mean that for every $v \geq 0$, q_0^vx survives to the E_r-page, q_0^vy is a permanent cycle, and $d_rq_0^vx =$

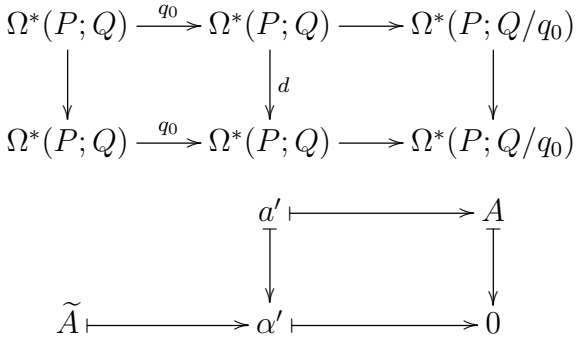
$q_0^{v+r}y$. In this case, x is said to *support* a d_r differential. If one of the differentials $d_r q_0^v x = q_0^{v+r}y$ is nontrivial, then x is said to *support a nontrivial differential.*

Lemma 3.2.4. *Suppose $x, y \in H^*(P; Q/q_0)$. Then $d_r x = y$ in the Q-BSS if and only if there exist a and b in $\Omega^*(P;Q)$ with $da = q_0^r b$ such that their images in $\Omega^*(P;Q/q_0)$ are cocycles representing x and y, respectively.*

Proof. Suppose that $d_r x = y$ in the Q-BSS. By definition 2.1.2 there exist $\tilde{x}$ and $\tilde{y}$ fitting into the following diagram.



Let $A \in \Omega^*(P;Q/q_0)$ be a representative for x and $\widetilde{B} \in \Omega^*(P;Q)$ be a representative for $\tilde{y}$. There exists an $\widetilde{A} \in \Omega^*(P;Q)$ representing $\tilde{x}$, and an a' and α' fitting into the following diagram.



Moreover, there exists $\widetilde{C} \in \Omega^*(P;Q)$ such that $\widetilde{A} = q_0^{r-1}\widetilde{B} + d\widetilde{C}$. Let $a = a' - q_0\widetilde{C}$.

We see that a, like a', gives a lift of A, and that

$$da = \alpha' - q_0 d\widetilde{C} = q_0(\widetilde{A} - d\widetilde{C}) = q_0(q_0^{r-1}\widetilde{B}) = q_0^r \widetilde{B}.$$

Taking $b = \widetilde{B}$ completes the "only if" direction.

The "if" direction is clear. □

3.3 The q_0^∞-Bockstein spectral sequence (q_0^∞-BSS)

Applying $H^*(P; -)$ to the short exact sequence of P-comodules

$$0 \longrightarrow Q/q_0 \longrightarrow Q/q_0^\infty \xrightarrow{q_0} Q/q_0^\infty \longrightarrow 0$$

gives a long exact sequence. We also have a trivial long exact sequence consisting of the zero group every three terms and $H^*(P; Q/q_0^\infty)$ elsewhere. Intertwining these long exact sequences gives an exact couple, the nontrivial part, of which, looks as follows ($v < 0$).

$$\begin{array}{ccccccc} H^{s,u}(P;[Q/q_0^\infty]^{t-v+r}) & \longleftarrow & H^{s,u}(P;[Q/q_0^\infty]^{t-v+r-1}) & \xleftarrow{q_0} & \cdots & \xleftarrow{q_0} & H^{s,u}(P;[Q/q_0^\infty]^{t-v}) \\ \downarrow & \nearrow & & & & & \downarrow \partial \\ H^{s,u}(P;[Q/q_0]^{t-v+r})\langle q_0^{v-r}\rangle & & & & & & H^{s,u}(P;[Q/q_0]^{t-v})\langle q_0^v\rangle \end{array}$$

Here ∂ raises the degree of s by one relative to what is indicated and the powers of q_0 are used to distinguish copies of $H^*(P; Q/q_0)$ from one another.

Definition 3.3.1. The spectral sequence arising from this exact couple is called the *q_0^∞-Bockstein spectral sequence (q_0^∞-BSS)*. It has E_1-page given by

$$E_1^{s,t,u,v}(q_0^\infty\text{-BSS}) = \begin{cases} H^{s,u}(P;[Q/q_0]^{t-v})\langle q_0^v\rangle & \text{if } v < 0 \\ 0 & \text{if } v \geq 0 \end{cases}$$

and d_r has degree $(1, 0, 0, r)$. The spectral sequence converges to $H^*(P; Q/q_0^\infty)$ and

the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{s,t,u,v}(q_0^\infty\text{-BSS}) = F^v H^{s,u}(P;[Q/q_0^\infty]^t)/F^{v+1}H^{s,u}(P;[Q/q_0^\infty]^t)$$

where $F^v H^*(P;Q/q_0^\infty) = \ker\left(q_0^{-v} : H^*(P;Q/q_0^\infty) \longrightarrow H^*(P;Q/q_0^\infty)\right)$ for $v \leq 0$. The identification is given by taking a permanent cycle in $H^*(P;Q/q_0)\langle q_0^v\rangle$, mapping it up to $H^*(P;Q/q_0^\infty)$ and pulling this element back to the 0^{th} copy of $H^*(P;Q/q_0^\infty)$.

Remark 3.3.2. One can describe the E_1-page of the q_0^∞-BSS more concisely as the $H^*(P;Q/q_0)[q_0]$-module

$$\left[H^*(P;Q/q_0)\left[q_0\right]\right]/q_0^\infty.$$

Notation 3.3.3. Suppose $x, y \in H^*(P;Q/q_0)$. We write $d_r x = y$ to mean that for all $v \in \mathbb{Z}$, $q_0^v x$ and $q_0^v y$ survive until the E_r-page and that $d_r q_0^v x = q_0^{v+r} y$. In this case, notice that $q_0^v x$ is a permanent cycle for $v \geq -r$ and that $q_0^v y$ is a permanent cycle for all $v \in \mathbb{Z}$.

Again, x is said to *support* a d_r differential. If one of the differentials $d_r q_0^v x = q_0^{v+r} y$ is nontrivial, then x is said to *support a nontrivial differential.*

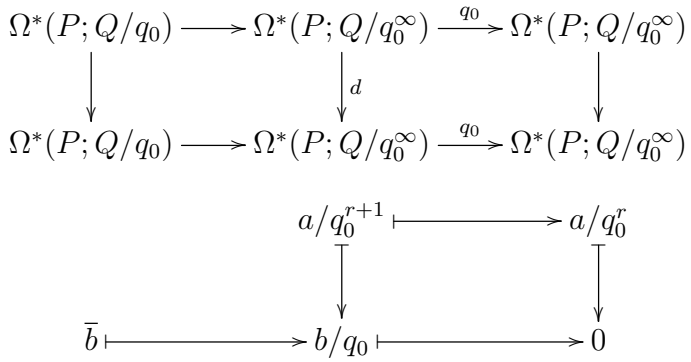
3.4 The Q-BSS and the q_0^∞-BSS: a relationship

Suppose that $x \in H^{s,u}(P;[Q/q_0]^t)$, $y \in H^{s+1,u}(P;[Q/q_0]^{t-r})$. 3.2.3 and 3.3.3 give meanings to the equation $d_r x = y$ in the Q-BSS and the q_0^∞-BSS, respectively. It appears, a priori, that the truth of the equation $d_r x = y$ depends on which spectral sequence we are working in. The following lemma shows that this is not the case.

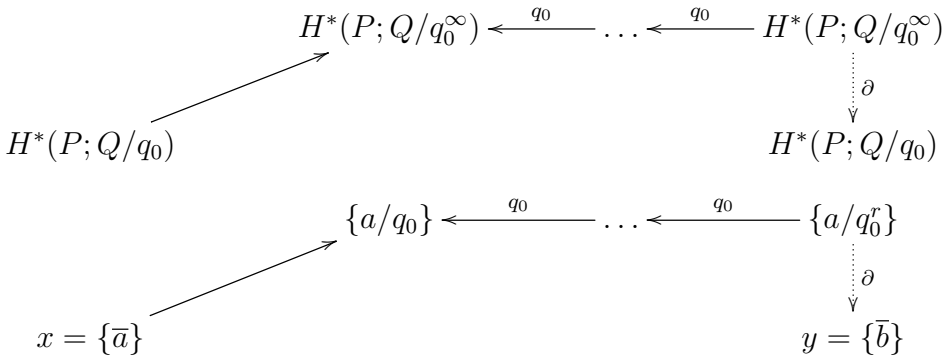
Lemma 3.4.1. *Suppose $x, y \in H^*(P;Q/q_0)$. Then $d_r x = y$ in the Q-BSS if and only if $d_r x = y$ in the q_0^∞-BSS.*

Proof. Suppose that $d_r x = y$ in the Q-BSS. By lemma 3.2.4, we find that there exist a and b in $\Omega^*(P;Q)$ with $da = q_0^r b$ such that their images in $\Omega^*(P;Q/q_0)$ are cocycles representing x and y, respectively. Let $\overline{a}$ and $\overline{b}$ be the images of a and b in $\Omega^*(P;Q/q_0)$, respectively.

Then we have



and so



giving $d_r x = y$ in the q_0^∞-BSS.

We prove the converse using induction on r. The result is clear for $r = 0$ since, by convention, d_0 is zero for both spectral sequences. For $r \geq 1$ we have

$$
\begin{array}{rrcll}
 & d_r x & = & y & \text{in the } q_0^\infty\text{-BSS} \\
\Longrightarrow & d_{r-1} x & = & 0 & \text{in the } q_0^\infty\text{-BSS} \qquad \text{(Lemma 2.1.4)} \\
\Longrightarrow & d_{r-1} x & = & 0 & \text{in the } Q\text{-BSS} \qquad \text{(Induction)} \\
\Longrightarrow & d_r x & = & y' & \text{in the } Q\text{-BSS} \quad \text{for some } y' \qquad \text{(Lemma 2.1.4)} \\
\Longrightarrow & d_r x & = & y' & \text{in the } q_0^\infty\text{-BSS} \qquad (1^{\text{st}} \text{ half of proof}) \\
\Longrightarrow & d_{r-1} x' & = & y' - y & \text{in the } q_0^\infty\text{-BSS} \quad \text{for some } x' \qquad \text{(Corollary 2.1.5)} \\
\Longrightarrow & d_{r-1} x' & = & y' - y & \text{in the } Q\text{-BSS} \qquad \text{(Induction)} \\
\Longrightarrow & d_r x & = & y & \text{in the } Q\text{-BSS} \qquad \text{(Corollary 2.1.5)}
\end{array}
$$

which completes the proof. □

3.5 The q_1^{-1}-Bockstein spectral sequence (q_1^{-1}-BSS)

We can mimic the construction of the q_0^∞-BSS using the following short exact sequence of P-comodules.

$$0 \longrightarrow q_1^{-1}Q/q_0 \longrightarrow q_1^{-1}Q/q_0^\infty \xrightarrow{q_0} q_1^{-1}Q/q_0^\infty \longrightarrow 0 \tag{3.5.1}$$

Definition 3.5.2. The spectral sequence arising from this exact couple is called the *q_1^{-1}-Bockstein spectral sequence (q_1^{-1}-BSS)*. It has E_1-page given by

$$E_1(q_1^{-1}\text{-BSS}) = \left[H^*(P; q_1^{-1}Q/q_0)\left[q_0\right]\right]/q_0^\infty$$

and d_r has degree $(1, 0, 0, r)$. The spectral sequence converges to $H^*(P; q_1^{-1}Q/q_0^\infty)$ and the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{s,t,u,v}(q_1^{-1}\text{-BSS}) = F^v H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t)/F^{v+1}H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t)$$

where, as in the q_0^∞-BSS, $F^v = \ker q_0^{-v}$ for $v \leq 0$. The identification is given by taking a permanent cycle in $H^*(P; q_1^{-1}Q/q_0)\langle q_0^v\rangle$, mapping it up to $H^*(P; q_1^{-1}Q/q_0^\infty)$ and pulling this element back to the 0^{th} copy of $H^*(P; q_1^{-1}Q/q_0^\infty)$.

We follow the notational conventions in 3.3.3.

We notice, that as a consequence of lemma 3.4.1, a d_r-differential in the q_0^∞-BSS can be validated using only elements in $\Omega^*(P; M_{r+1})$ (see definition 3.1.4). The same can be said of the q_1^{-1}-BSS and the proof is similar. The following lemma statement makes use of the connecting homomorphism in the long exact sequence coming from the short exact sequence of P-comodules

$$0 \longrightarrow q_1^{-1}Q/q_0 \longrightarrow q_1^{-1}M_{r+1} \xrightarrow{q_0} q_1^{-1}M_r \longrightarrow 0.$$

Lemma 3.5.3. *Suppose that $x, y \in H^*(q_1^{-1}Q/q_0)$ and that $d_r x = y$ in the q_1^{-1}-BSS. Then there exist $\tilde{x} \in H^*(P; q_1^{-1}M_1)$ and $\tilde{y} \in H^*(P; q_1^{-1}M_r)$ with the properties that $\tilde{x} = q_0^{r-1}\tilde{y}$, and under the maps*

$$H^*(P; q_1^{-1}Q/q_0) \xrightarrow{\cong} H^*(P; q_1^{-1}M_1),\ \partial : H^*(P; q_1^{-1}M_r) \longrightarrow H^*(q_1^{-1}Q/q_0),$$

x is mapped to $\tilde{x}$, and $\tilde{y}$ is mapped to y, respectively. We summarize this situation by saying that $d_r x = y$ in the M_{r+1}-zig-zag.

Proof. The result is clear for $r = 1$ and so we proceed by induction on r. For $r > 1$ we have

$$\begin{array}{rrcll}
 & d_r x & = & y & \text{in the } q_1^{-1}\text{-BSS} \\
\Longrightarrow & d_{r-1}x & = & 0 & \text{in the } q_1^{-1}\text{-BSS} & \text{(Lemma 2.1.4)} \\
\Longrightarrow & d_{r-1}x & = & 0 & \text{in the } M_r\text{-zig-zag} & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y' & \text{in the } M_{r+1}\text{-zig-zag for some } y' & \text{(Lemma 2.1.4)} \\
\Longrightarrow & d_r x & = & y' & \text{in the } q_1^{-1}\text{-BSS} \\
\Longrightarrow & d_{r-1}x' & = & y' - y & \text{in the } q_1^{-1}\text{-BSS for some } x' & \text{(Corollary 2.1.5)} \\
\Longrightarrow & d_{r-1}x' & = & y' - y & \text{in the } M_r\text{-zig-zag} & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y & \text{in the } M_{r+1}\text{-zig-zag} & \text{(Corollary 2.1.5)}
\end{array}$$

which completes the proof. □

We note the following simple result.

Lemma 3.5.4. *In the q_1^{-1}-BSS we have $d_{p^n-1}q_1^{\pm p^n} = 0$.*

Proof. One sees that $q_1^{\pm p^n}/q_0^{p^n} \in \Omega^*(P; q_1^{-1}Q/q_0^{\infty})$ is a cocycle. □

We have an evident map of spectral sequences

$$E_*^{*,*,*,*}(q_0^{\infty}\text{-BSS}) \longrightarrow E_*^{*,*,*,*}(q_1^{-1}\text{-BSS}).$$

3.6 Multiplicativity of the BSSs

The Q-BSS is multiplicative because $\Omega^*(P;Q) \longrightarrow \Omega^*(P;Q/q_0)$ is a map of DG algebras.

Lemma 3.6.1. *Suppose $x, x', y, y' \in H^*(P;Q/q_0)$ and that $d_r x = y$ and $d_r x' = y'$ in the Q-BSS. Then*

$$d_r(xx') = yx' + (-1)^{|x|}xy'.$$

Here $|x|$ and $|y|$ denote the cohomological gradings of x and y, respectively, since every element of P, Q and Q/q_0 has even u grading.

Proof. Suppose $d_r x = y$ and $d_r x' = y'$.

Lemma 3.2.4 tells us that there exist $a, a', b, b' \in \Omega^*(P;Q)$ such that their images in $\Omega^*(P;Q/q_0)$ represent x, x', y, y', respectively, and such that $da = q_0^r b$, $da' = q_0^r b'$. The image of $aa' \in \Omega^*(P;Q)$ in $\Omega^*(P;Q/q_0)$ represents xx' and the image of

$$ba' + (-1)^{|a|}ab' \in \Omega^*(P;Q)$$

in $\Omega^*(P;Q/q_0)$ represents $yx' + (-1)^{|x|}xy'$. Since $d(aa') = q_0^r(ba' + (-1)^{|a|}ab')$, lemma 3.2.4 completes the proof. □

Corollary 3.6.2. *We have a multiplication*

$$E_1^{s,t,u,v}(Q\text{-}BSS) \otimes E_1^{s',t',u',v'}(Q\text{-}BSS) \longrightarrow E_1^{s+s',t+t',u+u',v+v'}(Q\text{-}BSS)$$

restricting to the following maps.

$$\begin{array}{ccc}
\ker d_r \otimes \operatorname{im} d_r & \longrightarrow & \operatorname{im} d_r \\
\downarrow & & \downarrow \\
\ker d_r \otimes \ker d_r & \longrightarrow & \ker d_r \\
\uparrow & & \uparrow \\
\operatorname{im} d_r \otimes \ker d_r & \longrightarrow & \operatorname{im} d_r
\end{array}
\qquad
\begin{array}{ccc}
\bigcap_s \ker d_s \otimes \bigcup_s \operatorname{im} d_s & \longrightarrow & \bigcup_s \operatorname{im} d_s \\
\downarrow & & \downarrow \\
\bigcap_s \ker d_s \otimes \bigcap_s \ker d_s & \longrightarrow & \bigcap_s \ker d_s \\
\uparrow & & \uparrow \\
\bigcup_s \operatorname{im} d_s \otimes \bigcap_s \ker d_s & \longrightarrow & \bigcup_s \operatorname{im} d_s
\end{array}$$

Thus we have induced maps

$$E_r^{s,t,u,v}(Q\text{-}BSS) \otimes E_r^{s',t',u',v'}(Q\text{-}BSS) \longrightarrow E_r^{s+s',t+t',u+u',v+v'}(Q\text{-}BSS)$$

for $1 \leq r \leq \infty$. *Moreover,*

$$E_\infty^{s,t,u,*}(Q\text{-}BSS) \otimes E_\infty^{s',t',u',*}(Q\text{-}BSS) \longrightarrow E_\infty^{s+s',t+t',u+u',*}(Q\text{-}BSS)$$

is the associated graded of the map

$$H^{s,u}(P; Q^t) \otimes H^{s',u'}(P; Q^{t'}) \longrightarrow H^{s+s',u+u'}(P; Q^{t+t'}).$$

Lemma 3.4.1 means that we have the following corollary to the previous lemma.

Corollary 3.6.3. *Suppose* $x, x', y, y' \in H^*(P; Q/q_0)$ *and that* $d_r x = y$ *and* $d_r x' = y'$ *in the* q_0^∞*-BSS. Then*

$$d_r(xx') = yx' + (-1)^{|x|}xy'.$$

The q_0^∞-BSS is not multiplicative in the sense that we do not have a strict analogue of corollary 3.6.2. This is unsurprising because $H^*(P; Q/q_0^\infty)$ does not have an obvious algebra structure. However, we do have a pairing between the Q-BSS and the q_0^∞-BSS converging to the $H^*(P; Q)$-module structure map of $H^*(P; Q/q_0^\infty)$.

An identical result to lemma 3.6.1 holds for the q_1^{-1}-BSS.

Lemma 3.6.4. *Suppose* $x, x', y, y' \in H^*(P; q_1^{-1}Q/q_0)$ *and that* $d_r x = y$ *and* $d_r x' = y'$ *in the* q_1^{-1}*-BSS. Then*

$$d_r(xx') = yx' + (-1)^{|x|}xy'.$$

Proof. Suppose that $d_r x = y$ in the q_1^{-1}-BSS. We claim that for large enough k the elements $q_1^{kp^r}x$ and $q_1^{kp^r}y$ lift to elements X and Y in $H^*(P; Q/q_0)$ with the property that $d_r X = Y$ in the q_0^∞-BSS.

By lemma 3.5.3, we have $\tilde{x}$ and $\tilde{y}$ demonstrating that $d_r x = y$ in the M_{r+1}-zig-zag. Using definition 3.1.6 and the fact that filtered colimits commute with tensor products and homology, we can find a k such that $q_1^{kp^r}x$ and $q_1^{kp^r}\tilde{y}$ lift to $X \in H^*(P; Q/q_0)$ and

$\widetilde{Y} \in H^*(P; M_r)$, respectively, and such that their images in $H^*(P; M_1)$ coincide. Let $Y \in H^*(P; Q/q_0)$ be the image of $\widetilde{Y}$. Then Y lifts $q_1^{kp^r} y$ and $d_r X = Y$ in the q_0^∞-BSS, proving the claim.

Suppose that $d_r x = y$ and $d_r x' = y'$ in the q_1^{-1}-BSS. For large enough k, we obtain elements X, X', Y and Y' lifting $q_1^{kp^r} x$, $q_1^{kp^r} x'$, $q_1^{kp^r} y$ and $q_1^{kp^r} y'$, respectively, and differentials $d_r X = Y$ and $d_r X' = Y'$ in the q_0^∞-BSS. The previous corollary gives

$$d_r(XX') = YX' + (-1)^{|X|} XY'.$$

Mapping into the q_1^{-1}-BSS and using lemma 3.5.3 we obtain

$$d_r(q_1^{2kp^r}(xx')) = q_1^{2kp^r}(yx' + (-1)^{|x|} xy')$$

in the M_{r+1}-zig-zag. Dividing through by $q_1^{2kp^r}$ completes the proof. □

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Chapter 4

Vanishing lines and localization

In this chapter we prove some vanishing lines for $H^*(P;\mathfrak{Q})$ with various choices of $\mathfrak{Q}$. We also analyze the localization map $H^*(P;Q/q_0^\infty) \longrightarrow H^*(P;q_1^{-1}Q/q_0^\infty)$.

4.1 Vanishing lines

We make note of vanishing lines for $H^*(P;\mathfrak{Q})$ in the cases (see section 3.1 for definitions)

$$\mathfrak{Q} = Q/q_0,\ q_1^{-1}Q/q_0,\ M_n,\ q_1^{-1}M_n,\ Q/q_0^\infty,\ q_1^{-1}Q/q_0^\infty.$$

Notation 4.1.1. We write q for $|q_1| = 2p-2$.

Definition 4.1.2. For $s \in \mathbb{Z}_{\geq 0}$ let $U(2s) = pqs$ and $U(2s+1) = pqs+q$ and write $U(-1) = \infty$.

In [10] Miller uses the following result.

Lemma 4.1.3. *$H^{s,u}(P;[Q/q_0]^t) = 0$ when $u < U(s) + qt$.*

Since q_1 has (t,u) bigrading $(1,q)$ we obtain the following corollary.

Corollary 4.1.4. *$H^{s,u}(P;[q_1^{-1}Q/q_0]^t) = 0$ when $u < U(s) + qt$.*

Lemma 4.1.5. *For each $n \geq 1$, $H^{s,u}(P;[M_n]^t) = 0$ whenever $u < U(s) + q(t+1)$.*

Proof. We proceed by induction on n.

The previous corollary together with the isomorphism $H^*(P; Q/q_0) \cong H^*(P; M_1)$ gives the base case.

The long exact sequence associated to the short exact sequence of P-comodules $0 \longrightarrow M_1 \longrightarrow M_{n+1} \xrightarrow{q_0} M_n \longrightarrow 0$ shows $H^{s,u}(P; [M_{n+1}]^t)$ is zero provided that $H^{s,u}(P; [M_1]^t)$ and $H^{s,u}([M_n]^{t+1})$ are zero. Since $u < U(s) + q(t+1)$ implies that $u < U(s) + q((t+1)+1)$ the inductive step is complete. □

Corollary 4.1.6. *For $\mathfrak{Q} = M_n,\ q_1^{-1}M_n,\ Q/q_0^\infty,$ or $q_1^{-1}Q/q_0^\infty$ we have*

$$H^{s,u}(P; \mathfrak{Q}^t) = 0 \text{ whenever } u < U(s) + q(t+1).$$

Notation 4.1.7. We write (σ, λ) for $(s+t, u+t)$.

Since $(q+1)s - 1 \leq U(s)$ we have the following corollaries.

Corollary 4.1.8. *For $\mathfrak{Q} = Q/q_0$ or $q_1^{-1}Q/q_0$ we have*

$$H^{s,u}(P; \mathfrak{Q}^t) = 0 \text{ whenever } \lambda - \sigma < q\sigma - 1.$$

Corollary 4.1.9. *For $\mathfrak{Q} = M_n,\ Q/q_0^\infty,\ q_1^{-1}M_n,$ or $q_1^{-1}Q/q_0^\infty$ we have*

$$H^{s,u}(P; \mathfrak{Q}^t) = 0 \text{ whenever } \lambda - \sigma < q(\sigma+1) - 1.$$

Lemma 4.1.10. *For $n \geq 1$, $H^{s,u}(P; [M_n]^t) \longrightarrow H^{s,u}(P; [Q/q_0^\infty]^t)$ is*

1. *surjective when $\lambda - \sigma = p^{n-1}q$ and $\sigma \geq p^{n-1} - n$.*
2. *injective when $\lambda - \sigma = p^{n-1}q - 1$ and $\sigma \geq p^{n-1} - n + 1$;*

Proof. The previous corollary tells us that $H^{s,u}(P; [Q/q_0^\infty]^t) = 0$ when $\lambda - \sigma = p^{n-1}q$ and $\sigma \geq p^{n-1}$. The following exact sequence completes the proof.

$$H^{s-1,u}(P; [Q/q_0^\infty]^{t+n}) \rightarrow H^{s,u}(P; [M_n]^t) \rightarrow H^{s,u}(P; [Q/q_0^\infty]^t) \rightarrow H^{s,u}(P; [Q/q_0^\infty]^{t+n})$$

□

4.2 The localization map: the trigraded perspective

In this section we analyze the map $H^*(P; Q/q_0^\infty) \longrightarrow H^*(P; q_1^{-1}Q/q_0^\infty)$. In particular, we find a range in which it is an isomorphism. The result which allows us to do this follows. Throughout this section $s \geq 0$. Recall definition 4.1.2.

Proposition 4.2.1 ([10, pg. 81])**.** *The localization map*

$$H^{s,u}(P; [Q/q_0]^t) \longrightarrow H^{s,u}(P; [q_1^{-1}Q/q_0]^t)$$

1. *is injective if* $u < U(s-1) + (2p^2-2)(t+1) - q$*;*
2. *is surjective if* $u < U(s) + (2p^2-2)(t+1) - q$*.*

This allows us to prove the following lemma which explains how we can transfer differentials between the q_0^∞-BSS and the q_1^{-1}-BSS.

Lemma 4.2.2. *Suppose* $u < U(s)+(2p^2-2)(t+2)-q$ *so that proposition 4.2.1 gives a surjection* $E_1^{s,t,u,*}(q_0^\infty\text{-}BSS) \to E_1^{s,t,u,*}(q_1^{-1}\text{-}BSS)$ *and an injection* $E_1^{s+1,t,u,*}(q_0^\infty\text{-}BSS) \to E_1^{s+1,t,u,*}(q_1^{-1}\text{-}BSS)$*.*

Suppose $x \in E_1^{s,t,u,*}(q_0^\infty\text{-}BSS)$ *maps to* $\overline{x} \in E_1^{s,t,u,*}(q_1^{-1}\text{-}BSS)$ *and that* $d_r\overline{x} = y$ *in the* q_1^{-1}*-BSS. Then, in fact,* y *lies in* $E_1^{s+1,t,u,*}(q_0^\infty\text{-}BSS)$ *and* $d_r x = y$ *in the* q_0^∞*-BSS.*

Proof. We proceed by induction on r. The result is true in the case $r = 0$ where $d_0 = 0$ and the case $r = 1$ where d_r is a function. Suppose $r > 1$. Then

$$\begin{array}{rrcll}
 & d_r\overline{x} & = & y & \text{in the } q_1^{-1}\text{-BSS} \\
\Longrightarrow & d_{r-1}\overline{x} & = & 0 & \text{in the } q_1^{-1}\text{-BSS} & & \text{(Lemma 2.1.4)} \\
\Longrightarrow & d_{r-1}x & = & 0 & \text{in the } q_0^\infty\text{-BSS} & & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y' & \text{in the } q_0^\infty\text{-BSS} & \text{for some } y' & \text{(Lemma 2.1.4)} \\
\Longrightarrow & d_r\overline{x} & = & y' & \text{in the } q_1^{-1}\text{-BSS} & & \text{(Map of SSs)} \\
\Longrightarrow & d_{r-1}\overline{x}' & = & y'-y & \text{in the } q_1^{-1}\text{-BSS} & \text{for some } x' & \text{(Corollary 2.1.5)} \\
\Longrightarrow & d_{r-1}x' & = & y'-y & \text{in the } q_0^\infty\text{-BSS} & & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y & \text{in the } q_0^\infty\text{-BSS} & & \text{(Corollary 2.1.5)}
\end{array}$$

We remark that the statement about y lying in $E_1^{s+1,t,u,*}(q_0^\infty)$ is actually trivial: the map $E_1^{s+1,t,u,*}(q_0^\infty\text{-BSS}) \longrightarrow E_1^{s+1,t,u,*}(q_1^{-1}\text{-BSS})$ is an isomorphism since $s \geq 0$ implies $U(s) < U(s+1)$. □

Corollary 4.2.3. *$E_\infty^{s,t,u,*}(q_0^\infty\text{-BSS}) \longrightarrow E_\infty^{s,t,u,*}(q_1^{-1}\text{-BSS})$ is*

1. *injective if $u < U(s-1) + (2p^2-2)(t+2) - q$;*

2. *surjective if $u < U(s) + (2p^2-2)(t+2) - q$.*

Proof. Suppose $u < U(s) + (2p^2-2)(t+2) - q$ and that $y \in E_\infty^{s+1,t,u,*}(q_0^\infty\text{-BSS})$ maps to zero in $E_\infty^{s+1,t,u,*}(q_1^{-1}\text{-BSS})$. This says that $d_r\overline{x} = y$ for some x in $E_1^{s,t,u,*}(q_0^\infty\text{-BSS})$. By the previous lemma $d_r x = y$, which says that y is zero in $E_\infty^{s+1,t,u,*}(q_0^\infty\text{-BSS})$. This proves the first statement when $s > 0$. For $s = 0$, the result is clear since it holds at the E_1-page and the only boundary is zero.

Suppose $u < U(s)+(2p^2-2)(t+2)-q$ and we have an element of $E_\infty^{s,t,u,*}(q_1^{-1}\text{-BSS})$. We can write this element as $\overline{x}$ for $x \in E_1^{s,t,u,*}(q_0^\infty\text{-BSS})$. Moreover, since $d_r\overline{x} = 0$ for each r the previous lemma tells us that each $d_r x = 0$ for each r, i.e. x is a permanent cycle, as is required to prove the second statement. □

Proposition 4.2.4. *The localization map*

$$H^{s,u}(P; [Q/q_0^\infty]^t) \longrightarrow H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t)$$

1. *is injective if $u < U(s-1) + (2p^2-2)(t+2) - q$;*

2. *is surjective if $u < U(s) + (2p^2-2)(t+2) - q$.*

Proof. We have $H^*(P;\mathfrak{Q}) = \bigcup_v F^v H^*(P;\mathfrak{Q})$ and $F^0 H^*(P;\mathfrak{Q}) = 0$ when $\mathfrak{Q} = Q/q_0^\infty$ or $q_1^{-1}Q/q_0^\infty$ and so the result follows from the previous corollary. □

4.3 The localization map: the bigraded perspective

Recall the bigrading (σ, λ) of definition 4.1.7. We prove the analogues of the results of the last section with respect to this bigrading.

Proposition 4.3.1 ([10, 4.7(a)])**.** *The localization map*

$$H^{s,u}(P;[Q/q_0]^t) \longrightarrow H^{s,u}(P;[q_1^{-1}Q/q_0]^t)$$

1. *is a surjection if* $\sigma \geq 0$ *and* $\lambda < U(\sigma+1) - q - 1$*;*

2. *is an isomorphism if* $\sigma \geq 0$ *and* $\lambda < U(\sigma) - q - 1$*.*

Corollary 4.3.2. *The localization map*

$$H^{s,u}(P;[Q/q_0]^t) \longrightarrow H^{s,u}(P;[q_1^{-1}Q/q_0]^t)$$

1. *is a surjection if* $\lambda < p(p-1)\sigma - 1$*, i.e.* $\lambda - \sigma < (p^2-p-1)\sigma - 1$*;*

2. *is an isomorphism if* $\lambda < p(p-1)(\sigma-1) - 1$*,*

$$\textit{i.e. } \lambda - \sigma < (p^2-p-1)(\sigma-1) - 2.$$

Proof. Consider $g(\sigma) = p(p-1)\sigma - U(\sigma)$ for $\sigma \geq 0$. We have $g(1) = p(p-3)+2 \geq 0 = g(0)$ and $g(\sigma+2) = g(\sigma)$. Thus $p(p-1)\sigma - U(\sigma) \leq p(p-3)+2$ and so

$$p(p-1)(\sigma-1) - 1 \leq \Big[U(\sigma) + p(p-3) + 2\Big] - p(p-1) - 1 = U(\sigma) - q - 1.$$

Together with the previous proposition, this proves the claim for $\sigma \geq 0$.

When $\sigma < 0$, $H^{s,u}(P;[Q/q_0]^t) = 0$ and so the localization map is injective. We just need to prove that $H^{s,u}(P;[q_1^{-1}Q/q_0]^t) = 0$ whenever $\lambda - \sigma < (p^2-p-1)\sigma - 1$ and $\sigma < 0$. We can only have $[(\lambda-\sigma)+1]/(p^2-p-1) < \sigma < 0$ if $(\lambda-\sigma)+1 < 0$. But then $[(\lambda-\sigma)+1]/q < \sigma < 0$ and the vanishing line of corollary 4.1.8 gives the result. □

This allows us to prove bigraded versions of all the results of the previous subsection. In particular, we have the following proposition.

Proposition 4.3.3. *The localization map*

$$H^{s,u}(P;[Q/q_0^{\infty}]^t) \longrightarrow H^{s,u}(P;[q_1^{-1}Q/q_0^{\infty}]^t)$$

1. *is a surjection if* $\lambda < p(p-1)(\sigma+1)-2$, *i.e.* $\lambda-\sigma < (p^2-p-1)(\sigma+1)-1$;

2. *is an isomorphism if* $\lambda < p(p-1)\sigma-2$, *i.e.* $\lambda-\sigma < (p^2-p-1)\sigma-2$.

Chapter 5

Calculating the 1-line of the q-CSS; its image in $H^*(A)$

This chapter contains our main result. We calculate the q_1^{-1}-BSS

$$\left[H^*(P; q_1^{-1}Q/q_0)\left[q_0\right]\right]/q_0^\infty \stackrel{v}{\Longrightarrow} H^*(P; q_1^{-1}Q/q_0^\infty).$$

In the introduction we discussed "principal towers" and their "side towers." Our presentation of the results is divided up in this way, too.

5.1 The E_1-page of the q_1^{-1}-BSS

Our starting place for the calculation of the q_1^{-1}-BSS is a result of Miller in [10] which gives a description of $E_1(q_1^{-1}\text{-BSS})$.

Definition 5.1.1. Denote by P' the Hopf algebra obtained from P by quotienting out the ideal generated by the image of the p-th power map $P \longrightarrow P$, $\xi \longmapsto \xi^p$.

We can make $\mathbb{F}_p[q_1]$ into an algebra in P'-comodules by defining q_1 to be a comodule primitive. The map $Q/q_0 \longrightarrow Q/(q_0, q_2, q_3, \ldots) = \mathbb{F}_p[q_1]$ is an algebra map over the Hopf algebra map $P \longrightarrow P'$. Thus, we have the following induced map.

$$\Omega^*(P; q_1^{-1}Q/q_0) \longrightarrow \Omega^*(P'; \mathbb{F}_p[q_1^{\pm 1}]) \tag{5.1.2}$$

Theorem 5.1.3 (Miller, [10, 4.4]). *The map* $H^*(P; q_1^{-1}Q/q_0) \longrightarrow H^*(P'; \mathbb{F}_p[q_1^{\pm 1}])$ *is an isomorphism.*

$[\xi_n]$ and $\sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j}[\xi_n^j|\xi_n^{p-j}]$ are cocycles in $\Omega^*(P')$ and so they define elements $h_{n,0}$ and $b_{n,0}$ in $H^*(P'; \mathbb{F}_p)$. The cohomology of a primitively generated Hopf algebra is well understood and the following lemma is a consequence.

Lemma 5.1.4. $H^*(P'; \mathbb{F}_p) = E[h_{n,0} : n \geq 1] \otimes \mathbb{F}_p[b_{n,0} : n \geq 1]$.

Corollary 5.1.5. $H^*(P; q_1^{-1}Q/q_0) = \mathbb{F}_p[q_1^{\pm 1}] \otimes E[h_{n,0} : n \geq 1] \otimes \mathbb{F}_p[b_{n,0} : n \geq 1]$. *The* (s, t, u) *trigradings are as follows.*

$$|q_1| = (0, 1, 2p-2), \quad |h_{n,0}| = (1, 0, 2p^n - 2), \quad |b_{n,0}| = (2, 0, p(2p^n - 2)).$$

For our work it is convenient to change these exterior and polynomial generators by units.

Notation 5.1.6. For $n \geq 1$, let $p^{[n]} = \frac{p^n - 1}{p-1}$, $\epsilon_n = q_1^{-p^{[n]}} h_{n,0}$, and $\rho_n = q_1^{-p \cdot p^{[n]}} b_{n,0}$.

Let $p^{[0]} = 0$ and note that we have $p^{[n+1]} = p^n + p^{[n]} = p \cdot p^{[n]} + 1$ for $n \geq 0$.

Corollary 5.1.7. $H^*(P; q_1^{-1}Q/q_0) = \mathbb{F}_p[q_1^{\pm 1}] \otimes E[\epsilon_n : n \geq 1] \otimes \mathbb{F}_p[\rho_n : n \geq 1]$. *The* (s, t, u) *trigradings are as follows.*

$$|q_1| = (0, 1, 2p-2), \quad |\epsilon_n| = (1, -p^{[n]}, 0), \quad |\rho_n| = (2, 1 - p^{[n+1]}, 0).$$

We make note of some elements that lift uniquely to $H^*(P; Q/q_0)$.

Lemma 5.1.8. *The elements*

$$1,\ q_1^{2p^{n-1}}\epsilon_n,\ b_{1,0} = q_1^p \rho_1,\ q_1^{2p^n}\rho_n \in H^*(P; q_1^{-1}Q/q_0)$$

have unique lifts to $H^*(P; Q/q_0)$. *The same is true after multiplying by* q_1^n *as long as* $n \geq 0$.

Proof. We use proposition 4.2.1. The (s,t,u) trigradings of the elements in the lemma are

$$(0,0,0),\ (1,2p^{n-1}-p^{[n]},2qp^{n-1}),\ (2,0,qp),\ (2,2p^n-p^{[n+1]}+1,2qp^n),$$

respectively. In each case (s,t,u) satisfies $u < U(s-1)+(2p^2-2)(t+1)-q$ and $u < U(s)+(2p^2-2)(t+1)-q$; the key inequalities one needs are $q < 2p^2-2$ and

$$2qp^{n-1} < (2p^2-2)(2p^{n-1}-p^{[n]}+1)-q. \tag{5.1.9}$$

The latter inequality is equivalent to $(p+1)p^{[n]} < 2p^n+p$, which holds because $p \geq 3$. Since $q < 2p^2-2$ multiplication by a positive power of q_1 only makes things better. □

5.2 The first family of differentials, principal towers

5.2.1 Main results

Notation 5.2.1.1. We write $\doteq$ to denote equality up to multiplication by an element in $\mathbb{F}_p^\times$.

The main results of this section are as follows. The first concerns the q_1^{-1}-BSS and the second gives the corresponding result in the Q-BSS.

Proposition 5.2.1.2. *For $n \geq 1$ and $k \in \mathbb{Z}-p\mathbb{Z}$ we have the following differential in the q_1^{-1}-BSS.*

$$d_{p^{[n]}}q_1^{kp^{n-1}} \doteq q_1^{kp^{n-1}}\epsilon_n$$

Proposition 5.2.1.3. *Let $n \geq 1$. We have the following differential in the Q-BSS.*

$$d_{p^{n-1}}q_1^{p^{n-1}} \doteq h_{1,n-1}$$

Moreover, if $k \in \mathbb{Z}-p\mathbb{Z}$ and $k > 1$, $d_{p^{[n]}}q_1^{kp^{n-1}}$ is defined in the Q-BSS.

5.2.2 Quick proofs

The differentials in the q_1^{-1}-BSS are derivations (lemma 3.6.4) and $d_{p^{[n]}}q_1^{-p^n} = 0$ (lemma 3.5.4). This means that proposition 5.2.1.2 follows quickly from the following sub-proposition.

Proposition 5.2.2.1. *For $n \geq 1$ we have the following differential in the q_1^{-1}-BSS.*

$$d_{p^{[n]}}q_1^{p^{n-1}} \doteq q_1^{p^{n-1}}\epsilon_n$$

This is the consuming calculation of the section. Supposing this result for now, we prove proposition 5.2.1.3.

Proof of proposition 5.2.1.3. The formula $d\Big([\ ]q_1^{p^{n-1}}\Big) = [\xi_1^{p^{n-1}}]q_0^{p^{n-1}}$ in $\Omega^*(P;Q)$, together with lemma 3.2.4 proves the first statement.

By lemma 3.4.1, we can verify the second statement in the q_0^∞-BSS. We have

$$q_0^v q_1^{kp^{n-1}} \in E_1^{0,kp^{n-1}+v,kqp^{n-1},v}(q_0^\infty\text{-BSS})$$

and we will show that $q_0^{-p^{[n]}-1}q_1^{kp^{n-1}}$ survives to the $E_{p^{[n]}}$-page. Proposition 5.2.1.2 and lemma 4.2.2 say that it is enough to verify that $(s,t,u) = (0, kp^{n-1} - p^{[n]} - 1, kqp^{n-1})$ satisfies $u < U(s) + (2p^2-2)(t+2) - q$. Since $q < 2p^2 - 2$ the worst case is when $k = 2$ where the inequality is (5.1.9). □

5.2.3 The proof of proposition 5.2.2.1

We prove proposition 5.2.2.1 via the following cocycle version of the statement.

Proposition 5.2.3.1. *For each $n \geq 1$, there exist cocycles*

$$x_n \in \Omega^0(P; q_1^{-1}Q/q_0^\infty),\ y_n \in \Omega^1(P; q_1^{-1}Q/q_0)$$

such that

1. $q_0^{p^{[n]}-1}x_n = q_0^{-1}q_1^{p^{n-1}}$,

2. $y_n = q_0 d(q_0^{-1} x_n)$,

3. *the image of* y_n *in* $\Omega^1(P'; \mathbb{F}_p[q_1^{\pm 1}])$ *is* $(-1)^{n-1}[\xi_n] q_1^{-p^{[n-1]}}$.

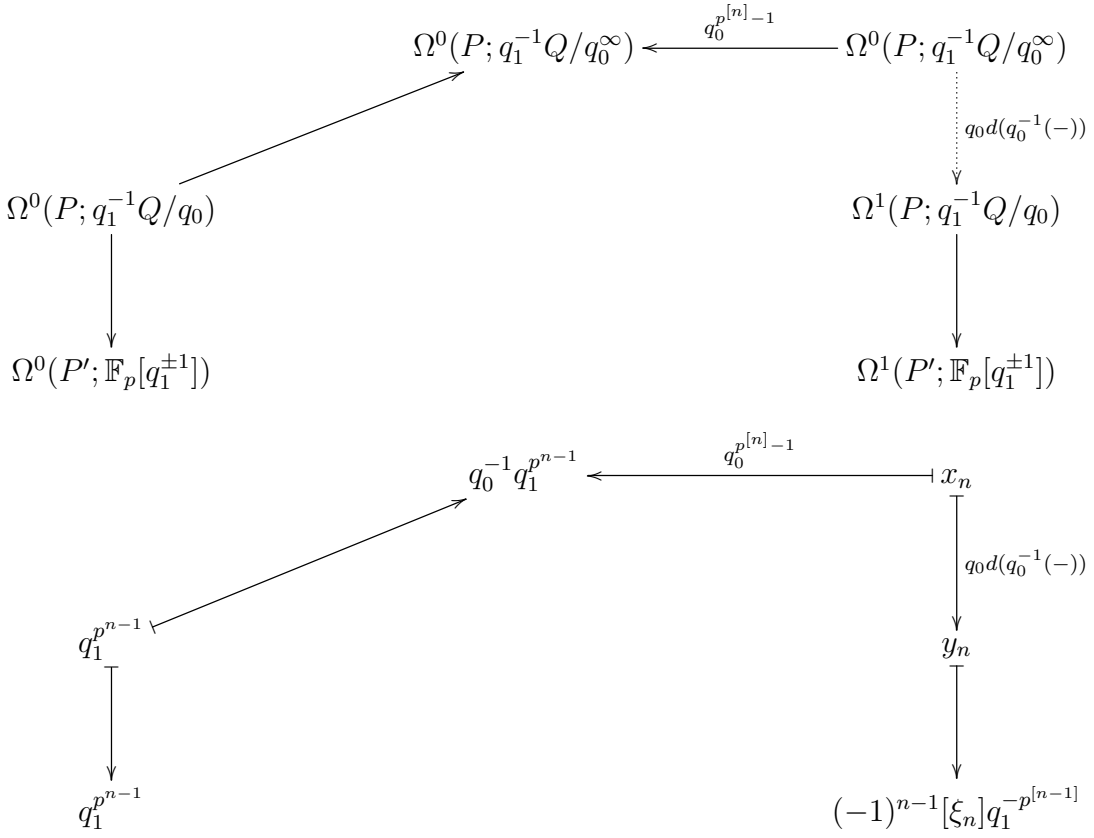
In the expression $q_0 d(q_0^{-1} x_n)$, $q_0^{-1} x_n$ denotes the element of $\Omega^0(P; q_1^{-1}Q/q_0^{\infty})$ with the following two properties:

1. multiplying by q_0 gives x_n;

2. the denominators of the terms in $q_0^{-1} x_n$ have q_0 raised to a power greater than or equal to 2.

Thus, $q_0 d(q_0^{-1} x_n)$ gives a particular representative for the image of the class of x_n under the boundary map $\partial : H^0(P; q_1^{-1}Q/q_0^{\infty}) \longrightarrow H^1(P; q_1^{-1}Q/q_0)$ coming from the short exact sequence (3.5.1).

To illuminate the statement of the proposition we draw the relevant diagrams.



Passing to cohomology and using theorem 5.1.3, we see that the proposition implies that $d_{p^{[n]}} q_1^{p^{n-1}} = (-1)^{n-1} q_1^{-p^{[n-1]}} h_{n,0} \doteq q_1^{p^{n-1}} \epsilon_n$, as required.

We note that for the $n = 1$ and $n = 2$ cases of the proposition we can take

$$x_1 = q_0^{-1}q_1,\ y_1 = [\xi_1],\ x_2 = q_0^{-p-1}q_1^p - q_0^{-1}q_1^{-1}q_2,\ y_2 = [\overline{\xi}_2]q_1^{-1} + [\xi_1]q_1^{-2}q_2.$$

Sketch proof of proposition 5.2.3.1. We proceed by induction on n. So suppose that we have cocycles x_n and y_n satisfying the statements in the proposition. Write $\tilde{P}^0 x_n$ and $\tilde{P}^0 y_n$ for the cochains in which we have raised every symbol to the p^{th} power. We claim that:

1. $\tilde{P}^0 x_n$ and $\tilde{P}^0 y_n$ are cocyles;
2. $q_0^{p^{[n+1]}-2}\tilde{P}^0 x_n = q_0^{-1}q_1^{p^n}$;
3. $\tilde{P}^0 y_n = q_0 d(q_0^{-1}\tilde{P}^0 x_n)$.

Since y_n maps to $(-1)^{n-1}[\xi_n]q_1^{-p^{[n-1]}}$ in $\Omega^1(P'; \mathbb{F}_p[q_1^{\pm 1}])$ and ξ_n^p is zero in P', $\tilde{P}^0 y_n$ maps to 0. By theorem 5.1.3, we deduce that there exists a $w_n \in \Omega^0(P; q_1^{-1}Q/q_0)$ with $dw_n = \tilde{P}^0 y_n$. We summarize some of this information in the following diagram.

$$\begin{array}{ccccc}
\Omega^*(P; q_1^{-1}Q/q_0) & \longrightarrow & \Omega^*(P; q_1^{-1}Q/q_0^{\infty}) & \longrightarrow & \Omega^*(P; q_1^{-1}Q/q_0^{\infty}) \\
w_n & & q_0^{-1}\tilde{P}^0 x_n & \longmapsto & \tilde{P}^0 x_n \\
\Big\downarrow d & & \Big\downarrow d & & \\
\tilde{P}^0 y_n & \longmapsto & q_0^{-1}\tilde{P}^0 y_n & &
\end{array}$$

Let $x_{n+1} = q_0^{-1}\tilde{P}^0 x_n - q_0^{-1}w_n$, a cocycle in $\Omega^0(P; q_1^{-1}Q/q_0^{\infty})$ and $y_{n+1} = q_0 d(q_0^{-1}x_{n+1})$, a cocycle in $\Omega^1(P; q_1^{-1}Q/q_0)$. We claim that:

1. $q_0^{p^{[n+1]}-1}x_{n+1} = q_0^{-1}q_1^{p^n}$;
2. $y_{n+1} = q_0 d(q_0^{-1}x_{n+1})$;
3. the image of y_{n+1} in $\Omega^1(P'; \mathbb{F}_p[q_1^{\pm 1}])$ is $(-1)^n[\xi_{n+1}]q_1^{-p^{[n]}}$.

The first claim follows from the claim above that $q_0^{p^{[n+1]}-2}\tilde{P}^0 x_n = q_0^{-1}q_1^{p^n}$, since then $q_0^{p^{[n+1]}-1}x_{n+1} = q_0^{p^{[n+1]}-1}[q_0^{-1}\tilde{P}^0 x_n - q_0^{-1}w_n] = q_0^{p^{[n+1]}-2}\tilde{P}^0 x_n = q_0^{-1}q_1^{p^n}$. The second claim holds by definition of y_{n+1}. □

In order to convert the sketch proof into a proof we must prove the first three claims and the final claim. The next lemma takes care of the first two claims.

Lemma 5.2.3.2. *Suppose that $x \in \Omega^0(P; q_1^{-1}Q/q_0^\infty)$ and $y \in \Omega^1(P; q_1^{-1}Q/q_0)$ are cocycles. Then $\tilde{P}^0 x$ and $\tilde{P}^0 y$ are cocyles, too. Moreover, $q_0^{p^{[n]}-1}x = q_0^{-1}q_1^{p^{n-1}}$ implies $q_0^{p^{[n+1]}-2}\tilde{P}^0 x = q_0^{-1}q_1^{p^n}$.*

Proof. The result is clear for $\tilde{P}^0 y$ since $\mathrm{Fr} : P \longrightarrow P$, $\xi \longmapsto \xi^p$ is a Hopf algebra map and $q_1^{-1}Q/q_0 \longrightarrow q_1^{-1}Q/q_0$, $\mathfrak{q} \longmapsto \mathfrak{q}^p$ is an algebra map over Fr.

Suppose that x and $\tilde{P}^0 x$ involve negative powers of q_0 at worst q_0^{-n} and that x involves negative powers of q_1 at worst q_1^{-k}. Then we have the following sequence of injections (recall definitions 3.1.4 through 3.1.7).

$$\begin{array}{ccccccc}
\Omega^*(P; M_n(k)) & \longrightarrow & \Omega^*(P; M_n(kp)) & \longrightarrow & \Omega^*(P; q_1^{-1}M_n) & \longrightarrow & \Omega^*(P; q_1^{-1}Q/q_0^\infty) \\
q_1^{kp^{n-1}}x & \longmapsto & x & \longmapsto & x & & \\
 & & q_1^{kp^n}\tilde{P}^0 x & \longmapsto & \tilde{P}^0 x & \longmapsto & \tilde{P}^0 x
\end{array}$$

Since x is a cocycle in $\Omega^0(P; q_1^{-1}Q/q_0^\infty)$, $q_1^{kp^{n-1}}x$ is a cocycle in $\Omega^0(P; M_n(k))$. Thus, $q_1^{kp^n}\tilde{P}^0 x$ is a cocycle in $\Omega^0(P; M_n(kp))$ and $\tilde{P}^0 x$ is a cocycle in $\Omega^0(P; q_1^{-1}Q/q_0^\infty)$. Also,

$$q_0^{p^{[n]}-1}x = q_0^{-1}q_1^{p^{n-1}} \implies q_0^{p\cdot p^{[n]}-p}\tilde{P}^0 x = q_0^{-p}q_1^{p^n} \implies q_0^{p\cdot p^{[n]}-1}\tilde{P}^0 x = q_0^{-1}q_1^{p^n}.$$

The proof is completed by noting that $p \cdot p^{[n]} - 1 = p^{[n+1]} - 2$. □

The next lemma takes care of the third claim.

Lemma 5.2.3.3. *Suppose $x \in \Omega^0(P; q_1^{-1}Q/q_0^\infty)$ is a cocycle and that*

$$q_0 d(q_0^{-1}x) = y \in \Omega^1(P; q_1^{-1}Q/q_0).$$

Then $q_0 d(q_0^{-1}\tilde{P}^0 x) = \tilde{P}^0 y \in \Omega^1(P; q_1^{-1}Q/q_0)$.

Proof. Suppose that $q_0^{-1}x$ and $q_0^{-p}\tilde{P}^0x$ involve negative powers of q_0 at worst q_0^{-n} and that $q_0^{-1}x$ involves negative powers of q_1 at worst q_1^{-k}. Then we have the following sequence of injections (recall definitions 3.1.4 through 3.1.7).

$$\begin{array}{ccccccc}
\Omega^*(P;M_n(k)) & \longrightarrow & \Omega^*(P;M_n(kp)) & \longrightarrow & \Omega^*(P;q_1^{-1}M_n) & \longrightarrow & \Omega^*(P;q_1^{-1}Q/q_0^{\infty}) \\
q_1^{kp^{n-1}}q_0^{-1}x & \longmapsto & q_0^{-1}x & \longmapsto & q_0^{-1}x & \longmapsto & q_0^{-1}x \\
 & & q_1^{kp^n}q_0^{-p}\tilde{P}^0x & \longmapsto & q_0^{-p}\tilde{P}^0x & \longmapsto & q_0^{-p}\tilde{P}^0x
\end{array}$$

We have

$$d(q_1^{kp^n}q_0^{-p}\tilde{P}^0x) = \tilde{P}^0d(q_1^{kp^{n-1}}q_0^{-1}x) \in \Omega^1(P;M_n(kp))$$

and so

$$d(q_0^{-p}\tilde{P}^0x) = q_1^{-kp^n}d(q_1^{kp^n}q_0^{-p}\tilde{P}^0x) = \tilde{P}^0\left[q_1^{-kp^{n-1}}d(q_1^{kp^{n-1}}q_0^{-1}x)\right] = \tilde{P}^0d(q_0^{-1}x) \tag{5.2.3.4}$$

in $\Omega^1(P;q_1^{-1}Q/q_0^{\infty})$. We obtain

$$q_0d(q_0^{-1}\tilde{P}^0x) = q_0d(q_0^{p-1}(q_0^{-p}\tilde{P}^0x)) = q_0^pd(q_0^{-p}\tilde{P}^0x) = \tilde{P}^0(q_0d(q_0^{-1}x)) = \tilde{P}^0y$$

where the penultimate equality comes from the preceding observation. □

Proof of proposition 5.2.3.1. We are just left with the final claim, that the image of y_{n+1} in $\Omega^1(P';\mathbb{F}_p[q_1^{\pm 1}])$ is $(-1)^n[\xi_{n+1}]q_1^{-p^{[n]}}$.

Recall that y_{n+1} is defined to be $q_0d(q_0^{-1}x_{n+1})$ and that x_{n+1} is $q_0^{-1}\tilde{P}^0x_n - q_0^{-1}w_n$. We summarize this in the following diagram.

$$\begin{array}{ccccc}
\Omega^*(P;q_1^{-1}Q/q_0) & \longrightarrow & \Omega^*(P;q_1^{-1}Q/q_0^{\infty}) & \longrightarrow & \Omega^*(P;q_1^{-1}Q/q_0^{\infty}) \\
 & & q_0^{-1}x_{n+1} = q_0^{-2}\tilde{P}^0x_n - q_0^{-2}w_n & \longmapsto & x_{n+1} \\
 & & \Big\downarrow{\scriptstyle d} & & \\
y_{n+1} & \longmapsto & q_0^{-1}y_{n+1} & &
\end{array}$$

When considering the image of y_{n+1} in $\Omega^1(P';\mathbb{F}_p[q_1^{\pm 1}])$ we can ignore contributions arising from $q_0^{-2}\tilde{P}^0 x_n$ since (5.2.3.4) gives

$$d(q_0^{-2}\tilde{P}^0 x_n) = q_0^{p-2}\tilde{P}^0 d(q_0^{-1}x_n)$$

and so all terms involve a ξ_j raised to a p-th power. Let

$$w'_n = w_n + (-1)^n q_1^{-p^{[n]}} q_{n+1} \in \Omega^0(P; q_1^{-1}Q/q_0)$$

so that

$$-q_0^{-2}w_n = (-1)^n q_0^{-2} q_1^{-p^{[n]}} q_{n+1} - q_0^{-2} w'_n \in \Omega^0(P; q_1^{-1}Q/q_0^{\infty}).$$

As an example, we recall that

$$x_1 = q_0^{-1}q_1,\ y_1 = [\xi_1],\ x_2 = q_0^{-p-1}q_1^p - q_0^{-1}q_1^{-1}q_2,\ y_2 = [\overline{\xi}_2]q_1^{-1} + [\xi_1]q_1^{-2}q_2;$$

we have

$$w_1 = q_1^{-1}q_2,\ w'_1 = 0,\ w_2 = q_1^{-2p-1}q_2^{p+1} - q_1^{-p-1}q_3,\ w'_2 = q_1^{-2p-1}q_2^{p+1}.$$

We consider the contributions from the two terms in the expression for $-q_0^{-2}w_n$ separately.

Lemma 5.2.3.5. *The only term of $d(q_0^{-2}q_1^{-p^{[n]}}q_{n+1})$, which is relevant to the image of y_{n+1} in $\Omega^1(P';\mathbb{F}_p[q_1^{\pm 1}])$, is $[\xi_{n+1}]q_0^{-1}q_1^{-p^{[n]}}$.*

Proof. Recall definitions 3.1.4 and 3.1.6. We have a P-comodule map

$$M_2(p^{[n-1]}+1) \longrightarrow q_1^{-1}M_2 \subset q_1^{-1}Q/q_0^{\infty}, \qquad q_0^{-2}q_1^{p-1}q_{n+1} \longmapsto q_0^{-2}q_1^{-p^{[n]}}q_{n+1}.$$

Under the coaction map $Q \longrightarrow P \otimes Q$, we have

$$q_1^{p-1} \longmapsto \sum_{i+j=p-1} (-1)^i \xi_1^i \otimes q_0^i q_1^j \quad \text{and} \quad q_{n+1} \longmapsto \sum_{r+s=n+1} \xi_r^{p^s} \otimes q_s.$$

Under the coaction map $q_0^{-1}Q \longrightarrow P \otimes q_0^{-1}Q$, we have

$$q_0^{-2}q_1^{p-1}q_{n+1} \longmapsto \sum_{i+j=p-1} \sum_{r+s=n+1} (-1)^i \xi_1^i \xi_r^{p^s} \otimes q_0^{i-2} q_1^j q_s$$

so that under the coaction map $q_1^{-1}Q/q_0^\infty \longrightarrow P \otimes q_1^{-1}Q/q_0^\infty$, we have

$$q_0^{-2}q_1^{-p^{[n]}}q_{n+1} \longmapsto \sum_{\substack{i+j=p-1 \\ i=0,1}} \sum_{r+s=n+1} (-1)^i \xi_1^i \xi_r^{p^s} \otimes q_0^{i-2} q_1^{j-p(p^{[n-1]}+1)} q_s.$$

We know that terms involving q_0^{-2} must eventually cancel in some way so we ignore these. Because we are concerned with an image in $\Omega^1(P'; \mathbb{F}_p[q_1^{\pm 1}])$ we ignore terms involving ξ_j's raised to a power greater than or equal to p and terms involving q_j's other than q_1 and q_0. Since $n \geq 1$, we are left with the term corresponding to $s = 0$, $r = n+1$, $i = 0$ and $j = p-1$: it is $\xi_{n+1} \otimes q_0^{-1} q_1^{-p^{[n]}}$. □

The proof of proposition 5.2.3.1 is almost complete. We just need to show that $d(q_0^{-2} w_n')$ contributes nothing to the image of y_{n+1} in $\Omega^1(P'; \mathbb{F}_p[q_1^{\pm 1}])$. Recall that

$$w_n' = w_n + (-1)^n q_1^{-p^{[n]}} q_{n+1} \in \Omega^0(P; q_1^{-1}Q/q_0)$$

and that $dw_n = \tilde{P}^0 y_n$.

Denote by P'' the Hopf algebra obtained from P by quotienting out the ideal generated by the image of the map $P \longrightarrow P$, $\xi \longmapsto \xi^{p^2}$.

Lemma 5.2.3.6.

$$dw_n' = \tilde{P}^0 y_n + (-1)^n \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} [\xi_i^{p^j}] q_1^{-p^{[n]}} q_j \in \Omega^1(P; q_1^{-1}Q/q_0)$$

is in the kernel of the map $\overline{P} \otimes q_1^{-1}Q/q_0 \longrightarrow \overline{P''} \otimes \mathbb{F}_p[q_1^{\pm 1}]$.

Proof. By the inductive hypothesis $y_n \equiv (-1)^{n-1}[\xi_n] q_1^{-p^{[n-1]}}$ modulo the kernel of $\overline{P} \otimes q_1^{-1}Q/q_0 \longrightarrow \overline{P'} \otimes \mathbb{F}_p[q_1^{\pm 1}]$. So $\tilde{P}^0 y_n \equiv (-1)^{n-1}[\xi_n^p] q_1^{-p^{[n]}+1}$ modulo the kernel of

$\overline{P} \otimes q_1^{-1}Q/q_0 \longrightarrow \overline{P''} \otimes \mathbb{F}_p[q_1^{\pm 1}]$. $(-1)^{n-1}[\xi_n^p]q_1^{-p^{[n]}+1}$ cancels with the $j = 1$ term of the summation in the lemma statement. $\square$

Corollary 5.2.3.7. *For each monomial W of w_n' not equal to a power of q_1, there exists a $j > 1$ such that q_j^p divides W.*

Proof. The map $q_1^{-1}Q/q_0 \longrightarrow P \otimes q_1^{-1}Q/q_0 \longrightarrow P \otimes \mathbb{F}_p[q_1^{\pm 1}]$ takes

$$q_{n_1}^{k_1} \cdots q_{n_r}^{k_r} \longmapsto \xi_{n_1-1}^{k_1 p} \cdots \xi_{n_r-1}^{k_r p} \otimes q_1^{\sum k_i}$$

and so it is injective with image $\mathbb{F}_p[\xi_1^p, \xi_2^p, \xi_3^p, \ldots] \otimes \mathbb{F}_p[q_1^{\pm 1}]$. One sees that elements $q_{n_1}^{k_1} \cdots q_{n_r}^{k_r}$ with $r \geq 2$, $1 = n_1 < \ldots < n_r$, $k_1 \in \mathbb{Z}$, $k_2, \ldots, k_r \in \{1, 2, \ldots, p-1\}$ are not sent to $\ker(P \longrightarrow P'') \otimes \mathbb{F}_p[q_1^{\pm 1}]$. By the previous lemma, each monomial of w_n' not equal to a power of q_1 must contain some q_j $(j > 1)$ raised to a power greater than or equal to p. $\square$

Since powers of $q_1 \in \Omega^0(P; q_1^{-1}Q/q_0)$ are cocycles we can assume that w_n and w_n' do not contain powers of q_1 as monomials.

Suppose no power of q_1 worse than q_1^{-kp} appears in w_n'. Making use of the map (see definitions 3.1.4 and 3.1.6)

$$\Omega^*(P; M_2(k)) \longrightarrow \Omega^*(P; q_1^{-1}M_2) \subset \Omega^*(P; q_1^{-1}Q/q_0^\infty), \qquad q_0^{-2}q_1^{kp}w_n' \longmapsto q_0^{-2}w_n'$$

we see that it is sufficient to analyze $d(q_0^{-2}q_1^{kp}w_n')$. Viewing $q_1^{kp}w_n'$ as lying in $\Omega^0(P; Q)$, we care about terms of $d(q_1^{kp}w_n')$ involving a single power of q_0. From the previous corollary we see that the boundary of every monomial in w_n' will involve terms which consist of either a ξ_j raised to a power greater than or equal to p or a q_j with $j > 1$. We conclude that the contribution from $d(q_0^{-2}w_n')$ is zero in $\Omega^1(P'; \mathbb{F}_p[q_1^{\pm 1}])$. $\square$

Proving the part of proposition 5.3.1.2 which is left to section 5.3.4 relies heavily on the ideas used in the previous proof. One may like to look ahead to that proof while the ideas are still fresh.

5.3 The second family of differentials, side towers

5.3.1 Main results

The main results of this section are as follows. The first concerns the q_1^{-1}-BSS and the second gives the corresponding result in the Q-BSS.

Proposition 5.3.1.1. *For $n \geq 1$ and $k \in \mathbb{Z}$ we have the following differential in the q_1^{-1}-BSS.*

$$d_{p^n-1} q_1^{kp^n} \epsilon_n \doteq q_1^{kp^n} \rho_n$$

Proposition 5.3.1.2. *Let $n \geq 1$. Then $q_1^{p^n} \epsilon_n \in H^*(P; q_1^{-1}Q/q_0)$ lifts to an element $H^*(P; Q/q_0)$ which we also denote by $q_1^{p^n} \epsilon_n$. We have the following differential in the Q-BSS.*

$$d_{p^n - p^{[n]}} q_1^{p^n} \epsilon_n \doteq b_{1,n-1}$$

Moreover, if $k \in \mathbb{Z}$ and $k > 1$, $d_{p^n-1} q_1^{kp^n} \epsilon_n$ is defined in the Q-BSS.

5.3.2 Quick proofs

The differentials in the q_1^{-1}-BSS are derivations (lemma 3.6.4) and $d_{p^n-1} q_1^{\pm p^n} = 0$ (lemma 3.5.4). This means that proposition 5.3.1.1 follows quickly from the following sub-proposition.

Proposition 5.3.2.1. *For $n \geq 1$ we have the following differential in the q_1^{-1}-BSS.*

$$d_{p^n-1} q_1^{p^n(p+1)} \epsilon_n \doteq q_1^{p^n(p+1)} \rho_n$$

In this subsection, we prove this proposition assuming the following Kudo transgression theorem.

Recall lemma 5.1.8, which says that $q_1^{p^{n-1}(p+1)} \epsilon_n$ and $q_1^{p^n(p+1)} \rho_n$ have unique lifts to $H^*(P; Q/q_0)$. We denote the lifts by the same name.

Proposition 5.3.2.2 (Kudo transgression)**.** *Suppose $x, y \in H^*(P; Q/q_0)$, x has cohomological degree 0, y has cohomological degree 1, and that $d_r x = y$ in the Q-BSS.*

Then we have $d_{(p-1)r}x^{p-1}y \doteq \langle y\rangle^p$*, where* $\langle y\rangle^p$ *will be defined in the course of the proof.*

Moreover, $\langle q_1^{p^{n-1}(p+1)}\epsilon_n\rangle^p = q_1^{p^n(p+1)}\rho_n$ *in* $H^*(P;Q/q_0)$.

The Kudo transgression theorem is the consuming result of this section. Supposing it for now, we prove proposition 5.3.2.1 and proposition 5.3.1.2, save for the claim about $d_{p^n-p^{[n]}}q_1^{p^n}\epsilon_n$.

Proof of proposition 5.3.2.1. Proposition 5.2.1.2, proposition 5.2.1.3 and lemma 5.1.8 tells us that

$$d_{p^{[n]}}q_1^{p^{n-1}(p+1)} \doteq q_1^{p^{n-1}(p+1)}\epsilon_n$$

in the Q-BSS. By the Kudo transgression theorem we have

$$d_{p^n-1}q_1^{p^n(p+1)}\epsilon_n \doteq q_1^{p^n(p+1)}\rho_n$$

in the Q-BSS and hence (lemma 3.4.1), the q_1^{-1}-BSS. □

Proof of part of proposition 5.3.1.2. By lemma 3.4.1, we can verify the last statement in the q_0^∞-BSS. We have

$$q_0^v q_1^{kp^n}\epsilon_n \in E_1^{1,kp^n-p^{[n]}+v,kqp^n,v}(q_1^\infty\text{-BSS}).$$

Consider the case $v = -p^n$. By proposition 5.3.1.1 and lemma 4.2.2, it is enough to show that $(s,t,u) = (1, kp^n - p^{[n]} - p^n, kqp^n)$ satisfies $u < U(s) + (2p^2-2)(t+2) - q$. The worst case is when $k = 2$ where the inequality is implied by (5.1.9). □

5.3.3 A Kudo transgression theorem

Suppose given a connected commutative Hopf algebra $\mathfrak{P}$, a commutative algebra $\mathfrak{Q}$ in $\mathfrak{P}$-comodules, and suppose that all nontrivial elements of $\mathfrak{P}$ and $\mathfrak{Q}$ have even degree. In order to prove proposition 5.3.2.2, we mimic theorem 3.1 of [9] to define natural operations

$$\beta\tilde{P}^0 : \Omega^0(\mathfrak{P};\mathfrak{Q}) \longrightarrow \Omega^1(\mathfrak{P};\mathfrak{Q}).$$

Once these operations have been defined and we have observed their basic properties the proof of the Kudo transgression proposition follows quickly.

The reader should refer to [10, pg. 75-76] for notation regarding twisting morphisms and twisted tensor products. We write τ for the universal twisting morphism instead of [ ].

The first step towards proving the existence of the operation $\beta\tilde{P}^0$ is to describe a map

$$\Phi : W \otimes \Omega^*(\mathfrak{P};\mathfrak{Q})^{\otimes p} \longrightarrow \Omega^*(\mathfrak{P};\mathfrak{Q}),$$

which acts as the θ appearing in [9, theorem 3.1]. This can be obtained by dualizing the construction in [9, lemma 11.3]. Conveniently, this has already been documented in [5, lemma 2.3].

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathfrak{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \psi_{\mathfrak{Q}}}\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & \mathfrak{P}\otimes\mathfrak{Q} & \xrightarrow{d} & \mathfrak{P}\otimes\overline{\mathfrak{P}}\otimes\mathfrak{Q} & \xrightarrow{d} & \mathfrak{P}\otimes\overline{\mathfrak{P}}\otimes\overline{\mathfrak{P}}\otimes\mathfrak{Q} & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle \epsilon\otimes 1}\ \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & \\
0 & \longrightarrow & \mathfrak{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Consider the diagram above. The top and bottom row are equal to the chain complex consisting of $\mathfrak{Q}$ concentrated in cohomological degree zero and the middle row is the chain complex $\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})$. We have the counit $\epsilon : \mathfrak{P} \longrightarrow \mathbb{F}_p$ and the coaction $\psi_{\mathfrak{Q}} : \mathfrak{Q} \longrightarrow \mathfrak{P} \otimes \mathfrak{Q}$. The definition of a $\mathfrak{P}$-comodule gives $1 - ri = 0$. We also have $1 - ir = dS + Sd$ where S is the contraction defined by

$$S(p_0[p_1|\dots|p_s]q) = \epsilon(p_0)p_1[p_2|\dots|p_s]q.$$

[Note that just for this section q no longer means $2p-2$.]

Let C_p denote the cyclic group of order p and let W be the standard $\mathbb{F}_p[C_p]$-free resolution of $\mathbb{F}_p$ (see [5, definition 2.2]). We are careful to note that the boundary map

in W decreases degree. Following Bruner's account in [5, lemma 2.3], we can extend the multiplication map displayed at the top of the following diagram and construct Φ, a C_p-equivariant map of DG $\mathfrak{P}$-comodules (with $\Phi(W_i \otimes [\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q}))^{\otimes p}]^j) = 0$ if $pi > (p-1)j$).

$$\begin{array}{ccc} \mathfrak{Q}^{\otimes p} & \longrightarrow & \mathfrak{Q} \\ \downarrow{\scriptstyle e_0 \otimes i^{\otimes p}} & & \downarrow{\scriptstyle i} \\ W \otimes (\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q}))^{\otimes p} & \xrightarrow{\Phi} & \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q}) \end{array}$$

Precisely, we make the following definition.

Definition 5.3.3.1.

$$\Phi : W \otimes (\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q}))^{\otimes p} \longrightarrow \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})$$

is the map obtained by applying [5, lemma 2.3] to the following set up:

1. $r = p$, $\rho = \langle (1\ 2\ \cdots\ p) \rangle = C_p$ and $\mathcal{V} = W$;
2. $(R, A) = (\mathbb{F}_p, \mathfrak{P})$, $M = N = \mathfrak{Q}$ and $K = L = \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})$;
3. $f : M^{\otimes r} \longrightarrow N$ is the iterated multiplication $\mathfrak{Q}^{\otimes p} \longrightarrow \mathfrak{Q}$.

Let's recall the construction. Bruner defines

$$\Phi_{i,j} : W_i \otimes [\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})^{\otimes p}]^j \longrightarrow \mathfrak{P} \otimes_\tau \Omega^{j-i}(\mathfrak{P};\mathfrak{Q})$$

inductively. The gradings here are all (co)homological gradings.

As documented in [16, pg. 325, A1.2.15] there is a natural associative multiplication

$$(\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})) \otimes^\Delta (\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})) \longrightarrow \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P};\mathfrak{Q})$$

$$p[p_1|\cdots|p_s]q \cdot p'[p'_1|\cdots p'_t]q' = \sum pp'_{(0)}[p_1p'_{(1)}|\cdots|p_sp'_{(s)}|q_{(1)}p'_1|\cdots|q_{(t)}p'_t]q_{(t+1)}q'. \tag{5.3.3.2}$$

Here, $\sum p'_{(0)} \otimes \cdots \otimes p'_{(s)} \in \mathfrak{P}^{\otimes(s+1)}$ is the s-fold diagonal of $p' \in \mathfrak{P}$ and $\sum q_{(1)} \otimes \cdots \otimes q_{(t+1)} \in \mathfrak{P}^{\otimes t} \otimes \mathfrak{Q}$ is the t-fold diagonal of $q \in \mathfrak{Q}$. Also, $\otimes^{\Delta}$ denotes the internal tensor product in the category of $\mathfrak{P}$-comodules as in [10, pg. 74]; one checks directly that the multiplication above is a $\mathfrak{P}$-comodule map.

Iterating this multiplication gives a map

$$(\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P}; \mathfrak{Q}))^{\otimes p} \longrightarrow \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P}; \mathfrak{Q})$$

which determines $\Phi_{0,*}$.

Suppose we have defined $\Phi_{i',j}$ for $i' < i$. Since $\Phi_{i,j} = 0$ for $j < i$ we may suppose that we have defined $\Phi_{i,j'}$ for $j' < j$. We define $\Phi_{i,j}$ using C_p-equivariance, the adjunction

$$\mathfrak{P}\text{-comodules} \underset{P\otimes(-)}{\overset{\text{forget}}{\rightleftarrows}} \mathbb{F}_p\text{-modules} \qquad\qquad f \longmapsto \tilde{f}$$

and the contracting homotopy

$$T = \sum_{i=1}^{p} (ir)^{i-1} \otimes S \otimes 1^{p-i}.$$

In particular, we define $\widetilde{\Phi_{i,j}}$ on $e_i \otimes x$ by

$$\widetilde{\Phi_{i,j}} = ([d\Phi_{i,j-1}]^{\sim} - [\Phi_{i-1,j-1}(d \otimes 1)]^{\sim})(1 \otimes T).$$

Our choice of Φ is natural in $\mathfrak{P}$ and $\mathfrak{Q}$ because we specified the multiplication determining $\Phi_{0,*}$ and the contracting homotopy T in a natural way.

Φ restricts to a natural C_p-equivariant DG homomorphism

$$\Phi : W \otimes \Omega^*(\mathfrak{P}; \mathfrak{Q})^{\otimes p} \longrightarrow \Omega^*(\mathfrak{P}; \mathfrak{Q}).$$

In the proof of proposition 5.3.2.2 we need the fact that Φ interacts nicely with $\mathfrak{P}$-comodule primitives.

Definition 5.3.3.3. Suppose that $x \in \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P}; \mathfrak{Q})$ and that $q \in \mathfrak{Q}$ is a $\mathfrak{P}$-comodule primitive. We write qx for $x \cdot 1[]q$.

Lemma 5.3.3.4. *Suppose that $q \in \mathfrak{Q}$ is $\mathfrak{P}$-comodule primitive. Then*

$$\Phi(e_i \otimes q^{i_1} x_1 \otimes \cdots \otimes q^{i_p} x_p) = q^{\sum_j i_j} \Phi(e_i \otimes x_1 \otimes \cdots \otimes x_p).$$

Proof. A special case of formula (5.3.3.2) gives

$$p'[p'_1| \cdots |p'_s]q' \cdot 1[]q = p'[p'_1| \cdots |p'_s]q'q.$$

Since $q \in \mathfrak{Q}$ is a $\mathfrak{P}$-comodule primitive we also obtain

$$1[]q \cdot p'[p'_1| \cdots p'_t]q' = p'[p'_1| \cdots |p'_t]q'q;$$

left and right multiplication by $1[]q$ agree. This observation proves the $i = 0$ case of the result since $\Phi_{0,*}(e_0 \otimes - \otimes \ldots \otimes -)$ is equal to the map $(\mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P}; \mathfrak{Q}))^{\otimes p} \longrightarrow \mathfrak{P} \otimes_\tau \Omega^*(\mathfrak{P}; \mathfrak{Q})$. We can now make use of the inductive formula

$$\widetilde{\Phi_{i,j}} = ([d\Phi_{i,j-1}]^\sim - [\Phi_{i-1,j-1}(d \otimes 1)]^\sim)(1 \otimes T).$$

$\psi_{\mathfrak{Q}}$, $\epsilon \otimes 1$, and S commute with multiplication by q and so $1 \otimes T$ commutes with multiplication by $1 \otimes q^{i_1} \otimes \ldots \otimes q^{i_p}$. By an inductive hypothesis we can suppose $\Phi_{i,j-1}$ and $\Phi_{i-1,j-1}$ have the required property. It follows that $d\Phi_{i,j-1}$ and $\Phi_{i-1,j-1}(d \otimes 1)$ have the required property. The same is true of their adjoints and so the result holds for the adjoint of $\Phi_{i,j}$ and thus for $\Phi_{i,j}$ itself. □

We finally define $\beta\tilde{P}^0 : \Omega^0(\mathfrak{P}; \mathfrak{Q}) \longrightarrow \Omega^1(\mathfrak{P}; \mathfrak{Q})$ and note a couple of its properties. One should read the proof of [9, theorem 3.1]; this definition mimics that of $\beta P_0 : K_0 \to K_{-1}$. In particular, we take $q = s = 0$ and the reader will note that we omit a $\nu(-1)$ in our definition.

Definition 5.3.3.5. Let $a \in \Omega^0(\mathfrak{P}; \mathfrak{Q})$. We define $\beta\tilde{P}^0 a \in \Omega^1(\mathfrak{P}; \mathfrak{Q})$ as follows.

1. Let $b = da \in \Omega^1(\mathfrak{P}; \mathfrak{Q})$.

2. We define $t_k \in \Omega^*(\mathfrak{P}; \mathfrak{Q})^{\otimes p}$ for $0 < k < p$.

   In the following two formulae juxtaposition denotes tensor product.

   Write $p = 2m + 1$ and define for $0 < k \leq m$

$$t_{2k} = (k-1)! \sum_I b^{i_1} a^2 b^{i_2} a^2 \cdots b^{i_k} a^2$$

   summed over all k-tuples $I = (i_1, \ldots, i_k)$ such that $\sum_j i_j = p - 2k$.

   Define for $0 \leq k < m$

$$t_{2k+1} = k! \sum_I b^{i_1} a^2 \cdots b^{i_k} a^2 b^{i_{k+1}} a$$

   summed over all $(k+1)$-tuples $I = (i_1, \ldots, i_{k+1})$ such that $\sum_j i_j = p - 2k - 1$.

3. Define $c \in W \otimes \Omega^*(\mathfrak{P}; \mathfrak{Q})^{\otimes p}$ by

$$c = \sum_{k=1}^{m} (-1)^k \left[e_{p-2k-1} \otimes t_{2k} - e_{p-2k} \otimes t_{2k-1} \right],$$

   so $dc = -e_{p-2} \otimes b^p$ [9, 3.1(8)].

4. $\beta \tilde{P}^0 a$ is defined to be Φc.

Naturality of $\beta \tilde{P}^0$ follows from the naturality of Φ. Using the observation made in part (3) of the definition we immediately obtain the following lemma.

Lemma 5.3.3.6. *Let $a \in \Omega^0(\mathfrak{P}; \mathfrak{Q})$. Then $d(\beta \tilde{P}^0 a) = -\Phi(e_{p-2} \otimes (da)^p)$.*

Moreover, we make the following definition.

Definition 5.3.3.7. Given $b \in \Omega^1(\mathfrak{P}; \mathfrak{Q})$, we define $\langle b \rangle^p$ to be the element

$$\Phi(e_{p-2} \otimes b^p) \in \Omega^2(\mathfrak{P}; \mathfrak{Q}).$$

If $y \in H^1(\mathfrak{P};\mathfrak{Q})$ is represented by b, then $\langle y\rangle^p \in H^2(\mathfrak{P};\mathfrak{Q})$ is defined to be the class of $\langle b\rangle^p$.

The fact that $\langle y\rangle^p$ is well-defined is used in [9, definition 2.2].

We are now ready to prove the first statement in the proposition.

Proof of the first part of proposition 5.3.2.2. By lemma 3.2.4 there exists a and b in $\Omega^*(P;Q)$ with $da = q_0^r b$ such that their images $\overline{a}$ and $\overline{b}$ in $\Omega^*(P;Q/q_0)$ are cocycles representing x and y, respectively.

Consider $\beta\tilde{P}^0 a$. To get a grasp on what this element looks like we need to go back to definition 5.3.3.5. Since $da = q_0^r b$ we should stare at the definition but replace b by $q_0^r b$. We note that the sum defining c involves $t_1, \ldots, t_{2m}$. t_{2m} is given by

$$(m-1)! \sum_{i=0}^{m-1} a^{2i}(q_0^r b)a^{2m-2i}.$$

There are only single $(q_0^r b)$'s in each term, whereas the terms in the sums defining $t_1, \ldots, t_{2m-1}$ all involve at least two $(q_0^r b)$'s. By lemma 5.3.3.4, $\beta\tilde{P}^0 a$ is divisible by q_0^r and the image of $A = (\beta\tilde{P}^0 a)/q_0^r$ in $\Omega^1(P;Q/q_0)$ is a unit multiple of the image of $\Phi(e_0 \otimes t_{2m})/q_0^r$ in $\Omega^1(P;Q/q_0)$. This latter image is equal to

$$\overline{A} = (m-1)! \sum_{i=0}^{m-1} \overline{a}^{2i}\ \overline{b}\ \overline{a}^{2m-2i},$$

where juxtaposition now denotes multiplication.

On the other hand, lemma 5.3.3.6, lemma 5.3.3.4 and definition 5.3.3.7 give

$$d(\beta\tilde{P}^0 a) \doteq \Phi(e_{p-2} \otimes (q_0^r b)^p) = q_0^{pr}\Phi(e_{p-2} \otimes b^p) = q_0^{pr}\langle b\rangle^p.$$

Letting $B = d(\beta\tilde{P}^0 a)/q_0^{pr}$ gives $dA = q_0^{(p-1)r}B$, and the image $\overline{B}$ of B in $\Omega^2(P;Q/q_0)$ is a unit multiple of $\langle\overline{b}\rangle^p$, which represents $\langle y\rangle^p$.

The formula for $\overline{A}$ above, shows that it represents a unit multiple of $x^{p-1}y$ and so we deduce from lemma 3.2.4 that $d_{(p-1)r}x^{p-1}y \doteq \langle y\rangle^p$. □

To complete the proof of proposition 5.3.2.2 we need the following lemma.

Lemma 5.3.3.8. *Let $\mathfrak{P}$ be the primitively generated Hopf algebra $\mathbb{F}_p[\xi]/(\xi^p)$ where the degree of ξ is even. Let h and b be classes in $H^*(\mathfrak{P};\mathbb{F}_p)$ which are represented in $\Omega^*\mathfrak{P}$ by $[\xi]$ and*

$$\sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j}[\xi^j|\xi^{p-j}],$$

respectively. Then $\langle h\rangle^p \doteq b$.

Proof. This follows from remarks 6.9 and 11.11 of [9]. Beware of the different use of notation: our $\langle y\rangle^p$ is May's $\beta\tilde{P}^0 y$ and May defines $\langle y\rangle^p$ using the $\cup_1$-product associated to $\Omega^*\mathfrak{P}$. □

Finishing the proof of proposition 5.3.2.2. The previous lemma gives $\langle h_{n,0}\rangle^p \doteq b_{n,0}$ in $H^*(\mathbb{F}_p[\xi_n]/(\xi_n^p);\mathbb{F}_p)$. Since q_1 is primitive, definition 5.3.3.7 and lemma 5.3.3.4 show that

$$\langle q_1^{p^{n-1}(p+1)}\epsilon_n\rangle^p = \langle q_1^{p^n-p^{[n-1]}}h_{n,0}\rangle^p \doteq q_1^{p^{n+1}-p\cdot p^{[n-1]}}b_{n,0} = q_1^{p^n(p+1)}\rho_n$$

in $H^*(\mathbb{F}_p[\xi_n]/(\xi_n^p);\mathbb{F}_p[q_1^{\pm 1}])$. We use naturality to transfer the required identity from $H^*(\mathbb{F}_p[\xi_n]/(\xi_n^p);\mathbb{F}_p[q_1^{\pm 1}])$ to $H^*(P;Q/q_0)$. We have homomorphisms

$$H^*(\mathbb{F}_p[\xi_n]/(\xi_n^p);\mathbb{F}_p[q_1^{\pm 1}]) \longrightarrow H^*(P';\mathbb{F}_p[q_1^{\pm 1}]) \longleftarrow H^*(P;q_1^{-1}Q/q_0) \longleftarrow H^*(P;Q/q_0).$$

The first is induced by the inclusion $\mathbb{F}_p[\xi_n]/(\xi_n^p) \longrightarrow P'$. Theorem 5.1.3 tells us that the second is an isomorphism. Lemma 5.1.8 says that $q_1^{p^{n-1}(p+1)}\epsilon_n$ and $q_1^{p^n(p+1)}\rho_n$ have unique lifts to $H^*(P;Q/q_0)$. This completes the proof. □

5.3.4 Completing the proof of proposition 5.3.1.2

We are left to show that $d_{p^n-p^{[n]}}q_1^{p^n}\epsilon_n \doteq b_{1,n-1}$ for $n \geq 1$. The $n = 1$ case

$$d_{p-1}q_1^{p-1}h_{1,0} \doteq b_{1,0}$$

is given by proposition 5.3.1.1 and lemma 5.1.8 or by noting the following formula in $\Omega^*(P;Q)$ and using lemma 3.2.4.

$$d\left[\sum_{j=1}^{p-1}\frac{(-1)^j}{j}[\xi_1^j]q_0^{j-1}q_1^{p-j}\right]=\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}[\xi_1^j|\xi_1^{p-j}]q_0^{p-1}$$

Suppose that for some $n\geq 1$ we have $a_n\in\Omega^1(P;Q)$ and $b_n\in\Omega^2(P;Q)$, such that

1. a_n maps to $(-1)^n[\xi_n]q_1^{p^n-p^{[n]}}$ in $\Omega^1(P';\mathbb{F}_p[q_1])$;
2. b_n maps to $\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}[\xi_1^{jp^{n-1}}|\xi_1^{(p-j)p^{n-1}}]$ in $\Omega^2(P;Q/q_0)$;
3. $da_n=q_0^{p^n-p^{[n]}}b_n$.

$\tilde{P}^0a_n$ lies in the injectivity range of proposition 4.2.1 and so using theorem 5.1.3 together with the diagram below we see that $\Omega^*(P;Q/q_0)\longrightarrow\Omega^*(P';\mathbb{F}_p[q_1])$ induces an injection on homology in this tridegree.

$$\begin{array}{ccc} H^*(P;Q/q_0) & \longrightarrow & H^*(P;q_1^{-1}Q/q_0) \\ \downarrow & & \downarrow\cong \\ H^*(P';\mathbb{F}_p[q_1]) & \longrightarrow & H^*(P';\mathbb{F}_p[q_1^{\pm 1}]) \end{array}$$

We note that $\tilde{P}^0a_n$ maps to zero in $\Omega^1(P';\mathbb{F}_p[q_1])$, and so, because $\Omega^*(P;Q)\longrightarrow\Omega^*(P;Q/q_0)$ is surjective, we can find a $w_n\in\Omega^0(P;Q)$ such that $dw_n=\tilde{P}^0a_n$ in $\Omega^1(P;Q/q_0)$. In particular, $\tilde{P}^0a_n-dw_n$ is divisible by q_0. Let

$$a_{n+1}=\frac{\tilde{P}^0a_n-dw_n}{q_0}.$$

We claim that a_{n+1} and $b_{n+1}=\tilde{P}^0b_n\in\Omega^*(P;Q)$ satisfy the following conditions.

1. a_{n+1} maps to $(-1)^{n+1}[\xi_{n+1}]q_1^{p^{n+1}-p^{[n+1]}}$ in $\Omega^1(P';\mathbb{F}_p[q_1])$;
2. b_{n+1} maps to $\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}[\xi_1^{jp^n}|\xi_1^{(p-j)p^n}]$ in $\Omega^2(P;Q/q_0)$;
3. $da_{n+1}=q_0^{p^{n+1}-p^{[n+1]}}b_{n+1}$.

The second condition is clear. To see the last condition, note that $da_n = q_0^{p^n - p^{[n]}} b_n$ implies $d\tilde{P}^0 a_n = q_0^{p^{n+1} - p \cdot p^{[n]}} \tilde{P}^0 b_n = q_0^{p^{n+1} - p^{[n+1]} + 1} b_{n+1}$, and so

$$da_{n+1} = d\left(\frac{\tilde{P}^0 a_n - dw_n}{q_0} \right) = \frac{d\tilde{P}^0 a_n}{q_0} = q_0^{p^{n+1} - p^{[n+1]}} b_{n+1}.$$

For the first condition, we note that $\tilde{P}^0 a_n$ will not contribute to the image of a_{n+1} in $\Omega^1(P'; \mathbb{F}_p[q_1])$. Moreover, since

$$dw_n = \tilde{P}^0 a_n = (-1)^n [\xi_n^p] q_1^{p^{n+1} - p \cdot p^{[n]}}$$

in $\overline{P''} \otimes \mathbb{F}_p[q_1]$, we see, as in the proof of proposition 5.2.3.1, that the only relevant term of w_n is $(-1)^n q_1^{p^{n+1} - p^{[n+1]}} q_{n+1}$, and that it contributes $(-1)^{n+1} [\xi_{n+1}] q_1^{p^{n+1} - p^{[n+1]}}$ to $-q_0^{-1} dw_n$.

The proof is complete by induction and lemma 3.2.4.

5.4 The E_∞-page of the q_1^{-1}-BSS

In this subsection we obtain all the nontrivial differentials in the q_1^{-1}-BSS. The main result is simple to prove as long as one has the correct picture in mind; otherwise, the proof may seem rather opaque. Figure 5-1 on page 77 displays some of Christian Nassau's chart [14] for $H^*(A)$ when $p = 3$. His chart tells us about the object we are trying to calculate in a range by proposition 4.2.4 and the facts that

$$H^*(P; Q/q_0^\infty) / \Big[\mathbb{F}_p [q_0]/q_\infty \Big] = H^*(P; Q) / \Big[\mathbb{F}_p [q_0] \Big]$$

and $H^*(P; Q) = H^*(A)$. A q_0-tower corresponds to a differential in the Q-BSS. Labels at the top of towers are the sources of the corresponding Bockstein differentials; labels at the bottom of towers are the targets of the corresponding Bockstein differentials. We note that the part of figure 5-1 in gray is not displayed in Nassau's charts and is deduced from the results of this chapter.

Recall from corollary 5.1.7 that $H^*(P; q_1^{-1} Q/q_0)$ is an exterior algebra tensored

with a polynomial algebra, and so we have a convenient $\mathbb{F}_p$-basis for it given by monomials in q_1, the ϵ_n's and the ρ_n's. We introduce the following notation.

Notation 5.4.1. Suppose given $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$, $K = (k_1, \ldots, k_s)$ such that $i_1 > \ldots > i_r \geq 1$, $j_1 > \ldots > j_s \geq 1$, and $k_a \geq 0$ for $a \in \{1, \ldots, s\}$. We write

1. $\epsilon[I]\rho[J, K]$ for the monomial $\epsilon_{i_1} \cdots \epsilon_{i_r} \rho_{j_1}^{k_1} \cdots \rho_{j_s}^{k_s}$;
2. $n[I]$ for $\sum_c p^{i_c - 1}$;
3. I_- for $(i_1, \ldots, i_{r-1})$ if $r \geq 1$;
4. K_- for $(k_1, \ldots, k_s - 1)$ if $s \geq 1$ and $k_s \geq 1$.

Notice that the indexing of a monomial in the ϵ_i's and ρ_j's by I, J and K is unique once we impose the additional condition that $k_a \geq 1$ for each $a \in \{1, \ldots, s\}$. Moreover, $\left\{ q_1^N \epsilon[I]\rho[J, K] \right\}$ gives a basis for $H^*(P; q_1^{-1}Q/q_0)$.

We have the following corollary to proposition 5.2.1.2 and proposition 5.3.1.1 and we shall see that it completely describes all the nontrivial differentials in the q_1^{-1}-BSS.

Corollary 5.4.2. *Suppose given $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$, $K = (k_1, \ldots, k_s)$ such that $i_1 > \ldots > i_r \geq 1$, $j_1 > \ldots > j_s \geq 1$, and $k_a \geq 1$ for $a \in \{1, \ldots, s\}$.*

Suppose $r \geq 1$, that either $s = 0$, or $s \geq 1$ and $i_r \leq j_s$, and that $k \in \mathbb{Z} - p\mathbb{Z}$. Then we have the following differential in the q_1^{-1}-BSS.

$$d_{p^{[i_r]}} \left[q_1^{kp^{i_r - 1}} \epsilon[I_-]\rho[J, K] \right] \doteq q_1^{kp^{i_r - 1}} \epsilon[I]\rho[J, K] \tag{5.4.3}$$

Suppose $s \geq 1$, that either $r = 0$, or $r \geq 1$ and $i_r > j_s$, and that $k \in \mathbb{Z}$. Then we have the following differential in the q_1^{-1}-BSS.

$$d_{p^{j_s} - 1} \left[q_1^{kp^{j_s}} \epsilon[I]\epsilon_{j_s}\rho[J, K_-] \right] \doteq q_1^{kp^{j_s}} \epsilon[I]\rho[J, K] \tag{5.4.4}$$

Proof. By proposition 5.2.2.1, proposition 5.3.1.1, lemma 2.1.6 and lemma 3.6.4 we see that $q_1^{n[I]} \epsilon[I]\rho[J, K]$ is a permanent cycle. In the first case lemma 3.5.4

gives $d_{p^{[i_r]}} q_1^{-n[I_-]} = 0$ and so the differential $d_{p^{[i_r]}} q_1^{kp^{i_r-1}} \doteq q_1^{kp^{i_r-1}} \epsilon_{i_r}$ completes the proof. In the second case lemma 3.5.4 gives $d_{p^{j_s}-1} q_1^{-n[I]} = 0$ and so the differential $d_{p^{j_s}-1} q_1^{kp^{j_s}} \epsilon_{j_s} \doteq q_1^{kp^{j_s}} \rho_{j_s}$ completes the proof. □

The content of the next proposition is that the previous corollary describes all of the nontrivial differentials in the q_1^{-1}-BSS.

Proposition 5.4.5. *The union*

$$\{1\} \cup \{x : x \text{ is a source of one of the differentials in corollary 5.4.2}\}$$
$$\cup \{y : y \text{ is a target of one of the differentials in corollary 5.4.2}\}$$

is a basis for $H^*(P; q_1^{-1}Q/q_0)$. *Moreover, the sources and targets of the differentials in corollary 5.4.2 are distinct and never equal to* 1.

Proof. We note that for any $N \neq 0$, q_1^N is the source of a differential like the one in (5.4.3).

Take I, J and K as in (5.4.3). We wish to show that $q_1^N \epsilon[I]\rho[J,K]$ is the source or target of one of the differentials in corollary 5.4.2. There are three cases (the second case is empty if $i_r = 1$):

1. $N = kp^{i_r-1}$ for some $k \in \mathbb{Z} - p\mathbb{Z}$.
2. $N = kp^{i_{r+1}-1}$ for some $k \in \mathbb{Z} - p\mathbb{Z}$ and some $i_{r+1} \geq 1$ with $i_r > i_{r+1}$.
3. $N = kp^{i_r}$ for some $k \in \mathbb{Z}$.

In the first case, $q_1^N \epsilon[I]\rho[J,K]$ is the target of the differential (5.4.3). In the second case, $q_1^N \epsilon[I]\rho[J,K]$ is the source of a differential like the one in (5.4.3). In the third case, $q_1^N \epsilon[I]\rho[J,K]$ is the source of a differential like the one in (5.4.4).

These cases are highlighted in figure 5-1 when $p = 3$, $I = (3)$, and J and K are empty. The three cases are:

1. $N = 9k$ for some $k \in \mathbb{Z} - 3\mathbb{Z}$.
2. $N = 3^{i-1}k$ for some $k \in \mathbb{Z} - 3\mathbb{Z}$ and some i with $1 \leq i < 3$.

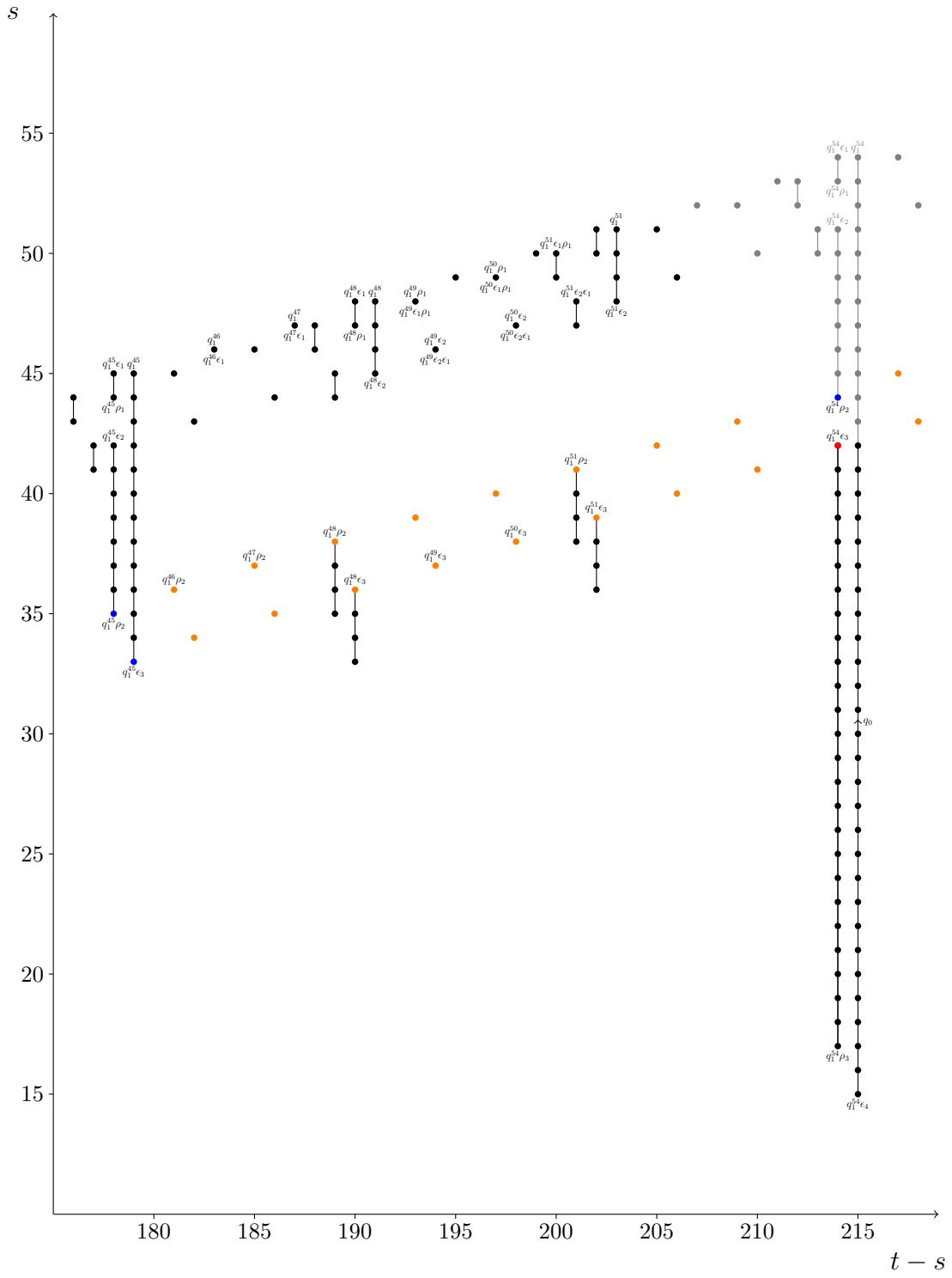


Figure 5-1: The relevant part of $H^{s,t}(A)$ when $p = 3$, in the range $175 < t - s < 219$. Vertical black lines indicate multiplication by q_0. The top and/or bottom of selected q_0-towers are labelled by the source and/or target, respectively, of the corresponding Bockstein differential.

3. $N = 27k$ for some $k \in \mathbb{Z}$.

The first case is highlighted in blue when $k = 5$; the second case is highlighted in orange and we see both the cases $i = 1$ and $i = 2$ occurring; the last case is highlighted in red when $k = 2$.

Take I, J and K as in (5.4.4). We wish to show that $q_1^N \epsilon[I]\rho[J, K]$ is the source or target of one of the differentials in corollary 5.4.2. There are two cases:

1. $N = kp^{j_s}$ for some $k \in \mathbb{Z}$.

2. $N = kp^{i_{r+1}-1}$ for some $k \in \mathbb{Z} - p\mathbb{Z}$ and some $i_{r+1} \geq 1$ with $i_{r+1} \leq j_s$.

In the first case, $q_1^N \epsilon[I]\rho[J, K]$ is the target of the differential (5.4.4). In the second case, $q_1^N \epsilon[I]\rho[J, K]$ is the source of a differential like the one in (5.4.3).

These cases are highlighted in figure 5-1 when $p = 3$, I is empty, $J = (2)$ and $K = (1)$. The two cases are:

1. $N = 9k$ for some $k \in \mathbb{Z}$.

2. $N = 3^{i-1}k$ for some $k \in \mathbb{Z} - 3\mathbb{Z}$ and some i with $1 \leq i \leq 2$.

The first case is highlighted in blue when $k = 5$ and $k = 6$; the second case is highlighted in orange and we see both the cases $i = 1$ and $i = 2$ occurring.

Since the empty sequences I, J and K together with those satisfying the conditions in (5.4.3) or (5.4.4) make up all choices of I, J and K, and since $\left\{q_1^N \epsilon[I]\rho[J, K]\right\}$ gives a basis for $H^*(P; q_1^{-1}Q/q_0)$ (corollary 5.1.7), we have proved the first claim.

Careful inspection of the previous argument shows that this also proves the second claim. □

This proposition allows us to determine an $\mathbb{F}_p$-basis of $E_\infty(q_1^{-1}\text{-BSS})$. We use the following lemma.

Lemma 5.4.6. *Suppose we have an indexing set A and an $\mathbb{F}_p$-basis*

$$\{1\} \cup \{x_\alpha\}_{\alpha \in A} \cup \{y_\alpha\}_{\alpha \in A}$$

of $H^*(P; q_1^{-1}Q/q_0)$ *such that each* x_α *supports a differential* $d_{r_\alpha}x_\alpha = y_\alpha$. *Then we have an* $\mathbb{F}_p$*-basis of* $E_\infty(q_1^{-1}$*-BSS) given by the classes of*

$$\Big\{q_0^v : v < 0\Big\} \cup \Big\{q_0^v x_\alpha : \alpha \in A,\ -r_\alpha \leq v < 0\Big\}.$$

In the above statement, we intend for 1, *the* x_α*'s and the* y_α*'s to be distinct as in proposition 5.4.5.*

Proof. Let $v < 0$. We see make some observations.

1. $E_1^{*,*,*,v} \cap \bigcup_{s<r} \operatorname{im} d_s$ has basis $\{q_0^v y_\alpha : \alpha \in A,\ r_\alpha < r\}$.
2. $\{q_0^v y_\alpha : \alpha \in A,\ r_\alpha = r\}$ is independent in $E_1^{*,*,*,v} / \left(E_1^{*,*,*,v} \cap \bigcup_{s<r} \operatorname{im} d_s\right)$.
3. $E_1^{*,*,*,v} \cap \bigcap_{s<r} \ker d_s$ has basis

$$\Big\{q_0^v\Big\} \cup \Big\{q_0^v x_\alpha : \alpha \in A,\ r_\alpha \geq \min\{r, -v\}\Big\} \cup \Big\{q_0^v y_\alpha : \alpha \in A\Big\}.$$

4. $E_\infty^{*,*,*,v} = \left(E_1^{*,*,*,v} \cap \bigcap_s \ker d_s\right) / \left(E_1^{*,*,*,v} \cap \bigcup_s \operatorname{im} d_s\right)$ has basis

$$\Big\{q_0^v\Big\} \cup \Big\{q_0^v x_\alpha : \alpha \in A,\ r_\alpha \geq -v\Big\}.$$

We see that q_0^v is a basis element for $E_\infty^{*,*,*,v}$ for all $v < 0$ and that $q_0^v x_\alpha$ is a basis element for $E_\infty^{*,*,*,v}$ as long as $-r_\alpha \leq v < 0$. This completes the proof. □

We state the relevant corollary, a description of the E_∞-page in the next section. Of course, this allows us to find a basis of $H^*(P; q_1^{-1}Q/q_0^\infty)$ if we wish.

5.5 Summary of main results

We have completely calculated the q_1^{-1}-BSS.

Theorem 5.5.1. *In the* q_1^{-1}*-BSS we have two families of differentials. For* $n \geq 1$,

1. $d_{p^{[n]}} q_1^{kp^{n-1}} \doteq q_1^{kp^{n-1}} \epsilon_n$, *whenever* $k \in \mathbb{Z} - p\mathbb{Z}$;

2. $d_{p^n-1}q_1^{kp^n}\epsilon_n \doteq q_1^{kp^n}\rho_n$, *whenever* $k \in \mathbb{Z}$.

Together with the fact that $d_r 1 = 0$ *for* $r \geq 1$, *these two families of differentials determine the* q_1^{-1}*-BSS.*

Corollary 5.5.2. *$E_\infty(q_1^{-1}$-BSS) has an $\mathbb{F}_p$-basis given by the classes of the following elements.*

$$\begin{aligned}
&\left\{ \; q_0^v \;:\; v < 0 \; \right\} \\
\cup &\left\{ \; q_0^v q_1^{kp^{i_r-1}}\epsilon[I_-]\rho[J,K] \;:\; I,J,K,k \text{ satisfy (5.4.3)}, \; -p^{[i_r]} \leq v < 0 \; \right\} \\
\cup &\left\{ \; q_0^v q_1^{kp^{j_s}}\epsilon[I]\epsilon_{j_s}\rho[J,K_-] \;:\; I,J,K,k \text{ satisfy (5.4.4)}, \; 1-p^{j_s} \leq v < 0 \; \right\}
\end{aligned}$$

We have also obtained useful information about the Q-BSS.

Lemma 5.5.3. *The elements*

$$1, \; q_1^{2p^{n-1}}\epsilon_n, \; q_1^{2p^n}\rho_n \in H^*(P; q_1^{-1}Q/q_0)$$

have unique lifts to $H^*(P; Q/q_0)$. *The same is true after multiplying by* q_1^n *as long as* $n \geq 0$.

We give the lifts the same name.

Theorem 5.5.4. *Let* $n \geq 1$. *We have the following differentials in the Q-BSS.*

1. $d_{p^n-1}q_1^{p^{n-1}} \doteq h_{1,n-1}$;

2. $d_{p^{[n]}}q_1^{kp^{n-1}} \doteq q_1^{kp^{n-1}}\epsilon_n$, *whenever* $k \in \mathbb{Z} - p\mathbb{Z}$ *and* $k > 1$;

3. $d_{p^n-p^{[n]}}q_1^{p^n}\epsilon_n \doteq b_{1,n-1}$;

4. $d_{p^n-1}q_1^{kp^n}\epsilon_n \doteq q_1^{kp^n}\rho_n$, *whenever* $k \in \mathbb{Z}$ *and* $k > 1$.

Chapter 6

The localized algebraic Novikov spectral sequence

In this chapter we calculate the localized algebraic Novikov spectral sequence

$$H^*(P; q_1^{-1}Q/q_0^\infty) \stackrel{t}{\Longrightarrow} H^*(BP_*BP; v_1^{-1}BP_*/p^\infty).$$

6.1 Algebraic Novikov spectral sequences

Recall that the coefficient ring of the *Brown-Peterson spectrum* BP is the polynomial algebra $\mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$ on the Hazewinkel generators. Moreover, $BP_*BP = BP_*[t_1, t_2, t_3, \ldots]$ together with BP_* defines a *Hopf algebroid* [13, §2].

BP_* admits a filtration by *invariant ideals*, powers of $I = \ker(BP_* \longrightarrow \mathbb{F}_p)$, and we have $Q = \mathrm{gr}^* BP_*$. Moreover, this allows us to filter the *cobar construction* $\Omega^*(BP_*BP)$ by setting $F^t\Omega^s(BP_*BP) = I^t\Omega^s(BP_*BP)$, and we have

$$\mathrm{gr}^t\Omega^s(BP_*BP) = \Omega^s(P; Q^t).$$

In this way we obtain the algebraic Novikov spectral sequence

$$E_1^{s,t,u}(\text{alg.NSS}) = H^{s,u}(P; Q^t) \stackrel{t}{\Longrightarrow} H^{s,u}(BP_*BP);$$

d_r has degree $(1, r, 0)$. This makes sense of the terminology "Novikov weight."

One motivation for using the algebraic Novikov spectral is to make comparisons with the Adams spectral sequence, and so we reindex it:

$$E_2^{s,t,u}(\text{alg.NSS}) = H^{s,u}(P; Q^t) \stackrel{t}{\Longrightarrow} H^{s,u}(BP_*BP)$$

and the degree of d_r is $(1, r-1, 0)$.

$p \in BP_*$ is a BP_*BP-comodule primitive and so BP_*/p^n and $p^{-1}BP_*$ are BP_*BP-comodules; define BP_*/p^∞ by the following exact sequence of BP_*BP-comodules.

$$0 \longrightarrow BP_* \longrightarrow p^{-1}BP_* \longrightarrow BP_*/p^\infty \longrightarrow 0$$

We find that $v_1^{p^{n-1}} \in BP_*/p^n$ is a BP_*BP-comodule primitive and so we may define BP_*BP-comodules $v_1^{-1}BP_*/p^n$ and $v_1^{-1}BP_*/p^\infty$ by mimicking the constructions in section 3.1.

By letting $F^t\Omega^s(BP_*BP; v_1^{-1}BP_*/p^\infty) = I^t\Omega^s(BP_*BP; v_1^{-1}BP_*/p^\infty)$ and reindexing, as above, we obtain the *localized algebraic Novikov spectral sequence (loc.alg.NSS)*

$$E_2^{s,t,u}(\text{loc.alg.NSS}) = H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t) \stackrel{t}{\Longrightarrow} H^{s,u}(BP_*BP; v_1^{-1}BP_*/p^\infty).$$

It has a pairing with the unlocalized algebraic Novikov spectral sequence converging to the $H^*(BP_*BP)$-module structure map of $H^*(BP_*BP; v_1^{-1}BP_*/p^\infty)$. Moreover, it receives a map from the *v_1-algebraic Novikov spectral sequence (v_1-alg.NSS)*

$$E_2^{s,t,u}(v_1\text{-alg.NSS}) = H^{s,u}(P; [q_1^{-1}Q/q_0]^t) \stackrel{t}{\Longrightarrow} H^{s,u}(BP_*BP; v_1^{-1}BP_*/p).$$

6.2 Evidence for the main result

In the introduction, we discussed "principal towers" and their "side towers" but said little about the other elements in $H^*(P; q_1^{-1}Q/q_0^\infty)$. Figure 6-1 is obtained from figure 5-1 by removing principal towers and their side towers. We see that the remaining

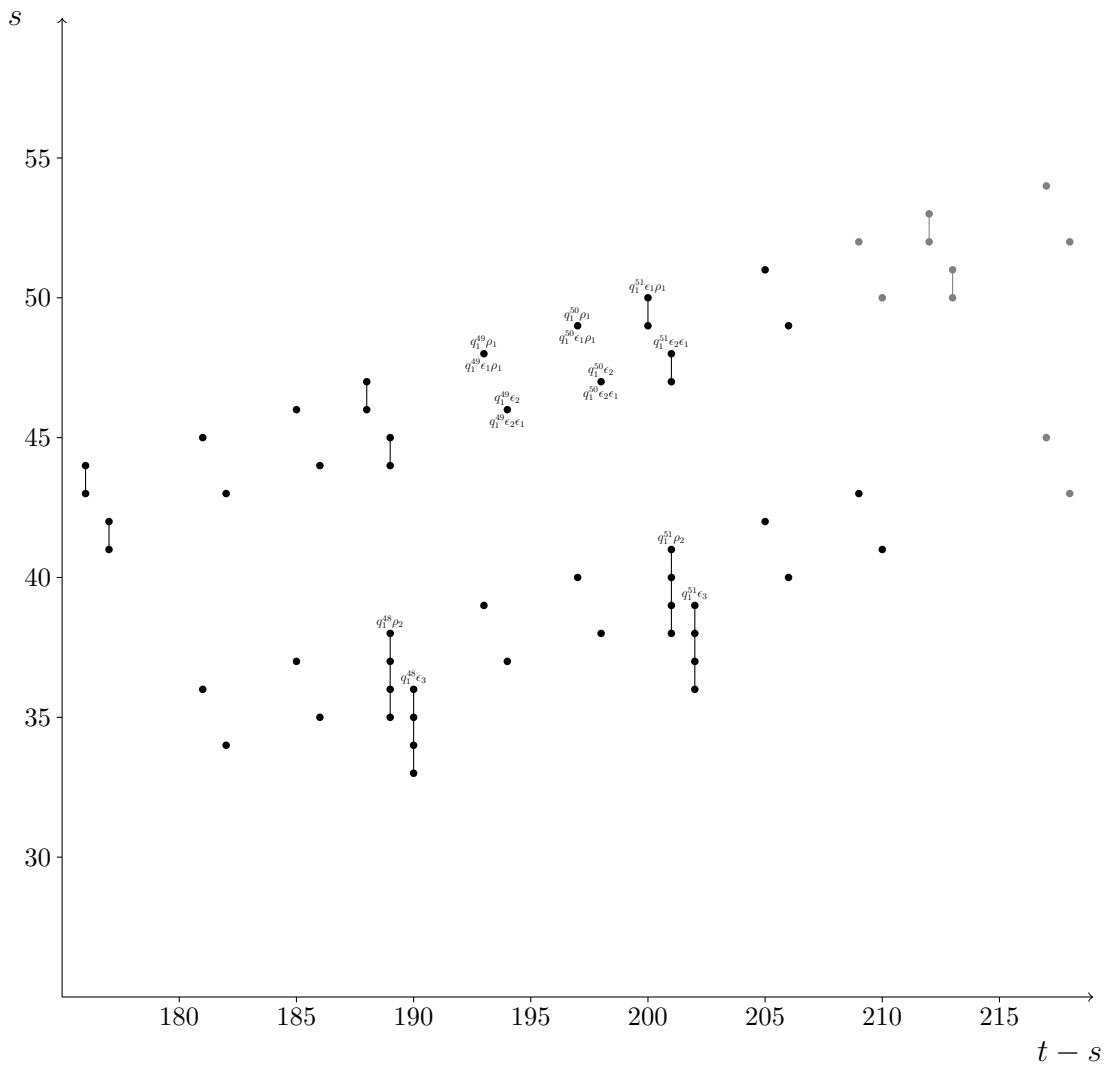


Figure 6-1: A part of $H^{s,t}(A)$ when $p = 3$, in the range $175 < t - s < 219$. Vertical black lines indicate multiplication by q_0. The top and/or bottom of selected q_0-towers are labelled by the source and/or target, respectively, of the corresponding Bockstein differential.

q_0-towers come in pairs, arranged perfectly so that there is a chance that they form an acyclic complex with respect to d_2. Moreover, the labelling at the top of the towers obeys a nice pattern with respect to this arrangement. The pattern of differentials we hope for can be described by the following equations.

$$q_1^{48}\epsilon_3 \longmapsto q_1^{48}\rho_2,\ q_1^{51}\epsilon_3 \longmapsto q_1^{51}\rho_2,\ q_1^{49}\epsilon_2 \longmapsto q_1^{49}\rho_1,\ q_1^{50}\epsilon_2 \longmapsto q_1^{50}\rho_1,\ q_1^{51}\epsilon_2\epsilon_1 \longmapsto q_1^{51}\epsilon_1\rho_1.$$

In each case, this comes from replacing an ϵ_{n+1} by ρ_n, which resembles a theorem of Miller.

Theorem 6.2.1 (Miller, [11, 9.19])**.** *In the v_1-alg.NSS*

$$H^*(P; q_1^{-1}Q/q_0) \overset{t}{\Longrightarrow} H^*(BP_*BP; v_1^{-1}BP_*/p)$$

we have, for $n \geq 1$, $d_2\epsilon_{n+1} \doteq \rho_n$.

This is precisely the theorem enabling the calculation of this chapter, which shows that the d_2 differentials discussed above do occur in the loc.alg.NSS.

6.3 The filtration spectral sequence (q_0-FILT)

Corollary 5.5.2 describes the associated graded of the E_2-page of the loc.alg.NSS with respect to the Bockstein filtration. Since

$$d_2^{\text{loc.alg.NSS}} : H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t) \longrightarrow H^{s+1,u}(P; [q_1^{-1}Q/q_0^\infty]^{t+1})$$

respects the Bockstein filtration, we have a *filtration spectral sequence (q_0-FILT)*

$$E_0^{s,t,u,v}(q_0\text{-FILT}) = E_\infty^{s,t,u,v}(q_1^{-1}\text{-BSS}) \overset{v}{\Longrightarrow} E_3^{s,t,u}(\text{loc.alg.NSS}).$$

The main result of this section is a calculation of the E_1-page of this spectral sequence. Recall corollary 5.5.2.

Theorem 6.3.1. *$E_1(q_0$-FILT) has an $\mathbb{F}_p$-basis given by the following elements.*

$$\left\{q_0^v : v<0\right\} \cup \left\{q_0^v q_1^{kp^{n-1}} : \ n\geq 1,\ k\in\mathbb{Z}-p\mathbb{Z},\ -p^{[n]}\leq v<0\right\}$$
$$\cup \left\{q_0^v q_1^{kp^n}\epsilon_n : \ n\geq 1,\ k\in\mathbb{Z},\ 1-p^n\leq v<0\right\}$$

We prove the theorem via the following proposition.

Proposition 6.3.2. *Fix, $i,j\geq 1$.*

$$d_0^{q_0\text{-}FILT} : E_\infty^{s,t,u,v}(q_1^{-1}\text{-}BSS) \longrightarrow E_\infty^{s+1,t+1,u,v}(q_1^{-1}\text{-}BSS)$$

restricts to an operation on the subspaces with bases given by the classes of the elements

$$\left\{\ q_0^v\ :\ v<0\ \right\},$$

$$\left\{\ q_0^v q_1^{kp^{i_r-1}}\epsilon[I_-]\rho[J,K]\ :\ I,J,K,k\ \textit{satisfy}\ (5.4.3),\ i_r=i,\ -p^{[i]}\leq v<0\ \right\},$$

and

$$\left\{\ q_0^v q_1^{kp^{j_s}}\epsilon[I]\epsilon_{j_s}\rho[J,K_-]\ :\ I,J,K,k\ \textit{satisfy}\ (5.4.4),\ j_s=j,\ 1-p^j\leq v<0\ \right\}.$$

Moreover, the respective homology groups have bases given by the elements

$$\left\{\ q_0^v\ :\ v<0\ \right\},$$

$$\left\{\ q_0^v q_1^{kp^{i-1}}\ :\ k\in\mathbb{Z}-p\mathbb{Z},\ -p^{[i]}\leq v<0\ \right\},$$

and

$$\left\{\ q_0^v q_1^{kp^j}\epsilon_j\ :\ k\in\mathbb{Z},\ 1-p^j\leq v<0\ \right\}.$$

Proof. Each of the maps in the exact couple defining the q_1^{-1}-BSS comes from a map of algebraic Novikov spectral sequences. This means that if $x\in H^*(P;q_1^{-1}Q/q_0)$ and $q_0^v x\in E_\infty(q_1^{-1}\text{-BSS})$ then $d_0^{q_0\text{-FILT}}(q_0^v x)=q_0^v d_2^{v_1\text{-alg.NSS}}x$. We understand $d_2^{v_1\text{-alg.NSS}}$ by

theorem 6.2.1. For the rest of the proof we write d_0 for $d_0^{q_0\text{-FILT}}$.

$d_0(q_0^v) = 0$ and so the claims concerning $\{q_0^v : v < 0\}$ are evident.

First, fix $i \geq 1$ and consider

$$x = q_0^v q_1^{kp^{i-1}} \epsilon[I_-]\rho[J, K]$$

where I, J, K and k satisfy (5.4.3), $i_r = i$, and $-p^{[i]} \leq v < 0$. If $r = 1$ then $d_0(x) = 0$ so suppose that $r > 1$ and let $c \in \{1, \ldots, r-1\}$. We wish to show that replacing ϵ_{i_c} by ρ_{i_c-1} in x gives an element x' of the same form as x. This is true because

$$x' = q_0^v q_1^{kp^{i-1}} \epsilon[I'_-]\rho[J', K']$$

where I', J', K' are determined by the following properties.

1. $\epsilon[I'_-]\rho[J', K']$ is obtained from $\epsilon[I_-]\rho[J, K]$ by replacing ϵ_{i_c} by ρ_{i_c-1};

2. $r' = r - 1$, $i'_1 > \ldots > i'_{r'} = i$;

3. $j'_1 > \ldots > j'_{s'} \geq 1$;

4. $k'_a \geq 1$ for all $a \in \{1, \ldots, s'\}$.

In particular, $i'_{r'} = i$ and I', J', K' and k satisfy (5.4.3) because $s' \geq 1$, and $j_s \geq i_r = i$ and $i_c > i_r = i$ implies that $j'_{s'} \geq i = i'_{r'}$. Since d_0 is a derivation, this observation shows that d_0 induces an operation on the second subspace of the proposition. The claim about the homology is true because the complex

$$\left(E[\epsilon_n : n > i] \otimes \mathbb{F}_p[\rho_n : n \geq i] : \partial\epsilon_{n+1} = \rho_n \right)$$

has homology $\mathbb{F}_p$.

Second, fix $j \geq 1$ and consider

$$y = q_0^v q_1^{kp^j} \epsilon[I]\epsilon_j\rho[J, K_-]$$

where I, J, K and k satisfy (5.4.4), $j_s = j$, and $1 - p^j \leq v < 0$.

First, we wish to show the term obtained from applying d_0 to ϵ_j is trivial. If $j = 1$ then $d_0(\epsilon_j) = 0$ so suppose, for now, that $j > 1$. Replacing ϵ_j by ρ_{j-1} gives

$$y' = q_0^v q_1^{(kp)p^{j-1}} \epsilon[I']\rho[J', K']$$

where I', J', K' are determined by the following properties.

1. $\epsilon[I']\rho[J', K'] = \epsilon[I]\rho[J, K_-]\rho_{j-1}$;
2. $I' = I$;
3. $j'_1 > \ldots > j'_{s'} = j - 1$;
4. $k'_a \geq 1$ for all $a \in \{1, \ldots, s'\}$.

$s' \geq 1$ and either $r = r' = 0$, or $r = r' \geq 1$ and $i'_{r'} = i_r > j_s = j > j - 1 = j'_{s'}$, so we see that I', J', K' and $k' = kp$ satisfy (5.4.4). This shows that y' is the source of a (5.4.4) q_1^{-1}-Bockstein differential, i.e. zero in $E_\infty(q_1^{-1}\text{-BSS}) = E_0(q_0\text{-FILT})$. We deduce that when applying d_0 the only terms of interest come from applying d_0 to the $\epsilon[I]\rho[J, K_-]$ part of y.

If $r = 0$ then $d_0(y) = 0$ so suppose that $r > 0$ and let $c \in \{1, \ldots, r\}$. We wish to show that replacing ϵ_{i_c} by $\rho_{i_c - 1}$ in y gives an element y'' of the same form as y.

$$y'' = q_0^v q_1^{kp^j} \epsilon[I']\epsilon_j\rho[J', K'_-]$$

where I', J', K' are determined by the following properties.

1. $\epsilon[I']\rho[J', K'_-]$ is obtained from $\epsilon[I]\rho[J, K_-]$ by replacing ϵ_{i_c} by $\rho_{i_c - 1}$;
2. $r' = r - 1$ and $i'_1 > \ldots > i'_{r'}$;
3. $j'_1 > \ldots > j'_{s'} = j$;
4. $k'_a \geq 1$ for all $a \in \{1, \ldots, s'\}$.

$i_c \geq i_r > j_s = j$ ensures that condition (3) can be met. $s' \geq s \geq 1$ and either $r' = 0$ or $r' \geq 1$ and $i'_{r'} \geq i_r > j_s = j = j'_{s'}$. Thus, I', J', K' and k satisfy 5.4.4 and y'' has the same form as y. Since d_0 is a derivation, this shows that d_0 induces an operation on the third subspace of the proposition. The claim about the homology is true because

$$\left(E[\epsilon_n : n > j] \otimes \mathbb{F}_p[\rho_n : n \geq j] : \partial\epsilon_{n+1} = \rho_n \right)$$

has homology $\mathbb{F}_p$. □

6.4 The E_∞-page of the loc.alg.NSS

One knows that $H^*(BP_*BP; v_1^{-1}BP_*/p^\infty)$ is nonzero, only in cohomological degree 0 and 1. $H^0(BP_*BP; v_1^{-1}BP_*/p^\infty)$ is generated as an abelian group by the elements

$$\left\{ \frac{1}{p^n} \ : \ n \geq 1 \right\} \cup \left\{ \frac{v_1^{kp^{n-1}}}{p^n} \ : \ n \geq 1, \ k \in \mathbb{Z} - p\,\mathbb{Z} \right\}.$$

These are detected in the loc.alg.NSS by the following elements of $H^*(P; q_1^{-1}Q/q_0^\infty)$.

$$\left\{ \frac{1}{q_0^n} \ : \ n \geq 1 \right\} \ \cup \ \left\{ \frac{q_1^{kp^{n-1}}}{q_0^n} \ : \ n \geq 1, \ k \in \mathbb{Z} - p\,\mathbb{Z} \right\}$$

An element of order p in $H^1(BP_*BP; v_1^{-1}BP_*/p^\infty) = \mathbb{Z}/p^\infty$ is given by the class of

$$-p^{-1}v_1^{-1}[t_1] \in \Omega^1(BP_*BP; v_1^{-1}BP_*/p^\infty)$$

in $H^1(BP_*BP; v_1^{-1}BP_*/p^\infty)$, which is detected by $q_0^{-1}\epsilon_1$ in the loc.alg.NSS. Theorem 6.3.1, degree considerations, and the fact that each q_0^v is a permanent cycle in the loc.alg.NSS, allow us to see that there are permanent cycles in the loc.alg.NSS, which are not boundaries, which are detected in the q_0-FILT spectral sequence by the elements

$$\left\{ q_0^v \epsilon_n \ : \ n \geq 1, \ 1 - p^n \leq v < 0 \right\}.$$

These elements must detect the elements of $H^1(BP_*BP; v_1^{-1}BP_*/p^\infty)$. In summary, we have the following proposition.

Proposition 6.4.1. *E_∞(loc.alg.NSS) has an $\mathbb{F}_p$-basis given by the following elements.*

$$\left\{ q_0^v : v < 0 \right\} \cup \left\{ q_0^v q_1^{kp^{n-1}} \ : \ n \geq 1,\ k \in \mathbb{Z} - p\mathbb{Z},\ -n \leq v < 0 \right\}$$
$$\cup \left\{ \overline{q_0^v \epsilon_n} \ : \ n \geq 1,\ 1 - p^n \leq v < 0 \right\}$$

Here, $\overline{q_0^v \epsilon_n}$ denotes the element of E_3(loc.alg.NSS) representing $q_0^v \epsilon_n \in E_1(q_0\text{-FILT})$.

Using theorem 6.3.1 together with this result, we see that the only possible pattern for the differentials between a principal tower and its side towers, in the loc.alg.NSS, is the one drawn in figure 1-1.

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Chapter 7

Some permanent cycles in the ASS

Our calculation of the q_1^{-1}-BSS gives information about the Adams E_2-page via the zig-zag (1.4.5). Our calculation of the loc.alg.NSS gives information about Adams d_2 differentials in a similar way (figure 1-2). We would like to learn about higher Adams differentials, but first, we say what we can about some permanent cycles in the Adams spectral sequence. We show that for each $n \geq 0$, $q_0^{p^n-n-1}h_{1,n}$ is a permanent cycle in the Adams spectral sequence and we give a homotopy class representing it. This is the odd primary analogue of a result of Davis and Mahowald appearing in [6].

7.1 Maps between stunted projective spaces

The maps we construct to represent the classes $q_0^{p^n-n-1}h_{1,n}$ make use of maps we have between skeletal subquotients of $(\Sigma^\infty B\Sigma_p)_{(p)}$. The analog of these spectra at $p = 2$ are the *stunted projective spaces* $\mathbb{R}P_n^m$ and so we use the same terminology. Throughout this thesis we write H for $H\mathbb{F}_p$, the mod p Eilenberg-Mac Lane spectrum.

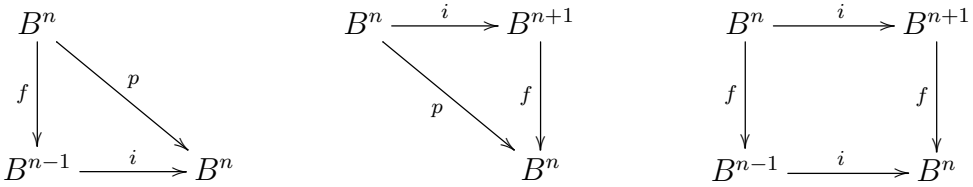
In [1] Adams shows that there is a CW spectrum B with one cell in each positive dimension congruent to 0 or -1 modulo $q = 2p - 2$ such that $B \simeq (\Sigma^\infty B\Sigma_p)_{(p)}$. In particular, B is built up from many copies of the *mod p Moore spectrum* S/p. The maps we construct between stunted projective spaces all come from the fact that multiplication by p is zero on S/p (since p is odd). For this reason, we emphasize the filtration by the copies of S/p over the skeletal filtration, and writing a superscript in

square brackets to denote the skeletal filtration, we use the following notation.

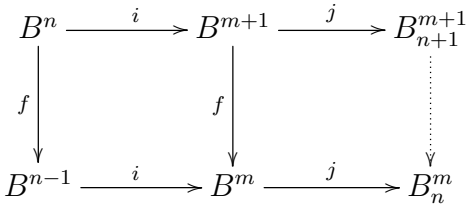
Notation 7.1.1. Write B for the spectrum of [1, 2.2]. For $n \geq 0$ let $B^n = B^{[nq]}$ and for $1 \leq n \leq m$ let $B^m_n = B^m/B^{n-1}$. Notice that $B^0 = *$ and so $B^n = B^n_1$. For $n > m$ let $B^m_n = *$.

We now proceed to construct compatible maps between stunted projective spaces of Adams filtration one. All proofs will be deferred until the end of the section.

Lemma 7.1.2. *For each $n \geq 1$ there exists a unique map $f : B^n \longrightarrow B^{n-1}$ such that the left diagram commutes. Moreover, the center diagram commutes so that the right diagram commutes.*

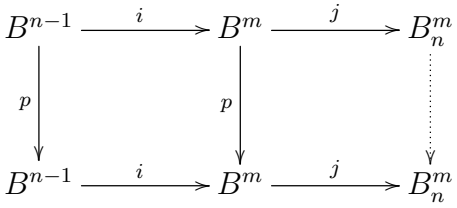


For $1 \leq n \leq m$ the filler for the diagram



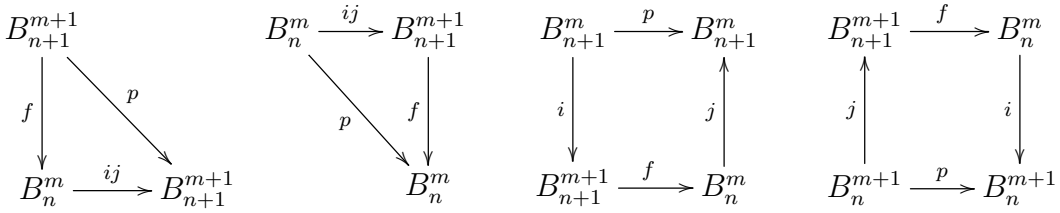
is unique and we call it f. The collection of such f are compatible.

For $1 \leq n \leq m$ the filler for the diagram



is unique and so equal to p.

The following diagrams commute for the appropriate values of m and n.



We wish to analyze the Adams filtrations of the maps that we have just constructed. First, we describe spectra which are more convenient than those appearing in the relevant H-canonical Adams towers, and for this we need to recall the structure of $H^*(B_+) = H^*(B\Sigma_p)$.

Proposition 7.1.3 ([1, 2.1])**.** *Let $i : C_p \longrightarrow \Sigma_p$ be the inclusion of a Sylow subgroup.*

1. *$H^*(BC_p) = E[x] \otimes \mathbb{F}_p[y]$ where $|x| = 1$, $|y| = 2$ and $\beta x = y$.*
2. *$H^*(B\Sigma_p) = E[x_{q-1}] \otimes \mathbb{F}_p[y_q]$ where $(Bi)^*(x_{q-1}) = xy^{p-2}$ and $(Bi)^*(y_q) = y^{p-1}$.*

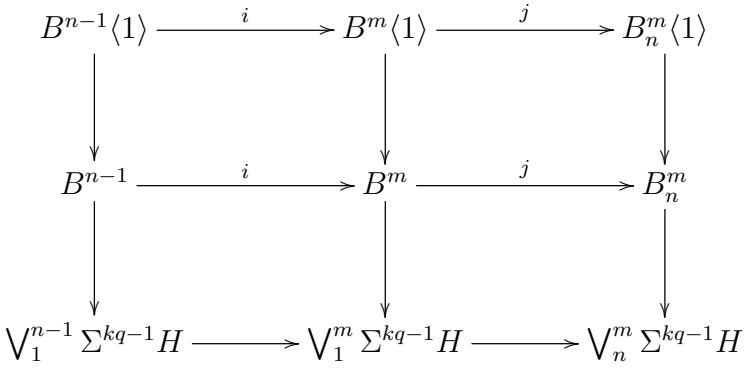
Notation 7.1.4. For $k \geq 1$ write e^k for $x_{q-1}y_q^{k-1} \in H^{kq-1}(B)$. Use the same notation for the corresponding elements in $H^*(B^m_n)$.

Definition 7.1.5. For $1 \leq n \leq m$ define $B^m_n\langle 1\rangle$ by the following cofibration sequence.

$$B^m_n\langle 1\rangle \longrightarrow B^m_n \xrightarrow{(e^n,\ldots,e^m)} \bigvee_n^m \Sigma^{kq-1}H$$

Let $B^0\langle 1\rangle = *$ and for $n \geq 1$ let $B^n\langle 1\rangle = B^n_1\langle 1\rangle$.

For $1 \leq n \leq m$, we have the following square of cofibration sequences.

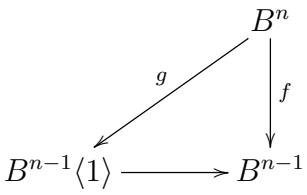


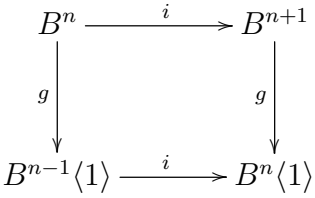
The purpose of the spectra just defined is highlighted by the following lemma.

Lemma 7.1.6. *A map to B^m_n can be factored through $B^m_n\langle 1\rangle$ if and only if it can be factored through $\overline{H} \wedge B^m_n = fib(B^m_n \longrightarrow H \wedge B^m_n)$.*

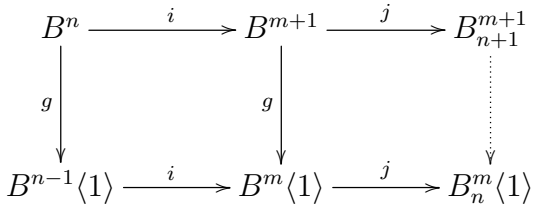
The following lemma shows that the maps we have constructed between stunted projective spaces have Adams filtration one.

Lemma 7.1.7. *For each $n \geq 1$ there exists a unique map $g : B^n \longrightarrow B^{n-1}\langle 1\rangle$ such that the left diagram commutes. Moreover, the right diagram commutes.*

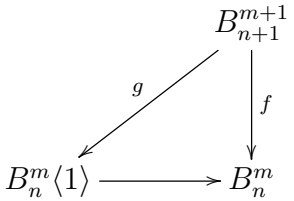




For $1 \leq n \leq m$ the filler for the diagram



is unique and we call it g. For $1 \leq n \leq m$ the following diagram commutes.



Before proving all the lemmas above, we make a preliminary calculation.

Lemma 7.1.8. *For $m, n \geq 1$ $[\Sigma B^{n-1}, B^m_n] = 0$, $[\Sigma B^n, B^m_n] = 0$, $[\Sigma B^n, B^m_n\langle 1\rangle] = 0$.*

Proof. The results are all obvious if $m < n$ so suppose that $m \geq n$.

The first follows from cellular approximation; the third does too, although we will give a different proof.

Cellular approximation gives $[\Sigma B^n, B_n^m] = [\Sigma B_n^n, B_n^n] = [\Sigma S/p, S/p]$. We have an exact sequence

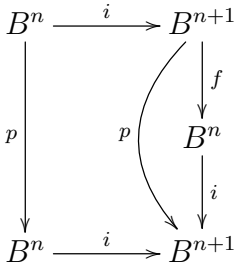
$$\pi_2(S/p) \longrightarrow [\Sigma S/p, S/p] \longrightarrow \pi_1(S/p)$$

and $\pi_1(S/p) = \pi_2(S/p) = 0$, which gives the second identification.

Since $[\Sigma B^n, \bigvee_n^m \Sigma^{kq-2} H] = 0$, $[\Sigma B^n, B_n^m\langle 1\rangle] \longrightarrow [\Sigma B^n, B_n^m]$ is injective and this completes the proof. □

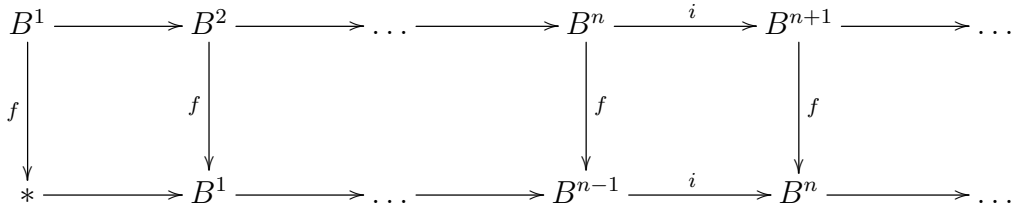
Proof of lemma 7.1.2. f exists because the composite $B^n \xrightarrow{p} B^n \longrightarrow B_n^n = \Sigma^{nq-1} S/p$ is null. f is unique because $[B^n, \Sigma^{-1} B_n^n] = 0$.

Since $[B^n, \Sigma^{-1} B_{n+1}^{n+1}] = 0$ the map $i_* : [B^n, B^n] \longrightarrow [B^n, B^{n+1}]$ is injective and so commutativity of the following diagram gives commutativity of the second diagram in the lemma.



Uniqueness of the fillers is given by the facts $[\Sigma B^n, B_n^m] = 0$ and $[\Sigma B^{n-1}, B_n^m] = 0$, respectively.

We turn to compatibility of the collection $\{f : B_{n+1}^{m+1} \longrightarrow B_n^m\}$. We already have compatibility of the collection $\{f : B^n \longrightarrow B^{n-1}\}$, i.e. the following diagram in the homotopy category commutes.



For concreteness, suppose that we a have pointset level model for this diagram in which each representative $\bar{i} : B^{n-1} \longrightarrow B^n$ is a cofibration between cofibrant spectra. By a spectrum, we mean an S-module [7], and so every spectrum is fibrant. The

"homotopy extension property" that S-modules satisfy says that we can make any of the squares strictly commute at the cost of changing the right map to a homotopic one. By proceeding inductively, starting with the left most square, we can assume that the representative $\overline{f}$'s are chosen so that each square strictly commutes. Let $\overline{f} : B^{m+1}_{n+1} \longrightarrow B^m_n$ be obtained by taking strict cofibers of the appropriate diagram. The homotopy class of $\overline{f}$ provides a filler for the diagram in the lemma and so is equal to f. It is clear that the $\overline{f}$'s are compatible and so the f's are compatible.

The deductions that each of the final four diagrams commute are similar and rely on the uniqueness of the second filler. We'll need the fourth diagram so we show this in detail. We have a commuting diagram.

$$\begin{array}{ccccc}
B^{n-1} & \longrightarrow & B^{m+1} & \longrightarrow & B^{m+1}_n \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle j} \\
B^{n} & \longrightarrow & B^{m+1} & \longrightarrow & B^{m+1}_{n+1} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
B^{n-1} & \longrightarrow & B^{m} & \longrightarrow & B^{m}_n \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
B^{n-1} & \longrightarrow & B^{m+1} & \longrightarrow & B^{m+1}_n
\end{array}$$

The vertical composites in the first two columns are p and so the third is too. □

Proof of lemma 7.1.6. $H^*(B^m_n)$ is free over $E[\beta]$ with basis $e^n, \ldots, e^m$. This basis allowed us to construct the top map in the following diagram.

$$\begin{array}{ccc}
B^m_n & \xrightarrow{(e^n,\ldots,e^m)} & \bigvee_n^m \Sigma^{kq-1} H \\
\downarrow & & \downarrow{\scriptstyle \bigvee_n^m \Sigma^{kq-1}(1,\beta)} \\
H \wedge B^m_n & \xrightarrow{\simeq} & \bigvee_n^m \left(\Sigma^{kq-1} H \vee \Sigma^{kq} H \right)
\end{array}$$

We have a map $(1, \beta) : H \longrightarrow H \vee \Sigma H$ which is used to construct the map on the

right. Since the target of this map is an H-module we obtain the bottom map and one can check that this is an equivalence. Thus, we obtain the map of cofibration sequences displayed at the top of the following diagram.

$$\begin{array}{ccccc}
B_n^m\langle 1\rangle & \longrightarrow & B_n^m & \xrightarrow{(e^n,\ldots,e^m)} & \bigvee_n^m \Sigma^{kq-1}H \\
\downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \simeq\circ\left[\bigvee_n^m \Sigma^{kq-1}(1,\beta)\right]} \\
\overline{H}\wedge B_n^m & \longrightarrow & B_n^m & \longrightarrow & H\wedge B_n^m \\
\downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \left[\bigvee_n^m \Sigma^{kq-1}(1,*)\right]\circ\simeq} \\
B_n^m\langle 1\rangle & \longrightarrow & B_n^m & \xrightarrow{(e^n,\ldots,e^m)} & \bigvee_n^m \Sigma^{kq-1}H
\end{array}$$

The bottom right square is checked to commute and so we obtain the map of cofibration sequences displayed at the bottom. This diagram shows that a map to B_n^m can be factored through $B_n^m\langle 1\rangle$ if and only if it can be factoring through $\overline{H}\wedge B_n^m$; this is also clear if one uses the more general theory of Adams resolutions discussed in [11].

[One sees that $(e_n,\ldots,e_m)$ is an H_*-isomorphism in dimensions which are strictly less than $(n+1)q-1$ so $B_n^m\langle 1\rangle$ is $((n+1)q-3)$-connected and hence $(nq+1)$-connected.] □

Proof of lemma 7.1.7. We have $[B^n,\Sigma^{nq-2}H]=0$ and so the map

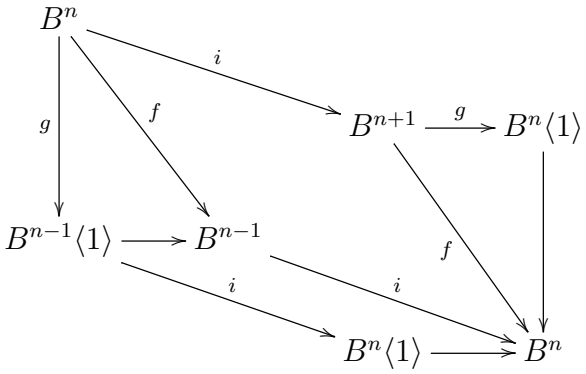
$$\left[B^n,\bigvee_1^{n-1}\Sigma^{kq-1}H\right]\longrightarrow\left[B^n,\bigvee_1^{n}\Sigma^{kq-1}H\right]$$

is injective. Since $if=p$ and $p=0$ on H, the following diagram proves the existence of g.

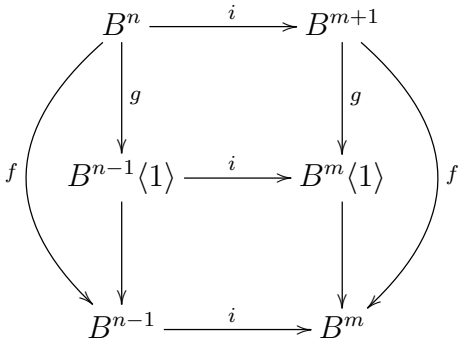
$$\begin{array}{ccccc}
 & & B^n & & \\
 & & \downarrow{\scriptstyle f} & & \\
B^{n-1}\langle 1\rangle & \longrightarrow & B^{n-1} & \longrightarrow & \bigvee_1^{n-1}\Sigma^{kq-1}H \\
 & & \downarrow{\scriptstyle i} & & \downarrow \\
 & & B^n & \longrightarrow & \bigvee_1^{n}\Sigma^{kq-1}H
\end{array}$$

Uniqueness of g is given by the fact that $[B^n, \bigvee_1^{n-1} \Sigma^{kq-2} H] = 0$.

Since $[B^n, \bigvee_1^{n} \Sigma^{kq-2} H] = 0$ the map $[B^n, B^n\langle 1\rangle] \longrightarrow [B^n, B^n]$ is injective and so commutativity of the following diagram gives commutativity of the second diagram.



The filler is unique because, by lemma 7.1.8, $[\Sigma B^n, B^m_n\langle 1\rangle] = 0$. The final diagram commutes because we have the following commutative diagram and a uniqueness condition on f as a filler.



□

7.2 Homotopy and cohomotopy classes in stunted projective spaces

Throughout this thesis we write $A = H_*H$ for the dual Steenrod algebra, and $A^* = H^*H$ for the Steenrod algebra.

To construct the homotopy class representing $q_0^{p^n-n-1}h_{1,n}$ in the Adams spectral

sequence we make use of a homotopy class in $\pi_{p^n q-1}(B^{p^n}_{p^n-n})$. First, we analyze the algebraic picture and identify the corresponding A-comodule primitive. Recall 7.1.4.

Notation 7.2.1. For $k \geq 1$ write e_k for the class in $H_{kq-1}(B)$ dual to $e^k \in H^{kq-1}(B)$. Use the same notation for the corresponding elements in $H_*(B^m_n)$.

Lemma 7.2.2. *For each $n \geq 0$, $e_{p^n} \in H_{p^n q-1}(B)$ is an A-comodule primitive.*

Proof. The result is clear when $n = 0$, since $H_k(B) = 0$ for $k < q-1$. Assume from now on that $n > 0$.

Since the (co)homology of B is concentrated in dimensions which are 0 or -1 congruent to q, the dual result is that $P^i e^j = 0$ whenever $i, j > 0$ and $i + j = p^n$.

Let $i : C_p \longrightarrow \Sigma_p$ be the inclusion of a Sylow subgroup. Since $(Bi)^*$ is injective it is enough to show that the equation is true after applying $(Bi)^*$. Writing this out explicitly, we must show that

$$P^i(xy^{(p-1)j-1}) = 0 \text{ whenever } i, j > 0 \text{ and } i + j = p^n.$$

Writing P for the total reduced p-th power we have $P(x) = x$ and $P(y) = y + y^p = y(1 + y^{p-1})$. Suppose that $i, j > 0$ and that $i + j = p^n$. Then

$$P(xy^{(p-1)j-1}) = xy^{(p-1)j-1}(1+y^{p-1})^{(p-1)j-1} = xy^{(p-1)j-1} \sum_{k=0}^{(p-1)j-1} \binom{(p-1)j-1}{k} y^{(p-1)k}$$

which gives

$$P^i(xy^{(p-1)j-1}) = \binom{(p-1)j-1}{i} xy^{(p-1)p^n-1}$$

as long as $i \leq (p-1)j - 1$ and $P^i(xy^{(p-1)j-1}) = 0$ otherwise. We just need to show that

$$p \,\Big|\, \binom{(p-1)(p^n-i)-1}{i}$$

whenever $0 < i \leq (p-1)(p^n - i) - 1$. The largest value of i for which we have $i \leq (p-1)(p^n - i) - 1$ is $(p-1)p^{n-1} - 1$ so write $i = sp^k$ for $0 \leq k < n$ and $s \not\equiv 0 \pmod{p}$. Let $m = (p-1)(p^n - i) - 1$ so that we are interested in $\binom{m}{i}$. $m - i \equiv -1$

(mod p^{k+1}) and so when we add $m-i$ to i in base p there is a carry. An elementary fact about binomial coefficients completes the proof. □

The relevant topological result is given by the following proposition.

Proposition 7.2.3. *For each $n \geq 0$, $e_{p^n} \in H_{p^n q-1}(B^{p^n}_{p^n-n})$ is in the image of the Hurewicz homomorphism.*

Proof. The result is clear for $n = 0$, since we have the map $S^{q-1} \longrightarrow \Sigma^{q-1}S/p = B^1_1$. For $n \geq 1$, setting $\epsilon = 0$, $i = n+1$, $j = p^n - n - 1$ and $k = iq - 1$ in [4, 2.9(v)] shows that

$$Z = B^{[p^n q-1]}/B^{[(p^n-n-1)q-1]}$$

has reductive top cell and we have an "include-collapse" map $Z \longrightarrow B^{p^n}_{p^n-n}$. □

To construct the homotopy class representing $q_0^{p^n-n-1}h_{1,n}$ in the Adams spectral sequence we also make use of the transfer map.

Definition 7.2.4. Write $t : B^\infty_1 \longrightarrow S^0$ for the transfer map of [1, 2.3(i)] and let C be the cofiber of $\Sigma^{-1}t$.

We need to analyze the affect of t algebraically.

Notation 7.2.5. We have a cofibration sequence $S^{-1} \longrightarrow C \longrightarrow B$. Abuse notation and write e^k and e_k for the elements in $H^*(C)$ and $H_*(C)$ which correspond to the elements of the same name in $H^*(B)$ and $H_*(B)$. Write U and u for the dual classes in $H^*(C)$ and $H_*(C)$ corresponding to generators of $H^{-1}(S^{-1})$ and $H_{-1}(S^{-1})$.

Lemma 7.2.6. *Suppose $n \geq 0$. Then $e_{p^n} \in H_{p^n q-1}(C)$ is mapped to $1 \otimes e_{p^n} + \dot{\xi}_1^{p^n} \otimes u$ under the A-coaction map.*

Proof. First, let's introduce some notation which will be useful for the proof. Write Sq^{kq}_p and Sq^{kq+1}_p for P^k and βP^k, respectively. Recall that the Steenrod algebra A^* has a $\mathbb{F}_p$-vector space basis given by admissible monomials

$$\mathcal{B} = \{\mathrm{Sq}^{i_1}_p \cdots \mathrm{Sq}^{i_r}_p : i_j \geq p i_{j+1},\ i_j \equiv 0 \text{ or } 1 \pmod q\}.$$

We claim that $\mathrm{Sq}_p^{p^n q} U \doteq e^{p^n}$, and that $bU = 0$ for any $b \in \mathcal{B}$ of length greater than 1. Here, length greater than one means that $r > 1$ and $i_r > 0$.

By lemma 7.2.2 we know that e_{p^n} is mapped, under the coaction map, to $1 \otimes e_{p^n} + a \otimes u$ for some $a \in A$. If we can prove the claim above then we will deduce that $a \doteq \xi_1^{p^n}$.

Take an element $b = \mathrm{Sq}_p^{i_1} \cdots \mathrm{Sq}_p^{i_r} \in \mathcal{B}$ of length greater than one. Let $k = \lfloor i_{r-1}/q \rfloor$ so that either $i_{r-1} = kq$ or $kq + 1$ and $\mathrm{Sq}_p^{i_{r-1}} = P^k$ or βP^k. We have

$$i_{r-1} \geq p i_r \implies i_{r-1} - 1 \geq p i_r - 1 > (p-1) i_r - (p-1) = q(i_r - 1)/2$$

and so $2k \geq 2(i_{r-1} - 1)/q > i_r - 1 = |\mathrm{Sq}_p^{i_r} U|$. Since $\mathrm{Sq}_p^{i_r} U$ comes from the cohomology of a space we deduce that $P^k \mathrm{Sq}_p^{i_r} U = 0$. Thus, $\mathrm{Sq}_p^{i_{r-1}} \mathrm{Sq}_p^{i_r} U = 0$ and $bU = 0$, which verifies the second part of the claim.

We are left to prove that $P^{p^n} U \doteq e^{p^n}$ for each $n \geq 0$. First, we prove the $n = 0$ case $P^1 U \doteq e^1$. This statement is equivalent to the claim that

$$S^{q-1} = B_1^1 \longrightarrow B_1^\infty \xrightarrow{t} S^0$$

is detected by a unit multiple of $h_{1,0}$ in the Adams spectral sequence. By cellular approximation a generator of $\pi_{q-1}(B_1^\infty)$ is given by $S^{q-1} = B_1^1 \longrightarrow B_1^\infty$. By definition $t : B_1^\infty \longrightarrow S^0$ is an isomorphism on π_{q-1}. By low dimensional calculations a generator of $\pi_{q-1}(S^0)$ is detected by $h_{1,0}$. This completes the proof of the $n = 0$ case.

To prove that $P^{p^n} U \doteq e^{p^n}$ it is enough to show that $\beta P^{p^n} U \doteq \beta e^{p^n}$. Notice that $|\beta e^1| = q$ and so

$$\beta e^{p^n} = (\beta e^1)^{p^n} = P^{p^{n-1} q/2} \cdots P^{pq/2} P^{q/2} \beta e^1 \doteq P^{p^{n-1} q/2} \cdots P^{pq/2} P^{q/2} \beta P^1 U.$$

We are left with proving that $P^{p^{n-1} q/2} \cdots P^{q/2} \beta P^1 U \doteq \beta P^{p^n} U$. We induct on n, the

result being trivial for $n = 0$. Suppose it is proven for some $n \geq 0$. Then we have

$$\begin{aligned} P^{p^n q/2} \dots P^{q/2} \beta P^1 U &\doteq P^{p^n q/2} \beta P^{p^n} U \\ &\doteq (\beta P^{p^n + p^n q/2} + \text{elements of } \mathcal{B} \text{ of length greater than 1}) U \\ &= \beta P^{p^{n+1}} U, \end{aligned}$$

which completes the inductive step and the proof of the lemma. □

7.3 A permanent cycle in the ASS

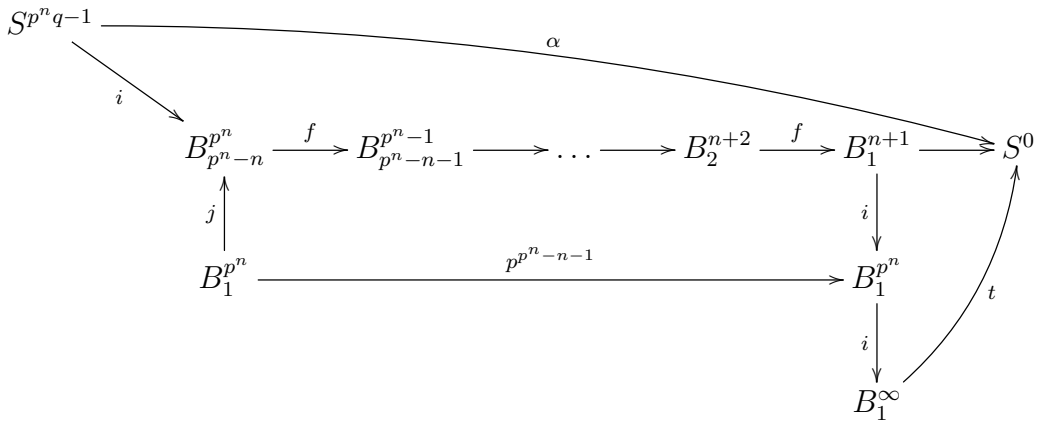
We are now ready to prove the main result of the chapter.

Theorem 7.3.1. *The element* $q_0^{p^n - n - 1} h_{1,n} \doteq \{[\tau_0]^{p^n - n - 1} [\xi_1^{p^n}]\} \in H^{p^n - n, p^n(q+1) - n - 1}(A)$ *is a permanent cycle in the Adams spectral sequence represented by the map*

$$\alpha : S^{p^n q - 1} \xrightarrow{i} B^{p^n}_{p^n - n} \xrightarrow{f} B^{p^n - 1}_{p^n - n - 1} \longrightarrow \dots \longrightarrow B^{n+2}_2 \xrightarrow{f} B^{n+1}_1 \xrightarrow{t} S^0.$$

Here, i comes from proposition 7.2.3, f comes from lemma 7.1.2, and t is the restriction of the transfer map.

Proof. By lemma 7.1.2 the following diagram commutes.



We look at the maps induced on E_2-pages.

By definition, $i : S^{p^n q - 1} \longrightarrow B^{p^n}_{p^n - n}$ is represented in the Adams spectral sequence

by $e_{p^n} \in H^{0,p^n q-1}(A; H_*(B^{p^n}_{p^n-n}))$, and by lemma 7.2.2, this element is the image of $e_{p^n} \in H^{0,p^n q-1}(A; H_*(B^{p^n}_1))$. Moreover,

$$q_0^{p^n-n-1} \cdot e_{p^n} \in H^{p^n-n-1,p^n(q+1)-n-2}(A; H_*(B^\infty_1)).$$

$t_* : E_2(B^\infty_1) \longrightarrow E_2(S^0)$ is described by the geometric boundary theorem. The cofibration sequence $S^{-1} \longrightarrow C \longrightarrow B$ induces a short exact sequence of A-comodules. The boundary map obtained by applying $H^*(A; -)$ is the map induced by t.

$$0 \longrightarrow \Omega^*(A; H_*(S^{-1})) \longrightarrow \Omega^*(A; H_*(C)) \longrightarrow \Omega^*(A; H_*(B)) \longrightarrow 0$$

$$[\tau_0]^{p^n-n-1} e_{p^n} \longmapsto [\tau_0]^{p^n-n-1} e_{p^n}$$

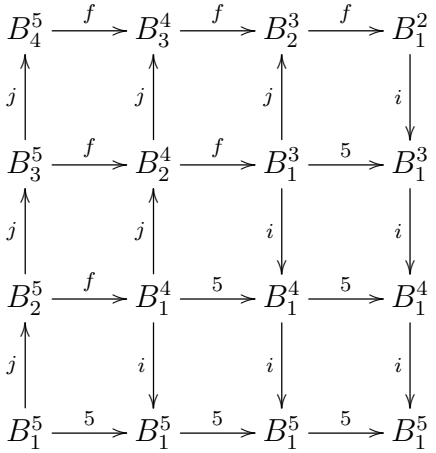
$$\downarrow .$$

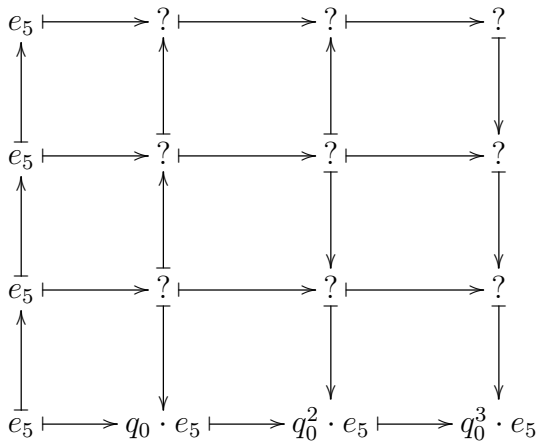
$$[\tau_0]^{p^n-n-1}[\xi_1^{p^n}] \longmapsto [\tau_0]^{p^n-n-1}[\xi_1^{p^n}]u$$

Thus, by using lemma 7.2.6, we see that

$$t_*(q_0^{p^n-n-1} \cdot e_{p^n}) \doteq q_0^{p^n-n-1} h_{1,n} \in H^{p^n-n,p^n(q+1)-n-1}(A).$$

This almost completes the proof. There is a subtlety, however. A map of filtration degree k only gives a well-defined map on E_{k+1} pages. To complete the proof we break the rectangle appearing in the first diagram up into $(p^n - n - 1)^2$ squares. We have demonstrated this for the case when $p = 5$ and $n = 1$ below.





Each square involves two maps of Adams filtration zero in the vertical direction and two maps of Adams filtration one in the horizontal direction. Each square commutes by lemma 7.1.2 and the maps induced on E_2-pages are well-defined. This completes the proof. $\square$

Chapter 8

Adams spectral sequences

In this chapter we set up and calculate the localized Adams spectral sequence for the *v_1-periodic sphere*. Along the way we construct Adams spectral sequences for calculating the homotopy of the mod p^n *Moore spectrum* S/p^n, the *Prüfer sphere*

$$S/p^\infty = \text{hocolim}(S/p \xrightarrow{p} S/p^2 \xrightarrow{p} S/p^3 \longrightarrow \ldots),$$

and we prove the final theorem stated in the introduction.

8.1 Towers and their spectral sequences

In this section we introduce some essential concepts and constructions: towers (definition 8.1.4), the smash product of towers and the spectral sequences associated with them. We provide important examples, which will be useful for the construction of the modified Adams spectral sequence for S/p^n and for verifying its properties. We also recall the main properties of the Adams spectral sequence.

Notation 8.1.1. We write $\mathscr{S}$ for the *stable homotopy category*.

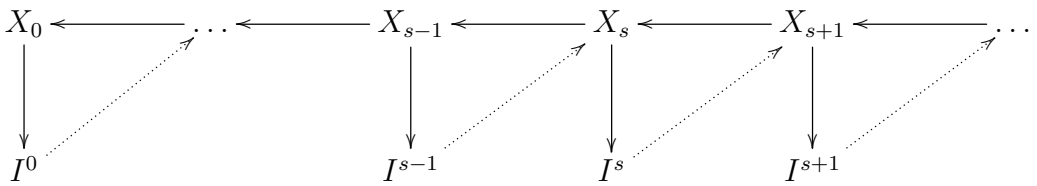
Definition 8.1.2. Write $\text{Ch}(\mathscr{S})$ for the category of non-negative *cochain complexes* in $\mathscr{S}$. An object $C^\bullet$ of this category is a diagram

$$C^0 \xrightarrow{d} C^1 \longrightarrow \ldots \longrightarrow C^s \xrightarrow{d} C^{s+1} \longrightarrow \ldots$$

in $\mathscr{S}$ with $d^2 = 0$. An *augmentation* $X \longrightarrow C^\bullet$ of a cochain complex $C^\bullet \in \mathrm{Ch}(\mathscr{S})$ is a map of cochain complexes from $X \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow * \longrightarrow \ldots$ to $C^\bullet$.

Notation 8.1.3. Let $\mathbb{Z}$ denote the category with the integers as objects and hom-sets determined by: $|\mathbb{Z}(n,m)| = 1$ if $n \geq m$, and $|\mathbb{Z}(n,m)| = 0$ otherwise. Write $\mathbb{Z}_{\geq 0}$ for the full subcategory of $\mathbb{Z}$ with the non-negative integers as objects.

Definition 8.1.4. An object $\{X_s\}$ of the diagram category $\mathscr{S}^{\mathbb{Z}_{\geq 0}}$ is called a *sequence*. A system of interlocking cofibration sequences



in $\mathscr{S}$ is called a *tower* and we use the notation $\{X, I\}$. Notice that a tower $\{X, I\}$ has an *underlying sequence* $\{X_s\}$ and an *underlying augmented cochain complex* $X_0 \longrightarrow \Sigma^\bullet I^\bullet$.

A *map of towers* $\{X, I\} \longrightarrow \{Y, J\}$ is a compatible collection of maps

$$\{X_s \longrightarrow Y_s\} \cup \{I^s \longrightarrow J^s\}.$$

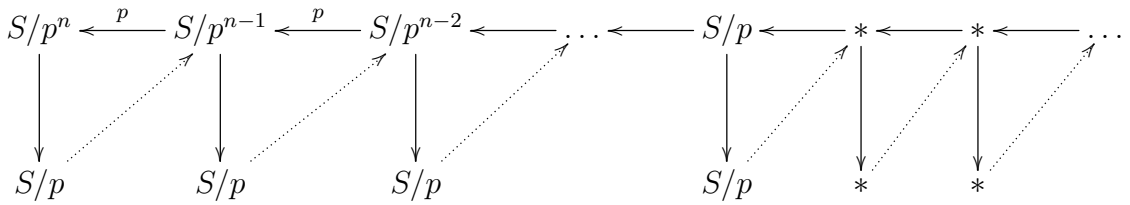
The following tower is important for us.

Definition 8.1.5. We write $\{S^0, S/p\}$ for the tower in which each of the maps in the underlying sequence is $p : S^0 \longrightarrow S^0$. The underlying augmented cochain complex is $S^0 \longrightarrow \Sigma^\bullet S/p$, where each differential is given by a suspension of the Bockstein $\beta : S/p \longrightarrow S^1 \longrightarrow \Sigma S/p$.

Definition 8.1.6. Suppose that $\{X, I\}$ is a tower. Then by changing $\{X, I\}$ up to an isomorphism we can find a pointset model in which each $X_{s+1} \longrightarrow X_s$ is a cofibration between cofibrant S-modules [7] and I^s is the strict cofiber of this map. We say that such a pointset level model is *cofibrant*.

By taking a cofibrant pointset level model $\{S^0, S/p\}$, we can construct another tower by collapsing the n-th copy of S^0.

Definition 8.1.7. Let $\{S/p^{n-*}, S/p\}$ be the tower obtained in this way.



To construct the multiplicative structure on the modified Adams spectral sequence for S/p^n we need to make sense of smashing towers together. A modern version of [5, definition 4.2] is as follows.

Definition 8.1.8. Suppose $\{X, I\}$ and $\{Y, J\}$ are towers and that we have chosen cofibrant models for them. Let

$$Z_s = \operatorname{colim}_{\substack{i+j\geq s \\ 0\leq i,j\leq s}} X_i \wedge Y_j.$$

The indexing category in this colimit is a full subcategory of $\mathbb{Z} \times \mathbb{Z}$ and the notation only indicates the objects. Let

$$K^s = \bigvee_{\substack{i+j=s \\ 0\leq i,j\leq s}} I^i \wedge J^j.$$

We have maps $Z_{s+1} \longrightarrow Z_s$ and $Z_s \longrightarrow K^s$, which give a cofibrant model for the *smash product of towers* $\{X, I\} \wedge \{Y, J\} = \{Z, K\}$. Moreover, the underlying augmented cochain complex of $\{X, I\} \wedge \{Y, J\}$ is the tensor product of the underlying augmented cochain complexes of $\{X, I\}$ and $\{Y, J\}$.

Note that the definition of $\{X, I\} \wedge \{Y, J\}$ depends on the choice of cofibrant models for $\{X, I\}$ and $\{Y, J\}$, but the following proposition shows that it is well-defined up to isomorphism.

Proposition 8.1.9. *Suppose we have maps of towers*

$$\{X, I\} \longrightarrow \{\mathcal{X}, \mathcal{I}\},\ \{Y, J\} \longrightarrow \{\mathcal{Y}, \mathcal{J}\}.$$

Then there exists a map of towers $\{X, I\} \wedge \{Y, J\} \longrightarrow \{\mathcal{X}, \mathcal{I}\} \wedge \{\mathcal{Y}, \mathcal{J}\}$ *such that the underlying map on augmented cochain complexes is the tensor product*

$$\left[\left(X_0 \to \Sigma^\bullet I^\bullet\right) \longrightarrow \left(\mathcal{X}_0 \to \Sigma^\bullet \mathcal{I}^\bullet\right)\right] \wedge \left[\left(Y_0 \to \Sigma^\bullet J^\bullet\right) \longrightarrow \left(\mathcal{Y}_0 \to \Sigma^\bullet \mathcal{J}^\bullet\right)\right].$$

Writing down the proof of the proposition carefully is a lengthy detour. I assure the reader that I have done this. Indeed, in the draft of my thesis all details were included and this will remain available on my website. However, I do not want the content of my thesis to be concerned with such technical issues. The point is that a map of towers restricts to a map of sequences. On the cofibrant pointset level models, we only know that each square commutes up to homotopy, but each homotopy is determined by the map on the respective cofiber and this is part of the data of the map of towers. Since the Z_s appearing in definition 8.1.8 is, in fact, a homotopy colimit, we can define maps $Z_s \longrightarrow \mathcal{Z}_s$ using these homotopies. One finds that this provides a map of sequences compatible with the tensor product of the underlying maps of augmented cochain complexes.

As an example of a smash product of towers and a map of towers, we would like to construct a map (recall definition 8.1.5)

$$\{S^0, S/p\} \wedge \{S^0, S/p\} \longrightarrow \{S^0, S/p\} \tag{8.1.10}$$

extending the multiplication $S^0 \wedge S^0 \longrightarrow S^0$. Using the terminology of [5, 11], we see that $\{S^0, S/p\}$ is the S/p-canonical resolution of S^0. Moreover, [5, 4.3(b)] tells us that $\{S^0, S/p\} \wedge \{S^0, S/p\}$ is an S/p-Adams resolution. Thus, the following lemma,

which is given in [11], means that it is enough to construct a map

$$\Big(S^0 \to \Sigma^\bullet S/p\Big) \wedge \Big(S^0 \to \Sigma^\bullet S/p\Big) \longrightarrow \Big(S^0 \to \Sigma^\bullet S/p\Big). \tag{8.1.11}$$

Lemma 8.1.12. *Suppose $\{X, I\}$ and $\{Y, J\}$ are E-Adams resolutions. Then any map of augmented cochain complexes $\Big(X_0 \to \Sigma^\bullet I^\bullet\Big) \longrightarrow \Big(Y_0 \to \Sigma^\bullet J^\bullet\Big)$ extends to a map of towers.*

The following lemma shows that we can construct the map (8.1.11) by using the multiplication $\mu : S/p \wedge S/p \longrightarrow S/p$ on every factor appearing in the tensor product.

Lemma 8.1.13. *The following diagram commutes, where $\mu : S/p \wedge S/p \longrightarrow S/p$ is the multiplication on the ring spectrum S/p.*

$$\begin{array}{ccc} S/p \wedge S/p & \xrightarrow{(\beta\wedge S/p,\, S/p\wedge\beta)} & (\Sigma S/p \wedge S/p) \vee (S/p \wedge \Sigma S/p) \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \Sigma(\mu,\mu)} \\ S/p & \xrightarrow{\qquad\qquad\beta\qquad\qquad} & \Sigma S/p \end{array}$$

Proof. $S/p \wedge S/p = S/p \vee \Sigma S/p$ and so to check commutativity of the diagram it is enough to restrict to each factor. We are then comparing maps in $[\Sigma S/p, \Sigma S/p] = [S/p, S/p]$ and $[S/p, \Sigma S/p]$. Both groups are cyclic of order p and generated by 1 and β, respectively. Since 1 and β are homologically non-trivial, the lemma follows from the fact that the diagram commutes after applying homology. □

Our motivation for constructing the map (8.1.10) was, in fact, to prove the following lemma.

Lemma 8.1.14. *The exists a map of towers*

$$\{S/p^{n-*}, S/p\} \wedge \{S/p^{n-*}, S/p\} \longrightarrow \{S/p^{n-*}, S/p\}$$

(recall definition 8.1.7) compatible with the map of (8.1.10).

Proof. Take the cofibrant pointset level model for $\{S^0, S/p\}$ which was used to define $\{S/p^{n-*}, S/p\}$ and consider the underlying map of sequences of (8.1.10). We use the "homotopy extension property" that S-modules satisfy, just like in the proof of lemma 7.1.2. It says that we can make any of the squares in the map of sequences strictly commute at the cost of changing the left map to a homotopic one. The homotopy we extend should be the one determined by the map on cofibers. By starting at the $(2n-1)$-st position, we can make the first $(2n-1)$ squares commute strictly. One obtains the map of the lemma by collapsing out the n-th copy of S^0 in $\{S^0, S/p\}$. □

For us, the purpose of a tower is to a construct spectral sequence.

Definition 8.1.15. The $\{X, I\}$-*spectral sequence* is the spectral sequence obtained from the exact couple got by applying $\pi_*(-)$ to a given tower $\{X, I\}$. For $s \geq 0$, it has

$$E_1^{s,t}(\{X, I\}) = \pi_{t-s}(I^s) = \pi_t(\Sigma^s I^s)$$

and $E_1^{\bullet,t} = \pi_t(\Sigma^\bullet I^\bullet)$ as chain complexes. It attempts to converge to $\pi_{t-s}(X_0)$.

The filtration is given by $F^s\pi_*(X_0) = \mathrm{im}(\pi_*(X_s) \longrightarrow \pi_*(X_0))$. Given an element in $F^s\pi_*(X_0)$ we can obtain a permanent cycle by lifting to $\pi_*(X_s)$ and mapping this lift down to $\pi_*(I^s)$.

Smashing together towers enables us to construct pairings of such spectral sequences.

Proposition 8.1.16 ([5, 4.4])**.** *We have a pairing of spectral sequences*

$$E_r^{s,t}(\{X, I\}) \otimes E_r^{s',t'}(\{Y, J\}) \longrightarrow E_r^{s+s',t+t'}(\{X, I\} \wedge \{Y, J\}).$$

At the E_1-page the pairing is given by the natural map

$$\pi_{t-s}(I^s) \otimes \pi_{t'-s'}(J^{s'}) \xrightarrow{\wedge} \pi_{(t+t')-(s+s')}(I^s \wedge J^{s'}) \longrightarrow \pi_{(t+t')-(s+s')}(K^{s+s'}),$$

where K is as in definition 8.1.8. If all the spectral sequences converge then the pairing converges to the smash product $\wedge : \pi_*(X_0) \otimes \pi_*(Y_0) \longrightarrow \pi_*(X_0 \wedge Y_0)$.

People often only talk about the Adams spectral sequence for a spectrum X from the E_2-page onwards. Our definition gives a functorial construction from the E_1-page.

Recall again, from [5, 11], the definition of the H-canonical resolution.

Notation 8.1.17. We write $\{\overline{H}^{\wedge *}, H^{[*]}\}$ for the H-canonical resolution of S^0. Here we mimic the notation used in [2], and intend for $H^{[s]}$ to mean $H \wedge \overline{H}^{\wedge s}$. The H-canonical resolution for a spectrum X is obtained by smashing with the tower whose underlying augmented cochain complex has augmentation $X \longrightarrow C^\bullet$ given by the identity; we write $\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge X$.

Definition 8.1.18. Suppose X is any spectrum. The *Adams spectral sequence for X* is the $\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge X$-spectral sequence.

The E_1-page of the Adams spectral sequence can be identified with the cobar complex $\Omega^\bullet(A; H_*(X))$ and there exists a map of towers

$$\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge \{\overline{H}^{\wedge *}, H^{[*]}\} \longrightarrow \{\overline{H}^{\wedge *}, H^{[*]}\}$$

such that the induced pairing on E_1-pages is the multiplication on $\Omega^\bullet(A)$. This gives the following properties of the Adams spectral sequence for X, which we list as a proposition.

Proposition 8.1.19. *The Adams spectral sequence is functorial in X and it has E_1-page given by $\Omega^\bullet(A; H_*(X))$. We have a pairing of Adams spectral sequences*

$$E_r^{s,t}(X) \otimes E_r^{s',t'}(Y) \longrightarrow E_r^{s+s',t+t'}(X \wedge Y)$$

which, at the E_1-page, agrees with the following multiplication (see [10, pg. 76]).

$$\Omega^\bullet(A; H_*(X)) \otimes \Omega^\bullet(A; H_*(Y)) \longrightarrow \Omega^\bullet(A; H_*(X) \otimes^\Delta H_*(Y)) = \Omega^\bullet(A; H_*(X \wedge Y))$$

Providing X is p-complete the Adams spectral sequence for X converges to $\pi_*(X)$ in the sense of definition 2.2.2, case 1. If each Adams spectral sequence converges then the pairing above converges to the smash product $\wedge : \pi_*(X) \otimes \pi_*(Y) \longrightarrow \pi_*(X \wedge Y)$.

8.2 The modified Adams spectral sequence for S/p^n

When one starts to calculate $H^*(A; H_*(S/p))$, the Adams E_2-page for the mod p Moore spectrum, the first step is to describe the A-comodule $H_*(S/p)$. It is the subalgebra of A in A-comodules given by $E[\tau_0]$. In particular, it has a nontrivial A-coaction. For $n \geq 2$, $H_*(S/p^n)$ has trivial A-coaction which means that $H^*(A; H_*(S/p^n))$ is two copies of the Adam E_2-page for the sphere. We would like the E_2-page to reflect that fact that the multiplication by p^n-map is zero on S/p^n. This is the case when we set up the modified Adams spectral sequence for S/p^n.

Recall 8.1.17 and definition 8.1.7.

Definition 8.2.1. The *modified Adams spectral sequence for S/p^n (MASS-n)* is the $\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge \{S/p^{n-*}, S/p\}$-spectral sequence.

Smashing the maps of towers (recall lemma 8.1.14)

$$\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge \{\overline{H}^{\wedge *}, H^{[*]}\} \longrightarrow \{\overline{H}^{\wedge *}, H^{[*]}\},$$

$$\{S/p^{n-*}, S/p\} \wedge \{S/p^{n-*}, S/p\} \longrightarrow \{S/p^{n-*}, S/p\}$$

and composing with the swap map, we obtain a map of towers

$$\Big[\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge \{S/p^{n-*}, S/p\}\Big]^{\wedge 2} \longrightarrow \{\overline{H}^{\wedge *}, H^{[*]}\} \wedge \{S/p^{n-*}, S/p\}$$

extending the multiplication $S/p^n \wedge S/p^n \longrightarrow S/p^n$. By proposition 8.1.16, the MASS-n is multiplicative.

We turn to the structure of the E_1-page. First, we note that the underlying chain complex of $\{S/p^{n-*}, S/p\}$ is a truncated version of $\Sigma^\bullet S/p$:

$$S/p \xrightarrow{\beta} \Sigma S/p \xrightarrow{\beta} \Sigma^2 S/p \longrightarrow \ldots \longrightarrow \Sigma^{n-1} S/p \longrightarrow * \longrightarrow * \longrightarrow \ldots$$

Definition 8.2.2. Write $\mathfrak{B}^\bullet$ for $H_*(\Sigma^\bullet S/p)$ and $\mathfrak{B}(n)^\bullet$ for the homology of the complex just noted. Write $1_j, \tau_{0,j}$ for the $\mathbb{F}_p$-basis elements of $\mathfrak{B}^j$, and also for their images

in $\mathfrak{B}(n)^j$. Note that 1_j and $\tau_{0,j}$ are zero in $\mathfrak{B}(n)$ for $j \geq n$.

Since $\Sigma^\bullet S/p$ and its truncation are ring objects in $\mathrm{Ch}(\mathscr{S})$ we see that $\mathfrak{B}^\bullet$ and $\mathfrak{B}(n)^\bullet$ are DG algebras over A. Moreover, using the same identification used for the Adams E_1-page, we see that $E_1^{\bullet,*}(\text{MASS-}n) = \Omega^\bullet(A; \mathfrak{B}(n)^\bullet)$, as DG algebras. This cobar complex has coefficients in a DG algebra. Such a set up is described in [10].

To describe the E_2-page we need the following theorem and lemma.

Theorem 8.2.3 ([10, pg. 80])**.** *For any differential A-comodule $\mathfrak{M}^\bullet$ which is bounded below we have a homology isomorphism*

$$\Omega^\bullet(A; \mathfrak{M}^\bullet) \longrightarrow \Omega^\bullet(P; Q \otimes_\theta \mathfrak{M}^\bullet).$$

Here, θ is a twisting homomorphism $E \longrightarrow Q$; E is the exterior part of A and θ takes $1 \longmapsto 0$, $\tau_n \longmapsto q_n$, and $\tau_{n_1} \cdots \tau_{n_r} \longmapsto 0$ when $r > 1$.

Lemma 8.2.4. *We have a homology isomorphism*

$$\Omega^\bullet(P; Q \otimes_\theta \mathfrak{B}(n)^\bullet) \longrightarrow \Omega^\bullet(P; Q/q_0^n).$$

Moreover, this is a map of differential algebras.

Proof. A short calculation in $Q \otimes_\theta \mathfrak{B}(n)^\bullet$ shows that

$$d(q \otimes 1_j) = 0 \text{ and } d(q \otimes \tau_{0,j}) = q_0 q \otimes 1_j - q \otimes 1_{j+1}.$$

[A sign might be wrong here but the end result will still be the same.] Define a map

$$Q \otimes_\theta \mathfrak{B}(n)^\bullet \longrightarrow Q/q_0^n$$

by $q \otimes 1_j \longmapsto q_0^j q$ and $q \otimes \tau_{0,j} \longmapsto 0$. This is a map of differential algebras over P, where the target has a trivial differential. In addition, it is a homology isomorphism and so the Eilenberg-Moore spectral sequence completes the proof. □

We should keeping track of the gradings under the maps we use:

$$\begin{aligned} E_1^{\sigma,\lambda}(\text{MASS-}n) &= \bigoplus_{i+j=\sigma} \Omega^{i,\lambda}(A; \mathfrak{B}(n)^j) \\ &\longrightarrow \bigoplus_{\substack{i+j=\sigma \\ s+\Xi=i \\ u+\Xi=\lambda}} \Omega^{s,u}(P; Q^{\Xi} \otimes_\theta \mathfrak{B}(n)^j) \\ &\longrightarrow \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \Omega^{s,u}(P; [Q/q_0^n]^t). \end{aligned}$$

We summarize what we have proved.

Proposition 8.2.5. *The modified Adams spectral sequence for S/p^n (MASS-n) is a multiplicative spectral sequence with E_1-page $\Omega^\bullet(A; \mathfrak{B}(n)^\bullet)$ and*

$$E_2^{\sigma,\lambda}(\textit{MASS-}n) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} H^{s,u}(P; [Q/q_0^n]^t).$$

We also make note of a modified Adams spectral sequence for $BP \wedge S/p^n$, which receives the BP-Hurewicz homomorphism from the MASS-n.

Definition 8.2.6. The *modified Adams spectral sequence for $BP \wedge S/p^n$ (MASS-BP-n)* is the $\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge BP \wedge \{S/p^{n-*}, S/p\}$-spectral sequence, where BP denotes the tower whose underlying augmented cochain complex has augmentation $BP \longrightarrow C^\bullet$ given by the identity.

In the identification of the E_1 and E_2-page of this spectral sequence $\mathfrak{B}^\bullet$ is replaced by $H_*(BP \wedge \Sigma^\bullet S/p) = P \otimes^\Delta \mathfrak{B}^\bullet$. By using a shearing isomorphism, we obtain the following proposition.

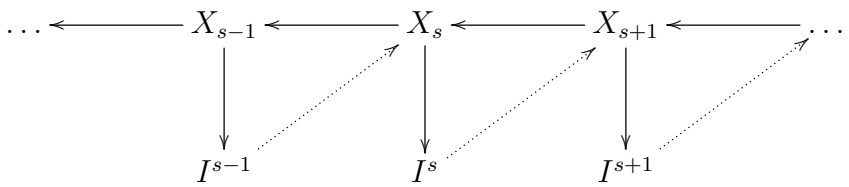
Proposition 8.2.7. *The modified Adams spectral sequence for $BP \wedge S/p^n$ (MASS-BP-n) is a multiplicative spectral sequence with E_1-page $\Omega^\bullet(A; P \otimes^\Delta \mathfrak{B}(n)^\bullet)$ and*

$$E_2^{\sigma,\lambda}(\textit{MASS-BP-}n) = E_\infty^{\sigma,\lambda}(\textit{MASS-BP-}n) = [Q/q_0^n]^{\sigma,\lambda-\sigma}.$$

8.3 The modified Adams spectral sequence for S/p^∞

The cleanest way to define our modified Adams spectral sequence for the Prüfer sphere involves defining a reindexed MASS-n. To make sense of the reindexing geometrically, we extend our definition of sequences and towers.

Definition 8.3.1. An object $\{X_s\}$ of the diagram category $\mathscr{S}^{\mathbb{Z}}$ is called a $\mathbb{Z}$-*sequence*. A system of interlocking cofibration sequences



in $\mathscr{S}$, where $s \in \mathbb{Z}$, is called a $\mathbb{Z}$-*tower* and we use the notation $\{X, I\}$. A $\mathbb{Z}$-tower is said to be *bounded below* if there is an $N \in \mathbb{Z}$ such that $I_s = *$ for $s < N$. Notice that a $\mathbb{Z}$-tower $\{X, I\}$ has an *underlying* $\mathbb{Z}$-*sequence* $\{X_s\}$.

A *map of* $\mathbb{Z}$-*towers* $\{X, I\} \longrightarrow \{Y, J\}$ is a compatible collection of maps

$$\{X_s \longrightarrow Y_s\} \cup \{I^s \longrightarrow J^s\}.$$

We can still smash together bounded below $\mathbb{Z}$-towers and they still give rise to a spectral sequence.

Definition 8.3.2. Let $\{S/p^{\min\{-*,n\}}, S/p\}$ be the bounded below $\mathbb{Z}$-tower obtained from $\{S/p^{n-*}, S/p\}$ by shifting it n positions to the left.

Definition 8.3.3. The *reindexed Adams spectral sequence for* S/p^n *(RASS-n)* is the $\{\overline{H}^{\wedge *}, H^{[*]}\} \wedge \{S/p^{\min\{-*,n\}}, S/p\}$-spectral sequence.

Recall definition 3.1.4. We see immediately from proposition 8.2.5 that

$$E_2^{\sigma,\lambda}(\text{RASS-}n) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} H^{s,u}(P; [M_n]^t).$$

Moreover, there are maps of bounded below $\mathbb{Z}$-towers

$$\{S/p^{\min\{-*,n\}}, S/p\} \longrightarrow \{S/p^{\min\{-*,n+1\}}, S/p\},$$

which give maps of spectral sequences from the RASS-n to the RASS-$(n+1)$. Chasing through the identification of the E_2-pages one see that the map at E_2-pages is induced by the inclusion $M_n \longrightarrow M_{n+1}$.

Definition 8.3.4. The *modified Adams spectral sequence for S/p^∞ (MASS-∞)* is the colimit of the reindexed Adams spectral sequences for S/p^n. It has

$$E_2^{\sigma,\lambda}(\text{MASS-}\infty) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} H^{s,u}(P; [Q/q_0^\infty]^t).$$

There are some technicalities to worry about when taking the colimit of spectral sequences. We resolve such issues in appendix B.

By definition, we have a map of spectral sequences from the RASS-n to the MASS-∞. Lemma A.4 provides the following corollary to lemma 4.1.10.

Corollary 8.3.5. *The map $E_\infty^{\sigma,\lambda}(RASS\text{-}(n+1)) \longrightarrow E_\infty^{\sigma,\lambda}(MASS\text{-}\infty)$ is surjective when $\lambda - \sigma = p^n q$ and $\sigma \geq p^n - n - 1$.*

8.4 A permanent cycle in the MASS-$(n+1)$

In order to localize the MASS-$(n+1)$ we need to find a permanent cycle detecting a v_1-self map

$$v_1^{p^n} : S/p^{n+1} \longrightarrow \Sigma^{-p^n q} S/p^{n+1}.$$

The following theorem provides such a permanent cycle.

Theorem 8.4.1. *The element $q_1^{p^n}$ in $H^{0,p^n q}(P; [Q/q_0^{n+1}]^{p^n})$ is a permanent cycle in the MASS-$(n+1)$.*

Proof. By definition, the RASS-$(n+1)$ is obtained by reindexing the MASS-$(n+1)$ and so it is equivalent to prove that $q_1^{p^n}/q_0^{n+1} \in H^{0,p^n q}(P; [M_{n+1}]^{p^n - n - 1})$ is a permanent

cycle in the RASS-$(n+1)$. Lemmas 4.1.10 and A.4 show that it is enough to prove that $q_1^{p^n}/q_0^{n+1} \in H^{0,p^n q}(P; [Q/q_0^\infty]^{p^n-n-1})$ is a permanent cycle in the MASS-∞. The map of spectral sequences induced by $\Sigma^{-1}S/p^\infty \longrightarrow S^0$ (proposition A.1) is an isomorphism in this range and so we are left with showing that $\partial(q_1^{p^n}/q_0^{n+1}) = q_0^{p^n-n-1}h_{1,n}$ is a permanent cycle in the ASS, but this is the content of theorem 7.3.1. □

Pick a representative for $q_1^{p^n}$ in $\pi_{p^n q}(S/p^{n+1})$. Using the map of spectral sequences from the MASS-$(n+1)$ to the MASS-BP-1 and the fact that $q_1^{p^n}$ in Q/q_0 has the highest monomial weight, i.e. modified Adams filtration, among elements of the same internal degree, we see that the image of the chosen representative in $BP_*(S/p)$ is $v_1^{p^n}$. Thus, tensoring up any representative for $q_1^{p^n}$ to a self-map $v_1^{p^n} : S/p^{n+1} \longrightarrow \Sigma^{-p^n q}S/p^{n+1}$ defines a v_1 self-map. Corollary 8.3.5 tells us that it is possible to refine our choice of a representative for $q_1^{p^n}$ so that it maps to the α of theorem 7.3.1 under $\Sigma^{-1}S/p^{n+1} \to S^0$. This completes the proof of the final theorem stated in the introduction.

8.5 The localized Adams spectral sequences

Since $q_1^{p^{n-1}}$ is a permanent cycle in the MASS-n, multiplication by $q_1^{p^{n-1}}$ defines a map of spectral sequences, which enables us to make the following definition.

Definition 8.5.1. The *localized Adams spectral sequence for $v_1^{-1}S/p^n$ (LASS-n)* is the colimit of the following diagram of spectral sequences.

$$E_*^{*,*}(\text{MASS-}n) \xrightarrow{q_1^{p^{n-1}}} E_*^{*,*}(\text{MASS-}n) \xrightarrow{q_1^{p^{n-1}}} E_*^{*,*}(\text{MASS-}n) \xrightarrow{q_1^{p^{n-1}}} \dots$$

It has

$$E_2^{\sigma,\lambda}(\text{LASS-}n) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} H^{s,u}(P; [q_1^{-1}Q/q_0^n]^t).$$

Since the MASS-n is multiplicative, the differentials in the LASS-n are derivations.

The following diagram commutes when $r = 2$.

$$\begin{array}{ccc} E_r^{*,*}(\text{RASS-}n) & \xrightarrow{q_1^{p^n}} & E_r^{*,*}(\text{RASS-}n) \\ \downarrow & & \downarrow \\ E_r^{*,*}(\text{RASS-}(n+1)) & \xrightarrow{q_1^{p^n}} & E_r^{*,*}(\text{RASS-}(n+1)) \end{array}$$

Taking homology, we see, inductively, that it commutes for all $r \geq 2$. This means that we have maps of spectral sequences between *reindexed* localized Adams spectral sequences for $v_1^{-1}S/p^n$ and so we can make the following definition.

Definition 8.5.2. The *localized Adams spectral sequence for the v_1-periodic sphere (LASS-∞)* is the colimit of the apparent reindexed localized Adams spectral sequences for $v_1^{-1}S/p^n$. It has

$$E_2^{\sigma,\lambda}(\text{LASS-}\infty) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} H^{s,u}(P; [q_1^{-1}Q/q_0^{\infty}]^t).$$

8.6 Calculating the LASS-∞

Our calculation of the LASS-∞ imitates that of the loc.alg.NSS. First, we note some permanent cycles in the MASS-∞ and LASS-∞.

Proposition 8.6.1. *For $n \geq 1$ and $k \geq 0$, $q_1^{kp^{n-1}}/q_0^n$ is a permanent cycle in the MASS-∞. For $n \geq 1$ and $k \in \mathbb{Z}$, $q_1^{kp^{n-1}}/q_0^n$ is a permanent cycle in the LASS-∞.*

Proof. In the first case, $q_1^{kp^{n-1}}$ is permanent cycle in the MASS-n and so $q_1^{kp^{n-1}}/q_0^n$ is a permanent cycle in the RASS-n and the MASS-∞. In the second case, $q_1^{kp^{n-1}}$ is permanent cycle in the LASS-n and the same argument gives the result. □

Corollary 5.5.2 describes the associated graded of the E_2-page of the LASS-∞ with respect to the Bockstein filtration and we claim that

$$d_2 : E_2^{\sigma,\lambda}(\text{LASS-}\infty) \longrightarrow E_2^{\sigma+2,\lambda+1}(\text{LASS-}\infty)$$

respects the Bockstein filtration.

Note that $q_0 \in H^*(P; q_1^{-1}Q/q_0^n)$ is a permanent cycle in the LASS-n. Because d_2 is a derivation in the LASS-n, multiplication by q_0 commutes with d_2. Thus, we find the same in the reindexed localized Adams spectral sequences for $v_1^{-1}S/p^n$ and hence, in the LASS-∞, too. This verifies the claim.

We conclude that we have a *filtration spectral sequence (q_0-FILT2)*

$$E_0^{\sigma,\lambda,v}(q_0\text{-FILT2}) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} E_\infty^{s,t,u,v}(q_1^{-1}\text{-BSS}) \overset{v}{\Longrightarrow} E_3^{\sigma,\lambda}(\text{LASS-}\infty).$$

Our calculation of this spectral sequence comes down to the calculation of the E_1-page of the q_0-FILT spectral sequence made in proposition 6.3.1.

In appendix A we show that each of the maps in the exact couple defining the q_1^{-1}-BSS comes from a map of localized Adams spectral sequences. This means that if $x \in H^*(P; q_1^{-1}Q/q_0)$ and $q_0^v x \in E_\infty(q_1^{-1}\text{-BSS})$ then $d_0^{q_0\text{-FILT2}}(q_0^v x) = q_0^v d_2^{\text{LASS-1}} x$. The S/p analog of theorem 1.4.7 therefore tells us that

$$d_0^{q_0\text{-FILT2}} : \bigoplus_{\substack{s\geq\hat{s} \\ s+t=\sigma \\ u+t=\lambda}} E_\infty^{s,t,u,v}(q_1^{-1}\text{-BSS}) \longrightarrow \bigoplus_{\substack{s\geq\hat{s}+1 \\ s+t=\sigma+2 \\ u+t=\lambda+1}} E_\infty^{s,t,u,v}(q_1^{-1}\text{-BSS})$$

and that if x lies in a single trigrading, then

$$d_0^{q_0\text{-FILT2}}(q_0^v x) \equiv q_0^v d_2^{v_1\text{-alg.NSS}} x = d_0^{q_0\text{-FILT}}(q_0^v x)$$

up to terms with higher s-grading. Thus, using a filtration spectral sequence with respect to the s-grading we deduce from proposition 6.3.1 that there is an $\mathbb{F}_p$-vector space isomorphism

$$E_1^{\sigma,\lambda,v}(q_0\text{-FILT2}) \cong \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} E_1^{s,t,u,v}(q_0\text{-FILT}).$$

In appendix B the LASS-∞ is shown to converge to $\pi_*(v_1^{-1}S/p^\infty)$. Moreover, since

the localized Adams-Novikov spectral for $\pi_*(v_1^{-1}S/p^\infty)$ is degenerate and convergent (the height 1 telescope conjecture is true), we know precisely what group the LASS-∞ converges to. From the bound on the size of the E_3-page given by our knowledge of $E_1(q_0\text{-FILT2})$ we can deduce the following proposition.

Proposition 8.6.2. *For $r \geq 3$, there exist isomorphisms*

$$E_r^{\sigma,\lambda}(\mathit{LASS}\text{-}\infty) \cong \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} E_r^{s,t,u}(\mathit{loc.alg.NSS})$$

compatible with differentials.

$E_\infty(\mathit{LASS}\text{-}\infty)$ has an $\mathbb{F}_p$-basis given by the classes of the following elements.

$$\left\{q_0^v : v < 0\right\} \cup \left\{q_0^v q_1^{kp^{n-1}} \ : \ n \geq 1,\ k \in \mathbb{Z} - p\mathbb{Z},\ -n \leq v < 0\right\}$$
$$\cup \left\{\overline{q_0^v \epsilon_n} \ : \ n \geq 1,\ 1 - p^n \leq v < 0\right\}$$

Here, $\overline{q_0^v \epsilon_n}$ denotes an element of $E_3(\mathit{LASS}\text{-}\infty)$ corresponding to the element of the same name in $E_3(\mathit{loc.alg.NSS})$ (see proposition 6.4.1).

8.7 The Adams spectral sequence

The following two corollaries show that our calculation of the LASS-∞ has implications for the Adams spectral sequence.

First, lemma A.4 provides the following corollary to proposition 4.3.3.

Corollary 8.7.1. *The localization map $E_3^{\sigma,\lambda}(\mathit{MASS}\text{-}\infty) \longrightarrow E_3^{\sigma,\lambda}(\mathit{LASS}\text{-}\infty)$*

1. *is a surjection if $\lambda < p(p-1)(\sigma+1) - 2$, i.e. $\lambda - \sigma < (p^2 - p - 1)(\sigma + 1) - 1$;*

2. *is an isomorphism if $\lambda - 1 < p(p-1)(\sigma-1) - 2$,*

$$\textit{i.e. } \lambda - \sigma < (p^2 - p - 1)(\sigma - 1) - 2.$$

Using the map of spectral sequences induced by $\Sigma^{-1}S/p^\infty \longrightarrow S^0$ (proposition A.1) we obtain the following corollary.

Corollary 8.7.2. *$E_3^{\sigma,\lambda}(ASS) \cong E_3^{\sigma-1,\lambda}(LASS\text{-}\infty)$ if $\lambda - \sigma < (p^2-p-1)(\sigma-2)-3$ and $\lambda - \sigma > 0$.*

The line of the corollary just stated is drawn in green in figure 1-1.

One has to be careful when discussing higher differentials in the Adams spectral sequence. Here is what we know:

- There are permanent cycles at the top of each principal tower, the images of the following elements under the map $\partial : H^*(P;Q/q_0^\infty) \longrightarrow H^*(P;Q) \cong H^*(A)$.

$$\left\{ q_0^v q_1^{kp^{n-1}} \ : \ n \geq 1,\ k \in \mathbb{Z} - p\mathbb{Z},\ k \geq 1,\ -n \leq v < 0 \right\}$$

  Every other element in a principal tower supports a nontrivial differential.

- An element of a side tower in the Adams spectral sequence cannot be hit by a shorter differential than the corresponding element of the LASS-∞.

- A non-permanent cycle in a principal tower in the Adams spectral sequence above the line of the previous corollary supports a differential of the expected length.

- A non-permanent cycle in a principal tower in the Adams spectral sequence cannot support a longer nontrivial differential than the corresponding element of the LASS-∞, but perhaps it supports a shorter one than expected, leaving an element of a side tower to detect a nontrivial homotopy class.

Looking at the charts of Nassau [14], we cannot find an example of the final phenomenon, but the class $b_{1,0} \in E_2^{2,pq}(\text{ASS})$ gives a related example. [Similarly, one could consider the potential $p = 3$ Arf invariant elements.] We describe this example presently.

Under the isomorphism $E_2^{2,pq}(\text{ASS}) \cong E_2^{1,pq}(\text{MASS-}\infty)$, $b_{1,0}$ is mapped to an element detected by $q_0^{-(p-1)}q_1^p\epsilon_1$ in the q_0^∞-BSS. Under the localization map this element maps to an element of $E_2^{1,pq}(\text{LASS-}\infty)$ detected by $q_0^{-(p-1)}q_1^p\epsilon_1$ in the q_1^{-1}-BSS.

The element

$$x = q_0^{-p-1}q_1^p - q_0^{-1}q_1^{-1}q_2 \in H^{0,pq}(P; [q_1^{-1}Q/q_0^\infty]^{-1}) \subset E_2^{-1,pq-1}(\text{LASS-}\infty)$$

is detected by $q_0^{-p-1}q_1^p$ in the q_1^{-1}-BSS. Our calculation of the LASS-∞ shows that $d_2x \in E_2^{1,pq}(\text{LASS-}\infty)$ is nonzero. Moreover, we find that $E_2^{1,pq}(\text{LASS-}\infty) = \mathbb{F}_p$ so that a unit multiple of x maps via d_2 to the localization of $b_{1,0}$. This demonstrates the well-known fact that $\beta \in \pi_{pq-2}(S^0)$ is not v_1-periodic.

Appendix A

Maps of spectral sequences

The most difficult result of this appendix is the following proposition.

Proposition A.1. *There are maps of spectral sequences*

$$\begin{array}{ccc} RASS\text{-}n & \longrightarrow & ASS \\ \downarrow & & \downarrow = \\ MASS\text{-}\infty & \longrightarrow & ASS \\ = \downarrow & & \downarrow \\ MASS\text{-}\infty & \longrightarrow & MASS\text{-}1 \end{array} \qquad \textit{induced by} \qquad \begin{array}{ccc} \Sigma^{-1}S/p^n & \longrightarrow & S^0 \\ \downarrow & & \downarrow = \\ \Sigma^{-1}S/p^\infty & \longrightarrow & S^0 \\ = \downarrow & & \downarrow \\ \Sigma^{-1}S/p^\infty & \longrightarrow & S/p. \end{array}$$

At E_2-pages we get the maps by taking the connecting homomorphisms in the long exact sequences got by applying $H^*(P;-)$ to the following short exact sequences of P-comodules (recall definition 3.1.4).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \longrightarrow & Q\langle q_0^{-n}\rangle & \longrightarrow & M_n & \longrightarrow & 0 \\ & & = \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & q_0^{-1}Q & \longrightarrow & Q/q_0^\infty & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle /q_0} & & \downarrow = & & \\ 0 & \longrightarrow & Q/q_0 & \longrightarrow & Q/q_0^\infty & \xrightarrow{q_0} & Q/q_0^\infty & \longrightarrow & 0 \end{array}$$

The previous proposition was required in the proof of theorem 8.4.1, which was necessary to construct the LASS-n and the LASS-∞. We record, for completeness, the following lemma.

Lemma A.2. *There are maps of spectral sequences*

$$\begin{array}{rcll}
MASS\text{-}(n+1) & \longrightarrow & MASS\text{-}n & \text{induced by } S/p^{n+1} \longrightarrow S/p^n \\
MASS\text{-}n & \longrightarrow & MASS\text{-}BP\text{-}n & \text{induced by } S/p^n \longrightarrow BP \wedge S/p^n \\
RASS\text{-}n & \longrightarrow & RASS\text{-}(n+1) & \text{induced by } S/p^n \xrightarrow{p} S/p^{n+1} \\
RASS\text{-}n & \longrightarrow & MASS\text{-}\infty & \text{induced by } S/p^n \longrightarrow S/p^\infty \\
MASS\text{-}(n+1) & \longrightarrow & MASS\text{-}(n+1) & \text{induced by } S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-p^n q} S/p^{n+1} \\
MASS\text{-}n & \longrightarrow & LASS\text{-}n & \text{induced by } S/p^n \longrightarrow v_1^{-1} S/p^n \\
RASS\text{-}(n+1) & \longrightarrow & RASS\text{-}(n+1) & \text{induced by } S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-p^n q} S/p^{n+1} \\
MASS\text{-}\infty & \longrightarrow & LASS\text{-}\infty & \text{induced by } S/p^\infty \longrightarrow v_1^{-1} S/p^\infty
\end{array}$$

To calculate the LASS-∞ we used the q_0-FILT2 spectral sequence. To calculate $E_1(q_0\text{-FILT2})$ we required the following proposition.

Proposition A.3. *There are maps of spectral sequences induced by the following cofibration sequences.*

$$S/p \longrightarrow S/p^\infty \longrightarrow S/p^\infty$$

$$v_1^{-1} S/p \longrightarrow v_1^{-1} S/p^\infty \longrightarrow v_1^{-1} S/p^\infty$$

At E_2-pages the maps are the ones in the exact couples defining the q_0^∞-BSS and the q_1^{-1}-BSS, respectively.

Often we have a map of spectral sequences and we know that on a given page, at various bidegrees, we have surjections and injections. The following lemma tells us that if we have a map of spectral sequences, a d_r-differential, and that the map on the E_r-pages is a surjection at the source of the differential and an injection at the target of the differential, then we have a surjection and an injection at the same positions on the E_{r+1}-page. The proof is a diagram chase.

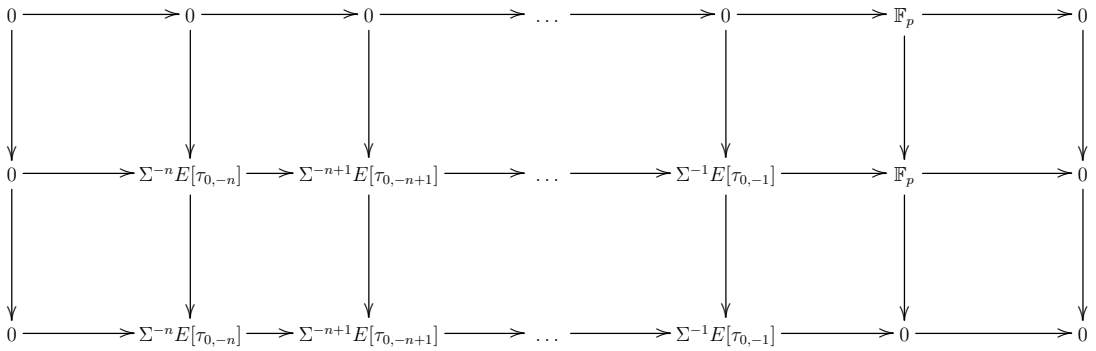
Lemma A.4. *Suppose $C^\bullet \longrightarrow D^\bullet$ is a map of cochain complexes (in abelian groups), that $C^n \longrightarrow D^n$ is surjective and $C^{n+1} \longrightarrow D^{n+1}$ is injective. Then $H^n(C^\bullet) \longrightarrow H^n(D^\bullet)$ is surjective and $H^{n+1}(C^\bullet) \longrightarrow H^{n+1}(D^\bullet)$ is injective.*

We now turn to the proofs of the two propositions.

Proof of proposition A.1. The map RASS-$n \longrightarrow$ MASS-∞ is given by definition. The map ASS $\longrightarrow$ MASS-1 is just a normal map of Adams spectral sequences induced by $S^0 \longrightarrow S/p$. The difficult map to construct is the one induced by $\Sigma^{-1}S/p^n \longrightarrow S^0$. We turn to this presently.

The idea is to start with the connecting homomorphism corresponding to the short exact sequence of P-comodules $0 \longrightarrow Q \longrightarrow Q\langle q_0^{-n}\rangle \longrightarrow M_n \longrightarrow 0$, and try to realize it geometrically.

Consider the following short exact sequence of cochain complexes in A-comodules.



The suspensions indicate cohomological degree. The first complex is concentrated in cohomological degree 0 and is just $\mathbb{F}_p$. The last complex is a shifted version of $\mathfrak{B}(n)^\bullet$, which we call $\mathfrak{B}^-(n)^\bullet$. We call the middle complex $\mathfrak{C}(n)^\bullet$. We will show that the connecting homomorphism of interest is the same as the connecting homomorphism corresponding to the short exact sequence of differential A-comodules

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathfrak{C}(n)^\bullet \longrightarrow \mathfrak{B}^-(n)^\bullet \longrightarrow 0.$$

Recall lemma 8.2.4 which helped us to identify the E_2-page of the MASS-n. We

used a map

$$Q \otimes_\theta \mathfrak{B}(n)^\bullet \longrightarrow Q/q_0^n$$

defined by $q \otimes 1_j \longmapsto q_0^j q$ and $q \otimes \tau_{0,j} \longmapsto 0$. Similarly, we have maps making the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q \otimes_\theta \mathbb{F}_p & \longrightarrow & Q \otimes_\theta \mathfrak{C}(n)^\bullet & \longrightarrow & Q \otimes_\theta \mathfrak{B}^-(n)^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & Q\langle q_0^{-n} \rangle & \longrightarrow & M_n & \longrightarrow & 0 \end{array}$$

Theorem 8.2.3 was also important in identifying the E_2-page of the MASS-n. Using it again, together with the maps just defined, we find that we have a diagram of cochain complexes.

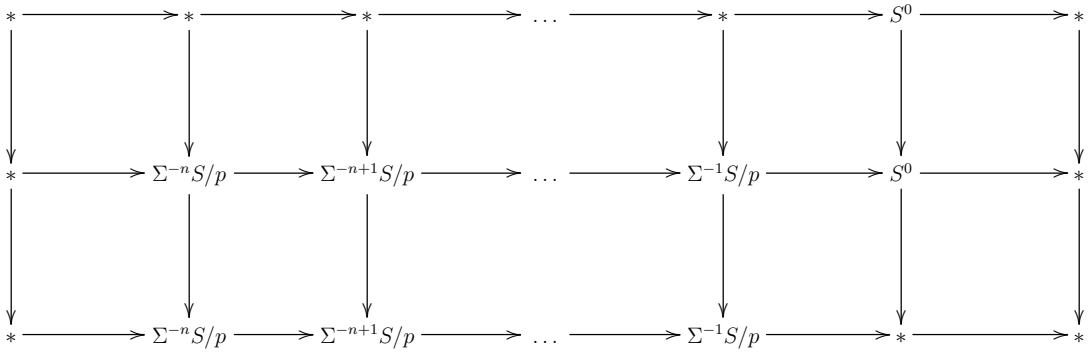
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet(A; \mathbb{F}_p) & \longrightarrow & \Omega^\bullet(A; \mathfrak{C}(n)^\bullet) & \longrightarrow & \Omega^\bullet(A; \mathfrak{B}^-(n)^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet(P; Q) & \longrightarrow & \Omega^\bullet(P; Q\langle q_0^{-n} \rangle) & \longrightarrow & \Omega^\bullet(P; M_n) & \longrightarrow & 0 \end{array}$$

Each of the vertical maps is a homology isomorphism and so we can calculate the connecting homomorphism of interest using, instead, the connecting homomorphism associated with $0 \longrightarrow \mathbb{F}_p \longrightarrow \mathfrak{C}(n)^\bullet \longrightarrow \mathfrak{B}^-(n)^\bullet \longrightarrow 0$.

The connecting homomorphism for this short exact sequence can be described even more explicitly than is usual. Lifting under the map $\mathfrak{C}(n)^\bullet \twoheadrightarrow \mathfrak{B}^-(n)^\bullet$ can done using the unique A-comodule splitting $\mathfrak{B}^-(n)^\bullet \hookrightarrow \mathfrak{C}(n)^\bullet$, which puts a zero in the $\mathbb{F}_p$ spot. Similarly, the map $\mathbb{F}_p \hookrightarrow \mathfrak{C}(n)^\bullet$ has a unique A-comodule splitting $\mathfrak{C}(n)^\bullet \twoheadrightarrow \mathbb{F}_p$.

The diagram on the next page realizes the cochain complexes $\mathbb{F}_p$, $\mathfrak{C}(n)^\bullet$, and $\mathfrak{B}^-(n)^\bullet$ geometrically. We note that the first cochain complex comes from the tower S^0 whose underlying $\mathbb{Z}$-sequence is S^0 in nonpositive degrees, $*$ in positive degrees, with identity structure maps where possible. The last cochain complex is the underlying cochain complex of $\{S/p^{\min\{-*,n\}}, S/p\}$, one of the towers used to construct the

RASS-n.



Label these cochain complexes in $\mathscr{S}$ by the same names as the cochain complexes obtained by applying $H_*(-)$ and recall the underlying cochain complex $\Sigma^\bullet H^{[\bullet]}$ of the canonical H-resolution of S^0. The snake lemma for calculating the connecting the homomorphism is realized geometrically by the following composite.

$$\left[\Sigma^\bullet H^{[\bullet]} \wedge \mathfrak{B}^-(n)^\bullet\right]^\sigma \xrightarrow{s} \left[\Sigma^\bullet H^{[\bullet]} \wedge \mathfrak{C}(n)^\bullet\right]^\sigma \xrightarrow{d} \left[\Sigma^\bullet H^{[\bullet]} \wedge \mathfrak{C}(n)^\bullet\right]^{\sigma+1} \xrightarrow{r} \left[\Sigma^\bullet H^{[\bullet]}\right]^{\sigma+1}$$

Here, s and r denote the respective splittings at the level of underlying spectra, d is the differential in the cochain complex $\Sigma^\bullet H^{[\bullet]} \wedge \mathfrak{C}(n)^\bullet$, and we have used that $\Sigma^\bullet H^{[\bullet]} \wedge \mathbb{F}_p = \Sigma^\bullet H^{[\bullet]}$.

To get the map of spectral sequences we just need to define a map of $\mathbb{Z}$-towers $\{S/p^{\min\{-*,n\}}, S/p\} \longrightarrow S^0$, which pairs with $\{\overline{H}^{\wedge *}, H^{[*]}\}$ to give the composite above. Such a map of $\mathbb{Z}$-towers has nonzero degree: it raises cohomological degree by 1. The underlying map of $\mathbb{Z}$-sequences takes $\Sigma^{-1}S/p^j$ to S^0 via the composite

$$\Sigma^{-1}S/p^j \longrightarrow \Sigma^{-1}S/p^\infty \longrightarrow S^0$$

and the map on underlying cochain complexes is the map $\mathfrak{B}^-(n)^\bullet \longrightarrow \mathbb{F}_p$ which, on homology, takes $\tau_{0,-1}$ to 1_0. This completes the construction of the map of spectral sequences RASS-$n \longrightarrow$ ASS induced by $\Sigma^{-1}S/p^n \longrightarrow S^0$.

Since the maps of towers we use are compatible with the maps

$$\{S/p^{\min\{-*,n\}}, S/p\} \longrightarrow \{S/p^{\min\{-*,n+1\}}, S/p\},$$

the maps just constructed pass to the colimit to give the map MASS-$\infty \longrightarrow$ ASS induced by $\Sigma^{-1}S/p^\infty \longrightarrow S^0$. The map MASS-$\infty \longrightarrow$ MASS-1 can be obtained by composition with the map ASS $\longrightarrow$ MASS-1. $\square$

Proof of proposition A.3. The map of spectral sequences induced by the connecting map $\Sigma^{-1}S/p^\infty \longrightarrow S/p$ was constructed in the previous proposition. The map induced by $S/p \longrightarrow S/p^\infty$ is the map MASS-1 $\cong$ RASS-1 $\longrightarrow$ MASS-∞.

The maps MASS-$(n+1) \longrightarrow$ MASS-n induced by $S/p^{n+1} \longrightarrow S/p^n$ can be reindexed to give maps RASS-$(n+1) \longrightarrow$ RASS-n of nonzero degree. Taking a colimit we obtain the map MASS-$\infty \longrightarrow$ MASS-∞ induced by $p : S/p^\infty \longrightarrow S/p^\infty$.

We turn to the localized version. The map induced by $v_1^{-1}S/p \longrightarrow v_1^{-1}S/p^\infty$ is obtained in an identical manner to the unlocalized one, passing through a reindexed localized Adams spectral sequence for $v_1^{-1}S/p$. Similarly, for the map $p : v_1^{-1}S/p^\infty \longrightarrow v_1^{-1}S/p^\infty$, the maps RASS-$(n+1) \longrightarrow$ RASS-n localize to give maps of reindexed localized Adams spectral sequences, and we can take a colimit.

Finally, for the connecting homomorphism we recall that the map RASS-$n \longrightarrow$ MASS-1 is constructed using the map of towers $\{S/p^{\min\{-*,n\}}, S/p\} \longrightarrow S^0 \longrightarrow S/p$, which makes use of the connecting homomorphism $\Sigma^{-1}S/p^\infty \longrightarrow S/p$. Each of the maps RASS-$n \longrightarrow$ MASS-1 localizes. The collection of localized maps is compatible and so defines the requisite map LASS-$\infty \longrightarrow$ LASS-1. $\square$

Appendix B

Convergence of spectral sequences

In this appendix we check that each of the spectral sequences used in this thesis converges in the sense of definition 2.2.2. In particular, we describe how to deal with the technicalities associated with taking the colimits of spectral sequences that appear in the definition of the MASS-∞, the LASS-n, and the LASS-∞.

In definition 8.1.15 we define the filtration of $\pi_*(X_0)$ that is relevant for the $\{X, I\}$-spectral sequence, and we describe a *detection map*

$$F^s\pi_{t-s}(X_0)/F^{s+1}\pi_{t-s}(X_0) \longrightarrow E_\infty^{s,t}(\{X, I\}).$$

This takes care of the filtration and detection map for the MASS-n and we will verify case 1 of definition 2.2.2 to prove the following proposition.

Proposition B.1. *The MASS-n converges to* $\pi_*(S/p^n)$.

Moreover, the filtration associated with the RASS-n is obtained by reindexing the filtration associated with the MASS-n, and our proof will verify case 3 of definition 2.2.2 to give the following corollary.

Corollary B.2. *The RASS-n converges to* $\pi_*(S/p^n)$.

Delaying the proof of these results for now, we note that we have not even defined the filtration or detection map for the MASS-∞, the LASS-n, and the LASS-∞. We

carefully discuss the situation for the MASS-∞ by turning straight to the proof of the following proposition.

Proposition B.3. *The MASS-∞ converges to $\pi_*(S/p^\infty)$.*

Proof. The purpose of a convergent spectral sequence is to identify the associated graded of an abelian group with respect to some convergent filtration. Thus, the most immediate aspects of the MASS-∞ to address are the associated filtration, the E_∞-page and the relationship between the two.

We have injections $F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^n)$, where the F denotes the filtration associated with the RASS-n. Since, the maps $p : S/p^n \longrightarrow S/p^{n+1}$ used to define S/p^∞ are compatible with these filtrations, and filtered colimits preserve exactness, we obtain an injection $\mathrm{colim}_n F^\sigma\pi_*(S/p^n) \longrightarrow \mathrm{colim}_n\pi_*(S/p^n) = \pi_*(S/p^\infty)$. We define

$$F^\sigma\pi_*(S/p^\infty) = \mathrm{im}\left(\mathrm{colim}_n\ F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)\right).$$

When we say that the MASS-∞ is the colimit of the reindexed Adams spectral sequences for S/p^n we mean that

$$E_r^{\sigma,\lambda}(\text{MASS-}\infty) = \mathrm{colim}_n\ E_r^{\sigma,\lambda}(\text{MASS-}n)$$

for each $r \geq 2$. Since filtered colimits commute with homology we have identifications $H^{\sigma,\lambda}(E_r^{*,*}(\text{MASS-}\infty), d_r) = E_{r+1}^{\sigma,\lambda}(\text{MASS-}\infty)$ for each $r \geq 2$, which justifies calling the MASS-∞ a spectral sequence.

Staying true to definition 2.1.9, the E_∞-page of the MASS-∞ is given by the permanent cycles modulo the boundaries. However, we had another choice for the definition of the E_∞-page:

$$E_\infty^{\sigma,\lambda}(\text{MASS-}\infty) = \mathrm{colim}_n\ E_\infty^{\sigma,\lambda}(\text{MASS-}n).$$

We show that the two definitions coincide presently.

The vanishing line of corollary 4.1.9 ensures that, for large r depending only on (σ, λ), not on n, we have maps $E_r^{\sigma,\lambda}(\text{RASS-}n) \longrightarrow E_{r+1}^{\sigma,\lambda}(\text{RASS-}n)$. Moreover, n is

allowed to be ∞. Thus,

$$\begin{aligned} E_\infty^{\sigma,\lambda}(\text{MASS-}\infty) &= \text{colim}_{r>>0}\ E_r^{\sigma,\lambda}(\text{MASS-}\infty) \\ &= \text{colim}_{r>>0}\ \text{colim}_n\ E_r^{\sigma,\lambda}(\text{RASS-}n) \\ &= \text{colim}_n\ \text{colim}_{r>>0}\ E_r^{\sigma,\lambda}(\text{RASS-}n) \\ &= \text{colim}_n\ E_\infty^{\sigma,\lambda}(\text{RASS-}n). \end{aligned}$$

The vanishing line makes sure that an element of the RASS-n cannot support longer and longer differentials as it is mapped forward into subsequent reindexed Adams spectral sequences, without eventually becoming a permanent cycle.

This observation is what allows us to make an identification

$$F^\sigma \pi_{\lambda-\sigma}(S/p^\infty)/F^{\sigma+1}\pi_{\lambda-\sigma}(S/p^\infty) = E_\infty^{\sigma,\lambda}(\text{MASS-}\infty).$$

Providing we have proved the previous corollary, that the RASS-n converges, we have the following short exact sequence.

$$0 \longrightarrow F^{\sigma+1}\pi_{\lambda-\sigma}(S/p^n) \longrightarrow F^\sigma \pi_{\lambda-\sigma}(S/p^{n+1}) \longrightarrow E_\infty^{\sigma,\lambda}(\text{MASS-}n) \longrightarrow 0 \qquad \text{(B.4)}$$

Taking colimits gives another short exact sequence. By our definition of $F^\sigma\pi_*(S/p^\infty)$, and the fact that the right term can be identified with $E_\infty^{\sigma,\lambda}(\text{MASS-}\infty)$, that short exact sequence gives the requisite identification.

We are just left with showing that $\bigcup_\sigma F^\sigma\pi_*(S/p^\infty) = \pi_*(S/p^\infty)$ and that for each u, there exists a σ with $F^\sigma\pi_u(S/p^\infty) = 0$. For the first part we note that

$$\begin{aligned} \bigcup\nolimits_\sigma F^\sigma\pi_*(S/p^\infty) &= \text{im}\left(\text{colim}_\sigma\ \text{colim}_n\ F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)\right) \\ &= \text{im}\left(\text{colim}_n\ \text{colim}_\sigma\ F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)\right) \\ &= \text{im}\left(\text{colim}_n\ \pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)\right) \\ &= \pi_*(S/p^\infty). \end{aligned}$$

For the second part, we use both the vanishing line of corollary 4.1.9 and the convergence of the RASS-n again. They tell us that $F^\sigma \pi_{\lambda-\sigma}(S/p^n)$ is zero for $\sigma > K$ where

$$K = \frac{(\lambda - \sigma) + 1}{q} - 1.$$

K is independent of n, so $F^\sigma \pi_{\lambda-\sigma}(S/p^\infty) = 0$ for $\sigma > K$.

The vanishing line makes sure that if an element of $\pi_*(S/p^n)$ has infinitely many filtration shifts as it is mapped forward into subsequent Moore spectra, then it must map to zero in $\pi_*(S/p^\infty)$.

We have proved that the MASS-∞ convergences in accordance with definition 2.2.2, case 3. □

Since the convergence of the LASS-n and LASS-∞ are similar we address them presently.

Proposition B.5. *The LASS-$(n+1)$ converges to $\pi_*(v_1^{-1}S/p^{n+1})$.*

Proof. This is essentially the same proof as just given for the MASS-∞. We just need to make a few remarks.

First, we have not said precisely what we mean by $v_1^{-1}S/p^{n+1}$. Theorem 8.4.1 tells us that $q_1^{p^n}$ is a permanent cycle in the MASS-$(n+1)$. Thus it detects some homotopy class, which we call

$$v_1^{p^n} : S^0 \longrightarrow \Sigma^{-p^n q} S/p^{n+1}.$$

Since S/p^{n+1} is a ring spectrum, we can "tensor up" to obtain a v_1 self-map, which we give the same name

$$v_1^{p^n} : S/p^{n+1} \longrightarrow \Sigma^{-p^n q} S/p^{n+1}.$$

$v_1^{-1}S/p^{n+1}$ is the homotopy colimit of the diagram

$$S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-p^n q} S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-2p^n q} S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-3p^n q} S/p^{n+1} \longrightarrow \dots$$

By construction we have $\pi_*(v_1^{-1}S/p^{n+1}) = (v_1^{p^n})^{-1}\pi_*(S/p^{n+1})$. This is what allows us

to use the multiplicative structure of the MASS-n to localize the spectral sequence as opposed to constructing maps of towers.

Let $[\sigma, k] = \sigma + p^n k$ and $[\lambda, k] = \lambda + p^n(q+1)k$. The vanishing line of corollary 4.1.9 is parallel to the multiplication-by-$q_1^{p^n}$-line and this ensures that for large r, depending only on (σ, λ), not k, we have maps

$$E_r^{[\sigma,k],[\lambda,k]}(\text{MASS-}(n+1)) \longrightarrow E_{r+1}^{[\sigma,k],[\lambda,k]}(\text{MASS-}(n+1)).$$

Thus, just as in the previous proof we have an identification

$$E_\infty^{\sigma,\lambda}(\text{LASS-}(n+1)) = \text{colim}_k\ E_\infty^{[\sigma,k],[\lambda,k]}(\text{MASS-}(n+1)).$$

where the maps in the system are multiplication by $q_1^{p^n}$.

The proof of convergence is now the same as for the MASS-∞. We define

$$F^\sigma \pi_u(v_1^{-1}S/p^{n+1}) = \text{im}\left(\text{colim}_k\ F^{[\sigma,k]}\pi_{u+p^n qk}(S/p^{n+1}) \longrightarrow \pi_u(v_1^{-1}S/p^{n+1})\right),$$

where the F on the right hand side of the equation denotes the MASS-$(n+1)$ filtration. We verify convergence in accordance with definition 2.2.2, case 3. □

Proposition B.6. *The LASS-∞ converges to $\pi_*(v_1^{-1}S/p^\infty)$.*

Proof. The proof is exactly the same as for the MASS-∞. We define

$$F^\sigma \pi_*(v_1^{-1}S/p^\infty) = \text{im}\left(\text{colim}_n\ F^\sigma \pi_*(v_1^{-1}S/p^n) \longrightarrow \pi_*(v_1^{-1}S/p^\infty)\right),$$

where the F on the right hand side of the equation denotes a reindexed localized Adams filtration; we use the convergence and vanishing lines of the reindexed localized Adams spectral sequences for $v_1^{-1}S/p^n$ instead of the convergence and vanishing lines for the RASS-n. The only subtlety is taking the colimit of the short exact sequence analogous to (B.4). This is intertwined with the issue of defining of $v_1^{-1}S/p^\infty$.

Corollary 3.8 of [8] tells us that there exists integers $i_1, i_2, i_3, \ldots$ such that the

following diagrams commute.

$$\begin{array}{ccc} S/p^n & \xrightarrow{\left[v_1^{p^{n-1}}\right]^{pi_n}} & \Sigma^{-i_n p^n q} S/p^n \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ S/p^{n+1} & \xrightarrow{\left[v_1^{p^n}\right]^{i_n}} & \Sigma^{-i_n p^n q} S/p^{n+1} \end{array}$$

This means that $p : S/p^n \longrightarrow S/p^{n+1}$ induces a map $p : v_1^{-1}S/p^n \longrightarrow v_1^{-1}S/p^{n+1}$. $v_1^{-1}S/p^\infty$ is the homotopy colimit of the diagram

$$v_1^{-1}S/p \xrightarrow{p} v_1^{-1}S/p^2 \longrightarrow \ldots \longrightarrow v_1^{-1}S/p^n \xrightarrow{p} v_1^{-1}S/p^{n+1} \longrightarrow \ldots$$

Moreover, the diagram below commutes, where the filtrations are those of the RASS-n and RASS-$(n+1)$.

$$\begin{array}{ccc} F^\sigma \pi_*(S/p^n) & \xrightarrow{\left[v_1^{p^{n-1}}\right]^{pi_n}} & F^{\sigma+i_n p^n} \pi_*(S/p^n) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ F^\sigma \pi_*(S/p^{n+1}) & \xrightarrow{\left[v_1^{p^n}\right]^{i_n}} & F^{\sigma+i_n p^n} \pi_*(S/p^{n+1}) \end{array}$$

Thus, $p : v_1^{-1}S/p^n \longrightarrow v_1^{-1}S/p^{n+1}$ respects the reindexed localized Adams filtrations.

□

The proof convergence of the loc.alg.NSS follows the same chain of ideas as for the LASS-∞.

Lemma B.7. *The loc.alg.NSS converges to* $H^*(BP_*BP; v_1^{-1}BP_*/p^\infty)$.

Proof. This is essentially the same proof as for the LASS-∞. The following algebraic Novikov spectral sequence converges in accordance with definition 2.2.2, case 1.

$$H^{s,u}(P; [Q/q_0^n]^t) \overset{t}{\Longrightarrow} H^{s,u}(BP_*BP; BP_*/p^n)$$

This is because the I-adic filtration of $\Omega^*(BP_*BP; BP_*/p^n)$ is finite in a fixed internal

degree: $F^{\lceil u/q \rceil + n}\Omega^{*,u}(BP_*BP; BP_*/p^n) = 0$. Because we have a vanishing line parallel to the multiplication-by-$q_1^{p^{n-1}}$-line, we deduce that each

$$H^{s,u}(P; [q_1^{-1}Q/q_0^n]^t) \stackrel{t}{\Longrightarrow} H^{s,u}(BP_*BP; v_1^{-1}BP_*/p^n)$$

converges in accordance with definition 2.2.2, case 3.

After the reindexing that occurs in constructing

$$H^{s,u}(P; [q_1^{-1}Q/q_0^\infty]^t) \stackrel{t}{\Longrightarrow} H^{s,u}(BP_*BP; v_1^{-1}BP_*/p^\infty)$$

the vanishing lines for these spectral sequences become independent of n and so we conclude convergence in accordance with definition 2.2.2, case 3. □

We now go back to the proof of the first proposition.

Proof of proposition B.1. We are in case 1 of definition 2.2.2. We need to check the following conditions.

- The map $F^\sigma\pi_*(S/p^n)/F^{\sigma+1}\pi_*(S/p^n) \longrightarrow E_\infty^{\sigma,*}(\text{MASS-}n)$ appearing in definition 8.1.15 is an isomorphism.
- $\bigcap_\sigma F^\sigma\pi_*(S/p^n) = 0$ and the map $\pi_*(S/p^n) \longrightarrow \lim_\sigma \pi_*(S/p^n)/F^\sigma\pi_*(S/p^n)$ is an isomorphism.

We will appeal to [15, theorem 3.6] but first we need to relate our construction of the MASS-n to the one given there.

Suppose $\{X, I\}$ and $\{Y, J\}$ are towers and that we have chosen cofibrant models for them. Write $F(-,-)$ for the internal hom functor in S-modules [7] and Q for a cofibrant replacement functor. Then we obtain a zig-zag

$$\operatorname*{colim}_{\substack{i+j\geq\sigma\\0\leq i,j\leq\sigma}} X_i \wedge Y_j \stackrel{\sim}{\longleftarrow} \operatorname*{hocolim}_{\substack{i+j\geq\sigma\\0\leq i,j\leq\sigma}} X_i \wedge Y_j \longrightarrow \operatorname*{hocolim}_{\substack{i+j\geq\sigma\\0\leq i,j\leq\sigma}} F(QF(Y_j, S^0), X_i).$$

As long as each Y_j is finite, this will be an equivalence. Taking $\{X, I\}$ to be $\{\overline{H}^{\wedge *}, H^{[*]}\}$ and $\{Y, J\}$ to be $\{S/p^{n-*}, S/p\}$, we that our MASS-n is the same as the one in [15,

definition 2.2] when one uses the dual of $\{S/p^{n-*}, S/p\}$ in the source.

p^j is zero on S/p^j and so [15, proposition 1.2(a)] tells us S/p^j is p-adically cocomplete. Moreover, each S/p^j is finite and connective. Thus, [15, theorem 3.6] applies (after suspending once): the first bullet point holds and $\bigcap_\sigma F^\sigma \pi_*(S/p^n) = 0$. Moreover, the vanishing line of corollary 4.1.9 gives a vanishing line for the MASS-n, and so we see that for each u, there exists a σ with $F^\sigma \pi_u(S/p^n) = 0$. We conclude that the map $\pi_*(S/p^n) \longrightarrow \lim_\sigma \pi_*(S/p^n)/F^\sigma \pi_*(S/p^n)$ is an isomorphism, as required. □

Proof of corollary B.2. We are in case 3 of definition 2.2.2 by the previous argument. □

It is far easier to show that the other spectral sequences we use converge.

Lemma B.8. *The Q-BSS, the q_0^∞-BSS and the q_1^{-1}-BSS converge.*

Proof. The relevant filtrations are given in 3.2.1, 3.3.1 and 3.5.2, as are the identifications $E_\infty^v = F^v/F^{v+1}$. For the Q-BSS we are in case 1 of definition 2.2.2:

$$F^0 H^*(P;Q) = H^*(P;Q) \quad \text{and} \quad F^{t+1} H^{s,u}(P;Q^t) = 0$$

and so the requisite conditions hold. For the q_0^∞-BSS and the q_1^{-1}-BSS we are in case 2 of definition 2.2.2. □

Corollary B.9. *The q_0-FILT and q_0-FILT2 spectral sequence converge to the E_3-pages of the loc.alg.NSS and the LASS-∞, respectively.*

Proof. Since the q_1^{-1}-BSS converges in accordance with definition 2.2.2, case 2, one finds that this means the q_0-FILT and q_0-FILT2 spectral sequences do, too. □

For our final convergence proof we need the following lemma.

Lemma B.10. *Fix (σ, λ). There are finitely many (s,t,u) with $s+t=\sigma$, $u+t=\lambda$ and $H^{s,u}(P;[q_1^{-1}Q/q_0]^t)$ nonzero.*

Proof. Multiplication by q_1 defines an isomorphism

$$\bigoplus_{\substack{s+t=\sigma\\u+t=\lambda}} H^{s,u}(P;[q_1^{-1}Q/q_0]^t) \longrightarrow \bigoplus_{\substack{s+t=\sigma+1\\u+t=\lambda+q+1}} H^{s,u}(P;[q_1^{-1}Q/q_0]^t).$$

Thus, it is equivalent to ask the question for $(\sigma+n, \lambda+n(q+1))$. By corollary 4.3.2 we can choose n so that

$$\bigoplus_{\substack{s+t=\sigma+n\\u+t=\lambda+n(q+1)}} H^{s,u}(P;[Q/q_0]^t) \longrightarrow \bigoplus_{\substack{s+t=\sigma+n\\u+t=\lambda+n(q+1)}} H^{s,u}(P;[q_1^{-1}Q/q_0]^t)$$

is an isomorphism. Nonzero elements in the left hand side have $s,t \geq 0$. There are finitely many (s,t,u) with $s+t=\sigma+n$ and $s,t\geq 0$ and so the result follows. □

This shows that the spectral sequence argument alluded to in the proof of [11, theorem 4.8] is valid. It also allows us to prove the following lemma.

Proposition B.11. *The s-filtration spectral sequence of section 8.6 converges to* $E_1(q_0$*-FILT2).*

Proof. To show that the spectral sequence converges in accordance with definition 2.2.2, case 1, we just need to show that for each (σ,λ,v), there are finitely many (s,t,u) with $s+t=\sigma$, $u+t=\lambda$ and $E_\infty^{s,t,u,v}(q_1^{-1}$-BSS) nonzero. This follows from the fact that for each (σ,λ,v), there are finitely many (s,t,u) with $s+t=\sigma$, $u+t=\lambda$ and $H^{s,u}(P;[q_1^{-1}Q/q_0]^{t-v})$ nonzero. □

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