ORIENTABLE SMOOTH MANIFOLDS ARE ESSENTIALLY QUASIGROUPS

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ABSTRACT. We introduce an n-dimensional analogue of the construction of tessellated surfaces from finite groups first described by Herman and Pakianathan. Our construction is functorial and associates to each n-ary alternating quasigroup both a smooth, flat Riemannian n-manifold which we dub the open serenation of the quasigroup in question, as well as a topological n-manifold (the serenation of the quasigroup) which is a subspace of the metric completion of the open serenation. We prove that every connected orientable smooth manifold is serene, in the sense that each such manifold is a component of the serenation of some quasigroup. We prove some basic results about the variety of alternating n-quasigroups and note connections between our construction and topics including Latin hypercubes, Johnson graphs, and Galois theory.

CONTENTS

1. INTRODUCTION

This work builds on the ideas behind the construction of tessellated surfaces from finite groups given by Herman and Pakianathan[8]. They produced a functor from a category of nonabelian finite groups equipped with certain homomorphisms preserving noncommutativity to a category of singular, oriented 2-manifolds. They then desingularized these manifolds into compact orientable surfaces. The group used in this construction (or a subquotient, or its automorphism group) would then act on the cells of a functorially-occurring polyhedral tessellation of the resulting manifold, yielding a faithful, orientation-preserving group action. Herman and

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Pakianathan's motivation was the manufacture of these actions and the study of finite groups through these associated surfaces.

Our motivation is the exact opposite. In higher dimensions the orientable manifolds are much less well understood so we reverse course by showing that the same method allows us to understand higher-dimensional manifolds by working with certain discrete algebraic structures. In order to extend this technique to dimensions 3 and above a radically different formalism is needed, although we will see in section 6 that when applied to a finite group it produces the same results Herman and Pakianathan saw. We hope that the reader will agree that it is morally the correct generalization. The appropriate analogue of a group must be found (which is not an n-ary group in the sense Post introduced in [12], as one might've guessed) and one must find a new technique for desingularization of singular n-manifolds. We supply these concepts.

We construct smooth n-dimensional manifolds from n-quasigroups. For each n we produce an *open serenation functor* $\mathbf{OSer}_n$ from a subcategory $\mathbf{NCAQ}_n$ of a variety of n-quasigroups to a category of smooth n-manifolds. Given any alternating quasigroup $\mathbf{A}$ we will find that $\mathbf{OSer}_n(\mathbf{A})$ is orientable and consists of a (perhaps uncountable) collection of second countable connected components. Moreover, $\mathbf{OSer}_n(\mathbf{A})$ inherits a metric from the quasigroup structure on $\mathbf{A}$ which makes $\mathbf{OSer}_n(\mathbf{A})$ into a Riemannian manifold. Certain homomorphisms $\mathbf{A} \to \mathbf{B}$ of n-quasigroups yield smooth maps $\mathbf{OSer}_n(\mathbf{A}) \to \mathbf{OSer}_n(\mathbf{B})$ which are local isometries everywhere with respect to these metrics.

We can use this metric structure to manufacture another functor, the *serenation functor* denoted by $\mathbf{Ser}_n$, from the same category of n-quasigroups to a category of topological manifolds. This functor is in a sense the metric completion of $\mathbf{OSer}_n$. Although in dimensions 4 and above we can't upgrade the codomain of this functor to again be a category of smooth manifolds, it does have a redeeming property, which is that every connected orientable triangulable topological n-manifold is a component of the serenation of a quasigroup. This is theorem 1, which has as an immediate corollary corollary 1, which says that every connected orientable smooth n-manifold is a component of the serenation of an alternating n-quasigroup.

The varieties of n-quasigroups which we utilize have not to our knowledge been studied before. However, there is a long history of interplay between certain varieties of n-quasigroups and both combinatorics and topology. On the combinatorial side, varieties of n-quasigroups often correspond to certain nice classes of Latin squares or Latin (hyper)cubes[4, 5]. Even when the algebraic viewpoint is not present, as in [9], we will see in section 7 that we encounter very similar problems in our setting. Historically the study of loops (quasigroups with identity) and finite projective planes have been closely intertwined[1]. Recently there has been a great deal of interplay between ternary quasigroups and knot theory[11]. The present work may be taken as evidence that the aforementioned connection between quasigroups and knot theory is an example of a general pattern in which topology and quasigroup theory inform each other, rather than a special quirk of the theory of knots.

Perhaps most evocatively, we propose a program for the classification of all smooth orientable n-manifolds up to homeomorphism. By corollary 1 we have that every orientable smooth manifold $\mathbf{M}$ is a component of the serenation $\mathbf{Ser}(\mathbf{A})$ for some alternating n-quasigroup $\mathbf{A}$. Since the category of alternating n-quasigroups

$\mathbf{AQ}_n$ is a variety of algebras we have that every member of the class of alternating n-quasigroups AQ_n can be expressed as a subdirect product of subdirectly irreducible alternating n-quasigroups. See section 3 for more discussion of the relevant algebraic notions. The upshot is that the subdirectly irreducible alternating n-quasigroups play a role somewhat analogous to that of simple groups in group theory so a classification of them amounts, via serenation, to a classification of the possible underlying topological manifolds of the smooth orientable n-manifolds.

The task of describing all subdirectly irreducible alternating n-quasigroups, including the infinite ones, may be unattainably difficult. In the case that $n = 2$ we would want to classify all subdirectly irreducible quasigroups, which include all the (even infinite) simple groups. This would appear to be an algebraic analogue of the situation where describing all possible noncompact orientable surfaces is confounded by the difficulty of classifying an uncountable possible collection of ends for a surface. The situation where the manifolds in question are compact is much mellower. Indeed, one may determine all compact orientable surfaces up to homeomorphism without knowing the classification of the finite simple groups, much less the classification of all subdirectly irreducible quasigroups.

Our problem 1 asks whether it is possible to realize every connected compact orientable smooth manifold as a homeomorphic copy of a component of the serenation of some alternating quasigroup. An affirmative answer to this question would be implied by that to a combinatorial question, which is our problem 2 and which is a generalized version of the Evans conjecture about the completion of partial Latin squares. In the case that the answer to both of these questions is "yes", then we may attain a great deal of insight into compact orientable manifolds by determining the finite subdirectly irreducible alternating quasigroups. The additional condition of finiteness brings us into a setting similar to the classification of the finite simple groups, which has been done. We also know that the classification of the compact surfaces was easier than that of the finite simple groups, so it is conceivable that we may find a few families of subdirect irreducibles which suffice to manufacture all compact manifolds without having to give a complete classification.

We are concerned with the actions of the symmetric and alternating groups on n-tuples from a set A. Here we give our conventions in this direction. We take $\mathbb{N} \coloneqq \{1, 2, 3, \dots\}$ and $\mathbb{W} \coloneqq \{0, 1, 2, \dots\}$. Given $n \in \mathbb{N}$ we define $[n] \coloneqq \{1, 2, 3, \dots, n\}$. Given a set S we denote by Perm_S (or $\mathrm{Perm}(S)$) the set of permutations of S, we denote by $\mathbf{Perm}_S$ (or $\mathbf{Perm}(S)$) the group of permutations of S, we denote by Alt_S the set of even permutations of S, and we denote by $\mathbf{Alt}_S$ the group of even permutations of S. When $S = [n]$ for some $n \in \mathbb{N}$ we write Perm_n, $\mathbf{Perm}_n$, Alt_n, and $\mathbf{Alt}_n$ rather than $\mathrm{Perm}_{[n]}$, $\mathbf{Perm}_{[n]}$, $\mathrm{Alt}_{[n]}$, and $\mathbf{Alt}_{[n]}$, respectively. Given a group $\mathbf{G}$ and an action $h\colon \mathbf{G} \to \mathbf{Perm}_S$ of $\mathbf{G}$ on a set S we denote by $\mathrm{Orb}(h)$ the collection of orbits of S under the action h and we denote by $\mathrm{Orb}_h(s)$ the orbit of $s \in S$ under the action h. When h is understood by context we write $\mathrm{Orb}_{\mathbf{G}}(S)$ (or even just $\mathrm{Orb}(S)$) instead of $\mathrm{Orb}(h)$. Similarly we write $\mathrm{Orb}_{\mathbf{G}}(s)$ (or even just $\mathrm{Orb}(s)$) instead of $\mathrm{Orb}_h(s)$ for $s \in S$ when context makes the action clear. We denote by $\mathrm{Sb}(S)$ the collection of all subsets of the set S and we denote by $|S|$ the cardinality of the set S. We write $\mathbf{A} \leq \mathbf{B}$ to indicate that an algebra $\mathbf{A}$ (such as a group) is a subalgebra of the algebra $\mathbf{B}$ (which is necessarily of the same signature). More (universal) algebra definitions are reviewed in section 3.

Here a *manifold* is a finite-dimensional real manifold without boundary which, while not necessarily second countable, is taken to consist of a collection (whose cardinality is unconstrained) of second countable connected components. A *smooth manifold* is a manifold in the previous sense equipped with a smooth atlas. A *Riemannian manifold* is a pair $(\mathbf{M}, g)$ where $\mathbf{M}$ is a smooth manifold in the previous sense and g is a smooth Riemannian metric on $\mathbf{M}$. We write Riem_n to indicate the class of n-dimensional Riemannian manifolds and we write $\mathbf{Riem}_n$ to indicate the category whose objects are n-dimensional Riemannian manifolds and whose morphisms are smooth maps which are local isometries everywhere.

We outline here the structure of the paper. In section 2 we give some preliminaries on simplicial complexes and pseudomanifolds. We do the same for algebraic structures in section 3. We introduce the algebras of primary concern for us, which are alternating quasigroups, in section 4. With the basic algebra and topology in place, we define and see examples of open serenation, our Riemannian manifold construction, in section 5. The following section, section 6, details the serenation functor, which is a sort of metric completion of the open serenation functor. It is in this section that we prove our theorem 1, which says that all connected orientable triangulable topological manifolds may be produced via this construction. In section 7 we examine similar questions to those answered in the preceding section, but with finiteness and compactness constraints added. This rolls into our closing discussion in section 8, where we consider the next steps that may be taken using the tools described in this paper.

2. Pseudomanifolds

In this section we detail the basic tools pertaining to pseudomanifolds which we will use in our construction. We refer to Munkres[10] for basic topological definitions but we deviate from his terminology in several places.

2.1. Simplicial complexes.

Pseudomanifolds are a family of particularly well-behaved simplicial complexes, so we begin by discussing this latter class of combinatorial objects.

Definition 1 (Simplicial complex)**.** Given a set S and a set $\Gamma \subset S$ we refer to $\mathbf{S} := (S, \Gamma)$ as a *simplicial complex* when

1. for each $\gamma \in \Gamma$ and each $\gamma' \subset \gamma$ we have that $\gamma' \in \Gamma$ and
2. for each $\gamma \in \Gamma$ we have that $|\gamma| \in \mathbb{W}$.

Our simplicial complexes are elsewhere known as *abstract simplicial complexes* which have only finite-dimensional faces. By this definition a simplicial complex may have faces of unbounded (but finite) dimension, but we will only consider examples where there is a finite global bound on the dimension of a face. See below where we recall the formal definition of dimension.

Definition 2 (Finite simplicial complex)**.** We say that a simplicial complex $\mathbf{S} := (S, \Gamma)$ is *finite* when S is a finite set.

We will need to consider simplicial complexes whose vertices can be any set, no matter how large, but finite simplicial complexes play an important role.

Definition 3 (Face)**.** Given a simplicial complex $\mathbf{S} := (S, \Gamma)$ we refer to an element $\gamma \in \Gamma$ as a *face* of $\mathbf{S}$. Given two faces $\gamma, \gamma' \in \Gamma$ we say that γ' is a *face* of γ when $\gamma' \subset \gamma$.

Our simplicial complexes always have the empty set as a face.

Definition 4 (Simplicial map)**.** Given simplicial complexes $\mathbf{S}_1 \coloneqq (S_1, \Gamma_1)$ and $\mathbf{S}_2 \coloneqq (S_2, \Gamma_2)$ we say that a function $h\colon S_1 \to S_2$ is a *simplicial map* from $\mathbf{S}_1$ to $\mathbf{S}_2$ and write $h\colon \mathbf{S}_1 \to \mathbf{S}_2$ when for each $\gamma \in \Gamma_1$ we have that $\{\, h(s) \in S_2 \mid s \in \gamma \,\}$ is a face of $\mathbf{S}_2$.

Simplicial maps are the morphisms in the category associated to simplicial complexes.

Definition 5 (Class of simplicial complexes)**.** We denote by SCmplx the class of all simplicial complexes.

Definition 6 (Category of simplicial complexes)**.** The *category of simplicial complexes* **SCmplx** is the category whose objects form the class SCmplx, whose morphisms are simplicial maps, and whose identities and composition are those induced by the underlying functions of those simplicial maps.

We will fix a natural number n and focus our attention on complexes which are in the following sense n-dimensional.

Definition 7 (Dimension of a face)**.** Given a simplicial complex $\mathbf{S} \coloneqq (S, \Gamma)$ and a face $\gamma \in \Gamma$ with $|\gamma| \in \mathbb{W}$ we say that the *dimension* of γ is $|\gamma| - 1$, that γ is an *$(|\gamma| - 1)$-face*, or that γ is *$(|\gamma| - 1)$-dimensional*.

In particular this means that every simplicial complex has exactly one (-1)-dimensional face, the empty face. On the opposite end of the size spectrum we have facets.

Definition 8 (Facet)**.** Given a simplicial complex $\mathbf{S} \coloneqq (S, \Gamma)$ we say that a face $\gamma \in \Gamma$ is a *facet* of $\mathbf{S}$ when γ is maximal with respect to subset inclusion among faces of $\mathbf{S}$.

Definition 9 (Facets of a simplicial complex)**.** We denote the set of all facets of a simplicial complex $\mathbf{S}$ by $\mathrm{Fct}(\mathbf{S})$.

It is possible that a simplicial complex has no facets or that a simplicial complex has some facets but there are faces which are not contained in any facet. These pathologies won't arise in the cases we actually consider.

Definition 10 (Pure simplicial complex)**.** We say that a simplicial complex $\mathbf{S} \coloneqq (S, \Gamma)$ is *pure* when

(1) given any $\gamma, \gamma' \in \mathrm{Fct}(\mathbf{S})$ we have that $|\gamma| = |\gamma'|$ and
(2) given any $\gamma \in \Gamma$ there exists some $\gamma' \in \mathrm{Fct}(\mathbf{S})$ such that $\gamma \subset \gamma'$.

Definition 11 (Dimension of a pure simplicial complex)**.** When $\mathbf{S} \coloneqq (S, \Gamma)$ is a pure simplicial complex such that each $\gamma \in \mathrm{Fct}(\mathbf{S})$ is n-dimensional we say that $\mathbf{S}$ is an *n-dimensional simplicial complex*.

2.2. **Pseudomanifolds.** Pseudomanifolds are an even more special class of simplicial complexes contained within the class of pure simplicial complexes. Intuitively, n-pseudomanifolds look like n-dimensional manifolds, except that their $(n-2)$-skeleta may be highly singular.

Definition 12 (Pseudomanifold)**.** We say that an n-dimensional simplicial complex $\mathbf{S} \coloneqq (S, \Gamma)$ is an *n-pseudomanifold* when given an $(n-1)$-face $\gamma_1 \in \Gamma$ there exist exactly two n-faces $\gamma_2, \gamma_3 \in \Gamma$ such that $\gamma_2 \cap \gamma_3 = \gamma_1$.

We want to consider morphisms of pseudomanifolds which respect the pseudomanifold structure, so general simplicial maps will not suffice.

Definition 13 (Pseudomanifold morphism)**.** Given pseudomanifolds $\mathbf{S}_1$ and $\mathbf{S}_2$ we say that a simplicial map $h\colon \mathbf{S}_1 \to \mathbf{S}_2$ is a *morphism* of pseudomanifolds when for each maximal face γ of $\mathbf{S}_1$ we have that $h(\gamma)$ is a maximal face of $\mathbf{S}_2$.

Pseudomanifolds and their morphisms form a category.

Definition 14 (Class of n-pseudomanifolds)**.** We denote by PMfld_n the class of all n-pseudomanifolds.

Definition 15 (Category of n-pseudomanifolds)**.** We refer to the subcategory of **SCmplx** whose objects form the class PMfld_n and whose morphisms are all pseudomanifold morphisms as the *category of n-pseudomanifolds*, which we denote by $\mathbf{PMfld}_n$.

2.3. **Geometric realization.** We collect here the terminology we need about geometric realization of simplicial complexes. We denote by Top the class of all topological spaces and **Top** the category whose objects are spaces, whose morphisms are continuous maps, whose identities are the usual identity functions, and whose composition is the usual composition of functions.

Given a set S we denote by $\mathbb{R}^S$ the S-fold Cartesian power of the real line equipped with the Euclidean topology. Given $X \subset \mathbb{R}^S$ we denote by $\mathrm{Cvx}(X)$ the (closed) convex hull of X and by $\mathrm{OCvx}(X)$ the interior of $\mathrm{Cvx}(X)$ in the affine hull of X. Given any $\gamma \subset S$ we identify each member of γ with the corresponding basis element of $\mathbb{R}^S$ so that $\mathrm{OCvx}(\gamma)$ is the interior (in the appropriate affine subspace) of the simplex in $\mathbb{R}^S$ whose vertices form the set γ.

We make use of a nonstandard geometric realization of pseudomanifolds, in the sense that this geometric realization is not the restriction of the usual geometric realization of simplicial complexes to the category of n-pseudomanifolds. Our geometric realization of pseudomanifolds drops the $(n-2)$-skeleton of a pseudomanifold in order to dispose of the singularities present there.

Definition 16 (Open geometric realization functor)**.** Fix $n \in \mathbb{N}$. The *open geometric realization functor*

$$\mathbf{OGeo}_n\colon \mathbf{PMfld}_n \to \mathbf{Top}$$

is defined as follows. Given an n-pseudomanifold $\mathbf{S} := (S, \Gamma)$ we define

$$\mathbf{OGeo}_n(\mathbf{S}) := (\mathrm{OGeo}_n(\mathbf{S}), \tau)$$

where $\mathrm{OGeo}_n(\mathbf{S}) \subset \mathbb{R}^S$ is given by

$$\mathrm{OGeo}_n(\mathbf{S}) := \bigcup_{\substack{\gamma \in \Gamma \\ |\gamma| \in \{n, n+1\}}} \mathrm{OCvx}(\gamma)$$

and τ is the subspace topology which $\mathrm{OGeo}_n(\mathbf{S})$ inherits from $\mathbb{R}^S$.

Given an n-pseudomanifold morphism $h\colon \mathbf{S}_1 \to \mathbf{S}_2$ we define

$$\mathbf{OGeo}_n(h)\colon \mathbf{OGeo}(\mathbf{S}_1) \to \mathbf{OGeo}_n(\mathbf{S}_2)$$

by linear extension of the rule

$$\mathbf{OGeo}_n(h)(s) := h(s)$$

for $s \in S_1$.

For a general simplicial complex we have the usual (closed) geometric realization functor.

Definition 17 (Geometric realization functor)**.** The *geometric realization functor*

$$\mathbf{Geo}\colon \mathbf{Sim} \to \mathbf{Top}$$

is defined as follows. Given a simplicial complex $\mathbf{S} := (S, \Gamma)$ we define

$$\mathbf{Geo}(\mathbf{S}) := (\mathrm{Geo}(\mathbf{S}), \tau)$$

where $\mathrm{Geo}(\mathbf{S}) \subset \mathbb{R}^S$ is given by

$$\mathrm{Geo}(\mathbf{S}) := \bigcup_{\gamma \in \Gamma} \mathrm{Cvx}(\gamma)$$

and τ is the subspace topology which $\mathrm{Geo}(\mathbf{S})$ inherits from $\mathbb{R}^S$.

Given a simplicial map $h\colon \mathbf{S}_1 \to \mathbf{S}_2$ we define

$$\mathbf{Geo}(h)\colon \mathbf{Geo}(\mathbf{S}_1) \to \mathbf{Geo}(\mathbf{S}_2)$$

by linear extension of the rule

$$\mathbf{Geo}(h)(s) := h(s)$$

for $s \in S_1$.

3. Algebraic preliminaries

The inputs in our construction are a family of algebras equipped with a particular class of homomorphisms between them. In this section we set up all the pertinent algebraic machinery.

3.1. **Magmas.**

We introduce the basic notation and terminology we will need with respect to magmas. We use Bergman[2], Smith and Romanowska[14], and Burris and Sankpanavar[3] as general references for universal algebra.

Definition 18 (Magma)**.** Given some $n \in \mathbb{N}$ we refer to an algebra $\mathbf{A} := (A, f)$ where $f\colon A^n \to A$ is an n-ary operation on the set A as a *magma* (or as an *n-magma* when we want to emphasize the arity of f).

Note that this definition is broader than what is traditionally meant by "binar", "groupoid", or "magma" (in the sense of Bourbaki), which all refer to only the case where $f\colon A^2 \to A$ is a binary operation.

Definition 19 (Class of magmas)**.** We denote by Mag_n the class of all n-ary magmas.

Magma homomorphisms are defined analogously to those for more familiar algebras.

Definition 20 (Magma homomorphism)**.** Given n-magmas $\mathbf{A} := (A, f)$ and $\mathbf{B} := (B, g)$ we say that a function $h\colon A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when for all $a_1, \ldots, a_n \in A$ we have that

$$h(f(a_1, \ldots, a_n)) = g(h(a_1), \ldots, h(a_n)).$$

We write $h\colon \mathbf{A} \to \mathbf{B}$ to indicate that h is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Magmas are the objects of a category whose morphisms are magma homomorphisms.

Definition 21 (Category of magmas)**.** The *category of n-magmas* $\mathbf{Mag}_n$ is the category whose objects form the class Mag_n, whose morphisms are n-ary magma homomorphisms, and whose identities and composition are those induced by the underlying functions of those homomorphisms.

As is standard in universal algebra, we fix a signature $\rho\colon I \to \mathbb{W}$ and write $t_1(x_1, \ldots, x_n) \approx t_2(x_1, \ldots, x_n)$ (or $t_1(x) \approx t_2(x)$, or even $t_1 \approx t_2$) to indicate the formal expression (or *identity*)

$$(\forall x_1, \ldots, x_n)(t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n))$$

where t_1 and t_2 are terms obtained by formally composing basic operation symbols $\{f_i\}_{i \in I}$ where the arity of f_i is $\rho(i)$.

Given an algebra $\mathbf{A}$ of signature ρ we write $f_i^{\mathbf{A}}$ to indicate the i^{th} basic operation of $\mathbf{A}$. If A is the universe of $\mathbf{A}$ then we say that $\mathbf{A}$ *models* $t_1(x) \approx t_2(x)$ when

$$(\forall x_1, \ldots, x_n \in A)(t_1^{\mathbf{A}}(x_1, \ldots, x_n) = t_2^{\mathbf{A}}(x_1, \ldots, x_n))$$

where $t_j^{\mathbf{A}}$ is the term t_j viewed as an actual operation on A obtained by replacing each f_i appearing in t_j by the basic operation $f_i^{\mathbf{A}}$ of $\mathbf{A}$. In the case that $\mathbf{A}$ models $t_1 \approx t_2$ we write $\mathbf{A} \models t_1 \approx t_2$.

We associate to each n-magma a particular group of permutations of $[n]$.

Definition 22 (Permutomorphism)**.** Given a magma $\mathbf{A} \coloneqq (A, f)$ we say that a permutation $\alpha \in \mathrm{Perm}_n$ is a *permutomorphism* of $\mathbf{A}$ when

$$\mathbf{A} \models f(x_1, \ldots, x_n) \approx f(x_{\alpha(1)}, \ldots, x_{\alpha(n)}).$$

Definition 23 (Set of permutomorphisms)**.** We denote by $\mathrm{Perm}(\mathbf{A})$ the set of all permutomorphisms of the magma $\mathbf{A}$.

One immediately verifies that $\mathrm{Perm}(\mathbf{A})$ is closed under taking composites and inverses and also contains the identity permutation id_A of A.

Definition 24 (Permutomorphism group)**.** Given an n-magma $\mathbf{A}$ we denote by $\mathbf{Perm}(\mathbf{A})$ the *permutomorphism group* of $\mathbf{A}$, which is the subgroup of $\mathbf{Perm}_n$ with universe $\mathrm{Perm}(\mathbf{A})$.

We write $\mathbf{Aut}(\mathbf{A})$ to indicate the automorphism group of $\mathbf{A}$, which is in general a different group from the permutomorphism group $\mathbf{Perm}(\mathbf{A})$ of $\mathbf{A}$. Elements of $\mathrm{Aut}(\mathbf{A})$ are permutations of A while elements of $\mathrm{Perm}(\mathbf{A})$ are permutations of $[n]$.

We are particularly interested in the cases where $\mathbf{Perm}(\mathbf{A})$ is either $\mathbf{Perm}_n$ or $\mathbf{Alt}_n$.

Definition 25 (Commutative magma)**.** We say that an n-magma $\mathbf{A}$ is *commutative* when $\mathbf{Perm}(\mathbf{A}) = \mathbf{Perm}_n$.

Definition 26 (Alternating magma)**.** We say that an n-magma $\mathbf{A}$ is *alternating* when $\mathbf{Perm}(\mathbf{A}) \geq \mathbf{Alt}_n$.

It will turn out that most of the examples of alternating magmas we care to look at do have $\mathbf{Perm}(\mathbf{A}) = \mathbf{Alt}_n$, but we only require that $\mathbf{Alt}_n$ is a subgroup of $\mathbf{Perm}(\mathbf{A})$ in order to say that $\mathbf{A}$ is alternating. In particular, every commutative magma is alternating.

3.2. **Quasigroups.** Intuitively, quasigroups are magmas in which division is always possible. We recall the definition of an n-quasigroup. Our definition is in analogy with Birkhoff's equational axioms for binary quasigroups[2, p.5].

Definition 27 (Quasigroup)**.** Given $n \in \mathbb{N}$ we say that an algebra

$$\mathbf{A} \coloneqq (A, f, g_1, \ldots, g_n)$$

of signature $(n, \ldots, n)$ is a *quasigroup* (or an *n-quasigroup* when we want to emphasize the arity of f) when for each $i \in [n]$ we have that

$$\mathbf{A} \models f(x_1, \ldots, x_{i-1}, g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), x_{i+1}, \ldots, x_n) \approx y$$

and

$$\mathbf{A} \models g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n)) \approx x_i.$$

Definition 28 (Class of quasigroups)**.** We denote by Quas_n the class of all n-quasigroups.

It follows immediately from the definition of an n-quasigroup that for each $n \in \mathbb{N}$ we have that Quas_n is an equational class and hence a variety.

We think of f as the multiplication of the n-quasigroup $\mathbf{A} \coloneqq (A, f, g_1, \ldots, g_n)$ and we think of g_i as the i^{th}-coordinate division operation. That is,

$$g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$$

is taken to indicate the division of y simultaneously by x_j in the j^{th} coordinate for each $j \neq i$. The impetus for this will be made clear by the following characterization of n-quasigroups.

Definition 29 (n-quasigroup magma)**.** Given $n \in \mathbb{N}$ we say that an n-ary magma $\mathbf{A} \coloneqq (A, f)$ is an *n-quasigroup magma* when given

$$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y \in A$$

there exists a unique $x_i \in A$ such that $f(x_1, \ldots, x_n) = y$.

The n-quasigroups are in bijective correspondence with the n-quasigroup magmas, allowing us to freely switch between these two definitions when it suits us.

Proposition 1. *Given an n-quasigroup $\mathbf{A} \coloneqq (A, f, g_1, \ldots, g_n)$ the n-ary magma reduct $\mathbf{B} \coloneqq (A, f)$ is an n-quasigroup magma. Conversely, given an n-quasigroup magma $\mathbf{B} \coloneqq (A, f)$ there exists a unique n-quasigroup expansion*

$$\mathbf{A} \coloneqq (A, f, g_1, \ldots, g_n)$$

of $\mathbf{B}$. In this expansion $g_i \colon A^n \to A$ is defined so that

$$g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$$

is the unique $x_i \in A$ such that $f(x_1, \ldots, x_n) = y$.

Proof. Suppose that $\mathbf{A} \coloneqq (A, f, g_1, \ldots, g_n)$ is an n-quasigroup. We show that $\mathbf{B} \coloneqq (A, f)$ is an n-quasigroup magma. Suppose that $f(x_1, \ldots, x_n) = y$ for some $x_1, \ldots, x_n, y \in A$. It suffices to show that $x_i = g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$ as this implies that x_i is uniquely determined by the x_j for $j \neq i$ and y. Observe that

$$\begin{aligned} x_i &= g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n)) \\ &= g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), \end{aligned}$$

as claimed.

On the other hand, consider an n-quasigroup magma $\mathbf{B} \coloneqq (A, f)$. Suppose that

$$\mathbf{A} \coloneqq (A, f, g_1, \ldots, g_n)$$

is an n-quasigroup expansion of $\mathbf{B}$. If $f(x_1, \ldots, x_n) = y$ for some $x_1, \ldots, x_n, y \in A$ we must have that

$$g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y) = x_i$$

by the n-quasigroup axioms. As there is a unique x_i for which $f(x_1, \ldots, x_n) = y$ the operation g_i is well-defined.

It remains to show that $\mathbf{A}$ with the g_i so defined does in fact satisfy all the n-quasigroup axioms. Given $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y \in A$ let

$$x_i \coloneqq g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y).$$

By definition x_i is the unique element of A such that $f(x_1, \ldots, x_n) = y$. This implies that

$$f(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), x_{i+1}, \ldots, x_n) = y.$$

Similarly, given $x_1, \ldots, x_n \in A$ let $y \coloneqq f(x_1, \ldots, x_n)$. By definition

$$g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$$

is the unique member of A such that

$$f(x_1, \ldots, x_{i-1}, g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), x_{i+1}, \ldots, x_n) = y.$$

Since we also have that $f(x_1, \ldots, x_n) = y$ it must be that

$$g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y) = x_i$$

and hence

$$g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n)) = x_i,$$

as desired. □

We will speak of an n-quasigroup and its n-quasigroup magma reduct as though they are the same object. Traditionally a quasigroup is assumed to be binary, but we will default to our more expansive terminology and emphasize when we are talking about a binary quasigroup as opposed to an n-quasigroup for a general n.

Since we can view n-quasigroups as magmas without losing any information we can take the variety of all n-quasigroups to be a subcategory of $\mathbf{Mag}_n$.

Definition 30 (Category of quasigroups)**.** The *category of n-quasigroups* $\mathbf{Quas}_n$ is the full subcategory of $\mathbf{Mag}_n$ whose objects form the class Quas_n.

4. Alternating quasigroups

We are ready to name the class of quasigroups we will study.

Definition 31 (Alternating quasigroup)**.** An *alternating quasigroup* is a quasigroup which is also an alternating magma.

Definition 32 (Class of alternating quasigroups)**.** We denote by AQ_n the class of all alternating n-quasigroups.

Note that since the classes of alternating n-magmas and n-quasigroups are defined by universally quantified equations we have that AQ_n is equational and hence a variety of algebras. When $n = 2$ we have that $\mathbf{Alt}_2$ is trivial and hence AQ_2 is the variety of binary quasigroups. Every group is thus a binary alternating quasigroup.

Since every variety of algebras forms a category whose morphisms are the usual homomorphisms of those algebras we have a category of alternating n-quasigroups for each $n \in \mathbb{N}$ whose structure doesn't depend on whether we consider our quasigroups as magmas or as algebras of signature $(n, \dots, n)$.

Definition 33 (Category of alternating quasigroups)**.** The *category of alternating n-quasigroups* $\mathbf{AQ}_n$ is the full subcategory of $\mathbf{Mag}_n$ (or of $\mathbf{Quas}_n$) whose objects form the class AQ_n.

Given any variety of algebras $\mathbf{V}$ (viewed as a category) and any set X there exists a free algebra $\mathbf{Fr}_{\mathbf{V}}(X)$, which is a member of the object class V of $\mathbf{V}$, satisfying

$$\hom_{\mathbf{V}}(\mathbf{Fr}_{\mathbf{V}}(X), \mathbf{A}) \cong \hom_{\mathbf{Set}}(X, A)$$

for any $\mathbf{A} \in V$. In particular, for any set X there exists a free alternating n-quasigroup $\mathbf{Fr}_{\mathbf{AQ}_n}(X)$ freely generated by X. See section 4.3 of [2] for more information.

Instead of looking at the whole category of alternating n-quasigroups, we restrict our attention to a subcategory whose morphism sets are restricted to those which preserve noncommutativity.

Definition 34 (Commuting tuple)**.** Given $\mathbf{A} := (A, f) \in \mathrm{AQ}_n$ we say that $a \in A^n$ *commutes* (or is a *commuting tuple*) in $\mathbf{A}$ when we have for each $\sigma \in \mathrm{Perm}_n$ that

$$f(a) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)}).$$

Definition 35 (Noncommuting tuple)**.** Given $\mathbf{A} := (A, f) \in \mathrm{AQ}_n$ we say that $a \in A^n$ *does not commute* (or is a *noncommuting tuple*) in $\mathbf{A}$ when a is not a commuting tuple in $\mathbf{A}$.

The collection of all noncommuting tuples will be important for us.

Definition 36 (Set of noncommuting tuples)**.** Given $\mathbf{A} := (A, f) \in \mathrm{AQ}_n$ we define the *noncommuting tuples* $\mathrm{NCT}(\mathbf{A})$ of $\mathbf{A}$ by

$$\mathrm{NCT}(\mathbf{A}) := \{\, a \in A^n \mid a \text{ does not commute in } \mathbf{A} \,\}.$$

The particular class of homomorphisms we will examine consists of those which preserve these sets of noncommuting tuples.

Definition 37 (NC homomorphism)**.** We say that a homomorphism $h \colon \mathbf{A}_1 \to \mathbf{A}_2$ of alternating quasigroups is an *NC homomorphism* (or a *noncommuting homomorphism*) when for each $a \in \mathrm{NCT}(\mathbf{A}_1)$ we have that

$$h(a) = (h(a_1), \dots, h(a_n)) \in \mathrm{NCT}(\mathbf{A}_2).$$

Note that while embeddings are always NC homomorphisms the converse is not true. A typical NC homomorphism is not injective (or surjective, for that matter).

We can now define the category of quasigroups we will be using in our construction of manifolds.

Definition 38 (Category of NC alternating quasigroups)**.** Given $n \in \mathbb{N}$ we define the *category of NC alternating n-quasigroups* to be the category $\mathbf{NCAQ}_n$ whose object class is AQ_n and whose morphisms are NC homomorphisms.

We associate to each alternating n-quasigroup two sets of elements, which we will use when we produce pseudomanifolds from quasigroups.

Definition 39 (Input elements)**.** Given a quasigroup $\mathbf{A}$ we define the *input elements* of $\mathbf{A}$ to be

$$\mathrm{In}(\mathbf{A}) \coloneqq \{\, a_i \in A \mid \exists (a_1, \ldots, a_n) \in \mathrm{NCT}(\mathbf{A}) \,\}\,.$$

Definition 40 (Output elements)**.** Given a quasigroup $\mathbf{A}$ we define the *output elements* of $\mathbf{A}$ to be

$$\mathrm{Out}(\mathbf{A}) \coloneqq \{\, f(a) \mid a \in \mathrm{NCT}(\mathbf{A}) \,\}\,.$$

Although we leave a more thorough analysis of the structure of the variety $\mathbf{AQ}_n$ for another time, we illustrate that notions from binary quasigroup theory may generalize nicely to the n-ary case by way of the proceeding proposition.

Proposition 2. *The variety of n-quasigroups is congruence permutable.*

Proof. We provide a Mal'cev term, which is an n-ary analogue of the Mal'cev term for binary quasigroups. The usual Mal'cev term for binary quasigroups is, in our notation,

$$p(x, y, z) \coloneqq f(g_1(g_2(y, y), x), g_2(y, z)).$$

This term is usually given as

$$p(x, y, z) \coloneqq (x/(y\backslash y))(y\backslash z),$$

as one can find in Burris and Sankpanavar[3, p.79].

Let

$$p(x, y, z) \coloneqq f(g_1(x, \ldots, x, g_n(y, \ldots, y), x), x, \ldots, x, g_n(y, \ldots, y, z)).$$

We claim that p is a Mal'cev term for $\mathbf{Quas}_n$, which means that for any $\mathbf{A} \in \mathbf{Quas}_n$ we have that

$$\mathbf{A} \models p(x, y, y) \approx x \approx p(y, y, x).$$

To see this, first observe that any n-quasigroup $\mathbf{A}$ models both

$$f(g_1(x, \ldots, x, g_n(x, \ldots, x), x), x, \ldots, x, g_n(x, \ldots, x)) \approx x$$

and

$$f(x, \ldots, x, g_n(x, \ldots, x)) \approx x,$$

which implies that

$$\mathbf{A} \models g_1(x, \ldots, x, g_n(x, \ldots, x), x) \approx x$$

as both $g_1(x, \ldots, x, g_n(x, \ldots, x), x)$ and x are candidates for the solution y to the equation

$$f(y, x, \ldots, x, g_n(x, \ldots, x)) = x.$$

We then compute that in $\mathbf{Quas}_n$ we have

$$\begin{aligned}
p(x, y, y) &\approx f(g_1(x, \ldots, x, g_n(y, \ldots, y), x), x, \ldots, x, g_n(y, \ldots, y, y)) \\
&\approx x \\
&\approx f(y, \ldots, y, g_n(y, \ldots, y, x)) \\
&\approx f(g_1(y, \ldots, y, g_n(y, \ldots, y), y), y, \ldots, y, g_n(y, \ldots, y, x)) \\
&\approx p(y, y, x),
\end{aligned}$$

as claimed. ∎

This implies that $\mathbf{AQ}_n$ is congruence permutable, as well. Since every congruence permutable variety is congruence modular we may apply the theory of commutators for congruence modular varieties to the variety $\mathbf{AQ}_n$. We refer the reader to the Freese and McKenzie text on this subject[6]. Commutators will appear again in section 8.

So far we have not investigated whether the quasigroups under consideration may be associative or not. Associative n-quasigroups are the n-ary groups introduced by Post[12]. Certainly it is possible for a binary alternating quasigroup to be a group without being commutative, as any noncommutative group is an example. For higher arities it turns out that associativity implies commutativity.

Proposition 3. *Every alternating n-ary group for $n \geq 3$ is commutative.*

Proof. Suppose that $\mathbf{A} := (A, f)$ is an alternating n-ary group where $n \geq 3$ and let $a_1, \ldots, a_{n+1} \in A$. Observe that

$$\begin{aligned} f(f(a_1, \ldots, a_n), a_{n-1}, \ldots, a_{n-1}, a_{n+1}) &= \\ f(a_1, a_2, \ldots, a_{n-2}, f(a_{n-1}, a_n, a_{n-1}, \ldots, a_{n-1}), a_{n+1}) &= \\ f(a_1, a_2, \ldots, a_{n-2}, f(a_n, a_{n-1}, a_{n-1}, \ldots, a_{n-1}), a_{n+1}) &= \\ f(f(a_1, a_2, \ldots, a_{n-2}, a_n, a_{n-1}), a_{n-1}, \ldots, a_{n-1}, a_{n+1}), & \end{aligned}$$

which implies that

$$f(a_1, \ldots, a_n) = f(a_1, \ldots, a_{n-2}, a_n, a_{n-1}),$$

so $\mathbf{A}$ is in fact commutative. □

4.1. **Examples of alternating quasigroups.** We give three classes of examples of alternating quasigroups:

(1) A particular order 5 ternary alternating quasigroup.
(2) Alternating n-quasigroups which are given as alternating products of commutative ones.
(3) Any binary quasigroup, which includes all groups.

After the authors laboriously produced (1) they prevailed upon Jonathan Smith for other examples of alternating n-quasigroups. Although no one had, to Smith's knowledge, studied the varieties of alternating n-quasigroups for $n > 2$ before he nonetheless provided a special case of (2), which we generalized. Other varieties of n-quasigroups have been investigated before, as discussed in the introduction as well as in section 6. Since binary quasigroups, and in particular groups, are well-known we won't discuss (3) any more in this section.

We are especially interested in alternating quasigroups which are not commutative. Although commutative quasigroups are an input for the alternating product construction in (2), we will focus on examples of noncommutative alternating quasigroups since these will yield nontrivial manifolds and appear more difficult to find.

Example 1. Take $S := (\mathbb{Z}/5\mathbb{Z})^3$ and define $h\colon \mathbb{Z}/5\mathbb{Z} \times \mathbf{Alt}_3 \to \mathbf{Perm}_S$ by

$$(h(k, \sigma))(x_1, x_2, x_3) := (x_{\sigma(1)} + k, x_{\sigma(2)} + k, x_{\sigma(3)} + k).$$

There are 7 members of $\mathrm{Orb}(h)$. One system of orbit representatives is:

$$\{000, 011, 022, 012, 021, 013, 031\}$$

where we follow the convention that xyz indicates (x, y, z) for $x, y, z \in \mathbb{Z}/5\mathbb{Z}$. Let $A \coloneqq \mathbb{Z}/5\mathbb{Z}$ and define a ternary operation $f\colon A^3 \to A$ so that

$$f((h(k,\sigma))(x_1, x_2, x_3)) = f(x_1, x_2, x_3) + k$$

and f is defined on the above set of orbit representatives as follows.

xyz	$f(x, y, z)$
000	0
011	0
022	0
012	3
021	4
013	4
031	2

One may verify that the above values and the symmetry imposed on f completely define a ternary alternating quasigroup operation on A.

Since ternary alternating quasigroups are an equational class the above example yields infinitely many others. For example, if we take $\mathbf{A}$ to be the ternary alternating quasigroup in example 1 then we have that any algebra of the form $\prod_{j \in J} \mathbf{A}_j$ where $\mathbf{A}_j \coloneqq \mathbf{A}$ for all $j \in J$ where J is any set is a ternary alternating quasigroup.

Our next class of examples makes use of a product-like construction which we now define.

Definition 41 (Alternating map)**.** Given sets A and B we say that a function $\alpha\colon A^n \to B$ is an *n-ary alternating map* from A to B when for each $\sigma \in \mathrm{Alt}_n$ and each $a \in A^n$ we have that

$$\alpha(a) = \alpha(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

The prototypical example of an alternating map is the determinant map

$$\det\colon (\mathbb{F}^n)^n \to \mathbb{F}$$

where $\mathbb{F}$ is any field. This is an n-ary alternating map from $\mathbb{F}^n$ to $\mathbb{F}$.

Definition 42 (Alternating product)**.** Given an n-ary commutative quasigroup $\mathbf{U} \coloneqq (U, g)$, an $(n+1)$-ary commutative quasigroup $\mathbf{V} \coloneqq (V, h)$, and an n-ary alternating map $\alpha\colon A^n \to B$ the *alternating product* of $\mathbf{U}$ and $\mathbf{V}$ with alternating map α is the alternating n-quasigroup

$$\mathbf{U} \boxtimes_\alpha \mathbf{V} \coloneqq (U \times V, g \boxtimes_\alpha h\colon (U \times V)^n \to U \times V)$$

where for $(u_1, v_1), \ldots, (u_n, v_n) \in U \times V$ we define

$$(g \boxtimes_\alpha h)((u_1, v_1), \ldots, (u_n, v_n)) \coloneqq (g(u), h(\alpha(u), v_1, \ldots, v_n))$$

where $u \coloneqq (u_1, \ldots, u_n)$.

Example 2. Let $\mathbb{F}$ be a field of odd characteristic or characteristic 0 and let $n \in \mathbb{N}$. Define $\mathbf{U} \coloneqq (\mathbb{F}^n, g)$ where

$$g(u_1, \ldots, u_n) \coloneqq \sum_{i=1}^{n} u_i,$$

when $u_1, \ldots, u_n \in \mathbb{F}_q^n$, take $\mathbf{V} \coloneqq (\mathbb{F}, h)$ where

$$h(v_1, \ldots, v_{n+1}) \coloneqq \sum_{i=1}^{n+1} v_i$$

when $v_1, \ldots, v_{n+1} \in \mathbb{F}$, and define $\alpha \coloneqq \det\colon (\mathbb{F}^n)^n \to \mathbb{F}$. The alternating n-quasigroup $\mathbf{U} \boxtimes_\alpha \mathbf{V}$, which we will also denote by $\mathbb{F}^{(n)}$, is the example Jonathan Smith indicated.

Note the similarity between this construction and the decomposition of an algebra $\mathbf{A}$ in a congruence modular variety as $\mathbf{Q} \otimes^T \mathbf{B}$ where $\mathbf{Q}$ is Abelian and $\mathbf{B} \coloneqq \mathbf{A}/\zeta_{\mathbf{A}}$[6, p.53].

We could have allowed $\mathbb{F}$ to have characteristic 2 as well, but the resulting alternating quasigroup $\mathbb{F}^{(n)}$ would again be commutative, which will not give us an interesting object when we proceed to construct manifolds from quasigroups, although of course such commutative quasigroups may be used as the inputs in another alternating product.

The smallest n-quasigroup of the form of example 2 we can produce has $\mathbb{F} = \mathbb{F}_3$ and hence has order 3^{n+1}. This is still slightly too large for us to work with "by hand", so we consider instead a more contrived example of smaller order.

Example 3. Define $\mathbf{U} \coloneqq (\mathbb{Z}/3\mathbb{Z}, g)$ where

$$g(u_1, u_2, u_3) \coloneqq u_1 + u_2 + u_3$$

for $u_1, u_2, u_3 \in \mathbb{Z}/3\mathbb{Z}$ and define $\mathbf{V} \coloneqq (\mathbb{Z}/2\mathbb{Z}, h)$ where

$$h(v_1, v_2, v_3, v_4) \coloneqq v_1 + v_2 + v_3 + v_4$$

for $v_1, v_2, v_3, v_4 \in \mathbb{Z}/2\mathbb{Z}$. Define an alternating map $\alpha\colon (\mathbb{Z}/3\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z}$ by

$$\alpha(x) \coloneqq \begin{cases} 1 & \text{when } x \in \mathrm{Orb}_{\mathbf{Alt}_3}(0, 2, 1) \\ 0 & \text{otherwise.} \end{cases}$$

We then have a ternary alternating quasigroup $\mathbf{U} \boxtimes_\alpha \mathbf{V}$ of order 6.

Observe that our example 1 cannot be given as an alternating product since the order of an alternating product of two nontrivial commutative quasigroups must be composite and if at least one of the input quasigroups $\mathbf{U}$ or $\mathbf{V}$ is trivial then the resulting quasigroup will be commutative.

5. Open serenation

We are now ready to begin our construction of manifolds from quasigroups. We have already described the functor $\mathbf{OGeo}_n\colon \mathbf{PMfld}_n \to \mathbf{Top}$. It remains to define a functor $\mathbf{Sim}_n\colon \mathbf{NCAQ}_n \to \mathbf{PMfld}_n$ and to give an appropriate smooth atlas to $(\mathbf{OGeo}_n \circ \mathbf{Sim}_n)(\mathbf{A})$ for each $\mathbf{A} \in \mathrm{AQ}_n$. These tasks are performed in the subsections on simplicization and open serenation, respectively.

5.1. Simplicization.

From each alternating quasigroup of arity n we obtain a pseudomanifold of dimension n.

Definition 43 (Simplicization functor)**.** We define a *simplicization functor*

$$\mathbf{Sim}_n\colon \mathbf{NCAQ}_n \to \mathbf{PMfld}_n$$

as follows. Given $\mathbf{A} \in \mathrm{AQ}_n$ we define

$$\mathbf{Sim}_n(\mathbf{A}) \coloneqq (\mathrm{Sim}(\mathbf{A}), \mathrm{SimFace}(\mathbf{A}))$$

where

$$\mathrm{Sim}(\mathbf{A}) \coloneqq \{\, \underline{a} \mid a \in \mathrm{In}(\mathbf{A}) \,\} \cup \{\, \overline{a} \mid a \in \mathrm{Out}(\mathbf{A}) \,\}$$

and

$$\mathrm{SimFace}(\mathbf{A}) \coloneqq \bigcup_{a \in \mathrm{NCT}(\mathbf{A})} \mathrm{Sb}\left(\left\{\underline{a}_1, \dots, \underline{a}_n, \overline{f(a)}\right\}\right).$$

Given an NC homomorphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ we define

$$\mathbf{Sim}_n(h)\colon \mathbf{Sim}_n(\mathbf{A}_1) \to \mathbf{Sim}_n(\mathbf{A}_2)$$

by

$$\mathbf{Sim}_n(h)(\underline{a}) \coloneqq \underline{h(a)}$$

and

$$\mathbf{Sim}_n(h)(\overline{a}) \coloneqq \overline{h(a)}.$$

We will begin to drop the subscript n going forward unless we need to explicitly refer to the arity or dimension under consideration. We refer to $\mathbf{Sim}(\mathbf{A})$ as the *simplicization* of $\mathbf{A}$ and use similar language for the simplicial maps $\mathbf{Sim}(h)$.

5.2. **Open serenation.** Finally we can give describe our functor taking $\mathbf{NCAQ}_n$ to $\mathbf{SMfld}_n$. We give the relevant coordinate charts first. The domain for all of our coordinate charts is a particular bipyrimid situated on the origin and standard basis points in $\mathbb{R}^n$.

Definition 44 (Bipyrimid)**.** The *standard open bipyrimid* (or just *bipyrimid*) in $\mathbb{R}^n$ is

$$\mathrm{Bipyr}_n \coloneqq \mathrm{OCvx}\left(\left\{(0, \dots, 0), \left(\frac{2}{n}, \dots, \frac{2}{n}\right)\right\} \cup \{e_1, \dots, e_n\}\right)$$

where e_i is the i^{th} standard basis vector of $\mathbb{R}^n$.

We have two types of charts, but they're quite similar to each other. Given $a \in \mathrm{NCT}(\mathbf{A})$ let $a' \in \mathrm{NCT}(\mathbf{A})$ be a permutation of a obtained by swapping two entries.

Definition 45 (Serene chart of input type)**.** Given an alternating n-quasigroup $\mathbf{A}$ and $a = (a_1, \dots, a_n) \in \mathrm{NCT}(\mathbf{A})$ the *serene chart of input type* for a is

$$\underline{\phi}_a\colon \mathrm{Bipyr}_n \to \mathrm{OSer}_n(\mathbf{A})$$

where we set

$$\underline{\phi}_a(u_1, \dots, u_n) \coloneqq \sum_{i=1}^n u_i \underline{a}_i + \left(1 - \sum_{i=1}^n u_i\right) \overline{f(a)}$$

when $\sum_{i=1}^n u_i \le 1$ and

$$\underline{\phi}_a(u_1, \dots, u_n) \coloneqq \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{n-2}{2} u_i - \sum_{j \ne i} u_j\right) \underline{a}_i + \left(-1 + \sum_{i=1}^n u_i\right) \overline{f(a')}$$

when $\sum_{i=1}^n u_i > 1$.

The reader may find that the formula for $\underline{\phi}_a(u_1, \dots, u_n)$ when $\sum_{i=1}^n u_i > 1$ is not entirely obvious. We sketch its derivation, which is visualized in figure 1.

A point $u \in \mathrm{Bipyr}_n$ either has

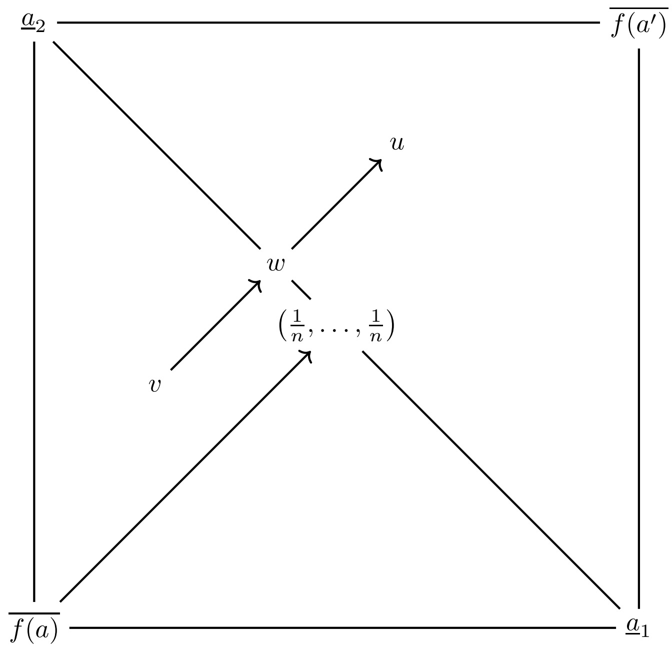

FIGURE 1. A bipyrimid

(1) $\sum_{i=1}^n u_i < 1$, in which case it is mapped to a point in the open convex hull of the $\underline{a}_i$ and $\overline{f(a)}$,
(2) $\sum_{i=1}^n u_i = 1$, in which case it is mapped to a point in the open convex hull of the $\underline{a}_i$, or
(3) $\sum_{i=1}^n u_i > 1$, in which case it is mapped to a point in the open convex hull of the $\underline{a}_i$ and $\overline{f(a')}$.

It is this last case which we need to consider carefully. Note that

$$\underline{\phi}_a^{-1}\left(\mathrm{OCvx}\left(\left\{\underline{a}_1, \ldots, \underline{a}_n, \overline{f(a')}\right\}\right)\right)$$

is the reflection of

$$\underline{\phi}_a^{-1}\left(\mathrm{OCvx}\left(\left\{\underline{a}_1, \ldots, \underline{a}_n, \overline{f(a)}\right\}\right)\right)$$

over the affine hyperplane through the point $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ orthogonal to the vector $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. A point $u \in \mathrm{Bipyr}_n$ with $\sum_{i=1}^n u_i > 1$ has a mirror image $v \in \mathrm{Bipyr}_n$ with $\sum_{i=1}^n v_i < 1$ where

$$u = v + \gamma\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$$

for some $\gamma \in \mathbb{R}$. Since we set

$$\underline{\phi}_a(v) = \sum_{i=1}^n v_i \underline{a}_i + \left(1 - \sum_{i=1}^n v_i\right)\overline{f(a)}$$

we should analogously set

$$\underline{\phi}_a(u) = \sum_{i=1}^{n} v_i \underline{a}_i + \left(1 - \sum_{i=1}^{n} v_i\right) \overline{f(a')}.$$

We would like to find a formula for the v_i in terms of the u_i. We accomplish this by finding the point $w \in \mathrm{OCvx}(\{e_1, \ldots, e_n\})$ which also lies on the line determined by u and v. We can then write $v = u - 2(u - w)$. For any $i \neq n$ we have that

$$(u - w) \cdot (e_i - e_n) = 0$$

so $w_i - w_n = u_i - u_n$. We also know that $\sum_{i=1}^{n} w_i = 1$ so taking the u_i as constants we have a system of n linear equations in the n unknowns $w_1, \ldots, w_n$. Solving this system yields that

$$w_i = \frac{1}{n}\left(1 + (n-1)u_i - \sum_{j \neq i} u_j\right).$$

Since $v = u - 2(u - w) = 2w - u$ we obtain

$$v_i = \frac{2}{n}\left(1 + (n-1)u_i - \sum_{j \neq i} u_j\right) - u_i = \frac{2}{n}\left(1 + \frac{n-2}{2}u_i - \sum_{j \neq i} u_j\right).$$

The coefficient of $\overline{f(a')}$ in $\underline{\phi}_a(u)$ is then

$$\begin{aligned} 1 - \sum_{i=1}^{n} v_i &= 1 - \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{n-2}{2}u_i - \sum_{j \neq i} u_j\right) \\ &= -1 + \sum_{i=1}^{n} u_i. \end{aligned}$$

We produce another family of charts. Given $a \in \mathrm{NCT}(\mathbf{A})$ let $a_{n+1} \in A$ be the unique solution to $f(a) = f(b)$ where

$$b_i := \begin{cases} a_i & \text{when } 1 \leq i \leq n-2 \\ a_{n+1} & \text{when } i = n-1 \\ a_{n-1} & \text{when } i = n \end{cases}.$$

Definition 46 (Serene chart of output type)**.** Given an alternating n-quasigroup $\mathbf{A}$ and $a = (a_1, \ldots, a_n) \in \mathrm{NCT}(\mathbf{A})$ the *serene chart of output type* for a is

$$\overline{\phi}_a \colon \mathrm{Bipyr}_n \to \mathrm{OSer}_n(\mathbf{A})$$

where we set

$$\overline{\phi}_a(u_1, \ldots, u_n) := \sum_{i=1}^{n-1} u_i \underline{a}_i + u_n \overline{f(a)} + \left(1 - \sum_{i=1}^{n} u_i\right) \underline{a}_n$$

when $\sum_{i=1}^{n} u_i \leq 1$ and

$$\overline{\phi}_a(u_1,\dots,u_n) \coloneqq \frac{2}{n}\sum_{i=1}^{n-1}\left(1+\frac{n-2}{2}u_i-\sum_{j\neq i}u_j\right)\underline{a}_i + \frac{2}{n}\left(1+\frac{n-2}{2}u_n-\sum_{j=1}^{n-1}u_j\right)\overline{f(a)} + \left(-1+\sum_{i=1}^{n}u_i\right)\underline{a}_{n+1}$$

when $\sum_{i=1}^{n} u_i > 1$.

Definition 47 (Open serenation functor)**.** We define an *open serenation functor*

$$\mathbf{OSer}_n\colon \mathbf{NCAQ}_n \to \mathbf{SMfld}_n$$

as follows. Given $\mathbf{A} \in \mathrm{AQ}_n$ we define

$$\mathbf{OSer}_n(\mathbf{A}) \coloneqq (\mathrm{OSer}_n(\mathbf{A}), \tau, \mathrm{SerAt}_n(\mathbf{A}))$$

where

$$(\mathrm{OSer}_n(\mathbf{A}), \tau) \coloneqq (\mathbf{OGeo}_n \circ \mathbf{Sim}_n)(\mathbf{A})$$

is the underlying topological space of $\mathbf{OSer}_n(\mathbf{A})$ and the atlas of $\mathbf{OSer}_n(\mathbf{A})$ is given by

$$\mathrm{SerAt}_n(\mathbf{A}) \coloneqq \bigcup_{a\in \mathrm{NCT}(\mathbf{A})}\left\{\underline{\phi}_a, \overline{\phi}_a\right\}.$$

Given an NC homomorphism $h\colon \mathbf{A}\to\mathbf{B}$ where $\mathbf{A},\mathbf{B} \in \mathrm{AQ}_n$ we define

$$\mathbf{OSer}_n(h) \coloneqq (\mathbf{OGeo}_n \circ \mathbf{Sim}_n)(h).$$

Note that by our definition $\mathrm{OSer}_n(\mathbf{A}) = \mathrm{OGeo}_n(\mathbf{Sim}_n(\mathbf{A}))$. As in the case of the simplicization functor we'll generally omit the dimension subscript unless it's germane to the discussion at hand.

The tangent spaces of $\mathbf{OSer}(\mathbf{A})$ can be described directly in terms of the quasigroup $\mathbf{A}$. Given a point x in a manifold we denote by T_x the set of tangent vectors at x and by $\mathbf{T}_x$ the tangent space at x.

Proposition 4. *Given a quasigroup* $\mathbf{A}$*,* $a \in \mathrm{NCT}(\mathbf{A})$*, and* $x \in \mathrm{OSer}(\mathbf{A})$ *with*

$$x \in \mathrm{OCvx}\left(\left\{\underline{a}_1,\dots,\underline{a}_n,\overline{f(a)}\right\}\right)$$

we have that

$$\mathbf{T}_x = \mathbf{Span}\left(\left\{\underline{a}_i - \overline{f(a)} \;\middle|\; i\in[n]\right\}\right).$$

Proof. We obtain the same tangent space with any chart so take $a = (a_1,\dots,a_n) \in \mathrm{NCT}(\mathbf{A})$ and consider that by our assumption

$$\underline{\phi}_a^{-1}(x) = (x_1,\dots,x_n)$$

where $\sum_{i=1}^{n} x_i < 1$. It follows that near $\underline{\phi}_a^{-1}(x)$ the map $\underline{\phi}_a$ is given by

$$\underline{\phi}_a(u_1,\dots,u_n) = \sum_{i=1}^{n} u_i\underline{a}_i + \left(1-\sum_{i=1}^{n}u_i\right)\overline{f(a)}.$$

Note that $\underline{\phi}_a$ is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^{\mathrm{Sim}(\mathbf{A})}$ near $\underline{\phi}_a^{-1}(x)$. For each $i \in [n]$ we have a tangent vector of the form

$$\frac{\partial}{\partial u_i}\underline{\phi}_a = \underline{a}_i - \overline{f(a)},$$

as claimed. □

The points addressed by the previous proposition cover almost all of $\mathbf{OSer}(\mathbf{A})$. We have a similar description of the tangent space to a point on one of the remaining "creases", of which there are two types: those which are the open convex hull of an $(n-1)$-face containing an output vertex and those which are the open convex hull of an $(n-1)$-face containing only input vertices.

Proposition 5. *Given a quasigroup* $\mathbf{A}$, $a \in \mathrm{NCT}(\mathbf{A})$, *and* $x \in \mathrm{OSer}(\mathbf{A})$ *with*

$$x \in \mathrm{OCvx}\left(\left\{\underline{a}_1, \ldots, \underline{a}_{n-1}, \overline{f(a)}\right\}\right)$$

we have that

$$\mathbf{T}_x = \mathbf{Span}\left(\left\{\underline{a}_1, \ldots, \underline{a}_{n-1}, \overline{f(a)}\right\}\right).$$

Proof. Fix some $k \in [n]$ and let $\epsilon > 0$ be sufficiently small that

$$x + te_k \in \mathrm{Bipyr}_n$$

for all $t \in (-\epsilon, \epsilon)$. Let $\gamma\colon (-\epsilon, \epsilon) \to \mathrm{OSer}_n(\mathbf{A})$ be given by

$$\gamma(t) \coloneqq \underline{\phi}_a(x + te_k).$$

We have a corresponding tangent vector

$$\left.\frac{\mathrm{d}(\underline{\phi}_a^{-1} \circ \gamma)}{\mathrm{d}t}\right|_{t=0} = e_k,$$

which we naturally identify with $\underline{a}_k$ when $k < n$ and with $\overline{f(a)}$ when $k = n$. □

Proposition 6. *Given a quasigroup* $\mathbf{A}$, $a \in \mathrm{NCT}(\mathbf{A})$, *and* $x \in \mathrm{OSer}(\mathbf{A})$ *with*

$$x \in \mathrm{OCvx}\left(\{\underline{a}_1, \ldots, \underline{a}_n\}\right)$$

we have that

$$\mathbf{T}_x = \mathbf{Span}\left(\{\underline{a}_1, \ldots, \underline{a}_n\}\right).$$

Proof. The argument here is identical to that for the preceding proposition with the label $\overline{f(a)}$ replaced with $\underline{a}_n$ and the chart $\underline{\phi}_a$ is replaced with $\overline{\phi}_a$. □

5.3. **Examples of open serenation.** We give some small examples of open serenation. Our first illustrates the distinction between open serenation and the construction for groups described by Herman and Pakianathan[8].

Example 4. Let $G \coloneqq \{\pm 1, \pm i, \pm j, \pm k\}$ and let $\mathbf{G}$ be the quaternion group of order 8 with universe G. We have that

$$\mathrm{NCT}(\mathbf{G}) = \left\{ (\pm u, \pm v) \;\middle|\; \{u, v\} \in \binom{\{i, j, k\}}{2} \right\}$$

so

$$\mathrm{In}(\mathbf{G}) = \{\pm i, \pm j, \pm k\}$$

and

$$\mathrm{Out}(\mathbf{G}) = \{\pm i, \pm j, \pm k\}.$$

$(a,b) \in \mathrm{NCT}(\mathbf{G})$	$\sigma \in \mathrm{SimFace}(\mathbf{G})$
(i,j)	$\{\underline{i}, \underline{j}, \overline{k}\}$
$(i,-j)$	$\{\underline{i}, \underline{-j}, \overline{-k}\}$
$(-i,j)$	$\{\underline{-i}, \underline{j}, \overline{-k}\}$
$(-i,-j)$	$\{\underline{-i}, \underline{-j}, \overline{k}\}$
(i,k)	$\{\underline{i}, \underline{k}, \overline{-j}\}$
$(i,-k)$	$\{\underline{i}, \underline{-k}, \overline{j}\}$
$(-i,k)$	$\{\underline{-i}, \underline{k}, \overline{j}\}$
$(-i,-k)$	$\{\underline{-i}, \underline{-k}, \overline{-j}\}$
(j,i)	$\{\underline{j}, \underline{i}, \overline{-k}\}$
$(j,-i)$	$\{\underline{j}, \underline{-i}, \overline{k}\}$
$(-j,i)$	$\{\underline{-j}, \underline{i}, \overline{k}\}$
$(-j,-i)$	$\{\underline{-j}, \underline{-i}, \overline{-k}\}$
(j,k)	$\{\underline{j}, \underline{k}, \overline{i}\}$
$(j,-k)$	$\{\underline{j}, \underline{-k}, \overline{-i}\}$
$(-j,k)$	$\{\underline{-j}, \underline{k}, \overline{-i}\}$
$(-j,-k)$	$\{\underline{-j}, \underline{-k}, \overline{i}\}$
(k,i)	$\{\underline{k}, \underline{i}, \overline{j}\}$
$(k,-i)$	$\{\underline{k}, \underline{-i}, \overline{-j}\}$
$(-k,i)$	$\{\underline{-k}, \underline{i}, \overline{-j}\}$
$(-k,-i)$	$\{\underline{-k}, \underline{-i}, \overline{j}\}$
(k,j)	$\{\underline{k}, \underline{j}, \overline{-i}\}$
$(k,-j)$	$\{\underline{k}, \underline{-j}, \overline{i}\}$
$(-k,j)$	$\{\underline{-k}, \underline{j}, \overline{i}\}$
$(-k,-j)$	$\{\underline{-k}, \underline{-j}, \overline{-i}\}$

FIGURE 2. Facets of $\mathbf{Sim}(\mathbf{G})$

We see that

$$\mathrm{Sim}(\mathbf{G}) = \{\, \underline{\pm u} \mid u \in \{i,j,k\} \,\} \cup \{\, \overline{\mp u} \mid u \in \{i,j,k\} \,\}\,.$$

The table in figure 2 gives the facet of $\mathbf{Sim}(\mathbf{G})$ associated with each $(a,b) \in \mathrm{NCT}(\mathbf{G})$.

The induced subcomplex on $\{\, \underline{\pm u} \mid u \in \{i,j,k\} \,\}$ is the graph pictured in figure 3. We may decompose this graph into three 4-cycles, which are

$$(\underline{i}, \underline{j}, \underline{-i}, \underline{-j}),\ (\underline{i}, \underline{k}, \underline{-i}, \underline{-k}),\ \text{and}\ (\underline{j}, \underline{k}, \underline{-j}, \underline{-k}).$$

Any pair of these 4-cycles intersect at two vertices and each 4-cycle can be viewed as the equator of an octohedron in $\mathbf{Sim}(\mathbf{G})$ whose poles are $\overline{\mp u}$ for some $u \in \{i,j,k\}$. For example, the first of the aforementioned 4-cycles bounds an octohedron whose poles are $\overline{\pm k}$.

Note that $\mathbf{Sim}(\mathbf{G})$ is the simplicial complex Herman and Pakianathan called $X(Q_8)$[8, p.18]. The geometric realization of $\mathbf{Sim}(\mathbf{G})$ consists of three 2-spheres, each pair of which is glued at two points. Since $\mathbf{OSer}(\mathbf{G})$ is the geometric realization of $\mathbf{Sim}(\mathbf{G})$ minus its 0-skeleton, we find that $\mathbf{OSer}(\mathbf{G})$ is the disjoint union of three copies of a 2-sphere which has had 6 points removed. This is not the desingularized complex $Y(Q_8)$ of Herman and Pakianathan, which is simply the disjoint union of three 2-spheres.

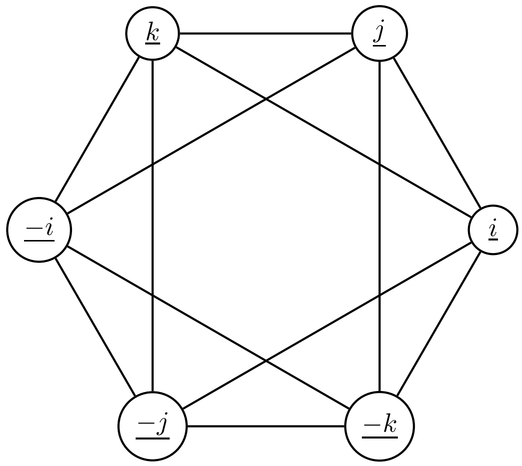

FIGURE 3. A subcomplex of $\mathbf{Sim}(\mathbf{G})$

Our next example uses the alternating 3-quasigroup from example 1.

Example 5. Let $\mathbf{A}$ be the alternating 3-quasigroup of order 5 introduced in example 1. We have that

$$\mathrm{NCT}(\mathbf{A}) = \left\{ (a,b,c) \;\middle|\; \{a,b,c\} \in \binom{A}{3} \right\}$$

so

$$\mathrm{In}(\mathbf{A}) = A = \mathrm{Out}(\mathbf{A}).$$

We see that

$$\mathrm{Sim}(\mathbf{A}) = \{\, \underline{u} \mid u \in A \,\} \cup \{\, \overline{u} \mid u \in A \,\}.$$

The table in figure 4 gives the facet of $\mathbf{Sim}(\mathbf{A})$ associated with each $(a,b,c) \in \mathrm{NCT}(\mathbf{A})$, up to the action of the alternating group. That is, if $(0,1,2)$ is listed then we don't also give the facet corresponding to $(1,2,0)$, as they are the same facet of $\mathbf{Sim}(\mathbf{A})$.

Observe that if

$$\{u_0,u_1,u_2,u_3,u_4\} = \{0,1,2,3,4\}$$

then $\mathbf{Sim}(\mathbf{A})$ has a subcomplex $\mathbf{S}$ whose facets form the set

$$\left\{ \underline{u}_i, \underline{u}_j, \underline{u}_k, \overline{u}_4 \;\middle|\; \{i,j,k\} \in \binom{0,1,2,3}{3} \right\}.$$

Note that $\mathbf{Geo}(\mathbf{S})$ is homeomorphic to the 3-simplex whose vertices are $\underline{u}_0$, $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$. It follows that $\mathbf{Sim}(\mathbf{A})$ is the subdivided complex obtained by adding a single vertex to the center of each facet of the boundary of the 4-simplex whose vertices are the $\underline{u}_i$ for $i \in \{0,1,2,3,4\}$. This implies that $\mathbf{Geo}(\mathbf{Sim}(\mathbf{A}))$ is homeomorphic to the 3-sphere. Thus, $\mathbf{OSer}(\mathbf{A})$ is homeomorphic to the 3-sphere minus the 1-skeleton of $\mathbf{Sim}(\mathbf{A})$. That 1-skeleton is the graph pictured in figure 5, which is homotopy equivalent to the join of 21 circles.

For our penultimate example we consider the order 6 alternating 3-quasigroup from example 3.

$(a,b,c) \in \mathrm{NCT}(\mathbf{A})$	$\sigma \in \mathrm{SimFace}(\mathbf{A})$
$(0,1,2)$	$\{\underline{0},\underline{1},\underline{2},\overline{3}\}$
$(1,2,3)$	$\{\underline{1},\underline{2},\underline{3},\overline{4}\}$
$(2,3,4)$	$\{\underline{2},\underline{3},\underline{4},\overline{0}\}$
$(3,4,0)$	$\{\underline{3},\underline{4},\underline{0},\overline{1}\}$
$(4,0,1)$	$\{\underline{4},\underline{0},\underline{1},\overline{2}\}$
$(0,2,1)$	$\{\underline{0},\underline{2},\underline{1},\overline{4}\}$
$(1,3,2)$	$\{\underline{1},\underline{3},\underline{2},\overline{0}\}$
$(2,4,3)$	$\{\underline{2},\underline{4},\underline{3},\overline{1}\}$
$(3,0,4)$	$\{\underline{3},\underline{0},\underline{4},\overline{2}\}$
$(4,1,0)$	$\{\underline{4},\underline{1},\underline{0},\overline{3}\}$
$(0,1,3)$	$\{\underline{0},\underline{1},\underline{3},\overline{4}\}$
$(1,2,4)$	$\{\underline{1},\underline{2},\underline{4},\overline{0}\}$
$(2,3,0)$	$\{\underline{2},\underline{3},\underline{0},\overline{1}\}$
$(3,4,1)$	$\{\underline{3},\underline{4},\underline{1},\overline{2}\}$
$(4,0,2)$	$\{\underline{4},\underline{0},\underline{2},\overline{3}\}$
$(0,3,1)$	$\{\underline{0},\underline{3},\underline{1},\overline{2}\}$
$(1,4,2)$	$\{\underline{1},\underline{4},\underline{2},\overline{3}\}$
$(2,0,3)$	$\{\underline{2},\underline{0},\underline{3},\overline{4}\}$
$(3,1,4)$	$\{\underline{3},\underline{1},\underline{4},\overline{0}\}$
$(4,2,0)$	$\{\underline{4},\underline{2},\underline{0},\overline{1}\}$

FIGURE 4. Facets of $\mathbf{Sim}(\mathbf{A})$

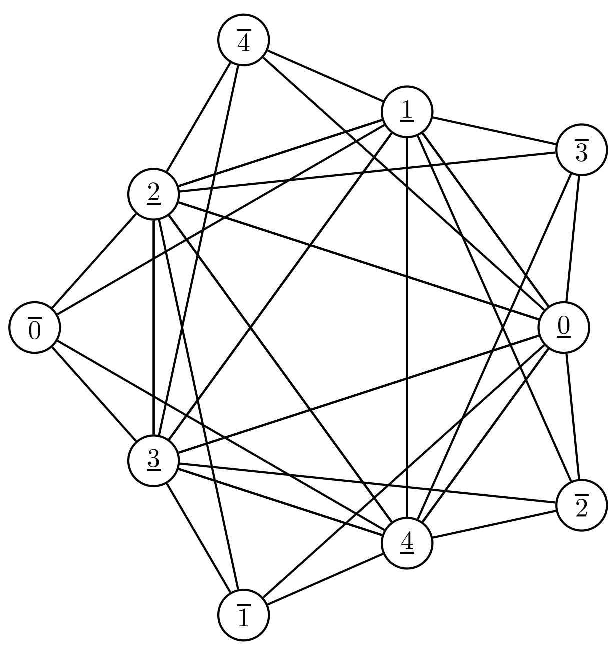

FIGURE 5. The 1-skeleton of $\mathbf{Sim}(\mathbf{A})$

$((u_1,v_1),(u_2,v_2),(u_3,v_3)) \in \mathrm{NCT}(\mathbf{A})$	$\sigma \in \mathrm{SimFace}(\mathbf{A})$
$((0,0),(1,0),(2,0))$	$\{\underline{00},\underline{10},\underline{20},\overline{00}\}$
$((0,0),(1,0),(2,1))$	$\{\underline{00},\underline{10},\underline{21},\overline{01}\}$
$((0,0),(1,1),(2,0))$	$\{\underline{00},\underline{11},\underline{20},\overline{01}\}$
$((0,0),(1,1),(2,1))$	$\{\underline{00},\underline{11},\underline{21},\overline{00}\}$
$((0,1),(1,0),(2,0))$	$\{\underline{01},\underline{10},\underline{20},\overline{01}\}$
$((0,1),(1,0),(2,1))$	$\{\underline{01},\underline{10},\underline{21},\overline{00}\}$
$((0,1),(1,1),(2,0))$	$\{\underline{01},\underline{11},\underline{20},\overline{00}\}$
$((0,1),(1,1),(2,1))$	$\{\underline{01},\underline{11},\underline{21},\overline{01}\}$
$((0,0),(2,0),(1,0))$	$\{\underline{00},\underline{20},\underline{10},\overline{01}\}$
$((0,0),(2,0),(1,1))$	$\{\underline{00},\underline{20},\underline{11},\overline{00}\}$
$((0,0),(2,1),(1,0))$	$\{\underline{00},\underline{21},\underline{10},\overline{00}\}$
$((0,0),(2,1),(1,1))$	$\{\underline{00},\underline{21},\underline{11},\overline{01}\}$
$((0,1),(2,0),(1,0))$	$\{\underline{01},\underline{20},\underline{10},\overline{00}\}$
$((0,1),(2,0),(1,1))$	$\{\underline{01},\underline{20},\underline{11},\overline{01}\}$
$((0,1),(2,1),(1,0))$	$\{\underline{01},\underline{21},\underline{10},\overline{01}\}$
$((0,1),(2,1),(1,1))$	$\{\underline{01},\underline{21},\underline{11},\overline{00}\}$

FIGURE 6. Facets of $\mathbf{Sim}(\mathbf{A})$

Example 6. Let $\mathbf{A}$ be the ternary alternating quasigroup $\mathbf{U} \boxtimes_\alpha \mathbf{V}$ introduced in example 3. We have that

$$\mathrm{NCT}(\mathbf{A}) = \left\{ ((u_1,v_1),(u_2,v_2),(u_3,v_3)) \in (U \times V)^3 \;\middle|\; \{u_1,u_2,u_3\} = \{0,1,2\} \right\}.$$

so

$$\mathrm{In}(\mathbf{A}) = A$$

and

$$\mathrm{Out}(\mathbf{A}) = \{(0,0),(0,1)\}.$$

We see that

$$\mathrm{Sim}(\mathbf{A}) = \{ \underline{uv} \mid u \in \mathbb{Z}/3\mathbb{Z} \text{ and } v \in \mathbb{Z}/2\mathbb{Z} \} \cup \{\overline{00},\overline{01}\}.$$

The table in figure 6 gives the facet of $\mathbf{Sim}(\mathbf{A})$ associated with each

$$((u_1,v_1),(u_2,v_2),(u_3,v_3)) \in \mathrm{NCT}(\mathbf{A}),$$

up to the action of the alternating group.

The induced subcomplex on $\{ \underline{uv} \mid u \in \mathbb{Z}/3\mathbb{Z} \text{ and } v \in \mathbb{Z}/2\mathbb{Z} \}$ is an octohedron, as one may see by taking as the equator the vertices $\underline{00}$, $\underline{20}$, $\underline{01}$, and $\underline{21}$, in that order, and taking the poles to be $\overline{10}$ and $\overline{11}$. It follows that $\mathbf{Geo}(\mathbf{Sim}(\mathbf{A}))$ is a 3-sphere whose hemispheres are the two cones over this octohedron with cone points $\overline{00}$ and $\overline{01}$. Thus, $\mathbf{OSer}(\mathbf{A})$ is homeomorphic to the 3-sphere minus the 1-skeleton of $\mathbf{Sim}(\mathbf{A})$. That 1-skeleton is the graph pictured in figure 7, which is homotopy equivalent to the join of 17 circles.

Our last example is the most degenerate, but it's worth noting that this corner case is still defined.

Example 7. Suppose that $\mathbf{A}$ is a commutative n-quasigroup. We have that $\mathbf{OSer}(\mathbf{A})$ is the empty manifold. To see this, note that $\mathrm{NCT}(\mathbf{A}) = \varnothing$, which

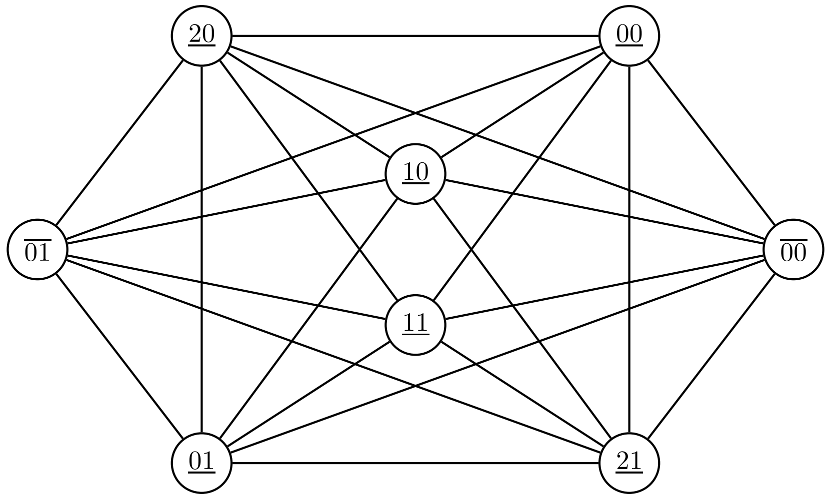

FIGURE 7. The 1-skeleton of $\mathbf{Sim}(\mathbf{A})$

implies that $\mathrm{In}(\mathbf{A}) = \varnothing$ and $\mathrm{Out}(\mathbf{A}) = \varnothing$. It follows that $\mathbf{Sim}(\mathbf{A}) = (\varnothing, \varnothing)$ and hence

$$(\mathbf{OGeo} \circ \mathbf{Sim})(\mathbf{A})$$

is the empty topological space with no points.

5.4. **The graph retract.** The manifolds $\mathbf{OSer}(\mathbf{A})$ are relatively unstructured up to homotopy.

Definition 48 (NC graph)**.** Given an alternating n-quasigroup $\mathbf{A}$ the *NC graph* of $\mathbf{A}$ is the simple graph

$$\mathbf{NCGrph}(\mathbf{A}) \coloneqq (\mathrm{NCVert}(\mathbf{A}), \mathrm{NCEdge}(\mathbf{A}))$$

where

$$\mathrm{NCVert}(\mathbf{A}) \coloneqq \mathrm{Orb}_{\mathbf{Alt}_n}(\mathrm{NCT}(\mathbf{A}))$$

and we define $\mathrm{NCEdge}(\mathbf{A})$ to consist of all pairs

$$\{a/\mathbf{Alt}_n, b/\mathbf{Alt}_n\} \in \binom{\mathrm{NCVert}(\mathbf{A})}{2}$$

such that either

(1) we have $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$ or
(2) $f(a) = f(b)$ and

$$|\{a_1, \dots, a_n\} \cap \{b_1, \dots, b_n\}| = n - 1.$$

Proposition 7. *For any alternating n-quasigroup $\mathbf{A}$ we have an embedding*

$$\iota\colon \mathbf{Geo}(\mathbf{NCGrph}(\mathbf{A})) \hookrightarrow \mathbf{OSer}(\mathbf{A})$$

such that $\mathrm{Im}(\iota)$ is a strong deformation retract of $\mathbf{OSer}(\mathbf{A})$.

Proof. Given

$$\{a/\mathbf{Alt}_n, b/\mathbf{Alt}_n\} \in \mathrm{NCEdge}(\mathbf{A})$$

and $\gamma \in \left[\frac{1}{2}, 1\right]$ we define

$$\iota(\gamma a/\,\mathbf{Alt}_n + (1-\gamma)b/\,\mathbf{Alt}_n) \coloneqq \frac{\gamma}{n+1}\left(\sum_{i=1}^{n} \underline{a}_i + \overline{f(a)}\right) + \frac{1-\gamma}{n}\sum_{i=1}^{n} c_i$$

where

$$\{c_1, \dots, c_n\} \coloneqq \left\{\underline{a}_1, \dots, \underline{a}_n, \overline{f(a)}\right\} \cap \left\{\underline{b}_1, \dots, \underline{b}_n, \overline{f(b)}\right\}.$$

Edges from NCEdge($\mathbf{A}$) are mapped by ι to piecewise linear curves between the midpoints of the geometric realization of facets in $\mathbf{Sim}(\mathbf{A})$ which intersect at an $(n-1)$-face.

Define a homotopy

$$h\colon\ \mathbf{OSer}(\mathbf{A}) \times [0,1] \to \mathbf{OSer}(\mathbf{A})$$

as follows. Let $\{b_1, \dots, b_{n+1}\}$ be the vertices of a facet of $\mathbf{Sim}(\mathbf{A})$ and consider the maximal flag

$$\{b_1\} \subset \{b_1, b_2\} \subset \dots \subset \{b_1, \dots, b_{n+1}\}$$

associated with the given labeling on the b_i. Define

$$c_k \coloneqq \frac{1}{k}\sum_{i=1}^{k} b_k$$

and note that the c_k are the vertices of the facet of the barycentric subdivision of the simplex with vertices $\{b_1, \dots, b_{n+1}\}$ corresponding to the flag in question. When

$$x \in \mathrm{Cvx}(\{c_1, \dots, c_{n+1}\})$$

let P be the affine span of

$$\{c_1, \dots, c_{n-1}, x\}$$

and let y be the unique point of intersection between P and $\mathrm{Cvx}(\{c_n, c_{n+1}\})$. We set

$$h(x,t) \coloneqq (1-t)x + ty.$$

Note that if $x \in \mathrm{Im}(\iota)$ then $y = x$ and $h(x,t) = x$ for all time t. □

5.5. **Examples of the graph retract.** We give several examples of $\mathbf{NCGrph}(\mathbf{A})$ for various choices of $\mathbf{A}$. We can be a bit more expansive than in the analogous subsection 5.3 as there are fewer topological considerations here.

Our first example is that of the quaternion group of example 4.

Example 8. Let $\mathbf{G}$ be the quaternion group of order 8. Note that we have one vertex of $\mathbf{NCGrph}(\mathbf{G})$ for each facet of $\mathbf{Sim}(\mathbf{G})$ so we have that

$$\mathrm{NCVert}(\mathbf{G}) = \left\{ (\pm u, \pm v)/\,\mathbf{Alt}_2 \;\middle|\; \{u,v\} \in \binom{\{i,j,k\}}{2} \right\}.$$

Choosing orbit representatives and suppressing the obvious isomorphism we take

$$\mathrm{NCVert}(\mathbf{G}) = \left\{ (\pm u, \pm v) \;\middle|\; \{u,v\} \in \binom{\{i,j,k\}}{2} \right\}.$$

We see that $\mathbf{NCGrph}(\mathbf{G})$ is 3-regular with the neighbors of (x,y) being

$$(y,x),\ (xyx^{-1}, x),\ \text{and}\ (y, y^{-1}xy).$$

The resulting graph is pictured in figure 8. We see that

$$\mathbf{NCGrph}(\mathbf{G}) \cong \mathbf{Q}_3 \sqcup \mathbf{Q}_3 \sqcup \mathbf{Q}_3$$

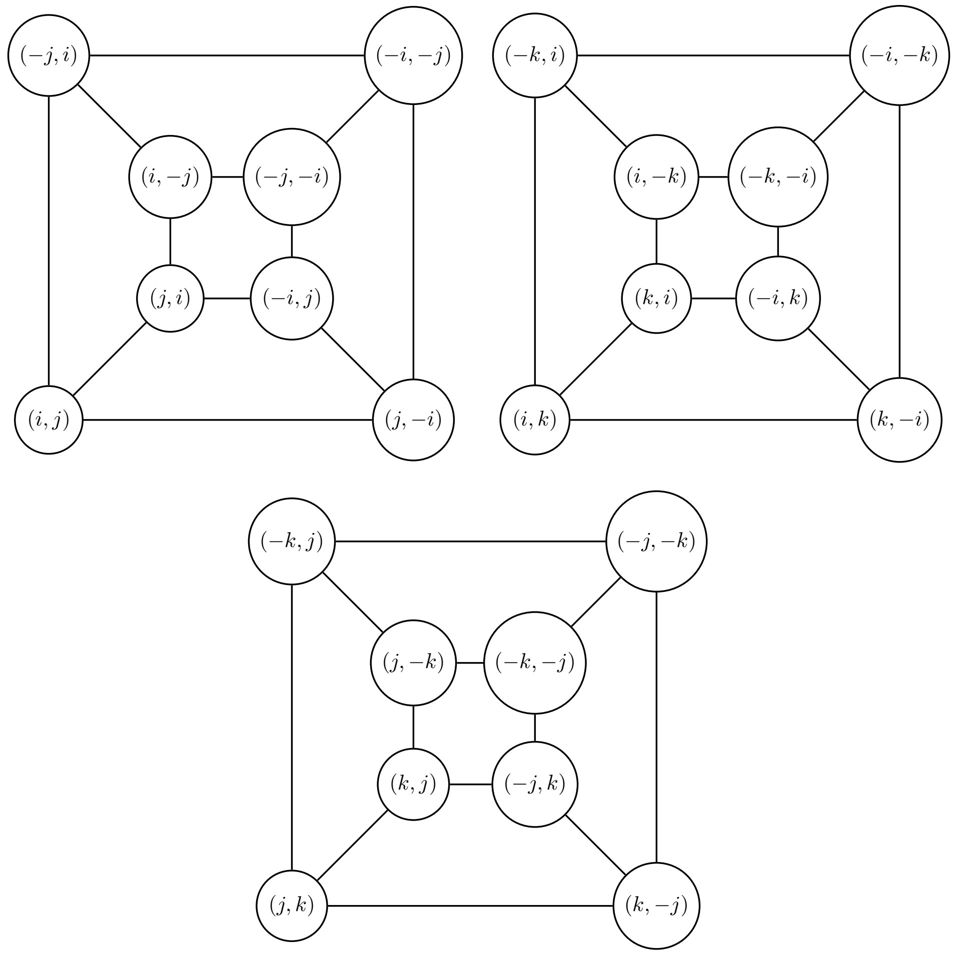

FIGURE 8. The graph $\mathbf{NCGrph}(\mathbf{G})$

where $\mathbf{Q}_3$ is the 3-cube graph.

Our next example concerns the order 5 quasigroup from example 1 and example 5.

Example 9. Let $\mathbf{A}$ be the alternating 3-quasigroup of order 5 from the examples indicated above. As in our previous example of and NC graph we choose orbit representatives under the action of $\mathbf{Alt}_3$ and suppress the obvious isomorphism to say that

$$\begin{aligned}\operatorname{NCVert}(\mathbf{A}) = \{&012, 123, 234, 340, 401, 021, 132, 243, 304, 410,\\ &013, 124, 230, 341, 402, 031, 142, 203, 314, 420\}.\end{aligned}$$

Since $\mathbf{A}$ is ternary we have that $\mathbf{NCGrph}(\mathbf{A})$ is 4-regular. We could give a sort of conjugacy formula for the neighbors of a vertex analogous to that from example 8 but in this case it's easier to just directly examine figure 4 in order to see that $\mathbf{NCGrph}(\mathbf{A})$ is the graph pictured in figure 9.

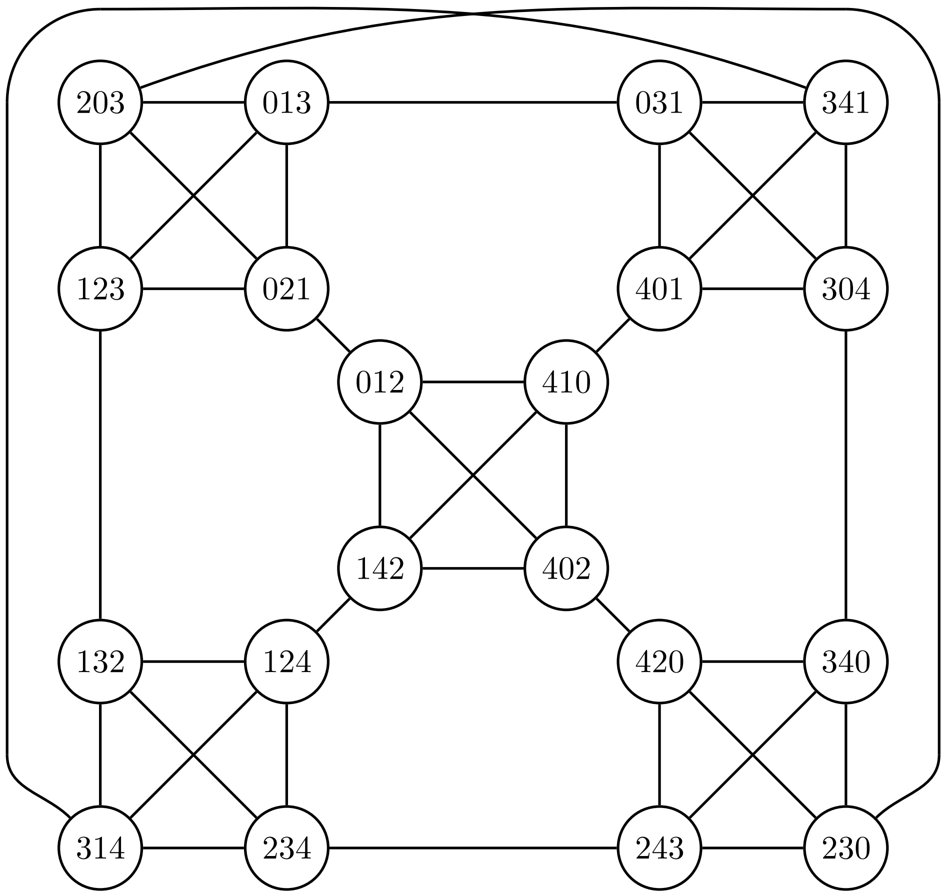

FIGURE 9. The graph $\mathbf{NCGrph}(\mathbf{A})$

The case of the order 6 quasigroup from example 3 and example 6 is similar.

Example 10. Let $\mathbf{A}$ be the alternating 3-quasigroup of order 6 from the examples indicated above. Again we choose orbit representatives and declare that

$$\begin{aligned}\mathrm{NCVert}(\mathbf{A}) = \{&(00,10,20),(00,10,21),(00,11,20),(00,11,21),\\&(01,10,20),(01,10,21),(01,11,20),(01,11,21),\\&(00,20,10),(00,20,11),(00,21,10),(00,21,11),\\&(01,20,10),(01,20,11),(01,21,10),(01,21,11)\}.\end{aligned}$$

In this case we see that $\mathbf{NCGrph}(\mathbf{A})$, which is pictured in figure 10, is isomorphic to $\mathbf{C}_4 \square \mathbf{C}_4$ where $\mathbf{C}_4$ is the 4-cycle graph and $\square$ is the Cartesian product of graphs. Note also that $\mathbf{C}_4 \square \mathbf{C}_4 \cong \mathbf{Q}_4$, where $\mathbf{Q}_4$ is the 4-cube graph.

Our last example concerns the quasigroups $\mathbb{F}^{(n)}$ of example 2.

Example 11. Fix an odd prime power q and observe that

$$((u_1,v_1),\ldots,(u_n,v_n)) \in \mathrm{NCT}(\mathbb{F}_q^{(n)})$$

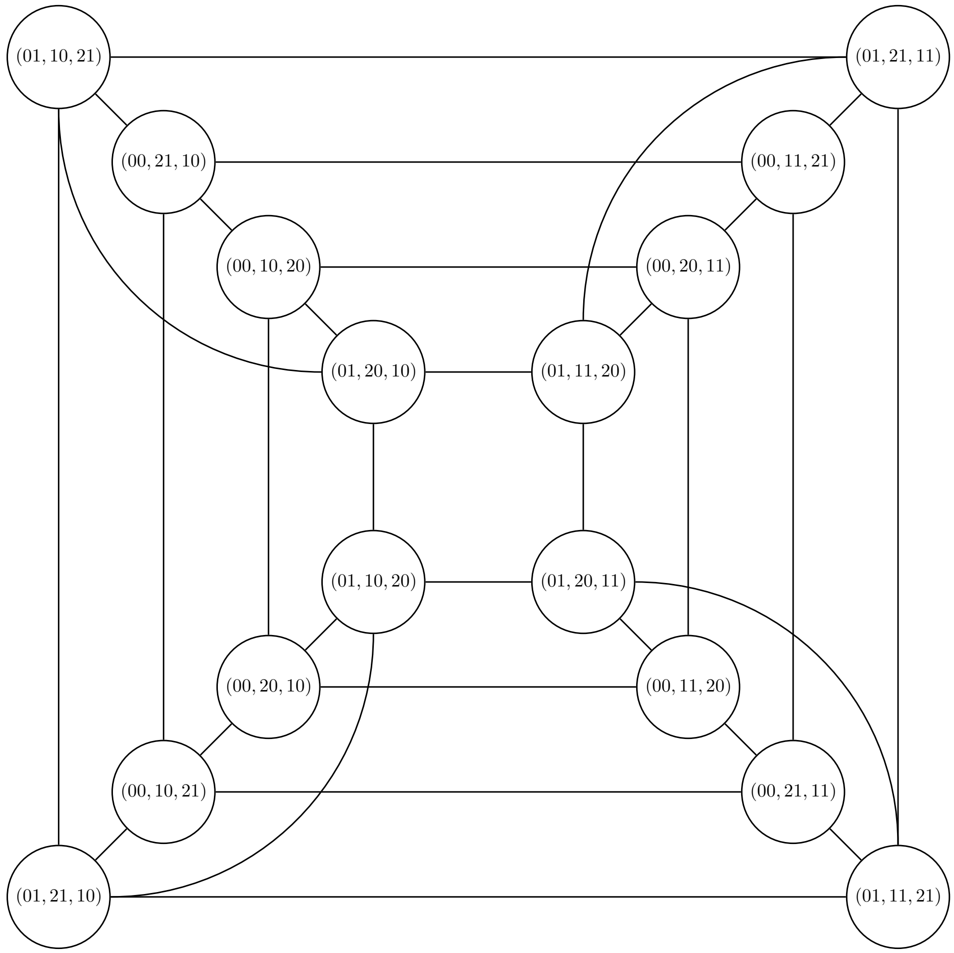

FIGURE 10. The graph **NCGrph**(**A**)

if and only if

$$\det(u_1, \ldots, u_n) \neq 0.$$

It follows that

$$\mathrm{NCT}(\mathbb{F}_q^{(n)}) \cong \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_q^n$$

where $\mathrm{GL}_n(\mathbb{F}_q)$ is the set of invertible $n \times n$ matrices with entries in $\mathbb{F}_q$. We find that $\mathbf{NCGrph}(\mathbb{F}_q^{(n)})$ is an $(n+1)$-regular graph with

$$\begin{aligned} \left|\mathrm{Orb}_{\mathbf{Alt}_n}(\mathrm{NCT}(\mathbb{F}_q^{(n)}))\right| &= \frac{2}{n!}\left|\mathrm{NCT}(\mathbb{F}_q^{(n)})\right| \\ &= \frac{2}{n!}\left|\mathrm{GL}_n(\mathbb{F}_q)\right|\left|\mathbb{F}_q^n\right| \\ &= \frac{2q^n}{n!}\prod_{k=1}^{n-1}(q^n - q^k) \end{aligned}$$

vertices.

5.6. **NC graphs and Johnson graphs.** Finite NC graphs are quite structured, as they are regular and are induced subgraphs of Johnson graphs. For an introduction to Johnson graphs, see [7]. Similar comments hold in the infinite case.

Proposition 8. *Let* $\mathbf{A}$ *be a finite alternating* n*-quasigroup and let*

$$s \coloneqq |\mathrm{Sim}(\mathbf{A})| = |\mathrm{In}(\mathbf{A})| + |\mathrm{Out}(\mathbf{A})| .$$

We have that $\mathbf{NCGrph}(\mathbf{A})$ *is an induced subgraph of the Johnson graph* $J(s, n+1, n)$*.*

Proof. Let $\psi\colon \mathrm{Sim}(\mathbf{A}) \to [s]$ be a bijection and define a graph homomorphism

$$h\colon \mathbf{NCGrph}(\mathbf{A}) \to J(s, n+1, n)$$

by

$$h(a/\, \mathbf{Alt}_n) \coloneqq \left\{\psi(\underline{a}_1), \dots, \psi(\underline{a}_n), \psi(\overline{f(a)})\right\}.$$

Observe that h is an embedding and that $\{a/\,\mathbf{Alt}_n, b/\,\mathbf{Alt}_n\} \in \mathrm{NCEdge}(\mathbf{A})$ if and only if $h(a/\,\mathbf{Alt}_n)$ and $h(b/\,\mathbf{Alt}_n)$ have exactly n elements in common. □

Since $\mathbf{NCGrph}(\mathbf{A})$ is an induced subgraph of $J(s, n+1, n)$ when $\mathbf{A}$ is a finite alternating n-quasigroup we can use interlacing[7, Theorem 9.1.1] to understand the spectrum of $\mathbf{NCGrph}(\mathbf{A})$ given the spectrum of $J(s, n+1, n)$. The graph $\mathbf{NCGrph}(\mathbf{A})$ has $|\mathrm{SimFace}(\mathbf{A})|$ vertices, so interlacing will give stronger results the larger the ratio

$$\frac{|\mathrm{SimFace}(\mathbf{A})|}{\left|\binom{\mathrm{Sim}(\mathbf{A})}{n+1}\right|}$$

is.

5.7. **The Riemannian metric.** For any alternating n-quasigroup $\mathbf{A}$ we have that $\mathbf{OSer}(\mathbf{A})$ carries a canonical metric. Let δ denote the Kronecker delta function.

Definition 49 (Standard metric)**.** Let $\mathbf{A}$ be an alternating n-quasigroup. The *standard metric* g on $\mathbf{OSer}(\mathbf{A})$ is given by bilinear extension of the following rules:

(1) When $x \in \mathrm{OCvx}\left(\left\{\underline{a}_1, \dots, \underline{a}_n, \overline{f(a)}\right\}\right)$ we set

$$g_x(\underline{a}_i - \overline{f(a)}, \underline{a}_j - \overline{f(a)}) \coloneqq 1 + \delta_{ij}.$$

(2) When $x \in \mathrm{OCvx}\left(\left\{\underline{a}_1, \dots, \underline{a}_{n-1}, \overline{f(a)}\right\}\right)$ we set

$$g_x(\underline{a}_i, \underline{a}_j) \coloneqq 1 + \delta_{ij},$$

$$g_x(\underline{a}_i, \overline{f(a)}) \coloneqq 1,$$

and

$$g_x(\overline{f(a)}, \overline{f(a)}) = 2.$$

(3) When $x \in \mathrm{OCvx}\left(\{\underline{a}_1, \dots, \underline{a}_n\}\right)$ we set

$$g_x(\underline{a}_i, \underline{a}_j) \coloneqq 1 + \delta_{ij}.$$

This metric isn't too exciting in the sense that it always makes $\mathbf{OSer}(\mathbf{A})$ flat. Intuitively, the Riemannian manifold $(\mathbf{OSer}(\mathbf{A}), g)$ consists of glued copies of regular simplices whose edges all have length $\sqrt{2}$.

Proposition 9. *The Riemannian manifold* $(\mathbf{OSer}(\mathbf{A}), g)$ *is flat for any alternating* n*-quasigroup* $\mathbf{A}$ *when* g *is the standard metric.*

$(v_k)_x$	$k<n$	$k=n$
$\sum_{i=1}^n u_i < 1$	$(\underline{a}_k - \overline{f(a)}) - (\underline{a}_n - \overline{f(a)})$	$-(\underline{a}_n - \overline{f(a)})$
$\sum_{i=1}^n u_i = 1$	$\underline{a}_k$	$\overline{f(a)}$
$\sum_{i=1}^n u_i > 1$	$(\underline{a}_i - \overline{f(a)}) - \frac{2}{n}\sum_{\ell=1}^{n-1}(\underline{a}_\ell - \overline{f(a)}) + (\underline{a}_{n+1} - \overline{f(a)})$	$-\frac{2}{n}\sum_{\ell=1}^{n-1}(\underline{a}_\ell - \overline{f(a)}) + (\underline{a}_{n+1} - \overline{f(a)})$

FIGURE 11. Output chart tangent vector fields

Proof. We show that g is constant on any given coordinate chart, which makes the metric tensor 0.

Given $a = (a_1, \dots, a_n) \in \mathrm{NCT}(\mathbf{A})$ consider the serene chart of input type $\underline{\phi}_a$. Given $k \in [n]$ define a tangent vector field $v_k \colon \operatorname{Im}(\underline{\phi}_a) \to T\,\mathbf{OSer}(\mathbf{A})$ to $\mathbf{OSer}(\mathbf{A})$ on $\operatorname{Im}(\underline{\phi}_a)$ by setting

$$(v_k)_x \coloneqq \underline{a}_k - \overline{f(a)}$$

when $x = \underline{\phi}_a(u)$ where $\sum_{i=1}^n u_i < 1$,

$$(v_k)_x \coloneqq \underline{a}_k$$

when $x = \underline{\phi}_a(u)$ where $\sum_{i=1}^n u_i = 1$, and

$$(v_k)_x \coloneqq (\underline{a}_k - \overline{f(a')}) - \frac{2}{n}\sum_{\ell=1}^{n}(\underline{a}_\ell - \overline{f(a')})$$

when $x = \underline{\phi}_a(u)$ where $\sum_{i=1}^n u_i > 1$.

Note that at each point $x \in \operatorname{Im}(\underline{\phi}_a)$ we have that $\{(v_1)_x, \dots, (v_n)_x\}$ is a basis for $\mathbf{T}_x$ and that the vector fields v_k pull back to the standard basis constant vector fields on Bipyr_n.

The matrix of g with respect to this coordinate chart $\underline{\phi}_a$ and basis is

$$[g_x((v_i)_x, (v_j)_x)] = [1 + \delta_{ij}] = J_n + I_n$$

where J_n is the $n \times n$ matrix whose entries are all 1 and I_n is the $n \times n$ identity matrix. Since the matrix of g is constant as a function of x we find that $\mathbf{OSer}(\mathbf{A})$ is flat when endowed with g in $\operatorname{Im}(\underline{\phi}_a)$.

Now consider the serene chart of output type $\overline{\phi}_a$. Given $k \in [n]$ define a tangent vector field $v_k \colon \operatorname{Im}(\overline{\phi}_a) \to T\,\mathbf{OSer}(\mathbf{A})$ to $\mathbf{OSer}(\mathbf{A})$ on $\operatorname{Im}(\overline{\phi}_a)$ as in figure 11 where $x = \overline{\phi}_a(u)$. By a slightly more involved calculation we again see that the matrix of g with respect to this coordinate chart $\overline{\phi}_a$ and basis is

$$[g_x((v_i)_x, (v_j)_x)] = [1 + \delta_{ij}] = J_n + I_n,$$

so g is indeed constant on any coordinate chart. □

Recall that one can define the distance between two points in $\mathbf{OSer}(\mathbf{A})$ with respect to such a metric by defining the length $L(\gamma)$ of a piecewise continuously differentiable curve $\gamma\colon [0,1] \to \mathbf{OSer}(\mathbf{A})$ to be

$$L(\gamma) \coloneqq \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t$$

and then defining the distance $d(x, y)$ from $x \in \mathrm{OSer}(\mathbf{A})$ to $y \in \mathrm{OSer}(\mathbf{A})$ to be

$$d(x, y) \coloneqq \inf(\{\, L(\gamma) \mid \gamma(0) = x \text{ and } \gamma(1) = y \,\}\,.$$

6. Serenation

Now we have almost all the tools we require to complete the process analogous to desingularization of $\mathbf{Sim}(\mathbf{A})$. Since $\mathbf{OSer}(\mathbf{A})$ only includes the necessarily nonsingular points of $\mathbf{Sim}(\mathbf{A})$ (that is, those which do not belong to the $(n-2)$-skeleton), we have already removed all singularities which may have been present. It remains to "fill in holes" in the appropriate fashion so that we obtain something which is topologically more interesting than a combinatorial graph.

The last ingredient is a slight modification of the usual notion of the completion of a metric space. Given a metric space (S, d) let $\mathrm{Cmplt}(S, d)$ denote the set of all points in the metric completion of (S, d). That is, $\mathrm{Cmplt}(S, d)$ is the set of all equivalences classes of Cauchy sequences of points in S under the equivalence relation induced by the metric d. Similarly, let $\mathbf{Cmplt}(S, d)$ denote the space obtained by equipping $\mathrm{Cmplt}(S, d)$ with the metric topology induced by d.

We say that a point x in a topological space $\mathbf{T}$ is *n-Euclidean (in $\mathbf{T}$)* when there exists a neighborhood U of x which is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 50 (Euclidean metric completion functor)**.** We define a *Euclidean metric completion functor*

$$\mathbf{EuCmplt}\colon\ \mathbf{Riem}_n \to \mathbf{Mfld}_n$$

as follows. Given a Riemannian n-manifold $(\mathbf{M}, g)$ consisting of a smooth n-manifold $\mathbf{M}$ and a Riemannian metric g whose corresponding metric on M is d, define $\mathbf{EuCmplt}(\mathbf{M}, g)$ to be the set of points

$$\mathrm{EuCmplt}(\mathbf{M}, g) \coloneqq \{\, x \in \mathrm{Cmplt}(M, d) \mid x \text{ is } n\text{-Euclidean in } \mathbf{Cmplt}(M, d) \,\}$$

equipped with the subspace topology inherited from $\mathbf{Cmplt}(M, d)$. Given a smooth map $h\colon \mathbf{M}_1 \to \mathbf{M}_2$ between Riemannian manifolds $(\mathbf{M}_1, g_1)$ and $(\mathbf{M}_2, g_2)$ which is a local isometry everywhere define

$$\mathbf{EuCmplt}(h)\colon\ \mathbf{EuCmplt}(\mathbf{M}_1, g_1) \to \mathbf{EuCmplt}(\mathbf{M}_2, g_2)$$

by

$$(\mathbf{EuCmplt}(h))(\{x_1, x_2, x_3, \dots\}/\sim_1) \coloneqq \{h(x_1), h(x_2), h(x_3), \dots\}/\sim_2$$

where $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence representing a member of $\mathrm{EuCmplt}(\mathbf{M}_1, g)$ and $\sim_1$ and $\sim_2$ are the equivalence relations identifying Cauchy sequences with distance 0 from each other in $(\mathbf{M}_1, g_1)$ and $(\mathbf{M}_2, g_2)$, respectively.

We give some motivating examples.

Example 12. Let $\mathbf{S}^n$ denote the n-sphere and let g be the usual metric on $\mathbf{S}^n$ inherited from $\mathbb{R}^{n+1}$. Take $\mathbf{M}$ to be $\mathbf{S}^n$ minus a finite set of points. We have that $\mathbf{EuCmplt}(\mathbf{M}, g) \cong \mathbf{S}^n$.

The following example illustrates that in general

$$\mathbf{EuCmplt}(\mathbf{M}, g) \neq \mathbf{Cmplt}(\mathbf{M}, d).$$

Example 13. Let $\mathbf{T}$ denote the 2-dimensional torus and let

$$\mathbf{C}(\mathbf{T}) \coloneqq ([0,1] \times \mathbf{T})/\sim$$

be cone over $\mathbf{T}$. View $\mathbf{C}(\mathbf{T})$ as embedded in $\mathbb{R}^5$ with

$$T = \{\, (\cos(\theta_1), \sin(\theta_1), \cos(\theta_2), \sin(\theta_2), 0) \mid \theta_1, \theta_2 \in \mathbb{R} \,\}$$

and cone point $(0,0,0,0,1)$. Take $\mathbf{M}$ to be $\mathbf{C}(\mathbf{T})$ minus the cone point $(0,0,0,0,1)$ and the points T of the original torus. We have that $\mathbf{M}$ is a smooth 3-manifold which may be endowed with the usual metric inherited from $\mathbb{R}^5$. We have that $\mathbf{EuCmplt}(\mathbf{M}, g) = \mathbf{M}$ while $\mathbf{Cmplt}(\mathbf{M}, d) = \mathbf{C}(\mathbf{T})$. The cone point $(0,0,0,0,1)$ is not included in $\mathrm{EuCmplt}(\mathbf{M}, g)$ because this point of the cone over the torus is not 3-Euclidean. To see this, note that if $(0,0,0,0,1)$ was 3-Euclidean in $\mathbf{C}(\mathbf{T})$ then removing it would not change the fundamental group. Since the space obtained by removing the cone point from $\mathbf{C}(\mathbf{T})$ deformation retracts to $\mathbf{T}$ we would have that $\pi_1(\mathbf{C}(\mathbf{T})) \cong \mathbb{Z}^2$. However, the cone over $\mathbf{T}$ is contractible so $\pi_1(\mathbf{C}(\mathbf{T}))$ is trivial, a contradiction. The points of T are not included in $\mathrm{EuCmplt}(\mathbf{M}, g)$ because they are also not Euclidean in the sense of having a neighborhood homeomorphic to an open set in $\mathbb{R}^3$. We emphasize here that the points of T would be Euclidean in the sense of a manifold with boundary, as they do have neighborhoods homeomorphic to a half-space in $\mathbb{R}^3$, but we do not allow that for our Euclidean metric completion.

Definition 51 (Serenation functor)**.** We define a *serenation functor*

$$\mathbf{Ser}_n \colon \mathbf{NCAQ}_n \to \mathbf{Mfld}_n$$

as follows. Given an alternating n-quasigroup $\mathbf{A}$ we define

$$\mathbf{Ser}(\mathbf{A}) \coloneqq \mathbf{EuCmplt}(\mathbf{OSer}(\mathbf{A}), g)$$

where g is the standard metric on $\mathbf{OSer}(\mathbf{A})$. Given an NC homomorphism $h \colon \mathbf{A} \to \mathbf{B}$ where $\mathbf{A}, \mathbf{B} \in \mathrm{AQ}_n$ we define

$$\mathbf{Ser}(h) \coloneqq \mathbf{EuCmplt}(h, g_{\mathbf{A}}, g_{\mathbf{B}})$$

where $g_{\mathbf{A}}$ and $g_{\mathbf{B}}$ are the standard metrics on $\mathbf{OSer}(\mathbf{A})$ and $\mathbf{OSer}(\mathbf{B})$, respectively.

Since every topological n-manifold for $n = 2$ or $n = 3$ carries a unique smooth structure we actually have functors which we might, by a slight abuse of notation, refer to as

$$\mathbf{Ser}_n \colon \mathbf{NCAQ}_n \to \mathbf{SMfld}_n$$

when $n = 2$ or $n = 3$.

It is perhaps too much to ask that any manifold $\mathbf{M}$ is of the form $\mathbf{Ser}(\mathbf{A})$ for some quasigroup $\mathbf{A}$, especially in light of the fact that any such manifold would have to be orientable and there exist nonorientable manifolds, but we would like to examine the slightly weaker condition that a connected orientable manifold $\mathbf{M}$ is a component of $\mathbf{Ser}(\mathbf{A})$ for some quasigroup $\mathbf{A}$.

Definition 52 (Serene manifold)**.** We say that a connected orientable n-manifold $\mathbf{M}$ is *serene* when there exists some alternating n-quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is a component of $\mathbf{Ser}(\mathbf{A})$.

This turns out to always be the case when $\mathbf{M}$ is triangulable. Our proof is constructive, provided a triangulation and orientation on the manifold in question.

Theorem 1. *Every connected orientable triangulable n-manifold is serene.*

Proof. Let $\mathbf{M}$ be a connected orientable triangulable n-manifold and let $\mathbf{S} = (S, \Gamma)$ be a simplicial complex with $\mathbf{Geo}(\mathbf{S}) = \mathbf{M}$. We effectively subdivide $\mathbf{S}$ and use the orientation on $\mathbf{M}$ in order to define the requisite quasigroup $\mathbf{A}$.

Take

$$\mathbf{Fr}(S) = (\mathrm{Fr}(S), f\colon (\mathrm{Fr}(S))^n \to \mathrm{Fr}(S))$$

to be the free alternating n-quasigroup generated by the set S, as discussed briefly in section 4. For each facet $\gamma = \{s_1, \ldots, s_{n+1}\}$ of $\mathbf{S}$ fix an orientation $[s_1, \ldots, s_{n+1}]$ so that the complex $\mathbf{S}$ is oriented. Let

$$\gamma_i \coloneqq (s_1, \ldots, s_{i-1}, \hat{s}_i, s_{i+1}, \ldots, s_{n+1})$$

denote the n-tuple of vertices of $\mathbf{S}$ obtained by deleting the i^{th} entry from the tuple $(s_1, \ldots, s_{n+1})$. Define $\mu_{\mathbf{S}} \subset (\mathrm{Fr}(S))^2$ by

$$\mu_{\mathbf{S}} \coloneqq \{\, (f(\gamma_{n+1}), f(\gamma_i)) \mid \gamma \in \Gamma,\ |\gamma| = n+1, \text{ and } i \in [n] \,\}.$$

We set $\theta_{\mathbf{S}} \coloneqq \mathrm{Cg}^{\mathbf{Fr}(S)}(\mu_{\mathbf{S}})$ and define

$$\mathbf{A} \coloneqq \mathbf{Fr}(S)/\theta_{\mathbf{S}}.$$

We claim that $\mathbf{M}$ is a component of $\mathbf{Ser}(\mathbf{A})$.

To see this, first consider the simplicial complex

$$\mathbf{S}' \coloneqq (S', \Gamma')$$

where

$$S' \coloneqq \{\, \underline{s} \mid s \in S \,\} \cup \{\, \overline{\gamma} \mid \gamma \in \mathrm{Fct}(\mathbf{S}) \,\}$$

and

$$\Gamma' \coloneqq \bigcup_{\substack{\gamma \in \mathrm{Fct}(\mathbf{S}) \\ s \in \gamma}} \mathrm{Sb}(\{\, \underline{s}' \mid s' \in \gamma \setminus \{s\} \,\} \cup \{\overline{\gamma}\}).$$

Note that $\mathbf{Geo}(\mathbf{S}') \cong \mathbf{Geo}(\mathbf{S})$. The new vertices $\overline{\gamma}$ correspond to the equivalence classes of the elements $f(\gamma_i)$ of $\mathrm{Fr}(S)$ under $\theta_{\mathbf{S}}$. Since $\mathbf{M}$ is connected so is $\mathbf{Geo}(\mathbf{S}')$, which means that there is a component of $\mathbf{OSer}(\mathbf{A})$ which is homeomorphic to

$$\mathbf{U} \coloneqq \mathbf{Geo}(\mathbf{S}') \setminus (\mathbf{Geo}(\mathbf{S}'))^{(n-2)},$$

which is the geometric realization of $\mathbf{S}'$ with its $(n-2)$-skeleton excised. Since $\mathbf{U}$ is still connected we have that $\mathbf{EuCmplt}(\mathbf{U}, g)$ (where g is the restriction of the standard metric on $\mathbf{OSer}(\mathbf{A})$ to $\mathbf{U}$) is a component of $\mathbf{Ser}(\mathbf{A})$. Since $\mathbf{U}$ is all but the $(n-2)$-skeleton of the manifold $\mathbf{M}$ we have that $\mathbf{EuCmplt}(\mathbf{U}, g) \cong \mathbf{M}$ and hence $\mathbf{M}$ is homeomorphic to a component of $\mathbf{Ser}(\mathbf{A})$. □

In the case of second countable smooth manifolds triangulation is always possible so we have the following corollary.

Corollary 1. *Every connected orientable smooth manifold is serene.*

Proof. Every second countable smooth manifold can be triangulated[15]. □

6.1. **Examples of serenation.** Our earlier work leaves little more to do in describing $\mathbf{Ser}(\mathbf{A})$ for those quasigroups which we have already visited in previous examples. We begin with the quaternion group considered in example 4 and example 8.

Example 14. Let $\mathbf{G}$ denote the quaternion group of order 8. By our analysis in example 4 we have that $\mathbf{OSer}(\mathbf{G})$ consists of three 2-spheres, each of which has had 6 points removed. It follows that $\mathbf{Ser}(\mathbf{G})$ is homeomorphic to the disjoint union of three 2-spheres. This is the same space as the desingularized complex $Y(Q_8)$ of Herman and Pakianathan[8, p.18].

The order 5 and 6 ternary quasigroups we've examined yield spheres as well.

Example 15. Let $\mathbf{A}$ be the order 5 alternating 3-quasigroup from example 1, example 5, and example 9. Since $\mathbf{OSer}(\mathbf{A})$ consists of a 3-sphere minus a 1-dimensional subcomplex we find that $\mathbf{Ser}(\mathbf{A})$ is homeomorphic to the 3-sphere.

Example 16. Let $\mathbf{A}$ be the order 6 alternating 3-quasigroup from example 3, example 6, and example 10. Since $\mathbf{OSer}(\mathbf{A})$ consists of a 3-sphere minus a 1-dimensional subcomplex we find that $\mathbf{Ser}(\mathbf{A})$ is homeomorphic to the 3-sphere.

7. Compact manifolds and Latin cubes

In general the construction in the proof of theorem 1 may be expected to yield an infinite alternating quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is a component of $\mathbf{Ser}(\mathbf{A})$. Necessarily this is the case when $\mathbf{M}$ is given with a triangulation $\mathbf{S} = (S, \Gamma)$ where S is infinite, for each member $s \in S$ becomes a distinct generator of $\mathbf{A}$. In the case that S is a finite set (or, topologically speaking, in the case that $\mathbf{M}$ is compact) can we always take $\mathbf{A}$ to be finite?

Definition 53 (Quasifinite manifold)**.** We say that a connected compact orientable smooth n-manifold $\mathbf{M}$ is *quasifinite* when there exists a finite alternating n-quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is homeomorphic to a component of $\mathbf{Ser}(\mathbf{A})$.

Problem 1. Is every connected compact orientable smooth manifold quasifinite?

This problem's solution would be implied by a positive solution of another, more combinatorial, problem which has the flavor of a standard question in the theory of (binary) quasigroups at large. The prototype for that standard question is the Evans Conjecture, which states that every partial Latin square of order n with at most $n-1$ entries may have its other entries filled in so as to obtain a complete Latin square. The Evans Conjecture was proven by Smetaniuk in 1981[13], and many similar results followed, including those on 3-dimensional Latin cubes in [9].

Definition 54 (Partial Latin cube)**.** Given a set A and some $n \in \mathbb{N}$ we say that $\theta \subset A^{n+1}$ is a *partial Latin n-cube* when for each $i \in [n]$ and each

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1} \in A^n$$

there exists at most one $a_i \in A$ so that

$$(a_1, \dots, a_{n+1}) \in \theta.$$

This is to say that a partial Latin cube is the graph of a partial function from A^n to A which satisfies the identities of an n-quasigroup wherever all the relevant operations are defined.

Definition 55 (Partial alternating Latin cube)**.** Given a set A and some $n \in \mathbb{N}$ we say that $\theta \subset A^{n+1}$ is a *partial alternating Latin n-cube* when θ is a partial Latin cube and for each $\alpha \in \mathrm{Alt}_n$ we have that if

$$(a_1, \ldots, a_n, b_1) \in \theta$$

and

$$(a_{\alpha(1)}, \ldots, a_{\alpha(n)}, b_2) \in \theta$$

then $b_1 = b_2$.

Definition 56 (Complete Latin cube)**.** We say that a partial Latin n-cube $\theta \subset A^{n+1}$ is a *complete Latin n-cube* when for each $i \in [n]$ and each

$$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in A^n$$

there exists at least one $a_i \in A$ so that

$$(a_1, \ldots, a_n) \in \theta.$$

That is, a complete Latin n-cube is the graph of an n-quasigroup operation. We also refer to complete Latin n-cubes simply as *Latin n-cubes*, the "complete" emphasizing the distinction from partial Latin cubes. Note that we may have defined a Latin cube without reference to the notion of a partial Latin cube as a relation $\theta \subset A^{n+1}$ such that for each

$$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in A^n$$

there exists a unique $a_i \in A$ so that

$$(a_1, \ldots, a_n) \in \theta.$$

Definition 57 (Finite partial Latin cube)**.** We say that a partial Latin cube $\theta \subset A^{n+1}$ is *finite* when A is a finite set.

Using this language, our Evans-like problem for alternating quasigroups may be stated as follows.

Problem 2. Given a finite partial alternating Latin cube $\theta \subset A^{n+1}$ does there always exist a finite complete alternating Latin cube $\psi \subset B^{n+1}$ such that $\theta \subset \psi$?

This problem is a bit weaker than the Evans conjecture for the case $n = 2$, as we don't posit a relationship between $|\theta|$ and $|B|$. In the $n = 2$ situation the veracity of the Evans conjecture implies that the answer to problem 2 is "yes" for $n = 2$. This in turn implies that the answer to problem 1 is "yes" for $n = 2$. Thus, we have a corollary to the Evans conjecture.

Corollary 2. *Every connected compact orientable surface is a component of the serenation of some finite binary quasigroup.*

This result appears to be significantly easier to establish than if we required the binary quasigroup in question be a group, as Herman and Pakianathan did. They were only able to show that an infinite family of such surfaces occurred as components of the serenation of some finite group[8, Corollary 3.5]. One may have asked the following question having only seen the construction of Herman and Pakianathan, but the preceding corollary adds weight to it.

Problem 3. Is every connected compact orientable surface a component of the serenation of some finite group?

8. Future directions

We conclude by discussing a number of possible directions for future research, beyond those questions already raised in the previous section.

8.1. **Classification of subdirect irreducibles.** As addressed in the introduction, we have by Birkhoff's Subdirect Representation Theorem[2, Theorem 3.24] that every (nontrivial) alternating n-quasigroup is isomorphic to a subdirect product of subdirectly irreducible alternating n-quasigroups. Moreover, in the case that the original quasigroup $\mathbf{A}$ in question is finite all of the subdirect factors will be finite as well since they are always quotients of $\mathbf{A}$.

This implies that, given an orientable smooth manifold $\mathbf{M}$ which appears as a component of $\mathbf{Ser}(\mathbf{A})$, we have that there exists a subdirect representation

$$h \colon \mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{B}_i$$

for some index set I and some subdirectly irreducible alternating quasigroups $\mathbf{B}_i$. Embeddings are always NC homomorphisms and it is fairly immediate that if h is injective then so is $\mathbf{Ser}(h)$. It follows that we have

$$\mathbf{M} \hookrightarrow \mathbf{Ser}(\mathbf{A}) \hookrightarrow \mathbf{Ser}\left(\prod_{i \in I} \mathbf{B}_i\right)$$

so the topology of $\mathbf{M}$ is determined by the algebraic structure of the $\mathbf{B}_i$.

We again lament that classifying even the infinite simple groups may be unreasonably difficult, which means that a full description of all subdirectly irreducible alternating n-quasigroups is likely unreachable. The finite simple groups were classified during the course of the twentieth century, however, so we may aspire to do the same for finite subdirectly irreducible alternating n-quasigroups. If the answer to problem 1 is "yes", then such a classification would be enough to gain some control over all compact orientable smooth manifolds via serenation.

Until now we have omitted an important aspect of understanding a manifold $\mathbf{M}$ by decomposing some $\mathbf{A}$ with $\mathbf{M} \hookrightarrow \mathbf{Ser}(\mathbf{A})$ into subdirect irreducibles. It is certainly not the case that $\mathbf{Ser}$ preserves even finite products, so a natural question is the following.

Problem 4. What can be said about the relationship between $\mathbf{Ser}\left(\prod_{i \in I} \mathbf{B}_i\right)$ and the individual $\mathbf{Ser}(\mathbf{B}_i)$?

One special case has a reasonable answer. In general we can't expect that the coordinate projection maps

$$p_k \colon \prod_{i \in I} \mathbf{B}_i \twoheadrightarrow \mathbf{B}_k$$

are NC homomorphisms, but we can when all factors other than the one under consideration are commutative.

Proposition 10. *Let $\mathbf{B} \coloneqq \prod_{i \in I} \mathbf{B}_i$ be a product of alternating n-quasigroups $\mathbf{B}_i$ and let $k \in I$. If $\mathbf{B}_i$ is commutative for all $i \neq k$ then we have that the projection homomorphism $p_k \colon \mathbf{B} \twoheadrightarrow \mathbf{B}_k$ is an NC homomorphism.*

Proof. Suppose that $b \coloneqq (b_1, \dots, b_n) \in \mathrm{NCT}(\mathbf{B})$. Since all the $\mathbf{B}_i$ are commutative it must be that

$$((b_1)_k, \dots, (b_n)_k) = (p_k(b_1), \dots, p_k(b_n)) = p_k(b)$$

is not a commuting tuple in $\mathbf{B}_k$. ☐

We then have that $\mathbf{Ser}(p_k)\colon \mathbf{Ser}(\mathbf{B}) \to \mathbf{Ser}(\mathbf{B}_k)$ is a continuous map for such alternating quasigroups $\mathbf{B}$ and $\mathbf{B}_k$. By the surjectivity of p_k we find that $\mathbf{Ser}(p_k)$ is also surjective.

8.2. **NC congruences.** We would be remiss if we did not mention that an analysis of the kernels of NC homomorphisms should take place, for these correspond to quotients of alternating quasigroups which actually pass through to surjective continuous maps between manifolds via serenation.

Definition 58 (NC congruence)**.** Given an alternating n-quasigroup $\mathbf{A}$ we say that $\theta \subset A^2$ is an NC congruence of $\mathbf{A}$ when $\theta = \ker(h)$ for some NC homomorphism h with domain $\mathbf{A}$.

NC congruences have the following property.

Proposition 11. *Given an NC congruence $\theta \in \mathrm{Con}(\mathbf{A})$ for an alternating n-quasigroup $\mathbf{A}$ we have for all $(a_1, a_2) \in \theta$ and all $b_1, \ldots, b_{n-2} \in A$ that the tuple*

$$(a_1, a_2, b_1, \ldots, b_{n-2})$$

is commuting in $\mathbf{A}$.

Proof. Suppose that θ is an NC congruence. This means that there exists an NC homomorphism

$$h\colon \mathbf{A} \to \mathbf{B}$$

for some alternating n-quasigroup $\mathbf{B}$ such that $\ker(h) = \theta$. Let $(a_1, a_2) \in \theta$. It follows that for any $b_1, \ldots, b_{n-2} \in A$ we have that

$$h(a_1, a_2, b_1, \ldots, b_{n-2}) = (h(a_1), h(a_1), h(b_1), \ldots, h(b_{n-2})),$$

which is a commuting tuple in $\mathbf{B}$. Thus, $(a_1, a_2, b_1, \ldots, b_{n-2})$ is a commuting tuple in $\mathbf{A}$. ☐

This property is also possessed by the center of any alternating quasigroup $\mathbf{A}$, which is the centralizer 1_A^* of the congruence 1_A identifying all members of A[2, p.197].

Proposition 12. *The center of an alternating n-quasigroup $\mathbf{A}$ has the property that if $(a_1, a_2) \in 1_A^*$ then for all $b_1, \ldots, b_{n-2} \in A$ we have that the tuple*

$$(a_1, a_2, b_1, \ldots, b_{n-2})$$

is commuting in $\mathbf{A}$.

Proof. Suppose that $(a_1, a_2) \in 1_A^*$. This means that for any n-ary term t and any $c, d \in A^{n-1}$ we have that $t(a_1, c) = t(a_1, d)$ if and only if $t(a_2, c) = t(a_2, d)$. Let $t \coloneqq f$,

$$c \coloneqq (a_2, b_1, b_2, \ldots, b_{n-2})$$

and

$$d \coloneqq (b_1, a_2, b_2, \ldots, b_{n-2}).$$

We have that

$$f(a_1, a_2, b_1, b_2, \ldots, b_{n-2}) = f(a_1, b_1, a_2, b_2, \ldots, b_{n-2})$$

if and only if

$$f(a_2, a_2, b_1, b_2, \ldots, b_{n-2}) = f(a_2, b_1, a_2, b_2, \ldots, b_{n-2}).$$

By alternativity this last equation always holds, so it is the case for all $b_1, \ldots, b_{n-2}$ that

$$(a_1, a_2, b_1, \ldots, b_{n-2})$$

commutes in **A**. □

One would hope that there is a strong relationship between the collection of all NC congruences of **A** and those congruences subordinate to the center of **A**, as is the situation for groups, but this remains to be seen.

8.3. **Field extensions.** The category $\mathbf{NCAQ}_n$ admits an interesting functor from the category **Field** whose objects are fields and whose morphisms are field homomorphisms. Recall that in example 2 we saw that given any field $\mathbb{F}$ (even those of even characteristic, which we now explicitly include) we may construct an alternating n-quasigroup $\mathbb{F}^{(n)}$.

Let

$$\mathbf{D}\colon \mathbf{Field} \to \mathbf{NCAQ}_n$$

be the functor given by

$$\mathbf{D}(\mathbb{F}) \coloneqq \mathbb{F}^{(n)}$$

when $\mathbb{F}$ is a field and

$$(\mathbf{D}(h))(u, v) \coloneqq ((h(u_1), \ldots, h(u_n)), h(v))$$

when h is a field homomorphism. Observe that if $\mathbb{K}/\mathbb{F}$ is a Galois extension then $\mathbf{Gal}(\mathbb{K}/\mathbb{F})$ acts on $\mathbb{K}$ through field automorphisms which fix $\mathbb{F}$ pointwise. It follows that for each n we have an action

$$\alpha\colon \mathbf{Gal}(\mathbb{K}/\mathbb{F}) \to \mathbf{Homeo}(\mathbf{Ser}_n(\mathbb{K}^{(n)}))$$

of $\mathbf{Gal}(\mathbb{K}/\mathbb{F})$ on $\mathbf{Ser}_n(\mathbb{K}^{(n)})$ which fixes the components of $\mathbf{Ser}_n(\mathbb{K}^{(n)})$ which belong to $\mathbf{Ser}_n(\mathbb{F}^{(n)})$ pointwise. Thus, a particular group **G** may only be the Galois group of an extension of $\mathbb{F}$ if there exists some extension field $\mathbb{K}$ of $\mathbb{F}$ for which **G** appears as a group of homeomorphisms of the manifold $\mathrm{Ser}_n(\mathbb{K}^{(n)})$ which fix $\mathrm{Ser}_n(\mathbb{F}^{(n)})$ pointwise.

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