HECKE OPERATORS AS OPERATIONS IN ELLIPTIC COHOMOLOGY †

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ABSTRACT *We construct stable operations*

$$\overline{\mathrm{T}}_n\colon E\ell\ell^*(\ ) \longrightarrow E\ell\ell(1/n)^*(\ ) \text{ for } n > 0$$

in the version of elliptic cohomology where the coefficient ring $E\ell\ell_*$ agrees with the ring of modular forms for $SL_2(\mathbb{Z})$ which are meromorphic at ∞, and $\overline{\mathrm{T}}_n$ restricts to the n th Hecke operator T_n on $E\ell\ell_*$.

In the past few years, the idea of *elliptic cohomology* has emerged from the combined efforts of a variety of mathematicians and physicists, and it is widely expected that it will play as important a rôle in global analysis and topology as K–theory and bordism have in the past. At present, there is no explicit geometric description of the cohomology theories that arise in this area, although there are several promising ideas which it is hoped will eventually lead to such a description. On the other hand, there are constructions of these theories based upon cobordism theories and for many purposes these seem to be adequate, at least for problems within the realm of *stable homotopy theory*. In particular, in this paper we will show that there are *stable operations* defined within a suitable version of elliptic cohomology and which restrict on the coefficient ring to the classical *Hecke operators* on modular forms. Together with the analogues of the *Adams operations* this gives a large collection of stable operations with which we can work. It is to be hoped that given a good model for elliptic cohomology these operations will have a more geometric definition as *unstable* operations and will be of great practical use. Because our operations are merely additive (and not multiplicative, although they are in a certain sense symmetrisations of multiplicative operations) they appear to be hard to compute explictly except in a few simple situations; however we anticipate their use even in their present form. Since this paper was written the author has been made aware of recent work of G. Nishida **[5]** on the double S^1 transfer which makes use of Hecke–like operations in a situation seemingly related to elliptic cohomology, although we do not understand the precise connection with our work.

I would like thank Peter Landweber for both introducing me to elliptic cohomology and for asking me (twice!) if the Hecke operators extended to operations. I also wish to thank Francis Clarke, Haynes Miller, Serge Ochanine, Doug Ravenel, Nige Ray, and Bob Stong for advice and encouragement in connection with this work.

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We begin by considering the *universal Weierstrass cubic* $\mathbf{Ell}/R_*$:

$$\mathbf{Ell}\colon Y^2 = 4X^3 - g_2X - g_3$$

where $R_* = \mathbb{Z}(1/6)[g_2, g_3]$ is the graded ring for which $|g_n| = 4n$. We can also assign gradings 4, 6 to X, Y respectively. Given any $\mathbb{Z}(1/6)$ algebra S_*, a homomorphism $\varphi\colon R_* \longrightarrow S_*$ induces a cubic

$$\varphi_*\mathbf{Ell}\ /S_*\colon Y^2 = 4X^3 - \varphi(g_2)X - \varphi(g_3).$$

If we ensure that the *discriminant*

$$\Delta_{\mathbf{Ell}}\ = g_2^3 - 27g_3^2$$

is mapped non-zero by φ then $\varphi_*\mathbf{Ell}$ is an *elliptic curve* over S_*. In that case we can define an abelian group structure on $\varphi_*\mathbf{Ell}$ when considered as a projective variety—see **[3]**, **[9]**. This has the unique point at infinity $\mathbf{O} = [0, 1, 0]$ as its zero. We can take the local parameter

$$T = -\frac{2X}{Y}$$

and then the group law on $\varphi_*\mathbf{Ell}$ yields a *formal group law* (commutative and 1 dimensional) $F^{\varphi_*\mathbf{Ell}}$ over S_*. This is explained in detail in for example **[9]**. Associated to this is an *invariant differential*

$$\omega_{\varphi_*\mathbf{Ell}} = \frac{dT}{\frac{\partial}{\partial Y}F^{\varphi_*\mathbf{Ell}}(T,0)} = \frac{dX}{Y}$$

which can also be written as

$$\omega_{\varphi_*\mathbf{Ell}} = d\log^{F^{\varphi_*\mathbf{Ell}}}(T).$$

Of course all of this applies in the case where φ is the identity map! We therefore consider this case from now on.

The formal group law $F^{\mathbf{Ell}}$ is classified by a unique homomorphism $\varphi\colon L_* \longrightarrow R_*$ where L_* is Lazard's universal ring (given its natural grading). But topologists know that L_* is isomorphic to MU_*, the coefficient ring of complex (co)bordism $MU^*(\ )$, and moreover the natural orientation for complex line bundles in this theory has associated to it a universal formal group law F^{MU}. This is all explained in for example **[1]**.

Hence we obtain a multiplicative *genus*

$$\varphi_{E\ell\ell}\colon MU_* \longrightarrow R_*$$

and upon localising R_* at the multiplicative set generated by $\Delta_{\mathbf{Ell}}$ we obtain a ring homomorphism

$$\varphi_{E\ell\ell}\colon MU_* \longrightarrow R_*[\Delta_{\mathbf{Ell}}^{-1}].$$

Using $\varphi_{E\ell\ell}$ we can define the following functor on the category $\mathbf{CW}^f$ of finite CW complexes:

$$E\ell\ell^*(\ ) = R_*[\Delta_{\mathbf{Ell}}^{-1}]\otimes_{MU_*}MU^*(\ ).$$

By the next theorem, this is a cohomology theory and is our version of *elliptic cohomology*.

THEOREM (1). *The functor $E\ell\ell^*(\ )$ is a multiplicative cohomology theory on $\mathbf{CW}^f$ and $\varphi_{E\ell\ell}$ extends to a multiplicative cohomology operation*

$$\overline{\varphi_{E\ell\ell}}\colon MU^*(\ ) \longrightarrow E\ell\ell^*(\ )$$

which complex orients $E\ell\ell^*(\ )$ in the sense of [**1**].

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Proof: See [**4**]. The key ingredient is the observation that over a field of characteristic $p \geq 2$ the formal group law associated to an elliptic curve has *height* 1 or 2.

■■

Now we proceed to identify the coefficient ring $E\ell\ell_* = R_*[\Delta_{\mathbf{Ell}}^{-1}]$ with a ring of modular forms.We begin by considering the ring of *finite tailed Laurent series*

$$\mathbb{Z}(1/6)((q)) = \{f : f(q) = \sum_{n=N}^{\infty} a_n q^n, a_n \in \mathbb{Z}(1/6), N \in \mathbb{Z}\}.$$

Now write

$$E_{2n} = E_{2n}(q) = 1 - \frac{4n}{B_{2n}} \sum_{k\geq 1} \sigma_{2n-1}(k) q^k$$

for the weight $2n$ *Eisenstein series*, where we set

$$\sigma_n(k) = \sum_{d|k} d^n$$

and B_n denotes the nth *Bernoulli number*. Then we have the *Tate curve*

$$\mathbf{Ell}_{Tate}(q)\colon Y^2 = 4X^3 - \frac{1}{12}E_4(q)X + \frac{1}{216}E_6(q)$$

which is defined over $\mathbb{Z}(1/6)((q))$ and induced by the homomorphism

$$\theta_{Tate}\colon R_*[\Delta_{\mathbf{Ell}}^{-1}] \longrightarrow \mathbb{Z}(1/6)((q))$$

with

$$\theta_{Tate}(g_2) = \frac{1}{12}E_4(q),$$

$$\theta_{Tate}(g_3) = -\frac{1}{216}E_6(q),$$

and

$$\theta_{Tate}(\Delta_{\mathbf{Ell}}) = \Delta = q\prod_{n\geq 1}(1-q^n)^{24}.$$

Now the rings

$$\theta_{Tate}(R_*[\Delta_{\mathbf{Ell}}^{-1}]) \subset \mathbb{Z}(1/6)((q))$$

and

$$\theta_{Tate}(R_*) \subset \mathbb{Z}(1/6)((q))$$

are rings of modular forms for $SL_2(\mathbb{Z})$ in the following sense.

We follow **[3]** and **[2]** in our account; in particular the relevant integrality statements are taken from the latter reference. Let

$$\mathfrak{H} = \{\tau \in \mathbb{C} : \quad \operatorname{im} \tau > 0\}$$

be the upper half plane. Then a holomorphic function $f : \mathfrak{H} \longrightarrow \mathbb{C}$ is a *modular form of weight* $k \in \mathbb{Z}$ if

$$\forall \tau \in \mathfrak{H}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

Let the Fourier series (= q-expansion) of f be

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n$$

where $q = e^{2\pi i \tau}$. If this is an element of $\mathbb{C}((q))$ then we say that f is a *meromorphic (at* $i\infty$*) modular form* and if it is in $\mathbb{C}[[q]]$ then we say that f is a *holomorphic (at* $i\infty$*) modular form*. We can then consider the sets of weight k meromorphic modular forms $M(\mathbb{C})_k$ and holomorphic modular forms $S(\mathbb{C})_k$, both of which are naturally considered as subsets of $\mathbb{C}((q))$. More generally, given any subring $K \subset \mathbb{C}$ containing $1/6$ we can consider the *weight* k *modular forms over* K:

$$M(K)_k = \{f \in M(\mathbb{C})_k : f = \sum_{n \geq N} a_n q^n, a_n \in K\}$$

and

$$S(K)_k = \{f \in S(\mathbb{C})_k : f = \sum_{n \geq 0} a_n q^n, a_n \in K\}.$$

From **[2]** we know that

$$M(K)_k = K \otimes M(\mathbb{Z})_k$$

and

$$S(K)_k = K \otimes S(\mathbb{Z})_k.$$

We can combine these groups to give the graded rings of meromorphic and holomorphic modular forms for $SL_2(\mathbb{Z})$ over K:

$$M(K)_* \text{ and } S(K)_*;$$

these are both K-algebras and the grading is the one for which weight k corresponds to grading $2k$.

THEOREM (2). *As graded K algebras,*

$$S(K)_* = K[E_4, E_6]$$

and

$$M(K)_* = K[E_4, E_6, \Delta^{-1}]$$

where $\Delta = (E_4^3 - E_6^2)/1728$.

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Proof: For $S(K)_*$ see **[2]**. For $M(K)_*$ note that if $f \in M(K)_k$, there is a non-negative integer d such that $\Delta^d f \in S(K)_{k+12d}$ since Δ is a cusp form (in fact $\Delta = q \prod_{n \geq 1}(1 - q^n)^{24}$). But then the result for $S(K)_*$ gives us this case too.

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COROLLARY (3). *As graded rings,*

$$E\ell\ell_* \cong M(\mathbb{Z}(1/6))_* \, .$$

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We will now discuss *Hecke operators* on $M(K)_*$ and $S(K)_*$. Again our principal references are **[2]** and the more elementary **[3]**, **[7]**; see also **[8]**.

The Tate curve $\mathbf{Ell}_{Tate}(q)$ can be interpreted as giving a *uniformisation* of the curve

$$Y^2 = 4X^3 - \frac{1}{12}E_4(q) + \frac{1}{216}E_6(q).$$

Given any $\tau \in \mathfrak{H}$, the lattice $L_\tau = <1, \tau> \subset \mathbb{C}$ yields a complex torus $\mathbb{C}/L_\tau$. Then the map

$$\underline{\wp}_{Tate}(\ \ ;q)\colon \mathbb{C}/L_\tau \longrightarrow \mathbf{Ell}_{Tate}(q) \subset \mathbb{C}P^2$$

for which

$$\underline{\wp}_{Tate}([z];q) \ = \left[\wp_{Tate}(2\pi\mathrm{i}z;q); \wp'_{Tate}(2\pi\mathrm{i}z;q), 1\right]$$

if $z \notin L_\tau$ and $\mathbf{O}$ otherwise, where

$$\wp_{Tate}(z;q) \ = \frac{e^z}{(e^z-1)^2} + \sum_{n\geq 1}\Big[\frac{q^n e^z}{(1-q^n e^z)^2} + \frac{q^n e^{-z}}{(1-q^n e^{-z})^2}\Big] + \frac{1}{12}E_2(q)$$

is Tate's renormalised version of the Weierstrass $\wp$-function as described in **[2]** for example.

Now let $n \geq 1$. Then for any lattice L' containing L_τ with index $[L'; L_\tau] = n$ we have an analytic surjection of groups

$$\mathbb{C}/L_\tau \longrightarrow \mathbb{C}/L'$$

with kernel L'/L_τ of order n. If we have $L' = <\omega'_1, \omega'_2>$ then we can assume that $\tau' = \omega'_2/\omega'_1 \in \mathfrak{H}$. Then we obtain a Tate curve evaluated at $q' = e^{2\pi\mathrm{i}\tau'}$,

$$\mathbf{Ell}_{Tate}(q')\colon Y^2 = 4X^3 - \frac{1}{12}E_4(q')X + \frac{1}{216}E_6(q')$$

defined over $\mathbb{Z}(1/6)((q')) \subset \mathbb{Z}(1/6,\zeta_n)((q^{1/n}))$, where $q^{1/n} = e^{2\pi\mathrm{i}\tau/n}$ and $\zeta_n = e^{2\pi\mathrm{i}/n}$ is a primitive nth root of 1. So for each L' we have a homomorphism $\psi_{L'}\colon \mathbb{Z}(1/6)((q)) \longrightarrow \mathbb{Z}(1/6,\zeta_n)((q^{1/n}))$ which in turn restricts to $\psi_{L'}\colon E\ell\ell_* \longrightarrow \mathbb{Z}(1/6,\zeta_n)((q^{1/n}))$. Now for any subring $K \subset \mathbb{C}$ we define the nth *Hecke operator* (for $n \geq 1$) by

$$\mathrm{T}_n\colon M(K)_* \longrightarrow K(1/6,\zeta_n)((q^{1/n}))$$

where

$$(\mathrm{T}_n f)(q) = \frac{1}{n} \sum_{\substack{L_\tau \subset L' \\ [L', L_\tau]=n}} {\omega'_1}^{-k} f(q')$$

and $f \in M(K)_k$, using the above notations.

THEOREM (4). *The function* T_n *gives rise to* K*-linear maps for each* $k \in \mathbb{Z}$*,*

$$\mathrm{T}_n\colon M(K)_k \longrightarrow M\big(K(1/n)\big)_k$$

and

$$\mathrm{T}_n\colon S(K)_k \longrightarrow S\big(K(1/n)\big)_k.$$

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Proof: See any of the cited references above, especially [**2**] which contains the integrality statements.

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Now the T_n satisfy some famous identities.

PROPOSITION (5). *For* K *a subring of* $\mathbb{C}$ *containing* $1/6$*, and for* $m, n \geq 1$ *we have*

(a) $\mathrm{T}_m\mathrm{T}_n = \mathrm{T}_n\mathrm{T}_m$.
(b) If m, n *are coprime then* $\mathrm{T}_m\mathrm{T}_n = \mathrm{T}_{mn}$.
(c) If p *is a prime and* $r \geq 1$ *then as operators on weight* k *modular forms,*

$$\mathrm{T}_{p^{r+1}} = \mathrm{T}_{p^r}\mathrm{T}_p - p^{k-1}\mathrm{T}_{p^{r-1}}.$$

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We can evaluate $\mathrm{T}_p f$ for p a prime in terms of the q-expansion of f.

PROPOSITION (6). *Let* p *be a prime and* $f \in M(K)_k$*. Then if* $f(q) = \sum a_n q^n$ *we have*

$$(\mathrm{T}_p f)(q) = \sum a_{np}q^n + p^{k-1}\sum a_n q^{np}.$$

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We now show how the T_n can be extended to stable operations in Elliptic Cohomology. If J is a set of (non-zero) primes let $E\ell\ell^*_J(\ )$ be the cohomology theory $E\ell\ell^*(\ )\otimes\mathbb{Z}(J^{-1})$, the localisation of $E\ell\ell^*(\ )$ at the multiplicative set generated by J. We will need the elliptic cohomology Adams operations

$$\psi^p_{E\ell\ell}\colon E\ell\ell^*(\ ) \longrightarrow E\ell\ell^*_J(\ ) \quad \text{for } p \in J.$$

Then $\psi^p_{E\ell\ell}$ is a multiplicative natural transformation constructed with the aid of the p-series $[p]_{E\ell\ell}(X)$ together with the definition of $E\ell\ell^*(\ )$ in terms of $MU^*(\ )$; if $\alpha \in E\ell\ell_{2k}$, we have

$$\psi^p_{E\ell\ell}(\alpha) = p^k\alpha$$

and this characterises $\psi^p_{E\ell\ell}$ as a multiplicative natural transformation.

THEOREM (7). *Let* J *be a set of primes. Then there is a family of stable cohomology operations of degree 0*

$$\overline{\mathrm{T}}_n\colon E\ell\ell^*(\ ) \longrightarrow E\ell\ell^*_J(\ ), 1/n \in \mathbb{Z}(J^{-1})$$

with the following properties:

(a) For all m, n *such that* $\overline{\mathrm{T}}_m, \overline{\mathrm{T}}_n$ *are defined,* $\overline{\mathrm{T}}_m\overline{\mathrm{T}}_n = \overline{\mathrm{T}}_n\overline{\mathrm{T}}_m$.
(b) For all coprime m, n *for which* $\overline{\mathrm{T}}_m, \overline{\mathrm{T}}_n$ *are defined,* $\overline{\mathrm{T}}_m\overline{\mathrm{T}}_n = \overline{\mathrm{T}}_{mn}$.
(c) For a prime p *with* $\overline{\mathrm{T}}_p$ *defined and* $r \geq 1$*,*

$$\overline{\mathrm{T}}_{p^{r+1}} = \overline{\mathrm{T}}_{p^r}\overline{\mathrm{T}}_p - \frac{1}{p}\psi^p_{E\ell\ell}\circ\overline{\mathrm{T}}_{p^{r-1}}.$$

(d) On the coefficient ring $E\ell\ell_* \cong M(\mathbb{Z}(1/6))_*$, each $\overline{\mathrm{T}}_n$ agrees with T_n.

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Proof: It suffices to construct $\overline{\mathrm{T}}_p$ for p an admissible prime and then use (b), (c) to *define* the remaining $\overline{\mathrm{T}}_n$; property (a) follows by a careful inspection of the construction and the proof of **PROPOSITION (5)** part (a).

Let $\tau \in \mathfrak{H}$ and consider the lattice $L = L_\tau =< 1, \tau >$ and any lattice L' containing L of index p. Then there are exactly $p+1$ possibilities for L'. Either $L' =< 1/p, \tau >= L_0$ or $L' =< 1, (r+\tau)/p >= L_r$ for $1 \leq r \leq p$. We will be forced to treat these two cases distinctly. The next result is crucial.

LEMMA (8). *There are strict isomorphisms of group laws*

$$h_0 \colon F^{\mathbf{Ell}(q^p)} \xrightarrow{\cong} F^{\mathbf{Ell}(q)}$$

and

$$h_j \colon F^{\mathbf{Ell}({\zeta_p}^j q^{1/p})} \xrightarrow{\cong} F^{\mathbf{Ell}(q)} \qquad \textit{for } 1 \leq j \leq p,$$

both defined over $\mathbb{Z}(1/6, \zeta_p)((q^{1/p}))$.

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Proof: The key to the proof lies in the observation that

$$\wp_{_{Tate}}(z;q) \in \big(\mathbb{Z}(1/6)((q))\big)((e^z - 1)),$$

which implies that the local parameter

$$t_q \;=\; -\frac{2\wp_{_{Tate}}(z;q)}{\wp'_{_{Tate}}(z;q)} \in \big(\mathbb{Z}(1/6)((q))\big)[[e^z - 1]]$$

provides a strict isomorphism

$$\hat{G}_m \xrightarrow{\cong} F^{\mathbf{Ell}(q)}$$

defined over $\mathbb{Z}(1/6)((q))$, where $\hat{G}_m$ is the multiplicative group law

$$\hat{G}_m(X,Y) \;=\; X + Y + XY.$$

The rest of the proof can now be safely left to the reader! ∎∎

Now recall from **[6]** that the ring MU_*MU (defined as the complex bordism of the spectrum MU) is in fact universal for strict isomorphisms of formal group laws. This ring has the structure of a *Hopf algebroid* and as part of this structure there are the left and right units

$$\eta_L, \eta_R \colon MU_* \longrightarrow MU_*MU$$

which are ring homomorphisms. Because of this, the $p+1$ isomorphisms h_j give rise to ring homomorphisms

$$H_j \colon MU_*MU \longrightarrow \mathbb{Z}(1/6p, \zeta_p)((q^{1/p}))$$

for which $H_j\eta_L$ classifies $F^{\mathbf{Ell}(q^p)}$ if $j = 0$ and $F^{\mathbf{Ell}({\zeta_p}^j q^{1/p})}$ if $1 \leq j \leq p$; also $H_j\eta_R$ classifies $F^{\mathbf{Ell}(q)}$ in all cases.

The power series

$$P(X) = \frac{1}{p}[p]_{F^{\mathbf{Ell}(q^p)}}(X) \in \big(\mathbb{Z}(1/6p)((q))\big)[[X]]$$

gives rise to a new formal group law

$$F'(X,Y) = P\big(F^{\mathbf{Ell}(q^p)}(P^{-1}(X), P^{-1}(Y))\big)$$

over $\mathbb{Z}(1/6p)((q^p))$ which is classified by a homomorphism H_0' factoring as

$$H_0' \colon MU_* \xrightarrow{\psi^p_{MU}} MU(1/p)_* \longrightarrow \mathbb{Z}(1/6p, \zeta_p)((q^p))$$

where

$$\psi^p_{MU}(x) = p^d x \quad \text{if } x \in MU_{2d}$$

and the second map classifies $F^{\mathbf{Ell}(q^p)}$.

Using these facts we can form the sum

$$\overline{H} = \frac{1}{p}\Big[H_0' + \sum_{1\leq j\leq p} H_j\Big] \colon MU_*MU \longrightarrow \mathbb{Z}(1/6p, \zeta_p)((q^{1/p}))$$

which is a *right* MU_* module map. But now recall that

$$MU_*MU = \eta_R(MU_*)[B_n : n \geq 1]$$

with

$$B(X) = \sum_{0\leq n} B_n X^{n+1}$$

being the *universal isomorphism* from the left to the right universal group law over MU_*MU, as described, for example, in **[6]**. Then we see that

$$H_j\big(B(X)\big) = h_j(X)$$

and

$$H_0'\big(B(X)\big) = \frac{1}{p}[p]_{F^{\mathbf{Ell}(q^p)}}\big(h_0(X)\big).$$

It is now clear that for any element $u \in MU_*MU$, $\overline{H}(u) \in \mathbb{Z}(1/6p)((q))$ since it is invariant under $\zeta_p \mapsto {\zeta_p}^k$ for each $k \not\equiv 0 \pmod{p}$.

As there is no torsion in our rings, we can work rationally and we then find that $MU_*MU\otimes\mathbb{Q}$ is the $\mathbb{Q}$ algebra generated by the subrings $\eta_L(MU_*)$ and $\eta_R(MU_*)$. Hence for any element $u \in MU_*MU$ we can write

$$u = \sum_n a_n b_n \text{ where } a_n \in \eta_L(MU_*) \text{ and } b_n \in \eta_R(MU_*).$$

For $v \in MU_* \otimes \mathbb{Q}$, let $v(q) \in \mathbb{Q}((q))$ denote the image of v under the map $MU_* \otimes \mathbb{Q} \longrightarrow \mathbb{Q}((q))$ classifying $F^{\mathbf{Ell}(q)}$. Then

$$H_j(a_n) = a_n({\zeta_p}^j q^{1/p}) \quad \text{if } 1 \leq j \leq p$$

and

$$H'_0(a_n) \;=\; p^d a_n(q^p) \quad \text{if } j=0.$$

Hence we have

$$\overline{H}(a_n) \;=\; \frac{1}{p}\sum_{1\le j\le p} a_n({\zeta_p}^j q^{1/p}) + p^{d-1}a_n(q^p).$$

Similarly we obtain

$$\overline{H}(b_n) \;=\; \left(1+\frac{1}{p}\right) b_n(q).$$

Now for each $1\le j\le p$, H_j extends to a right MU_* and left $E\ell\ell_*$ module map

$$H_j\colon E\ell\ell_*\otimes_{MU_*}MU_*MU \longrightarrow \mathbb{Z}(1/6p,\zeta_p)((q^{1/p}))$$

and similarly for H'_0. Hence we also have an extension

$$\overline{H}_j\colon E\ell\ell_*\otimes_{MU_*}MU_*MU \longrightarrow \mathbb{Z}(1/6p,\zeta_p)((q^{1/p}))$$

But an element $\alpha\otimes v$ can be expanded as

$$\alpha\otimes v \;=\; \sum_n \alpha_n\otimes v_n$$

where $\alpha_n\in E\ell\ell_*\otimes\mathbb{Q}$ and $v_n\in\eta_R(MU_*\otimes\mathbb{Q})$. It is easily seen that

$$\overline{H}(\alpha_n\otimes v_n) \;=\; \Big[\; p^{d-1}\alpha_n(q^p) \;+\; \frac{1}{p}\sum_{1\le j\le p}\alpha_n({\zeta_p}^j q^{1/p})\;\Big] v_n(q)$$

and hence by **PROPOSITION (6)** we see that

$$\overline{H}(\alpha_n\otimes v_n) \;=\; \mathrm{T}_p(\alpha_n)\otimes v_n.$$

Therefore, we certainly have

$$\overline{H}(E\ell\ell_*\otimes_{MU_*}MU_*MU) \;\subset\; E\ell\ell_*\otimes\mathbb{Q} \;\subset\; \mathbb{Q}((q)).$$

But we already know that

$$\overline{H}(E\ell\ell_*\otimes_{MU_*}MU_*MU) \;\subset\; \mathbb{Z}(1/6p)((q))$$

and thus we have shown that

$$\overline{H}(E\ell\ell_*\otimes_{MU_*}MU_*MU) \;\subset\; E\ell\ell_*.$$

Since $\overline{H}$ is a right MU_* module map we can form the composite

$$\begin{aligned} E\ell\ell_*\otimes_{MU_*}MU^*(\;) &\xrightarrow{1\otimes\overline{mu}} E\ell\ell_*\otimes_{MU_*}(MU\wedge MU)^*(\;)\\ &\xrightarrow{\cong} E\ell\ell_*\otimes_{MU_*}MU_*MU\otimes_{MU_*}MU^*(\;)\\ &\xrightarrow{\overline{H}\otimes 1} E\ell\ell_*\otimes_{MU_*}MU^*(\;)\end{aligned}$$

where

$$\overline{mu}\colon MU^*(\ ) \cong (MU \wedge S)^*(\ ) \longrightarrow (MU \wedge MU)^*(\ )$$

is the *Boardman map* induced by the unit $mu\colon S \longrightarrow MU$ and the isomorphism is described in **[1]**. This composite is our required operation $\overline{\mathrm{T}}_p$.

■■

We end with some remarks.

(i) Genera taking values in the ring of "level N" modular forms have been investigated by F. Hirzebruch and it seems likely that there are versions of elliptic cohomology related to these. As Hecke operators are also defined for such modular forms (see **[2]**), there are presumably analogous constructions of $\overline{\mathrm{T}}_n$ provided N and n are coprime. The case of level 2 corresponds to the original version of Landweber, Ravenel and Stong **[4]**.

(ii) If we wish to work locally at p rather than with p inverted, there is a theory of p-adic modular forms (see **[2]** and **[8]**) in which the Hecke operator T_p is replaced by Atkin's operator U_p, given on q-expansions by

$$\mathrm{U}_p\Big(\sum a_n q^n\Big) = \sum a_{np} q^n.$$

There is a p-adic version of elliptic cohomology but as well as having $\Delta_{\mathbf{Ell}}$ inverted this also has E_{p-1} as a unit. In fact this theory is K-theoretic in the sense that it is a product of copies of the p-completion of Adams' summand of K-theory. We describe this in a second paper, *"Elliptic cohomology, p-adic modular forms and Atkin's operator U_p"* to appear in the proceedings of the International Conference on Homotopy Theory at Evanston, 1988 .

(iii) The version of elliptic cohomology that we use is more fundamental than the one in **[4]** from the point of view of stable homotopy theory, since Landweber's version is a sum of copies of ours. We will describe in more detail the relationship and also versions at the primes 2 and 3 in a future paper.

REFERENCES

[1] J.F.Adams, *"Stable Homotopy and Generalised Homology"*, University of Chicago Press (1974).

[2] N.Katz, "p-adic properties of modular schemes and modular forms ", in *"Modular Functions of One Variable III"*, *Springer Lecture Notes in Mathematics* **350** (1973) 69-190.

[3] N.Koblitz, *"Elliptic Curves and Modular Forms"*, Springer-Verlag (1984).

[4] P.S.Landweber, "Elliptic cohomology and modular forms", in *"Elliptic Curves and Modular Functions in Algebraic Topology"*, *Springer Lecture Notes in Mathematics* **1326** (1988).

[5] G. Nishida, "Modular forms and the double transfer for BT^2", preprint (1989).

[6] D.C.Ravenel, *"Complex Cobordism and the Stable Homotopy Groups of spheres"*, Academic Press (1986).

[7] J-P.Serre, *"Cours d'Arithmétique"*, Presses Universitaires de France (1977). English edition: Springer-Verlag (1973) .

[8] J-P.Serre, "Formes modulaires et fonctions zeta p-adiques", in *"Modular Functions of One Variable III"*, *Springer Lecture Notes in Mathematics* **350** (1973) 191-268.

[9] J.Silverman, *"The Arithmetic of Elliptic Curves"*, Springer-Verlag (1986).