Homotopy operations and homotopy groups.

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The basic sequences.

The homotopy groups of X (+ base point) with coefficients A are the track groups $[S^nA, X] = \pi_n^A(X)$. A cofibration $Q = A/B$ yields the exact track group sequence*

$$\ldots \to [S^nQ, X] \to [S^nA, X] \to [S^nB, X] \to [S^{n-1}Q, X] \to \ldots$$

which, with a suitable filtration of A, e.g. by skeletons, provides a spectral sequence for determining $[S^nA, X]$ from the cohomology of A with coefficients in the ordinary homotopy groups of X. This exact sequence was generalised to relative and stable forms by Spanier and J. H. C. Whitehead,* the inventors of the stable groups.

Any of the ordinary homotopy groups, including those of pairs, triads, n-ads, can be converted to coefficients A by smashing the argument space with A; exact sequences become exact sequences. A caveat has to be entered for a conditionally exact sequence like the important SHP sequence of G. W. Whitehead

$$\longrightarrow \pi_n^A(X) \xrightarrow{S} \pi_{n+1}^A(SX) \xrightarrow{H} \pi_{n+1}^A(SX*X) \xrightarrow{P} \pi_{n-1}^A(X) \longrightarrow$$

where the range of exactness (the meta-stable range) has to be reduced by the dimension of A. This leads at once to the SHP lattice of a pair (X, Y), of great use in finding the structure of the homotopy groups of certain complexes in metastable range.

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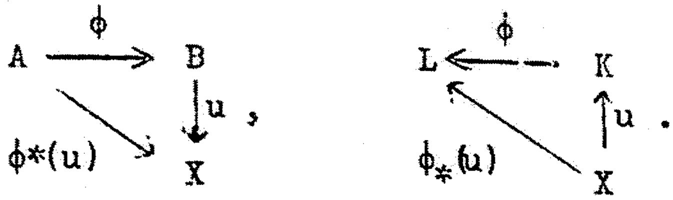
*Barratt [1952], Spanier and Whitehead [1955]. The spectral sequence was also independently discovered by Federer [1956], Berstein [1957], and in other language by S. T. Hu [1949]. The exact sequence has also been called the Puppe sequence.

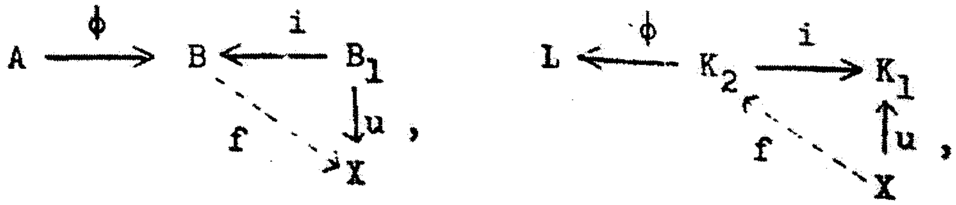
There are two generalizations: the first is James' remarkable theorem that when X is a sphere, the SHP sequence exists and is exact on the 2-component, for all n. This is an adequate instrument for computing the 2-component of the homotopy groups of spheres, with the aid of enough homotopy operations. The p-components for odd primes can be found from the other generalisation, the spectral sequence of a suspension, a particular case of the spectral sequence of an inclusion*, and too complex to explain briefly here. This is a generalisation in the sense that it reduces to the SHP sequence in meta-stable range. Strictly this is the spectral sequence for a desuspension, but spectral sequences are frequently used to compute some of the E_1 terms from the others and the E_∞.

Operations.

In order to compute with these sequences, it is important that the operations used be natural. The natural operations in a category are derived from the primary operation of composition: the dual forms are

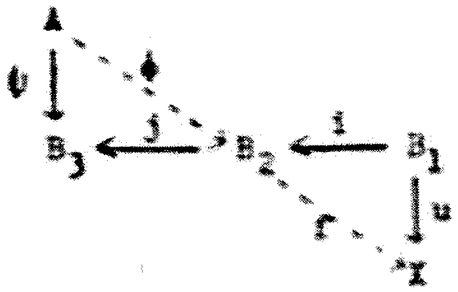


The natural operations (with values in sets) can be represented generically by



*Barratt, Aarhus Coll: [1962].

the value of the operation ϕ on u being the set of maps $f \circ \phi$ or $\phi \circ f$ for all f through which u factors. In practice it may be convenient to increase the ambiguities for ease of calculation and handling: the operation ϕ is replaced by the union of a set of operations sharing some common property, e.g. a



common value under a dual operation. The description of an operation then becomes a recipe for constructing objects and maps of a desired nature. The important Toda brackets are of this kind.

In applications the operations are classified according to the number of obstructions met in attempting to lift the map u into a map f: these are usually the results of other operations on u. The advantage of this is that when suitable group structures are about, the ambiguities of the operations may be expressible in terms of other operations.

The importance of the naturality of the operations is that their values under the homomorphisms used in computing can be determined. Thus, if $H(u) \neq 0$, $H\phi^*(u) = \phi^* H(u)$, and usually $H\phi^*(u)$ can be computed in terms of operations derived from ϕ when $H(u) = 0$.

Relations.

Homotopy operations are built up from relations. Thus, if the composition

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} B_1$$

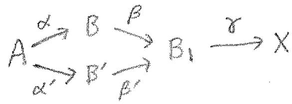
is zero, there is a secondary operation

$$SA \xrightarrow{\phi} B_1 \underset{\beta}{\cup} CB \xleftarrow{i} B_1$$

where ϕ is allowed to be any element such that $\mu \circ \phi = S\alpha$, where μ pinches B_1 to a point. This is an operation defined on the kernel of the operation β^*, and defines of course the Toda bracket $\{\gamma, \beta, \alpha\}$ for any $\gamma \varepsilon \operatorname{Ker} \beta^*$.

The relations can be more complex: thus from an equation

$$\alpha \circ \beta + \alpha' \circ \beta' = 0: A \to B_1$$



where α, α' map a suspension A to B, B' respectively, may be derived a secondary operation

$$SA \xrightarrow{\phi} B_1 \cup C(B \vee B') \xleftarrow{i} B_1$$

where ϕ is allowed to be any element such that $\mu \circ \phi = S\alpha + S\alpha'$. This is defined on the intersection of the kernels of β^*, β'^*, and defines a generalised Toda bracket $\left\{\gamma \begin{smallmatrix}\beta \\ \beta'\end{smallmatrix}, \begin{smallmatrix}\alpha \\ \alpha'\end{smallmatrix}\right\}$. Little ingenuity is required to proliferate methods of constructing secondary and higher operations of increasing complexity, and there seems little point in classifying the methods.

A variety of techniques are available for determining relations: apart from general theorems concerning commutation of composition, and Toda bracket identities, ad hoc methods can be used when the elements involved are given by operations. Some of the more subtle relations can only be discovered by subsequent calculations; some relations in the 8-stem, for example, can only be found by computing in the next 3 stems. But usually it is necessary to find most of the relations in a given stem to provide the operations needed to compute higher stems.

Constructions.

The crucial question in computing the 2-components of homotopy groups of spheres with the SHP sequence is that of determining the homomorphism P and constructing elements with assigned Hopf invariants. The method is inductive on the dimensions of the spheres, for since the image of P is the kernel of E, there is usually a relation on the next higher sphere which cannot be desuspended. This can be used to construct maps with a known Hopf invariant, and a suitable stable operation may be found to yield a construction of elements with other Hopf invariants.

Example 1. The Hopf class η on S^2 has the property that $2\eta = P(\iota)$, the Whitehead product. Therefore $2\eta = 0$ on S^3, and a map

$$S^3RP^2 = S^4 \cup_{2\iota} e^5 \xrightarrow{\eta^*} S^3$$

can be constructed which is not a suspension element. The Hopf invariant of $\eta^\#$ is therefore the nonzero element of $[S^3RP^2, S^5] = Z_2$ ($\eta^\#$ has Hopf invariant 1). Thus $H\{\eta, 2\iota, \theta\} = 3\theta$. It was by contemplating this example that Toda invented Toda brackets.

Example 2. The Hopf class ν on S^4 has the property that $\nu\eta$ suspends to the Whitehead product P( ) on S^5. Therefore there is a map

$$S^7CP^2 = S^9 \cup_\eta e^{11} \xrightarrow{\nu^\#} S^6$$

whose Hopf invariant is 1 in some sense. Thus $H\{\nu, \eta, \theta\} = S\theta$.

The principle involved here can be formalised. Toda first proved in general that

$$P(\lambda) = \alpha \circ \beta \Rightarrow H\{\alpha, \beta, \gamma\} = \lambda \circ S_\gamma.$$

This can be generalised, but the present formulation is messy.

Periodicities.

One of the objects of calculating a lot of homotopy groups of spheres is to provide material for conjecture about regularities of behaviour. Two types of regularity can be identified, or looked for. One is the presence of periodic families, on which stringent requirements are imposed that the elements should be defined on the same spheres, in similar ways, with similar invariants. Another is the presence of recursive elements, defined on periodic spheres with the same invariants, and possessing the same properties. As an example of the latter we have

Proposition. $\delta'[k]$, $\delta[k]$ in the 4k-stem generating Z_2 summands on S^{4k} such that

(a) $\delta'[k]$ has nonzero Hopf invariant on S^{4k-1} and suspends nontrivially to the image of P.

(b) $\delta[k]$ has nonzero Hopf invariant on S^{4k} and suspends nontrivially to the image of P.

Thus the last unstable groups in the 4k-stem are $G \oplus Z_2$, $G \oplus Z_2 \oplus Z_2$, $G \oplus Z_2$, where G is the stable group. More recursive elements can be found whenever periodicity of P can be established.

The periodic families known are largely but not exclusively in the image of J. The simplest case is the stable family α_r in the p-component on S^3, defined from α_1 recursively by $\alpha_r = \{\alpha_1, p, \alpha_{r-1}\}$: here α_1 is the first element of order p, in $\pi_{2p}(S^3)$. They have the property $d_2\alpha_r = \alpha_{r-1}$ in the spectral sequence, and in stable range are related to the generator of the p-component of the image of J, by

$$(k+1)\{\alpha_1, \alpha_k, p\} = \alpha_{k+1}.$$

In the 2 component the following stable families are linked: let η, ν, σ be the Hopf classes in the 1, 3 and 7 stems respectively. These are exceptional elements in the families, and occur earlier than they should.

<u>Stem 8n + 1</u>. $\mu[0] = \eta$ on S^2; $\mu[n] = \{\eta, 2\iota, a[n-1]\}$ on S^3, with $H\mu[n] = a[n-1]$, $2\mu[n] = 0$. These are not in the image of J.

<u>Stem 8n + 3</u>. $\bar{\mu}[n] = \{\eta, 2, \mu[n]\}$ on S^3, with $H\bar{\mu}[n] = \mu[n]$.
$2\bar{\mu}[n] = \mu[n]\eta\eta \neq 0$.
$\zeta[0] =$ on S^4;
$\zeta[n] = \{\nu, 8\iota, c[n]\}$ on S^5 with $H\zeta[n] = a[n-1]$,
$2\zeta[n] = \bar{\mu}[n]$. This generates the $Z_8 \subset \mathrm{Im}\, J$.

<u>Stem 8n + 7</u>. $a[0] = \{\nu, 8\iota, \nu\}$, $a[n] = \{a[1], 2, a[n-1]\}$, all on S^5
with $Ha[n] = \mu[n]\eta\eta$, $2a[n] = 0$
$b[0] = \begin{Bmatrix}\nu & \omega & 2\iota \\ & \eta & \eta\eta\end{Bmatrix}$, $b[n] = \{b[1], 4, b[n-1]\}$, all on S^6
with $Hb[n] = \mu[n]\eta$, $2b[n] = a[n]$.
$c[0] = \langle\iota, \iota\rangle$, $c[n] = \{c[1], 8, c[n-1]\}$, all on S^7
with $Hc[n] = \mu[n]$, $2c[n] = b[n]$

$d[0] = \sigma$ on S^8; $d[n] = \{d[1], 16, d[n-1]\}$ on S^9, with $H d[n] = a[n-1]$, $2d[n] = c[n]$. This generates the $Z_{16} \subset \mathrm{Im}\, J$.

Other members can be defined depending on the composition of n.