NON-TRIVIAL EXTENSIONS IN EQUIVARIANT COHOMOLOGY WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper, we prove some computational results about equivariant cohomology over the cyclic group C_{p^n} of prime power order. We show that there is an inductive formula when the dimension of the C_p -fixed points of the grading is large. Among other calculations, we also show the existence of non-trivial extensions when $n \geq 3$.

1. INTRODUCTION

The equivariant stable homotopy category has a rich structure coming from the desuspension along representation spheres. This equips equivariant cohomology groups with a grading over the virtual representations RO(G). The resulting structure is usually difficult to compute even in the case of ordinary cohomology. Their computations give interesting results and also have surprising consequences.

The coefficients of ordinary cohomology are Mackey functors, and the important ones are the Burnside ring Mackey functor \underline{A} , and the constant Mackey functor $\underline{\mathbb{Z}}$. For the group C_p , the $RO(C_p)$ -graded commutative rings $\underline{\pi}_{-\bigstar}^{C_p} H\underline{A} \cong \underline{H}_{C_p}^{\bigstar}(S^0;\underline{A})$ was computed by Lewis [10], and analogous results for $\underline{\mathbb{Z}}$, $\underline{\mathbb{Z}}/p$ were computed by Stong and Lewis. There are computations for a few other groups (see [3], [4], [15] [9]), however, for most ones, even among Abelian groups, very little is known.

In this paper, we study the Mackey functors $\underline{\pi}_{-\alpha}^{C_{p^n}} H\underline{\mathbb{Z}} \cong \underline{H}_{C_{p^n}}^{\alpha}(S^0;\underline{\mathbb{Z}})$ for $\alpha \in RO(C_{p^n})$, which form the additive structure for equivariant cohomology over the group C_{p^n} . We prove the following result.

Theorem A. If $|\alpha^{C_p}| \leq -2n+2$ or $|\alpha^{C_p}| \geq 2n$, the Mackey functor $\underline{H}^{\alpha}_{C_{p^n}}(S^0;\underline{\mathbb{Z}})$ can be computed directly from the Mackey functor $\underline{H}^{\alpha^{C_p}}_{C_{p^{n-1}}}(S^0;\underline{\mathbb{Z}})$. (see Table 4.1)

We also point out various computations of these Mackey functors not covered by the above theorem. The formulas that appear here are mostly written as a direct sum of those of the form $\underline{\mathbb{Z}}_T$ and $\underline{B}_{T,S}$, a notation inspired from [8]. A consequence of these results are the complete calculation of the additive structure for the group C_{p^2} (see Table 5.1). A new feature of these groups starting from $n \geq 2$ is the existence of a_{λ} -periodic classes. For the group C_p , the classes were either part of a polynomial algebra or were a_{λ} -torsion.

In these computations, we also point out a non-trivial extension of Mackey functors. These extensions first occur in the case n = 3, and hence also for higher n. In cohomology over the Burnside ring \underline{A} , one has extensions $\underline{A}[d]$ for the group C_p of the form $0 \to \langle \mathbb{Z} \rangle \to$ $\underline{A}[d] \to \mathbb{Z} \to 0$, that occur in the additive structure. On the other hand, over \mathbb{Z} coefficients, the Mackey functors occuring in the additive structure over C_p are a direct sum of those of the form \mathbb{Z}_T and $\underline{B}_{T,S}$. For the group C_{p^n} , many special cases have been shown to be of this type (see for example [8, Theorem 5.7]). However, we point out through examples that

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this does not happen in general, and in these cases, the a_{λ_i} -multiplication gives non-trivial extensions.

1.1. Organization. In Section 2, we recall some preliminaries on equivariant cohomology, and their computational methods. In Section 3, we discuss the category of Z-modules, and their extensions, constructing important examples used in later sections. In Section 4, we compute the equivariant cohomology at large C_p -fixed points. In Section 5, we compute the equivariant cohomology over C_{p^2} and use them to point out the non-trivial extensions.

Notation 1.2. Throughout this paper, G denotes the cyclic group C_{p^n} of order p^n , where m is odd, and g denotes a fixed generator of G. For an orthogonal G-representation V, S(V) denotes the unit sphere, D(V) the unit disk, and S^V the one-point compactification $\cong D(V)/S(V).$

2. Equivariant cohomology

Ordinary cohomology theories are defined for Abelian groups, and these are represented by spectra with homotopy concentrated in degree 0. In the equivariant world, the analogous role is played by *Mackey functors*. In this section we briefly recall their definition, and relate them to equivariant cohomology (see [12] for details).

The Burnside category Burn_G is the category with objects finite G-sets and each morphism set $Mor_{Burnc}(S,T)$ is the group completion of the hom-set of spans between S and T in the category of finite G-sets. It is a fact that $Burn_G$ is self dual, that is, the duality map $D: \operatorname{\mathsf{Burn}}_{\mathsf{G}} \to \operatorname{\mathsf{Burn}}_{\mathsf{G}}^{\operatorname{op}}$ that is identity on objects and switches the legs of the spans, is an isomorphism of categories.

Definition 2.1. A functor $\underline{M} : \mathsf{Burn}_{\mathsf{G}}^{\mathsf{op}} \to \mathsf{Ab}$ from Burnside category into Abelian groups is called Mackey functor.

In this paper we restrict our attention to $G = C_{p^n}$. For the remainder of the paper G will always refer to this group. Explicitly, a G-Mackey functor¹ \underline{M} is a collection of commutative $W_G(H)$ -groups $\underline{M}(G/H)$ one for each subgroup $H \leq \overline{G}$, each accompanied by transfer $\operatorname{tr}_{K}^{H}: \underline{M}(G/K) \to \underline{M}(G/H)$ and restriction $\operatorname{res}_{K}^{H}: \underline{M}(G/H) \to \underline{M}(G/K)$ for $K \leq H \leq G$ such that

(1) $\operatorname{tr}_{J}^{H} = \operatorname{tr}_{J}^{H} \operatorname{tr}_{J}^{K}$ and $\operatorname{res}_{J}^{H} = \operatorname{res}_{J}^{K} \operatorname{res}_{K}^{H}$ for all $J \leq K \leq H$. (2) $\operatorname{tr}_{K}^{H}(\gamma.x) = \operatorname{tr}_{K}^{H}(x)$ for all $x \in \underline{M}(G/K)$ and $\gamma \in W_{H}(K)$. (3) $\gamma.\operatorname{res}_{K}^{H}(x) = \operatorname{res}_{K}^{H}(x)$ for all $x \in \underline{M}(G/H)$ and $\gamma \in W_{H}(K)$. (4) $\operatorname{res}_{K}^{H}\operatorname{tr}_{K}^{J}(x) = \sum_{\gamma \in W_{H}(K)} \gamma.\operatorname{tr}_{J\cap K}^{K}(x)$ for all subgroups $J, H \leq K$.

We will often write $\underline{M}(H)$ for $\underline{M}(G/H)$.

A morphism between two Mackey functors is given by natural transformations. We denote the category of Mackey functors of the group G, by $Mack_G$. It is a fact that Mack_G is an Abelian category. The Burnside ring Mackey functor is representable functor $\operatorname{Burn}_{\mathsf{G}}(-, \mathsf{G}/\mathsf{G})$. This is denoted by <u>A</u>. For an Abelian group C, the constant Mackey functor <u>C</u> is described as $\underline{C}(G/H) = C$ with the res_K^H = Id, and tr_K^H = multiplication by [H: K]. Following Lewis [10], the data of a Mackey functor for the group C_p may be organized in a diagram as demonstrated below.

$$\underline{A}: \begin{bmatrix} 1 & p \end{bmatrix} \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \underline{\mathbb{Z}}: \qquad \operatorname{Id} \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{bmatrix} p$$

¹This is a simplification in the case G is Abelian. Otherwise the double coset formula (4) has a slightly more complicated expression.

The top row gives the value of the Mackey functor at C_p/C_p , and the bottom row gives the Mackey functor at C_p/e . For the groups C_{p^n} , there are analogous diagrams arranged vertically with n + 1 different levels.

Example 2.2. For a G-spectrum X, the equivariant homotopy groups forms a Mackey functor $\underline{\pi}_n(X)$, defined by the formula

$$\underline{\pi}_n(X)(G/H) := \pi_n(X^H).$$

For a Mackey functor \underline{M} one may define an Eilenberg-MacLane spectrum [6] $H\underline{M}$ such that $\underline{\pi}_n(H\underline{M})$ is concentrated in degree 0 where it is the Mackey functor \underline{M} . This constructs a RO(G)-graded cohomology theory by the formula

$$H^{\alpha}_{G}(X;\underline{M}) \cong$$
 Ho-G-spectra $(X, \Sigma^{\alpha} H \underline{M}).$

Recall that RO(G) denotes the group completion of the monoid of irreducible representations of G. A general element $\alpha \in RO(G)$ can be represented as a formal difference $\alpha = V - W$ for G-representations V, W. For a unitary representation V of G, we denote by S^V the one-point compactification of V. Analogously for a virtual representation $\alpha = V - W$, S^{α} denotes the G-spectrum $\Sigma^{-W}S^V$. Using the functor $G/H \mapsto X \wedge G/H_+$, the cohomology groups are part of a Mackey functor denoted by $\underline{H}^{\alpha}_{G}(S^0; \underline{M})$.

One may put a symmetric monoidal structure on the category $\mathsf{Mack}_{\mathsf{G}}$ by using the *box* product. For two Mackey functors \underline{M} and $\underline{N} \in \mathsf{Mack}_{\mathsf{G}}$, the box product $\underline{M} \Box \underline{N}$ is the left Kan extension of tensor product of Abelian groups along the functor $\times : \mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}} \times \mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}} \to \mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}}$ given by $(S,T) \mapsto S \times T$. The Burnside ring Mackey functor \underline{A} plays the role of unit object in the symmetric monoidal structure of $\mathsf{Mack}_{\mathsf{G}}$.

Definition 2.3. A (*commutative*) Green functor for G, is a (commutative) monoid in the symmetric monoidal category $Mack_G$ defined as above.

Both <u>A</u> and <u>Z</u> are examples of commutative Green functors. Given a commutative Green functor R, an <u>R</u>-module is a Mackey functor <u>M</u> equipped with $\mu_{\underline{M}} : \underline{R} \Box \underline{M} \to \underline{M}$ satisfying the usual relations. The category of <u>R</u>-modules will be denoted by <u>R</u>-Mod_G. This in turn has the structure of a symmetric monoidal category with the induced box product. For a commutative Green functor R, the corresponding equivariant cohomology has a graded commutative ring structure.

Notation 2.4. The representation ring $RO(C_{p^n})$ is generated by the trivial representation 1, and the 2-dimension representation $\lambda(k)$ given by the rotation by the angle $\frac{2\pi k}{p^n}$ for $k = 1, \dots, \frac{p^n-1}{2}$. Denote the representation $\lambda(p^m)$ by λ_m . Write $RO_0(C_{p^n}) \subset RO(C_{p^n})$ of those α such that the dimension of α is zero.

We now describe some equivariant cohomology classes in $\underline{H}_{G}^{\bigstar}(S^{0};\underline{\mathbb{Z}})(G/G) \cong \pi_{-\bigstar}^{G}(H\underline{\mathbb{Z}})$. The generators used are defined in [7, Section 3] which we now recall.

Definition 2.5. Let V be a G-representation. We have the G-map $S^0 \to S^{V2}$ which induces

$$S^0 \to S^V \wedge S^0 \to S^V \wedge H\underline{\mathbb{Z}}$$

which we call $a_V \in \underline{H}_G^V(S^0; \underline{\mathbb{Z}})$. If V is an oriented G-representation, a choice of orientation gives a class

$$u_V \in H_G^{V-\dim(V)}(S^0;\underline{\mathbb{Z}}) \cong \mathbb{Z}.$$

²This is given by the inclusion of $\{0, \infty\} \subseteq S^V$.

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We also have relations among these generators namely $u_V u_W = u_{V \oplus W}$ and $a_V a_W = a_{V \oplus W}$. It follows that these classes are products of $u_{\lambda(m)}$ and $a_{\lambda(m)}$ for $0 \le m < n$. These classes satisfy relations

$$(2.6) p^{n-i}a_{\lambda_i} = 0,$$

and

(2.7)
$$u_{\lambda_i} a_{\lambda_j} = p^{j-i} u_{\lambda_j} a_{\lambda_i} \text{ if } i < j.$$

Definition 2.8. Observe that the map $a: S^{\lambda_i} \to S^{\lambda_j}$ for i < j < n, described as the map on one-point compactifications induced by $z \mapsto z^{p^{j-i}}$, satisfies $a \circ a_{\lambda_i} = a_{\lambda_j}$. This class is denoted by a_{λ_j/λ_i} . In the case j = n, this construction also makes sense and gives a class which after multiplication with u_{λ_i} gives p^{n-i} , so it is denoted $[p^{n-i}u_{\lambda_i}^{-1}]$.

In this notation, for $G = C_p$, the equivariant cohomology of S^0 in gradings $n + m\lambda_0$ is given by (see [10], [15])

$$\underline{H}_{C_p}^{\cdot,\cdot,\lambda_0}(S^0;\underline{\mathbb{Z}})(C_p/C_p) \cong \mathbb{Z}[u_{\lambda_0},a_{\lambda_0}] \oplus_{j\geq 1} \mathbb{Z}\{[pu_{\lambda_0}^{-j}]\} \oplus_{j,k\geq 1} \Sigma^{-1}\mathbb{Z}/p\{u_{\lambda_0}^{-j}a_{\lambda_0}^{-k}\}.$$

For $G = C_{p^n}$, the ring $\underline{H}_{C_{p^n}}^{\bigstar}(S^0; \underline{\mathbb{Z}})(C_{p^n}/C_{p^n})$ in gradings which are combinations of integers and positive multiples of λ_i is described as [8, Remark 4.6]

$$\mathbb{Z}[a_{\lambda_0}, u_{\lambda_0}, \cdots, a_{\lambda_{n-1}}, u_{\lambda_{n-1}}]/(p^{n-i}a_{\lambda_i}, u_{\lambda_i}a_{\lambda_{i+k}} - p^k u_{\lambda_{i+k}}a_{\lambda_i}, i, k \ge 0).$$

Notation 2.9. Recall [13] for $H \leq G$, there is the *restriction* functor

$$\downarrow_{H}^{G}: \mathsf{Mack}_{\mathsf{G}} \to \mathsf{Mack}_{\mathsf{H}}$$

given by $\downarrow_{H}^{G}(\underline{N})(H/L) := \underline{N}(G \times_{H} H/L)$ where $\underline{N} \in \mathsf{Mack}_{\mathsf{G}}$ and $L \leq H$. Given a Mackey functor \underline{M} , one defines $\operatorname{Hom}_{L}(\underline{M}, \mathbb{Z})$ as the composition of the functors

$$\mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}} \xrightarrow{D} \mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}} \xrightarrow{\underline{M}} \mathsf{Ab} \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Z})} \mathsf{Ab}$$

and similarly $\operatorname{Ext}_L(\underline{M}, \mathbb{Z})$ as the composition of the functors

$$\mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}} \xrightarrow{D} \mathsf{Burn}_{\mathsf{G}}^{\mathrm{op}} \xrightarrow{\underline{M}} \mathsf{Ab} \xrightarrow{\operatorname{Ext}_{\mathbb{Z}}^{1}(-,\mathbb{Z})} \mathsf{Ab} .$$

We often denote $\operatorname{Ext}_L(\underline{M}, \mathbb{Z})$ by \underline{M}^E . The Mackey functor $\operatorname{Hom}_L(\underline{\mathbb{Z}}, \mathbb{Z})$ has the same groups as $\underline{\mathbb{Z}}$ with the restriction and transfer maps switched and is denoted by $\underline{\mathbb{Z}}^*$.

These Mackey functors play crucial role in the equivariant analog of the universal coefficient theorem discussed below along the lines of [1].

2.10. Anderson duality. Let $I_{\mathbb{Q}}$ and $I_{\mathbb{Q}/\mathbb{Z}}$ be the spectra representing the cohomology theories given by $X \mapsto \operatorname{Hom}(\pi_{-*}^G(X), \mathbb{Q})$ and $X \mapsto \operatorname{Hom}(\pi_{-*}^G(X), \mathbb{Q}/\mathbb{Z})$ respectively. The natural map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces the spectrum map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$, and the homotopy fibre is denoted by $I_{\mathbb{Z}}$. For a *G*-spectrum *X*, the *Anderson dual* $I_{\mathbb{Z}} X$ of *X*, is the function spectrum Fun $(X, I_{\mathbb{Z}})$. For $X = H\underline{\mathbb{Z}}$, one easily computes $I_{\mathbb{Z}} H\underline{\mathbb{Z}} \cong \Sigma^{2-\lambda_0} H\underline{\mathbb{Z}}$.

In general, for G-spectra E, X, and $\alpha \in RO(G)$, there is short exact sequence

(2.11)
$$0 \to \operatorname{Ext}_{L}(\underline{E}_{\alpha-1}(X), \mathbb{Z}) \to \operatorname{I}_{\mathbb{Z}}(E)^{\alpha}(X) \to \operatorname{Hom}_{L}(\underline{E}_{\alpha}(X), \mathbb{Z}) \to 0.$$

In particular, for $E = H\mathbb{Z}$ and $X = S^0$, we have the equivalence $\underline{E}_{\alpha}(X) \cong \underline{H}_{G}^{-\alpha}(S^0;\underline{\mathbb{Z}})$. Therefore, one may rewrite (2.11) as

$$(2.12) \quad 0 \to \operatorname{Ext}_{L}(\underline{H}_{G}^{3-\lambda_{0}-\alpha}(S^{0};\underline{\mathbb{Z}}),\mathbb{Z}) \to \underline{H}_{G}^{\alpha}(S^{0};\underline{\mathbb{Z}}) \to \operatorname{Hom}_{L}(\underline{H}_{G}^{2-\lambda_{0}-\alpha}(S^{0};\underline{\mathbb{Z}}),\mathbb{Z}) \to 0$$

for each $\alpha \in RO(G)$.

Anderson duality provides a relation in the equivariant cohomology ring of S^0 . A naive method to give another such relation is to build up S^V using a filtration such that the filtration quotients are computable, and then use this to relate $\underline{H}_G^{\alpha}(S^0)$ to $\underline{H}_G^{\alpha-V}(S^0)$. This method is commonly used (see for example [10], [3], [4]). More explicitly, for each $m \leq n$, we have homotopy cofibration sequences in C_{p^n} -spectra,

(2.13)
$$C_{p^n}/C_{p^m} \xrightarrow{1-g} C_{p^n}/C_{p^m} \xrightarrow{} S(\lambda_m) \xrightarrow{}$$

and

(2.14)
$$S(\lambda_m)_+ \to S^0 \to S^{\lambda_m}.$$

Here g is a chosen generator for the quotient group C_{p^n}/C_{p^m} .

For a non-negative integer $0 \leq k \leq n$, and a Mackey functor $\underline{M} \in \mathsf{Mack}_{\mathsf{C}_{p^k}}$ define $\Theta_k : \mathsf{Mack}_{\mathsf{C}_{p^k}} \to \mathsf{Mack}_{\mathsf{C}_{p^n}}$ as

$$\Theta_k(\underline{M})(G/H) = \begin{cases} \underline{\mathbb{Z}}(G/H) \otimes_{\mathbb{Z}} \underline{M}(C_{p^k}/C_{p^k}) & \text{if } C_{p^k} \subseteq H \\ \underline{M}(C_{p^k}/H) & \text{otherwise.} \end{cases}$$

The restrictions and transfers are clear from this description. In similar fashion, we define another functor $\Theta_k^* : \mathsf{Mack}_{\mathsf{C}_{\mathsf{p}^{\mathsf{n}}}} \to \mathsf{Mack}_{\mathsf{C}_{\mathsf{p}^{\mathsf{n}}}}$ as

$$\Theta_k^*(\underline{M})(G/H) = \begin{cases} \underline{\mathbb{Z}}^*(G/H) \otimes_{\mathbb{Z}} \underline{M}(C_{p^k}/C_{p^k}) & \text{if } C_{p^k} \subseteq H \\ \underline{M}(C_{p^k}/H) & \text{otherwise.} \end{cases}$$

Proposition 2.15. For each non-negative integer $m \leq n$, $\underline{M} \in \mathsf{Mack}_{\mathsf{C}_{p^n}}$, and $\alpha \in RO(C_{p^n})$, there is short exact sequence

$$\underline{0} \to \Theta_m^*(\underline{H}_{C_pm}^{\alpha-1}(S^0;\downarrow_{p^m}^{p^n}\underline{M})) \to \underline{H}_{C_pn}^{\alpha}(S(\lambda_m)_+;\underline{M}) \to \Theta_m(\underline{H}_{C_pm}^{\alpha}(S^0;\downarrow_{p^m}^{p^n}\underline{M})) \to \underline{0}$$

in $Mack_{C_{p^n}}$.

Proof. The cofiber sequence (2.13) yields the cohomology long exact sequence

$$\cdots \underline{H}^{\alpha-1}_{C_{p^n}}(C_{p^n}/C_{p^m}_+) \longrightarrow \underline{H}^{\alpha}_{C_{p^n}}(S(\lambda_m)_+) \longrightarrow \underline{H}^{\alpha}_{C_{p^n}}(C_{p^n}/C_{p^m}_+) \xrightarrow{(1-g)^*} \underline{H}^{\alpha}_{C_{p^n}}(C_{p^n}/C_{p^m}_+) \cdots$$

An immediate computation gives $\ker((1-g)^*) \cong \Theta_m(\underline{H}^{\alpha}_{C_{p^m}}(S^0; \downarrow_{p^m}^{p^n} \underline{M}))$ and the cokernel of $(1-g)^*$ is $\Theta^*_m(\underline{H}^{\alpha-1}_{C_{p^m}}(S^0; \downarrow_{p^m}^{p^n} \underline{M}))$. (See Proposition 4.3. and §4.5 in [4] for an analogous computation.) Hence the result follows.

Example 2.16. Observe that $\Theta_0(C)$ for an Abelian group C is the constant Mackey functor \underline{C} , while $\Theta_0^*(C)$ is the dual \underline{C}^* . In Proposition 2.15, $\underline{H}_e^{\alpha}(S^0;\underline{\mathbb{Z}})$ is 0 for $|\alpha| \neq 0$, and \mathbb{Z} for $|\alpha| = 0$. It follows that

(2.17)
$$\underline{H}^{\alpha}_{C_{p^n}}(S(\lambda_0)_+;\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } |\alpha| = 0\\ \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 1\\ 0 & \text{otherwise.} \end{cases}$$

For each m < n, consider the following long exact sequence in cohomology associated to (2.14)

(2.18)(m)

$$\cdots \underline{H}^{\alpha+\lambda_m-1}_{C_{p^n}}(S(\lambda_{m+1})) \longrightarrow \underline{H}^{\alpha}_{C_{p^n}}(S^0) \xrightarrow{\cdot a_{\lambda_m}} \underline{H}^{\alpha+\lambda_m}_{C_{p^n}}(S^0) \longrightarrow \underline{H}^{\alpha+\lambda_m}_{C_{p^n}}(S(\lambda_{m+1})) \cdots$$

At m = 0, (2.17) implies that multiplication by a_{λ_0} is an isomorphism unless $|\alpha| = -2, -1, 0$. At $|\alpha| = -2$, multiplication by a_{λ_0} is injective while at $|\alpha| = -1, 0$, multiplication by a_{λ_0} is surjective. The injectivity at $|\alpha| = -1$ requires some additional argument which is essentially the same as [4, Proposition 4.5].

3. $\underline{\mathbb{Z}}$ -modules & extensions involving them

In this section, we discuss the category of $\underline{\mathbb{Z}}$ -modules and a few particular \mathbb{Z} -modules that are important in the rest of the paper. Along the way we study certain extensions in the additive category $\underline{\mathbb{Z}}$ -Mod_G. We first note that the $\underline{\mathbb{Z}}$ -modules satisfy an additional condition on the restriction and transfer maps.

Remark 3.1. For any $\underline{M} \in \underline{\mathbb{Z}}$ -Mod_G, $\operatorname{tr}_{K}^{H} \operatorname{res}_{K}^{H}$ equals the multiplication by index [H:K] for $K \leq H \leq G$ [14, Theorem 4.3].

This condition puts certain restriction on which Mackey functors may actually be $\underline{\mathbb{Z}}$ -modules. One such example is the lemma below.

Lemma 3.2. If \underline{M} is a $\underline{\mathbb{Z}}$ -Mod_G satisfying $\underline{M}(G/e) = 0$, then $\underline{M}(G/H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $H \leq G$.

Proof. Applying Remark 3.1 to each $x \in \underline{M}(G/H)$, $|H|x = \operatorname{tr}_e^H \operatorname{res}_e^H(x) = \operatorname{tr}_e^H(0) = 0$. This implies each element in $\underline{M}(G/H)$ is torsion.

We now interpret Lemma 3.2 for equivariant cohomology with $\underline{\mathbb{Z}}$ -coefficients.

Corollary 3.3. Let $\alpha \in RO(G) \setminus RO_0(G)$, then the Mackey functor $\underline{H}^{\alpha}_G(S^0; \underline{\mathbb{Z}})$ is torsion. *Proof.* Note that the Mackey functors $\underline{H}^{\alpha}_G(S^0; \underline{\mathbb{Z}})$ are all $\underline{\mathbb{Z}}$ -modules. For α with $|\alpha| \neq 0$, one sees $\underline{H}^{\alpha}_G(S^0; \underline{M})(G/e) \cong \tilde{H}^{|\alpha|}(S^0; \underline{M}(G/e))$, hence is zero. By Lemma 3.2, the result follows.

We now observe that $\underline{M} \in \underline{\mathbb{Z}}$ -Mod_G satisfying $\underline{M}(G/e) = 0$, then Hom_L($\underline{M}, \mathbb{Z}$) = 0. Applying this fact to Corollary 3.3 using (2.12), we note

(3.4)
$$|\alpha| \neq 0 \implies \underline{H}^{\alpha}_{C_{p^n}}(S^0; \underline{\mathbb{Z}}) \cong \operatorname{Ext}_L(\underline{H}^{3-\lambda_0-\alpha}_{C_{p^n}}(S^0; \underline{\mathbb{Z}}), \mathbb{Z}).$$

The following proposition allows us to reduce the grading from $RO(C_{p^n})$ to the linear combinations of λ_k .

Proposition 3.5 ([15], Proposition 4.25). There is an equivalence $H\underline{\mathbb{Z}} \wedge S^{\lambda(p^k)} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda(rp^k)}$ whenever $p \nmid r$.

The above implies the existence of invertible classes in $H\underline{\mathbb{Z}}$ -cohomology in grading $\lambda(p^k) - \lambda(rp^k)$ for $p \nmid r$. One may make a coherent choice of these units, so that the $H\underline{\mathbb{Z}}$ -cohomology is, up to some units and their inverses, the part which lies in the graded pieces given by linear combinations of $1, \lambda_0, \dots, \lambda_{n-1}$. From now onwards, we assume that $\alpha \in RO(G)$ means that $\alpha = c + \sum_{k>0} a_k \lambda_k$.

Definition 3.6. Let $S \subseteq \overline{n} := \{1, \dots, n\}$. Denote by $\underline{\mathbb{Z}}_S$ the Mackey functor

$$\underline{\mathbb{Z}}_{S}(C_{p^{n}}/H) = \mathbb{Z} \text{ for } H \leq C_{p^{n}},$$
$$\operatorname{res}_{C_{p^{i-1}}}^{C_{p^{i}}} = p^{\chi_{S}(i)} \text{ for } 1 \leq i \leq n.$$

Here χ_S is the characteristic function on S.

We note that $\underline{\mathbb{Z}}_{\emptyset} = \underline{\mathbb{Z}}$, and $\underline{\mathbb{Z}}_{\overline{n}} = \underline{\mathbb{Z}}^*$. For subsets $S \subseteq T \subseteq \overline{n}$, there is a unique map $f_{T,S} : \underline{\mathbb{Z}}_T \to \underline{\mathbb{Z}}_S$ such that it is the identity at C_{p^n}/e^3 . Then the Mackey functor structure

³If $S \not\subseteq T$, there is no Mackey functor morphism from $\underline{\mathbb{Z}}_T \to \underline{\mathbb{Z}}_S$ which induces identity at C_{p^n}/e .

suggests $f_{T,S}(C_{p^n}/C_{p^k})$ is the multiplication by $p^{\alpha_{T\setminus S,k}}$, where $\alpha_{T\setminus S,k} := \#((T\setminus S)\cap \overline{k})$. Denoting the cokernel of $f_{T,S}$ by $\underline{B}_{T,S}$, we get a exact sequence

$$(3.7) \qquad \underline{0} \to \underline{\mathbb{Z}}_T \to \underline{\mathbb{Z}}_S \to \underline{B}_{T,S} \to \underline{0}.$$

where the Mackey functor $\underline{B}_{T,S}$ is given by

$$\underline{B}_{T,S}(C_{p^n}/C_{p^k}) = \frac{\mathbb{Z}}{p^{\alpha_{T\setminus S,k}}\mathbb{Z}}, \operatorname{res}_{C_{p^{k-1}}}^{C_{p^k}} = \begin{cases} p & \text{if } k \in S \\ 1 & \text{if } k \in S^c, \end{cases} \operatorname{tr}_{C_{p^{k-1}}}^{C_{p^k}} = \begin{cases} 1 & \text{if } k \in S \\ p & \text{if } k \in S^c. \end{cases}$$

For $k \leq \min(S \cup T)$, we readily observe that $\underline{B}_{T,S} = \underline{B}_{T \cup \overline{k}, S \cup \overline{k}}$. Applying the functor $\operatorname{Hom}_{L}(-,\mathbb{Z})$ to the short exact sequence (3.7) yields the long exact sequence

$$\operatorname{Hom}_{L}(\underline{B}_{T,S},\mathbb{Z}) \to \operatorname{Hom}_{L}(\underline{\mathbb{Z}}_{S},\mathbb{Z}) \to \operatorname{Hom}_{L}(\underline{\mathbb{Z}}_{T},\mathbb{Z}) \to \operatorname{Ext}_{L}(\underline{B}_{T,S},\mathbb{Z}) \to \operatorname{Ext}_{L}(\underline{\mathbb{Z}}_{S},\mathbb{Z}) \cdots$$

One readily observes that the first and last terms of the long exact sequence above are zero, which simplifies the expression to the short exact sequence

$$\underline{0} \to \underline{\mathbb{Z}}_{S^c} \to \underline{\mathbb{Z}}_{T^c} \to \operatorname{Ext}_L(\underline{B}_{T,S}, \mathbb{Z}) \to \underline{0}.$$

Comparing the short exact sequence above with (3.7), we deduce $\operatorname{Ext}_L(\underline{B}_{T,S},\mathbb{Z}) = \underline{B}_{S^c,T^c}$.

Proposition 3.8. Let \underline{M} be a torsion free $\underline{\mathbb{Z}}$ -module for the group C_{p^n} that fits into the short sequence

$$\underline{0} \to \underline{\mathbb{Z}}_S \to \underline{M} \to \underline{B}_{S,T} \to \underline{0}.$$

Then, there is an isomorphism $\underline{M} \cong \underline{\mathbb{Z}}_T$ of $\underline{\mathbb{Z}}$ -modules.

Proof. Since \underline{M} is torsion free, that is, for each $0 \leq k \leq n$, the Abelian group $\underline{M}(C_{p^n}/C_{p^k})$ has no torsion. Thus $\underline{M}(C_{p^n}/C_{p^k}) \cong \mathbb{Z}$. Therefore, there is a subset $T' \subseteq \overline{n}$ such that $\underline{M} \cong \underline{\mathbb{Z}}_{T'}$, and then the cokernel of $\underline{\mathbb{Z}}_S \to \underline{M}$ is $\underline{B}_{S,T'}$. However from the formula of $\underline{B}_{S,T}$ we easily observe that, $\underline{B}_{S,T'} \cong \underline{B}_{S,T}$ implies T = T'.

We demonstrate an exact sequence as in Proposition 3.8 using Mackey functor diagrams for the group C_{p^3} .



Proposition 3.9. Let $k \leq n$.

1) A map $f: \underline{B}_{\overline{n},\overline{k}^c} \to \underline{B}_{\overline{1},\emptyset}$ is uniquely determined by $f(C_{p^n}/C_p)$. 2) If $f(C_{p^n}/C_p)$ is an isomorphism, then $Ker(f) \cong \underline{B}_{\overline{1}^c}, \underline{1+k^c}$, and $Coker(f) \cong \underline{B}_{\{k+1\},\emptyset}$.

Proof. The proof relies on a careful examination of the restriction and transfer maps of $\underline{B}_{\overline{n},\overline{k}^c}$. These are given by

$$\underline{B}_{\overline{n},\overline{k}^c}(C_{p^n}/C_{p^r}) = \frac{\mathbb{Z}}{p^{\min(r,k)}\mathbb{Z}}, \ \operatorname{res}_{C_{p^{r-1}}}^{C_{p^r}} = \begin{cases} 1 & \text{if } r \le k \\ p & \text{if } r > k, \end{cases} \operatorname{tr}_{C_{p^{r-1}}}^{C_{p^r}} = \begin{cases} p & \text{if } r \le k \\ 1 & \text{if } r > k. \end{cases}$$

The unique extension of f from level C_{p^n}/C_p is guaranteed by the fact that the restriction maps in $\underline{B}_{\overline{1},\emptyset}$ are the identity above this level. If $f(C_{p^n}/C_p)$ is an isomorphism, we have

$$f(C_{p^n}/C_{p^l}) \text{ is } \begin{cases} \text{onto} & \text{ if } l \leq k \\ 0 & \text{ if } l \geq k+1. \end{cases}$$

This implies the required conclusion about the cokernel of f. Note that the part of f between the levels C_{p^n}/C_{p^k} and $C_{p^n}/C_{p^{k+1}}$ may be described as

Therefore, $\downarrow_{p^{k+1}}^{p^n} \ker(f) \cong \underline{B}_{\overline{1}^c,\emptyset}$. The part of this Mackey functor above the level $C_{p^n}/C_{p^{k+1}}$ is unchanged, that is, same with $\underline{B}_{\overline{n},\overline{k}^c}$. Hence the result follows.

3.10. Pullback Mackey functors. The $\underline{\mathbb{Z}}$ -modules may also be defined as Abelian group-valued functors $\underline{M} : \mathcal{B}\underline{\mathbb{Z}}_{G}^{\mathrm{op}} \to \mathsf{Ab}$. The category $\mathcal{B}\underline{\mathbb{Z}}_{G}$ has finite *G*-sets as objects and $\mathrm{Mor}_{\mathcal{B}\underline{\mathbb{Z}}_{G}}(S,T) := \mathrm{Mor}_{G}(\mathbb{Z}[S],\mathbb{Z}[T])$ (see [15, Proposition 2.15]). Suppose *N* is a normal subgroup of *G*. The quotient map $G \to G/N$ induces $\phi : \mathcal{B}\underline{\mathbb{Z}}_{G/N}^{\mathrm{op}} \to \mathcal{B}\underline{\mathbb{Z}}_{G}^{\mathrm{op}}$. Define $\Phi_{N}^{*} :$ $\underline{\mathbb{Z}}$ -Mod_{*G/N*} $\to \underline{\mathbb{Z}}$ -Mod_{*G*} as $\Phi_{N}^{*}(\underline{M}) := Lan_{\phi}(\underline{M})$, the left Kan extension of \underline{M} along ϕ . For $G = C_{p^{n}}$ we write, $\Phi_{p^{m}}^{*}$ for $\Phi_{C_{p^{m}}}^{*}$. The Mackey functor $\Phi_{N}^{*}\underline{M}$ is given by the formula (3.11)

$$\Phi_N^* \underline{M}(G/H) := \underset{(G/H \to G/L) \in \operatorname{Mor}_{\mathcal{B}_G}}{\operatorname{colim}} \underline{M}((G/N)/(L/N)) = \begin{cases} \underline{M}((G/N)/(H/N)) & \text{if } N \subseteq H \\ \underline{M}((G/N)/(N/N)) & \text{if } H \subset N. \end{cases}$$

and the restriction $\operatorname{res}_{e}^{N}$ is Id. From this formula, we note that Φ_{p}^{*} commutes with Ext_{L} on the Z-modules which are 0 at $C_{p^{n}}/e$.

Example 3.12. The formula above implies $\Phi_N^* \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$. Also, $\Phi_{p^m}^*(\underline{B}_{T,S}) \cong \underline{B}_{T^{(m)},S^{(m)}}$, where $T^{(m)}$ is the image of T under the map $\overline{n-m} \to \overline{n}$ given by $r \mapsto r+m$.

We now point out a non-trivial extension of $\underline{\mathbb{Z}}$ -modules that arise in equivariant cohomology computations over C_{p^2} .



A change of basis gives another isomorphic form of $\underline{T}(2)$ occuring in the diagram above.

This generalizes to C_{p^n} Mackey functors $\underline{T}(n)$ obtained by repeating top part of the diagram above. This gives a non-trivial extension of Mackey functors

$$0 \to \underline{B}_{\overline{n},\overline{1}^c} \to \underline{T}(n) \to \underline{\mathbb{Z}}_{\overline{1}} \to 0.$$

4. Computations for large C_p -fixed point dimensions

This section deals with computations of $\underline{H}_{C_{p^n}}^{\alpha}(S^0,\underline{\mathbb{Z}})$ when the dimension of the C_p -fixed points of α is large. More precisely, we prove that if $|\alpha^{C_p}| \in (-2n+2,2n)^c$, then the Mackey functor $\underline{H}_{C_{p^n}}^{\alpha}(S^0;\underline{\mathbb{Z}})$ can be computed explicitly in terms of $|\alpha|$ and $\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0;\underline{\mathbb{Z}})$ (see Table 4.1). We now drop $\underline{\mathbb{Z}}$ in the notation to write $\underline{H}_{C_{p^n}}^{\alpha}(S^0)$ for $\underline{H}_{C_{p^n}}^{\alpha}(S^0;\underline{\mathbb{Z}})$, and throughout assume $n \geq 2$.

Lemma 4.1. Let $\alpha \in im(RO(C_{p^n}/C_{p^m}) \to RO(C_{p^n}))$. Then there is an equivalence

$$\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \Phi^*_{p^m}(\underline{H}^{\alpha}_{C_{p^n}/C_{p^m}}(S^0)).$$

Proof. The hypothesis implies that α and hence, also $S^{-\alpha}$ are C_{p^m} -fixed. Suppose $q: C_{p^n} \to C_{p^n}/C_{p^m}$, which induces adjunctions [11, Proposition V.3.10]

$$\{q^*(S^{-\alpha^{C_pm}}), H\underline{\mathbb{Z}}\}^{C_pn} \cong \{S^{-\alpha^{C_pm}}, H\underline{\mathbb{Z}}^{C_pm}\}^{C_pn/C_pm}.$$

The result now follows from $H\underline{\mathbb{Z}}^{C_{p^m}} \simeq H\underline{\mathbb{Z}}$, and a calculation of the restriction and transfers as in the proof of [4, Proposition 6.12].

Now apply Lemma 4.1 together with the fact that there are invertible classes in degree $\lambda(rp^k) - \lambda(p^k)$ for $p \nmid r$. This implies that the conclusion is valid once $|\alpha^K|$ are all equal for $K \leq C_{p^m}$. Further applying the techniques of Example 2.16, the hypothesis may be weakened to the fact $|\alpha^K|$ are all of the same sign for $K \leq C_{p^m}$.

Proposition 4.2. Suppose $\alpha \in RO(C_{p^n})$ such that the collection of numbers $|\alpha^{C_{p^k}}|$ for $k \leq m$, are either all 0 or are all of the same sign, then $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \Phi^*_{p^m}(\underline{H}^{\alpha^{C_{p^m}}}_{C_{p^n}/C_{p^m}}(S^0)).$

We now readily deduce that in a range of degrees, the cohomology is 0, starting from the fact that for the trivial group, the cohomology is 0 if the degree is non-zero.

Proposition 4.3. Let $\alpha \in RO(C_{p^n})$. (a) If $|\alpha^{C_{p^m}}|$ positive for all m, or negative for all m, then, $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \underline{0}$. (b) If $|\alpha^{C_{p^m}}| \leq 1$ for all m, then, $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \underline{0}$. (c) For α odd satisfying $|\alpha^{C_{p^m}}| < 1 \implies |\alpha^H| \leq 1 \forall C_{p^m} \subseteq H, \underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \underline{0}$.

Proof. Proposition 4.2 directly implies (a). For (b), the result follows from (3.4) and (a) as, $|\alpha| \leq -1, |\alpha^{H}| \leq 1 \implies |(3 - \lambda_0 - \alpha)^{H}| > 0$ for all H. But for $|\alpha| = 1, |(3 - \lambda_0 - \alpha)| = 0$, and all the other fixed point dimensions are > 0. Using (2.18)(0) along with (a) imply $\underline{H}^{3-\lambda_0-\alpha}_{C_n^n}(S^0) \cong \underline{\mathbb{Z}}^*$, so that Ext_L is trivial. Hence (b) follows. For (c), Example 2.16 implies that a_{λ_0} multiplication is an isomorphism if $|\alpha| \neq -1$ and is surjective if $|\alpha| = -1$. We now assume that (c) is true for $G = C_{p^k}$, k < n. For $|\alpha| > 0$ and $|\alpha^{C_p}| > 0$, Proposition 4.2 applies to prove the result. For $|\alpha| > 0$ and $|\alpha^{C_p}| < 0$, using the a_{λ_0} -multiplication, we get a surjective map from $\underline{H}^{\beta}_{C_{p^n}}(S^0)$ with $|\beta| < 0$ and $|\beta^{C_{p^l}}| = |\alpha^{C_{p^l}}|$ for l > 1, so that Proposition 4.2 implies the result again. In the remaining cases, using the a_{λ_0} -multiplication, it suffices to prove for $|\alpha| = -1$, $|\alpha^{C_p}| = 1$. The proof is now completed by repeated applications of Proposition 4.2 and Anderson duality(3.4) as

$$\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \operatorname{Ext}_L(\underline{H}^{3-\lambda_0-\alpha}_{C_{p^n}}(S^0), \mathbb{Z}) \cong \operatorname{Ext}_L(\Phi^*_p \underline{H}^{3-\alpha^{C_p}}_{C_{p^{n-1}}}(S^0), \mathbb{Z}) \cong \Phi^*_p \underline{H}^{\alpha^{C_p}-\lambda_0}_{C_{p^{n-1}}}(S^0) \cong 0.$$

Proposition 4.3 has many applications in showing that certain even equivariant cell complexes have free cohomology. Examples include complex projective spaces and Grassmannians [10], [3], and linear actions on simply connected 4-manifolds [2].

Proposition 4.4. Suppose $\alpha \in RO_0(C_{p^n})$ satisfying $|\alpha^{C_{p^k}}| > 0$ for k > 1. Then,

$$\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \begin{cases} \underline{\mathbb{Z}}_{1-\frac{|\alpha^{C_p}|}{2}}^c & \text{if } 4-2n \le |\alpha^{C_p}| \le 0\\ \underline{\mathbb{Z}}^{} & \text{if } |\alpha^{C_p}| \le 2-2n\\ \underline{\mathbb{Z}}^* & \text{if } |\alpha^{C_p}| \ge 2. \end{cases}$$

Proof. From the proof of Proposition 4.3 (b), observe that $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \underline{\mathbb{Z}}^*$ for $|\alpha^{C_p}| > 0$. We use induction on $F(\alpha) = 1 - \frac{|\alpha^{C_p}|}{2}$. We now assume the result for $F(\alpha) < k$, and prove this first for those with $F(\alpha) = k \leq n$.

Let $\beta \in RO_0(C_{p^n})$ with $|\beta^{C_{p^k}}| > 0$ for k > 1. Note that $F(\beta + \lambda_1 - \lambda_0) = F(\beta) - 1$ and $\beta + \lambda_1 - \lambda_0 \in RO_0(C_{p^n})$, so that it satisfies the induction hypothesis. Hence, using (2.18)(0) for index $\beta + \lambda_1 - \lambda_0$, we obtain the long exact sequence

$$\cdots \underline{H}_{C_{p^n}}^{\beta+\lambda_1-1}(S(\lambda_0)_+) \to \underline{H}_{C_{p^n}}^{\beta+\lambda_1-\lambda_0}(S^0) \to \underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S^0) \to \underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S(\lambda_0)_+) \cdots$$

In the above, we have (2.17) $\underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S(\lambda_0)_+) = 0$ and $\underline{H}_{C_{p^n}}^{\beta+\lambda_1-1}(S(\lambda_0)_+) = \underline{\mathbb{Z}}^*$. The term $\underline{H}_{C_{p^n}}^{\beta+\lambda_1-\lambda_0}(S^0) \cong \underline{\mathbb{Z}}_{\overline{F(\beta)}-1^c}$. Hence (3.7), $\underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S^0) \cong \underline{B}_{\overline{n},\overline{F(\beta)}-1^c}$. Making use of (2.18)(1), we obtain

The identifications at the two ends of the sequence are $\underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S(\lambda_1)_+) \cong \Theta_1(\underline{B}_{\overline{1},\emptyset}) \cong \underline{B}_{\overline{1},\emptyset}$, and $\underline{H}_{C_{p^n}}^{\beta+\lambda_1-1}(S(\lambda_1)_+) \cong \Theta_1^*(\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}_{\overline{1}^c}$. Proposition 3.9 computes the kernel of $\underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S^0) \to \underline{H}_{C_{p^n}}^{\beta+\lambda_1}(S(\lambda_1)_+)$ as $\underline{B}_{\overline{1}^c}, \overline{F(\beta)}^c$. Hence we deduce the short exact sequence

(4.5) $\underline{0} \to \underline{\mathbb{Z}}_{\overline{1}^c} \to \underline{H}^{\beta}_{C_{p^n}}(S^0) \to \underline{B}_{\overline{1}^c, \overline{F(\beta)}^c} \to \underline{0}.$

Using (2.18)(0) and Corollary 4.2, it follows $\underline{H}^{\beta}_{C_{p^n}}(S^0) \to \underline{H}^{\beta}_{C_{p^n}}(S(\lambda_0)_+) \cong \underline{\mathbb{Z}}$ is injective, hence $\underline{H}^{\beta}_{C_{p^n}}(S^0)$ is torsion free. Thus (4.5) along with Proposition 3.8 imply the middle term $\underline{H}^{\beta}_{C_{p^n}}(S^0) \cong \underline{\mathbb{Z}}_{\overline{F(\beta)}}^c$. If $F(\alpha) = n$, then by the above computation $\underline{H}^{\alpha}_{C_{p^n}}(S^0; \underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$. Working as above we get, $\underline{H}^{\alpha+\lambda_0}_{C_{p^n}}(S^0) \cong \underline{B}_{\overline{n},\emptyset}$ (using (2.18)(0)), and that the kernel $\underline{B}_{\overline{n},\emptyset} \to \underline{B}_{\overline{1},\emptyset}$ is $\underline{B}_{\overline{1}^c,\emptyset}$. Analogously we obtain $\underline{H}^{\alpha-\lambda_1+\lambda_0}_{C_{p^n}}(S^0) \cong \underline{\mathbb{Z}}$.

$ \alpha $ even	$\underline{H}^{\alpha}_{C_{p^n}}(S^0)$	$ \alpha $ odd	$\underline{H}^{\alpha}_{C_{p^n}}(S^0)$
$ \alpha > 0, \alpha^{C_p} \le -2n+2$	$\underline{B}_{\overline{n},\emptyset} \oplus \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$	$ \alpha < 0, \alpha^{C_p} \le -2n+3$	$\Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$
$ \alpha = 0, \alpha^{C_p} \le -2n+2$	$\underline{\mathbb{Z}} \oplus \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$	$ \alpha > 0, \ \alpha^{C_p} \le -2n+3$	$\Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$
$ \alpha < 0, \alpha^{C_p} \le -2n+2$	$\Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$	$ \alpha > 0, \alpha^{C_p} \ge 2n+1$	$\Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$
$ \alpha \neq 0, \alpha^{C_p} \ge 2n$	$\Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$	$ \alpha < 0, \alpha^{C_p} \ge 2n+1$	$\underline{B}_{\overline{n},\emptyset} \oplus \Phi_p^*(\underline{H}_{C_p^{n-1}}^{\alpha^{C_p}}(S^0))$
$ \alpha = 0, \alpha^{C_p} \ge 2n$	$\underline{\mathbb{Z}}^* \oplus \Phi_p^*(\underline{H}^{\alpha^{C_p}}_{C_{p^{n-1}}}(S^0))$		

TABLE 4.1. Formula for $\underline{H}^{\alpha}_{C_{n^n}}(S^0)$ at large C_p -fixed points.

Incorporating the values obtained in Proposition 4.4 into the long exact sequence (2.18)(0), we derive

Corollary 4.6. Suppose $\alpha \in RO(C_{p^n})$ satisfying $|\alpha^H| > 0$ for all $H \neq C_p$. Then,

$$\underline{H}^{\alpha}_{C_{p^{n}}}(S^{0};\underline{\mathbb{Z}}) \cong \begin{cases} \underline{B}_{\overline{n},\overline{1-\frac{|\alpha^{C_{p}}|}{2}}^{c}} & \text{if } 4-2n \leq |\alpha^{C_{p}}| \leq 0\\ \underline{B}_{\overline{n},\emptyset} & \text{if } |\alpha^{C_{p}}| \leq 2-2n\\ 0 & \text{if } |\alpha^{C_{p}}| \geq 2. \end{cases}$$

We next use the multiplicative structure to derive a computation with $|\alpha| = 0$ and $|\alpha^{C_p}|$ a sufficiently large negative number.

Theorem 4.7. If $\alpha \in RO_0(C_{p^n})$ with $|\alpha^{C_p}| \leq -2n+2$, then the torsion-free part of $\underline{H}^{\alpha}_{C_{p^n}}(S^0)$ is $\underline{\mathbb{Z}}$, and there is a decomposition

$$\underline{H}_{C_{p^n}}^{\alpha}(S^0) \cong \underline{\mathbb{Z}} \oplus \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0)).$$

$$\alpha| < 0 \text{ and } |\alpha^{C_p}| \le -2n+2 \text{ even, } \underline{H}_{C_{p^n}}^{\alpha}(S^0) \cong \Phi_p^*(\underline{H}_{C_{n^{n-1}}}^{\alpha^{C_p}}(S^0))$$

 $For \mid$

Proof. The last statement follows from Proposition 4.2. In the rest of the proof, we use the $\underline{H}^{\bigstar}_{C_{p^n}}(S^0)$ -module structure on $\underline{H}^{\bigstar}_{C_{p^n}}(S(\lambda_0)_+)$ which we denote by $\mu_{S(\lambda_0)}$. The multiplication in $\underline{H}^{\bigstar}_{C_{p^n}}(S^0)$ is denoted by μ .

Proposition 4.4 implies that $\underline{H}_{C_{pn}}^{\alpha}(S^{0}) \cong \underline{\mathbb{Z}}$ for $\alpha \in RO_{0}(C_{p^{n}})$ satisfying $|\alpha^{H}| > 0$ for all $H \neq e, C_{p}$ and $|\alpha^{C_{p}}| \leq 2 - 2n$. Let $\beta \in RO_{0}(C_{p^{n}})$ satisfying $|\beta^{C_{p}}| = 0$ and $|\beta^{H}| \leq 0$ for all $H \neq e, C_{p}$. Applying Anderson duality (2.12) and Propositions 4.4, 4.3 (a), we deduce $\underline{H}_{C_{pn}}^{\beta}(S^{0};\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$.

We have the commutative diagram $(\pi : S(\lambda_0)_+ \to S^0$ is the map induced by adding disjoint base-points to $S(\lambda_0) \to *$)

Note that $\pi^* \Box_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}}$ and $\mu_{S(\lambda_0)}$ are isomorphisms, so that μ is a section for π^* up to isomorphism. This yields the decomposition (using (2.18)(0))

$$\underline{H}_{C_{p^n}}^{\alpha+\beta}(S^0) \cong \underline{\mathbb{Z}} \oplus \underline{H}_{C_{p^n}}^{\alpha+\beta-\lambda_0}(S^0,\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}} \oplus \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{(\alpha+\beta)^{C_p}}(S^0,\underline{\mathbb{Z}}))$$
(by Corollary 4.2).

Hence the result follows as for each $H \neq e, C_p, |(\alpha + \beta)^H|$ can be made arbitrary.

For α odd positive case, the following is a direct consequence of Theorem 4.7.

Corollary 4.8. Suppose $\alpha \in RO(C_{p^n})$ satisfying $|\alpha| > 0$ odd and $|\alpha^{C_p}| \leq -2n+3$, then $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \Phi_p^*(\underline{H}^{\alpha^{C_p}}_{C_{p^{n-1}}}(S^0)).$

Proof. From Example 2.16, it suffices to prove for $|\alpha| = 1$. Then by (2.18)(0), one obtains

$$\begin{array}{cccc} & \cdots \underline{H}^{\alpha-1}_{C_{p^n}}(S^0) \longrightarrow \underline{H}^{\alpha-1}_{C_{p^n}}(S(\lambda_0)_+) \longrightarrow \underline{H}^{\alpha-\lambda_0}_{C_{p^n}}(S^0) \longrightarrow \underline{H}^{\alpha}_{C_{p^n}}(S^0) \longrightarrow \underline{H}^{\alpha}_{C_{p^n}}(S(\lambda_0)_+) \cdots \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \underline{\mathbb{Z}} \oplus \Phi^*_p(\underline{H}^{\alpha^{C_p}-1}_{C_{p^{n-1}}}(S^0,\underline{\mathbb{Z}})) & & & & \underline{\mathbb{Z}} \end{array}$$

The map $\underline{H}_{C_{p^n}}^{\alpha-1}(S^0) \to \underline{H}_{C_{p^n}}^{\alpha-1}(S(\lambda_0)_+)$ has a section according to the proof of Theorem 4.7. Thus, there is an isomorphism $\underline{H}_{C_{p^n}}^{\alpha-\lambda_0}(S^0) \xrightarrow{\cong} \underline{H}_{C_{p^n}}^{\alpha}(S^0)$. The result follows from $\underline{H}_{C_{p^n}}^{\alpha-\lambda_0}(S^0) \cong \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$ using Corollary 4.2.

The following corollary is a direct consequence of the Anderson duality and Theorem 4.7. **Corollary 4.9.** The Mackey functor $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \underline{\mathbb{Z}}^* \oplus \Phi^*_p(\underline{H}^{\alpha}_{C_{p^{n-1}}}(S^0))$ if $|\alpha| = 0$ and $|\alpha^{C_p}| \ge 2n$.

Proof. Using Theorem 4.7 and Corollary 4.8, the short exact sequence (2.12) turns out to be

$$\underline{0} \to \Phi_p^*(\underline{H}^{3-\alpha^{C_p}}_{C_{p^{n-1}}}(S^0))^E \to \underline{H}^{\alpha}_{C_{p^n}}(S^0) \to \underline{\mathbb{Z}}^* \to \underline{0}.$$

The map $\underline{\mathbb{Z}}^* \cong \underline{H}_{C_{p^n}}^{\alpha+\lambda_0-1}(S^0) \to \underline{H}_{C_{p^n}}^{\alpha}(S^0)$ in (2.18)(0) serves as a splitting for this sequence. Applying (3.4) to $\underline{H}_{C_{p^{n-1}}}^{3-\alpha^{C_p}}(S^0)$, we obtain the result using the identifications in Example 2.16.

The following result is readily deduced from Corollary 4.8 and Anderson duality (3.4) **Corollary 4.10.** Suppose $\alpha \in RO(C_{p^n})$ satisfying $|\alpha| < 0$ even and $|\alpha^{C_p}| \geq 2n$, then $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \Phi_p^*(\underline{H}^{\alpha^{C_p}}_{C_{p^{n-1}}}(S^0)).$

The following result is obtained by applying (2.18)(0) to the results of Theorem 4.7.

Corollary 4.11. Suppose $\alpha \in RO(C_{p^n})$ satisfying $|\alpha| > 0$ even and $|\alpha^{C_p}| \leq -2n+2$, then $\underline{H}^{\alpha}_{C_{p^n}}(S^0) \cong \underline{B}_{\overline{n},\emptyset} \oplus \Phi_p^*(\underline{H}^{\alpha^{C_p}}_{C_{p^{n-1}}}(S^0)).$

Proof. From Example 2.16, it suffices to assume $|\alpha| = 2$. We then have the following reduction of (2.18)(0)

$$\underline{0} \to \underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \oplus \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0)) \to \underline{H}_{C_{p^n}}^{\alpha}(S^0) \to \underline{0}.$$

As $\underline{\mathbb{Z}}^*$ is generated by transfers on the element $1 \in \mathbb{Z} = \underline{\mathbb{Z}}^*(C_{p^n}/e)$, the image of $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \oplus \Phi_p^*(\underline{H}_{C_{p^{n-1}}}^{\alpha^{C_p}}(S^0))$ sits inside $\underline{\mathbb{Z}}$, uniquely defined by the fact that it is an isomorphism at C_{p^n}/e . The result follows as the cokernel of $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}}$ is $\underline{B}_{\overline{n},\emptyset}$.

For the group C_p , the equivariant cohomology ring (in grading $n + m\lambda_0$) consists of a polynomial algebra $\mathbb{Z}[u_{\lambda_0}, a_{\lambda_0}]$, plus modules $\mathbb{Z}\{[pu_{\lambda_0}^{-j}]\}$ and $\mathbb{Z}/p\{u_{\lambda_0}^{-j}a_{\lambda_0}^{-k}\}$. The latter part are all a_{λ_0} -torsion, and the former part gives a region where multiplication by a_{λ_0} is injective, but we do not have any a_{λ_0} -periodic pieces. For $n \geq 2$, the Mackey functor $\Phi_p^*(\underline{H}_{C_pn-1}^{\alpha C_p}(S^0))$ forms a a_{λ_0} -periodic piece as observed in the proofs of Theorem 4.7 and Corollary 4.11, and in Corollary 4.9 and Corollary 4.10. Using Lemma 4.1, this also constructs a_{λ_i} -periodic pieces over C_{p^n} whenever $n \geq i+2$.

Example 4.12. For the group C_{p^2} , an example where a_{λ_0} -periodicity occurs is in degrees $2\lambda_1 + c\lambda_0$, where the periodic piece is the Mackey functor $\underline{B}_{\{2\},\emptyset}$ which is \mathbb{Z}/p at C_{p^2}/C_{p^2} , and 0 at other levels. The Mackey functor in degree $2\lambda_1 - 2\lambda_0$ is demonstrated in Table 4.2, where the symbols for the generating classes are written alongside. The class $[p^2 u_{\lambda_0}^{-2}]$ is a_{λ_0} -torsion, and the periodic piece is represented in degree $2\lambda_1$ by $a_{\lambda_0}^2 a_{\lambda_1/\lambda_0}^2 = a_{\lambda_1}^2$. This construction easily generalizes so that over C_{p^n} for $n \ge 2$, the class $a_{\lambda_1}^n$ is a_{λ_0} -periodic, and for $n \ge i+2$, $a_{\lambda_{i+1}}^{n-i}$ is a_{λ_i} -periodic.

Mackey functor diagram	Generating elements
$\mathbb{Z}\oplus\mathbb{Z}/p$	$u_{\lambda_1}^2[p^2u_{\lambda_0}^{-2}], (a_{\lambda_1/\lambda_0})^2 - u_{\lambda_1}^2[p^2u_{\lambda_0}^{-2}]$
$\begin{bmatrix} p & 0 \end{bmatrix} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\$	$[pu_{\lambda_0}^{-2}]$
Ž Z	1

TABLE 4.2. Formula for $\underline{H}_{C_{p^2}}^{2\lambda_1-2\lambda_0}(S^0)$.

Note that $\operatorname{Ext}_{L}(\underline{B}_{\overline{n},\emptyset},\mathbb{Z}) \cong \underline{B}_{\overline{n},\emptyset}$, hence (3.4) along with Corollary 4.11 readily implies **Corollary 4.13.** For an element $\alpha \in RO(C_{p^{n}})$ satisfying $|\alpha| < 0$ odd and $|\alpha^{C_{p}}| \geq 2n + 1$, $\underline{H}^{\alpha}_{C_{p^{n}}}(S^{0}) \cong \underline{B}_{\overline{n},\emptyset} \oplus \Phi_{p}^{*}(\underline{H}^{\alpha^{C_{p}}}_{C_{p^{n-1}}}(S^{0},\underline{\mathbb{Z}})).$

5. Examples of non-trivial extensions

In this section, we compute the coefficient Mackey functor $\underline{H}_{C_{p^2}}^{\bigstar}(S^0;\underline{\mathbb{Z}})$ completely, and using that we observe non-trivial extensions for the group C_{p^n} when $n \geq 3$. The ring structure of $\underline{H}_{C_{p^2}}^{\bigstar}(S^0;\underline{\mathbb{Z}})(C_{p^2}/C_{p^2})$ is completely determined in [15] using the Tate square. Our computations follow mainly from the general results of §4. For C_p , the $\underline{\mathbb{Z}}$ -cohomology has the following additive structure [5, Corollary B.10]

(5.1)
$$\underline{H}^{\alpha}_{C_{p}}(S^{0};\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } |\alpha| = 0, |\alpha^{C_{p}}| \leq 0\\ \underline{\mathbb{Z}}^{*} & \text{if } |\alpha| = 0, |\alpha^{C_{p}}| > 0\\ \underline{B}_{\overline{1},\emptyset} & \text{if } |\alpha| > 0, |\alpha^{C_{p}}| \leq 0 \text{ even}\\ \underline{B}_{\overline{1},\emptyset} & \text{if } |\alpha| < 0, |\alpha^{C_{p}}| \geq 3 \text{ odd}\\ 0 & \text{otherwise.} \end{cases}$$

$ \alpha $ even & positive	$\underline{H}^{\alpha}_{C_{p^2}}(S^0)$	$ \alpha = 0$	$\underline{H}^{\alpha}_{C_{p^2}}(S^0)$
$ \alpha^{C_p} > 0, \alpha^{C_{p^2}} > 0$	0	$ \alpha^{C_p} \le 0, \alpha^{C_{p^2}} \le 0$	$\underline{\mathbb{Z}}$
$ \alpha^{C_p} > 0, \alpha^{C_{p^2}} \le 0$	$\underline{B}_{\overline{1}^c,\emptyset}$	$ \alpha^{C_p} > 0, \alpha^{C_{p^2}} > 0$	$\underline{\mathbb{Z}}^*$
$ \alpha^{C_p} = 0, \alpha^{C_{p^2}} \le 0$	$\underline{B}_{\overline{2},\emptyset}$	$ \alpha^{C_p} \ge 4, \alpha^{C_{p^2}} \le 0$	$\underline{B}_{\overline{1}^c,\emptyset} \oplus \underline{\mathbb{Z}}^*$
$ \alpha^{C_p} = 0, \alpha^{C_{p^2}} > 0$	$\underline{B}_{\overline{2},\overline{1}^c}$	$ \alpha^{C_p} = 2, \alpha^{C_{p^2}} \le 0$	$\underline{\mathbb{Z}}_{\overline{1}}$
$ \alpha^{C_p} < 0, \alpha^{C_{p^2}} > 0$	$\underline{B}_{\overline{2},\emptyset}$	$ \alpha^{C_p} = 0, \alpha^{C_{p^2}} > 0$	$\underline{\mathbb{Z}}_{\overline{1}^c}$
$ \alpha^{C_p} \le 0, \alpha^{C_{p^2}} \le 0$	$\underline{B}_{\overline{2},\emptyset}$	$ \alpha^{C_p} < 0, \alpha^{C_{p^2}} > 0$	$\underline{\mathbb{Z}}$
$ \alpha $ odd & negative	$\underline{H}^{\alpha}_{C_{p^2}}(S^0)$	$ \alpha $ even & negative	$\underline{H}^{\alpha}_{C_{p^2}}(S^0)$
$ \alpha^{C_p} \le 1, \alpha^{C_{p^2}} \le 1$	0	$ \alpha^{C_p} \ge 4, \alpha^{C_{p^2}} \le 0$	$\underline{B}_{\overline{1}^c,\emptyset}$
$ \alpha^{C_p} \le 1, \alpha^{C_{p^2}} > 1$	$\underline{B}_{\overline{1}^c,\emptyset}$	otherwise	0
$ \alpha^{C_p} = 3, \alpha^{C_{p^2}} > 1$	$\underline{B}_{\overline{2},\emptyset}$	$ \alpha $ odd & positive	$\underline{H}^{\alpha}_{C_{p^2}}(S^0)$
$ \alpha^{C_p} = 3, \alpha^{C_{p^2}} \le 1$	$\underline{B}_{\overline{1},\emptyset}$	$ \alpha^{C_p} \le -1, \alpha^{C_{p^2}} > 1$	$\underline{B}_{\overline{1}^c,\emptyset}$
$ \alpha^{C_p} \ge 5$	$\underline{B}_{\overline{2},\emptyset}$	otherwise	0

5.2. The additive structure of C_{p^2} **cohomology.** We compute the Mackey functors $\underline{H}^{\alpha}_{C_{-2}}(S^0;\underline{\mathbb{Z}})$ for each $\alpha \in RO(C_{p^2})$. The entire computation is summarized in Table 5.1.

TABLE 5.1. Formula for $\underline{H}^{\alpha}_{C_{n^2}}(S^0;\underline{\mathbb{Z}})$.

Theorem 5.3. The Mackey functors $\underline{H}^{\alpha}_{C_{p^2}}(S^0;\underline{\mathbb{Z}})$ are as demonstrated in Table 5.1.

Proof. The starting point is the application Proposition 4.2 to (5.1), which gives the result whenever $|\alpha|$ and $|\alpha^{C_p}|$ are either both 0, or both have the same sign. The remaining cases with $|\alpha| = 0$ follow from Proposition 4.4, Theorem 4.7, and Corollary 4.9. For $|\alpha| > 0$ odd, apart from the above, the remaining follow from Corollary 4.8. Applying Anderson duality (3.4) to the cases with $|\alpha| > 0$ odd, we obtain the computations for $|\alpha| < 0$ even.

We next consider the case $|\alpha| > 0$ even. Summarizing the results from §4, we note that other than $|\alpha^{C_p}| = 0$, $|\alpha^{C_{p^2}}| \leq 0$, the results follow from Proposition 4.2, Corollary 4.6 and Corollary 4.11. Using the calculations in Example 2.16, it suffices to consider $|\alpha| = 2$, in which case we already have computed the cohomology at the grading $\alpha - \lambda_0$ to be $\underline{\mathbb{Z}}$ (5.1). The short exact sequence

$$\underline{0} \to \underline{\mathbb{Z}}^* \to \underline{H}^{\alpha-\lambda_0}_{C_{p^2}}(S^0;\underline{\mathbb{Z}}) \to \underline{H}^{\alpha}_{C_{p^2}}(S^0;\underline{\mathbb{Z}}) \to \underline{0}.$$

computes $\underline{H}^{\alpha}_{C_{p^2}}(S^0) \cong \underline{B}_{\overline{2},\emptyset}$. Finally for $|\alpha| < 0$ odd, the result follows from Anderson duality (3.4).

5.4. Examples of non-trivial extensions. We now point out cohomology computations where the Mackey functors which arise are not a direct sum of copies of $\underline{\mathbb{Z}}_T$ and $\underline{B}_{T,S}$. We start by assuming $n \geq 3$ and $\alpha \in RO_0(C_{p^n})$ satisfying $|\alpha^{C_p}| = 2n - 2$ and $|\alpha^H| \leq 0$ for all $H \neq e, C_p$. By Proposition 2.15, there is a short exact sequence

(5.5)
$$\underline{0} \to \underline{B}_{\overline{n},\overline{1}^c} \cong \Theta_1^*(\underline{B}_{\overline{1}}) \to \underline{H}^{\alpha}_{C_{p^n}}(S(\lambda_1)_+;\underline{\mathbb{Z}}) \to \Theta_1(\underline{\mathbb{Z}}_{\overline{1}}) \cong \underline{\mathbb{Z}}_{\overline{1}} \to \underline{0}$$

in $\mathsf{Mack}_{\mathsf{C}_{p^n}}$. At each level C_{p^n}/H , the short exact sequence (5.5) splits as the right end is free. We note that there does not exists any splitting in $\mathsf{Mack}_{\mathsf{C}_{p^n}}$. For if it did, then $\downarrow_{p^2}^{p^n} \underline{H}^{\alpha}_{C_{p^n}}(S(\lambda_1)_+;\underline{\mathbb{Z}}) \cong \underline{B}_{\overline{2},\overline{1}^c} \oplus \underline{\mathbb{Z}}_{\overline{1}}$. Now we may apply (2.18)(1) substituting the values from Table 5.1 to obtain the following exact sequence

$$\underline{0} \to \underline{B}_{\overline{1}^c,\emptyset} \to \underline{\mathbb{Z}}^* \oplus \underline{B}_{\overline{1}^c,\emptyset} \to \underline{B}_{\overline{2},\overline{1}^c} \oplus \underline{\mathbb{Z}}_{\overline{1}} \to \underline{H}_{C_{p^2}}^{\alpha-\lambda_1+1}(S^0) \to \underline{0}.$$

The restriction $\operatorname{res}_{C_p}^{C_{p^2}}$ in $\underline{B}_{\overline{2},\overline{1}^c}$ is 0 and in $\underline{\mathbb{Z}}_{\overline{1}}$ is divisible by p, on the other hand, the restriction in $\underline{H}_{C_{p^2}}^{\alpha-\lambda_1+1}(S^0) \cong \underline{B}_{\overline{2},\emptyset}$ (Table 5.1) is the usual quotient $\mathbb{Z}/p^2 \to \mathbb{Z}/p$. Thus, the short exact sequence (5.5) does not split in $\operatorname{Mack}_{C_n}$.

Assuming (5.5) does not split, a direct computation implies that up to isomorphism, $\underline{H}^{\alpha}_{C_{p^n}}(S(\lambda_1)_+)$ has only one choice which we call $\underline{T}(n)$. The groups are given by

$$\underline{T}(n)(C_{p^n}/H) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/p & \text{if } H \neq e \\ \mathbb{Z} & \text{if } H = e, \end{cases}$$

and the restrictions and transfers are given by

$$\operatorname{res}_{C_{p^{i}-1}}^{C_{p^{i}}} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } 2 \le i \le n \\ \begin{pmatrix} p & 0 \end{pmatrix} & \text{for } i = 1. \end{cases} \text{ and } \operatorname{tr}_{C_{p^{i-1}}}^{C_{p^{i}}} = \begin{cases} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} & \text{for } 2 \le i \le n \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{for } i = 1. \end{cases}$$

Now, we restrict out attention to the group C_{p^3} and $\alpha \in RO_0(C_{p^3})$ satisfying $|\alpha^{C_p}| = 4$, $|\alpha^{C_{p^2}}| \leq 0$, and $|\alpha^{C_{p^3}}| \leq 0$. Consider the long exact sequence

$$(5.6) \qquad \cdots \underline{H}^{\alpha-\lambda_1}_{C_{p^3}}(S^0) \to \underline{H}^{\alpha}_{C_{p^3}}(S^0) \to \underline{H}^{\alpha}_{C_{p^3}}(S(\lambda_1)_+) \to \underline{H}^{\alpha-\lambda_1+1}_{C_{p^3}}(S^0) \to \underline{H}^{\alpha+1}_{C_{p^3}}(S^0) \cdots$$

From (3.4) and Proposition 4.3 (a), we get $\underline{H}_{C_{p^3}}^{\alpha-\lambda_1}(S^0) \cong \underline{0}$. Using (3.4), we obtain $\underline{H}_{C_{p^3}}^{\alpha-\lambda_1+1}(S^0) \cong \operatorname{Ext}_L(\underline{H}_{C_{p^3}}^{2-\lambda_0+\lambda_1-\alpha}(S^0), \mathbb{Z})$, which is $\underline{B}_{\overline{3},\overline{1}^c}^E \cong \underline{B}_{\overline{1},\emptyset}$ by Corollary 4.6. Finally, $\underline{H}_{C_{p^3}}^{\alpha+1}(S^0) = 0$ by Proposition 4.3 (b). Thus (5.6) reduces to the short exact sequence

$$\underline{0} \to \underline{H}^{\alpha}_{C_{p^{3}}}(S^{0}) \to \underline{T}(3) \to \underline{B}_{\overline{1}, \emptyset} \to \underline{0}.$$

A direct computation of the kernel of the map $\underline{T}(3) \to \underline{B}_{\overline{1},\emptyset}$ gives

$$\begin{bmatrix}
 \mathbb{Z} \oplus \mathbb{Z}/p \\
 \begin{bmatrix}
 1 & 0 \\
 0 & 0
 \end{bmatrix} \begin{pmatrix}
 \end{pmatrix} \begin{bmatrix}
 p & 0 \\
 0 & 1
 \end{bmatrix}$$

$$\begin{bmatrix}
 \mathbb{Z} \oplus \mathbb{Z}/p \\
 \begin{bmatrix}
 p & 0
 \end{bmatrix} \begin{pmatrix}
 \end{bmatrix} \begin{bmatrix}
 1 \\
 -1
 \end{bmatrix}$$

$$\begin{bmatrix}
 p \\
 p \\
 \end{bmatrix}^{p} \begin{pmatrix}
 \\
 \end{bmatrix}^{1}$$

Applying a change of basis $\epsilon_1 \mapsto \epsilon_1 - \epsilon_2$ and $\epsilon_2 \mapsto \epsilon_2$ at C_{p^3}/C_{p^2} , we may rewrite the Mackey functor above as

Note that $\downarrow_{p^2}^{p^3} \underline{H}^{\alpha}_{C_{p^3}}(S^0) \cong \underline{\mathbb{Z}}^* \oplus \underline{B}_{\overline{1}^c,\emptyset}$ as in Table 5.1. We now look at (5.6)

$$\underline{0} \to \underline{H}^{\alpha+\lambda_0-1}_{C_{p^3}}(S(\lambda_0)_+) \to \underline{H}^{\alpha}_{C_{p^3}}(S^0) \to \underline{H}^{\alpha+\lambda_0}_{C_{p^3}}(S^0) \to \underline{H}^{\alpha+\lambda_0}_{C_{p^3}}(S(\lambda_0)_+) \cong \underline{0}.$$

and put in the values to get



a non-trivial extension. One may compute and check that $\underline{H}^{\alpha}_{C_{p^3}}(S^0)$ does not split as a direct sum of Mackey functors of the type $\underline{\mathbb{Z}}_T$ and $\underline{B}_{T,S}$.

Mackey functor diagram	Generating elements
$\mathbb{Z}\oplus\mathbb{Z}/p$	$(a_{\lambda_1/\lambda_0})^2, p.(a_{\lambda_1/\lambda_0})^2 - u_{\lambda_1}^2 [p^3 u_{\lambda_0}^{-2}]$
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\$	$u_{\lambda_{1}}^{2}[p^{2}u_{\lambda_{0}}^{-2}], (a_{\lambda_{1}/\lambda_{0}})^{2} - u_{\lambda_{1}}^{2}[p^{2}u_{\lambda_{0}}^{-2}]$ $[pu_{\lambda_{0}}^{-2}]$ 1

TABLE 5.2. Formula for $\underline{H}_{C_{p^3}}^{2\lambda_1-2\lambda_0}(S^0)$.

Example 5.7. An example of α as the above is $2\lambda_1 - 2\lambda_0$. The diagram (Table (5.2)) compares the Mackey functor diagram with the corresponding generating classes.

References

- [1] D. W. ANDERSON, Universal coefficient theorems for k-theory. 1969.
- [2] S. BASU, P. DEY, AND A. KARMAKAR, Equivariant homology decompositions for definite 4-manifolds, 2021.
- [3] S. BASU AND S. GHOSH, Computations in C_{pq}-Bredon cohomology, Math. Z., 293 (2019), pp. 1443–1487.
- [4] _____, Equivariant cohomology for cyclic groups of square-free order, 2020. available at https://arxiv.org/abs/2006.09669.
- [5] K. K. FERLAND, On the RO(G)-graded equivariant ordinary cohomology of generalized G-cell complexes for G = Z/p, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–Syracuse University.
- [6] J. P. C. GREENLEES AND J. P. MAY, *Equivariant stable homotopy theory*, in Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 277–323.
- [7] M. A. HILL, M. J. HOPKINS, AND D. C. RAVENEL, On the nonexistence of elements of Kervaire invariant one, Ann. of Math. (2), 184 (2016), pp. 1–262.
- [8] M. A. HILL, M. J. HOPKINS, AND D. C. RAVENEL, The slice spectral sequence for certain $RO(C_{p^n})$ graded suspensions of $H\underline{Z}$, Bol. Soc. Mat. Mex. (3), 23 (2017), pp. 289–317.
- [9] I. KRIZ AND Y. LU, On the RO(G)-graded coefficients of dihedral equivariant cohomology, Math. Res. Lett., 27 (2020), pp. 1109–1128.
- [10] L. G. LEWIS, JR., The RO(G)-graded equivariant ordinary cohomology of complex projective spaces with linear Z/p actions, in Algebraic topology and transformation groups (Göttingen, 1987), vol. 1361 of Lecture Notes in Math., Springer, Berlin, 1988, pp. 53–122.
- [11] M. A. MANDELL AND J. P. MAY, Equivariant orthogonal spectra and S-modules, Mem. Amer. Math. Soc., 159 (2002), pp. x+108.
- [12] J. P. MAY, Equivariant homotopy and cohomology theory, vol. 91 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.
- [13] P. WEBB, A guide to Mackey functors, in Handbook of algebra, Vol. 2, vol. 2 of Handb. Algebr., Elsevier/North-Holland, Amsterdam, 2000, pp. 805–836.
- [14] T. YOSHIDA, On G-functors. II. Hecke operators and G-functors, J. Math. Soc. Japan, 35 (1983), pp. 179–190.
- [15] M. ZENG, Equivariant Eilenberg-MacLane spectra in cyclic p-groups, 2017. available at https://arxiv.org/abs/1710.01769.

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