TOWARDS $\pi_*L_{K(2)}V(0)$ AT $p = 2$

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ABSTRACT. Let $V(0)$ be the mod 2 Moore spectrum and let $C$ be the super-singular elliptic curve over $\mathbb{F}_4$ defined by the Weierstrass equation $y^2 + y = x^3$. Let $F_C$ be its formal group law and $E_C$ be the spectrum classifying the deformations of $F_C$. The group of automorphisms of $F_C$, which we denote by $S_C$, acts on $E_C$. Further, $S_C$ admits a norm whose kernel we denote by $S^1_C$. The cohomology of $S^1_C$ with coefficients in $(E_C)^* V(0)$ is the $E_2$-term of a spectral sequence converging to the homotopy groups of $E_C^{\mathbb{A}^S} \wedge V(0)$, a spectrum closely related to $L_{K(2)}V(0)$. In this paper, we use the algebraic duality resolution spectral sequence to compute an associated graded for $H^*(S_C^1; (E_C)^* V(0))$. These computations rely heavily on the geometry of elliptic curves made available to us at chromatic level 2.

CONTENTS

1. Introduction 2
  1.1. Background 2
  1.2. Statement of Results 5
  1.3. Organization of the paper 6
  1.4. Acknowledgements 7
2. Morava $E$-theory and Elliptic Curves 7
  2.1. Deformations 7
  2.2. The action of $\text{Aut}(\Gamma)$ 8
  2.3. The super-singular elliptic curve 9
  2.4. The automorphisms of $C$ 11
3. The Morava stabilizer group 12
  3.1. The isomorphism of $S_2$ and $S_C$ 12
  3.2. The filtration and the norm 13
  3.3. The action of the Morava stabilizer group 14
4. The algebraic duality resolution spectral sequence 15
  4.1. Preliminaries 15
  4.2. Some extra structure 16
  4.3. The $E_1$-term 18
  4.4. Approximate $\Delta$-linearity 21
5. Computation of the $E_\infty$-Term 24
  5.1. The differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ 25
  5.2. The differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ 26
  5.3. The differential $d_1 : E_1^{2,0} \to E_1^{3,0}$ 30
  5.4. The differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$ 36
  5.5. Higher Differentials 40
6. Appendix I: Some projective resolutions 48
1. Introduction

This paper can be read as a sequel to [2]. For this reason, this section builds upon the deeper discussion of [2, §2]. We give an overview of the tools that were not introduced in the prequel and state our results. The reader who wants more detail on background and motivation should refer to [2].

1.1. Background. Throughout this paper, I will work at the prime \( p = 2 \). Recall that Morava \( K \)-theory \( K(2) \) is the unique ring spectrum with coefficients

\[
K(2)_* = \mathbb{F}_2[v_2^{\pm 1}],
\]

for \( v_2 \) in degree 6, and with formal group law the Honda formal group law \( F_2 \) of height 2. The group \( S_2 \) is the group of automorphisms of \( F_2 \) over \( \mathbb{F}_4 \). The extended Morava stabilizer group \( G_2 \) is the extension of \( S_2 \) by the Galois group. Morava \( E \)-theory \( E_2 \) is the spectrum which classifies isomorphism classes of deformations of \( F_2 \). Its homotopy groups can be described as follows. Let \( \zeta \) be a primitive third root of unity and let

\[
\mathbb{W} := W(\mathbb{F}_4) \cong \mathbb{Z}_2[\zeta]
\]

be the Witt vectors on \( \mathbb{F}_4 \). Then

\[
(E_2)_* = \mathbb{W}[u_1][u^{\pm 1}],
\]

where \( u_1 \) has degree zero and \( u \) has degree \(-2\). The group \( G_2 \) acts on the spectrum \( E_2 \). For any finite spectrum \( X \), there is a weak equivalence

\[
L_{K(2)}X \simeq E_2^{hG_2} \wedge X.
\]

Further, for closed subgroups \( G \) of \( G_2 \) and finite spectra \( X \), there are descent spectral sequences

\[
E_2^{s,t} := H^s(G, (E_2)_tX) \Rightarrow \pi_{t-s}(E_2^{hG} \wedge X),
\]

The groups \( S_2 \) and \( G_2 \) both admit a norm induced by the determinant of a general linear representation of \( S_2 \). The elements of norm one form normal subgroups denoted \( S_2^1 \) and \( G_2^1 \) respectively. Further,

\[
S_2 \cong S_2^1 \times \mathbb{Z}_2,
\]

and

\[
G_2 \cong G_2^1 \times \mathbb{Z}_2.
\]

The group \( S_2 \) has a unique conjugacy class of maximal finite subgroups, which can be described as follows. The automorphism of \( F_2 \) given by \([-1]_{F_2}(x)\) generates a central subgroup \( C_2 \). The power series

\[
\omega(x) = \zeta x
\]
generates a subgroup of order three in $S_2$, denoted $C_3$. Define
\[ C_6 := C_2 \times C_3. \]
This group is contained in a subgroup
\[ G_{24} := Q_8 \times C_3 \]
for a quaternion group $Q_8$. The group $G_{24}$ is a maximal finite subgroup of $S_2$.

The subgroups $C_6$ and $G_{24}$ are contained in $S_1^3$. However, $S_1^3$ has two conjugacy classes of maximal finite subgroups. A representative for the other conjugacy class is given by
\[ G'_{24} = \pi G_{24} \pi^{-1} \]
for $\pi$ a topological generator of $\mathbb{Z}_2$ in (1.3).

The following result is Theorem 1.8 and Theorem 1.10 of [2].

**Theorem 1.4** (Goerss, Henn, Mahowald, Rezk). There is an exact sequence of complete $S_1^3$-module
\[
0 \to \mathcal{C}_3 \xrightarrow{\partial_1} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,
\]
where $\mathcal{C}_0 \cong \mathbb{Z}_2[[S_1^3/G_{24}]]$, $\mathcal{C}_1 \cong \mathcal{C}_2 \cong \mathbb{Z}_2[[S_1^3/C_6]]$ and $\mathcal{C}_0 \cong \mathbb{Z}_2[[S_1^3/G'_{24}]]$. Further, for any finitely generated complete $S_1^3$-module $M$, there is a first quadrant spectral sequence,
\[
E_1^{p,q} = \text{Ext}_{\mathbb{Z}_2[[S_1^3]]}^q(\mathcal{C}_p,M) \Rightarrow H_{\pi}^{p+q}(S_2^1,M).
\]
The differentials have degree
\[
d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},
\]
and
\[
E_1^{p,q} \cong \begin{cases} 
H^q(G_{24};M) & \text{if } p = 0; \\
H^q(G_6;M) & \text{if } p = 1,2; \\
H^q(G'_2;M) & \text{if } p = 3.
\end{cases}
\]

The exact sequence of Theorem 1.4 is called the algebraic duality resolution because it satisfies a certain duality. This is described in Theorem 1.9 of [2]. The associated spectral sequence is called the algebraic duality spectral sequence.

Let $V(0)$ be the mod 2 Moore spectrum. It is defined by the cofiber sequence
\[
S^2 \xrightarrow{\partial} S \to V(0).
\]
The goal of this paper is to compute the $E_{\infty}$-term of the algebraic duality spectral sequence when $M$ is the module $(E_2)_\ast V(0)$. We obtain an associated graded for $H^\ast(S_1^3; (E_2)_\ast V(0))$. By taking the Galois fixed points of the $E_{\infty}$-term, one obtains an associated graded for the cohomology $H^\ast(S_1^3; (E_2)_\ast V(0))$. Therefore, this computation gives the $E_2$-page of the descent spectral sequence (1.2) when $G = S_1^3$ and $X = V(0)$, that is
\[
H^\ast(S_1^3; (E_2)_\ast V(0)) \Rightarrow \pi_{t-\ast} E_2^G \land V(0).
\]

Because there is a fiber sequence
\[
L_{K(2)} V(0) \to E_2^G \land V(0) \to E_2^G \land V(0),
\]
computing $H^\ast(S_1^3; (E_2)_\ast V(0))$ is a first step for computing $\pi_{\ast} L_{K(2)} V(0)$.

The computations will be done using the fact that, at chromatic level $n = 2$, one can replace Morava $K$-theory $K(2)$ by a spectrum $K_C$ whose formal group law
is the formal group law of a super singular elliptic curve $C$. This allows us to use the geometry of elliptic curves to get a better understanding of the action of the Morava stabilizer group $S_2$ on $(E_2)_*$. Before stating the results, I will explain this point of view.

Let $C$ be the unique super-singular elliptic curve over $\mathbb{F}_4$, with Weierstrass equation

$$C : y^2 - y = x^3.$$  \hfill (1.5)

Let $F_C$ be the formal group law of $C$. It satisfies

$$[-2]_{F_C}(x) = x^4.$$  

Let $K_C$ denote the complex oriented ring spectrum whose ring of coefficients is $(K_C)_* = \mathbb{F}_4[u^{\pm 1}]$, where $u$ is in degree $-2$, and whose formal group law is $F_C$. In this paper, $E_C := E(\mathbb{F}_4, F_C)$ will denote the spectrum which represents isomorphism classes of deformations of $F_C$. There is an isomorphism $(E_C)_* \cong (E_2)_*$.  

(Note that the isomorphism cannot be realized by a map of $E_\infty$-ring spectra. Such a map would induce an $\mathbb{F}_4$-isomorphism on the formal group laws $F_C$ and the 2-periodic extension of $F_2$. However, these formal group laws are not isomorphic over $\mathbb{F}_4$. They become isomorphic after passing to the algebraic closure.)

Let $S_C$ be the group of automorphisms of $F_C$ over $\mathbb{F}_4$,

$$S_C := \text{Aut}(F_C).$$

The groups $S_2$ and $S_C$ are isomorphic. An explicit isomorphism is constructed in Theorem 3.2. The group $S_C$ admits an action of the Galois group and the group $G_C$ is the extension of $S_C$ by this action. The group $G_C$ acts on the deformations. By the Goerss-Hopkins-Miller theorem $[8, \S 7]$, it acts on $E_C$ by maps of $E_\infty$-ring spectra. The isomorphism of Section 3 does not extend to an isomorphism of the groups $G_2$ and $G_C$. In fact, these groups are not isomorphic. However, over an algebraic closure of $\mathbb{F}_2$, the formal group laws $F_2$ and $F_C$ are isomorphic. Therefore, the Bousfield classes of $K(2)$ and $K_C$ are the same. Their localization functors are weakly equivalent, so that

$$L_{K(2)}X \simeq L_{K_C}X.$$  

It follows from the work of Devinatz and Hopkins in $[7]$ that for $X$ a finite spectrum

$$L_{K_C}X \cong E_C^{hG_C} \wedge X.$$  

Further, for any closed subgroup $G$ of $G_C$ and any finite spectrum $X$, there is a spectral sequence analogous to (1.2). Therefore, for any finite $X$, there is a convergent spectral sequence

$$E_2^{s,t} := H^s(G_C,(E_C)_tX) \Longrightarrow \pi_{t-s}L_{K_C}(X) \cong \pi_{t-s}L_{K(2)}(X),$$

where

$$(E_C)_*X := \pi_*L_{K_C}(E_C \wedge X).$$
The groups $S_C$ and $G_C$ also admit a norm induced by a general linear representation of $S_C$. The groups $S_C^1$ and $G_C^1$ are defined to be the norm one subgroups. Further,
\[ S_C \cong S_C^1 \times \mathbb{Z}_2, \]
and
\[ G_C \cong G_C^1 \times \mathbb{Z}_2. \]

Since $S_C$ is isomorphic to $S_2$, the results of [2] also hold for $S_C$. In particular, the resolution of Theorem 1.4 can be constructed using $S_C^1$. Further, the algebraic duality resolution gives rise to a spectral sequence
\[ E_1^{p,q} = \text{Ext}^2_{\mathbb{Z}_2[[u^\pm 1]]}(\mathcal{E}_p, (E_C)_* V(0)) \Rightarrow H^{p+q}(S_C^1; (E_C)_* V(0)). \]

This spectral sequence is isomorphic to the spectral sequence of Theorem 1.4. In this paper, we compute the $E_\infty$-term of (1.6).

The main advantage of using $S_C$ is that the maximal finite subgroup of $S_C$ corresponds to those automorphisms of $F_C$ which are induced by automorphisms of the elliptic curve $C$. For the super-singular curve, Tate has shown that the natural map
\[ \rho : \text{End}(C) \otimes \mathbb{Z}_2 \to \text{End}(F_C). \]
is an isomorphism (see [3] or [19]). Therefore, the group $\text{Aut}(C)$ injects into $S_C$. Further,
\[ \text{Aut}(C) \cong G_{24}, \]
so that $\text{Aut}(C)$ is a choice of maximal finite subgroup of $S_C$. For the remainder of this paper, we let $G_{24}$ denote $\text{Aut}(C)$ in $S_C$

Using level three structures, Strickland has computed the action of $G_{24}$ on $(E_C)_*$. Strickland’s results are used heavily in the computation of (1.6). They are not in print and will be described in Section 2.4.

1.2. Statement of Results. In order to state the results, we will describe the $E_1$-term of (1.6). It follows from (1.1) that
\[ (E_C)_* V(0) = \mathbb{F}_4[[u_1]][u^\pm 1] \]
where $u_1$ has degree 0 and $u$ has degree $-2$. Let $v_1 = u_1 u^{-1}$ in $(E_C)_* V(0)$. Let $F_{E_C}$ be the graded formal group law of $E_C$. Then
\[ [2]_{F_{E_C}}(x) \equiv v_1 x^2 + \ldots \mod (2). \]
The element $v_1$ is invariant under the action of $S_C$ on $(E_C)_* V(0)$ that is,
\[ v_1 \in H^0(S_C^1; (E_C)_* V(0)). \]
Let $\beta$ be the Bockstein homomorphism associated to the exact sequence
\[ 0 \to (E_2)_* \overset{2}{\to} (E_2)_* \to (E_2)_* V(0) \to 0. \]
Let $h_1 = \beta(v_1)$ and $v_2 = u^{-3}$. Then
\[ H^*(C_6; (E_C)_* V(0)) \cong \mathbb{F}_4[[u_3]][v_1, v_2^\pm 1, h]/(v_2^{-1} v_1^3 = u_1^3), \]
for a class $h \in H^1(C_6; (E_C)_* V(0))$ satisfying $h_1 = hv_1$. In particular,
\[ H^0(C_6; (E_C)_* V(0)) = \mathbb{F}_4[[u_3]]/[v_1, v_2^3]/(v_2^{-1} v_1^3 = u_1^3). \]
Therefore, a set of $\mathbb{F}_4[v_1]$ generators of $H^k(C_6; (E_C)_* V(0))$ is given by
\[ \{ h^k v_2^n \}_{n \in \mathbb{Z}}. \]
The cohomology $G_{24}$ is harder to describe. It is related to the cohomology of the Hopf algebroid classifying Weierstrass curves over $F_4$ with their strict isomorphisms (see [1]). In particular, the $G_{24}$ fixed points are related to modular forms modulo 2. In fact,

$$H^0(G_{24}, (E_C)_{\ast} V(0)) \cong F_4[[v_1, \Delta \pm 1]]/(j \Delta = v_1^2),$$

where $v_1$ as defined above is the Hasse invariant, $\Delta$ is the determinant and $j$ is the $j$-invariant. The higher cohomology is described in Section 4 and is depicted in Figure 4.1. A set of $F_4[v_1]$ generators for $H^0(G_{24}, (E_C)_{\ast} V(0))$ is given by

$$\{\Delta^n\}_{n \in \mathbb{Z}}.$$

**Theorem 1.8.** The algebraic duality resolution spectral sequence converging to $H^\ast(S_2, (E_C)_{\ast} V(0))$ collapses at the $E_2$-term. The spectral sequence is a module over $F_4[v_1, h_1]$. There exist $F_4[v_1]$-generators $a_n \in E_1^{0,0}$, $b_n \in E_1^{1,0}$, $c_n \in E_1^{2,0}$ and $d_n \in E_1^{3,0}$ with

$$b_n \equiv c_n \equiv v_1^2 \mod (v_1)$$

$$a_n \equiv d_n \equiv \Delta^n \mod (v_1)$$

and such that, for $k \geq 0$ and $t \in \mathbb{Z}$,

$$d_1(a_n) = \begin{cases} v_1^{3-2k} b_{2k+1}(1+4t) & n = 2k(1+2t); \\ 0 & n = 0. \end{cases}$$

$$d_1(b_n) = \begin{cases} v_1^{3-2k} c_{2k+1}(1+2t) & n = 2k(3+4t); \\ v_1^{3-2k-1} c_{1+2k+1} & n = 1 + 2k+2 + t2^{k+3}; \\ 0 & \text{otherwise}. \end{cases}$$

$$d_1(c_n) = \begin{cases} v_1^{3(2^{k+1}+1)} d_{2k}(1+2t) & n = 1 + 2k+1 + 2^{k+2} + t2^{k+3}; \\ 0 & \text{otherwise}. \end{cases}$$

A differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is non-zero if and only if it is forced by $h_1$-linearity. All differentials $d_r : E_r^{p,q} \to E_r^{p+1,q}$ for $r \geq 2$ are zero, so that $E_2 = E_\infty$.

It is worth mentioning here that the related computation of the $E_2$-page of the Johnson-Wilson $E(2)$-local Adams-Novikov spectral sequence converging to $L_2 S$ was done by Shimomura and Wang in [17]. Their work is impressive, although it is hard to understand and verify. Our computations were done independently. However, historically, they depend on the work of Shimomura and Wang. Indeed, results similar to those of Theorem 1.8 can be extracted from [17], and it is using Shimomura and Wang’s computation that Mahowald conjectured the existence of the duality resolution for the $K(2)$-local sphere.

1.3. **Organization of the paper.** In Section 2.1, we review the deformation theory of formal group laws. In Section 2.2, we recall how the action of the group of automorphisms of a formal group law acts on the theory classifying its deformations. In Section 2.3, we describe the universal deformation $U$ of the super-singular elliptic curve $C$. This choice of deformation is due to Strickland. This allows us to define $E_C$. In Section 2.4, we describe the group of automorphisms of $C$ and give explicit formulas for its action on $E_C$. The author learned these results from unpublished notes of Strickland. The proofs given here are either his or constructed using his results.
Section 3 is dedicated to describing the structure of $S_C$. In Section 3.1, we give an explicit isomorphism between the group of automorphisms of the Honda formal group law $S_2$ and the group $S_C$. In Section 3.2, we recall the standard filtration on $S_C$. In Section 3.3, we give the information about the action $S_C$ on $(E_C)_*$ that will be used in the computation of $H^*(S_C^1, (E_C)_* V(0))$. The proofs are postponed to Section 8.

The goal of Section 4 is to introduce the algebraic duality resolution spectral sequence (ADRSS) for $S_C$ and to give the information necessary to begin the computation. In Section 4.1, we recall the construction of the ADRSS that was given in [2]. The ADRSS is not multiplicative, but it has some nice properties which we describe in Section 4.2. In Section 4.3, we give a detailed description of the $E_1$-term. The discriminant $\Delta$ of the curve $C_U$ has useful linearity properties which are given in Section 4.4.

The bulk of the paper is the computation of the $E_\infty$-term of the ADRSS with coefficients in $(E_C)_* V(0)$. This is done in Section 5. In Sections 5.1, 5.2 and 5.3, we compute the differentials $d_1 : E_1^{p, 0} \to E_1^{p+1, 0}$. In Section 5.4, we compute the differentials $d_q : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$. In Section 5.5, we prove that all differentials $d_r : E_r^{p,q} \to E_r^{p,1}$ for $r \geq 2$ are zero.

This paper has three appendices. Section 6 describes some projective resolutions which are used in the above computations. These are $C_3$-equivariant analogues of some of the classical projective resolutions which can be found in [6]. Although we do not give references, we believe these results are folklore. Section 7 describes the $v_1$-Bockstein spectral sequence whose construction can be found in [6, §1]. We also use this spectral sequence to compare the cohomology of $H^*(G_{24}, (E_C)_* V(0))$ and $H^*(A_4, (E_C)_* V(0))$, where $A_4 = G_{24}/C_2$. This comparison is used in the computations of Section 5. In Section 8, we describe the action of $S_C$ on $(E_C)_*$. First, we give formulas for the minus two series of $C$ and $C_U$. We then use these formulas to give estimates for the action.

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2. Morava $E$-theory and Elliptic Curves

In this section, I will explain how the spectrum $E_C$ arises from deformation theory of the super singular elliptic curve $C$. I will explain how this is used to compute the action of the automorphisms of $C$ on the coefficients $(E_C)_*$, results which are due to Strickland.

### 2.1. Deformations.
Let $k$ be a perfect field of characteristic $p > 0$, and $\Gamma$ be a formal group law of height $n$ over $k$. Let $R$ be the category of complete Noetherian local rings with continuous homomorphisms. Let $B \in R$ with maximal ideal $m$ and projection $\pi : B \to B/m$. Then $\text{Def}_\pi(B)$ is the groupoid whose objects are pairs $(G, i)$ where $G$ is a formal group law over $B$ and $i$ is an isomorphism

$$i : k \to B/m$$

such that

$$\pi_* G \cong i_* \Gamma.$$
A morphism between two pairs \((G_1, i)\) and \((G_2, i)\) with the same structure morphism \(i\) is an isomorphism \(f : G_1 \to G_2\) of formal group laws such that \(\pi_* f\) induces the identity on \(i_* \Gamma\). These are called \(\ast\)-isomorphisms. This defines a functor

\[ \text{Def}_\Gamma(-) : \mathcal{R} \to \mathcal{G} \]

where \(\mathcal{G}\) denotes the category of groupoids. The Lubin-Tate theorem describes the representability of this functor.

**Theorem 2.1** (Lubin, Tate). There exists a complete local ring \(R(k, \Gamma)\) and a formal group law \(F_R\) over \(R(k, \Gamma)\) that represents the functor \(\pi_0 \text{Def}_\Gamma(-)\) in the following sense. For \(B\) in \(\mathcal{R}\),

\[ \text{Hom}_\mathcal{R}(R(k, \Gamma), B) \cong \pi_0(\text{Def}_\Gamma(B)). \]

Given a representative \((G, i)\) of a \(\ast\)-isomorphism class in \(\pi_0 \text{Def}_\Gamma(B)\), there is a unique ring homomorphism \(\phi : R(k, \Gamma) \to B\) and a unique \(\ast\)-isomorphism

\[ f : \phi^* F_R \to G. \]

Further, if \(W(k)\) denotes the Witt vectors on \(k\), then

\[ R(k, \Gamma) \cong W(k)[[u_1, \ldots, u_n]]. \]

Let \(u\) be in degree \(-2\). Then

\[ F_E := uF_R(u^{-1}x, u^{-1}y) \]

defines a graded formal group law over

\[ E(k, \Gamma)_* := R(k, \Gamma)[u^\pm 1]. \]

This gives \(E(k, \Gamma)_*\) the structure of a Landweber exact \(MU_*\)-module (see, for example, [14, §6]). The associated homology theory is complex oriented and two periodic. It is represented by a ring spectrum \(E(k, \Gamma)\) such that

\[ E(k, \Gamma)_* \simeq W(k)[[u_1, \ldots, u_n]][u^\pm 1]. \]

By the Goerss-Hopkins-Miller theorem, \(E(k, \Gamma)\) is an \(E_\infty\)-ring spectrum (see [8]).

**Definition 2.2.** Let \(E_C = E(\mathbb{F}_4, F_C)\), where \(F_C\) is the formal group law of the super-singular elliptic curve \(C\) defined in (1.5).

2.2. **The action of** \(\text{Aut}(\Gamma)\). The group \(\text{Aut}(\Gamma)\) acts on \(R(k, \Gamma)\) as follows. An element \(\gamma \in \text{Aut}(\Gamma)\) is a power series in \(k[[x]]\). Let \(g\) in \(R(k, \Gamma)[[x]]\) be a lift of \(\gamma\). Define a new formal group law by

\[ F_g(x, y) = g^{-1}F_R(g(x), g(y)). \]

Then \(F_g\) is a deformation of \(\Gamma\) over \(R(k, \Gamma)\). By Theorem 2.1, there exists a unique ring isomorphism

\[ \phi_\gamma : R(k, \Gamma) \to R(k, \Gamma) \]

(2.3)

and a unique \(\ast\)-isomorphism

\[ f_g : (\phi_\gamma)_* F_R \to F_g \]

which classify \(F_g\). If \(h\) is another lift of \(\gamma\), then

\[ h^{-1}gf_g : (\phi_\gamma)_* F_R \to F_h \]

is a \(\ast\)-isomorphism. Therefore, \(\phi_\gamma\) is independent of the choice of lift \(g\). This gives an action of \(\text{Aut}(\Gamma)\) on \(R(k, \Gamma)\).
To extend this to an action on $E(k, \Gamma)\ast$, let $f_\gamma$ be the composite

$$(\phi_\gamma)_* F_R \xrightarrow{f_\gamma} F_g \xrightarrow{g} F_R.$$ 

Define

$$(2.4) \quad \phi_\gamma(u) := f_\gamma(0)u.$$ 

This is extends the action $\text{Aut}(\Gamma)$ to $E(k, \Gamma)\ast$ (see [14, §6]). By the Goerss-Hopkins-Miller theorem, this action can be realized through maps of $E_\infty$-ring spectra on $E(k, \Gamma)$ (see [8]). Further, $\text{Gal}(k/\mathbb{F}_p)$ acts on $W(k)$; hence, it acts on the coefficients $E(k, \Gamma)\ast$. If $\Gamma$ is fixed by $\text{Gal}(k/\mathbb{F}_p)$, this extends the action of $\text{Aut}(\Gamma)$ to an action of

$$\text{Aut}(\Gamma) \rtimes \text{Gal}(k/\mathbb{F}_p).$$

2.3. The super-singular elliptic curve. Elliptic curves over fields of characteristic $p > 0$ admit a theory of deformations which is analogous to that of formal group laws. In fact, for the super-singular elliptic curve, there is an equivalence of groupoids

$$(2.5) \quad \text{Def}_{\hat{E}}(B) \simeq \text{Def}_C(B),$$

which can be explained as follows.

Let $\hat{C}$ be the formal group of the elliptic curve $C$. The map which sends a deformation of $F_C$ to its associated formal group is an equivalence of groupoids

$$(2.6) \quad \text{Def}_{F_C}(B) \simeq \text{Def}_{\hat{C}}(B).$$

For super-singular curves over fields of characteristic $p > 0$, there is an isomorphism

$$(2.7) \quad \hat{C} \cong C[p^\infty],$$

where $C[p^\infty]$ denotes the $p$-divisible subgroup of $C$. Finally, the Serre-Tate Theorem relates the deformations of $C[p^\infty]$ to the deformations of the curve $C$ (see [15, §2.9]).

**Theorem 2.8** (Serre,Tate). Let $B$ be in $\mathcal{R}$. There is an equivalence of groupoids

$$\text{Def}_C(B) \simeq \text{Def}_{C[p^\infty]}(B),$$

which sends a deformation $E/B$ of $C$ to its $p$-divisible group $E[p^\infty]$.

Theorem 2.8 together with (2.6) and (2.7) imply (2.5). For the super-singular curve

$$C : y^2 - y = x^3,$$

these facts are made concrete by the following theorem.

**Theorem 2.9.** The formal group law of the elliptic curve

$$C_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

defined over $W[[u_1]]$ is a universal deformation of $F_C$. This specifies an isomorphism

$$(E_C)_* \cong W[[u_1]][u^\pm 1],$$

and a formal group law

$$(2.10) \quad F_{E_C} = uF_{C_U}(u^{-1}x, u^{-1}y),$$

where $F_{C_U}$ denotes the formal group law of the curve $C_U$. 

This choice of universal deformation $C_U$ is due to Strickland. In order to prove Theorem 2.9, we will use the following facts about isomorphisms of Weierstrass curves. Let $E$ be an elliptic curve defined over $R$ by a Weierstrass equation
\begin{equation}
E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.
\end{equation}
If $E'$ is a Weierstrass curve with coefficients $a'_i$, an isomorphism $f : E \to E'$ is given by a change of coordinates of the form
\begin{equation}
(x, y) = \left(l^2 x' + r, l^3 y' + l^2 sx' + t\right),
\end{equation}
where $(l, r, s, t)$ is a tuple in $R$ and $l$ is a unit (see [18, §III]). This forces the following relations on $(l, r, s, t)$:
\begin{align*}
l a'_1 &= a_1 + 2s \\
l^2 a'_2 &= a_2 - sa_1 + 3r - s^2 \\
l^3 a'_3 &= a_3 + ra_1 + 2t \\
l^4 a'_4 &= a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st \\
l^6 a'_6 &= a_6 + ra_4 + r^2 a_2 + r^3 - ta_3 - t^2 - rta_1.
\end{align*}

**Proof of Theorem 2.9.** The equivalence (2.5) implies that there exists a ring $R$ and an elliptic curve $C_R$ whose formal group law $F_{C_R}$ is a universal deformation of $F_C$. By the Lubin-Tate theorem, there is a non-canonical isomorphism $R \cong \mathbb{W}[[u_1]]$. The curve $C_U$ is a deformation of $C$, hence there is a ring homomorphism
\begin{equation}
\phi : R \to \mathbb{W}[[u_1]]
\end{equation}
such that
\begin{equation}
\phi^* C_R \cong C_U.
\end{equation}
It is sufficient to show that $\phi$ is an isomorphism. For a complete local ring $B$ with maximal ideal $m_B$, the tangent space of $B$ is defined by
\begin{align*}
\tau_B &= m_B / m_B^2. \\
A \text{ morphism of power series rings is an isomorphism if and only if it induces an isomorphism on tangent spaces. Let } \tau_R \text{ and } \tau_W \text{ denote the tangent spaces of } R \text{ and } \mathbb{W}[[u_1]] \text{ respectively. Then } \tau_R \text{ and } \tau_W \text{ are } \mathbb{F}_4 \text{ vector spaces of the same dimension and, therefore, the induced map }
\tau_\phi : \tau_R \to \tau_W,
\end{align*}
is an isomorphism if and only if it is surjective. The curve $C_R$ is given by a Weierstrass equation
\begin{equation}
y^2 + a_1 xy + a_3 y^2 = x^3 + a_2 x^2 + a_4 x + a_6,
\end{equation}
and the coefficients of the Weierstrass equation of $\phi^* C_R$ are $\phi(a_i)$. The isomorphism (2.13) is given by a change of coordinates of the form (2.12), where $(l, r, s, t)$ are elements of $\mathbb{W}[[u_1]]$ and $l$ is a unit. This imposes the relation
\begin{equation}
\phi^*(a_1) = l^{-1}(3u_1 + 2s).
\end{equation}
Hence
\begin{equation}
\phi^*(a_1) \equiv u_1 + 2s \mod m_W^2,
\end{equation}
Towards $\pi_* L_{K(2)} V(0)$ at $p = 2$

where $\pi \in \mathbb{F}_4$. But $\mathbb{F}_4$ is in the image of the induced map

$$\tau_\phi : \tau_R \to \tau_W,$$

which implies that $\tau_\phi$ is surjective. \qed

2.4. The automorphisms of $C$. In unpublished notes, Strickland has computed the action of the group $\text{Aut}(C)$ on $(E_C)_0$. We explain his results in this section.

The relations on the coefficients $a_i$ for an isomorphism of Weierstrass curves $E$ over $R$ can be used to compute the automorphisms of $E$. For the super-singular curve $C$, this is done in [18, Appendix A]. I give the results here.

Fix a primitive third root of unity $\zeta \in \mathbb{F}_4$. For the curve $C$ over $\mathbb{F}_4$, the group $\text{Aut}(C)$ is generated by the elements

$$\omega := (\zeta^2, 0, 0, 0),$$
$$i := (1, 1, 1, \zeta),$$

The subgroup $C_3 := \langle \omega \rangle$ is cyclic of order three, so that $\omega^{-1} = \omega^2$. The element $i$ satisfies $i^2 = -1$. Let

$$j := \omega i \omega^2,$$
$$k := \omega^2 i \omega.$$

Then $ij = k$, so that $i$ and $j$ generate a normal subgroup isomorphic to the quaternions $Q_8$ and

$$\text{Aut}(C) \cong Q_8 \rtimes C_3.$$

An automorphism of $C$ induces an automorphism of $F_C$. This gives a map

$$\rho : \text{Aut}(C) \to \text{Aut}(F_C),$$

which Tate has shown is injective (see [3]). Define

$$G_{24} := \rho(\text{Aut}(C)).$$

Let $\gamma$ be in $G_{24}$ and $\phi_{\gamma}$ be as in (2.3). The curve $\phi_{\gamma}^* C_U$ is a deformation of $C$, so there exists a unique isomorphism of elliptic curves

$$f_{\gamma} : \phi_{\gamma}^* C_U \to C_U,$$

which covers $\gamma$. Strickland has constructed $\phi_{\omega}$ and $\phi_i$ by using level three structures on the curves $C$ and $C_U$. First, he constructs isomorphisms $\phi_{\omega}$ and $\phi_i$ of $(E_C)_0$ given by

$$\phi_{\omega}(\zeta) = \zeta \quad \phi_i(\zeta) = \zeta$$
$$\phi_{\omega}(u_1) = \zeta u_1 \quad \phi_i(u_1) = \frac{u_1 + 2}{u_1 - 1}.$$

Let $a_i'$ be the coefficients of $C_U$ and $a_i = \phi_{\gamma}(a_i')$ the coefficients of $\phi_{\gamma}^* C_U$. The relations on the $a_i'$s determine the tuples $(l, r, s, t)$:

$$f_\omega = (\zeta^2, 0, 0, 0)$$
$$f_i = \left(\frac{\zeta^2 - \zeta}{u_1 - 1}, \frac{1 - u_1^2}{u_1 - 1} - \frac{\zeta^2 u_1 - 1}{u_1 - 1}, \frac{u_3 - 1}{u_1 - 1} - \frac{3}{4}((1 - \zeta) + (1 - \zeta^2)u_1)\right).$$

Note that $\zeta^2 - \zeta$ is a square root of $-3$. This choice is unique up to the action of the Galois group, which preserves $C$. The maps $f_\omega$ and $f_i$ lift $\omega$ and $i$ and the isomorphisms $\phi_{\omega}$ and $\phi_i$ generate the action of $G_{24}$ on $(E_2)_0$. 

Finally, if $F_{\phi^*C_U}$ denotes the formal group law associated to the curve $\phi^*C_U$, then

$$F_{\phi^*C_U} = \phi^*F_{C_U}$$

and the induced isomorphism on formal group laws satisfies $f'_\gamma(0) = l$. By (2.4),

$$(2.14) \quad \phi_\omega(u) = \zeta u \quad \phi_\gamma(u) = u \frac{\zeta^2 - \zeta}{u_1 - 1}.$$  

3. The Morava stabilizer group

The Morava stabilizer group $S_2$ is the group of automorphisms of the Honda formal group law $F_2$, which is the $p$-typical formal group law over $\mathbb{F}_4$ specified by the series


The standard presentation for $S_2$ is the non-commutative extension

$$S_2 \cong \left( \mathbb{W} \langle S \rangle / (S^2 = 2, aS = Sa^\sigma) \right)^\times,$$

where $S$ is the automorphism $S(x) = x^2$ and $a \in \mathbb{W}$ (see [13, Appendix A2] or [2] for more details.) In this section, I will specify an isomorphism $S_2 \cong S_C$, whose construction I owe to Henn. I will also recall some of the key properties of the structure of the group $S_2$, which transfer to properties of $S_C$ via this isomorphism.

3.1. The isomorphism of $S_2$ and $S_C$. As opposed to the Honda formal group law, it is the $[-2]$-series of the formal group law $F_C$ which has a nice presentation. The following result is proved in Proposition 8.11 of Section 8.

**Lemma 3.1.** Let $C$ be the super-singular elliptic curve defined by (1.5). If $F_C$ is the associated formal group law, then

$$[-2]F_C(x) = x^4.$$  

The curve $C$ and its formal group law $F_C$ are defined over $\mathbb{F}_2$. Therefore,

$$T(x) = x^2$$

is an endomorphism of $F_C$. Lemma 3.1 implies that $T(T(x)) = [-2](x)$. The element $\omega$ defined in Section 2.4 induces the isomorphism

$$\omega(x) = \zeta x$$

of $F_C$, so that $\omega T = T \omega^\sigma$. This shows that

$$\mathbb{W} \langle T \rangle / (T^2 = -2, \omega T = T \omega^\sigma) \subseteq \text{End}(F_C).$$

By Proposition 21.8.7 of [10], this must be an equality. Therefore,

$$S_C \cong \left( \mathbb{W} \langle T \rangle / (T^2 = -2, \omega T = T \omega^\sigma) \right)^\times.$$  

Let $\sigma$ be the Fröhbenius element in $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. The action of Gal on $\mathbb{W}$ induces an action on $S_C$ defined by

$$a + bT \mapsto a^\sigma + b^\sigma T.$$  

In [2], we constructed an element $\alpha$ in $\mathbb{W}^\times$ defined as

$$\alpha := \frac{1 - 2\omega}{\sqrt{-1}},$$

where $\omega$ is the generator of $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$.
TOWARDS $\pi_* L_K(2) V(0)$ AT $p = 2$

so that $\alpha = 1 + \omega T^2 + T^4 + \ldots$ and 

$$\alpha\alpha^\sigma = -1.$$ 

Theorem 3.2. The groups $\mathbb{S}_2$ and $\mathbb{S}_C$ are isomorphic.

Proof. Each element $\gamma \in \mathbb{S}_C$ can be expressed uniquely as $a + bT$ for $a$ and $b$ in $\mathbb{W}$ and $a$ a unit. On the other hand, the elements of $\mathbb{S}_2$ admit a similar representation as $a + bS$. The map $\mathbb{S}_C \to \mathbb{S}_2$,

$$a + bT \mapsto a + b(\alpha S),$$

is an isomorphism. \(\square\)

3.2. The filtration and the norm. Theorem 3.2 implies that all the results of [2] can be restated for the group $\mathbb{S}_C$ instead of $\mathbb{S}_2$. Here, I briefly review those results which will be important for the computations of this paper.

As in [2], any element $\gamma \in \mathbb{S}_C$ can be expressed as a power series $\gamma = \sum_{n=0}^{\infty} a_n T^n$, where the $a_i$'s satisfy the equation $x^4 - x = 0$ and $a_0 \neq 0$. Let $F_{n/2}\mathbb{S}_C := \mathbb{S}_C$. For $n > 0$, let

$$F_{n/2}\mathbb{S}_C := \{ \gamma \in \mathbb{S}_C \mid \gamma \equiv 1 \mod T^n \}.$$ 

Define

$$\mathbb{S}_C := F_{1/2}\mathbb{S}_C.$$ 

Then $\mathbb{S}_C$ is the 2-Sylow subgroup of $\mathbb{S}_C$. This filtration is compatible with the 2-adic filtration on $\mathbb{W}^\times$. Further, $\{F_{n/2}\mathbb{S}_C\}_{n \geq 0}$ forms a system of open subgroups and $\mathbb{S}_C$ is a profinite topological group.

Recall the following result follows from Theorem 2.29 of [2].

Proposition 3.4. The subgroup generated by $G_{24}$, $\pi$ and $\alpha$ is dense in $\mathbb{S}_C$.

The group $\mathbb{S}_C$ acts on $\text{End}(F_C)$ by right multiplication. This gives rise to a representation $\rho : \mathbb{S}_C \to GL_2(\mathbb{W})$, given by

$$\rho(a + bT) = \begin{pmatrix} a & b \\ -2b^\sigma & a^\sigma \end{pmatrix}.$$ 

The restriction of the determinant to $\mathbb{S}_C$ is given by

$$\det(a + bT) = aa^\sigma + 2bb^\sigma.$$ 

Therefore, the determinant induces a map $\det : \mathbb{S}_C \to \mathbb{Z}_2^\times$. The norm is defined as the composite

$$N : \mathbb{S}_C \xrightarrow{\det} \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2.$$ 

The norm is split surjective. Indeed, let

$$\pi = 1 + 2\omega.$$ 

Then $\det(\pi) = 3$ projects to a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$. The subgroup $\mathbb{S}_C^1$ is then defined by the short exact sequence,

$$1 \to \mathbb{S}_C^1 \to \mathbb{S}_C \xrightarrow{N} \mathbb{Z}_2^\times/\{\pm 1\} \to 1,$$
and

\[ S_C \cong S_C^1 \times \mathbb{Z}_2^2 / \{ \pm 1 \} . \]

Note that \( \mathbb{Z}_2^2 / \{ \pm 1 \} \cong \mathbb{Z}_2 \) is torsion-free; hence, \( G_{24} \) is a subgroup of \( S_C^1 \).

The following result was shown in Lemma 2.27 of [2], based on results of [4].

**Proposition 3.5.** The group \( S_C \) contains a unique conjugacy class of maximal finite subgroups isomorphic to \( G_{24} \). Further, \( S_C^1 \) contains two conjugacy classes of maximal finite subgroups, represented by \( G_{24} \) and \( G'_{24} = \pi G_{24} \pi^{-1} \).

### 3.3. The action of the Morava stabilizer group

In order to compute the cohomology of \( S_C \), it is necessary to understand its action on \((E_C)_*\). The action of the elements of \( G_{24} \) was computed by Strickland, and his results were explained in Section 2.4. By Proposition 3.4, it thus suffices to understand the action of \( \alpha \) and \( \pi \) on \((E_C)_*\), to approximate the action of any element of \( S_C \) on \((E_C)_*\).

A concrete method for approximating the action of \( S_C \) on \((E_C)_*\) was developed in [11]. We describe it in Section 8. We state the key results here and prove them there.

**Theorem 3.6.** For \( \gamma \in S_C \), there exists \( t_0(\gamma) \) in \((E_C)^0\) and \( t_1(\gamma) \) in \((E_C)_0\) such that

\[
\phi_\gamma(u) = t_0(\gamma)u, \\
\phi_\gamma(u_1) = t_0(\gamma)u_1 + \frac{2}{3} t_1(\gamma) .
\]

In particular, modulo (2),

\[
\phi_\gamma(u_1) \equiv t_0(\gamma)u_1, \\
\phi_\gamma(u) \equiv t_0(\gamma)u.
\]

Therefore, \( v_1 = u_1u^{-1} \) is fixed by the action of \( S_C \) modulo (2).

**Theorem 3.7.** Let \( \gamma = 1 + \sum_{i=1}^\infty a_i T^i \) be in \( F_1/2S_C \). Then

\[ t_0(\gamma) \equiv 1 \mod (2,v_1), \]

so that \( \phi_\gamma \equiv id \mod (2,v_1) \). For \( \gamma = 1 + a_2 T^2 + \ldots \) in \( F_2/2S_C \), modulo \((4,2v_1^2,v_1^0)\),

\[ t_0(\gamma) \equiv 1 + 2a_2 + 2a_3^2u_1 + (a_2 + a_3^2)u_1^3 + a_3u_1^5 + a_3u_1^8 + (a_2 + a_3^2 + a_4 + a_4^2)u_1^9. \]

and

\[ t_1(\gamma) \equiv a_2^2u_1 \mod (2,v_1^3). \]

If \( \gamma = 1 + a_4 T^4 + \ldots \) is in \( F_4/2S_C \), then

\[ \phi_\gamma \equiv id \mod (2,v_1^2). \]

We will also use the following result. Recall that the action of \([-1]_{E_C}(x) \in S_C \) and \( \omega \in C_3 \) is given by

\[
\phi_{-1}(u_1) = u_1, \quad \phi_\omega(u_1) = \zeta u_1, \\
\phi_{-1}(u) = -u, \quad \phi_\omega(u) = \zeta u,
\]

where \( \zeta \) is a primitive third root of unity. Hence,

\[
(E_C)^0_0 = \mathbb{W}[u_1]^C_0 = \mathbb{W}[u_1]^C_0 = \mathbb{W}[u_1^3].
\]
Lemma 3.8. Let $\gamma$ in $S_C$ be an element which commutes with $\omega$ in $C_3$. Then
t_0(\gamma) \in W[[u_1^2]].

Proof. Since $\gamma \omega = \omega \gamma$, we have
$$\phi \gamma \circ \phi \omega (u) = \phi \omega \circ \phi \gamma (u).$$
By (2.14), $\phi \omega (u) = \zeta u$. This forces
$$\phi \omega (t_0(\gamma)) = t_0(\gamma).$$
Therefore, $t_0(\gamma)$ is in $W[[u_1]]C_3$, where the action of $C_3$ is the $W$-linear map determined by $\phi \omega (u_1) = \zeta u_1$. This implies that $t_0(\gamma) \in W[[u_1^2]]$. \hfill $\square$

We now apply these results to study the action of
$$\alpha \equiv 1 + \omega T^2 \mod T^4.$$

Lemma 3.9. The unit $t_0(\alpha)$ is an element of $(E_C)^{C_3}_0$. For $\epsilon_0$ and $\epsilon$ in $(E_C)_0 V(0)^{C_6}$,
$$t_0(\alpha) \equiv 1 + u_1^3 + \epsilon_0 u_1^6 \mod (2),$$
and for $v_2 = u^{-3}$
$$\phi \alpha (v_2) = v_2 + v_1^3 + v_2^{-1} v_1^6 \epsilon \mod (2).$$
Further
$$\phi \alpha \equiv \phi \alpha^{-1} \mod (2, v_1^3).$$

Proof. The element $\alpha$ is in $W$. Therefore, it commutes with $\omega$. Lemma 3.8 implies that $t_0(\gamma)$ is in $(E_C)_0 V(0)^{C_6} = W[[u_1^2]]$. The identities for $t_0(\alpha)$ and $\phi \alpha (v_2)$ follows from Theorem 3.7, using the fact that, for $\alpha$, the coefficient $a_2 = \omega$ and $a_3 = 0$. Finally, since $\alpha^2 \in F_{2/2}S_C$, it follows from Theorem 3.7 that $\phi \alpha^2$ is the identity modulo $(2, v_1^3)$. Then, the claim follows from the fact that $\phi \alpha^{-1} = \phi \alpha$ and that $\phi \alpha^2 = \phi \alpha \circ \phi \alpha$. \hfill $\square$

Lemma 3.10. Let $\pi = 1 + 2 \omega$. Then
$$\phi \pi \equiv \text{id} \mod (2, u_1^3).$$

Proof. This follows from Theorem 3.7 since $\pi \in F_{2/2}S_C$ and, for $\pi$, $a_2 = \omega$. \hfill $\square$

4. The algebraic duality resolution spectral sequence

4.1. Preliminaries. The results in the following theorem were shown in [2] for the group $S^3_2$. I restate them here for the group $S^3_1$. The construction of the resolution is due to Goerss, Henn, Mahowald and Rezk. The descriptions of the maps $\partial_1$ and $\partial_2$ are due to the author and Henn.

Theorem 4.1. Let $Z_2$ be the trivial $S^3_1$-module. There is an exact sequence of complete $S^1_C$-modules
$$0 \to \mathcal{C}_1^0_2 \to \mathcal{C}_2^0_2 \to \mathcal{C}_3^0_2 \to \mathcal{C}_0^0_2 \to Z_2 \to 0,$$
where $\mathcal{C}_0^0_2 \cong Z_2[[S^1_C/G_{24}]]$, $\mathcal{C}_3^0_2 \cong Z_2[[S^1_C/G'_{24}]]$ and $\mathcal{C}_1^0_2 \cong \mathcal{C}_2^0_2 \cong Z_2[[S^1_C/C_6]]$. Let $e$ be the unit in $Z_2[[S^1_C]]$ and $e_p$ be the resulting generator of $\mathcal{C}_p$. The maps $\partial_p$ can be chosen to satisfy:

(i) $\partial_1(e_1) = (e - \alpha)e_0$,
(ii) \( \partial_2(e_2) = \Theta e_1 \) for \( \Theta \in \mathbb{Z}_2[[S^1_c]] \) such that
\[
\Theta \equiv e + \alpha \mod (2, (IS^1_c)_2).
\]

Further, there are isomorphisms of \( S^1_c \)-modules \( g_p : \mathcal{C}_p \to \mathcal{C}_p \) and an exact sequence
\[
0 \to \mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}_0 \to \mathbb{Z}_2 \to 0
\]
such that
\[
\partial'_3(e_3) = \pi(e + i + j + k)(e - \alpha^{-1})^{-1} e_2.
\]

The duality resolution gives rise to a spectral sequence which computes the cohomology of \( S^1_c \). Indeed, let \( M \) be a finitely generated complete \( S^1_c \)-module. There is a first quadrant spectral sequence,
\[
(4.2) \quad E^{p,q}_1 = \text{Ext}^{q}_{\mathbb{Z}_2[[S^1_c]]}(\mathcal{C}_p, M) \Rightarrow H^{p+q}(S^1_c, M).
\]

The differentials in (4.2) have degree
\[
d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r,
\]
and
\[
E^{p,q}_1 \equiv \begin{cases} 
H^q(G_{24}, M) & \text{if } p = 0; \\
H^q(C_6, M) & \text{if } p = 1, 2; \\
H^q(G'_24, M) & \text{if } p = 3.
\end{cases}
\]

4.2. Some extra structure. In our computation, we will need to use some additional structure in the algebraic duality resolution. We record that here. For any complete \( S^1_c \)-modules \( A \) and \( B \), let
\[
\text{Ext}(A, B) := \text{Ext}_{\mathbb{Z}_2[[S^1_c]]}(A, B).
\]
If \( B \) is an \( S^1_c \)-module which is free over the 2-adics \( \mathbb{Z}_2 \), then \( B/2 \) is defined by
\[
(4.3) \quad 0 \to B \xrightarrow{2} B \to B/2 \to 0.
\]
Let \( \beta : \text{Ext}(A, B) \to \text{Ext}(A, B/2) \) be the Bockstein homomorphism, that is, the reduction modulo 2 of the connecting homomorphism of the long exact sequence for (4.3). The algebraic duality resolution
\[
0 \to \mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}_0 \to \mathbb{Z}_2 \to 0
\]
is obtained from splicing exact sequences
\[
(4.4) \quad 0 \to N_p \to \mathcal{C}_p \to N_{p-1} \to 0.
\]
with $\mathcal{C}_3 = N_2$ and $N_{-1} = \mathbb{Z}_2$ (see [2]). The exact couple
\begin{equation}
\begin{array}{c}
\text{Ext}(N_\ast, B/2) \\
\text{Ext}(N_{-1}, B/2) \\
\end{array}
\begin{array}{c}
\delta_\ast \\
r_\ast \\
\end{array}
\begin{array}{c}
\text{Ext}(\mathcal{C}_\ast, B/2) \\
\end{array}
\end{equation}

(4.5)

gives rise to the algebraic duality resolution spectral sequence. Here, the dotted arrows are the connecting homomorphisms for the exact sequences (4.4), thus they increase the cohomological degree.

**Lemma 4.6.** Let $x \in E_\ast \mathcal{C}_\ast$ in the algebraic duality resolution spectral sequence. Then $\beta(x) \in E_{\ast-1} \mathcal{C}_\ast$ and $\delta_\ast(\beta(x)) = \beta(d_\ast(x)).$

**Proof.** The maps $d_\ast, r_\ast, i_\ast$ and $\delta_\ast$ in the exact couple (4.5) commute with $\beta$. A diagram chase shows that $d_\ast(\beta(x)) = \beta(d_\ast(x))$. \hfill \Box

**Lemma 4.7.** Let $R$ be an $S_1 \mathcal{C}_\ast$-module which is also a ring. Suppose that the action of $S_1 \mathcal{C}_\ast$ is given by ring homomorphisms. The algebraic duality resolution with coefficients $R$ is a module over the cohomology $H^\ast(S_1 \mathcal{C}_\ast; R)$.

**Proof.** Note that $\text{Ext}(A, R)$ is a module over $\text{Ext}(\mathbb{Z}_2, R)$ for any $S_1 \mathcal{C}_\ast$-module $A$. Further, the maps in the algebraic duality resolution are maps of $\mathbb{Z}_2$-modules. Therefore, the maps $d_\ast, r_\ast, i_\ast$ and $\delta_\ast$ in the exact couple giving rise to the algebraic duality resolution are morphisms of $\text{Ext}(\mathbb{Z}_2, R)$-modules. This implies that the differentials in the algebraic duality resolution are linear over $\text{Ext}(\mathbb{Z}_2, R)$. \hfill \Box

Recall that $(E_\mathcal{C}_\ast, V(0)) = (E_\mathcal{C}_\ast)/(2) = \mathbb{F}_4[[u_1]][u_1^{\pm 1}]$. In Theorem 2.9, it was shown that
\[ F_{E_\mathcal{C}_\ast} = u F_{C_\mathcal{C}_\ast}(u^{-1}x, u^{-1}y), \]
where $C_\mathcal{C}_\ast$ was defined by
\[ C_\mathcal{C}_\ast : y^2 + 3u_1 xy + (u_1^3 - 1)y = x^3. \]
It follows from [18, §IV.1] that
\[ [2] F_{E_\mathcal{C}_\ast}(x) = u^{-1}x u_1 x^2 + u^{-3}(u_1^4 + 1)x^4 + \ldots \mod (2). \]
Therefore, we can define
\[ v_1 := u^{-1}u_1 \]
and
\[ v_2 := u^{-3}. \]
Note that the element $v_1$ is uniquely determined modulo (2), but $v_2$ is only defined modulo $(2, v_1)$. The element $v_1$ is invariant under the action of $S_1 \mathcal{C}_\ast$ on $(E_\mathcal{C}_\ast, V(0))$, and $v_1$ is an element of $H^0(S_1 \mathcal{C}_\ast, (E_\mathcal{C}_\ast, V(0)))$. However, it does not lift to an invariant in $(E_\mathcal{C}_\ast)_\ast$. Therefore, $v_1$ has a non-zero Bockstein. Define
\begin{equation}
\begin{array}{c}
h_1 := \beta(v_1). \\
\end{array}
\end{equation}

**Lemma 4.9.** The algebraic duality resolution spectral sequence is a spectral sequence of modules over $\mathbb{F}_4[v_1, h_1]$. 

(4.8)
Proof. By Lemma 4.7, the duality resolution is a module over $H^*(S^1_+(E_C), V(0))$ and $H^*(S^1_+(E_C), V(0))$ is a module over $\mathbb{F}_4[v_1, h_1]$. 

4.3. The $E_1$-term. I will now describe the $E_1$-term of the algebraic duality spectral sequence. If $p: \mathcal{W} \to \mathbb{F}_4$ is the projection, then $p^*C_U$ is defined over $(E_C), V(0)$ and classifies deformations of $C$ to complete local $\mathbb{F}_4$-algebras. Let $E$ be a Weierstrass curve with coefficients $C$ and classifies deformations of $E$ in terms of $\omega_f$ be a generator for the module of invariant differentials on $E$. If $E$ is defined over an $\mathbb{F}_4$-algebra, then $a_1\omega_f$ is the Hasse invariant of $E$. For $p^*C_U$, the element $u^{-1} \in (E_C)_{2} V(0)$ generates the invariant differentials (see [9, §2.2]), and therefore the Hasse invariant is precisely

$$v_1 \equiv 3u_1u^{-1} \mod (2).$$

Further, the discriminant of $C_U$ is $\Delta_{C_U} = 27(u_3^3 - 1)^3$, so that

$$(4.10) \quad \Delta := u^{-1}27(u_3^3 - 1)^3 \equiv v_2(v_2 + v_3^3) \mod (2)$$

is invariant under the action of Aut($C$) (see [18, §III.1]). Finally, the element $j = v_1^2\Delta^{-1}$ is the $j$-invariant, not to be confused with the quaternion element. Theorem 4.11 follows from [9, §4.4]. Its computational content is originally due to Hopkins and Mahowald, but the best reference is Bauer, [1, §7]. We have used the notation of [1] in the statement below.

![Figure 4.1](image-url)

**Figure 4.1.** The cohomology $H^*(G_{24}, (E_C), V(0))$, drawn in the Adams grading $(t-s, s)$. It is periodic of period $t = 24$ with respect to the element $\Delta$. It is periodic of period $4$ with respect to the element $g$. $\Delta$ denotes a copy of $\mathbb{F}_4$. Lines of slope $1$ denote multiplication by $h_1$ and lines of slope $1/3$ denote multiplication by $h_2$. Horizontal lines denote multiplication by $v_1$. Classes attached to horizontal arrows are free over $\mathbb{F}_4[v_1]$.

**Theorem 4.11.** There is an isomorphism

$$H^*(G_{24}, (E_C), V(0)) \cong \mathbb{F}_4[[j]][v_1, \Delta\pm 1, h_1, h_2, x, y, g]/(\sim),$$

where the degrees $(s, t)$ (for $s$ the cohomological grading, and $t$ the internal grading) are given by

$$|v_1| = (0, 2), \quad |\Delta| = (0, 24), \quad |h_1| = (1, 2), \quad |h_2| = (1, 4)$$

$$|x| = (1, 8), \quad |y| = (1, 16), \quad |g| = (4, 24), \quad |j| = (0, 0),$$

and “$\sim$” denotes the following relations:

$$v_1 h_2 = 0, \quad v_1 y = 0, \quad v_1^2 x = 0, \quad h_1 h_2 = 0, \quad h_2^3 = h_1^2 x, \quad h_1 v_1 x = h_2 x \quad h_1^2 x^2 = 0, \quad h_1 y = v_1 x^2, \quad h_2^3 y = 0, \quad x^3 = 0, \quad v_1^2 g = h_1^4 \Delta, \quad \Delta^{-1} v_1^{12} = j.$$
Proof. Let \( \mathcal{M}_{\text{Weier}} \) denote the stack of Weierstrass curves and \( \omega \) be the canonical quasi-coherent sheaf of invariant 1-forms on \( \mathcal{M}_{\text{Weier}} \). Let \((A, \Gamma)\) be the Hopf algebroid classifying Weierstrass curves and their strict isomorphisms. Then

\[
H^*(\mathcal{M}_{\text{Weier}}, \omega^{\otimes *}) \cong H^*(A, \Gamma).
\]

The smooth locus of \( \mathcal{M}_{\text{sm}} \subseteq \mathcal{M}_{\text{Weier}} \) is given by the points where the determinant \( \Delta \) is invertible, hence

\[
H^*(\mathcal{M}_{\text{sm}}, \omega^{\otimes *}) \cong H^*(A, \Gamma)[\Delta^{-1}].
\]

Let \( \widehat{\mathcal{M}}_{\text{ss}} \) be the formal neighborhood of the super-singular locus in \( \mathcal{M}_{\text{sm}} \otimes \mathbb{F}_2 \). Then

\[
H^*(\widehat{\mathcal{M}}_{\text{ss}}, \omega^{\otimes *}) \cong (H^*(A \otimes \mathbb{F}_2, \Gamma \otimes \mathbb{F}_2)[\Delta^{-1}])_{(j)}^\wedge.
\]

Let \( G_{48} = G_{24} \times \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \). By Lubin-Tate theory,

\[
H^*(\widehat{\mathcal{M}}_{\text{ss}}, \omega^{\otimes *}) \cong H^*(G_{48}, (E_C)_*V(0)).
\]

The groups \( G_{48} \) and \( G_{24} \) differ by the action of the Galois group on the coefficients \( \mathbb{F}_4 \), so that

\[
H^*(G_{24}; (E_C)_*V(0)) \cong (H^*(A \otimes \mathbb{F}_2, \Gamma \otimes \mathbb{F}_2)[\Delta^{-1}])_{(j)}^\wedge \otimes_{\mathbb{F}_2} \mathbb{F}_4,
\]

where the completion is done degree-wise. Finally, in [1, §7], Bauer computes \( H^*(A \otimes \mathbb{F}_2, \Gamma \otimes \mathbb{F}_2) \). The result then follows from his computation. \( \square \)

The cohomology of \( C_6 \) can be computed using the formulas of Section 2.4.

**Lemma 4.12.** The cohomology of \( C_6 \) with coefficients in \( (E_C)_*V(0) \) is given by

\[
H^*(C_6; (E_C)_*V(0)) = \mathbb{F}_4[[u_1^3]][v_1, v_2^\pm 1, h]/(v_1^3 = v_2u_1^3),
\]

where \(|h| = (1,0), |v_2| = (0,6), |v_1| = (0, 2) \) and \(|u_1| = (0, 0) \). Further, the action of \( h_1 \) is determined by

\[
(4.13) \quad h_1 \cdot 1 = v_1 h.
\]

**Proof.** Recall that \( C_2 = \{\pm 1\} \) denotes the center of \( G_{24} \) and that \( C_6 = C_2 \times C_3 \). Because \( C_2 \) acts trivially on \((E_C)_*V(0)\),

\[
H^*(C_2; (E_C)_*V(0)) = ((E_C)_*V(0))[h],
\]

where \( h \) is in \((s,t)\) degree \((1, 0)\). The order of \( C_3 \) is coprime to \( 2 \), so that

\[
H^*(C_6; (E_C)_*V(0)) \cong H^*(C_2; (E_C)_*V(0))^{C_3}
= (E_C)_*V(0)^{C_3}[h]
= \mathbb{F}_4[[u_1^3]][v_1, v_2^\pm 1, h]/(v_1^3 = v_2u_1^3).
\]

To prove (4.13), first note that \( C_2 \) acts on \( (E_C)_*V = \mathbb{W}[u_1][u^\pm 1] \) by

\[
\phi_{-1}(u) = -u
\]

\[
\phi_{-1}(u_1) = u_1.
\]

This follows from (2.4) and the fact that \(-1\) fixes the curve \( C_U \). Using this and the standard resolution of the trivial \( C_2 \)-module \( \mathbb{Z}_2 \), one can compute that the Bockstein

\[
\beta : (E_C)_*V(0) \to (E_C)_*V(0)
\]
Figure 4.2. The $E_1$-term for the ADRSS with coefficients $(E_C)_*V(0)$. The rows represent $E_1^{p,*}$, where the top row corresponds to $p = 3$. The grading is given by $(t - q - p, q)$, where $t$ is the internal grading, so that $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ decreases the horizontal grading by 1. A ⋄ denotes a copy of $\mathbb{F}_4$. Dashed horizontal lines denote multiplication by $v_1$, and a ○ denotes a copy of $\mathbb{F}_4[v_1]$. A ★ is a copy of Figure 4.3.
induced by

\[ 0 \to (E_C)_* \xrightarrow{2} (E_C)_* \to (E_C)_* V(0) \to 0, \]

satisfies

\[ \beta(v_1) = v_1 h. \]

The identity (4.13) follows by naturality and the fact that \( \beta(v_1) = h_1 \) (see (4.8)). \( \square \)

**Figure 4.3.** The pattern \( \star \) in Figure 4.2.

**Lemma 4.14.** Let \( \pi = 1 + 2\omega \) in \( S_C \). Let \( G'_{24} = \pi G_{24} \pi^{-1} \). Let \( \phi_\pi : (E_C)_* \to (E_C)_* \) give the action of \( \pi \) on \( (E_C)_* \). Then \( \phi_\pi \) induces an \( F[4][v_1, h_1] \)-linear isomorphism

\[ H^*(G_{24}, (E_C)_* V(0)) \cong H^*(G'_{24}, (E_C)_* V(0)). \]

**Proof.** For \( M \) an \( S_C \)-module, define a map \( F_\pi : M \to M \) by \( m \mapsto \pi \cdot m \). Although this is not a morphism of \( S_C \)-modules, it induces a natural isomorphism

\[ F_\pi : (-)^{G_{24}} \to (-)^{G'_{24}}. \]

Indeed, for another \( S_C \)-module \( N \) and a morphism of \( S_C \)-modules \( f : M \to N \),

\[ (M)^{G_{24}} \xrightarrow{F_\pi} (M)^{G'_{24}} \quad \text{and} \quad (N)^{G_{24}} \xrightarrow{F_\pi} (N)^{G'_{24}} \]

is commutative. Therefore, \( F_\pi \) induces an isomorphism on the right derived functors, i.e., on group cohomology. For \( (E_C)_* V(0) \), the map \( F_\pi \) is induced by \( \phi_\pi \). The linearity follows from the fact that \( v_1 \) is invariant under the action of \( \pi \), and \( h_1 = \beta(v_1) \). \( \square \)

To avoid ambiguities, define \( \Delta' := \phi_\pi(\Delta) \) and \( j' := \phi_\pi(j) \).

**4.4. Approximate \( \Delta \)-linearity.** In this section, I explain some additional properties of the action of \( S_C \). These will be used in the computations of the differentials \( d_1 : E_1^{p,0} \to E_1^{p+1,0} \). Recall from (4.10) that there is an element \( \Delta \in (E_C)_*^{G_{24}} \) such that

\[ \Delta = 27v_2(v_1^3 + v_2)^3 \equiv v_2(v_2 + v_1^3)^3 \mod (2). \]

As in Theorem 4.11, we abuse notation by denoting

\[ \Delta = v_2(v_2 + v_1^3)^3 \]

in \( (E_C)_* V(0)^{G_{24}} \). The key observation in the computation of Section 5 is that the action of \( (IS_1^1)^2 \) is **approximately \( \Delta \)-linear**. The following theorem makes this precise.
Theorem 4.16. Let $x$ be in $(E_C)_*V(0)$. Let $\sum a_{g,h}(e-g)(e-h)$ be an element of $(IS_C^1)^2$, where $a_{g,h}$ is in $\mathbb{Z}_2[[S_C]]$. Then, modulo $(2, v_1^{1+3.2^{k+1}})$,

\begin{equation}
\sum a_{g,h}(id - \phi_g)(id - \phi_h)(x\Delta^{2^{k+1+2t}}) \equiv \sum a_{g,h}(id - \phi_g)(id - \phi_h)(x)\Delta^{2^{k+1+2t}}.
\end{equation}

Further,

\begin{equation}
\sum a_{g,h}(id - \phi_g)(id - \phi_h)(\Delta) \equiv 0 \mod (2, v_1^8).
\end{equation}

The next results are needed to prove Theorem 4.16.

Lemma 4.19. The action of $\alpha$ on $\Delta$ is given by

\begin{equation}
\phi_\alpha(\Delta) \equiv (1 + v_2^{-2}v_1^6) \mod (2, v_1^9).
\end{equation}

Proof. By (3.9)

\begin{equation}
\phi_\alpha(\Delta) \equiv (v_2 + v_1^3 + v_1^6\epsilon)(v_2 + v_1^6\epsilon)^3
\equiv v_2^3(v_2 + v_1^3) \mod (2, v_1^9),
\end{equation}

so that

\begin{equation}
\phi_\alpha(\Delta)\Delta^{-1} \equiv v_2^2(v_2 + v_1^3)^{-2}
\equiv 1 + v_2^{-2}v_1^6 \mod (2, v_1^9).
\end{equation}

Lemma 4.21. The group $G_{24}$ acts as the identity on $\phi_\alpha(\Delta)$ modulo $(2, v_1^8)$.

Proof. First note that $\Delta$ itself is fixed by $G_{24}$. The group $G_{24}$ is generated by $\omega$ and $i$. As $\alpha$ and $\omega$ commute,

\begin{equation}
\phi_\omega(\phi_\alpha(\Delta)) = \phi_\alpha(\phi_\omega(\Delta))
= \phi_\alpha(\Delta).
\end{equation}

Further, it follows from Strickland’s computations, which were described in Section 2.4, that, for $t_0(i)$ as in Theorem 3.6,

\begin{equation}
t_0(i) \equiv (1 + u_1)^{-1} \mod (2).
\end{equation}

Therefore,

\begin{equation}
\phi_i(v_2) = (1 + u_1)^3v_2.
\end{equation}

It follows that

\begin{equation}
\phi_i(\phi_\alpha(\Delta)) = \Delta(1 + v_2^{-2}(1 + u_1)^{-6}v_1^6)
= \Delta(1 + v_2^{-2}(1 + u_2v_1^2)^{-3}v_1^6)
\equiv \Delta(1 + v_2^{-2}v_1^6)
\equiv \phi_\alpha(\Delta) \mod (2, v_1^8).
\end{equation}

Lemma 4.22. Let $\gamma$ be in $F_{3/2}S_C$. Then $\gamma$ acts trivially on $\Delta$ and $\phi_\alpha(\Delta)$ modulo $(2, v_1^8)$. 

\[\square\]
Proof. First suppose that $\gamma$ is in $F_{3/2}S_C$. By Theorem 3.7, the action of $\gamma$ is trivial modulo $(2, v_1^8)$. This implies that

$$\phi_\gamma(v_2) \equiv v_2 + v_2^{-1}v_1^6\epsilon_1 \mod (2, v_1^8).$$

Hence, modulo $(2, v_1^8)$,

$$\phi_\gamma(\Delta) \equiv (v_2 + v_2^{-1}v_1^6\epsilon_1)(v_2 + v_1^3 + v_2^{-1}v_1^6\epsilon_1)^3$$
$$\equiv (v_2 + v_2^{-1}v_1^6\epsilon_1)((v_2 + v_1^3)^3 + (v_2 + v_1^3)^2v_2^{-1}v_1^6\epsilon_1)$$
$$\equiv (v_2 + v_2^{-1}v_1^6\epsilon_1)(v_2 + v_1^3)^3 + 2v_2v_1^6\epsilon_1$$
$$\equiv \Delta \mod (2, v_1^8).$$

Now suppose that $\gamma \in F_{3/2}S_C$. It was shown in [2] that $F_{3/2}S_C/F_{4/2}S_C$ is generated by $\alpha_i$ and $\alpha_j$, where

$$\alpha_\tau = \tau\alpha\tau^{-1}\alpha^{-1}.$$}

Thus, $\gamma = \alpha_i\gamma_0$, $\gamma = \alpha_j\gamma_0$ or $\gamma = \alpha_i\alpha_j\gamma_0$, where $\gamma_0 \in F_{4/2}S_C$. But $\alpha_j = \omega\alpha_i\omega^2$ and both $\omega$ and $\gamma_0$ fix $\Delta$ modulo $(2, v_1^8)$, so it is enough to verify the case when $\gamma = \alpha_i$. Using Lemma 3.9 and Lemma 4.21,

$$\phi_{\alpha_i}(\Delta) \equiv \phi_i \circ \phi_\alpha \circ \phi_{i^{-1}} \circ \phi_{\alpha^{-1}}(\Delta)$$
$$\equiv \phi_i \circ \phi_\alpha \circ \phi_{i^{-1}}(\phi_\alpha(\Delta))$$
$$\equiv \phi_i \circ \phi_{\alpha^2}(\Delta)$$
$$\equiv \Delta \mod (2, v_1^8).$$

To prove that $\phi_\gamma(\phi_\alpha(\Delta)) = \phi_\alpha(\phi_\gamma(\Delta)) \equiv \phi_\alpha(\Delta) \mod (2, v_1^8)$.□

Lemma 4.23. Let $\gamma$ be in $S_C^1$. Modulo $(2, v_1^8)$, the action of $\gamma$ either permutes $\Delta$ and $\phi_\alpha(\Delta)$, or it fixes both.

Proof. Write $\gamma = \gamma_0\tau$ where $\gamma_0 \in K^1$ and $\tau \in G_{24}$. That such a representation is possible follows from the fact that $S_C^1 \cong K^1 \rtimes G_{24}$. It is sufficient to show that, modulo $(2, v_1^8)$,

$$\phi_\tau(\Delta) \equiv \begin{cases} 
\phi_\alpha(\Delta) & \text{if } \gamma_0 \notin F_{3/2}S_C^1; \\
\Delta & \text{if } \gamma_0 \in F_{3/2}S_C^1.
\end{cases}$$

Because $\tau$ acts trivially on $\Delta$,

$$\phi_\gamma(\Delta) \equiv \phi_{\gamma_0}(\Delta) \mod (2, v_1^8).$$

If $\gamma_0$ is not in $F_{3/2}S_C^1$, then $\gamma_0 = \alpha_1\gamma_1$ for $\gamma_1 \in F_{3/2}S_C^1$. Modulo $(2, v_1^8)$, the element $\gamma_1$ acts trivially on $\Delta$, so that $\phi_{\gamma_0}(\Delta) \equiv \phi_\alpha(\Delta)$. The same proof works for $\phi_\alpha(\Delta)$. □
Proof of Theorem 4.16. Elements of \((I S_1^1)^2\) are possibly infinite linear combinations of elements of the form \((e - g)(e - h)\) for \(g\) and \(h\) in \(S_1^1\). It suffices to show that, for these generators,

\[
(4.24) \quad (id + \phi_g)(id + \phi_h)(x \Delta^{2^k(1+2t)}) \equiv (id + \phi_g + \phi_h + \phi_{gh})(x) \cdot \Delta^{2^k(1+2t)}
\]

modulo \((2, v_1^{1+3.2^{k+1}})\). By Lemma 4.23, the elements \(g\) and \(h\) either fix \(\Delta\) and \(\phi_\alpha(\Delta)\) or permute them modulo \((2, v_1^k)\). If both permute \(\Delta\) and \(\phi_\alpha(\Delta)\), then \(gh\) fixes them. Therefore, up to relabeling, one can assume that \(h\) fixes \(\Delta\) and \(\phi_\alpha(\Delta)\).

There are two cases depending on the action of \(g\).

If \(g\) and \(h\) fix \(\Delta\) modulo \((2, v_1^k)\), then they fix \(\Delta^{2^k(1+2t)}\) modulo \((2, v_1^{k+3})\). Hence, modulo \((2, v_1^{k+3})\),

\[
(4.25) \quad (id + \phi_h)(id + \phi_g)(x \Delta^{2^k(1+2t)}) \equiv (x + \phi_h(x) + \phi_g(x) + \phi_{gh}(x)) \Delta^{2^k(1+2t)}.
\]

This implies (4.18) for the case of \(x = 1\) and \(k, t = 0\). Further, since \(2^{k+3} > 1 + 3 \cdot 2^{k+1}\),

(4.25) trivially implies (4.17).

If \(g\) permutes \(\Delta\) and \(\phi_\alpha(\Delta)\) modulo \((2, v_1^k)\), then it permutes \(\Delta^{2^k(1+2t)}\) and \(\phi_\alpha(\Delta)^{2^k(1+2t)}\) modulo \((2, v_1^{k+3})\). Therefore,

\[
(id + \phi_h)(id + \phi_g)(x \Delta^{2^k(1+2t)}) \equiv (x + \phi_h(x)) \Delta^{2^k(1+2t)} + (\phi_g(x) + \phi_{gh}(x)) \phi_\alpha(\Delta)^{2^k(1+2t)}.
\]

But

\[
(\phi_g(x) + \phi_{gh}(x))\phi_\alpha(\Delta)^{2^k(1+2t)} \equiv (\phi_g(x) + \phi_{gh}(x))\Delta^{2^k(1+2t)}(1 + v_2^{-2^{k+1}} v_1^{3 \cdot 2^{k+1}})
\]

\[
\equiv (\phi_g(x) + \phi_{gh}(x)) + \phi_g(x + \phi_h(x)) v_2^{-2^{k+1}} v_1^{3 \cdot 2^{k+1}} \Delta^{2^k(1+2t)}.
\]

When \(x = 1\) and \(k, t = 0\), this implies (4.18). Because \(h\) is in \(S_1^1\), Theorem 3.7 implies that

\[x + \phi_h(x) \equiv 0 \mod (2, v_1),\]

so that

\[
(\phi_g(x) + \phi_{gh}(x))\phi_\alpha(\Delta)^{2^k(1+2t)} \equiv (\phi_g(x) + \phi_{gh}(x))\Delta^{2^k(1+2t)} \mod (2, v_1^{1+3 \cdot 2^{k+1}}).
\]

Therefore, modulo \((2, v_1^{1+3 \cdot 2^{k+1}}),

\[
(id + \phi_h)(id + \phi_g)(x \Delta^{2^k(1+2t)}) \equiv (x + \phi_g(x) + \phi_h(x) + \phi_{gh}(x)) \Delta^{2^k(1+2t)}.
\]

\[
\square
\]

5. Computation of the \(E_\infty\)-Term

Now we turn to the computation of the algebraic duality resolution spectral sequence

\[
(5.1) \quad E_1^{p,q} = \text{Ext}_\mathbb{Z}[\mathbb{S}_1^1]^{n,n}(\xi_p, (E_C)_s V(0)) \implies H^{p+q}(\mathbb{S}_1^1, (E_C)_s V(0)),
\]
Recall that compute, Theorem 4.11. Let Theorem 5.2. The differentials induce by the map \( \partial \) given by 
\[
d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}
\]
are thus induced by 
\[
d_1 = \text{Ext}^2_{\mathbb{Z}_2[[\mathcal{C}]]}(\partial_{p+1}, (E_\mathcal{C}), V(0)).
\]
The morphisms \( \partial_{p+1} \) were described in Theorem 4.1. We will use these descriptions together with our partial knowledge of the action of \( \mathbb{S}_\mathcal{C} \) on \( (E_\mathcal{C})_\ast \) to compute the \( d_1 \) differentials.

Recall that \( E_1^{0,0} \cong (E_\mathcal{C})^{G_{24}} \) and \( E_1^{p,0} \cong (E_\mathcal{C})^{G_0} \) for \( p = 1 \) and \( p = 2 \). Since there is an inclusion
\[
(E_\mathcal{C})^{G_{24}} \rightarrow (E_\mathcal{C})^{G_0},
\]
there is an action of \( (E_\mathcal{C})^{G_{24}} \) on \( E_1^{p,0} \) for \( 0 \leq p \leq 2 \). Therefore, it will make sense to talk about the image of \( \Delta \) defined in Theorem 4.11 in \( E_1^{p,0} \). To avoid ambiguity, we will use the convention 
\[
\Delta^k[p] = \Delta^k \cdot 1 \in E_1^{p,0}
\]
in the statement of the results. However, in the proofs, we will assume that the context is sufficient to determine which elements are meant. Similarly, \( v_2 \in (E_\mathcal{C})^{G_0} \).

To distinguish between \( E_1^{1,0} \) and \( E_1^{2,0} \), we let 
\[
v_1^k[p] = v_1^k \cdot 1 \in E_1^{p,0}.
\]

Finally, recall that the differentials are \( v_1 \)-linear. This will be used without mention.

5.1. The differential \( d_1 : E_1^{0,0} \rightarrow E_1^{1,0} \). The differential \( d_1 : E_1^{0,0} \rightarrow E_1^{1,0} \) is induced by the map 
\[
\partial_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_0,
\]
given by \( \partial_1(\gamma e_1) = \gamma(e - \alpha)e_0 \). Here, \( e_i \) is the canonical generator of \( \mathcal{C}_i \). Therefore, 
\[
d_1 = id + \phi_\alpha : E_1^{0,0} \rightarrow E_1^{1,0}.
\]
Recall from Theorem 4.11 that the powers of the element 
\[
\Delta = v_2(v_2 + v_1^3)^3
\]
generate \( H^0(G_{24}, (E_\mathcal{C}), V(0)) \cong (E_\mathcal{C})^{G_{24}} \) as an \( \mathbb{F}_4[v_1] \)-module. So it is sufficient to compute \( d_1 \) on \( \Delta^n[0] \) for \( n \in \mathbb{Z} \).

Theorem 5.2. Let \( n = 2^k(2t + 1) \), then 
\[
d_1(\Delta^n[0]) = v_1^{6-2^k}v_2^{3^{k+1}(4t+1)[1]} \mod (2, v_1^{9}2^k).
\]
Proof. Recall that \( d_1 \) is induced by \( id + \phi_\alpha \). Using Lemma 4.19, one computes
\[
\Delta^n + \phi_\alpha(\Delta^n) = (v_2(v_2 + v_1^3)^3)^n + (v_2(v_2 + v_1^3)^3(1 + v_2^{-2}v_1^6 + v_1^9))^n
\]
\[
= (v_2(v_2 + v_1^3)^3)^n(1 + (1 + v_2^{-2}v_1^6 + v_1^9)^{2^{k+1}} + v_1^{9}2^k(2t+1))
\]
\[
\equiv v_2^{3^{k+1}(4t+1)}v_1^{6-2^k} \mod (2, v_1^{9}2^k).
\]
5.2. The differential $d_1 : E_1^{1,0} \to E_1^{2,0}$. The differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ is induced by the map
$$\partial_2 : \mathcal{C}_2 \to \mathcal{C}_1.$$  
Recall from Theorem 4.1 that
$$\partial_2(\gamma e_2) = \gamma \Theta e_1$$
for $\Theta \in \mathbb{Z}_2[[S_1^1]]$ such that
$$\Theta \equiv e + \alpha \mod (2, (IS_2^1)^2).$$
Let
$$\Theta = e + \alpha + \varepsilon,$$
where $\varepsilon = \sum a_{g,h}(e-g)(e-h)$ is in $(IS_1^1)^2$ and is to be thought of as the error. Further, let
$$\phi_{\varepsilon} = \sum a_{g,h}(id - \phi_g)(id - \phi_h).$$
The goal of this section is to prove the following theorem:

**Theorem 5.5.** Let $n = 2^k(1 + 2t)$ where $t \in \mathbb{Z}$ and $k \geq 0$. There exist homogenous elements $b_n$, such that
$$b_n \equiv v_2^n[1] \mod (2, v_1).$$
The elements $b_n$ satisfy
$$d_1(\Delta^n[0]) = \begin{cases} v_1^{2k} b_{2k+1(1+4t)} & n = 2^k(1 + 2t) \\ 0 & n = 0 \end{cases}$$
and
$$d_1(b_n) = \begin{cases} v_1^{3-2k} - v_2^{1+2k+1} [2] \mod (2, v_1^{3-2k+3}) & n = 2^k(3 + 4t) \\ v_1^{3-2k+1} - v_2^{m-2k+1} [2] \mod (2, v_1^{3-2k+1+3}) & n = 1 + 2^{k+2} + 2^{k+3}t \\ 0 & n = 0, 1 \text{ and } 2^{k+1}(1 + 4t). \end{cases}$$

The idea for the next theorem comes from Mahowald and Rezk's computations of the homotopy of $\text{TMF}$, specifically, Corollary 6.2 in [12]. The idea is to consider the spectral sequence

$$\tilde{E}_1^{p,q} = \text{Ext}_{\mathbb{Z}_2[[S_1^1]]}^q(\mathcal{C}_p, (E_C)_*) \Rightarrow H^{p+q}(S^1, (E_C)_*).$$

Let
$$f : \tilde{E}_1^{p,q} \to E_1^{p,q}$$
be the map of spectral sequences induced by the map $(E_C)_* \to (E_C)_* V(0)$ on the coefficients. We will show that there is a permanent cycle $B_1 \in \tilde{E}_1^{1,0}$ such that
$$f(B_1) \equiv v_1 v_2 \mod (2, v_1).$$
This will allow us to define a permanent cycle $b_1 \in E_1^{1,0}$ by
$$b_1 = v_1^{-1} f(B_1).$$

**Theorem 5.8.** There is an element $b_1 \in E_1^{1,0}$ such that
$$b_1 \equiv v_2[1] \mod (2, v_1^3),$$
and
$$d_1(b_1) = 0.$$
Proof. There is a modular form \( c_4 \) in \((E_\mathcal{C})_2^{24} \) given by

\[
c_4 = 9(v_1^4 + 8v_1v_2) = 9u^{-4}u_1(u_1^4 + 8)
\]

(see, for example, [18, §III.1]). I claim that there is an element \( B_1 \in (E_\mathcal{C})_2^n \) such that

\[
B_1 \equiv v_1v_2 \mod (2, v_1^2)
\]

and

\[
d_1(c_4) = c_4 - \phi_\alpha(c_4) = 16B_1,
\]

where \( d_1 \) here denotes the differential in the spectral sequence \( \tilde{E}_p^r \) defined by (5.6).

The first step is to show that

\[
d_1(c_4) \equiv 0 \mod (16).
\]

Let \( t_0 = t_0(\alpha) \) and \( t_1 = t_1(\alpha) \) as defined in Theorem 3.6. A direct computation using Theorem 3.6 implies that

\[
d_1(c_4) \equiv 8u^{-4} \left( t_1 + 3u_1 \frac{t_1^3}{t_0^3} + 2t_1 \frac{t_1^4}{t_0^4} + 2t_1^2 \frac{t_1^4}{t_0^4} + 2t_1 \frac{t_1^4}{t_0^4} \right) \mod (32).
\]

Let \( A = d_1(c_4)/(8u^{-4}) \). Then

\[
A = u_1 t_0^{-4} (t_0 + t_1^3 + u_1^2 t_1 t_0^2) \mod (2).
\]

It follows from Proposition 8.21 of Section 8 that

\[
t_0 \equiv t_0^3 + u_1^2 t_1 t_0^2 \mod 2.
\]

This proves that \( d_1(c_4) \equiv 0 \mod 16 \).

The next step is to show that

\[
A \equiv 2u_1 \mod (4, u_1^2).
\]

Theorem 3.7 applied to \( \alpha \) gives

\[
t_0 \equiv 1 + 2\omega \mod (4, u_1^2),
\]

\[
t_1 \equiv u_1 \omega^2 \mod (2, u_1^2).
\]

This implies that, modulo \((4,u_1^2)\),

\[
A \equiv u_1 + 3u_1 \frac{t_1^3}{t_0^3} + 2t_1 \frac{t_1^4}{t_0^4} + 2t_1^2 \frac{t_1^4}{t_0^4} \equiv 2u_1.
\]

Define

\[
B_1 = \frac{d_1(c_4)}{16}.
\]

Because \( v_1v_2 = u_1u^{-4} \), this implies that

\[
B_1 \equiv v_1 v_2 \mod (2, v_1^2)
\]

so that

\[
d_1(c_4) \equiv 16(v_1v_2 + \ldots) \mod (32).
\]

Let \( f : \tilde{E}_1^{p,q} \to E_1^{p,q} \) be the map defined in (5.7). Since \( f(B_1) \) is divisible by \( v_1 \), we can define an element \( b_1 \in E_1^{1,0} \) by

\[
b_1 := v_1^{-1} f(B_1).
\]
Then \( b_1 \equiv v_2 \) modulo \((2, v_1)\). But \( b_1 \) is an element of 
\[(E_C)_6 V(0)^{C_o} = \mathbb{F}_4[[u_1^3]]\{v_2\}.
\]
This forces the congruence 
\[b_1 \equiv v_2 \pmod{(2, v_1^3)}.
\]
It remains to show that \( d_1(b_1) = 0 \). In the spectral sequence \( \tilde{E}^{p,q}_r \), we have 
\[d_1^2(c_4) = d_1(16B_1) = 16d_1(B_1).
\]
Since \( d_1^2 = 0 \), and there is no torsion in \( (E_C)^{C_o} \), this implies that
\[d_1(B_1) = 0.
\]
in \( \tilde{E}^{2,0}_1 \). Therefore, 
\[d_1(f(B_1)) = 0.
\]
Since \( B_1 = v_1b_1 \), this implies that
\[d_1(v_1b_1) = 0
\]
in \( E^{2,0}_1 \). But the differential \( d_1 : \tilde{E}^{1,0}_1 \to \tilde{E}^{2,0}_1 \) is \( v_1 \)-linear, so that 
\[d_1(v_1b_1) = v_1d_1(b_1) = 0.
\]
Because there is no \( v_1 \)-torsion in \( E^{2,0}_1 \), we must have \( d_1(b_1) = 0 \). This finishes the proof. 
\[\square
\]
\[\text{Lemma 5.10.} \quad \text{For } d_1 : \tilde{E}^{1,0}_1 \to \tilde{E}^{2,0}_1 \n d_1(\Delta[1]) \equiv v_1^6v_2^3[2] \pmod{(2, v_1^8)}.
\]
\[\text{Proof.} \quad \text{Using Theorem 4.16, with } \phi_E \text{ as defined by (5.4),}
\]
\[d_1(\Delta) = \Delta + \phi_0(\Delta) + \phi_E(\Delta)
\]
\[\equiv \Delta + \Delta(1 + v_2^{-2}v_1^6)
\]
\[\equiv v_2^2v_1^6 \pmod{(2, v_1^8)}.\]
\[\square
\]
\[\text{Lemma 5.11.} \quad \text{For } d_1 : \tilde{E}^{1,0}_1 \to \tilde{E}^{2,0}_1 \n d_1(v_2^3[1]) \equiv v_1^3v_2^3[2] \pmod{(2, v_1^5)}.
\]
\[\text{Proof.} \quad \text{Because } b_1 \equiv v_2 \text{ modulo } (2, v_1^3),
\]
\[\Delta \equiv b_1^4 + v_2^3v_1^3 + b_1^2v_1^6 \pmod{(2, v_1^8)}.
\]
Hence, 
\[d_1(\Delta) \equiv d_1(b_1^4 + v_2^3v_1^3 + b_1^2v_1^6)
\]
\[\equiv d_1(b_1)^4 + v_1^3d_1(v_2^3) + v_1^6d_1(b_1)^2
\]
\[\equiv v_1^3d_1(v_2^3) \pmod{(2, v_1^5)}.
\]
It thus follows from Lemma 5.10 that 
\[v_1^3d_1(v_2^3) \equiv v_1^6v_2^2 \pmod{(2, v_1^8)}.
\]
As there is no \( v_1 \)-torsion in \( E^{2,0}_1 \), this proves the claim. \[\square\]
Lemma 5.12. For $\sum a_g g$ in $\mathbb{Z}_2[[S_k]]$, where $a_g \in \mathbb{Z}_2$,
$$\sum a_g \phi_g(v_2^{3+4t}) \equiv \sum a_g \phi_g(v_2^3 v_2^{4t}) \mod (2, v_1^4).$$

Proof. As $g \in S_k^0$, it follows from Theorem 3.7 that $t_0(g) \equiv 1$ modulo $(2, v_1)$. Hence, $t_0(g)^{4t} \equiv 1$ modulo $(2, v_1)$, and
$$\sum a_g \phi_g(v_2^{3+4t}) \equiv \sum a_g \phi_g(v_2^3 \phi_g(v_2^{4t})) \equiv \sum a_g \phi_g(v_2^3) t_0(g)^{4t} v_2^{4t} \equiv \sum a_g \phi_g(v_2^3) v_2^{4t} \mod (2, v_1^4).$$

Proof of Theorem 5.5. Let $t \in \mathbb{Z}$ and $k \geq 0$
$$b_n := \begin{cases} 
\begin{array}{ll}
\frac{b^n}{v_2^n} & n = 0, 1; \\
\frac{b_1}{v_2} \Delta^{2k+2k+i} & n = 1 + 2^k + 2^{k+1}; \\
\frac{v_2^{-6-2k}}{d_1} \left(\Delta^{2k(2i+1)}\right) & n = 2^{k+1}(4t + 1).
\end{array}
\end{cases}$$

The element $b_n$ is in degree $6n$ and
$$b_n \equiv v_2^n \mod (2, v_1^3).$$

By Theorem 5.5, for $n = 0, 1$,
$$d_1(b_n) = 0.$$ 

Let $n = 2^{k+1}(1 + 4t)$. Then
$$d_1(v_1^{6-2k} b_n) = d_1^2 \left(\Delta^{2k(2t+1)}\right) = 0.$$ 

The map $d_1$ is $v_1$-linear and there is no $v_1$-torsion in $E^{2,0}_1$; hence, $d_1(b_n) = 0$.

Next, let $n = 2^k(3 + 4t)$, so that $b_n = (v_2^{3+4t}) 2^k$. For any $b$ in $E^{p,q}$,
$$d_r(b^2) = d_r(b)^2 \mod (2),$$
so it is sufficient to prove the claim when $k = 0$. Let $\phi$ be as in (5.4). It follows from Lemma 5.11 and Lemma 5.12 that, modulo $(2, v_1^4)$,
$$d_1(v_2^{3+4t}) = v_2^{3+4t} + \phi_\alpha(v_2^{3+4t}) + \phi_\phi(v_2^{3+4t}) \equiv v_2^{3+4t} + \phi_\alpha(v_2^3) v_2^{4t} + \phi_\phi(v_2^3) v_2^{4t} \equiv (v_2^3 + \phi_\alpha(v_2^3) + \phi_\phi(v_2^3)) v_2^{4t} \equiv d_1(v_2^3) v_2^{4t} \equiv v_1^4 v_2^{2+4t} \mod (2, v_1^4).$$

Finally, let $n = 1 + 2^{k+2} + 2^{k+3} t$, so that $b_n = b_1 \Delta^{2k(1+2t)}$. By Theorem 4.16,
$$\phi_\phi(b_n) \equiv \phi_\phi(b_1) \Delta^{2k(1+2t)} \mod v_1^{1+3+2^{k+1}}.$$
Therefore,
\[ d_1(b_n) = b_n + \phi_\alpha(b_n) + \phi_\beta(b_n) \]
\[ \equiv b_1\Delta^{2k(1+2t)} + \phi_\alpha(b_1)\Delta^{2k(1+2t)}(1 + v_2^{-2}v_1^2)\Delta^{2k(1+2t)} + \phi_\beta(b_1)\Delta^{2k(1+2t)} \]
\[ \equiv b_1\Delta^{2k(1+2t)} + \phi_\alpha(b_1)\Delta^{2k(1+2t)}(1 + v_2^{-2k+1}v_1^3) + \phi_\beta(b_1)\Delta^{2k(1+2t)} \]
\[ \equiv (b_1 + \phi_\alpha(b_1) + \phi_\beta(b_1))\Delta^{2k(1+2t)} + \phi_\alpha(b_1)v_2^{-2k+1}v_1^3\Delta^{2k(1+2t)} \]
\[ \equiv d_1(b_1)\Delta^{2k(1+2t)} + \phi_\alpha(b_1)v_2^{-2k+1}v_1^3\Delta^{2k(1+2t)} \mod (2, v_1^{1+3k+1}). \]

But \( d_1(b_1) = 0 \) and
\[ \phi_\alpha(b_1) \equiv v_2 \mod (2, v_1^3). \]

Furthermore, \( \Delta^{2k(1+2t)} \equiv v_2^{2k+2+2k+3t} \), so that
\[ d_1(b_n) \equiv v_1^{3k+1}v_2^{-2k+1+2k+3t} \equiv v_1^{3k+1}v_2^{2k+1+2k+3t} \mod (2, v_1^{1+3k+1}). \]

This complete the proof of Theorem 5.5. \( \square \)

5.3. The differential \( d_1 : E_1^{2,0} \rightarrow E_1^{3,0} \). Recall that
\[ E_1^{2,0} \cong H^\text{c}(G_2^t, (E_2)_c, V(0)) = F_4[[j]][v_1, \Delta^t]/(j' = v_1^2\Delta^t). \]

We let
\[ \Delta'[3] = \Delta^t \cdot 1 \in E_1^{3,0}. \]

The next goal will be to prove:

**Theorem 5.13.** Let \( n = 2^k(1+2t) \) where \( t \in \mathbb{Z} \) and \( k \geq 0 \). There exist homogenous elements \( c_n \) such that
\[ c_n \equiv v_2^n[2] \mod (2, v_1) \]
and
\[ d_1(b_n) = \begin{cases} 
    v_1^{3k+1}v_2^{2k+1} & n = 2^k(3+4t) \\
    v_1^{3k+1}v_2^{2k+1+2k+3} & n = 1 + 2^k+2t+2k+3 \\
    0 & \text{otherwise.}
\end{cases} \]

Further,
\[ d_2(c_n) \equiv v_1^{3(1+2k+1)}\Delta^{2k(1+2t)}[3] \mod (2, v_1^{3(1+2k+1)+12}) \]
if \( n = 1 + 2^{k+1}(3+4t) \) and is zero otherwise.

Let \( (F_\alpha'^{s,t}, d_\alpha') \) be the duality resolution spectral sequence associated to the resolution
\[ 0 \rightarrow \mathcal{E}_3 \xrightarrow{\partial'} \mathcal{E}_2 \xrightarrow{\partial'} \mathcal{E}_1 \xrightarrow{\partial'} \mathcal{E}_0 \rightarrow \mathbb{Z}_2 \rightarrow 0, \]
described in Theorem 4.1. Recall also that there are isomorphisms
\[ g_p : \mathcal{E}_p \rightarrow \mathcal{E}_p, \]
which induce an isomorphism of resolutions

\[
0 \longrightarrow \mathcal{C}_3 \xrightarrow{d_3} \mathcal{C}_2 \xrightarrow{d_2} \mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \longrightarrow 0
\]

\[
0 \longrightarrow \mathcal{C}_3' \xrightarrow{d_3'} \mathcal{C}_2' \xrightarrow{d_2'} \mathcal{C}_1' \xrightarrow{d_1'} \mathcal{C}_0' \xrightarrow{\varepsilon} \mathbb{Z}_2 \longrightarrow 0,
\]

As \( g_p \) is an isomorphism, the map \( F_2 \otimes_{\mathbb{Z}_2[[S]]} g_p : F_2 \to F_2 \) is non-zero, so that

\[(5.16) \quad g_p(e_p) = (e + IS_1^1)e_p.\]

Let \( g_p^\ast \) be the map induced by \( g_p \)

\[g_p^\ast := \text{Hom}_{\mathbb{Z}_2[[S]]}(g_p, (E_C)_*V(0)).\]

Theorem 4.1 implies that there is an isomorphism of complexes,

\[(5.17) \quad 0 \longrightarrow F^{0,0}_1 \xrightarrow{d_1'} F^{1,0}_1 \xrightarrow{d_1} F^{2,0}_1 \xrightarrow{d_1} F^{3,0}_1 \longrightarrow 0,\]

and Theorem 3.7 together with (5.16) implies that

\[(5.18) \quad g_p^\ast \equiv \text{id} \mod (2, v_1).\]

**Proof of Theorem 5.13.** I construct the \( c_n \) inductively. For \( n = 2^k(3 + 4t) \) and \( n = 1 + 2^{k+2} + 2^{k+3}t \), define \( c_n \) by the identities

\[d_1(b_n) = \begin{cases} v_3^{2^k} \cdot c_{2k+1(1+2t)} & n = 2^k(3 + 4t); \\ v_1^{2^k+1} \cdot c_{1+2^{k+1}+2^{k+3}t} & n = 1 + 2^{k+2} + 2^{k+3}t. \end{cases}\]

Then \( c_n \) satisfies equation (5.14) and

\[d_1(c_n) = 0.\]

Now note that the morphism \( \phi_\pi \) for \( \pi = 1 + 2\omega \) restricts to an isomorphism of \( F_1^{2,0} \cong ((E_C)_*V(0))^C_0 \). The isomorphism of complexes (5.17) implies that, for \( m := 1 + 2^{k+1} + 2^{k+3}t \), there exist \( c'_m \) in \( F_1^{2,0} \) and \( x_m \) in \( E_1^{1,0} \) such that

\[g_2^\ast(\phi_\pi(c'_m)) = c_m + d_1(x_m).\]

But, for \( x_m \in E_1^{1,0} \),

\[d_1(x_m) \equiv 0 \mod (2, v_1^3),\]

so that

\[(5.19) \quad c'_m \equiv v_3^{1+2^{k+1}+2^{k+3}t} \mod (2, v_1).\]

For \( n = 1 + 2^{k+1} + 2^{k+2} + t2^{k+3} \), let

\[c'_n := c'_m \Delta^{2^k}\]

and define

\[c_n := g_2^\ast(\phi_\pi(c'_m)).\]
Because $\Delta^{2^k} \equiv v_2^{2^{k+1}}$ modulo $(2, v_1)$, the elements $c_n$ satisfy (5.14). Further, using the fact that $d_1g_2 = g_3d_1',
 d_1(c_n) = g_3^*(d_1'(\phi_n(c'_n))).$

By Theorem 4.1, the map $d_1': F_1^{2,0} \to F_1^{3,0}$ is given by $\phi_n(id + \phi_i + \phi_j + \phi_k)(id + \phi_\alpha^{-1})\phi^{-1}$, so that
\[
d_1'(\phi_n(c'_n)) = \phi_n(id + \phi_i + \phi_j + \phi_k)(id + \phi_\alpha^{-1})(c'_n).
\]
I will compute this in three steps.

By Lemma 3.9,
\[
\phi_\alpha^{-1}(\Delta) = \Delta(1 + v_2^{-2}v_1^6 + v_1^9\epsilon).
\]

Hence,
\[
(id + \phi_\alpha^{-1})(c'_m) = c'_m + \phi_\alpha^{-1}(c'_m)
\]
\[
\equiv c'_m\Delta^{2^k} + \phi_\alpha^{-1}(c'_m)\Delta^{2^k}(1 + v_2^{-2}v_1^6 + v_1^9\epsilon)2^{k}
\]
\[
\equiv c'_m\Delta^{2^k} + \phi_\alpha^{-1}(c'_m)\Delta^{2^k}(1 + v_2^{-2k+1}v_1^32^{k+1} + v_1^39^2\epsilon 2^{k})
\]
\[
= (id + \phi_\alpha^{-1})(c'_m)\cdot \Delta^{2^k} + \phi_\alpha^{-1}(c'_m)(v_2^{-2k^{k+1}v_1^32^{k+1} + v_1^39^2\epsilon 2^{k})\Delta^{2^k}.
\]

Now note that $i, j$ and $k$ fix $\Delta$, so that
\[
\phi_\pi \left((id + \phi_i + \phi_j + \phi_k) \left((id + \phi_\alpha^{-1})(c'_m)\Delta^{2^k}\right)\right) = d_1'(\phi_n(c'_m))\phi_\pi(\Delta^{2^k}) = 0.
\]

The second equality follows from the fact that $g_3^*d_1'(\phi_n(c'_m)) = d_1(c_m) = 0$
and $g_3^*$ is an isomorphism, so is injective. Therefore,
\[
d_1'(\phi_n(c'_m)) = \phi_\pi \left((id + \phi_i + \phi_j + \phi_k) \left((\phi_{\alpha^{-1}}(c'_m)(v_2^{-2k^{k+1}v_1^32^{k+1} + v_1^39^2\epsilon 2^{k})\Delta^{2^k})\right)\right).
\]

The morphism $\phi_\alpha^{-1}(c'_m) \equiv c'_m$ modulo $(2, v_1^3)$, so that (5.19) implies that
\[
\phi_\alpha^{-1}(c'_m) = v_2^m + v_3^3v_2^{2k+1}v_1^2\epsilon_0,
\]
and,
\[
d_1'(\phi_n(c'_m)) = \phi_\pi \left((id + \phi_i + \phi_j + \phi_k) \left((v_2^m + v_3^3v_2^{2k+1}v_1^2\epsilon_0)\right)\right)
\]
\[
\cdot (v_2^{-2k+1}v_1^32^{k+1} + v_1^39^2\epsilon 2^{k})\Delta^{2^k}
\]
\[\equiv \phi_\pi \left((id + \phi_i + \phi_j + \phi_k) \left(v_2^{1+2k+3}v_1^32^{k+1} + v_1^3(1+2k^{k+1}v_2^{2k+3}\epsilon_1\Delta^{2^k}.
\]

Here, I have used the fact that $3(2^{k+1} + 1) \leq 9 \cdot 2^{k} < 9 \cdot 2^{k} + 3$. Because $i, j$ and $k$ are in $S^2_3$, they act as the identify modulo $(2, v_1)$. Further,
\[
\omega(e + i + j + k)\omega^{-1} = e + i + j + k.
\]
Hence, for \( x \in (E_C)_* V(0)^{C_0} \),
\[
\phi_\omega((id + \phi_i + \phi_j + \phi_k)(x)) = (id + \phi_i + \phi_j + \phi_k)(x)
\]
so that
\[
(id + \phi_i + \phi_j + \phi_k)(x) \in (E_C)_* V(0)^{C_0}.
\]
Therefore,
\[
(id + \phi_i + \phi_j + \phi_k)(v_2^{2^k+3} \epsilon_1) \equiv 0 \pmod{(2, v_1^3)}.
\]
Hence, modulo \((2, v_1^{3(1 + 2^{k+1})+3})\),
\[
d'_1(\phi_\pi(c'_n)) \equiv \phi_\pi((id + \phi_i + \phi_j + \phi_k)(v_2^{1+2^k+3}t) v_1^{3 \cdot 2^{k+1}} \Delta^{2^k}).
\]
Finally, using Strickland’s formulas from Section 2.4,
\[
t_0(i)^{-1} = 1 + u_1
\]
\[
t_0(j)^{-1} = 1 + \zeta u_1
\]
\[
t_0(k)^{-1} = 1 + \zeta^2 u_1,
\]
and
\[
d'_1(\phi_\pi(c'_n)) \equiv (1 + t_i^{-3(1+2^{k+3})} + t_j^{-3(1+2^{k+3})} + t_k^{-3(1+2^{k+3})}) v_2^{1+2^{k+3}} v_1^{3 \cdot 2^{k+3}} \Delta^{2^k}.
\]
Note that
\[
3(1 + 2^{k+3}t) = 1 + 2 + 2^{k+3}t + 2^{k+4}t \equiv 1 + 2 \pmod{8}.
\]
Modulo 2, the binomial coefficients satisfy,
\[
\begin{align*}
(a_0 + 2a_1 + 2^2a_2 + \ldots + 2^n a_n) & \equiv (a_0) \cdot (a_1) \cdot (a_2) \ldots (a_n),
\end{align*}
\]
This implies that the binomial coefficients
\[
\binom{3(1 + 2^{k+3})}{i} \equiv \begin{cases} 1 & \text{if } 0 \leq i \leq 3; \\ 0 & \text{if } 3 < i \leq 6.
\end{cases}
\]
Hence, modulo \(v_1^{3(1 + 2^{k+1})+3}\),
\[
d'_1(\phi_\pi(c'_n)) \equiv \phi_\pi \left( 1 + \sum_{s=0}^{2} (1 + \zeta^s u_1 + \zeta^{2s} u_1^2 + u_1^3) v_2^{1+2^{k+3}} v_1^{3 \cdot 2^{k+1}} \Delta^{2^k} \right)
\]
\[
\equiv \phi_\pi \left( u_1^{3} v_2^{1+2^{k+1}} v_1^{3 \cdot 2^{k+1}} \Delta^{2^k} \right)
\]
\[
\equiv v_1^{3(1+2^{k+1})} v_2^{2^k+3} \phi_\pi(\Delta^{2^k}).
\]
The last equivalence uses the fact that \(\phi_\pi \equiv id \pmod{(2, v_1^3)}\), which was shown in Lemma 3.10.

Now, recall from Lemma 4.14 that the powers of \(\Delta' := \phi_\pi(\Delta)\) form a set of \(F_4[v_1]-\)generators of \(H^0(G_{24}, (E_C)_* V(0))\). Because, \(d'_1(\phi_\pi(c'_n)) \) is \(G'_{24}\)-invariant, it must be a linear combination of powers of \(\Delta'^{\pm 1}\) and powers of \(v_1\). This implies that
\[
d'_1(\phi_\pi(c'_n)) \equiv v_1^{3(1+2^{k+1})} \Delta^{2^k+2^{k+1}} \pmod{(2, v_1^{3(1+2^{k+1})+3})}.
\]
Finally, 
\[ d_1(c_n) = g_1^s(d_1^i(\phi_s(c_n))) \]
must satisfy the same congruence by (5.18).

The only element \( c_n \) which has not been constructed is \( c_1 \). Its existence follows from Lemma 5.20 below. \( \square \)

**Lemma 5.20.** There exists a sequence of elements \( \{c_{1,n}\} \) such that

1. \( c_{1,n} \equiv v_2 \) modulo \( (2, v_1^6) \),
2. \( d_1(c_{1,n}) \equiv 0 \) modulo \( (2, v_1^{3(1+4n)}) \),
3. \( c_{1,n+1} - c_{1,n} \equiv 0 \) modulo \( (2, v_1^{6n}) \).

If \( (E_C)_0V(0) \) is given the topology induced by the maximal ideal
\[ \mathfrak{m} = (v_1), \]
then the limit
\[ c_1 := \lim_{n \to \infty} c_{1,n} \]
exists. The element \( c_1 \) satisfies equation (5.14) and
\[ d_1(c_1) = 0. \]

**Proof.** The construction of \( \{c_{1,n}\} \) is by induction on \( n \). First, define
\[ c_{1,1} := v_2 \]
and note that
\[ c_{1,1} + \phi_{\alpha-1}(c_{1,1}) \equiv v_1^3 + v_1^6 \epsilon. \]
The \( \mathbb{F}_4 \)-vector space with basis
\[ \{v_1^3, v_1^{3-9} \Delta^{-1}, v_1^{3-9} \Delta^{-2}, \ldots, v_1^{3(1+4s)} \Delta^{-s}, \ldots\} \]
is dense in \( ((E_C)_0V(0))^{G_2s} \). Hence,
\[ d_1(c_{1,1}) \equiv 0 \mod (2, v_1^6). \]

Now suppose that \( c_{1,n} \) has been defined. If \( d_1(c_{1,n}) = 0 \), then let \( c_{1,N} := c_{1,n} \) for all \( N \geq n \). Otherwise,
\[ d_1(c_{1,n}) = v_1^{3+12s_n} \Delta^{-s_n} + \ldots \]
for \( s_n \geq n \). Let \( s_n = 2^{k_n}(1 + 2t_n) \) and let \( m_n = 3 \cdot 2^{k_n+1}(1 + 4t_n) \). Then
\[ m_n \geq 6n. \]
For
\[ r_n = 1 + 2^{k_n+1} + 2^{k_n+2} + 2^{k_n+3}(-t_n - 1), \]
(5.21) together with the fact that
\[ d_1(c_{r_n}) = v_1^{3(1+2^{k_n+1})} \Delta^{2^{k_n}(1+2(-t_n-1))} + \ldots, \]
implies that
\[ d_1(c_{1,n}) = v_1^{m_n}d_1(c_{r_n}) + \ldots \]
Define
\[ c_{1,n+1} := c_{1,n} + v_1^{m_n}c_{r_n}. \]
Then \( c_{1,n+1} \) satisfies properties (1), (2) and (3).
Now consider the sequence \( \{c_{1,n}\} \). Since \( m_{n+k} \geq 6n \) for \( k \geq 0 \),
\[
c_{1,n+k} - c_{1,n} = v_1^{m_{n+1}+1}c_{r_{n+1}} + \ldots + v_1^{m_{n+k}+1}c_{r_{n+k}} \in (v_1)^{6n}.
\]
The sequence \( \{c_{1,n}\} \) is Cauchy in the topology generated by \( m \). Since \((E_C)_0(V(0))^C_6 \) is complete with respect to \( m \), the limit
\[
c_1 := \lim_{n \to \infty} c_{1,n}
\]
exists. The map \( d_1 \) is continuous, so that,
\[
d_1(c_1) = \lim_{n \to \infty} d_1(c_{1,n}).
\]
But \( d_1(c_{1,n}) \in \mathfrak{m}^{3(1+4N)} \) for all \( n \geq N \), which implies that
\[
d_1(c_1) \in \bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0.
\]
\[\square\]

We can now combine the results of this section to prove the first part of Theorem 1.8. We restate it here for convenience.

**Theorem 5.22.** The algebraic duality resolution spectral sequence converging to \( H^*(S^1, (E_C)_0V(0)) \) collapses at the \( E_2 \)-term. The spectral sequence is a module over \( F_4[[v_1, h_1]] \). There exist \( F_4[[v_1]] \)-generators \( a_n \in E_1^{0,0} \), \( b_n \in E_1^{1,0} \), \( c_n \in E_1^{2,0} \) and \( d_n \in E_1^{3,0} \) with
\[
\begin{align*}
a_n &\equiv \Delta^n[0] \pmod{(v_1)} \\
b_n &\equiv v_2^n[1] \pmod{(v_1)} \\
c_n &\equiv v_2^n[2] \pmod{(v_1)} \\
d_n &\equiv \Delta^n[3] \pmod{(v_1)}
\end{align*}
\]
and such that, for \( k \geq 0 \) and \( t \in \mathbb{Z} \),
\[
\begin{align*}
d_1(a_n) &= \begin{cases} v_1^{6 \cdot 2^k} b_{2^{k+1}(1+4t)} & n = 2^k(1 + 2t) \\ 0 & n = 0 \end{cases} \\
d_1(b_n) &= \begin{cases} v_1^{3 \cdot 2^{k+1} + 1} c_{1+2^{k+1}+t2^{k+3}} & n = 2^k(3 + 4t) \\ v_1^{3 \cdot 2^{k+1}} c_{1+2^{k+1}+t2^{k+3}} & n = 1 + 2^{k+2} + t2^{k+3} \\ 0 & \text{otherwise} \end{cases} \\
d_1(c_n) &= \begin{cases} v_1^{3 \cdot 2^{k+1} + 1} d_{2^k(1+2t)} & n = 1 + 2^{k+1} + 2^{k+2} + t2^{k+3} \\ 0 & \text{otherwise} \end{cases}
\end{align*}
\]

**Proof.** Define
\[
\begin{align*}
a_n &:= \begin{cases} \Delta^n[0] & n = 2^k(1 + 2t) \\ 1 \cdot [0] & n = 0, \end{cases} \\
d_n &:= \begin{cases} v_1^{-3(1+2^k)} d_{1}(c_{1+2^{k+1}+2^{k+2}+2^{k+3}+4}) & n = 2^k(1 + 2t) \\ 1 \cdot [3] & n = 0. \end{cases}
\end{align*}
\]

Then Theorem 5.2, Theorem 5.5 and Theorem 5.13 together prove the theorem. \(\square\)
5.4. The differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$. The goal of this section is to compute the remaining $d_1$ differentials and obtain the $E_2$-term.

Although $V(0)$ is not a ring spectrum,  
$$(E_C)_* V(0) \cong (E_C)_*/2,$$
and a canonical generator is given by the image of the unit in $(E_C)_0$ in the long exact sequence  
$$\ldots \to (E_C)_* \xrightarrow{\alpha} (E_C)_* \to (E_C)_* V(0) \to \ldots$$
Thus, we can give $(E_C)_* V(0)$ the ring structure induced by that of $(E_C)_*$. Then, Lemma 4.7 implies that the algebraic duality resolution is a module over the cohomology $H^*(S_C, (E_C)_* V(0))$. The canonical inclusion  
$$F_4 \to (E_C)_* V(0)$$
induces a map  
$$H^*(S_C, F_4) \to H^*(S_C, (E_C)_* V(0)).$$
Therefore, the algebraic duality resolution spectral sequence for $(E_C)_* V(0)$ is also a module over $H^*(S_C, F_4)$.

Let  
$$(5.23) F_1^{p,q} = \text{Ext}_{Z_2[[S_C]]}^2(\mathcal{E}_p, F_4) \Rightarrow H^{p+q}(S_C^1, F_4).$$
Let $g_0 \in F_1^{0,4}$ be the periodicity generator for the cohomology of $G_{24},$  
$$g_0 \in H^4(G_{24}, F_4) = H^4(Q_8, F_4)^{C_3}. $$
The extension  
$$1 \to K^1 \to S_C^1 \to G_{24} \to 1$$
is split. Therefore, the map  
$$H^*(S_C^1, F_4) \to H^*(G_{24}, F_4)$$
induced by the inclusion of $G_{24}$ in $S_C^1$ is split surjective. This implies that the image of $g_0$ is a permanent cycle in $F_1^{0,4}$. Therefore, it represents a class  
$$g_0 \in H^4(S_C; F_4),$$
and the differentials in the algebraic duality spectral sequence commute with the action of $g_0$. To make sense of this, we must compute the action of $g_0$ on $E_1^{p,q}$. First, note that $g_0$ acts by multiplication by $\Delta_1^2$ in $E_1^{0,q}$ and by $\Delta_t^{-1} g'$ in $E_1^{3,q}$. Further, the map  
$$H^*(S_C^1; F_4) \to H^*(C_6; (E_C)_* V(0))$$
factors through the map  
$$H^*(G_{24}; (E_C)_* V(0)) \to H^*(C_6; (E_C)_* V(0))$$
induced by the inclusion $C_6 \to G_{24}$. Therefore, $g_0$ acts by multiplication by $h^4$ on $E_1^{p,q}$ for $p = 1$ and $p = 2$.

We collect these remarks in the following lemma.

**Lemma 5.24.** The differentials in the algebraic duality resolution are $g_0$-linear, where the action of $g_0$ is given by multiplication by $\Delta^{-1} g$ on $E_1^{0,q}$, by multiplication by $\Delta_t^{-1} g'$ on $E_1^{3,q}$, and by multiplication by $h^4$ on $E_1^{p,q}$ for $p = 1, 2$.

This lemma will allow us to compute some of the differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$ based on our results for $q = 0$. 
Lemma 5.25. Let \( x \in E_1^{0, q} \). The differential \( d_1 : E_1^{0, q} \to E_1^{1, q} \) is zero unless \( x = h_1^t \Delta^s \) or \( x = g^t \Delta^s \), in which case it is given by
\[
d_1(h_1^t \Delta^s) = h_1^t d_1(\Delta^s)
\]
and
\[
d_1(g^t \Delta^s) = h^t d_1(\Delta^{s+t}).
\]

Proof. There is no \( v_1 \)-torsion in \( E_1^{1, q} \), and \( d_1 \) is \( v_1 \)-linear. Therefore, if \( x \) is \( v_1 \)-torsion, we must have \( d_1(x) = 0 \). The only classes in \( E_1^{0, q} \) which are not \( v_1 \)-torsion are of the form \( x = h_1^t \Delta^s \) or \( x = g^t \Delta^s \). The statement for \( x = h_1^t \Delta^s \) follows from \( h_1 \)-linearity of the differentials. For \( x = g^t \Delta^s \), rewrite \( x \) as \( g^t \Delta^{s-1} \Delta^s \). The statement then follows from Lemma 5.24. \( \square \)

Lemma 5.26. Let \( x \in E_1^{1, q} \). The differential \( d_1 : E_1^{1, q} \to E_1^{2, q} \) satisfies
\[
h^k d_1(x) = d_1(h^k x).
\]

Proof. This follows from the fact that the differentials are \( h_1 = hv_1 \) and \( v_1 \)-linear. Indeed, since \( h_1 = v_1 h \), we have the following equalities
\[
v_1^i h^k d_1(x) = h^k d_1(x) = d_1(h^k x) = d_1(v_1^i h^k x) = v_1^i d_1(h^k x).
\]
Since there is no \( v_1 \)-torsion and no \( h \)-torsion in \( E_1^{1, q} \) and \( E_1^{2, q} \), \( h^k d_1(x) = d_1(h^k x) \).

Understanding the differential \( d_1 : E_1^{2, q} \to E_1^{3, q} \) is more subtle as there is \( v_1 \)-torsion in \( E_1^{3, q} \) for \( q > 0 \). We will use the following result. Its proof is postponed until the end of the section.

Lemma 5.27. Let \( x \in E_1^{2, q} \). There exists \( y \in E_1^{3, q} \) such that \( d_1(x) = v_1^3 y \).

Lemma 5.28. Let \( x \in E_1^{2, 0} \). Consider \( d_1 : E_1^{2, q} \to E_1^{3, q} \). Then
\[
d_1(h^i x) = v_1^{-i} h_1 d_1(x),
\]
where we make sense of division by \( v_1 \) as follows: if \( d_1(x) = v_1^k y \), then
\[
v_1^{-i} h_1 d_1(x) := v_1^{k-i} h_1 y.
\]
This includes the case when \( y = 0 \), in which case the formula reads as \( d_1(h^i x) = 0 \).

Proof. Since the differentials are \( g_0 \)-linear and the action of \( g_0 \) on \( x \) is given by multiplication by \( h^4 \), it suffices to consider the cases \( d_1(h^i x) \) for \( 1 \leq i \leq 3 \).

Now, suppose instead that \( x \in E_1^{2, 0} \). Note that since \( d_1 \) is \( \mathbb{F}_4[v_1, h_1] \)-linear and \( h_1 = v_1 h \), we have
\[
v_1^i d_1(h^i x) = h_1 d_1(x).
\]
We will treat the cases \( d_1(x) = 0 \) and \( d_1(x) \neq 0 \) separately.

First, suppose that \( d_1(x) \neq 0 \). By Lemma 5.27, there is some \( y \in E_1^{3, 0} \) such that
\[
d_1(x) = v_1^k y
\]
for \( k \geq 3 \). Hence,
\[
v_1^i d_1(h^i x) = h_1 d_1(x) = v_1^{k-i} h_1 y.
\]
Since \( E_1^{3, 0} \) is free over \( \mathbb{F}_4[v_1, h_1] \), \( d_1(x) \) is not annihilated by \( v_1 \) or \( h_1 \). Hence, neither is \( y \), so that
\[
d_1(h^i x) = v_1^{k-i} h_1 y.
\]
Suppose that \( d_1(x) = 0 \). We must show that \( d_1(h^i x) = 0 \) for \( i = 1, 2, 3 \). Since \( d_1 \) is \( h_1 \)-linear,
\[
v_1^i d_1(h^i x) = d_1(h_1^i x) = 0.
\]
Hence, in this case, \( d_1(h^i x) \) must be \( v_1 \)-torsion. By Lemma 5.27, if \( z \in E_1^{2,q} \), then \( d_1(z) = v_1^3 y \) for some \( y \in E_1^{3,q} \). Letting \( z = h^i x \), this implies that \( d_1(h^i z) \) is divisible by \( v_1^3 \). However, there are no non-zero elements in \( E_1^{3,q} \) which are both \( v_1 \)-torsion and divisible by \( v_1^3 \). Indeed, all the \( v_1 \)-torsion is annihilated by \( v_1^2 \). This finishes the proof.

To prove Lemma 5.27, we will use the following fact. To state it, recall that
\[
x_0 = i + j + k, \quad x_1 = i + \zeta j + \zeta^2 k, \quad x_2 = i + \zeta^2 j + \zeta k.
\]
Further, let \( B_* \to \mathbb{W} \) be the projective resolution of the trivial \( C_6 \)-module \( \mathbb{W} \) which was constructed in Lemma 6.2. Let \( C_* \to \mathbb{W} \) be the projective resolution of the trivial \( G_{24} \)-module \( \mathbb{W} \) which was constructed in Lemma 6.1.

**Lemma 5.30.** Let \( c_0 \) denote the canonical generator of \( \mathbb{W}[S_6^1 / G_{24}] \) and \( e_1 \) denote the canonical generator of \( \mathbb{W}[S_6^1 / C_6] \). The map
\[
F : \text{Ind}_{G_{24}}^{S_6^1}(\mathbb{W}) \to \text{Ind}_{C_6}^{S_6^1}(\mathbb{W})
\]
given by
\[
F(\gamma c_0) = \gamma(e + i + j + k)(e - \alpha^{-1})e_1
\]
has a lift \( F_* \)
\[
\begin{array}{ccc}
\text{Ind}_{G_{24}}^{S_6^1}(C_*) & \xrightarrow{F_*} & \text{Ind}_{C_6}^{S_6^1}(B_*) \\
\downarrow & & \downarrow \\
\text{Ind}_{G_{24}}^{S_6^1}(\mathbb{W}) & \xrightarrow{F} & \text{Ind}_{C_6}^{S_6^1}(\mathbb{W}).
\end{array}
\]
such that, for \( \gamma, \gamma_1 \) and \( \gamma_2 \) in \( \mathbb{W}[S_6^1] \),
\[
F_0(\gamma c_{0,0}) = \gamma(e + x_0)(e - \alpha^{-1})b_0
\]
\[
F_1(\gamma_1 c_{1,1}, \gamma_2 c_{1,2}) = -(\gamma_1 \zeta^2 x_1 + \gamma_2 \zeta x_2)(e - \alpha^{-1})b_1
\]
\[
F_2(\gamma_1 c_{2,1}, \gamma_2 c_{2,2}) = (\gamma_1 (\zeta^2 - \zeta)x_2 + \gamma_2 (\zeta - \zeta^2)x_1)(e - \alpha^{-1})b_2
\]
\[
F_3(\gamma c_{3,0}) = -3\gamma x_0(e - \alpha^{-1})b_3
\]
\[
F_4(\gamma c_{4,0}) = -3\gamma x_0(e + x_0)^2(e - \alpha^{-1})b_4.
\]
Further, for \( 0 \leq k < 4 \), if we define \( F_{4t+k} \otimes_W \mathbb{F}_4 = F_k \otimes_W \mathbb{F}_4 \), then
\[
F_* \otimes_W \mathbb{F}_4 : \text{Ind}_{G_{24}}^{S_6^1}(C_*) \otimes_W \mathbb{F}_4 \to \text{Ind}_{C_6}^{S_6^1}(B_*) \otimes_W \mathbb{F}_4
\]
is a periodic lift of \( F \otimes_W \mathbb{F}_4 \).

**Proof.** The chain complexes \( \text{Ind}_{G_{24}}^{S_6^1}(C_*) \) and \( \text{Ind}_{C_6}^{S_6^1}(B_*) \) are projective resolutions of \( S_6^1 \)-module of \( \text{Ind}_{G_{24}}^{S_6^1}(C_*) \equiv \mathbb{W}[S_6^1 / G_{24}] \) and \( \text{Ind}_{C_6}^{S_6^1}(B_*) \equiv \mathbb{W}[S_6^1 / C_6] \) respectively. A direct computation shows that for \( 0 \leq k \leq 4 \), \( F_{k-1}d_k = d_kF_k \). Since these are complexes of projective \( S_6^1 \)-modules, there exists \( F_k, k > 4 \) such that \( F_* \) lifts \( F \).

Finally, note that
\[
-3x_0(e + x_0)^2 \equiv (x_0 + x_0^2)^2 \equiv (e + x_0) \mod (2).
\]
Therefore,
\[-3x_0(e + x_0)^2(e - \alpha^{-1}) \equiv (e + x_0)(e - \alpha^{-1}) \mod (2)\]
and we can choose \(F\), so that \(F \otimes \mathbb{F}_4\) is periodic. \(\square\)

**Proof of Lemma 5.27.** Let \(M = (E_C)_* \mathbb{V}(0)\). Recall from Theorem 4.1 that there is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{C}_3 & \xrightarrow{\partial_3} & \mathcal{C}_2 \\
\downarrow{g_2} & & \downarrow{g_2} \\
\mathcal{C}_3 & \xrightarrow{\partial'_3} & \mathcal{C}_2,
\end{array}
\]
where
\[
\partial'_3(\gamma e'_3) = \gamma \pi(e + i + j + k)(e - \alpha^{-1})\pi^{-1}(e'_2).
\]
Let \(g_\pi : \mathcal{C}_3 \to \mathcal{C}_0\) be the map of \(S_{\mathbb{L}}^1\)-modules such that
\[g_\pi(e_3) = \pi e_0.\]
This is well-defined. Indeed, if \(\tau' \in G'_{24}\), then \(\tau' = \pi \tau \pi^{-1}\) for \(\tau \in G_{24}\). Hence,
\[
g_\pi(\tau' e_3) = \tau' g_\pi(e_3) = \pi \tau \pi^{-1} \pi e_0 = \pi e_0.
\]
Similarly, the map \(g_{\pi^{-1}} : \mathcal{C}_1 \to \mathcal{C}_2\) of \(S_{\mathbb{L}}^1\)-modules given by
\[g_{\pi}(e_1) = \pi^{-1} e_2\]
is well-defined because \(\pi\) commutes with the elements of \(C_6\). Let
\[f : \mathcal{C}_0 \to \mathcal{C}_1\]
be the map of \(S_{\mathbb{L}}^1\)-modules given by
\[f(\gamma e_0) = \gamma (e + i + j + k)(e - \alpha^{-1})e_1.\]
Then, \(\partial_1 = g_2^{-1} f g_{\pi} g_3\), that is, it is the composite
\[
\begin{array}{cccc}
\mathcal{C}_3 & \xrightarrow{g_2} & \mathcal{C}_3 & \xrightarrow{g_2} \mathcal{C}_0 & \xrightarrow{f} \mathcal{C}_1 & \xrightarrow{g_{\pi^{-1}}} \mathcal{C}_2 & \mathcal{C}_2
\end{array}
\]
Since \(g_{\pi} g_3\) and \(g_2^{-1} g_{\pi^{-1}}\) are isomorphisms and the map they induce on \(M\) are \(v_1\)-linear, it is sufficient to prove that the map
\[f^* : \text{Ext}_{Z_2[[S_{\mathbb{L}}^1]]}(\mathcal{C}_1, M) \to \text{Ext}_{Z_2[[S_{\mathbb{L}}^1]]}(\mathcal{C}_0, M)\]
induced by \(f\) has image divisible by \(v_1^3\). Further, note that
\[
\text{Ext}_{\mathbb{W}[[S_{\mathbb{L}}^1]]}(\mathbb{W}[[S_{\mathbb{L}}^1/H]], M) \cong \text{Ext}_{Z_2[[S_{\mathbb{L}}^1]]}(Z_2[[S_{\mathbb{L}}^1/H]], M).
\]
Therefore, it is sufficient to prove that the map
\[F^* : \text{Ext}_{\mathbb{W}_2[[S_{\mathbb{L}}^1]]}(\mathbb{W}_2[[S_{\mathbb{L}}^1/C_6]], M) \to \text{Ext}_{\mathbb{W}_2[[S_{\mathbb{L}}^1]]}(\mathbb{W}_2[[S_{\mathbb{L}}^1/G_{24}]], M)\]
induced by
\[F = \mathbb{W} \otimes_{Z_2} f : \mathbb{W}[[S_{\mathbb{L}}^1/G_{24}]] \to \mathbb{W}[[S_{\mathbb{L}}^1/C_6]]\]
has image divisible by \(v_1^3\).

To compute \(F^*\), we use the lift described in Lemma 5.30. Let
\[F_k^* = \text{Hom}_{\mathbb{W}_2[[S_{\mathbb{L}}^1]]}(F_k, M).\]
Let $E_{\lambda}$ be the $\lambda$-eigenspace with respect to the action of $\omega$. Note that $E_{\lambda}$ is an $F_4[v_1]$-module. Define

\[ \phi_{x_0} = \phi_i + \phi_j + \phi_k, \quad \phi_{x_1} = \phi_i + \zeta \phi_j + \zeta^2 \phi_k, \quad \phi_{x_2} = \phi_i + \zeta^3 \phi_j + \zeta \phi_k. \]

Let

\[ G_0^* = (\phi_e + \phi_{x_0}) : E_1 \to E_1, \]
\[ G_1^* = - (\zeta^2 \phi_{x_1}, \zeta \phi_{x_2}) : E_1 \to E_{\zeta^2} \oplus E_{\zeta}, \]
\[ G_2^* = ((\zeta^2 - \zeta) \phi_{x_2}, (\zeta - \zeta^2) \phi_{x_1}) : E_1 \to E_{\zeta} \oplus E_{\zeta^2}, \]
\[ G_3^* = - \phi_3 \phi_{x_0} : E_1 \to E_1, \]
\[ G_4^* = - \phi_3 \phi_{x_0} (\phi_e + \phi_{x_0}) \phi_{x_2} : E_1 \to E_1 \]

so that

\[ F_k^* = G_k^* (\phi_e - \phi_{\alpha}^-), \]

where $(\phi_e - \phi_{\alpha}^-) : E_1 \to E_1$. Let $x$ be an element of $\text{Ext}^k_{W[[S_G]]} (\mathbb{W}[[S_C/C_0]], M)$. Choose a representative $\tilde{x} \in \text{Hom}_{W[[S_G]]} (\text{Ind}_{C_0}^G (B_k), M)$. Since $(\phi_e - \phi_{\alpha}^-) (\tilde{x}) = v_2^3 \tilde{x}'$, and $G_k^*$ is a $v_1$-linear map, we have

\[ F_k^* (\tilde{x}) = v_2^3 G_k^* (\tilde{x}') \]

Let $\tilde{y} = G_k^* (\tilde{x}')$. Since $\tilde{x}$ is a cocycle, so is $v_2^3 \tilde{y}$. Since, the differential $d$ of $\text{Hom}_{W[[S_G]]} (\text{Ind}_{G_{24}}^G (C_*), M)$ is $v_1$-linear, and $\text{Hom}_{W[[S_G]]} (\text{Ind}_{G_{24}}^G (C_*), M)$ has no $v_1$-torsion, $\tilde{y}$ is also a cocycle. Let

\[ y \in \text{Ext}^k_{W[[S_G]]} (\mathbb{W}[[S_C/G_{24}]], M) \]

be the class detected by $\tilde{y}$. Then $F^* (x) = v_2^3 y$. \hfill \Box

This completes the computation of the $E_2$-page. A small sample is shown in Figure 5.1.

5.5. **Higher Differentials.** In this section, we prove that all differentials $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ for $r \geq 2$ are zero. Because of the sparsity of the spectral sequence, the only differentials $d_r$ for $r \geq 2$ which do not have a zero target are

\[ d_2 : E_2^{0,q} \to E_2^{2,q-1}, \quad q \geq 2 \]
\[ d_2 : E_2^{1,q} \to E_2^{3,q-1}, \quad q \geq 2 \]
\[ d_3 : E_3^{0,q} \to E_3^{3,q-2}, \quad q \geq 3. \]

The proof of the following result is a direct computation. A similar computation is done in [12, §4], and our notation corresponds to theirs.

**Lemma 5.31.** Let $v_1$ have degree $(s, t) = (2, 0)$, $v_2$ have degree $(6, 0)$, and $h$ have degree $(0, 1)$. Let $x = v_3^2 h$. Then

\[ H^* (C_6; (E_C)_*) \cong \mathbb{W}[[u_1^3]] [v_1^2, v_1 v_2, v_2^2, v_2^3, x] / (2x). \]

**Lemma 5.32.** All differentials $d_2 : E_2^{1,q} \to E_2^{3,q-1}$ are zero.

**Proof.** Let $b_n$ be as in Theorem 5.5. The set

\[ B = \{ h^k b_n \mid n = 0, 1, 2^r (1 + 4t), \quad 0 \leq k \leq 3, \quad 0 \leq s \} \]

generates $E_3^{1,*}$ as an $F_4[v_1, g_0]$-module, for $g_0$ as in Lemma 5.24. Because the differentials are $F_4[v_1, g_0]$-linear, it suffices to show that the $d_2$ differentials on the
The $E_2$-term of the ADRSS with coefficients $(E^c_\ast)_{V}(0)$. The notation and grading is as in Figure 5.1. In addition, a $\odot$ is a copy of $\mathbb{F}_4[v_1]/(v^c_1)$. In Section 5.5, we prove that $E_2 \cong E_\infty$. Therefore, this is also the $E_\infty$-term of the ADRSS.
elements of $B$ are zero. First, note that $d_2(b_n) = 0$ for all $n$, since the targets of these differentials are zero. Hence, it suffices to show that $d_2(h^k b_n) = 0$ for $1 \leq k \leq 3$.

The first remark is that, if $d_2(h^k b_n) = 0$, then

$$v_1 d_2(h^{k+1} b_n) = d_2(v_1 h^{k+1} b_n) = d_2(h^1 h^k b_n) = h_1 d_2(h^k b_n) = 0.$$

Hence, if $d_2(h^k b_n) = 0$, then $v_1 d_2(h^{k+1} b_n) = 0$. Further,

$$v_1^t d_2(h^k b_n) = d_2(h_1^t b_n) = h_1^t d_2(b_n).$$

Since $d_2(b_n) = 0$, we must have that $v_1^t d_2(h^k b_n) = 0$ for all $k \geq 0$.

Let $1 \leq k \leq 3$. Then $d_2(h^k b_0)$ is an element of internal degree $t = 0$ in $E_2^{3,k-1}$. Since $d_2(b_0) = 0$, $v_1 d_2(h b_0) = 0$. However, there is no $v_1$-torsion in $(E_2^{3,0})_0$, hence $d_2(h b_0) = 0$. Further, $(E_2^{3,1})_0$ and $(E_2^{3,2})_0$ are zero and $d_2(h b_0) = 0$ for $k = 2, 3$ for degree reasons.

Next, consider the elements of the form $h^k b_1$ for $1 \leq k \leq 3$. Since $d_2(h^k b_1)$ is in $E_2^{3,k-1}$ and there is no $v_1$-torsion in $E_2^{3,k-1}$ for $1 \leq k \leq 3$, these differentials must be zero.

The classes $h^kB_{2s+1}(1+4t)$ have internal degree $3 \cdot 2^{s+2}(1+4t)$. Hence, their degree is congruent to zero modulo 3.

First, consider the case when $k = 1$. The possible targets for the $d_2$ differentials on these classes are in $E_2^{3,0}$ and must be annihilated by $v_1$. Therefore, they must be of the form

$$v_1^t d_2(h^k b_n) = d_2(h_1^t b_n) = h_1^t d_2(b_n).$$

However, such classes have internal degree congruent to 1 modulo 3, since the degree of $d_2(h^k b_1)$ is $24 \cdot 2^s(1+2t)$ and the degree of $v_1$ is 2. Hence, there is no appropriate target for these differentials. Further, this implies that $d_2(h^2 b_n)$ is annihilated by $v_1$.

The classes which are annihilated by $v_1$ in $E_2^{3,1}$ are of one of the forms

$$v_1^t d_2(h^k b_n) = d_2(h_1^t b_n) = h_1^t d_2(b_n).$$

$h_2 d_2$, $v_1 x d_2$, or $y d_2$. Here, $h_2$ has internal degree 4, $x$ has internal degree 8 and $y$ has internal degree 16. Again, such classes have internal degree congruent to 1 modulo 3, so there is no possible target for the differentials. This, in turn, implies that $d_2(h^3 b_n)$ is annihilated by $v_1$.

The classes in $E_2^{3,2}$ which are annihilated by $v_1$ are of one of the forms

$$v_1^t d_2(h^k b_n) = d_2(h_1^t b_n) = h_1^t d_2(b_n).$$

$h_2^2 d_2$, $v_1 h_1 x d_2$, $h_1 y d_2$, or $h_2 y d_2$. Of these classes, $v_1 h_1 x d_2$ and $h_1 y d_2$ have internal degree congruent to 0 modulo 3, so we must make a more careful analysis.

Note that $3 \cdot 2^{s+2}(1+4t) \equiv 0 \pmod{24}$ if $s \geq 1$, and $3 \cdot 2^2(1+4t) \equiv 12 \pmod{24}$. Since the internal degree of $h_1 y d_2$ is 18 modulo 24, it cannot be hit by a differential. Further, the internal degree of $v_1 h_1 x d_2$ is 12 modulo 24. Therefore, the only possible differentials are $d_2(h^3 b_{2(1+4t)})$, with target $v_1 h_1 x d_2$.

Let $\beta$ be the Bockstein homomorphism described in Lemma 4.6. To finish the proof, we show that

$$\beta(d_2(h^3 b_{2(1+4t)})) = 0.$$
and that
\[ \beta(v_1h_1xd_{2t}) \neq 0. \]
Since \( b_{2(1+4t)} = v_2^{2(1+4t)}f \) for some power series \( f \in \mathbb{F}_4[[u_1^3]] \), there is a class in \( H^0(C_6, (E_C)_*) \) which reduces to \( b_{2(1+4t)} \). Hence,
\[ \beta(b_{2(1+4t)}) = 0. \]
Since
\[ \beta(h^{2t+1}) = h^{2t+2}, \]
it follows that
\[ \beta(h^3b_{2(1+4t)}) = \beta(h^3)b_{2(1+4t)} + h^3\beta(b_{2(1+4t)}) = h^4b_{2(1+4t)}. \]
By Lemma 4.6,
\[ \beta(d_2(h^3b_{2(1+4t)})) = d_2(\beta(h^3b_{2(1+4t)})) = d_2(h^4b_{2(1+4t)}) = g_0d_2(b_{2(1+4t)}) = 0. \]
The cohomology \( H^*(G_{24}, (E_C)_*) \) can be obtained from [1] (see Figure 5.2). The classes \( v_1h_1xd_{2t} \) are not integral cohomology class. Further, \( \beta(v_1h_1xd_{2t}) = h_2^3d_{2t} \). In particular, \( \beta(v_1h_1xd_{2t}) \neq 0 \). Therefore,
\[ d_2(h^3b_{2(1+4t)}) \neq v_1h_1xd_{2t}, \]
and we must have \( d_2(h^3b_{2(1+4t)}) = 0 \). \( \square \)

The next few results will be necessary to prove that all remaining higher differentials are zero. First, note that the algebraic duality resolution is a resolution of \( PS^1_{C*}-\text{modules} \), where \( \mathcal{C}_0 \cong \mathbb{Z}_2[[PS^1_{C}/A_4]] \), \( \mathcal{C}_1 \cong \mathcal{C}_2 \cong \mathbb{Z}_2[[PS^1_{C}/C_3]] \) and \( \mathcal{C}_3 \cong \mathbb{Z}_2[[PS^1_{C}/A'_4]] \). There is a corresponding algebraic duality resolution spectral sequence. Let \( F^{p,q}_r \) be this spectral sequence for coefficients in \( (E_C)_*V(0) \). This spectral sequence converges to \( H^*(PS^1_{C}, (E_C)_*V(0)) \). There is a map of spectral sequences \( F^{p,q}_r \rightarrow E^{p,q}_r \) induced by the projection \( S^1_{C} \rightarrow PS^1_{C} \). The induced map \( F^{0,q}_1 \rightarrow E^{0,q}_1 \) is the map
\[ \varphi : H^q(A_4, (E_C)_*V(0)) \rightarrow H^q(G_{24}, (E_C)_*V(0)). \]

**Theorem 5.33.** The map
\[ \varphi : H^*(A_4, (E_C)_*V(0)) \rightarrow H^*(G_{24}, (E_C)_*V(0))/(h_1) \]
induced by the projection \( G_{24} \rightarrow G_{24}/C_2 \cong A_4 \) is surjective in degrees \( * \leq 3 \). In particular,
\[ H^0(A_4, (E_C)_*V(0)) \cong H^0(G_{24}, (E_C)_*V(0)), \]
and the classes \( h_2, h_3^2, x, v_1x, x^2, v_1x^2, y, h_2y, h_3^2y \) (defined in Theorem 4.11) and their translations by powers of \( \Delta \) are in the image of \( \varphi \).

**Proof.** The 2-Sylow subgroups of \( G_{24} \) and \( A_4 \) are \( Q_8 \) and \( V = Q_8/C_2 \) respectively. The \( E_{\infty} \)-terms of the \( v_1 \)-Bockstein spectral sequences for \( Q_8 \) and \( V \) are computed in Proposition 7.1 and Proposition 7.6. Since \( C_2 \) acts trivially on \( (E_C)_*V(0) \) and on \( Q_8 \), the projection \( p : Q_8 \rightarrow V \) induces a morphism of \( v_1 \)-Bockstein spectral sequences \( E^{p,t,w}_*(V) \rightarrow E^{e_*t,w}_*(Q_8) \). Further, this morphism is induced by the map constructed in Lemma 6.4. Using this, one can compute the image of the projection \( p \) on the associated graded. Taking \( C_3 \)-fixed points finishes the proof. \( \square \)
Figure 5.2. The cohomology $H^q(G_{24}, (E_C)_*)$ for $q \leq 5$. It is periodic of period 24 on $\Delta$. A $\Box$ denotes a copy of $W[[j]]$. A $\bullet$ denotes a copy of $F_4$. A $\odot$ denotes an extension by multiplication by 2.

Figure 5.3. The cohomology $H^*(A_4, (E_C)_*, V(0))$ in degrees $* \leq 3$. Gray classes go to zero under the map $\varphi : H^*(A_4, (E_C)_*, V(0)) \to H^*(G_{24}, (E_C)_*, V(0))$. 
Theorem 5.33 is depicted in Figure 5.3. It implies that, modulo the image of multiplication by $h_1$, the map $\varphi$ is surjective in degrees $q \leq 3$. All classes of degree $q \geq 4$ in $E^{0,q}_1$ are multiples of $g_0$, so their differentials will be determined by differentials on classes of degree $q \leq 3$. Further, by $h_1$-linearity it suffices to show that the differentials on the classes in the image of $\varphi$ are zero. It is therefore sufficient to compute of some differentials $d_r : F^{0,q}_r \to F^{r,q-r+1}_r$ for $q \leq 3$. The advantage of this method is that the spectral sequence $F^{p,q}_r$ is sparser than $E^{p,q}_r$. Indeed, $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective $\mathbb{P}^1\mathbb{C}$-modules. Hence, for $p = 1$ or $p = 2$,

$$F^{1,q}_1 \cong \text{Ext}^q_{\mathbb{Z}_2[[\mathbb{P}^1\mathbb{C}]]}(\mathbb{Z}_2[[\mathbb{P}^1\mathbb{C}/C_3]], (E_C)_0, V(0))$$

is zero when $q > 0$. Hence, $F^{p,q}_1 = 0$ when $q \geq 0$ for $p = 1$ or and $p = 2$. Further, $E^{p,0}_1 \cong F^{p,0}_1$ for all $p$, so the computation of $F^{p,q}_2$ follows immediately from that of $E^{p,q}_2$,

$$F^{p,q}_2 \cong \begin{cases} E^{p,q}_2 & q = 0 \\ E^{p,q}_1 & q > 0. \end{cases}$$

The following results are generalizations of results that can be found in [11, §6]. The first result we state is Lemma 6.1 of [11].

**Lemma 5.34** (Henn-Karamanov-Mahowald). Let $R$ be a $\mathbb{Z}_2$-algebra and $M$ be an $R$-module. Let

$$0 \to \mathcal{C}_3 \xrightarrow{\partial_3} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \mathbb{Z}_2 \to 0$$

be an exact sequence of $R$-modules such that $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective. Define $N_i$ recursively by $0 \to N_i \to \mathcal{C}_i \xrightarrow{\partial_i} N_{i-1} \to 0$, and let $E^{n,t}_r$ be the first quadrant spectral sequence of the exact couple

$$\text{Ext}_R(N_i, M) \quad \text{Ext}_R(N_{i-1}, M)$$

Then $E^{p,q}_1 = 0$ for $0 < p < 3$ and $q > 0$. Further, there are isomorphisms

$$\text{Ext}_R^q(N_0, M) \cong \begin{cases} \ker(E^{1,0}_1 \to E^{2,0}_1) & q = 0 \\ E^{q+1,0}_2 \cong E^{r+1,0}_3 & q = 1, 2 \\ E^{q-2,0}_3 & q \geq 3. \end{cases}$$

Let $j : N_0 \to \mathcal{C}_0$ be the inclusion. The only possible non-zero higher differentials are of the form $d_r : E^{0,q}_r \to E^{r,q-r+1}_r$, and they can be identified with the map $\text{Ext}_R^p(\mathcal{C}_0, M) \to \text{Ext}_R^p(N_0, M)$ induced by $j$.

Let $P_\ast = \mathbb{Z}_2[[\mathbb{P}^1\mathbb{C}]] \otimes_{\mathbb{Z}_2[A_d]} D_\ast$ for $D_\ast$ as defined in Lemma 6.3 below, so that $P_\ast$ is a projective resolution of $\mathbb{Z}_2[[\mathbb{P}^1\mathbb{C}/A_d]]$. Let $P'_\ast$ be the analogous projective resolution of $\mathbb{Z}_2[[\mathbb{P}^1\mathbb{C}/A'_d]]$. Let $N_0$ be defined by the exact sequence $0 \to N_0 \to \mathcal{C}_0 \xrightarrow{\gamma} \mathbb{Z}_2 \to 0$. One can splice $P'_\ast$ with the algebraic duality resolution to obtain a $\mathbb{P}^1\mathbb{C}$-projective resolution $Q_\ast$ of $N_0$.

**Lemma 5.35.** There is a map $\phi : Q_\ast \to P_\ast$ such that

$$\phi_0 : Q_0 \to P_0$$

covers the map $j : N_0 \to \mathcal{C}_0$ which sends $e_1 \to (e - \alpha)e_0$. 

\[\text{TOWARDS } \pi_\ast L_{K(2)}(V(0)) \text{ AT } p = 2 \]
Proof: Note that $Q_0 \cong P_0 \cong \mathbb{F}_4[[P_{S4}^2/C_3]]$. So the map which sends the generator of $e \otimes 1 \in Q_0$ to $(e - \alpha) \otimes 1 \in P_0$ is well defined and covers $j$. By the theory of acyclic models, this extends to a chain map $\phi$. □

The following is an observation in [11, §6]. It follows from Lemma 5.34.

**Lemma 5.36.** Let $T_{*,*}$ be the double complex satisfying $T_{*,0} = P_*$ and $T_{*,1} = Q_*$ with vertical differentials $\delta v$ and horizontal differentials $\phi_s : Q_s \to P_s$. Up to reindexing, the filtration of the spectral sequence of this double complex agrees with that of the algebraic duality resolution spectral sequence.

The following result is an adaptation of part of Lemma 6.5 of [11].

**Lemma 5.37.** Let $s > 0$. Let $z \in H^s(A_4, (E_{C}), V(0))$ be an element of internal degree $2t$ such that

$$v_1^t z = 0.$$ 

Let $c \in \text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(P_*, M)$ be a cocycle which represents $x$. Choose an element $h$ in $\text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(P_{s-1}, M)$ such that

$$\delta_P(h) = v_1^t c.$$ 

Let

$$\phi^* : \text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(P_*, M) \to \text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(Q_*, M)$$

be induced by $\phi$. Then there are elements $d$ and $d'$ in $\text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(Q_{s-1}, M)$ and an element $d''$ in $\text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(Q_s, M)$ such that

$$\phi^*_{s-1}(h) = d' + v_1^k d$$

and

$$\delta_Q(d') = v_1^k d''.$$ 

For $d''$ as above,

$$j^*(z) = [d''] \in \text{Ext}_{\mathbb{F}_4[[P_{S4}^2]]}^s((N_0, M)).$$

**Proof.** Let $M = (E_{C}), V(0)$. Recall that $E_\lambda$ denotes the $\lambda$-eigenspace with respect to the action of $\omega$ in $C_3$. Consider $\phi^*_{s-1}(h)$ in $\text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(Q_{s-1}, M)$. Identify

$$\text{Hom}_{\mathbb{F}_4[[P_{S4}^2]]}(Q_{s-1}, M) \cong \begin{cases} E_1 & s = 1, 2 \\ \bigoplus_{n+m=s-3} E_{\zeta^{2n+m}} & s \geq 3. \end{cases}$$

Since $x$ has degree $2t$, so do $h$ and $\phi^*_{s-1}(h)$. Hence,

$$\phi^*_{s-1}(h) \cong \left\{ \begin{array}{ll} (E_1)_{2t} & s = 1, 2 \\ \bigoplus_{n+m=s-3} (E_{\zeta^{2n+m}})_{2t} & s \geq 3. \end{array} \right.$$

Therefore, it is the sum of terms of the form

$$u^{-t} \sum_{i=0}^\infty a_i u_1^i = u^{-t} \sum_{i=0}^{k-1} a_i u_1^k + v_1^k \left( u^{-t+k} \sum_{i=k}^\infty a_i u_1^{i-k} \right),$$

and we can write

$$\phi^*_{s-1}(h) = d' + v_1^k d.$$
To prove (5.38), note that
\[
\delta_Q(d') + v_k^i \delta_Q(d) = \delta_Q(d' + v_k^i d)
\]
\[
= \delta_Q(\phi_{k-1}(h))
\]
\[
= \phi_{k}(\delta_P(h))
\]
\[
= \phi_{k}^*(v_k^i c)
\]
\[
= v_k^i \phi_{k}^*(c).
\]
Hence, \(\delta_Q(d') \equiv 0 \mod (v_k^i)\), that is,
\[
\delta_Q(d') = v_k^i d''
\]
for some \(d''\). So the first claim holds.

Now, note that
\[
v_k^i j^*(c) = v_k^i \phi_{k}^*(c)
\]
\[
= \phi_{k}(v_k^i c)
\]
\[
= \phi_{k}(\delta_P(h))
\]
\[
= \delta_Q(\phi_{k-1}(h))
\]
\[
= \delta_Q(d' + v_k^i d)
\]
\[
= v_k^i d'' + v_k^i \delta_Q(d).
\]
Since there is no \(v_1\)-torsion in the double complex \(\text{Hom}_{F_4[[[P_{S^1}]]]}(T_{*,*}, M)\), we must have
\[
j^*(c) = d'' + \delta_Q(d).
\]
This reduces to
\[
j^*(z) = [d''] \in \text{Ext}_{F_4[[[P_{S^1}]]]}^s((N_0, M)).
\]
\[\square\]

**Lemma 5.39.** Let \(z\) be in \(F_2^{0,q}\). Then \(d_2(z) = 0\).

**Proof.** If \(q > 1\), then \(d_2(z) = 0\) since the target of the differential is zero. Suppose that \(q = 1\). Then \(z\) is \(v_1\)-torsion. Let \(k\) be the smallest integer such that \(v_k^i z = 0\). The computations in Section 7 show that \(k = 1\) or \(k = 2\) (see Figure 7.4). Choose \(h\) as in Lemma 5.37 and write
\[
\phi_0(h) = (e - \phi_{\alpha})(h) = d' + v_1^k d.
\]
However,
\[
\phi_{\alpha} \equiv \text{id} \mod v_1^3.
\]
So we must have
\[
d' = 0.
\]
By Lemma 5.34 and Lemma 5.37, this implies that \(d_2(z) = 0\) in the algebraic duality spectral sequence for \(P_{S^1}^\vee\). \(\square\)

**Corollary 5.40.** All differentials \(d_2 : E_2^{0,q} \to E_2^{2,q-1}\) are zero.

**Lemma 5.41.** All differentials \(d_3 : E_3^{0,q} \to E_3^{3,q-1}\) are zero.
Proof. Differentials $d_3 : E_3^{0,q} \to E_3^{3,q-2}$ are zero for degree reasons if $0 \leq q < 2$. By Corollary 5.40, the classes $h_2 \Delta^k$ survive to the $E_3$-term, and hence they must be permanent cycles. Thus, they represent cohomology classes in $H^*(S^3, (E^*), V(0))$. By Lemma 4.7, the differentials are $h_2 \Delta^k$-linear for all $k \in \mathbb{Z}$. Using this fact and linearity with respect to $h_1$ and $v_1$, the problem reduces to verifying the claim for $x^2 \Delta^k$. However, by the same argument, $x$ is a permanent cycle and $d_3(x^2 \Delta^k) = x d_3(x \Delta^k) = 0$.

□

Lemma 5.42. All differentials $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ are zero.

Proof. By Lemma 5.39 and Lemma 5.41, $E^{*,*}_4 \cong E^{*,*}_4$. The spectral sequence collapses at the $E_4$-term since the targets of all higher differentials are zero. □

6. Appendix I: Some projective resolutions

The group $G_{24}$ contains the central subgroup $C_2$. Further,
$$G_{24}/C_2 \cong A_4,$$
where $A_4$ is the alternating group on four letters. It will be necessary for the computation to know which classes are in the image of the map induced by the projection $G_{24} \to A_4$. To do this computation, one can use explicit projective resolutions for these groups, and a morphism of resolutions in order to compare their cohomology. I construct these resolutions and this map in this section. The resolutions described are $C_3$-equivariant versions of the classical projective resolutions for the finite groups involved. Having these resolutions, their cohomology with coefficients in $(E^*_G) V(0)$ can be computed using $v_1$-Bockstein spectral sequences as described in [6]. This is carried out in Section 7.

Choose generators $i$ and $j$ for $Q_8$ such that $\omega i \omega^{-1} = j$, and $\omega^2 i \omega^{-2} = ij$. Let $e$ be the identity. For any subgroup $H$ of $G_{24}$ containing $i^2 = -1$, I will call this element $i^2$ even if the group $H$ does not contain $i$. I do this in order to avoid confusion with the coefficients $-1 \in \mathbb{Z}_2$.

Let $\chi_{\zeta}$ be the representation of $C_3$ whose underlying module is $W$ and such that $\omega \in C_3$ acts by multiplication by $\zeta^s$. For a representation $\chi$ of $C_3$ and a group $G$ which contains $C_3$, we can form the induced module
$$\text{Ind}_{C_3}^G(\chi) := W[G] \otimes_{\mathbb{W}[C_3]} \chi.$$

The modules $\text{Ind}_{C_3}^{G_{24}}(\chi_{\zeta})$ are projective $Q_8$-modules. Define the following elements of $\mathbb{W}[G_{24}]$:

$$x_0 = i + j + ij,$$
$$x_1 = i + \zeta j + \zeta^2 ij,$$
$$x_2 = i + \zeta^2 j + \zeta ij.$$

These are eigenvectors for the action of $\omega$ with eigenvalues 1, $\zeta^2$ and $\zeta$ respectively.

The following proposition gives a periodic projective resolution of the trivial $G_{24}$-module $\mathbb{W}$. In essence, this is a $C_3$-equivariant version of the Cartan-Eilenberg resolution for $Q_8$ described in [5, XII§7]. For any ring $R$, let $R\{x\}$ denote the free $R$-module on the generator $x$. 
Lemma 6.1. There is a periodic projective resolution of the trivial $G_{24}$-module $\mathbb{W}$ given by

\[
C_k = \begin{cases}
\mathbb{W}[G_{24}/C_3]\{c_{k,0}\} & k \equiv 0, 3 \mod 4 \\
\text{Ind}_{C_3}^{G_{24}}(\chi^2)\{c_{k,1}\} \oplus \text{Ind}_{C_3}^{G_{24}}(\chi)\{c_{k,2}\} & k \equiv 1 \mod 4, \\
\text{Ind}_{C_3}^{G_{24}}(\chi)\{c_{k,1}\} \oplus \text{Ind}_{C_3}^{G_{24}}(\chi^2)\{c_{k,2}\} & k \equiv 2 \mod 4.
\end{cases}
\]

where the differentials

\[d_{4k+i} : C_{4k+i} \rightarrow C_{4k+i-1}\]

are given by

\[
d_{4k+1}(c_{4k+1,0}) = x_1c_{4k,0} + x_2c_{4k,2} \\
d_{4k+2}(c_{4k+2,1}) = -x_1c_{4k+1,1} + (e+i^2)c_{4k+1,2} \\
d_{4k+2}(c_{4k+2,2}) = (e+i^2)c_{4k+1,1} - x_2c_{4k+1,2} \\
d_{4k+3}(c_{4k+3,1}) = x_1c_{4k+2,0} \\
d_{4k+3}(c_{4k+3,2}) = x_2c_{4k+2,0} \\
d_{4k+4}(c_{4k+4,0}) = (e+x_0)(e+i^2)c_{4k+3,0}.
\]

That is, the differentials are given by the right action of the following matrices:

\[
d_{4k+1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad d_{4k+2} = \begin{pmatrix} -x_1 & e+i^2 \\ e+i^2 & -x_2 \end{pmatrix}, \\
d_{4k+3} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad d_{4k+4} = (e+x_0)(e+i^2).
\]

Lemma 6.2. There is a periodic projective resolution of the trivial $C_6$-module $\mathbb{W}$ given by

\[B_k = \mathbb{W}[C_6/C_3]\{b_k\},\]

and whose differentials are $d_{k+1} : B_{k+1} \rightarrow B_k$ given by

\[d_{k+1}(b_{k+1}) = (e+(-1)^k i^2)b_k.\]

Applying $\mathbb{Z}_2[[S_i]] \otimes_{\mathbb{Z}_2[G_{24}]}$ to $C_*$ and $\mathbb{Z}_2[[S_{1/2}]] \otimes_{\mathbb{Z}_2[C_6]}$ to $B_\bullet$ gives $C_3$-equivariant projective resolutions of the $C_{1/2}^1$-modules of $\mathbb{Z}_2[[S_{1/2}/G_{24}]]$ and $\mathbb{Z}_2[[S_{1/2}/C_6]]$.

The following is a $C_3$-equivariant analogue of the Cartan-Eilenberg resolution of the trivial $\mathbb{F}_4[C_2 \times C_2]$-module $\mathbb{F}_4$.

Lemma 6.3. Let $A_4 \cong G_{24}/C_2$. Let $\chi = \chi \otimes_{\mathbb{W}} \mathbb{F}_4$. Denote also by $x_1$ and $x_2$ the image of their projections under the natural map $\mathbb{W}[G_{24}] \rightarrow \mathbb{F}_4[A_4]$. There is a projective resolution of the trivial $\mathbb{F}_4[A_4]$-module $\mathbb{F}_4$ which, in degree $k$, is given by

\[D_k = \bigoplus_{s+t=k} \text{Ind}_{C_3}^{A_4}(\chi_{2s+1})\{d_{s,t}\}.
\]

The differential on $D_*$ is given by the unique $A_4$-linear maps determined by

\[d(d_{s,t}) = x_1d_{s-1,t} + x_2d_{s,t-1}.
\]

Proof. The modules $\text{Ind}_{C_3}^{A_4}(\chi_{2s+1})$ are projective $\mathbb{F}_4[A_4]$-modules. Using the fact that $x_1^2 = 0$, a direct computation shows that $d^2 = 0$. Further, the differential commutes with the left action of $C_3$. It remains to show that $(D_*, d)$ is an exact chain complex.

Recall that

\[A_4 \cong (C_2 \times C_2) \rtimes C_3.
\]
To prove that the complex $D_*$ is exact, we will prove that it is exact as a complex of $\mathbb{F}_4[C_2 \times C_2]$-modules. As left $\mathbb{F}_4[C_2 \times C_2]$-modules,

$$\text{Ind}_{C_3}^{A_4}(\chi_{2s}) \cong \mathbb{F}_4[C_2 \times C_2].$$

Let $E(x_1, x_2)$ be the exterior sub-algebra of $\mathbb{F}_4[C_2 \times C_2]$ generated by $x_1$ and $x_2$. The natural inclusion

$$\iota : E(x_1, x_2) \to \mathbb{F}_4[C_2 \times C_2]$$

induces an isomorphism of left $\mathbb{F}_4[C_2 \times C_2]$-modules. Let $\Gamma[\gamma]$ denote the divided power algebra generated by $\gamma$. Consider the projective resolution of the trivial $E(x_1, x_2)$-module $\mathbb{F}_4$ given by

$$X = E(x_1, x_2) \otimes \Gamma[\gamma_1, \gamma_2]$$

and differential $d(\gamma_1^{s} \gamma_2^{t}) = x_1 \gamma_1^{s-1} \gamma_2^{t} + x_2 \gamma_1^{s} \gamma_2^{t-1}$. Then $X$ is a projective resolution of the trivial $E(x_1, x_2)$-module $\mathbb{F}_4$. Further, the map of chain complexes

$$\varphi : X \to D_*$$

determined by

$$\varphi(a(\gamma_1^{s} \gamma_2^{t})) = \iota(a)d_{s,t}$$

is an isomorphism of chain complexes. $\square$

**Lemma 6.4.** The complex $D_*$ is a complex of $G_{24}$-modules via restriction along the natural map $G_{24} \to A_4$. The map $\phi : C_* \to D_*$ determined by

$$\phi(c_{k,i}) = \begin{cases} 
  d_{0,0} & k = i = 0 \\
  d_{1,0} & k = i = 1, \\
  d_{0,1} & k = 1, i = 2, \\
  d_{2,0} & k = 2, i = 1, \\
  d_{0,2} & k = i = 2, \\
  d_{3,0} + d_{0,3} & k = 3, i = 0 \\
  0 & \text{otherwise}
\end{cases}$$

lifts the canonical map $\mathbb{W} \to \mathbb{F}_4$.

To compute with these resolutions, it is necessary to understand the eigenspaces of $(E_\mathcal{C})_* V(0) = \mathbb{F}_4[[u_1]][u^k]$ with respect to the $C_3$ action.

**Lemma 6.5.** Let $\mathcal{E}_\lambda$ be the $\lambda$-eigenspace of $(E_\mathcal{C})_* V(0)$ with respect to the action of the generator $\omega \in C_3$. For $k \in \mathbb{Z}$,

$$\mathcal{E}_\lambda = \begin{cases} 
  \mathbb{F}_4[v_1]\{u^{3k}\}_{k \in \mathbb{Z}} & \lambda = 1 \\
  \mathbb{F}_4[v_1]\{u^{3k+1}\}_{k \in \mathbb{Z}} & \lambda = \zeta \\
  \mathbb{F}_4[v_1]\{u^{3k+2}\}_{k \in \mathbb{Z}} & \lambda = \zeta^2
\end{cases}$$

Further,

$$(E_\mathcal{C})_* V(0) \cong \mathcal{E}_1 \oplus \mathcal{E}_\zeta \oplus \mathcal{E}_{\zeta^2}$$

as $C_3$-modules.

**Proof.** This holds since $\phi_\omega(u_1) = \zeta u_1$ and $\phi_\omega(u) = \zeta u$. $\square$
7. Appendix II: The $v_1$-Bockstein spectral sequence

Our next goal is to introduce the $v_1$-Bockstein spectral sequence described in [6, §1]. Recall that $S_C$ is the $2$-Sylow subgroup of $S_C$. Fix a closed subgroup $G$ of either $S_C$ or $PS_C$, where

$$PS_C = S_C/C_2.$$ 

Consider the exact couple

$$H^*(G; \mathbb{F}_4[[u]](u^\pm 1)/(u_1^n)) \xrightarrow{v_1^n} H^*(G; \mathbb{F}_4[[u]](u_1^n)) \rightarrow H^*(G; \mathbb{F}_4[[u]](u^\pm 1)/(u_1))$$

Let $s$ denote the cohomological degree, $t$ denote the internal degree and $w$ denote the filtration degree. The above exact couple gives rise to a strongly convergent tri-graded spectral sequence

$$\bigoplus_{w \geq 0} H^*(G; \mathbb{F}_4[u^\pm 1]) \Rightarrow H^*(G; \mathbb{F}_4[[u]](u^\pm 1)).$$

We will show in Theorem 3.7 that the action of $S_C$ on $(E_2)_*$ is trivial modulo $(2, v_1)$. Therefore, for any subgroup $G$ of $S_C$ or $PS_C$

$$E^{s,t,w}_r = H^s(G; \mathbb{F}_4[u^\pm 1]) \cong H^s(G; \mathbb{F}_4) \otimes \mathbb{F}_4[u^\pm 1].$$

The differentials are given by

$$d_r: E^{s,t,w}_r \to E^{s+1,t-2r,w+2r}_r.$$ 

They can be computed using any $\mathbb{F}_4[G]$-projective resolution $P_*$ of $\mathbb{F}_4$ as follows.

Let $\partial_s: P_s \to P_{s-1}$ denote the differentials of $P_*$. Let $x$ be in $H^s(G; \mathbb{F}_4[u^\pm 1])$. Choose a representative $\tilde{x}$ in the complex $\text{Hom}_G(P_s, \mathbb{F}_4[u^\pm 1])$. Let $\tilde{x}'$ be a lift of $\tilde{x}$ in the complex $\text{Hom}_G(P_s, \mathbb{F}_4[[u]])[u^\pm 1])$. Let

$$\partial^{s+1}: \text{Hom}_G(P_s, \mathbb{F}_4[[u]])[u^\pm 1]) \to \text{Hom}_G(P_{s+1}, \mathbb{F}_4[[u]])[u^\pm 1])$$

be the morphism induced by $\partial_{s+1}$. Then

$$\partial^{s+1}(\tilde{x}') = v_1^n \tilde{y}$$

for some $\tilde{y}$ in $\text{Hom}_G(P_s, \mathbb{F}_4[[u]])[u^\pm 1])$. Let $y$ be the class in $H^{s+1}(G; \mathbb{F}_4[u^\pm 1])$ detected by the image of $\tilde{y}$ in $\text{Hom}_G(P_{s+1}, \mathbb{F}_4[[u]])[u^\pm 1])$. The differential $d_r$ is defined by

$$d_r(x) = y.$$ 

The differentials are $v_1$-linear and this is a spectral sequence of modules over $H^*(G; \mathbb{F}_4)$.

**Proposition 7.1.** Let $E^{s,t,w}_r(Q_8)$ be the $v_1$-Bockstein spectral sequence computing $H^*(Q_8; (E_C)_*V(0))$. Then

$$E^{s,*,*}_\infty = \mathbb{F}_4[g_0, u^{-4}] \otimes \left( \mathbb{F}_4[v_1] \{1, h_{1,0}u^{-1}, h_{1,0}u^{-2}, h_{1,0}u^{-3}\} \right.$$

$$\oplus \mathbb{F}_4[v_1]/(v_1^2) \{h_{1,0}, h_{1,0}^2, h_{1,0}^3 u^{-1}, h_{1,0}^3 u^{-1}\}$$

$$\oplus \mathbb{F}_4 \{h_{1,1}, h_{1,1}^2, h_{1,1}^3, h_{1,1} u^{-2}, h_{1,1} u^{-2}, h_{1,1}^3 u^{-2}\}.$$
A class is named by the name of the class which detects it, and
\begin{equation}
H^*(Q_8, F_4) \cong F_4[g_0, h_{1,0}, h_{1,1}]/(h_{1,0}h_{1,1}, h_{1,0}^3 + h_{1,1}^3)
\end{equation}
where \( h_{1,i} \) has cohomological degree 1 and \( g_0 \) has cohomological degree 4. Further, the action of \( C_3 \) is determined by
\begin{align*}
\phi_\omega(u) &= \omega u \\
\phi_\omega(h_{1,i}) &= \zeta^2 h_{1,i}.
\end{align*}

**Proof.** Let \( \tilde{h}_{1,0} \) be the canonical generator of \( \text{Hom}_{Q_8}(\text{Ind}_{C_3}^{G_2}(\chi), F_4) \) where
\begin{equation}
\text{Hom}_{Q_8}(\text{Ind}_{C_3}^{G_2}(\chi), F_4) \subseteq \text{Hom}_{Q_8}(C_1, F_4)
\end{equation}
and let \( \tilde{h}_{1,1} \) be the canonical generator of \( \text{Hom}_{Q_8}(\text{Ind}_{C_3}^{G_2}(\chi), F_4) \) where
\begin{equation}
\text{Hom}_{Q_8}(\text{Ind}_{C_3}^{G_2}(\chi), F_4) \subseteq \text{Hom}_{Q_8}(C_1, F_4).
\end{equation}

Let \( h_{1,i} \) be the corresponding cohomology classes in \( H^1(Q_8, F_4) \). Then \( H^*(Q_8, F_4) \) is given by (7.2). Note that
\begin{equation}
\phi_\omega(h_{1,i}) = \zeta^2 h_{1,i}.
\end{equation}
The cohomology \( H^*(Q_8, (E_C)_* V(0)) \) is computed by the following periodic resolution:
\begin{equation}
E_1 \xrightarrow{\begin{pmatrix} \phi_{x_1} \\ \phi_{x_2} \end{pmatrix}} E_{1} \oplus E_{1} \xrightarrow{\begin{pmatrix} \phi_{x_1} & 0 \\ 0 & \phi_{x_2} \end{pmatrix}} E_{1} \oplus E_{1} \xrightarrow{\begin{pmatrix} \phi_{x_1} & \phi_{x_2} \end{pmatrix}} E_{1} \xrightarrow{0} E_{1}.
\end{equation}
Therefore, the cohomology of \( H^*(Q_8, (E_C)_* V(0)) \) is periodic of period 4 with respect to the cohomological grading \( s \). The periodicity generator is given by the image of \( g_0 \) in \( H^4(Q_8, (E_C)_* V(0)) \). It is also periodic of period 8 with respect to its internal grading. Indeed, define the \( Q_8 \)-invariant
\begin{equation}
\delta = (u \cdot \phi(u) \cdot \phi(u) \cdot \phi(u))^{-1} = u^{-4}(1 + u_1^3) \mod (2).
\end{equation}
Note that \( \delta^3 = \Delta \). Since \( \delta \) is invertible, it induces an isomorphism
\begin{equation}
H^*(Q_8, (E_C)_* t) \to H^*(Q_8, (E_C)_{t+8}).
\end{equation}
The map induced by \( \delta \) on the associated graded
\begin{equation}
\bigoplus E_{1}^{s,t,w} \to \bigoplus E_{1}^{s,t+8,w}
\end{equation}
is given by multiplication by \( u^{-4} \). Therefore, all differentials in the \( v_1 \)-Bockstein spectral sequence are \( u^{-4} \)-linear.

This reduces the computation to a few simple verifications. It is sufficient to compute the differentials for \( u^{-r} \) when \( r = 0, 1, 2, 3 \) and the differential on \( h_{1,1}u^{-3} \).

Let \( \phi_{x_i} \) denote the action of \( x_i \) on \( (E_2)_* V(0) \). When \( r = 0 \), all differentials are zero. If \( r = 1 \), we have
\begin{equation}
\phi_{x_1}(u^{-r}) = \begin{cases} 
0 & r = 0 \\
0 & r = 1 \\
v_1^2 & r = 2 \\
u^{-1}v_1^2 & r = 3.
\end{cases}
\end{equation}
This gives the following key differentials
\[ d_1(u^{-1}) = v_1 h_{1,1}, \quad d_1(u^{-3}) = u^{-2} v_1 h_{1,1} \]
and
\[ d_2(u^{-2}) = v_2^2 h_{1,0}, \quad d_2(u^{-3} h_{1,0}) = u^{-1} v_1^2 h_{1,0}^2. \]
All other differentials are determined by \( v_1 \) and \( u^{-4} \) linearity.

Figure 7.1 illustrates the \( E_\infty \)-term. Classes have been named according to the element which detects them. Taking \( C_3 \)-fixed points, we obtain an associated graded for \( H^*(G_{24}, (E_C^*), V(0)) \), which is depicted in Figure 7.3.

**Proposition 7.6.** Let \( V = Q_8/C_2 \). Let \( E_\infty^{*,*,*} (V) \) be the \( v_1 \)-Bockstein spectral sequence computing \( H^*(V; (E_C^*), V(0)) \). Then,

\[
E_\infty^{*,*,*} = \mathbb{F}_4[u^{-4}] \otimes \left( \mathbb{F}_4[v_1] \{1\} \oplus \mathbb{F}_4[v_1]/(v_1^2) \{h_{1,0}, h_{1,0}^2, h_{1,1}, h_{1,1}^2\} \right)
\]
\[
\oplus \mathbb{F}_4\{h_{1,1}^2, h_{1,0}^2 h_{1,1}, h_{1,0} h_{1,1}^2, h_{1,1}^2, h_{1,1}^3, \}
\]
\[
h_{1,1} u^{-2}, h_{1,1}^2 u^{-2}, h_{1,0} h_{1,1} u^{-2}, h_{1,0}^2 h_{1,1} u^{-2}, h_{1,1}^3 u^{-2}, h_{1,0} h_{1,1}^2 u^{-2}\}.
\]

A class is named by the name of the class which detects it, where

\[
H^*(V, \mathbb{F}_4) \cong \mathbb{F}_4[h_{1,0}, h_{1,1}]
\]
with \( h_{1,i} \) of cohomological degree 1. Further, the action of \( C_3 \) is determined by

\[
\phi_\omega(u) = \omega u
\]
\[
\phi_\omega(h_{1,i}) = \zeta^2 h_{1,i}.
\]

**Proof.** Let \( \bar{h}_{1,0} \) be the canonical generator of \( \text{Hom}_V(\text{Ind}_{C_3}^A(\chi_{C_2}), \mathbb{F}_4) \) where

\[
\text{Hom}_V(\text{Ind}_{C_3}^A(\chi_{C_2}), \mathbb{F}_4) \subseteq \text{Hom}_V(\text{Ind}_{C_3}^A(D_1, \mathbb{F}_4)).
\]

Let \( \bar{h}_{1,1} \) the canonical generator of \( \text{Hom}_V(\text{Ind}_{C_3}^A(\chi_{C_2}), \mathbb{F}_4) \) where

\[
\text{Hom}_V(\text{Ind}_{C_3}^A(\chi_{C_2}), \mathbb{F}_4) \subseteq \text{Hom}_V(\text{Ind}_{C_3}^A(D_1, \mathbb{F}_4)).
\]

Let \( h_{1,i} \) be the corresponding cohomology classes in \( H^1(V, \mathbb{F}_4) \). Note that under the morphism of chain complexes induced by the map \( C_3 \to D_3 \) described in Lemma 6.4, the element \( \bar{h}_{1,i} \) in \( \text{Hom}_V(D_3, \mathbb{F}_4) \) is sent to the element of the same name in \( \text{Hom}_Q^1(C_1, \mathbb{F}_4) \). This justifies our choice of notation. Then \( H^*(V, \mathbb{F}_4) \) is given by

\[
(7.7) \quad H^*(V, \mathbb{F}_4) \cong \mathbb{F}_4[h_{1,0}, h_{1,1}]
\]
with \( h_{1,i} \) of cohomological degree 1. Further, the action of \( C_3 \) is determined by

\[
\phi_\omega(u) = \omega u
\]
\[
\phi_\omega(h_{1,i}) = \zeta^2 h_{1,i}.
\]
The element $\delta$ defined by (7.3) is again invariant and invertible. Therefore, the cohomology $H^\ast(V, (E_C)_\ast V(0))$ is also periodic of period 8 with respect to its internal grading. Multiplication by $\delta$ induces an isomorphism
\[
\bigoplus E_{s,t,w}^{s,t} \to \bigoplus E_{s,t+8,w}^{s,t+8}
\]
in the associated $v_1$-Bockstein spectral sequence given by multiplication by $u^{-4}$. The formulas given by (7.4) and (7.5) give the two key differentials:
\[
d_1(u^{-1}) = v_1 h_{1,1}, \quad d_2(u^{-2}) = v_2^2 h_{1,0}.
\]
All other differentials follow from linearity under multiplication by $v_1, u^{-4}$ and elements of $H^\ast(V, F_4)$. \hfill \Box

The $E_\infty$-term is drawn in Figure 7.2, where classes are named according to the class that detects them in the $E_1$-term of the spectral sequence. Taking $C_3$-fixed points gives the associated graded for $H^\ast(A_4, (E_C)_\ast V(0))$, which is depicted in Figure 7.4.
Figure 7.1. The $E_\infty$-term for the $v_1$-Bockstein spectral sequence computing $H^*(Q_8, (E_C)_*V(0))$. Horizontal lines denote multiplication by $v_1$. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The $E_\infty$-term is periodic of period 8 with respect to the internal grading $t$ on a class $\delta = u^{-4}(1 + u_1^3)$, which is detected by $u^{-4}$ in degree $(t - s, s) = (8, 0)$. It is periodic of period 4 with respect to the cohomological grading $s$ on a class detected by $g_0$ (see (7.2)) in degree $(t - s, s) = (-4, 4)$.

Figure 7.2. The $E_\infty$-term for the $v_1$-Bockstein spectral sequence computing $H^*(V, (E_C)_*V(0))$. Horizontal lines denote multiplication by $v_1$. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The $E_\infty$-term is periodic of period 8 with respect to the internal grading $t$ on a class $\delta = u^{-4}(1 + u_1^3)$, which is detected by $u^{-4}$ in degree $(t - s, s) = (8, 0)$. 
Figure 7.3. The associated graded for $H^*(G_{24}, (E_C)_*V(0))$. Horizontal lines denote multiplication by $v_1$. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The associated graded is periodic of period 24 with respect to the internal grading $t$ on the class $\Delta = u^{-12}(1 + u_1^3)^3$, which is detected by $u^{-12}$ in degree $(t - s, s) = (24, 0)$. It is periodic of period 4 with respect to the cohomological degree $s$ on a class detected by $g_0$ in degree $(t - s, s) = (-4, 4)$. Dotted lines denote extensions which are known from [1, §7]. The element $h_{1,0}u^{-1}$ detects $h_1$.

Figure 7.4. The associated graded for $H^*(A_4, (E_C)_*V(0))$. Horizontal lines denote multiplication by $v_1$. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The associated graded is periodic of period 24 with respect to the internal grading $t$ on the class $\Delta = u^{-12}(1 + u_1^3)^3$, which is detected by $u^{-12}$ in degree $(t - s, s) = (24, 0)$. Gray classes map to zero under the map $H^*(A_4, (E_C)_*V(0)) \to H^*(G_{24}, (E_C)_*V(0))$. 
8. Appendix III: The action of the Morava stabilizer group

The goal of this appendix is to approximate the action of elements of $S_C$ on $(E_C)_\ast$. Some of our results are stronger than needed for the computations of this paper, but the more precise estimates are necessary for future computations. Note that the results of this section prove Theorems 3.6 and 3.7.

8.1. The formal group laws. Let $E$ be an elliptic curve with Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$ 

Let $F_z(z_1, z_2)$ be the formal group law of $E$, where the coordinates $(z, w)$ at the origin are chosen so that

$$w(z) = z^3 + a_1zw(z) + a_2z^2w(z) + a_3w(z)^2 + a_4zw(z)^2 + a_6w(z)^3. \tag{8.1}$$

That the group $S_C$ acts on $(E_C)_\ast$ is a consequence of the fact that the formal group law $F_{E_C}$ of $E_C$ is a universal deformation of the formal group law $F_C$ of the elliptic curve

$$C : y^2 + y = x^3$$

defined over any field extension of $\mathbb{F}_2$. Further, $F_{E_C}$ is the formal group law of an elliptic curve, namely

$$C_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

defined over $(E_C)_0$. That is,

$$F_{E_C} = F_{C_U}.$$

The goal of this section is to gather information about $F_{C_U}$. These results will be used in the next section to compute the action of $S_C$ on $(E_C)_\ast$. We will also compute information about the formal group law of the curve

$$C_Z : y^2 - y = x^3$$

defined over $\mathbb{Z}$. The curve $C_Z$ is a lift of $C$ to $\mathbb{Z}$, and $C_U$ reduces to $C_Z$ modulo $u_1$. Therefore, we will derive information about $F_{C_U}$ from information about $F_{C_Z}$.

The following results are proved using the methods described in [18, §4]. We recall the key tools here. We restrict to elliptic curves $E$ with homogenous Weierstrass equation of the form

$$E : y^2z + a_1xyz + a_3yz^2 = x^3.$$ 

Let $z = -\frac{z}{w}$ and $w = -\frac{z}{w}$, so that $(z, w(z))$ is a coordinate chart of $E$ at the origin, with

$$w(z) = z^3 + a_1zw(z) + a_3w(z)^2.$$ 

This can be used to write $w(z)$ as a power series in $z$. Letting

$$\lambda(z_1, z_2) = \frac{w(z_2) - w(z_1)}{z_2 - z_1},$$

the line through the points $(z_1, w(z_1))$ and $(z_2, w(z_2))$ has equation

$$w(z) = \lambda(z_1, z_2)z + w(z_1) - \lambda(z_1, z_2)z_1. \tag{8.2}$$

(Note that there is a sign mistake in [18, §4.1] in the equation of the connecting line. This was pointed out to the author by Hans-Werner Henn.) Substituting (8.2) in (8.1), we obtain a monic cubic polynomial whose roots are $z_1$, $z_2$ and
\[\text{Proposition 8.5.} \quad \text{The formal group law } F_{C_U} \text{ has } [-2]\text{-series}
\]
\[[-2]_{F_{C_U}}(z) = -2z - 9z \frac{zu_1 - 2z^2u_1^2 + z^3(u_1^3 - 1)}{1 - 6zu_1 + 9z^2u_1^4 - 4z^3(u_1^3 - 1)},\]

so that
\[\text{(8.6) } [-2]_{F_{C_U}}(z) = -2z - 9u_1z^2 - 36u_1^2z^3 + 9z^4 - 144u_1^3z^4 + O(z^5).\]

**Proof.** For the curve \(C_U\), we have
\[w'(z) = \frac{3(z^2 + u_1w(z))}{1 - 3zu_1 - 2(u_1^3 - 1)w(z)}.\]

Using the fact that
\[[-2]_{F_{C_U}}(z) = -2z - 3u_1w'(z) - (u_1^3 - 1)w'(z)^2\]

and the fact that
\[(u_1^3 - 1)w(z)^2 = w(z) - z^3 - 3u_1zw(z),\]

one expands and obtains the formula for \([-2]_{F_{C_U}}(z)\). Computing the Taylor expansion gives the estimate (8.6). \(\square\)

**Corollary 8.7.**
\[[-2]_{F_{C_U}}(x) \equiv u_1x^2 + \sum_{k \geq 0} u_1^{2k}x^{2k+4} \mod (2).\]

**Proof.** It follows from Proposition 8.5 that modulo 2,
\[\text{mod } (2), \quad [\text{-2}]_{F_{C_U}}(z) \equiv \frac{u_1z^2 + u_1^3z^4 + z^4}{1 + u_1^2z^2}.\]

Therefore, modulo 2,
\[\text{mod } (2), \quad [\text{-2}]_{F_{C_U}}(z) \equiv (u_1z^2 + u_1^3z^4 + z^4) \sum_{k \geq 0} u_1^{2k}z^{2k}
\equiv u_1z^2 + \sum_{k \geq 0} u_1^{2k}z^{2k+4}.\]
Some of the key ingredients for the proof of the next result were brought to the author’s attention by Inna Zakharevich. Let
\[ C_k = \frac{(2k)!}{k!(k+1)!}. \]
be the \( k \)'th Catalan number. Let
\[ C(y) = \sum_{k \geq 0} C_k y^k \]
be their generating series. Let \( D(y) = yC(y) \). It is a standard fact that
\[ D(y) = \frac{1 - \sqrt{1 - 4y}}{2}. \]

**Proposition 8.10.** Let \( C_\mathbb{Z} \) be the elliptic curve defined over \( \mathbb{Z} \) by the Weierstrass equation
\[ C_\mathbb{Z} : y^2 - y = x^3. \]
Then
\[ [-2]C_\mathbb{Z}(z) = -2z + 9z^4 \sum_{n \geq 0} (-1)^n 4^n z^{3n}. \]
For \((z, w(z))\) a coordinate chart at the origin with \( w(z) = z^3 - w(z)^2 \),
\[ w(z) = -D((-z)^3) = \sum_{n \geq 0} (-1)^n C_n z^{3(n+1)} = \frac{\sqrt{1 + 4z^3} - 1}{2}. \]

Further,
\[ [-1]C_\mathbb{Z}F_{C_\mathbb{Z}}(z_1, z_2) = -z_1 - z_2 + (z_1^3 + z_2^3) + D((-z_1^3 + z_2^3 + 4z_1^3 z_2^3)) \]
\[ (z_2 - z_1)^2 \]

**Proof.** It follows from Proposition 8.5 that, modulo \( u_1 \),
\[ [-2]C_\mathbb{Z}(z) = -2z + 9z^4 \frac{1}{1 + 4z^3}. \]
This proves the first claim. The second claim is equivalent to showing that
\[ w(z) = z^3 C((-z)^3) \]
It is a standard result that
\[ C(z) = 1 + zC(z)^2. \]
Therefore,
\[ C((-z)^3) = 1 + (-z)^3 C((-z)^3)^2, \]
so that
\[ z^3 C((-z)^3) = z^3 - (z^3 C((-z)^3))^2. \]
Since \( w(z) \) and \( z^3 C((-z)^3) \) satisfy the same functional equation, they must be equal. Further, this implies that
\[ w(z) = \frac{\sqrt{1 + 4z^3} - 1}{2}. \]
Finally, note that
\[
\lambda(z_1, z_2) = \frac{1}{z_2 - z_1} \left( \frac{\sqrt{1 + 4z_2^3} - 1}{2} - \frac{\sqrt{1 + 4z_1^3} - 1}{2} \right)
\]
\[
= \frac{\sqrt{1 + 4z_2^3} - \sqrt{1 + 4z_1^3}}{2(z_2 - z_1)}.
\]
Therefore,
\[
[-1]_{\mathcal{C}_z} (F_{\mathcal{C}_z}(z_1, z_2)) = -z_1 - z_2 + \lambda(z_1, z_2)^2
\]
\[
= -z_1 - z_2 + \left( \frac{\sqrt{1 + 4z_2^3} - \sqrt{1 + 4z_1^3}}{2(z_2 - z_1)} \right)^2
\]
\[
= -z_1 - z_2 + \frac{2(z_1^3 + z_2^3) + 1 - \sqrt{1 + 4(z_1^3 + z_2^3 + 4z_1^3z_2^3)}}{2(z_2 - z_1)^2}
\]
\[
= -z_1 - z_2 + \frac{(z_1^3 + z_2^3) + D(- (z_1^3 + z_2^3 + 4z_1^3z_2^3))}{(z_2 - z_1)^2}.
\]

\[
\square
\]

**Proposition 8.11.** Let \( \mathcal{C} \) be the elliptic curve defined over \( \mathbb{F}_4 \) by the Weierstrass equation
\[
\mathcal{C} : y^2 + y = x^3.
\]

The local uniformizer at the origin \( w(z) = z^3 + w(z)^2 \), satisfies
\[
w(z) = \sum_{k \geq 0} z^{3 \cdot 2^k}.
\]

Further,
\[
[-2]_{\mathcal{C}} (z) = z^4,
\]
and
\[
(8.12) \quad [-1]_{\mathcal{C}} (F_{\mathcal{C}}(z_1, z_2)) = z_1 + z_2 + \sum_{k \geq 1} \sum_{n=0}^{3 \cdot 2^{k-1} - 1} (z_1^{2(3 \cdot 2^{k-1} - 1 - n)} z_2^{2n}).
\]

Further,
\[
[-1]_{\mathcal{C}} (z) = \sum_{k \geq 0} z^{3 \cdot 2^k - 2},
\]
so that
\[
F_{\mathcal{C}}(z_1, z_2) = z_1 + z_2 + z_1^2 z_2^2 + z_1^6 z_2^4 + z_1^4 z_2^6 + z_1^8 z_2^8 + z_1^{12} z_2^{12} + z_1^{14} z_2^{14} + z_1^{16} z_2^{16} + z_1^{18} z_2^{18} + \cdots
\]
where the next term has order 34.

**Proof.** One can compute directly that \( w(z) = \sum_{k \geq 0} z^{3 \cdot 2^k} \). This implies that \( C_n \neq 0 \) modulo 2 if and only if \( n + 1 \) is odd. Therefore, we have the following identity of power series
\[
D(y) = \sum_{n \geq 0} C_n y^{n+1} = \sum_{k \geq 0} y^{2^k}.
\]
Hence,
\[
[-1]_{F_{C}} (F_{C}(z_1, z_2)) = \frac{z_1 z_2 (z_1 + z_2) + D(z_1^3 + z_2^3)}{(z_1 + z_2)^2}
\]
\[
= \frac{1}{(z_1 + z_2)^2} \left( z_1 z_2 (z_1 + z_2) + \sum_{k \geq 0} (z_1^{3-2^k} + z_2^{3-2^k}) \right)
\]
\[
= \frac{1}{(z_1 + z_2)^2} \left( z_1 z_2 + \sum_{k \geq 0} \sum_{r=0}^{3-2^k-1} (z_1^{3-2^k-1-r} z_2^r) \right).
\]

The key observation is that if \( n \) is odd, then
\[
\sum_{r=0}^{n} (z_1^{n-r} z_2^r) = \sum_{r=0}^{n-1} (z_1 z_2)^r (z_1^{n-2r} + z_2^{n-2r})
\]
\[
= (z_1 + z_2) \sum_{r=0}^{n-1} \sum_{s=0}^{n-1-2r} (z_1^{n-1-r-s} z_2^{r+s})
\]

Therefore,
\[
[-1]_{F_{C}} (F_{C}(z_1, z_2)) = \frac{1}{(z_1 + z_2)^2} \left( z_1^2 + 2z_1 z_2 + z_2^2 + \sum_{k \geq 1} \sum_{r=0}^{3-2^k-1} (z_1^{3-2^k-1-r} z_2^r) \right)
\]
\[
= z_1 + z_2 + \sum_{k \geq 1} \sum_{r=0}^{3-2^k-1} 2(3 \cdot 2^{k-1} - 1) (z_1^{3-2^k-2-(r+s)} z_2^{r+s}).
\]

To simplify this expression, we must count the ways in which an integer \( n \) such that \( 0 \leq n \leq 2(3 \cdot 2^{k-1} - 1) \) can be written as a linear combination \( r + s \) where
\[
0 \leq r \leq 3 \cdot 2^{k-1} - 1
\]
and
\[
0 \leq s \leq 2(3 \cdot 2^{k-1} - 1 - r)
\]
If \( n \leq 3 \cdot 2^{k-1} - 1 \), there are \( n+1 \) combinations. If \( 3 \cdot 2^{k-1} - 1 < n \leq 2(3 \cdot 2^{k-1} - 1) \), there are \( 2(3 \cdot 2^{k-1} - 1 - n) + 1 \) combinations. Therefore, the coefficient of \( z_1^{3-2^k-2-n} z_2^n \) is zero if and only if \( n \) is even. This implies that
\[
[-1]_{F_{C}} (F_{C}(z_1, z_2)) = z_1 + z_2 + \sum_{k \geq 1} \sum_{n=0}^{3-2^k-1} \sum_{m=0}^{(3 \cdot 2^{k-1} - 1 - n) + 1} (z_1^{2(3 \cdot 2^{k-1} - 1 - n)} z_2^n).
\]

To prove the given estimate of \( F_{C}(z_1, z_2) \), note that since \( w(z) = z^3 + w(z)^2 \), we have that
\[
[-1]_{F_{C}} (z) = \frac{w(z)^2}{z^2} = \frac{z^3 + w(z)^2}{z^2} = \frac{w(z)}{z^2} = \sum_{k \geq 0} z^{3 \cdot 2^k - 2}.
\]
A direct computation using (8.12) proves the claim.

8.2. **The technique for computing the action of** \( S_C \). The method presented here is an adaptation of the techniques used in [11], which we describe here. Let \( \gamma \) be in \( S_C \). Then \( \gamma \in \mathbb{F}_4[[x]] \) is a power series which satisfies

\[
\gamma F_C(x, y) = F_C(\gamma(x), \gamma(y)).
\]

Recall from Section 2.4 that \( \gamma \) gives rise to an isomorphism \( \phi_\gamma : (E_C)_* \to (E_C)_* \) and a lift of \( \gamma \),

\[
f_\gamma : \phi_\gamma^* F_{E_C} \to F_{E_C},
\]

where

\[
f_\gamma \in (E_C)_0[[x]].
\]

The action of \( \gamma \) on \( (E_C)_* \) is given precisely by \( \phi_\gamma \).

The isomorphism \( \phi_\gamma \) is linear over \( \mathcal{W} \); hence it is sufficient to specify \( \phi_\gamma(u) \) and \( \phi_\gamma(u_1) \). The morphism \( f_\gamma \) is given by a power series

\[
f_\gamma(x) = t_0(\gamma)x + t_1(\gamma)x^2 + t_2(\gamma)x^3 + \ldots
\]

where

\[
t_i \in (E_C)_0 = \mathcal{W}[[u_1]].
\]

By (2.4)

\[(8.13)\]

\[
\phi_\gamma(u) = f_\gamma'(0)u = t_0(\gamma)u,
\]

which gives the action of \( \gamma \) on \( u \).

The morphism \( f_\gamma \) must satisfy

\[(8.14)\]

\[
f_\gamma([-2]_{\phi_\gamma^* F_{E_C}}(x)) = [-2]_{F_{E_C}}(f_\gamma(x)).
\]

This imposes a set of relations on the parameters \( t_i(\gamma) \) and \( \phi_\gamma(u_1) \). Further, \( f_\gamma \) is a lift of \( \gamma \), so that

\[(8.15)\]

\[
f_\gamma \equiv \gamma \mod (2, u_1).
\]

This specifies the parameters \( t_i(\gamma) \) modulo \( (2, u_1) \). With this information, the relations imposed by (8.14) are sufficient to approximate \( \phi_\gamma \). Before executing this program, we prove a preliminary result.

**Proposition 8.16.** If \( \gamma \in \mathbb{Z}_2^\times \cap S_C \), so that \( \gamma = \sum_{i \geq 0} a_i T^{2^i} \), for \( a_i \in \{0, 1\} \). Let

\[
\ell = \sum_{i \geq 0} a_i 2^i
\]

in \( \mathbb{Z}_2^\times \subseteq (E_C)_0 \). Then \( \phi_\gamma(u_1) = u_1 \) and \( \phi_\gamma(u) = \ell u \).

**Proof.** The element \( \gamma \) is given by

\[
\gamma(x) = a_0x + F_{E_C} a_1[-2]_{F_{E_C}}(x) + F_{E_C} a_2[4]_{F_{E_C}}(x) + F_{E_C} \ldots
\]

Let \( g \) be the lift for \( \gamma \) given by

\[
g(x) = a_0x + F_{E_C} a_1[-2]_{F_{E_C}}(x) + F_{E_C} a_2[4]_{F_{E_C}}(x) + F_{E_C} \ldots
\]

Then \( g \) is an automorphism of \( F_{E_C} \), hence \( \phi_g : (E_C)_0 \to (E_C)_0 \) is the identity. Therefore, the automorphism \( f_g \) described in Section 2.2 is the identity, and hence

\[
f_\gamma(x) = g(x).
\]

Since

\[
g(x) = \ell x + \ldots,
\]

\[
\gamma F_C(x, y) = F_C(\gamma(x), \gamma(y)).
\]
we have \( g'(0) = \ell \). Hence, \( \phi_\gamma(u) = \ell u \).

\[ \text{□} \]

**Theorem 8.17.** Let \( \gamma \in \mathbb{S}_C \) and \( t_i = t_i(\gamma) \). Then

\[ \phi_\gamma(u_1) = u_1 t_0 + \frac{2 t_1}{3 t_0}. \]  

**Proof.** Recall from (8.14) that

\[ f_\gamma([-2]_\phi^* \mathcal{F}_{E_C}(x)) = [-2]_\phi^* \mathcal{F}_\gamma(x). \]

Using (8.6), one obtains the following relation on the coefficients of \( x^2 \),

\[ -9 \phi_\gamma(u_1) t_0 + 4 t_1 = -9 u_1 t_0^2 - 2 t_1. \]

Because \( \phi_\gamma \) is an isomorphism, \( t_0 \) is invertible. Isolating \( \phi_\gamma(u_1) \) and dividing both sides by \(-9t_0\) proves the claim. \( \text{□} \)

Therefore, to approximate the action of an element \( \gamma \) in \( \mathbb{S}_C \) on \( (E_C)_{\ast} \), it suffices to approximate the parameters \( t_0(\gamma) \) and \( t_1(\gamma) \).

### 8.3. Approximations for the parameters \( t_i(\gamma) \)

In this section, we use the technique described in Section 8.2 to approximate the parameters \( t_i(\gamma) \). Our goal is to give an approximation of the action of \( \gamma \) modulo \((2, u_1^3)\). By Theorem 8.17, in order to do this, we must approximate \( t_0(\gamma) \) modulo \((2, u_1^3)\) and \( t_1(\gamma) \) modulo \((2, u_1^3)^2\) for \( \gamma \) in \( \mathbb{S}_C \), the 2-Sylow subgroup of \( \mathbb{S}_C \). The goal of this section is to obtain these estimates.

**Corollary 8.19.** Modulo \((2, u_1^3)\),

\[ t_s \equiv t_s^4 + u_1 t_s^2 + \left( \frac{s + 2}{2} \right) t_s^7 + \sum_{i=0}^{s-1} u_1^2 t_s^4 - \frac{s + 2}{3} t_s^4 \]

\[ + \left( \frac{s}{1} \right) t_s^4 + \left( \frac{s}{2} \right) t_s^4 + \left( \frac{s + 3}{4} \right) t_s^4 + \left( \frac{s + 2}{1} \right) t_s^4 \]

**Proof.** Let \( f_\gamma(x) = \sum_{i=0}^{\infty} t_i x^{i+1} \). Using Corollary 8.7, we obtain

\[ f_\gamma([-2]_\phi^* \mathcal{F}_{E_C}(x)) = \sum_{i=0}^{\infty} t_i \left( t_0 u_1 x^2 + x^4 + \sum_{i=1}^{\infty} (t_0 u_1)^{i+1} x^{i+1} \right) \]

\[ \equiv \sum_{i=0}^{\infty} t_i \left( t_0 u_1 x^2 + x^4 + t_0^2 u_1^2 x^6 + t_0^3 u_1^3 x^8 \right)^{i+1} \]

\[ \equiv \sum_{i=0}^{\infty} t_i \left( x^{4(i+1)} + \left( \frac{i+1}{1} \right) t_0 u_1 x^{4i+2} + t_0^2 u_1^2 x^{4i+6} + t_0^3 u_1^3 x^{4i+8} \right) \]

\[ + \left( \frac{i+1}{2} \right) t_0^2 u_1^2 x^{4i+1} + t_0^4 u_1 x^{4i+8} \]

\[ + \left( \frac{i+1}{3} \right) t_0^3 u_1^3 x^{4i-2} + t_0^4 u_1^4 x^{4i+2} + t_0^5 u_1^5 x^{4i+6} \]

\[ + \left( \frac{i+1}{4} \right) t_0^4 u_1^4 x^{4i-4} + \left( \frac{i+1}{5} \right) t_0^5 u_1^5 x^{4i-6} \].
Further,

\[-2 \bar{F}_c (f_\gamma(x)) = u_1 \left( \sum_{i=0}^{\infty} t_i x^{i+1} \right)^2 + \left( \sum_{i=0}^{\infty} t_i x^{i+1} \right)^4 + \sum_{k=1}^{\infty} u_1^{2k} \left( \sum_{i=0}^{\infty} t_i x^{i+1} \right)^{2k+4} \equiv \sum_{i=0}^{\infty} \left( u_1 t_i^2 x^{2(i+1)} + t_i^4 x^{4(i+1)} + u_1 t_i x^{8(i+1)} \right) + u_1^2 \left( \sum_{i=0}^{\infty} t_i^2 x^{2(i+1)} \right)^3 \]

Next, note that

\[ \left( \sum_{i \geq 0} a_i x^i \right)^3 \equiv \sum_{k \geq 0} \sum_{2i+j=k} a_i^2 a_j x^k. \]

Therefore,

\[ u_1^2 \left( \sum_{i=0}^{\infty} t_i^2 x^{2(i+1)} \right)^3 \equiv \sum_{k \geq 0} \sum_{2i+j=k} u_1^2 t_i^4 t_j^2 x^{2k+6} \]

Now, using (8.14), the coefficient of \( x^{4(s+1)} \) gives the relation

\[ t_s \equiv t_s^4 + u_1 t_s^2 t_{2s+1} + \left( \frac{s+2}{2} \right) t_0^2 t_{s+1} + \sum_{2i+j=2s-1} u_1^2 t_i t_j^2 \]
\[ + \left( \frac{s}{1} + \frac{s}{2} \right) t_0^4 t_{s-1} + \left( \frac{s+3}{4} \right) t_0^4 t_{s+2} x^4 + \sum_{0 \leq r < s} t_r t_s x^{2r+2s} \]

(Note that the coefficient of the last term is chosen to be zero when \( s \) is even, so that when \( t_{s-1} \) has a non-zero coefficient, \( (s-1)/2 \) is an integer.)

**Proposition 8.20.** For \( t_i = t_i(\gamma) \) where \( \gamma \in \mathcal{S}_c \), then

\[ t_i \equiv t_i^4 + u_1 t_i^2 t_{2i+1} + 2t_{4i+3} + 2 \sum_{r+s=2i \atop 0 \leq r < s} t_r^2 t_s^2 \mod (2, u_1)^2 \]

**Proof.** Modulo \((4, u_1), \) we have

\[ [-2]_{F_{cU}}(x) \equiv 2x + x^4. \]

This gives

\[ f_\gamma([-2]_{\phi_7 F_{cU}}(x)) \equiv \sum_{i=0}^{\infty} t_i \left( x^{4(i+1)} + \left( \frac{i+1}{1} \right) x^{4i+1} \right) \]

and

\[ [-2]_{F_{cU}}(f_\gamma(x)) \equiv \sum_{i=0}^{\infty} 2t_i x^{i+1} + \left( \sum_{i=0}^{\infty} t_i x^{i+1} \right)^4 \equiv \sum_{i=0}^{\infty} 2t_i x^{i+1} + \sum_{i=0}^{\infty} t_i^4 x^{4(i+1)} + \sum_{i=1}^{\infty} x^{4+2i} \sum_{r+s=i} t_r^2 t_s^2. \]
Using (8.14), the coefficient of \( x^{4(i+1)} \) gives the relation
\[
t_i = 2t_{4i+3} + t_i^4 + 2 \sum_{r+s=2i, 0 \leq r < s} t_i^r t_i^s \mod (4, u_1).
\]
The claim then follows from Corollary 8.19.

\[\text{Proposition 8.21. Modulo (4) (8.22)}
\]
\[
t_0 \equiv t_0^4 + 2t_3 + 3t_1^2u_1 + 2t_0t_2u_1 + 3t_0^2t_1u_1^2.
\]
Modulo (2),
\[
t_1 \equiv t_1^4 + t_3^2u_1 + t_0^4t_1^2u_1 + t_0^2t_2u_1 + t_0^6u_1 + t_0^3u_1 + t_0^3t_3u_1^4.
\]
Proof. Modulo (8), the coefficient of \( x^4 \) in \( f_r([-2] \circ \gamma \mathcal{F}_{\mathcal{E}_c}(x)) \) is given by
\[
t_0 + \phi_\gamma(u_1^2)t_1
\]
and the coefficient of \([-2] \mathcal{F}_{\mathcal{E}_c}(f_r(x)) \) is given by
\[
t_0 + t_0^2t_1u_1^2 \equiv t_1^4 + 2t_3 + 3t_1^2u_1 + 2t_0t_2u_1.
\]
Recall from Theorem 8.17 that \( \phi_\gamma(u_1) = u_1t_0 + \frac{t_0^4}{t_0^4} \). This and (8.14) imply that
\[
t_0 + t_0^2t_1u_1^2 \equiv t_1^4 + 2t_3 + 3t_1^2u_1 + 2t_0t_2u_1.
\]
Isolating \( t_0 \) proves the first claim.
Similarly, the coefficients of \( x^8 \) give the desired relation for \( t_1 \).

Recall that \( \gamma \in \mathbb{S}_C \) has an expansion of the form
\[
\gamma = a_0 + a_1T + a_2T^2 + a_3T^3 + \ldots
\]
Here the \( a_i \) are solutions to the equation \( x^4 - x = 0 \). Recall from Section 3 that if \( \omega^\bullet \in \mathrm{End}(F_C) \) is a solution to the equation \( x^4 - x = 0 \), then it corresponds the automorphism
\[
\omega^\bullet(x) = \zeta^\bullet x,
\]
where \( \zeta \in \mathbb{F}_4 = (E_C)_{\gamma}(2, v_1) \). Recall that there is a canonical copy of \( \mathbb{F}_4 \) in \( \mathrm{End}(F_C) \) given by the ring generated by the automorphism \( \omega(x) \). Further, \( (E_C)_{\gamma}(2, v_1) \) is isomorphic to \( \mathbb{F}_4 \), with canonical generator the image of \( \zeta \). Define a map
\[
f : \mathbb{F}_4 \subseteq \mathrm{End}(F_C) \rightarrow (E_C)_{\gamma}(2, v_1) \cong \mathbb{F}_4
\]
by
\[
f(\omega^\bullet(x)) = \zeta^\bullet.
\]
If \( \gamma \) is as above, using the fact that \( T(x) = x^2 \),
\[
\gamma(x) = f(a_0)x + F_C f(a_1)x^2 + F_C f(a_2)x^4 + F_C f(a_3)x^8 + \ldots
\]
For simplicity, we will identify \( a_i \) with \( f(a_i) \) and write
\[\text{(8.23)} \quad \gamma(x) = a_0x + F_C a_1x^2 + F_C a_2x^4 + F_C a_3x^8 + \ldots
\]

\[\text{Proposition 8.24. For } \gamma \in \mathbb{S}_C, \]
\[
\gamma(x) = x + a_1x^2 + a_2x^4 + a_3x^8 + a_4^4x^{10} + a_1^2a_2^2x^{12} + a_1x^{14} + (a_3^3 + a_4)x^{16}
\]
modulo \( (x^{18}) \).
Proof. This is a direct computation using (8.23) and the formal group law of Proposition 8.11, noting that for \( \gamma \in \mathbb{S}_C, a_0 = 1 \).
For the remainder of this section, we assume that $\gamma \in F_{2/2S_C}$.

**Corollary 8.25.** Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2S_C}$. Modulo $(2, u_1)$,

\[
t_0 \equiv 1 \quad t_3 \equiv a_2, \quad t_7 \equiv a_3, \quad t_9 \equiv a_2^2, \quad t_{15} \equiv a_4,
\]

and $t_1$, $t_5$, $t_{11}$, $t_{13}$ and $t_{24}$ for $0 < i < 8$ are zero modulo $(2, u_1)$.

**Proof.** Since $\gamma \in F_{2/2S_C}$, $a_1 = 0$. The claim follows from Proposition 8.24, noting that $t_1$ is congruent to the coefficient of $x^{i+1}$ modulo $(2, u_1)$. \qed

**Proposition 8.26.** Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2S_C}$. Modulo $(2, u_1^2)$,

\[
t_0 \equiv 1 \quad t_2 \equiv 0 \quad t_4 \equiv 2u_1 \quad t_6 \equiv 0
\]

\[
t_1 \equiv a_2^2u_1 \quad t_3 \equiv a_2 + a_3^2u_1 \quad t_5 \equiv 0 \quad t_7 \equiv a_3 + a_2^2u_1.
\]

**Proof.** This follows from Proposition 8.20 and Corollary 8.25. \qed

**Proposition 8.27.** Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2S_C}$. Modulo $(2, u_1^3)$,

\[
\begin{align*}
t_0 &\equiv 1 + (a_2 + a_3^2)u_1^3, \\
t_3 &\equiv a_2 + a_3^2u_1 + a_4u_1^3,
\end{align*}
\]

Modulo $(2, u_1^3)$,

\[
\begin{align*}
t_1 &\equiv a_2^2u_1, \\
t_5 &\equiv (a_2 + a_3^3)u_1^2.
\end{align*}
\]

Modulo $(2, u_1^4)$,

\[
t_2 \equiv a_2^3u_1^2 + a_3u_1^4 + (a_2 + a_3^3)u_1^5.
\]

**Proof.** It follows from Corollary 8.19 that, modulo $(2, u_1^4)$,

\[
\begin{align*}
t_3 &\equiv t_1^4 + t_2^4u_1 + t_2^4u_1^3 + t_1^4t_1^4u_1^2 + t_1^4t_1^2u_1^2 \\
t_5 &\equiv t_1^4 + t_2^4u_1 + t_2^4u_1^3 + t_1^4t_1^4u_1^2 + t_1^4t_1^2u_1^2 + t_1^4t_1^2u_1^2 + t_1^4t_1^2u_1^2.
\end{align*}
\]

The results for $t_3$ and $t_5$ then follow from Corollary 8.25 and Proposition 8.26. It also follows from Corollary 8.19 that, modulo $(2, u_1^4)$,

\[
t_2 \equiv t_2^4 + t_2^4u_1 + t_2^4u_1^3 + t_2^4t_1^4u_1^2 + t_2^4t_1^2u_1^2 + t_2^4t_1^2u_1^2.
\]

The identity for $t_2$ then follows from the Corollary 8.25 and Proposition 8.26, using the identity for $t_3$ modulo $(2, u_1^3)$. \qed

**Proposition 8.28.** Let $\gamma \in F_{2/2S_C}$. Modulo $(2, u_1^8)$,

\[
t_1(\gamma) \equiv a_2^2u_1 + a_3u_1^3 + a_3u_1^5 + a_3u_1^6 + (a_2^2 + a_3^5 + a_2 + a_2^3 + a_3^5)u_1^7.
\]

**Proof.** The estimate for $t_1$ follows from Propositions 8.21 and 8.27. The estimate for $t_0$ follow from Proposition 8.21 using the result for $t_1$. \qed

**Proposition 8.29.** Let $\gamma \in F_{2/2S_C}$. Modulo $(4, 2u_1^7, u_1^9)$,

\[
t_0(\gamma) \equiv 1 + 2a_2 + 2a_2^2u_1 + (a_2 + a_2^2)u_1^3 + a_3u_1^5 + a_3u_1^6 + (a_2 + a_2^3 + a_4 + a_4^2)u_1^9.
\]

**Proof.** This follows from Propositions 8.21, 8.26 and 8.28. \qed
Proof of Theorem 3.6. The claim follows from Theorem 8.17, noting that
\[ \phi(u_1) \equiv t_0 u_1 \pmod{2} \]
\[ \square \]

Proof of Theorem 3.7. The claim for \( t_0(\gamma) \) is Proposition 8.29. The claim for \( t_1(\gamma) \) follows from Proposition 8.28 by reducing the identity for \( t_1 \) modulo \( (2, u_3^1) \). If \( \gamma \) is in \( F_{4/2}S_0 \), that the action of \( \gamma \) is trivial modulo \( (2, v_9^1) \) follows from the fact that \( t_0(\gamma) \equiv 1 \pmod{(2, u_9^1)} \). \[ \square \]

References