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Towards the homotopy of the $K(2)$-local Moore spectrum at $p = 2$

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ABSTRACT

Let $V(0)$ be the mod 2 Moore spectrum and let $\mathcal{C}$ be the supersingular elliptic curve over $\mathbb{F}_4$ defined by the Weierstrass equation $y^2 + y = x^3$. Let $F_{\mathcal{C}}$ be its formal group law and $E_{\mathcal{C}}$ be the spectrum classifying the deformations of $F_{\mathcal{C}}$. The group of automorphisms of $F_{\mathcal{C}}$, which we denote by $\mathbb{S}_{\mathcal{C}}$, acts on $E_{\mathcal{C}}$. Further, $\mathbb{S}_{\mathcal{C}}$ admits a surjective homomorphism to $\mathbb{Z}_2$ whose kernel we denote by $\mathbb{S}^1_{\mathcal{C}}$. The cohomology of $\mathbb{S}^1_{\mathcal{C}}$ with coefficients in $(E_{\mathcal{C}})_*V(0)$ is the E_2-term of a spectral sequence converging to the homotopy groups of $E_{\mathcal{C}}^{h\mathbb{S}^1_{\mathcal{C}}} \wedge V(0)$, a spectrum closely related to $L_{K(2)}V(0)$. In this paper, we use the algebraic duality resolution spectral sequence to compute an associated graded for $H^*(\mathbb{S}^1_{\mathcal{C}}; (E_{\mathcal{C}})_*V(0))$. These computations rely heavily on the geometry of elliptic curves made available to us at chromatic level 2.

1. Introduction

This paper can be read as a sequel to [2]. For this reason, this section builds upon the deeper discussion of [2, Section 2]. We give an overview of the tools that were not introduced in the prequel and state our results. More background and motivation can be found in [2].

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1.1. Background

In this paper, we work at the prime $p = 2$. Recall that Morava K-theory $K(2)$ is the unique ring spectrum with coefficients $K(2)_* = \mathbb{F}_2[v_2^{\pm 1}]$, for v_2 in degree 6, and with formal group law the Honda formal group law F_2 of height 2. The group $\mathbb{S}_2$ is the group of automorphisms of F_2 over $\mathbb{F}_4$. The extended Morava stabilizer group $\mathbb{G}_2$ is the extension of $\mathbb{S}_2$ by the Galois group. Morava E-theory E_2 is the Landweber exact spectrum for which $\pi_0 E_2$ corepresents isomorphism classes of deformations of F_2. Its homotopy groups can be described as follows. Let ζ be a primitive third root of unity and let

$$\mathbb{W} := W(\mathbb{F}_4) \cong \mathbb{Z}_2[\zeta]$$

be the Witt vectors on $\mathbb{F}_4$. Then $(E_2)_* \cong \mathbb{W}[\![u_1]\!][u^{\pm 1}]$, where u_1 has degree zero and u has degree -2. The group $\mathbb{G}_2$ acts on the spectrum E_2. For a finite spectrum X, $L_{K(2)}X \simeq E_2^{h\mathbb{G}_2} \wedge X$. Further, for closed subgroups G of $\mathbb{G}_2$ and finite spectra X, there are descent spectral sequences

$$E_2^{s,t} := H^s(G, (E_2)_t X) \Longrightarrow \pi_{t-s}(E_2^{hG} \wedge X). \tag{1.1.1}$$

The groups $\mathbb{S}_2$ and $\mathbb{G}_2$ both admit a surjective homomorphism to $\mathbb{Z}_2$ whose kernels are denoted by $\mathbb{S}_2^1$ and $\mathbb{G}_2^1$ respectively and

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2, \qquad \mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2. \tag{1.1.2}$$

The group $\mathbb{S}_2$ has a unique conjugacy class of maximal finite subgroups, which can be described as follows. The automorphism of F_2 given by $[-1]_{F_2}(x)$ generates a central subgroup C_2. The power series $\omega(x) = \zeta x$ generates a subgroup of order three in $\mathbb{S}_2$, denoted C_3. The group C_3 acts on a quaternion subgroup Q_8 of $\mathbb{S}_2$ whose center is C_2. The semi-direct product $G_{24} = Q_8 \rtimes C_3$ is a maximal finite subgroup of $\mathbb{S}_2$. The subgroup $C_6 = C_2 \times C_3$ of G_{24} will also play a central role.

Both C_6 and G_{24} are contained in $\mathbb{S}_2^1$. However, $\mathbb{S}_2^1$ has two conjugacy classes of maximal finite subgroups. A representative for the other conjugacy class is given by $G_{24}' = \pi G_{24} \pi^{-1}$ for π a topological generator of $\mathbb{Z}_2$ in the decomposition $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$.

The next theorem follows from [2, Theorem 1.2.1, Theorem 1.2.4, Corollary 3.4.6]. (See Section 4 below for more details.)

Theorem 1.1.1. *Let $\mathbb{Z}_2$ be the trivial $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$-module. There is an exact sequence of complete left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$-modules*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

where $\mathscr{C}_0 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}]\!]$, $\mathscr{C}_3 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!]$ and $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!]$. Let e be the unit in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ and e_p be the resulting generator of $\mathscr{C}_p$. The maps ∂_p can be chosen to satisfy:

(a) $\partial_1(e_1) = (e - \alpha)e_0$,
(b) $\partial_2(e_2) \equiv (e + \alpha + \mathscr{E})e_1$ *for* $\mathscr{E} \in (2, (IS_2^1)^2)$.
(c) $\partial_3(e_3) = \pi(e + i + j + k)(e - \alpha^{-1})\pi^{-1}e_2$.

Let $F_0 = G_{24}$, $F_1 = F_2 = C_6$ *and* $F_3 = G_{24}'$. *For a profinite* $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$*-module* M, *there is a first quadrant spectral sequence*

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, M) \cong H^q(F_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M)$$

with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Remark 1.1.2. The exact sequence of Theorem 1.1.1 is called the *algebraic duality resolution* because it satisfies a certain duality. This is described in Theorem 1.2.2 of [2]. The associated spectral sequence is called the *algebraic duality spectral sequence* which we abbreviate as ADSS.

Let $V(0)$ be the mod 2 Moore spectrum, that is, the cofiber of multiplication by 2 on the sphere spectrum S^0. The goal of this paper is to compute the E_∞-term of the ADSS for $M = (E_2)_*V(0)$. We obtain an associated graded for $H^*(\mathbb{S}_2^1; (E_2)_*V(0))$. Taking the Galois fixed points of the E_∞-term gives an associated graded for the cohomology $H^*(\mathbb{G}_2^1; (E_2)_*V(0))$. Therefore, this computation can be used to understand the E_2-term of the descent spectral sequence (1.1.1) when $G = \mathbb{G}_2^1$ and $X = V(0)$, that is

$$H^s(\mathbb{G}_2^1; (E_2)_tV(0)) \Longrightarrow \pi_{t-s}(E_2^{h\mathbb{G}_2^1} \wedge V(0)).$$

Further, recall that there is a fiber sequence

$$L_{K(2)}V(0) \to E_2^{h\mathbb{G}_2^1} \wedge V(0) \to E_2^{h\mathbb{G}_2^1} \wedge V(0).$$

Hence, computing $H^*(\mathbb{S}_2^1; (E_2)_*V(0))$ is a first step for computing $\pi_*L_{K(2)}V(0)$.

The computations will be done using the fact that, at chromatic level $n = 2$, one can replace Morava K-theory $K(2)$ by a spectrum $K_{\mathcal{C}}$ whose formal group law is the formal group law of a supersingular elliptic curve $\mathcal{C}$. This allows us to use the geometry of elliptic curves to get a better understanding of the action of the Morava stabilizer group $\mathbb{S}_2$ on $(E_2)_*$. Before stating the results, we explain this point of view.

Consider the supersingular elliptic curve $\mathcal{C} : y^2 + y = x^3$ defined over $\mathbb{F}_4$. Let $F_{\mathcal{C}}$ be the formal group law of $\mathcal{C}$. It satisfies $[-2]_{F_{\mathcal{C}}}(x) = x^4$. Let $K_{\mathcal{C}}$ denote the complex oriented ring spectrum whose ring of coefficients is $(K_{\mathcal{C}})_* = \mathbb{F}_4[u^{\pm 1}]$, where u is in degree -2, and whose formal group law is $F_{\mathcal{C}}$.

In this paper, $E_{\mathcal{C}} := E(\mathbb{F}_4, F_{\mathcal{C}})$ will denote the complex oriented Landweber exact spectrum for which $\pi_0 E_{\mathcal{C}}$ corepresents isomorphism classes of deformations of $F_{\mathcal{C}}$. There is an abstract isomorphism $(E_{\mathcal{C}})_* \cong (E_2)_*$, but it cannot be realized by a map of E_∞-ring spectra. Such a map would induce an $\mathbb{F}_4$-isomorphism on the formal group laws $F_{\mathcal{C}}$ and F_2. These formal group laws are not isomorphic over $\mathbb{F}_4$, but become isomorphic after passing to the algebraic closure of $\mathbb{F}_2$.

Let $\mathbb{S}_{\mathcal{C}} := \mathrm{Aut}(F_{\mathcal{C}})$ be the group of automorphisms of $F_{\mathcal{C}}$ over $\mathbb{F}_4$. The groups $\mathbb{S}_2$ and $\mathbb{S}_{\mathcal{C}}$ are isomorphic. An explicit isomorphism is constructed in Lemma 3.1.2. The group $\mathbb{S}_{\mathcal{C}}$ admits an action of the Galois group and the group $\mathbb{G}_{\mathcal{C}}$ is the extension of $\mathbb{S}_{\mathcal{C}}$ by this action. The group $\mathbb{G}_{\mathcal{C}}$ acts on the deformations. By the Goerss–Hopkins–Miller Theorem (see Goerss and Hopkins [5, Section 7]), it acts on $E_{\mathcal{C}}$ by maps of E_∞-ring spectra.

The isomorphism of Lemma 3.1.2 does not extend to an isomorphism of the groups $\mathbb{G}_2$ and $\mathbb{G}_{\mathcal{C}}$. In fact, these groups are not isomorphic. However, over an algebraic closure of $\mathbb{F}_2$, the formal group laws F_2 and $F_{\mathcal{C}}$ are isomorphic. Therefore, the Bousfield classes of $K(2)$ and $K_{\mathcal{C}}$ are the same. Their localization functors are weakly equivalent, so that $L_{K(2)}X \simeq L_{K_{\mathcal{C}}}X$. As before, it follows from the work of Devinatz and Hopkins in [3] that for X a finite spectrum $L_{K_{\mathcal{C}}}X \simeq E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}} \wedge X$. Further, for any closed subgroup G of $\mathbb{G}_{\mathcal{C}}$, there is a spectral sequence analogous to (1.1.1).

The groups $\mathbb{S}_{\mathcal{C}}$ and $\mathbb{G}_{\mathcal{C}}$ also admit a surjective homomorphism to $\mathbb{Z}_2$ and $\mathbb{S}^1_{\mathcal{C}}$ and $\mathbb{G}^1_{\mathcal{C}}$ are defined to be the kernel of this homomorphism as before. Further, since $\mathbb{S}_{\mathcal{C}}$ is isomorphic to $\mathbb{S}_2$, the results of [2] also hold for $\mathbb{S}_{\mathcal{C}}$. In particular, the resolution of Theorem 1.1.1 can be constructed using $\mathbb{S}^1_{\mathcal{C}}$ and the algebraic duality resolution gives rise to an algebraic duality resolution spectral sequence

$$E_1^{p,q} \cong H^q(F_p, M) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}; (E_{\mathcal{C}})_*V(0)). \tag{1.1.3}$$

In this paper, we compute the E_∞-term of (1.1.3).

The main advantage of using $\mathbb{S}_{\mathcal{C}}$ is that the elliptic curve $\mathcal{C}$ has a large automorphism group. In fact, $\mathrm{Aut}(\mathcal{C})$ is isomorphic to G_{24} and it injects into $\mathrm{Aut}(F_{\mathcal{C}})$. Its image is a choice of maximal finite subgroup. Using level structures, Strickland has computed the action of $\mathrm{Aut}(\mathcal{C})$ on $(E_{\mathcal{C}})_*$. We use this result and, since it is not in print, we describe it Section 2.2. From now on, we will let G_{24} denote the image of $\mathrm{Aut}(\mathcal{C})$ in $\mathbb{S}_{\mathcal{C}}$.

1.2. Statement of results

In order to state the results, we will describe the E_1-term of (1.1.3). First, note that $(E_{\mathcal{C}})_*V(0) \cong \mathbb{F}_4[\![u_1]\!][u^{\pm 1}]$, where u_1 has degree 0 and u has degree -2. Let $F_{E_{\mathcal{C}}}$ be the graded formal group law of $E_{\mathcal{C}}$. Then

$$[2]_{F_{E_{\mathcal{C}}}}(x) \equiv u_1 u^{-1} x^2 + \dots \mod (2)$$

(see Section 6), hence we define $v_1 = u_1 u^{-1}$ in $(E_{\mathcal{C}})_*V(0)$. The element v_1 is invariant under the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*V(0)$. Let δ be the connecting homomorphism associated to the exact sequence

$$0 \to (E_2)_* \xrightarrow{2} (E_2)_* \to (E_2)_*V(0) \to 0.$$

Let $\eta = \delta(v_1)$ and $v_2 = u^{-3}$. Then

$$H^*(C_6; (E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4[\![u_1^3]\!][v_1, v_2^{\pm 1}, h]/(v_2^{-1}v_1^3 = u_1^3),$$

for a class h in $H^1(C_6; (E_{\mathcal{C}})_*V(0))$ satisfying $\eta = hv_1$. In particular, $\{v_2^n h^s\}_{n\in\mathbb{Z}}$ is a set topological generators of $H^s(C_6; (E_{\mathcal{C}})_*V(0))$ as an $\mathbb{F}_4[v_1]$-module. That is, in the category of profinite graded $\mathbb{F}_4[v_1]$-modules, there is an isomorphism

$$H^s(C_6; (E_{\mathcal{C}})_*V(0)) \cong \prod_{n\in\mathbb{Z}} \mathbb{F}_4[v_1]\{v_2^n h^s\}.$$

The cohomology G_{24} is related to the cohomology of the Hopf algebroid classifying Weierstrass curves over $\mathbb{F}_4$ with their strict isomorphisms, a computation originally due to Hopkins and Mahowald and presented by Bauer in [1]. In particular, the G_{24} fixed points are related to modular forms modulo 2. However, we have included a self-contained computation of $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$ in an appendix (see Appendix A). This computation is based on unpublished notes of Hans-Werner Henn. In Appendix A, it is shown that there is an isomorphism

$$H^*(G_{24}, (E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4[\![j]\!][v_1, \Delta^{\pm 1}, k, \eta, \nu, x, y]/(\sim)$$

where Δ of degree $(0, 24)$ is the reduction modulo (2) of the discriminant of a universal deformation of $\mathcal{C}$ over $(E_{\mathcal{C}})_*$ (see Theorem 2.1.1), j of degree $(0, 0)$ is the j-invariant of this deformation. The element k in $H^4(G_{24}, (E_{\mathcal{C}})_*V(0))$ is the image of the periodicity generator in $H^4(G_{24}, \mathbb{F}_4) \cong H^4(Q_8, \mathbb{F}_4)^{C_3}$ under the natural inclusion of $\mathbb{F}_4$ into $(E_{\mathcal{C}})_0V(0)$. The class ν has degree $(1, 4)$, x has degree $(1, 8)$ and y has degree $(1, 16)$. The relations $(\sim)$ contain $\eta^4 = v_1^4 k$ and $v_1^{12} = j\Delta$. We refer the reader to Theorem 4.2.2 for the complete ideal of relations.

A set of topological $\mathbb{F}_4[v_1]$-module generators for $H^0(G_{24}, (E_{\mathcal{C}})_*V(0))$ is given by $\{\Delta^n\}_{n\in\mathbb{Z}}$ so that, in the category of profinite graded $\mathbb{F}_4[v_1]$-modules,

$$H^0(G_{24}; (E_{\mathcal{C}})_*V(0)) \cong \prod_{n\in\mathbb{Z}} \mathbb{F}_4[v_1]\{\Delta^n\}.$$

Note that conjugation by π induces an $\mathbb{F}_4[v_1, \eta]$-linear isomorphism

$$H^*(G'_{24}, (E_{\mathcal{C}})_*V(0)) \cong H^*(G_{24}, (E_{\mathcal{C}})_*V(0)).$$

We let Δ' and j' be the image of Δ and j under this isomorphism. For z in positive cohomological dimension in $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$, we abuse notation and denote its image under the conjugation isomorphism by the same name. In this spirit, letting k act on the left via this isomorphism, we will treat $H^*(G'_{24}, (E_{\mathcal{C}})_*V(0))$ as an $\mathbb{F}_4[v_1, \eta, k]$-module (see Remark 4.2.5).

Finally, $H^*(C_6, (E_\mathcal{C})_*V(0))$ is an $\mathbb{F}_4[v_1, \eta, k]$-module where k acts by multiplication by h^4 and η by multiplication by $v_1 h$.

In the following result, we adopt notation similar to that of Henn, Karamanov and Mahowald [7, Theorem 1.2].

Theorem 1.2.1. *The ADSS converging to $H^*(\mathbb{S}_2^1, (E_\mathcal{C})_*V(0))$ collapses at the E_2-term. The spectral sequence is an $\mathbb{F}_4[v_1, \eta, k]/(\eta^4 - v_1^4 k)$-module. There exist $\mathbb{F}_4[v_1]$-generators $\Delta_n \in E_1^{0,0}$, $b_n \in E_1^{1,0}$, $\overline{b}_n \in E_1^{2,0}$ and $\overline{\Delta}_n \in E_1^{3,0}$ with*

$$\Delta_n \equiv \Delta^n \qquad b_n \equiv \overline{b}_n \equiv v_2^n, \qquad \overline{\Delta}_n \equiv (\Delta')^n$$

where the congruences are modulo the ideal (v_1) and such that, for $r \geq 0$ and $t \in \mathbb{Z}$,

$$d_1(\Delta_n) = \begin{cases} v_1^{6\cdot 2^r} b_{2^{r+1}(1+4t)} & n = 2^r(1+2t) \\ 0 & n = 0 \end{cases}$$

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^r} \overline{b}_{2^{r+1}(1+2t)} & n = 2^r(3+4t) \\ v_1^{3\cdot 2^{r+1}} \overline{b}_{1+2^{r+1}(1+4t)} & n = 1 + 2^{r+2}(1+2t) \\ 0 & \textit{otherwise} \end{cases}$$

$$d_1(\overline{b}_n) = \begin{cases} v_1^{3(2^{r+1}+1)} \overline{\Delta}_{2^r(1+2t)} & n = 1 + 2^{r+1}(3+4t) \\ 0 & \textit{otherwise.} \end{cases}$$

For $q > 0$, a the differential $d_1 : E_1^{p,q} \to E_1^{p,q}$ is non-zero if and only if it is forced by η-linearity and the $\mathbb{F}_4[v_1]$-module structure. All differentials $d_r : E_r^{p,q} \to E_r^{p+1,q}$ for $r \geq 2$ are zero, so that $E_2 = E_\infty$.

Theorem 1.2.2 below gives an explicit description of the E_∞-term for the interested reader. Theorem 1.2.1 and Theorem 1.2.2 are also displayed in Fig. 1 and Fig. 3.

Theorem 1.2.2. *As an $\mathbb{F}_4[v_1, k]$-module, the E_∞-term of the ADSS with coefficients in $(E_\mathcal{C})_*V(0)$ is isomorphic to a direct sum of cyclic modules generated by the following elements and with the following annihilator ideals.*

(a) *For $E_\infty^{0,*}$,*

$\eta^s \Delta_0$	$0 \leq s \leq 3$	
$\nu^s \Delta_t$	$1 \leq s \leq 3,\ t \in \mathbb{Z}$	(v_1)
$\eta^s x^r \Delta_t$	$1 \leq r \leq 2,\ 0 \leq s \leq 1,\ t \in \mathbb{Z}$	(v_1^2)
$\nu^s y \Delta_t$	$0 \leq s \leq 2,\ t \in \mathbb{Z}$	(v_1)

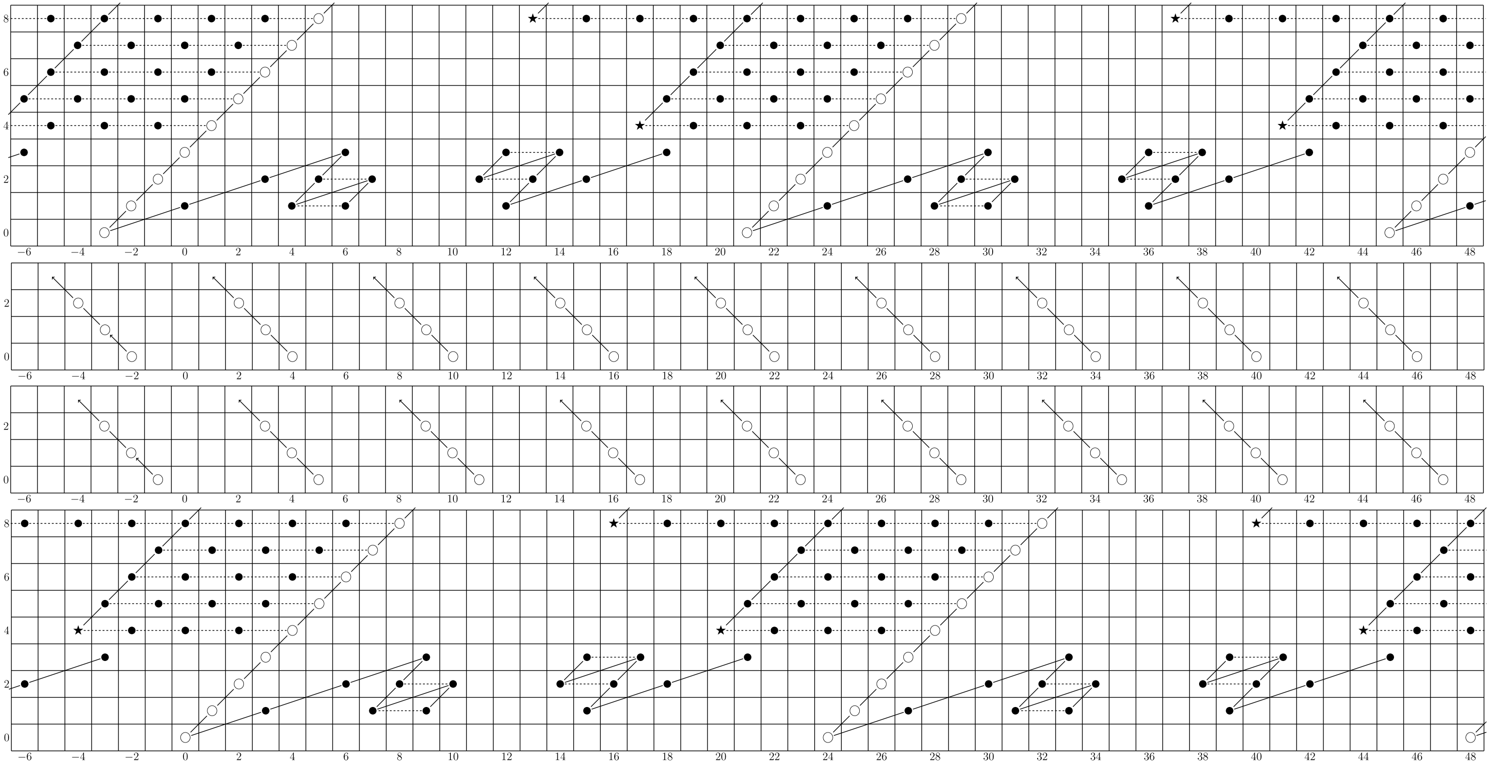

Fig. 1. The E_1-term for the ADSS with coefficients $(E_{\mathcal{C}})_*V(0)$. The rows represent $E_1^{p,*}$, the top row corresponding to $p = 3$. The grading is given by $(t-q-p, q)$, where t is the internal grading, so that $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ decreases the horizontal grading by 1. A $\bullet$ denotes a copy of $\mathbb{F}_4$. Dashed horizontal lines denote multiplication by v_1, and a $\bigcirc$ denotes a copy of $\mathbb{F}_4[v_1]$. A $\bigstar$ is a copy of Fig. 2.

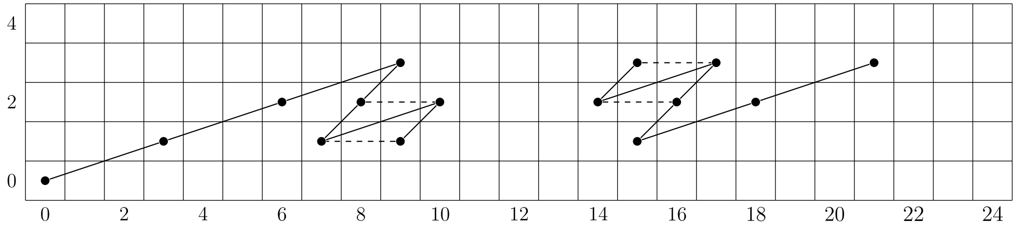

Fig. 2. The pattern ★ in Fig. 1.

(b) *For* $E_\infty^{1,*}$,

$$\begin{array}{lll} h^q b_s & 0 \le s \le 1,\ 0 \le q \le 3 & \\ h^q b_{2^{r+1}(1+4t)} & 0 \le r,\ 0 \le q \le 3 & (v_1^{6\cdot 2^r + \overline{q}}) \end{array}$$

(c) *For* $E_\infty^{2,*}$,

$$\begin{array}{lll} h^q \overline{b}_s & 0 \le s \le 1,\ 0 \le q \le 3 & \\ h^q \overline{b}_{2^{r+1}(1+2t)} & 0 \le r,\ t \in \mathbb{Z},\ 0 \le q \le 3 & (v_1^{3\cdot 2^r}) \\ h^q \overline{b}_{1+2^{r+1}(1+4t)} & 0 \le r,\ t \in \mathbb{Z},\ 0 \le q \le 3 & (v_1^{3\cdot 2^{r+1}}) \end{array}$$

(d) *For* $E_\infty^{3,*}$,

$$\begin{array}{lll} \eta^q \overline{\Delta}_0 & 0 \le q \le 3 & \\ \eta^q \overline{\Delta}_{2^r(1+2t)} & 0 \le r,\ t \in \mathbb{Z},\ 0 \le q \le 3 & (v_1^{3\cdot(2^{r+1}+1)-q}) \\ \nu^s \overline{\Delta}_t & 1 \le s \le 3,\ t \in \mathbb{Z} & (v_1) \\ \eta^s x^r \overline{\Delta}_t & 1 \le r \le 2,\ 0 \le s \le 1,\ t \in \mathbb{Z} & (v_1^2) \\ \nu^s y \overline{\Delta}_t & 0 \le s \le 2,\ t \in \mathbb{Z} & (v_1) \end{array}$$

If one inverts v_1, the situation is much simpler. The following result is an immediate consequence of Theorem 1.2.1 and Theorem 1.2.2.

Corollary 1.2.3. *The spectral sequence*

$$v_1^{-1} E_1^{p,q} = v_1^{-1} H^q(F_p, (E_{\mathcal{C}})_* V(0)) \Longrightarrow v_1^{-1} H^{p+q}(\mathbb{S}_{\mathcal{C}}^1; (E_{\mathcal{C}})_* V(0))$$

collapses at the E_2*-term. As an* $\mathbb{F}_4[v_1^{\pm 1}, \eta]$*-module,*

$$v_1^{-1} H^*(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_* V(0)) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta]\{\Delta_0, b_0, b_1, \overline{b}_0, \overline{b}_1, \overline{\Delta}_0\}.$$

Let s *be the cohomological degree and* t *be the internal degree. Then the* (s,t)*-degrees of the generators are* $|\Delta_0| = (0,0)$, $|b_0| = (1,0)$, $|b_1| = (1,6)$, $|\overline{b}_0| = (2,0)$, $|\overline{b}_1| = (2,6)$ *and* $|\overline{\Delta}_0| = (3,0)$.

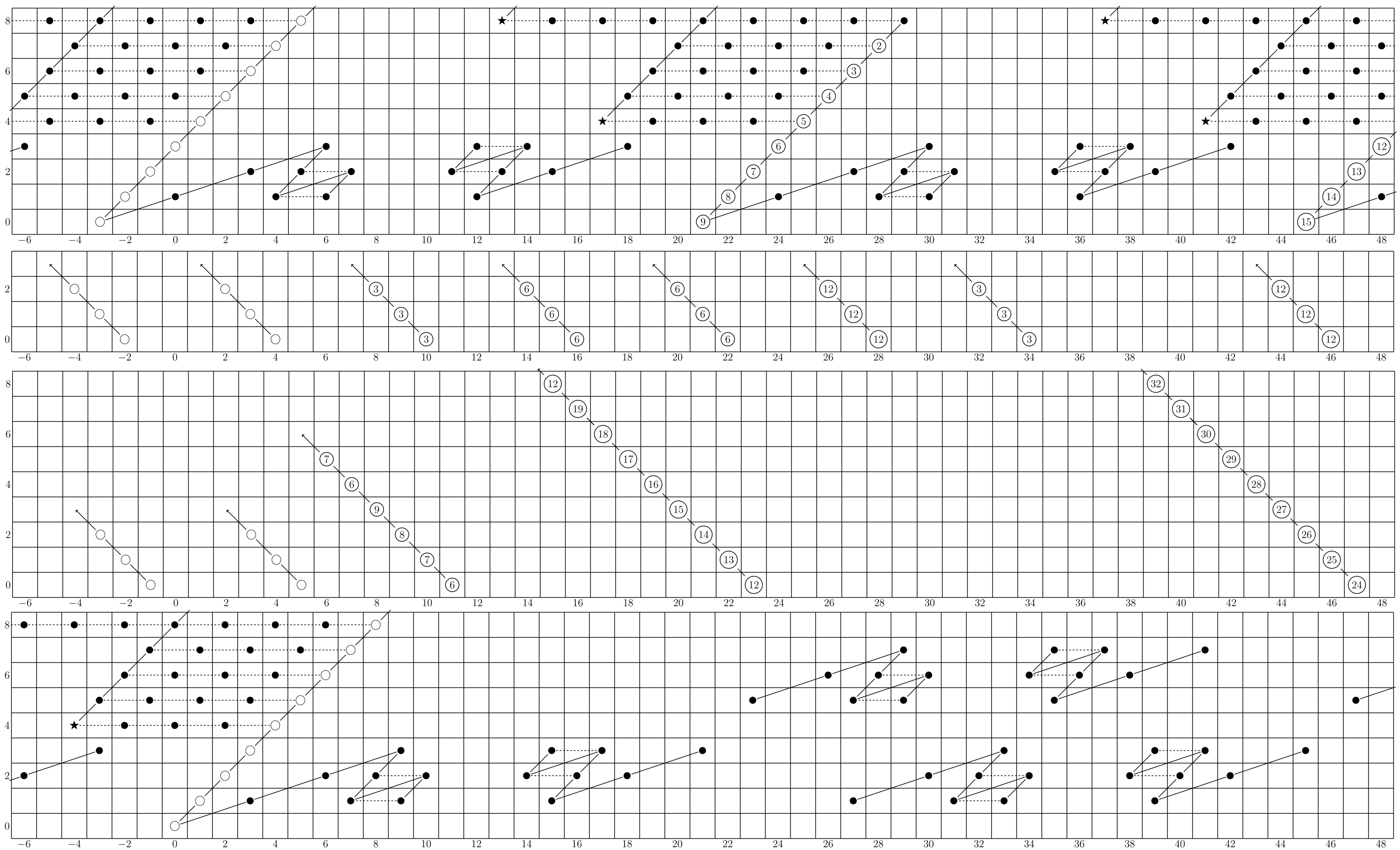

Fig. 3. The E_2-term of the ADSS with coefficients $(E_{\mathcal{C}})_*V(0)$. The notation and grading is as in Fig. 1. In addition, a © is a copy of $\mathbb{F}_4[v_1]/(v_1^c)$. In Section 5.5, we prove that $E_2 \cong E_\infty$. Therefore, this is also the E_∞-term of the ADSS.

This result is important because $v_1^{-1}H^*(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_*V(0))$ is the E_2-term of a spectral sequence that computes the homotopy groups of $L_1(E_2^{h\mathbb{S}_{\mathcal{C}}^1} \wedge V(0))$. This spectrum plays a central role in the study of the chromatic splitting conjecture at $n = p = 2$.

It is worth mentioning here that the related computation of the E_2-page of the Johnson–Wilson $E(2)$-local Adams–Novikov spectral sequence converging to L_2S was done by Shimomura and Wang in [13]. These computations were done independently. However, historically, they depend on the work of Shimomura and Wang. Indeed, results similar to those of Theorem 1.2.1 can be extracted from [13], and it is using Shimomura and Wang's computation that Mahowald conjectured the existence of the duality resolution for the $K(2)$-local sphere.

1.3. Organization of the paper

In Section 2.1, we describe a choice of universal deformation $F_{\mathcal{C}_U}$ of $F_{\mathcal{C}}$, where $F_{\mathcal{C}_U}$ is the formal group law of an elliptic curve $\mathcal{C}_U$. This allows us to define $E_{\mathcal{C}}$. The choice for the curve $\mathcal{C}_U$ is due to Strickland and, in Section 2.2, we present his formulas for the right action of $\mathrm{Aut}(\mathcal{C})$ on $(E_{\mathcal{C}})_*$. In Section 2.3, we tie this to the right action of $\mathrm{Aut}(F_{\mathcal{C}})$ on $(E_{\mathcal{C}})_*$. In Section 2.4 we adopt conventions that allow us to use the corresponding left action as in Henn, Karamanov and Henn [7] and settle our notation for the rest of the paper.

Section 3 is dedicated to describing the structure of $\mathbb{S}_{\mathcal{C}}$. In Section 3.1, we give an explicit isomorphism between the group of automorphisms $\mathbb{S}_2$ of the Honda formal group law and the group $\mathbb{S}_{\mathcal{C}}$. In Section 3.2, we recall the standard filtration on $\mathbb{S}_{\mathcal{C}}$. In Section 3.3, we give the information about the action $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ that will be used in the computation of $H^*(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_*V(0))$.

The goal of Section 4 is to introduce the ADSS for $\mathbb{S}_{\mathcal{C}}$ and to give the information necessary to begin the computation. The ADSS is not multiplicative, but it has some nice properties which we describe in Section 4.1. In Section 4.2, we describe the E_1-term. The discriminant Δ of the curve $\mathcal{C}_U$ has useful linearity properties which are given in Section 4.3.

The bulk of the paper is the computation of the E_∞-term of the ADSS with coefficients in $(E_{\mathcal{C}})_*V(0)$. This is done in Section 5. In Section 5.1, Section 5.2 and Section 5.3, we compute the differentials $d_1 : E_1^{p,0} \to E_1^{p+1,0}$. In Section 5.4, we compute the differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$. In Section 5.5, we prove that all differentials $d_r : E_r^{p,q} \to E_r^{p,1}$ for $r \geq 2$ are zero.

In Section 6, we describe the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ and deduce the formulas used in our computations. We give formulas for the minus two series of $\mathcal{C}$ and $\mathcal{C}_U$ and use them to give estimates for the action.

Appendix A gives a self-contained computation of the cohomology of G_{24} with coefficients in $(E_{\mathcal{C}})_*V(0)$. It consists of unpublished notes of by Hans-Werner Henn, which were edited by the author. The author is grateful for his blessing to include them in this document.

2. Morava E-theory and elliptic curves

In this section, we define the spectrum $E_{\mathcal{C}}$ and compute the action of $\mathrm{Aut}(\mathcal{C})$ on $(E_{\mathcal{C}})_*$. This computation is due to Strickland. For the deformation theory of formal group laws, we refer the reader to Rezk [11] or Goerss and Hopkins [5, Section 7]. In particular, if F is a formal group law over a field k, then $E(F,k)$ denotes the associated Lubin–Tate spectrum.

2.1. The supersingular elliptic curve

Consider the elliptic curve $\mathcal{C} : y^2 + y = x^3$ over $\mathbb{F}_4$. It is a standard fact that the formal group law of $\mathcal{C}$ has height 2 (see Silverman [14, Section V.4] or Proposition 6.1.4 below) so that $\mathcal{C}$ is a supersingular elliptic curve. Elliptic curves over fields of characteristic $p > 0$ admit a theory of deformations which is analogous to that of formal group laws. It follows from the Serre–Tate theorem that the deformation theory of a supersingular elliptic curve is equivalent to that of its formal group law. However, in our case, we make this concrete by the following result.

Theorem 2.1.1. *The formal group law $F_{\mathcal{C}_U}$ of the elliptic curve*

$$\mathcal{C}_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

defined over $\mathbb{W}[\![u_1]\!]$ is a universal deformation of $F_{\mathcal{C}}$. This specifies a Lubin–Tate spectrum $E_{\mathcal{C}} = E(F_{\mathcal{C}}, \mathbb{F}_4)$ with an isomorphism $(E_{\mathcal{C}})_* \cong \mathbb{W}[\![u_1]\!][u^{\pm 1}]$, where u has degree -2 and u_1 has degree zero, and a graded formal group law

$$F_{E_{\mathcal{C}}} = uF_{\mathcal{C}_U}(u^{-1}x, u^{-1}y).$$

Proof. We can verify directly that $F_{\mathcal{C}_U}$ satisfies the criteria for a universal deformation (see Lubin and Tate [8, Proposition 1.1]). Indeed, in Proposition 6.1.1 below, we compute that

$$F_{\mathcal{C}_U}(x,y) \equiv x + y - 3u_1xy - 2(u_1^3 - 1)xy(x^2 + y^2) - 3(u_1^3 - 1)x^2y^2 \mod (x,y)^5$$

so that

$$F_{\mathcal{C}_U}(x,y) \equiv x + y + u_1xy \mod (2, (x,y)^3)$$
$$F_{\mathcal{C}_U}(x,y) \equiv x + y + x^2y^2 \mod (2, u_1, (x,y)^5). \quad \square$$

Remark 2.1.2. A more obvious choice of universal deformation of $\mathcal{C}$ is

$$\mathcal{C}'_U : y^2 + u_1xy + y = x^3$$

defined over $\mathbb{W}[\![u_1]\!]$. Lubin and Tate [8, Section 3.5] prove that the formal group law associated to this curve is a universal deformation of $F_{\mathcal{C}}$. However, the choice of the curve $\mathcal{C}_U$, due to Strickland, yields nice formulas for the action of $\mathrm{Aut}(\mathcal{C})$ on $(E_{\mathcal{C}})_*$. For φ the $\mathbb{W}$-linear isomorphism of $\mathbb{W}[\![u_1]\!]$ determined by $\varphi(u_1) = -3u_1(1-u_1^3)^{-\frac{1}{3}}$, the change of coordinates

$$x = (u_1^3 - 1)^{-\frac{2}{3}}x' \qquad y = (u_1^3-1)^{-1}y'$$

is an isomorphism from $\mathcal{C}_U$ to $\varphi^*\mathcal{C}_{U'}$, where $(1-u_1^3)^{-\frac{1}{3}}$ is interpreted as its Taylor expansion.

2.2. *The automorphisms of* $\mathcal{C}$

The group $\mathrm{Aut}(F_{\mathcal{C}})$ acts on $(E_{\mathcal{C}})_*$ and, hence, so does its subgroup $\mathrm{Aut}(\mathcal{C})$. In unpublished notes, Strickland has computed the right action of the group $\mathrm{Aut}(\mathcal{C})$ on $(E_{\mathcal{C}})_*$. We explain his results in this section.

The automorphisms of the supersingular curve $\mathcal{C}$ are computed in Silverman [14, Appendix A]. The results are stated here without proof. Fix a primitive third root of unity $\zeta \in \mathbb{F}_4$. For the curve $\mathcal{C}$ over $\mathbb{F}_4$, the group $\mathrm{Aut}(\mathcal{C})$ is generated by the maps on points in the affine chart given by

$$\begin{aligned} a(x,y) &= (\zeta^2 x, \zeta^3 y), \\ b(x,y) &= (x+1, y+x+\zeta^2). \end{aligned}$$

The elements a and b generate a group of order 24 which will be described in more details at the end of Section 2.4.

Let $\psi_a, \psi_b : (E_{\mathcal{C}})_0 \to (E_{\mathcal{C}})_0$ be the $\mathbb{W}$-linear isomorphisms determined by

$$\psi_a(u_1) = \zeta u_1, \qquad \psi_b(u_1) = \frac{u_1+2}{u_1-1}. \tag{2.2.1}$$

Since a and b are generators of $\mathrm{Aut}(\mathcal{C})$, this determines a right action of $\mathrm{Aut}(\mathcal{C})$ on $(E_{\mathcal{C}})_0$. The action of γ is denoted by ψ_γ.

For ℓ, r, s and t in $\mathbb{W}[\![u_1]\!]$ and ℓ a unit, let

$$A(\ell,r,s,t) : \mathbb{P}^2(\mathbb{W}[\![u_1]\!]) \to \mathbb{P}^2(\mathbb{W}[\![u_1]\!])$$

be the automorphism

$$A(\ell,r,s,t)[x:y:z] = [\ell^2 x + rz : \ell^3 y + \ell^2 sx + tz : z].$$

Let

$$f_a = A(\zeta, 0, 0, 0)$$
$$f_b = A\left(\frac{\zeta^2-\zeta}{u_1-1}, 3\frac{1-u_1^3}{(u_1-1)^3}, 3\frac{\zeta^2 u_1 - 1}{u_1-1}, 3\frac{u_1^3-1}{(u_1-1)^4}((1-\zeta)+(1-\zeta^2)u_1)\right).$$

This determines a left action of $\mathrm{Aut}(\mathcal{C})$ on $\mathbb{P}^2(\mathbb{W}[\![u_1]\!])$ and, for γ in $\mathrm{Aut}(\mathcal{C})$, let $f_\gamma = A(\ell_\gamma, r_\gamma, s_\gamma, t_\gamma)$ be the corresponding automorphism. Then f_γ induces an isomorphism $f_\gamma : \mathcal{C}_U \to \psi_\gamma^* \mathcal{C}_U$ which pulls back to γ over $\mathbb{F}_4$.

The pair (f_γ, ψ_γ) induces an isomorphism of formal group laws. (For the formal group law associated to an elliptic curve, see Silverman [14, Chapter IV]). We abuse notation and denote this isomorphism

$$f_\gamma : F_{\mathcal{C}_U} \to \psi_\gamma^* F_{\mathcal{C}_U}$$

by (f_γ, ψ_γ) for f_γ the corresponding power series in $(E_{\mathcal{C}})_0[\![x]\!]$. One can verify directly by chasing through the change of coordinates in [14, Chapter IV] that

$$f_\gamma'(0) = \ell_\gamma^{-1}. \tag{2.2.2}$$

2.3. The right action of $\mathbb{S}_{\mathcal{C}}$

We first describe a right action $\mathbb{S}_{\mathcal{C}} = \mathrm{Aut}(F_{\mathcal{C}})$ on $E_{\mathcal{C}_U}$, following Rezk [11] as this meshes well with the geometry of the elliptic curves. However, it will be convenient in our computations to have a left action of $\mathbb{S}_{\mathcal{C}}$ and the reader should be warned that in Section 2.4, we will make the switch from a right to a left action and adopt the conventions of Henn, Karamanov and Mahowald in [7, Section 4].

Throughout, for $\psi : R_1 \to R_2$ a ring homomorphism and F a formal group law over a ring R_1, then

$$\psi^* F(x, y) = \sum_{i,j} \psi(a_{i,j}) x^i y^j$$

is a formal group law over R_2.

An element $\gamma \in \mathbb{S}_{\mathcal{C}}$ is a power series in $\mathbb{F}_4[\![x]\!]$. Let g be any lift of γ in $(E_{\mathcal{C}})_0[\![x]\!]$. Define a new formal group law by

$$F_g(x, y) = g F_{\mathcal{C}_U}(g^{-1}(x), g^{-1}(y)).$$

Then F_g is a deformation of $F_{\mathcal{C}}$ over $(E_{\mathcal{C}})_0$ and there is a unique ring isomorphism $\psi_\gamma : (E_{\mathcal{C}})_0 \to (E_{\mathcal{C}})_0$ and a unique $\star$-isomorphism $f_g : F_g \to (\psi_\gamma)^* F_{\mathcal{C}_U}$ which classify F_g. If g' is another lift of γ, then

$$f_{g'} = f_g \circ (g(g')^{-1}) : F_{g'} \to (\psi_\gamma)^* F_{\mathcal{C}_U}$$

is a $\star$-isomorphism. Therefore, ψ_γ is independent of the choice of lift g. The composite

$$F_{\mathcal{C}_U} \xrightarrow{g} F_g \xrightarrow{f_g} (\psi_\gamma)^* F_{\mathcal{C}_U}$$

is also independent of g and is denoted by f_γ. This gives a right action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_0$ where γ acts via ψ_γ. This action extends to an action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ with

$$\psi_\gamma(u) := f'_\gamma(0)^{-1} u \tag{2.3.1}$$

(see Rezk [11, Section 6.7], noting that u_{F_i} in this reference is in degree 2, while our u is in degree -2 and corresponds to the inverse of u_{F_i}). Note that by the Goerss–Hopkins–Miller theorem, this action can be realized through maps of E_∞-ring spectra on $E_{\mathcal{C}_U}$ (see Goerss and Hopkins [5]).

Further, $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ acts on $\mathbb{W}$; hence, it acts on the coefficients $(E_{\mathcal{C}_U})_*$. Since $F_{\mathcal{C}}$ is fixed by $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, this extends the action of $\mathbb{S}_{\mathcal{C}}$ to an action of

$$\mathbb{G}_{\mathcal{C}} = \mathbb{S}_{\mathcal{C}} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

Now, let γ in $\mathbb{S}_{\mathcal{C}} = \mathrm{Aut}(F_{\mathcal{C}})$ be induced by an element of $\mathrm{Aut}(\mathcal{C})$. The action of γ is classified by the pair (f_γ, ψ_γ) described in Section 2.2. Hence, the action of ψ_γ is determined by the formulas (2.2.1), (2.2.2) and (2.3.1). In particular,

$$\psi_\gamma(u) = f'_\gamma(0)^{-1} u = \ell_\gamma u.$$

2.4. The left action of $\mathbb{S}_{\mathcal{C}}$

The left action of an element γ in $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ is naturally given by $\psi_{\gamma^{-1}}$. For convenience, we thus adopt the notation

$$\phi_\gamma = \psi_{\gamma^{-1}} \qquad h_\gamma = (f_{\gamma^{-1}})^{-1}$$

so that $h_\gamma : \phi_\gamma^* F_{\mathcal{C}_U} \to F_{\mathcal{C}_U}$. With these conventions, we have a left action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ which satisfies

$$\phi_\gamma(u) = h'_\gamma(0) u. \tag{2.4.1}$$

These correspond to the conventions of Henn, Karamanov and Mahowald in [7, Section 4].

Finally, we fix our notation for the group $G_{24} = \mathrm{Aut}(\mathcal{C})$. We let

$$\omega = a^{-1} \qquad i = b^{-1}$$

so that $\phi_\omega = \psi_a$ and $\phi_i = \psi_b$. It then follows that

$$\phi_\omega(u_1) = \zeta u_1, \qquad \phi_\omega(u) = \zeta u$$

$$\phi_i(u_1) = \frac{u_1 + 2}{u_1 - 1}, \qquad \phi_i(u) = u\frac{\zeta^2 - \zeta}{u_1 - 1}.$$

The element ω has order 3 and, from now on, we denote the group it generates by C_3. The element i has order four and $i^2 = -1$, the inversion of the curve $\mathcal{C}$. We let

$$j := \omega i \omega^2, \qquad k := \omega^2 i \omega$$

and note that $ij = k$. The elements i and j generate a normal subgroup isomorphic to the quaternions Q_8 and hence $G_{24} \cong \mathrm{Aut}(\mathcal{C}) = Q_8 \rtimes C_3$.

3. The Morava stabilizer group

The Morava stabilizer group $\mathbb{S}_2$ is the group of automorphisms of the Honda formal group law F_2, which is the p-typical formal group law over $\mathbb{F}_4$ specified by the series $[2]_{F_2}(x) = x^4$. The standard presentation for $\mathbb{S}_2$ is the non-commutative extension

$$\mathbb{S}_2 \cong \left(\mathbb{W}\langle S\rangle/(S^2 = 2, aS = Sa^\sigma)\right)^\times,$$

where S is the automorphism $S(x) = x^2$, $a \in \mathbb{W}$ and σ is the Frobenius (see Ravenel [10, Appendix A2] for more details). In this section, we specify an isomorphism $\mathbb{S}_2 \cong \mathbb{S}_{\mathcal{C}}$, whose construction is due to Henn. We also recall some of the key properties of the structure of the group $\mathbb{S}_2$, which transfer to properties of $\mathbb{S}_{\mathcal{C}}$ via this isomorphism.

3.1. The isomorphism of $\mathbb{S}_2$ and $\mathbb{S}_{\mathcal{C}}$

As opposed to the Honda formal group law, it is the $[-2]$-series of the formal group law $F_{\mathcal{C}}$ which has a nice presentation. The following result is proved in Proposition 6.1.4 of Section 6.

Lemma 3.1.1. *Let $\mathcal{C} : y^2 + y = x^3$ be defined over a field of characteristic two. If $F_{\mathcal{C}}$ is the associated formal group law, then $[-2]_{F_{\mathcal{C}}}(x) = x^4$.*

The curve $\mathcal{C}$ and its formal group law $F_{\mathcal{C}}$ are defined over $\mathbb{F}_2$. Therefore, $T(x) = x^2$ is an endomorphism of $F_{\mathcal{C}}$. Lemma 3.1.1 implies that $T(T(x)) = [-2](x)$. The element ω defined in Section 2.2 induces the isomorphism $\omega(x) = \zeta x$ of $F_{\mathcal{C}}$, so that $\omega T = T\omega^\sigma$. This shows that

$$\mathbb{W}\langle T\rangle/(T^2 = -2, \omega T = T\omega^\sigma) \subseteq \mathrm{End}(F_{\mathcal{C}})$$

and this is in fact an equality (see for example Hazewinkel [6, Proposition 21.8.17]). Therefore,

$$\mathbb{S}_{\mathcal{C}} \cong \left(\mathbb{W}\langle T\rangle/(T^2 = -2, \omega T = T\omega^\sigma)\right)^\times.$$

The action of the Galois group on $\mathbb{W}$ induces an action on $\mathbb{S}_{\mathcal{C}}$ defined by $\sigma(a + bT) = a^\sigma + b^\sigma T$.

Let $\alpha = (1-2\omega)(-7)^{-\frac{1}{2}}$, where $(-7)^{\frac{1}{2}}$ in $\mathbb{W}$ is chosen to be congruent to 1 modulo (4). It follows that $\alpha = 1 + \omega T^2 + T^4$ modulo (T^6) and $\alpha\alpha^\sigma = -1$. The proof of the following result follows immediately.

Lemma 3.1.2. *The map $\phi : \mathbb{S}_{\mathcal{C}} \to \mathbb{S}_2$ given by $\phi(a+bT) = a + b(\alpha S)$ is an isomorphism.*

3.2. The filtration and the norm

Lemma 3.1.2 implies that all the results of [2] can be restated for the group $\mathbb{S}_{\mathcal{C}}$ instead of $\mathbb{S}_2$. Here, we briefly review the results which will be important for the computations of this paper.

As in [2], any element $\gamma \in \mathbb{S}_{\mathcal{C}}$ can be expressed as a power series

$$\gamma = \sum_{n=0}^{\infty} a_n T^n,$$

where the a_i's satisfy the equation $x^4 - x = 0$ and $a_0 \neq 0$. Let $F_{0/2}\mathbb{S}_{\mathcal{C}} := \mathbb{S}_{\mathcal{C}}$. For $n > 0$, let

$$F_{n/2}\mathbb{S}_{\mathcal{C}} := \{\gamma \in \mathbb{S}_{\mathcal{C}} \mid \gamma \equiv 1 \mod T^n \}.$$

Define $S_{\mathcal{C}} := F_{1/2}\mathbb{S}_{\mathcal{C}}$. Then $S_{\mathcal{C}}$ is the 2-Sylow subgroup of $\mathbb{S}_{\mathcal{C}}$. This filtration is compatible with the 2-adic filtration on $\mathbb{W}^\times$. Further, $\{F_{n/2}\mathbb{S}_{\mathcal{C}}\}_{n\geq 0}$ forms a system of open subgroups and $\mathbb{S}_{\mathcal{C}}$ is a profinite topological group.

The group $\mathbb{S}_{\mathcal{C}}$ acts on $\mathrm{End}(F_{\mathcal{C}})$ by right multiplication. This gives rise to a representation $\rho : \mathbb{S}_{\mathcal{C}} \to GL_2(\mathbb{W})$, given by

$$\rho(a+bT) = \begin{pmatrix} a & -2b^\sigma \\ b & a^\sigma \end{pmatrix}.$$

The restriction of the determinant to $\mathbb{S}_{\mathcal{C}}$ is given by $\det(a+bT) = aa^\sigma + 2bb^\sigma$. Therefore, the determinant induces a map $\det : \mathbb{S}_{\mathcal{C}} \to \mathbb{Z}_2^\times$. The *norm* is defined as the composite

$$N : \mathbb{S}_{\mathcal{C}} \xrightarrow{\det} \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2.$$

The norm is split surjective. Indeed, let $\pi = 1 + 2\omega$. Then $\det(\pi) = 3$ projects to a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$. The subgroup $\mathbb{S}_{\mathcal{C}}^1$ is then defined by the short exact sequence,

$$1 \to \mathbb{S}_{\mathcal{C}}^1 \to \mathbb{S}_{\mathcal{C}} \xrightarrow{N} \mathbb{Z}_2^\times/\{\pm 1\} \to 1,$$

and $\mathbb{S}_{\mathcal{C}} \cong \mathbb{S}_{\mathcal{C}}^1 \rtimes \mathbb{Z}_2^\times/\{\pm 1\}$. Note that $\mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2$ is torsion-free; hence, G_{24} is a subgroup of $\mathbb{S}_{\mathcal{C}}^1$.

As discussed in [2, Lemma 2.27], the group $\mathbb{S}_{\mathcal{C}}$ has a unique conjugacy class of maximal finite subgroups isomorphic to G_{24}. The group $\mathbb{S}_{\mathcal{C}}^1$ has two, and they are represented by G_{24} and $G'_{24} = \pi G_{24} \pi^{-1}$.

3.3. The action of the Morava stabilizer group

In order to compute the cohomology of $\mathbb{S}_{\mathcal{C}}$, we must have some understanding of its action on $(E_{\mathcal{C}})_*$. Recall from [2, Theorem 2.29] that $\mathbb{S}_{\mathcal{C}}$ is topologically generated by G_{24}, π and α. Since the action of G_{24} was described in Section 2.2, it remains to study the action of α and π.

A concrete method for approximating the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ is explained by Henn, Karamanov and Mahowald in [7, Section 4]. We describe it in Section 6 and give detailed proofs of the results needed for the following computations. For the sake of exposition, we recall the key points here.

It follows from Theorem 6.2.2 that, for γ in $\mathbb{S}_{\mathcal{C}}$, there exists continuous functions $t_0 : \mathbb{S}_{\mathcal{C}} \to (E_{\mathcal{C}})_0^{\times}$ and $t_1 : \mathbb{S}_{\mathcal{C}} \to (E_{\mathcal{C}})_0$ such that

$$\phi_\gamma(u) = t_0(\gamma)u, \qquad \phi_\gamma(u_1) = t_0(\gamma)u_1 + \frac{2}{3}\frac{t_1(\gamma)}{t_0(\gamma)}. \tag{3.3.1}$$

In particular, modulo (2), $\phi_\gamma(u_1) \equiv t_0(\gamma)u_1$ and $\phi_\gamma(u) \equiv t_0(\gamma)u$. Therefore, $v_1 = u_1 u^{-1}$ is fixed by the action of $\mathbb{S}_{\mathcal{C}}$ modulo (2).

Any $\gamma \in S_{\mathcal{C}}$ can be expressed as $\gamma = 1 + \sum_{i=1}^{\infty} a_i(\gamma)T^i$ for $a_i(\gamma) \in \mathbb{W}$ satisfying $a_i(\gamma)^4 - a_i(\gamma) = 0$. It follows from Corollary 6.3.7 that

$$t_0(\gamma) \equiv 1 + a_1(\gamma)^2 u_1 + a_1(\gamma)u_1^2 \mod (2, u_1^3). \tag{3.3.2}$$

In particular, $\phi_\gamma \equiv id \mod (2, u_1)$.

For $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$, we obtain better approximations. We prove in Proposition 6.3.10 that, modulo $(4, 2u_1^2, u_1^9)$,

$$t_0(\gamma) \equiv 1 + 2a_2(\gamma) + 2a_3(\gamma)^2 u_1 + (a_2(\gamma) + a_2(\gamma)^2)u_1^3 + a_3(\gamma)u_1^5 + a_3(\gamma)u_1^8.$$

It also follows from Proposition 6.3.9 that $t_1(\gamma) \equiv a_2(\gamma)^2 u_1$ modulo $(2, u_1^3)$.

We apply this to study the action of α and π. Modulo (T^6), we have

$$\alpha \equiv 1 + \omega T^2 + T^4, \qquad \pi \equiv 1 + \omega T^2 + \omega T^4.$$

Proposition 3.3.1. *Let $\gamma = \alpha$ or $\gamma = \pi$. The unit $t_0(\gamma)$ satisfies:*

$$t_0(\gamma) \equiv 1 + 2\omega + u_1^3 \mod (4, 2u_1^2, u_1^9)$$
$$t_1(\gamma) \equiv \omega^2 u_1 \mod (2, u_1^3).$$

Therefore, for $v_2 = u^{-3}$

$$\phi_\gamma(v_2) = v_2 + v_1^3 \mod (2, u_1^9).$$

Further

$$\phi_\gamma \equiv \phi_{\gamma^{-1}} \mod (2, u_1^9).$$

Proof. Since $\pi \equiv \alpha \mod T^4$, it follows from $a_i(\alpha) = a_i(\pi)$ for $i = 1$ and $i = 2$. Therefore, modulo $(2, u_1^9)$, they have the same action. The claim for $t_0(\gamma)$ follows immediately using the fact that for γ either α or π, the coefficient $a_2(\gamma) = \omega$ and $a_3(\gamma) = 0$. Modulo (2),

$$\begin{aligned}\phi_\gamma(v_2) &\equiv t_0(\gamma)^{-3} v_2 \\ &\equiv t_0(\gamma) t_0(\gamma)^{-4} v_2 \\ &\equiv t_0(\gamma) v_2 \mod (u_1^{12}),\end{aligned}$$

which proves the identity for $\phi_\gamma(v_2)$.

Note that $\gamma^2 \in F_{4/2}\mathbb{S}_{\mathcal{C}}$. It then follows from the formula for $t_0(\gamma^2)$ that $t_0(\gamma^2) \equiv 1$ modulo $(2, u_1^9)$. Hence, $\phi_{\gamma^2} = \phi_\gamma \circ \phi_\gamma \equiv id$ modulo $(2, u_1^9)$. □

4. The algebraic duality resolution spectral sequence

4.1. Preliminaries

The groups $\mathbb{S}_2^1$ and $\mathbb{S}_{\mathcal{C}}^1$ are isomorphic. Further, the isomorphism we constructed restricts to the identity on $\mathbb{W}$, so it preserves α and π (see Lemma 3.1.2). Therefore, Theorem 1.1.1 holds if we replace $\mathbb{S}_2^1$ by $\mathbb{S}_{\mathcal{C}}^1$.

Remark 4.1.1. This is a good place to justify the slight differences between Theorem 1.1.1 and the results of [2]. The existence of the resolution is [2, Theorem 1.2.1], but for the isomorphic group $\mathbb{S}_2^1$. However, the descriptions of the maps is different from [2, Theorem 1.2.6] and this requires an explanation. The map ∂_1 is unchanged. However, in the notation of [2, Theorem 1.2.6], we replace ∂_2 with $g_2^{-1} \circ \partial_2$ and ∂_3 with ∂_3'. The resulting chain complex of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$-modules is isomorphic to that of [2, Theorem 1.2.1]. As g_2^{-1} is an isomorphism, the map $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![S_2^1]\!]} g_2^{-1} : \mathbb{F}_2 \to \mathbb{F}_2$ is non-zero, so that

$$g_2^{-1}(e_2) = e_2 \mod (2, IS_2^1). \tag{4.1.1}$$

Since $e + \alpha \in (2, IS_2^1)$, the description of the middle map follows from the fact that $\Theta \equiv e + \alpha$ modulo $(2, (IS_2^1)^2)$. The last map clearly satisfies (c).

In our computation, we will need to use some additional structure in the algebraic duality resolution spectral sequence (ADSS). We record that here. For any complete $\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]$-modules A and B, let

$$\mathrm{Ext}(A,B) := \mathrm{Ext}_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(A,B).$$

If B is an $\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]$-module which is free over the 2-adics $\mathbb{Z}_2$, then the Bockstein

$$\beta : \mathrm{Ext}^*(A, B/2) \to \mathrm{Ext}^{*+1}(A, B/2)$$

is the connecting homomorphism associated to the exact sequence

$$0 \to B/2 \to B/4 \to B/2 \to 0.$$

The algebraic duality resolution

$$0 \to \mathscr{C}_3 \to \mathscr{C}_2 \to \mathscr{C}_1 \to \mathscr{C}_0 \to \mathbb{Z}_2 \to 0$$

is obtained from splicing exact sequences

$$0 \to N_p \to \mathscr{C}_p \to N_{p-1} \to 0 \tag{4.1.2}$$

with $\mathscr{C}_3 = N_2$ and $N_{-1} = \mathbb{Z}_2$ (see [2]). For B a profinite $\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]$-module, the spectral sequence

$$E_r^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, B) \cong H^q(F_p, B) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, B)$$

associated to the exact couple

$$\begin{array}{ccc} \mathrm{Ext}(N_*, B) & \overset{\delta_*}{\cdots\cdots\cdots} & \mathrm{Ext}(N_{*-1}, B) \\ & {\scriptstyle i^*}\nwarrow \quad \swarrow{\scriptstyle r_*} & \\ & \mathrm{Ext}(\mathscr{C}_*, B) & \end{array} \tag{4.1.3}$$

is the ADSS with coefficients in B. In (4.1.3), the dotted arrows are the connecting homomorphisms for the exact sequences (4.1.2).

Lemma 4.1.2. *For $x \in E_r^{p,q}$, $\beta(x) \in E_r^{p,q+1}$ and $d_r(\beta(x)) = \beta(d_r(x))$.*

Proof. The maps r_*, i_* and δ_* in the exact couple (4.1.3) commute with β. A diagram chase shows that $d_r(\beta(x)) = \beta(d_r(x))$. □

Lemma 4.1.3. *Let R be an $\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]$-module which is also a ring. Suppose that the action of $\mathbb{S}^1_{\mathcal{C}}$ is given by ring homomorphisms. The ADSS with coefficients R is a module over the cohomology $H^*(\mathbb{S}^1_{\mathcal{C}}; R)$.*

Proof. Note that $\mathrm{Ext}(A, R)$ is a module over $\mathrm{Ext}(\mathbb{Z}_2, R)$ for any $\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]$-module A. Further, the maps in the algebraic duality resolution are maps of $\mathbb{Z}_2$-modules. Therefore, the maps r_*, i_* and δ_* in (4.1.3) are morphisms of $\mathrm{Ext}(\mathbb{Z}_2, R)$-modules, hence so are the differentials in the ADSS. □

Recall that

$$(E_{\mathcal{C}})_*V(0) \cong (E_{\mathcal{C}})_*/(2) \cong \mathbb{F}_4[\![u_1]\!][u^{\pm 1}].$$

The spectrum $E_{\mathcal{C}}$ was chosen so that $F_{E_{\mathcal{C}}} = uF_{\mathcal{C}_U}(u^{-1}x, u^{-1}y)$, where $\mathcal{C}_U$ is the curve

$$\mathcal{C}_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

(see Theorem 2.1.1). It follows from Silverman [14, Section IV.1] that

$$[2]_{F_{E_{\mathcal{C}}}}(x) \equiv u^{-1}u_1x^2 + u^{-3}(u_1^3 + 1)x^4 + \dots \mod (2).$$

We adopt the notation

$$v_1 = u^{-1}u_1, \qquad v_2 = u^{-3}. \tag{4.1.4}$$

Warning 4.1.4. The reader should note that the formal group law $F_{E_{\mathcal{C}}}$ is not 2-typical. The image of the Araki generators "v_1" and "v_2" under the map $\varphi : BP_* \to (E_{\mathcal{C}})_*$ which classifies the 2-typification of $F_{E_{\mathcal{C}}}$ does not correspond to our choice of notation (4.1.4).

The element v_1 is invariant under the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*V(0)$ so it is an element of $H^0(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$. However, it does not lift to an invariant in $(E_{\mathcal{C}})_*$. Therefore, the image of v_1 in $H^1(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*)$ under the connecting homomorphism δ for the exact sequence

$$0 \to (E_{\mathcal{C}})_* \xrightarrow{2} (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_*V(0) \to 0 \tag{4.1.5}$$

is non-zero. Let $\eta = \delta(v_1)$ in $H^1(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*)$. We also call the image of η in $H^1(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$ by the same name and note that η is the image of v_1 under the Bockstein $\beta : H^0(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0)) \to H^1(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$, i.e., $\eta = \beta(v_1)$.

Lemma 4.1.5. *The ADSS is a spectral sequence of modules over* $\mathbb{F}_4[v_1, \eta]$.

Proof. The ADSS is a module over $H^*(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$ (see Lemma 4.1.3), which is a module over $\mathbb{F}_4[v_1, \eta]$. □

4.2. The E_1-term

The input for the ADSS is the group cohomology of G_{24} and C_6 with coefficients in $(E_{\mathcal{C}})_*V(0)$. These cohomology groups are described in this section.

The computation of $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$ is related to that of $H^*(A, \Gamma)$ for (A, Γ) the Hopf algebroid classifying Weierstrass curves and their strict isomorphisms. This result is originally due to Hopkins and Mahowald and can be found in Bauer in [1, Section 7]. A self-contained presentation of the computation of $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$ is included in Appendix A.

Some invariants of the curve $\mathcal{C}_U$ play a central role. The following classes of $(E_{\mathcal{C}})_*$ are invariant under the action of $\mathrm{Aut}(\mathcal{C})$. The reader may refer to either Appendix A or to Silverman [14, Section III.1]).

$$\Delta = 27v_2^3(v_1^3 - v_2)^3 \qquad c_4 = 9(v_1^4 + 8v_1v_2)$$
$$c_6 = -27(8v_2^2 + 20v_1^3v_2 - v_1^6) \qquad j = \frac{c_4^3}{\Delta}.$$

Warning 4.2.1. The reader must be careful not to confuse the j-invariant above and the element of G_{24}. Similarly, c_4 and c_6 differ from the elements of Theorem 1.2.1. Our meaning should be clear from the context.

We abuse notation and call the corresponding invariants in $(E_{\mathcal{C}})_*V(0)$ by the same name, so that $\Delta \equiv v_2^3(v_1^3 + v_2)^3$, $j \equiv v_1^{12}\Delta^{-1}$, $c_4 \equiv v_1^4$ and $c_6 \equiv v_1^6$ in $(E_{\mathcal{C}})_*V(0)$.

The main result of Appendix A is the following theorem.

Theorem 4.2.2. *There is an isomorphism*

$$H^*(G_{24}, (E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4[\![j]\!][v_1, \Delta^{\pm 1}, k, \eta, \nu, x, y]/(\sim)$$

where $(\sim)$ *is the ideal generated by the relations*

$$\begin{array}{llll} v_1^{12} = j\Delta & v_1\nu = 0, & v_1^2x = 0, & v_1y = 0, \\ \eta\nu = 0, & \nu x = v_1\eta x, & \eta y = v_1x^2, & xy = 0, \\ \eta^2x = \nu^3, & x^3 = \nu^2y, & y^2 = \nu^2\Delta, & \eta^4 = v_1^4k \end{array}$$

and degrees (s, t), *for* s *the cohomological grading and* t *the internal grading,*

$$\begin{array}{llll} |j| = (0,0), & |v_1| = (0,2), & |\Delta| = (0,24), & |\eta| = (1,2), \\ |\nu| = (1,4) & |x| = (1,8), & |y| = (1,16) & |k| = (4,0). \end{array}$$

Lemma 4.2.3. *The cohomology of* C_6 *with coefficients in* $(E_{\mathcal{C}})_*V(0)$ *is given by*

$$H^*(C_6; (E_{\mathcal{C}})_*V(0)) = \mathbb{F}_4[\![u_1^3]\!][v_1, v_2^{\pm 1}, h]/(v_1^3 = v_2u_1^3),$$

where $|h| = (1,0)$, $|v_2| = (0,6)$, $|v_1| = (0,2)$ *and* $|u_1^3| = (0,0)$. *Further, the action of* η *is determined by*

$$\eta \cdot 1 = v_1 h.$$

Proof. Recall that $C_2 = \{\pm 1\}$ denotes the center of G_{24} and that $C_6 = C_2 \times C_3$. Because C_2 acts trivially on $(E_{\mathcal{C}})_* V(0)$ and C_3 has order coprime to 2,

$$\begin{aligned} H^*(C_6, (E_{\mathcal{C}})_* V(0)) &\cong H^*(C_2; (E_{\mathcal{C}})_* V(0))^{C_3} \\ &= ((E_{\mathcal{C}})_* V(0))^{C_3}[h] \\ &= \mathbb{F}_4[\![u_1^3]\!][v_1, v_2^{\pm 1}, h]/(v_1^3 = v_2 u_1^3), \end{aligned}$$

where h is in (s,t) degree $(1,0)$. To prove that $\eta = v_1 h$, note that the action of C_2 on $(E_{\mathcal{C}})_* = \mathbb{W}[\![u_1]\!][u^{\pm 1}]$ is given by $\phi_{-1}(u) = -u$ and $\phi_{-1}(u_1) = u_1$. One computes that $\delta(v_1) = v_1 h$ for δ the connecting homomorphism associated to (4.1.5). The claim follows from the fact that $\delta(v_1) = \eta$ (see Section 4.1). □

Lemma 4.2.4. *Let* $\pi = 1 + 2\omega$ *in* $\mathbb{S}_{\mathcal{C}}$. *Let* $G'_{24} = \pi G_{24} \pi^{-1}$. *Let* $\phi_\pi : (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_*$ *give the action of* π *on* $(E_{\mathcal{C}})_*$. *Then* ϕ_π *induces an* $\mathbb{F}_4[v_1, \eta]$*-linear isomorphism*

$$H^*(G_{24}, (E_{\mathcal{C}})_* V(0)) \cong H^*(G'_{24}, (E_{\mathcal{C}})_* V(0)).$$

Proof. Conjugation by any element of $\mathbb{S}_{\mathcal{C}}$ induces an isomorphism on cohomology. The linearity follows from the fact that v_1 is invariant under the action of π, and $\eta = \beta(v_1)$. □

Remark 4.2.5. To avoid ambiguities, define $\Delta' := \phi_\pi(\Delta)$ and $j' := \phi_\pi(j)$. For an element z of positive dimension in the cohomology of G_{24}, we will abuse notation and denote $\phi_\pi(z)$ by z since there will be little room for confusion. Further, we let k act on $H^*(G'_{24}, (E_{\mathcal{C}})_* V(0))$ via $\phi_\pi(k)$ and treat the isomorphism of Lemma 4.2.4 as one of $\mathbb{F}_4[v_1, \eta, k]$-modules.

4.3. Approximate Δ*-linearity*

Here, we make the key observation for the computations of Section 5. Namely, that the action of $(IS^1_{\mathcal{C}})^2$ on $(E_{\mathcal{C}})_* V(0)$ is *approximately* Δ*-linear.*

Theorem 4.3.1. *Let* x *be in* $(E_{\mathcal{C}})_* V(0)$. *Let* g *and* h *be elements of* $S_{\mathcal{C}}$. *Then,*

(a) $\phi_h(\Delta) \equiv \Delta$ *modulo* $(2, u_1^6)$,
(b) $(id - \phi_g)(id - \phi_h)(\Delta) \equiv 0$ *modulo* $(2, u_1^8)$,
(c) $(id - \phi_g)(id - \phi_h)\left(x \Delta^{2^k(1+2t)}\right) \equiv (id - \phi_g)(id - \phi_h)(x) \Delta^{2^k(1+2t)}$ *modulo* $(2, u_1^{1+3\cdot 2^{k+1}})$.

Proof. For (a), note that by (3.3.2),

$$t_0(h) \equiv 1 + a_1(h)^2 u_1 + a_1(h) u_1^2 \mod (2, u_1^3),$$

where $a_i(h)^4 = a_i(h)$. It follows that $t_0(h)^{16} \equiv 1 \mod (2, u_1^{16})$. Since $\Delta = u^{-12}(1+u_1^3)^3$, modulo $(2, u_1^6)$ we compute

$$\begin{aligned}(id - \phi_h)(\Delta) &\equiv u^{-12}(1 - t_0(h)^{-12}) + u_1^3 u^{-12}(1 - t_0(h)^{-9}) \\ &\equiv u^{-12}(1 - t_0(h)^4) + u_1^3 u^{-12}(1 - t_0(h)^7) \\ &\equiv u^{-12}(a_1(h)^8 u_1^4) + u_1^3 u^{-12}(a_1(h)^2 u_1 + a_1(h) u_1^2 + a_1(h)^4 u_1^2) \\ &\equiv 0 \mod (2, u_1^6).\end{aligned}$$

To prove (b) Since $id - \phi_g$ applied to the ideal (u_1^6) is contained in the ideal (u_1^8), the claim follows from (a). Finally, it follows from (a) that for $h \in S_{\mathcal{C}}$, there exists y_h such that $\phi_h(\Delta^{2^k(1+2t)}) = \Delta^{2^k(1+2t)} + v_1^{6\cdot 2^k} y_h$. Hence,

$$(id - \phi_h)(\Delta^{2^k(1+2t)}) = \Delta^{2^k(1+2t)}(id - \phi_h)(x) + v_1^{6\cdot 2^k} y_h \phi_h(x).$$

Therefore,

$$\begin{aligned}(id - \phi_g)(id - \phi_h)(\Delta^{2^k(1+2t)}) &= (id - \phi_g)(\Delta^{2^k(1+2t)}(id - \phi_h)(x) + v_1^{6\cdot 2^k} y_h \phi_h(x)) \\ &= \Delta^{2^k(1+2t)}(id - \phi_g)(id - \phi_h)(x) \\ &\quad + v_1^{6\cdot 2^k} y_g \phi_g((id - \phi_h)(x)) + v_1^{6\cdot 2^k}(id - \phi_g)(y_h \phi_h(x))).\end{aligned}$$

Since h and g are in $S_{\mathcal{C}}$,

$$(id - \phi_h)(x) \equiv (id - \phi_g)(y_h \phi_h(x))) \equiv 0 \mod (2, u_1).$$

This proves (c). □

5. Computation of the E_∞-Term

Now we turn to the computation of the ADSS

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_* V(0)) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_* V(0)),$$

with $E_1^{p,q} \cong H^q(F_p, (E_{\mathcal{C}})_* V(0))$, whose construction was described in Section 4. Recall also that the spectral sequence comes from a resolution

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

Further, $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is induced by $\mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\partial_{p+1}, (E_{\mathcal{C}})_*V(0))$ for ∂_{p+1} as described in Theorem 1.1.1. We use these descriptions together with our partial knowledge of the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ to compute the d_1-differentials.

Recall that $E_1^{0,0} \cong (E_{\mathcal{C}})_*^{G_{24}}$ and $E_1^{p,0} \cong (E_{\mathcal{C}})_*^{C_6}$ for $p = 1$ and $p = 2$. Since there is an inclusion

$$(E_{\mathcal{C}})_*^{G_{24}} \to (E_{\mathcal{C}})_*^{C_6},$$

there is an action of $(E_{\mathcal{C}})_*^{G_{24}}$ on $E_1^{p,0}$ for $0 \le p \le 2$. Therefore, it will make sense to talk about the image of Δ defined in Theorem 4.2.2 in $E_1^{p,0}$. To avoid ambiguity, we will use the convention

$$\Delta^k[p] = \Delta^k \cdot 1 \in E_1^{p,0}$$

in the statement of the results. However, in the proofs, we will assume that the context is sufficient to determine which elements are meant. Similarly, for v_2 in $(E_{\mathcal{C}})_*^{C_6}$, to distinguish between $E_1^{1,0}$ and $E_1^{2,0}$, we let

$$v_2^k[p] = v_2^k \cdot 1 \in E_1^{p,0}.$$

Although the differentials d_1 are not ring homomorphisms, they are induced by the action of elements in $\mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}]\!]$. Since $\mathbb{Z}_2[\![\mathbb{S}_{\mathcal{C}}]\!]$ is generated by ring homomorphisms and $(E_{\mathcal{C}})_*V(0)$ is an $\mathbb{F}_2$-vector, it follows that the differentials commute with the squaring operation. That is, for any b in $E_1^{p,q}$,

$$d_1(b^2) = d_1(b)^2 \mod (2). \tag{5.0.1}$$

5.1. The differential $d_1 : E_1^{0,0} \to E_1^{1,0}$

The differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ is induced by the map $\partial_1 : \mathscr{C}_1 \to \mathscr{C}_0$, given by $\partial_1(\gamma e_1) = \gamma(e - \alpha)e_0$. Here, e_i is the canonical generator of $\mathscr{C}_i$. Therefore,

$$d_1 = id + \phi_\alpha : E_1^{0,0} \to E_1^{1,0}.$$

Recall from Theorem 4.2.2 that the powers of $\Delta = v_2(v_2 + v_1^3)^3$ generate $H^0(G_{24}, (E_{\mathcal{C}})_*V(0)) \cong (E_{\mathcal{C}})_*^{G_{24}}$ as an $\mathbb{F}_4[v_1]$-module. So it is sufficient to compute d_1 on $\Delta^n[0]$ for $n \in \mathbb{Z}$. Therefore, we begin by recording a result on the action of α on the powers of Δ.

Proposition 5.1.1. *Let $n = 2^k(2t + 1)$, then*

$$\phi_\alpha(\Delta^n) \equiv \Delta^n(1 + v_1^{6\cdot 2^k} v_2^{-2^{k+1}}) \equiv \Delta^n + v_1^{6\cdot 2^k} v_2^{2^{k+1}(4t+1)} \mod (2, u_1^{9\cdot 2^k}).$$

Proof. By Proposition 3.3.1, $\phi_\alpha(v_2) \equiv v_2 + v_1^3$ modulo $(2, u_1^9)$. Since $\phi_\alpha(v_1) \equiv v_1$ modulo (2),

$$\begin{aligned}\phi_\alpha(\Delta) &\equiv \phi_\alpha(v_2)(\phi_\alpha(v_2)+v_1^3)^3\\ &\equiv (v_2+v_1^3)v_2^3 \mod (2,u_1^9)\\ &\equiv \Delta(1+v_2^{-2}v_1^6) \mod (2,u_1^9).\end{aligned}$$

It suffices to prove the claim for $n = 2t+1$ odd as the more general statement then follows from (5.0.1). Using $\Delta^n \equiv v_2^{4n}$ modulo $(2, u_1)$, we have

$$\begin{aligned}\phi_\alpha(\Delta^n) &= (\Delta(1+v_2^{-2}v_1^6))^n \mod (2,u_1^9)\\ &\equiv \Delta^n(1+n\cdot v_2^{-2}v_1^6) \mod (2,u_1^9)\\ &\equiv \Delta^n + v_2^{2(4t+1)}v_1^6 \mod (2,u_1^9). \quad \square\end{aligned}$$

Corollary 5.1.2. *Let $n = 2^k(2t+1)$, then for $d_1 : E_1^{0,0} \to E_1^{1,0}$,*

$$d_1(\Delta^n[0]) = v_1^{6\cdot 2^k} v_2^{2^{k+1}(4t+1)}[1] \mod (u_1^{9\cdot 2^k}).$$

5.2. The differential $d_1 : E_1^{1,0} \to E_1^{2,0}$

The goal of this section is to prove the following result.

Proposition 5.2.1. *Let $n = 2^k(1+2t)$ where $t \in \mathbb{Z}$ and $k \geq 0$. There exist homogeneous elements b_n, such that $b_n \equiv v_2^n[1]$ modulo (u_1). The elements b_n satisfy*

$$d_1(\Delta^n[0]) = \begin{cases} v_1^{6\cdot 2^k} b_{2^{k+1}(1+4t)} & n = 2^k(1+2t)\\ 0 & n = 0,\end{cases}$$

and

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} v_2^{2^{k+1}(1+2t)}[2] \mod (2, u_1^{3\cdot 2^k+3}) & n = 2^k(3+4t)\\ v_1^{3\cdot 2^{k+1}} v_2^{m-2^{k+1}}[2] \mod (2, u_1^{3\cdot 2^{k+1}+3}) & n = 1+2^{k+2}+2^{k+3}t\\ 0 & n = 0,1 \text{ and } 2^{k+1}(1+4t).\end{cases}$$

We will break up the proof into a series of propositions.

The differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ is induced by the map $\partial_2 : \mathscr{C}_2 \to \mathscr{C}_1$. Recall from Theorem 1.1.1 that

$$\partial_2(\gamma e_2) = \gamma(e + \alpha + \mathscr{E})e_1$$

where $\mathscr{E} \in (2, (I\mathbb{S}_{\mathcal{C}}^1)^2)$. Therefore, modulo (2), $\mathscr{E} = \sum a_{g,h}(e-g)(e-h)$ is in $(IS_{\mathcal{C}}^1)^2$. It is to be thought of as the *error*. Let

$$\phi_{\mathscr{E}} = \sum a_{g,h}(id - \phi_g)(id - \phi_h). \tag{5.2.1}$$

We first construct the d_1-cycle b_1. The idea for its construction comes from Mahowald and Rezk [9, Corollary 6.2]. We need the following result.

Lemma 5.2.2. *Let c_4 in $(E_{\mathcal{C}})_8^{G_{24}}$ be given by*

$$c_4 = 9(v_1^4 + 8v_1v_2) = 9u^{-4}u_1(u_1^3 + 8)$$

as in Section 4.2. For any γ in $\mathbb{G}_{\mathcal{C}}$,

$$\phi_\gamma(c_4) \equiv c_4 \mod (16).$$

Further, if an element γ in $\mathbb{S}_{\mathcal{C}}$ has the form $\gamma \equiv 1 + a_2(\gamma)T^2$ modulo T^3 for $a_2(\gamma)$ as in Section 3.3, then

$$c_4 - \phi_\gamma(c_4) \equiv 16(a_2(\gamma) + a_2(\gamma)^2)u_1u^{-4} \mod (32, 16u_1^2).$$

Proof. The first step is to show that $\phi_\gamma(c_4) \equiv c_4$ modulo 16. Since the Galois group acts trivially on c_4, it suffices to prove the claim for γ in $\mathbb{S}_{\mathcal{C}}$. Let $t_0 = t_0(\gamma)$ and $t_1 = t_1(\gamma)$ as defined in (3.3.1). A direct computation using (3.3.1) implies that

$$\begin{aligned} c_4 - \phi_\gamma(c_4) &\equiv 8u^{-4}\left(u_1 + \frac{3u_1}{t_0^3} + \frac{t_1^2u_1^2}{t_0^4} + \frac{t_1u_1^3}{t_0^2} + \frac{2t_1}{t_0^5} + \frac{2t_1^4}{t_0^8}\right) \mod (32) \\ &\equiv 8u^{-4}u_1t_0^{-4}\left(t_0^4 + t_0 + u_1t_1^2 + u_1^2t_1t_0^2\right) \mod (16) \end{aligned}$$

It follows from Proposition 6.3.3 of Section 6 that

$$t_0 \equiv t_0^4 + u_1t_1^2 + u_1^2t_1t_0^2 \mod (2).$$

This proves that $c_4 - \phi_\gamma(c_4) \equiv 0$ modulo (16).

Let $a_i = a_i(\gamma)$ as in Section 3.3. By Proposition 6.3.9 and Proposition 6.3.10 applied to γ, we have

$$t_0 \equiv 1 + 2a_2 + 2a_3^2u_1 \mod (4, u_1^2), \qquad t_1 \equiv a_2^2u_1 \mod (2, u_1^2).$$

Therefore, $t_0^4 \equiv 1$ modulo $(4, u_1^2)$ and $t_1^4 \equiv 0$ modulo $(4, u_1^2)$ so that

$$\begin{aligned} c_4 - \phi_\gamma(c_4) &\equiv 8u^{-4}\left(u_1 + 3u_1t_0 + 2t_1t_0^3\right) \mod (32, u_1^2) \\ &\equiv 8u^{-4}\left(u_1 + 3u_1(1 + 2a_2 + 2a_3^2u_1) + 2a_2^2u_1\right) \mod (32, u_1^2) \\ &\equiv 16(a_2 + a_2^2)u^{-4}u_1 \mod (32, u_1^2) \end{aligned}$$

This implies that $c_4 - \phi_\gamma(c_4) = 16((a_2 + a_2^2)u^{-4}u_1 + \ldots)$. □

Consider the spectral sequence

$$\widetilde{E}_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_*) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*). \tag{5.2.2}$$

Let

$$f : \widetilde{E}_1^{p,q} \to E_1^{p,q} \tag{5.2.3}$$

be the map of spectral sequences induced by the reduction modulo (2) on the coefficients. Let $d_1 : \widetilde{E}_1^{0,0} \to \widetilde{E}_1^{1,0}$ denotes the differential in the spectral sequence $\widetilde{E}_r^{p,q}$. Since $d_1(x) = x - \phi_\alpha(x)$, and $\alpha \equiv 1 + \omega T^2$ modulo T^3, it follows from Lemma 5.2.2 that

$$d_1(c_4) = 16(v_1 v_2 + \ldots).$$

Definition 5.2.3. Let $B_1 \in \widetilde{E}_1^{1,0}$ be defined by $B_1 = \frac{d_1(c_4)}{16}$. Since $\widetilde{E}_1^{1,0}$ is torsion free, this specifies B_1 uniquely and

$$B_1 \equiv v_1 v_2 \mod (2, u_1^2).$$

Proposition 5.2.4. *There is an element $b_1 \in E_1^{1,0}$ specified by the identity $f(B_1) = v_1 b_1$ such that $b_1 \equiv v_2[1]$ modulo (u_1^3) and $d_1(b_1) = 0$.*

Proof. Let B_1 be as in Definition 5.2.3. Then, $f(B_1)$ is divisible by v_1. Therefore, we can define an element $b_1 \in E_1^{1,0}$ by $b_1 := v_1^{-1} f(B_1)$. Since $E_1^{1,0}$ is v_1-torsion free, this specifies b_1 uniquely. Further, it implies that $b_1 \equiv v_2$ modulo $(2, u_1)$. Since b_1 is an element of

$$(E_{\mathcal{C}})_6 V(0)^{C_6} = \mathbb{F}_4[\![u_1^3]\!]\{v_2\}.$$

This forces the congruence $b_1 \equiv v_2$ modulo $(2, u_1^3)$.

Finally, in the spectral sequence $\widetilde{E}_r^{p,q}$, we have $d_1^2(c_4) = 16 d_1(B_1)$. Since there is no torsion in $\widetilde{E}_1^{p,0}$ and $d_1^2 = 0$, $d_1(B_1) = 0$ so that $d_1(v_1 b_1) = 0$ in $E_1^{2,0}$. Since there is no v_1-torsion in $E_1^{p,0}$, this implies that $d_1(b_1) = 0$. □

Proposition 5.2.5. *Let $n = 2^k(3 + 4t)$. Define $b_n = v_2^n$. Then*

$$d_1(b_n) \equiv v_1^{3\cdot 2^k} v_2^{2^{k+1}(1+2t)} \mod (2, u_1^{6\cdot 2^k}).$$

Proof. By (5.0.1) is sufficient to prove the claim when $k = 0$. First, note that if $h \in S_{\mathcal{C}}$, by (3.3.2),

$$t_0(h) \equiv 1 + a_1(h)^2 u_1 + a_1(h) u_1^2 \mod (2, u_1^3)$$

where $a_1(h)^4 = a_1(h)$. Therefore, $t_0(h)^4 \equiv 1$ modulo $(2, u_1^4)$ and

$$
\begin{aligned}
(id - \phi_h)(v_2^{3+4t}) &= v_2^{3+4t} + t_0(h)^{-3(3+4t)} v_2^{3+4t} \\
&\equiv v_2^{3+4t} + t_0(h)^3 v_2^{3+4t} \mod (2, u_1^3) \\
&\equiv v_2^{3+4t} + v_2^{3+4t}(1 + a_1(h)^2 u_1 + a_1(h) u_1^2 + a_1(h)^4 u_1^2) \mod (2, u_1^3) \\
&\equiv v_2^{3+4t} a_1(h)^2 u_1 \mod (2, u_1^3).
\end{aligned}
$$

For any $g \in S_{\mathcal{C}}$, the image of $(2, u_1^3)$ under $id - \phi_g$ is in $(2, u_1^4)$. Further, since $t_0(g)^4 \equiv 1 \mod (2, u_1^4)$

$$
\begin{aligned}
(id - \phi_g)(v_2^{3+4t} u_1) &\equiv (1 - t_0(g)^{-8-12t}) v_2^{3+4t} u_1 \\
&\equiv 0 \mod (2, u_1^5).
\end{aligned}
$$

Hence, $(id - \phi_g)(id - \phi_h)(v_2^{3+4t}) \equiv 0$ modulo $(2, u_1^4)$. For $\phi_{\mathscr{E}}$ be as in (5.2.1), it follows that

$$
\phi_{\mathscr{E}}(v_2^{3+4t}) \equiv 0 \mod (2, u_1^4).
$$

Hence, since $\phi_\alpha(v_2) \equiv v_2 + v_1^3 \mod (2, u_1^9)$ (see Proposition 3.3.1),

$$
\begin{aligned}
d_1(v_2^{3+4t}) &= v_2^{3+4t} + \phi_\alpha(v_2^{3+4t}) + \phi_{\mathscr{E}}(v_2^{3+4t}) \\
&\equiv v_2^{3+4t} + (v_2 + v_1^3)^{3+4t} \\
&\equiv v_1^3 v_2^{2+4t} \mod (2, u_1^4).
\end{aligned}
$$

Since $d_1(b_n)$ is C_6-invariant, the congruence can be improved to $d_1(b_n) \equiv v_1^3 v_2^{2(1+2t)}$ modulo $(2, u_1^6)$. □

Proposition 5.2.6. *Let $n = 1 + 2^{k+2}(1+2t)$. Define $b_n = b_1 \Delta^{2^k(1+2t)}$. Then*

$$
d_1(b_n) \equiv v_1^{3 \cdot 2^{k+1}} v_2^{1 + 2^{k+1}(1+4t)} \mod (u_1^{3 \cdot 2^{k+1} + 3}).
$$

Proof. By Theorem 4.3.1,

$$
\phi_{\mathscr{E}}(b_n) \equiv \phi_{\mathscr{E}}(b_1) \Delta^{2^k(1+2t)} \mod (2, u_1^{1 + 3 \cdot 2^{k+1}}).
$$

Therefore,

$$
\begin{aligned}
d_1(b_n) &\equiv b_n + \phi_\alpha(b_n) + \phi_{\mathscr{E}}(b_n) \\
&\equiv b_1 \Delta^{2^k(1+2t)} + \phi_\alpha(b_1) \Delta^{2^k(1+2t)} (1 + v_2^{-2} v_1^6)^{2^k(1+2t)} + \phi_{\mathscr{E}}(b_1) \Delta^{2^k(1+2t)} \\
&\equiv b_1 \Delta^{2^k(1+2t)} + \phi_\alpha(b_1) \Delta^{2^k(1+2t)} (1 + v_2^{-2^{k+1}} v_1^{3 \cdot 2^{k+1}}) + \phi_{\mathscr{E}}(b_1) \Delta^{2^k(1+2t)} \\
&\equiv (b_1 + \phi_\alpha(b_1) + \phi_{\mathscr{E}}(b_1)) \Delta^{2^k(1+2t)} + \phi_\alpha(b_1) v_2^{-2^{k+1}} v_1^{3 \cdot 2^{k+1}} \Delta^{2^k(1+2t)} \\
&\equiv d_1(b_1) \Delta^{2^k(1+2t)} + \phi_\alpha(b_1) v_2^{-2^{k+1}} v_1^{3 \cdot 2^{k+1}} \Delta^{2^k(1+2t)} \mod (2, u_1^{1 + 3 \cdot 2^{k+1}}).
\end{aligned}
$$

But $d_1(b_1) = 0$ and $\phi_\alpha(b_1) \equiv v_2$ modulo $(2, u_1^3)$. Furthermore, $\Delta^{2^k(1+2t)} \equiv v_2^{2^{k+2}+2^{k+3}t}$, so that

$$\begin{aligned} d_1(b_n) &\equiv v_1^{3\cdot 2^{k+1}} v_2^{1-2^{k+1}+2^{k+2}+2^{k+3}t} \\ &\equiv v_1^{3\cdot 2^{k+1}} v_2^{1+2^{k+1}+2^{k+3}t} \quad \bmod (2, u_1^{3\cdot 2^{k+1}+1}). \end{aligned}$$

Since $d_1(b_n)$ is C_6 invariant, the congruence holds modulo $(2, u_1^{3\cdot 2^{k+1}+3})$. □

Proof of Proposition 5.2.1. Let $t \in \mathbb{Z}$ and $k \geq 0$

$$b_n := \begin{cases} b_1^n & n = 0, 1; \\ v_2^n & n = 2^k(3+4t); \\ b_1 \Delta^{2^k+2^{k+1}t} & n = 1 + 2^{k+2}(1+2t); \\ v_1^{-6\cdot 2^k} d_1\left(\Delta^{2^k(2t+1)}\right) & n = 2^{k+1}(4t+1). \end{cases}$$

The element b_n is in degree $6n$ and $b_n \equiv v_2^n$ modulo $(2, u_1^3)$. That $d_1(b_0) = 0$ follows from the fact that it is invariant under the action of $\mathbb{S}_{\mathcal{C}}$. It is the content of Proposition 5.2.4 that $d_1(b_1) = 0$. Let $n = 2^{k+1}(1+4t)$. Since d_1 is v_1-linear and there is no v_1-torsion in $E_1^{2,0}$, it follows from

$$d_1(v_1^{6\cdot 2^k} b_n) = d_1^2\left(\Delta^{2^k(2t+1)}\right) = 0,$$

that $d_1(b_n) = 0$. The remaining claims follow from Proposition 5.2.5 and Proposition 5.2.6. □

5.3. The differential $d_1 : E_1^{2,0} \to E_1^{3,0}$

Recall that

$$E_1^{3,0} \cong H^q(G'_{24}, (E_{\mathcal{C}})_* V(0)) = \mathbb{F}_4[\![j']\!][v_1, \Delta']/(j' = v_1^{12}\Delta'^{-1})$$

where $\Delta' = \phi_\pi(\Delta)$. We let $\Delta'[3] = \Delta' \cdot 1 \in E_1^{3,0}$. The next goal is to prove:

Proposition 5.3.1. *Let $n = 2^k(1+2t)$ where $t \in \mathbb{Z}$ and $k \geq 0$. There exist homogeneous elements $\overline{b}_n$ such that*

$$\overline{b}_n \equiv v_2^n[2] \mod (u_1) \tag{5.3.1}$$

and

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} \overline{b}_{2^{k+1}(1+2t)} & n = 2^k(3+4t) \\ v_1^{3\cdot 2^{k+1}} \overline{b}_{1+2^{k+1}(1+4t)} & n = 1 + 2^{k+2}(1+2t) \\ 0 & \textit{otherwise.} \end{cases}$$

Further,

$$d_1(\overline{b}_n) = v_1^{3(1+2^{k+1})}\Delta'^{\,2^k(1+2t)}[3] \mod (u_1^{3(1+2^{k+1})+12})$$

if $n = 1 + 2^{k+1}(3 + 4t)$ *and is zero otherwise.*

Proof of Proposition 5.3.1. For $n = 2^k(3 + 4t)$ and $n = 1 + 2^{k+2}(1 + 2t)$, define $\overline{b}_n$ by the identities

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k}\overline{b}_{2^{k+1}(1+2t)} & n = 2^k(3+4t); \\ v_1^{3\cdot 2^{k+1}}\overline{b}_{1+2^{k+1}(1+4t)} & n = 1+2^{k+2}(1+2t). \end{cases}$$

The classes $\overline{b}_{2^{k+1}(1+2t)}$ and $\overline{b}_{1+2^{k+1}(1+4t)}$ are well-defined since $E_1^{2,0}$ is torsion free. Further, the $\overline{b}_n$ satisfies equation (5.3.1) and $d_1(\overline{b}_n) = 0$.

Let $m = 1 + 2^{k+1}(1 + 4t)$. For $n = 1 + 2^{k+1}(3 + 4t)$, define

$$\overline{b}_n = \overline{b}_m(\Delta')^{2^k}.$$

Because $(\Delta')^{2^k} \equiv v_2^{2^{k+2}}$ modulo $(2, u_1)$, the elements $\overline{b}_n$ satisfy (5.3.1). We will prove that

$$d_1(\overline{b}_n) \equiv v_1^{3(2^{k+1}+1)}(\Delta')^{2^k(2t+1)} \mod (u_1^{3(2^{k+1}+1)+1}). \tag{5.3.2}$$

Because $d_1(\overline{b}_n)$ is G'_{24} invariant, if (5.3.2) holds, then the congruence also holds modulo $(u_1^{3(1+2^{k+1})+12})$. This will finish the proof of the theorem.

By Theorem 1.1.1, the map $d_1 : E_1^{2,0} \to E_1^{3,0}$ is given by

$$\phi_\pi(id + \phi_i + \phi_j + \phi_k)(id + \phi_\alpha^{-1})\phi_\pi^{-1}.$$

Since $\phi_\pi^{-1}(\Delta') = \Delta$,

$$d_1(\overline{b}_n) = \phi_\pi(id + \phi_i + \phi_j + \phi_k)(id + \phi_\alpha^{-1})(\phi_\pi^{-1}(\overline{b}_m)\Delta^{2^k}).$$

By Proposition 3.3.1, we can $\phi_\alpha = \phi_{\alpha^{-1}}$ modulo u_1^9. By Proposition 5.1.1, this implies that

$$\phi_{\alpha^{-1}}(\Delta^{2^k}) = \Delta^{2^k}(1 + v_1^{6\cdot 2^k}v_2^{-2^{k+1}}) \mod (2, u_1^{9\cdot 2^k}).$$

Hence, modulo $(2, u_1^{9\cdot 2^k})$,

$$\begin{aligned}(id + \phi_{\alpha^{-1}})(\phi_\pi^{-1}(\overline{b}_m)\Delta^{2k}) &\equiv \phi_\pi^{-1}(\overline{b}_m)\Delta^{2^k} + \phi_{\alpha^{-1}}(\phi_\pi^{-1}(\overline{b}_m))\Delta^{2^k}(1 + v_1^{6\cdot 2^k}v_2^{-2^{k+1}}) \\ &\equiv (id + \phi_{\alpha^{-1}})(\phi_\pi^{-1}(\overline{b}_m)) \cdot \Delta^{2^k} \\ &\quad + \phi_{\alpha^{-1}}(\phi_\pi^{-1}(\overline{b}_m))(v_1^{6\cdot 2^k}v_2^{-2^{k+1}})\Delta^{2^k}.\end{aligned}$$

We treat both terms separately. First, note that i, j and k fix Δ, so that

$$\phi_\pi\left((id+\phi_i+\phi_j+\phi_k)\left((id+\phi_{\alpha^{-1}})(\phi_\pi^{-1}(\overline{b}_m))\cdot\Delta^{2^k}\right)\right)=d_1(\overline{b}_m)\cdot(\Delta')^{2^k}=0.$$

Next, note that $\phi_{\alpha^{-1}}\phi_{\pi^{-1}}(\overline{b}_m)=\phi_{(\pi\alpha)^{-1}}(\overline{b}_m)$. Since $\pi\alpha\in F_{2/2}\mathbb{S}_{\mathcal{C}}$, it follows from Proposition 3.3.1 that

$$\phi_{\alpha^{-1}}\phi_{\pi^{-1}}(\overline{b}_m)\equiv\overline{b}_m \mod (2,u_1^3).$$

Since $\overline{b}_m\equiv v_2^{1+2^{k+1}(1+4t)} \mod (2,u_1^3)$, this implies that

$$\phi_{\alpha^{-1}}(\phi_\pi^{-1}(\overline{b}_m))(v_1^{6\cdot 2^k}v_2^{-2^{k+1}})\equiv v_1^{6\cdot 2^k}v_2^{1+2^{k+3}t} \mod (2,u_1^{3(2^{k+1}+1)})$$

Further, $(2,u_1^{9\cdot 2^k})\subseteq(2,u_1^{3(2^{k+1}+1)})$ and, since $(id+\phi_i+\phi_j+\phi_k)$ is in $IS_{\mathcal{C}}$ it maps $(2,u_1^{3(2^{k+1}+1)})$ to the ideal $(2,u_1^{3(2^{k+1}+1)+1})$, we can ignore the error terms. Hence,

$$d_1(\overline{b}_n)=\phi_\pi((id+\phi_i+\phi_j+\phi_k)(v_2^{1+2^{k+3}t}))\cdot v_1^{6\cdot 2^k}(\Delta')^{2^k} \mod (2,v_1^{3(2^{k+1}+1)+1}).$$

From Section 2.2,

$$t_0(i)^{-1}=1+u_1 \qquad t_0(j)^{-1}=1+\zeta u_1 \qquad t_0(k)^{-1}=1+\zeta^2u_1.$$

We have $t_0(i)^{-8}\equiv t_0(j)^{-8}\equiv t_0(k)^{-8}\equiv 1$ modulo $(2,u_1^8)$. Modulo $(2,u_1^8)$,

$$\begin{aligned}
t_0(i)^{-3(1+2^{k+3}t)}&\equiv(1+u_1)^3\equiv 1+u_1+u_1^2+u_1^3\\
t_0(j)^{-3(1+2^{k+3}t)}&\equiv(1+\zeta u_1)^3\equiv 1+\zeta u_1+\zeta^3u_1^2+u_1^3\\
t_0(k)^{-3(1+2^{k+3}t)}&\equiv(1+\zeta^2u_1)^3\equiv 1+\zeta^2u_1+\zeta u_1^2+u_1^3.
\end{aligned}$$

Since $\phi_\gamma(v_2^n)=t_0(\gamma)^nv_2^n$,

$$(id+\phi_i+\phi_j+\phi_k)(v_2^{1+2^{k+3}t})\equiv v_1^3v_2^{2^{k+3}t} \mod (2,u_1^8).$$

Hence,

$$\begin{aligned}
d_1(\overline{b}_n)&\equiv v_1^{3(1+2^{k+1})}\phi_\pi(v_2^{2^{k+3}t})(\Delta')^{2^k} \mod (2,v_1^{3(2^{k+1}+1)+1})\\
&\equiv v_1^{3(1+2^{k+1})}(\Delta')^{2^k(1+2t)} \mod (2,v_1^{3(2^{k+1}+1)+1}).
\end{aligned}$$

The only element $\overline{b}_n$ which has not been constructed is $\overline{b}_1$. Its existence follows from Lemma 5.3.2 below. □

Lemma 5.3.2. *There exists a sequence of elements* $\{\overline{b}_{1,n}\}$ *such that*

(1) $\overline{b}_{1,n} \equiv v_2$ *modulo* (u_1^6),
(2) $d_1(\overline{b}_{1,n}) \equiv 0$ *modulo* $(u_1^{3(1+4n)})$,
(3) $\overline{b}_{1,n+1} - \overline{b}_{1,n} \equiv 0$ *modulo* (u_1^{6n}).

If $(E_{\mathcal{C}})_6V(0)$ *is given the topology induced by the maximal ideal* $\mathfrak{m} = (u_1)$, *then the limit*

$$\overline{b}_1 := \lim_{n\to\infty} \overline{b}_{1,n}$$

exists. The element $\overline{b}_1$ *satisfies equation (5.3.1) and* $d_1(\overline{b}_1) = 0$.

Proof. The construction of $\{\overline{b}_{1,n}\}$ is by induction on n. First, define $\overline{b}_{1,1} := v_2$ and note that

$$\overline{b}_{1,1} + \phi_{\alpha^{-1}}(\overline{b}_{1,1}) \equiv v_1^3 + u_1^6\epsilon.$$

The $\mathbb{F}_4$-vector space with basis

$$\{v_1^3, v_1^{3\cdot 5}\Delta'^{-1}, v_1^{3\cdot 9}\Delta'^{-2}, \ldots, v_1^{3(1+4s)}\Delta'^{-s}, \ldots\}$$

is dense in $((E_{\mathcal{C}})_6V(0))^{G'_{24}}$. Hence, $d_1(\overline{b}_{1,1}) \equiv 0$ modulo (u_1^6).

Suppose that $\overline{b}_{1,n}$ has been defined. If $d_1(\overline{b}_{1,n}) = 0$, then let $\overline{b}_{1,N} := \overline{b}_{1,n}$ for all $N \geq n$. Otherwise,

$$d_1(\overline{b}_{1,n}) = v_1^{3+12s_n}\Delta'^{-s_n} + \ldots \tag{5.3.3}$$

for $s_n \geq n$. Let $s_n = 2^{k_n}(1+2t_n)$ and let $m_n = 3\cdot 2^{k_n+1}(1+4t_n)$. Then $m_n \geq 6n$. For

$$r_n = 1 + 2^{k_n+1} + 2^{k_n+2} + 2^{k_n+3}(-t_n - 1),$$

(5.3.3) together with the fact that

$$d_1(\overline{b}_{r_n}) = v_1^{3(1+2^{k_n+1})}\Delta'^{2^k_n(1+2(-t_n-1))} + \ldots,$$

implies that $d_1(\overline{b}_{1,n}) = v_1^{m_n}d_1(\overline{b}_{r_n}) + \ldots$ Define $\overline{b}_{1,n+1} := \overline{b}_{1,n} + v_1^{m_n}\overline{b}_{r_n}$. Then $\overline{b}_{1,n+1}$ satisfies properties (1), (2) and (3).

Now consider the sequence $\{\overline{b}_{1,n}\}$. Since $m_{n+k} \geq 6n$ for $k \geq 0$,

$$\overline{b}_{1,n+k} - \overline{b}_{1,n} = v_1^{m_{n+1}}\overline{b}_{r_{n+1}} + \ldots + v_1^{m_{n+k}}\overline{b}_{r_{n+k}} \in (u_1)^{6n},$$

so the sequence $\{\overline{b}_{1,n}\}$ is Cauchy. Since $((E_{\mathcal{C}})_6V(0))^{C_6}$ is complete with respect to $\mathfrak{m}$, the limit exists and $\overline{b}_1$ is well-defined. The map d_1 is continuous, so that,

$$d_1(\overline{b}_1) = \lim_{n\to\infty} d_1(\overline{b}_{1,n}).$$

But $d_1(\overline{b}_{1,n}) \in \mathfrak{m}^{3(1+4N)}$ for all $n \geq N$, which implies that

$$d_1(\overline{b}_1) \in \bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0. \quad \square$$

Remark 5.3.3. Define

$$\Delta_n := \begin{cases} \Delta^n[0] & n = 2^k(1+2t) \\ 1 \cdot [0] & n = 0, \end{cases}$$

$$\overline{\Delta}_n := \begin{cases} v_1^{-3(1+2^{k+1})} d_1(\overline{b}_{1+2^{k+1}+2^{k+2}+2^{k+3}t}) & n = 2^k(1+2t) \\ 1 \cdot [3] & n = 0. \end{cases}$$

Combining the results of this section to proves the first part of Theorem 1.2.1.

An analysis of the definition of the elements shows that the congruences stated in Theorem 1.2.1 can be improved as follows:

$$\Delta_n = \Delta^n[0]$$

$$b_n = \begin{cases} v_2^n[1] & n = 0 \text{ or } 2^k(3+4t) \\ v_2^n[1] \mod (u_1^3) & n = 1 \text{ or } 1 + 2^{k+2}(1+2t) \\ v_2^n[1] \mod (2, u_1^{3 \cdot 2^k}) & n = 2^{k+1}(4t+1) \end{cases}$$

$$\overline{b}_n = \begin{cases} v_2^n[2] & n = 0 \\ v_2^n[2] \mod (u_1^{3 \cdot 2^k}) & n = 1 \text{ or } n = 2^{k+1}(1+2t) \\ v_2^n[2] \mod (u_1^3) & n = 1 + 2^{k+1}(1+4t) \text{ or } n = 1 + 2^{k+1}(3+4t) \end{cases}$$

$$\overline{\Delta}_n = \begin{cases} 1 \cdot [3] & n = 0 \\ \Delta'^n[3] \mod (u_1^{12}) & n \neq 0 \end{cases}$$

5.4. *The differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$*

Although $V(0)$ is not a ring spectrum, $(E_{\mathcal{C}})_* V(0) \cong (E_{\mathcal{C}})_*/2$, and a canonical generator is given by the image of the unit in $(E_{\mathcal{C}})_0$ in the long exact sequence

$$\ldots \to (E_{\mathcal{C}})_* \xrightarrow{2} (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_* V(0) \to \ldots$$

Thus, $(E_{\mathcal{C}})_* V(0)$ inherits a ring structure from $(E_{\mathcal{C}})_*$. Lemma 4.1.3 implies that the ADSS for $(E_{\mathcal{C}})_* V(0)$ is a module over $H^*(\mathbb{S}_{\mathcal{C}}, (E_{\mathcal{C}})_* V(0))$. The canonical inclusion of $\mathbb{F}_4$ into $(E_{\mathcal{C}})_* V(0)$ induces a map

$$H^*(\mathbb{S}_{\mathcal{C}}^1, \mathbb{F}_4) \to H^*(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_* V(0))$$

and the ADSS for $(E_{\mathcal{C}})_* V(0)$ is also a module over $H^*(\mathbb{S}_{\mathcal{C}}^1, \mathbb{F}_4)$.

Let

$$F_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, \mathbb{F}_4) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, \mathbb{F}_4). \tag{5.4.1}$$

Let $k \in F_1^{0,4}$ be the periodicity generator for the cohomology of G_{24}, (see Lemma A.1). The extension

$$1 \to K^1 \to \mathbb{S}^1_{\mathcal{C}} \to G_{24} \to 1$$

is split. Therefore, the map

$$H^*(\mathbb{S}^1_{\mathcal{C}}, \mathbb{F}_4) \to H^*(G_{24}, \mathbb{F}_4)$$

induced by the inclusion of G_{24} in $\mathbb{S}^1_{\mathcal{C}}$ is split surjective. This implies that the image of k is a permanent cycle in $F_1^{0,4}$. Therefore, it represents a class

$$k \in H^4(\mathbb{S}_{\mathcal{C}}; \mathbb{F}_4),$$

and the differentials in the ADSS commute with the action of k. To make sense of this, we compute the action of k on $E_1^{p,q}$.

First, k acts by multiplication by the element of the same name in $E_1^{0,q}$ and $E_1^{3,q}$. Further, the map

$$H^*(\mathbb{S}^1_{\mathcal{C}}; \mathbb{F}_4) \to H^*(C_6; (E_{\mathcal{C}})_* V(0))$$

factors through the map

$$H^*(G_{24}; (E_{\mathcal{C}})_* V(0)) \to H^*(C_6; (E_{\mathcal{C}})_* V(0))$$

induced by the inclusion of C_6 in G_{24}. Therefore, k acts by multiplication by h^4 on $E_1^{p,q}$ for $p = 1$ and $p = 2$. We collect these remarks in the following lemma.

Lemma 5.4.1. *The differentials in the ADSS are k-linear, where the action of k is given by multiplication by k on $E_r^{0,*}$ and $E_r^{3,*}$, and by multiplication by h^4 on $E_r^{p,*}$ for $p = 1, 2$.*

This will allow us to compute some of the differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$ based on our results for $q = 0$.

Lemma 5.4.2. *Let $x \in E_1^{0,q}$. The differential $d_1 : E_1^{0,q} \to E_1^{1,q}$ is zero unless $x = v_1^r \eta^s \Delta^t$ or $x = v_1^r k^s \Delta^t$, in which case it is given by*

$$d_1(v_1^r \eta^s \Delta^t) = v_1^r \eta^s d_1(\Delta^t)$$

and

$$d_1(v_1^r k^s \Delta^t) = v_1^r h^{4s} d_1(\Delta^t).$$

Proof. There is no v_1-torsion in $E_1^{1,q}$, and d_1 is v_1-linear. Therefore, if x is v_1-torsion, we must have $d_1(x) = 0$. The only classes in $E_1^{0,q}$ which are not v_1-torsion are of the form $x = v_1^r \eta^s \Delta^t$ or $x = v_1^r k^s \Delta^t$. The statement then follows from the η and k-linearity of the differentials. □

Lemma 5.4.3. *Let $x \in E_1^{1,q}$. The differential $d_1 : E_1^{1,q} \to E_1^{2,q}$ satisfies*

$$h^k d_1(x) = d_1(h^k x).$$

Proof. This follows from the fact that the differentials are $\eta = hv_1$ and v_1-linear. Indeed, since $\eta = v_1 h$, we have the following equalities

$$v_1^k h^k d_1(x) = \eta^k d_1(x) = d_1(\eta^k x) = d_1(v_1^k h^k x) = v_1^k d_1(h^k x).$$

There is no v_1 or h-torsion in $E_1^{1,q}$ and $E_1^{2,q}$, so $h^k d_1(x) = d_1(h^k x)$. □

Understanding the differential $d_1 : E_1^{2,q} \to E_1^{3,q}$ is more subtle as there is v_1-torsion in $E_1^{3,q}$ for $q > 0$. We will use the following result. Its proof is postponed until the end of the section.

Lemma 5.4.4. *For $x \in E_1^{2,0}$, there is a unique $y \in E_1^{3,0}$ with $d_1(x) = v_1^3 y$.*

Proof. Let $\tau' = \pi\tau\pi^{-1}$. Recall that $d_1 : E_1^{2,q} \to E_1^{3,q}$ is given by

$$\phi_\pi(id + \phi_i + \phi_j + \phi_k)(e - \phi_\alpha^{-1})\phi_\pi^{-1} = (id + \phi_{i'} + \phi_{j'} + \phi_{k'})(e - \phi_\alpha^{-1}).$$

Further, this factors as the composite

$$E_1^{2,q} \xrightarrow{e-\phi_\alpha^{-1}} E_1^{2,q} \xrightarrow{(id+\phi_{i'}+\phi_{j'}+\phi_{k'})} E_1^{3,q}.$$

Let $x \in E_1^{2,0}$. It follows from Proposition 3.3.1 that there exists $z \in E_1^{2,q}$ such that $(e - \phi_\alpha^{-1})(x) = v_1^3 z$. Then, by v_1-linearity,

$$d_1(x) = v_1^3(id + \phi_{i'} + \phi_{j'} + \phi_{k'})(z),$$

and $y = (id + \phi_{i'} + \phi_{j'} + \phi_{k'})(z)$. This element is uniquely determined since $E_1^{3,0}$ is v_1-torsion free. □

Lemma 5.4.5. *Let x be an element of $E_1^{2,0}$. Consider $d_1 : E_1^{2,q} \to E_1^{3,q}$. Let $q = 4t + s$ for $0 \le s \le 3$. Then $d_1(h^q x) = k^t \eta^s (v_1^{-s} d_1(x))$, where $(v_1^{-s} d_1(x))$ is uniquely determined since $E_1^{3,0}$ is v_1-torsion free.*

Proof. Let y be as in Lemma 5.4.4 so that $d_1(x) = v_1^3 y$. Since α is in the centralizer of C_6 in $\mathbb{S}_2$, it has a trivial action on $H^*(C_6, \mathbb{F}_4) \cong \mathbb{F}_4[h]$. Hence, $e - \phi_\alpha^{-1} : E_1^{2,*} \to E_1^{2,*}$ is h-linear. Let $q = 4t + s$ as in the statement of the result. Recall that $\eta = v_1 h$, that $k = h^4$ and that all maps are v_1, k and η-linear. Therefore,

$$\begin{aligned} d_1(h^q x) &= (id + \phi_{i'} + \phi_{j'} + \phi_{k'})(e - \phi_{\alpha^{-1}})(h^q x) \\ &= k^t (id + \phi_{i'} + \phi_{j'} + \phi_{k'})(h^s (e - \phi_{\alpha^{-1}})(x)) \\ &= k^t (id + \phi_{i'} + \phi_{j'} + \phi_{k'})(\eta^s v_1^{3-s} z) \\ &= k^t \eta^s v_1^{3-s} y \\ &= k^t \eta^s v_1^{-s} d_1(x). \quad \square \end{aligned}$$

This completes the computation of the E_2-term (see Fig. 3).

5.5. *Higher differentials*

In this section, we prove that all differentials $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ for $r \geq 2$ are zero. Because of the sparsity of the spectral sequence, the only differentials d_r for $r \geq 2$ which do not have a zero target are

$$\begin{aligned} d_2 &: E_2^{0,q} \to E_2^{2,q-1}, \ q \geq 2 \\ d_2 &: E_2^{1,q} \to E_2^{3,q-1}, \ q \geq 2 \\ d_3 &: E_3^{0,q} \to E_3^{3,q-2}, \ q \geq 3. \end{aligned}$$

The proof of the following result is a direct computation similar to that of Lemma 4.2.3.

Lemma 5.5.1. *Let v_1 have degree $(s,t) = (0,2)$, v_2 have degree $(0,6)$, and h have degree $(1,0)$. Then*

$$H^*(C_6; (E_{\mathcal{C}})_*) \cong \mathbb{W}[\![u_1^3]\!][v_1^2, v_1 v_2, v_2^{\pm 1}, h]/(2h).$$

As in Lemma 4.1.2, let β be the connecting homomorphism for the long exact sequence in cohomology associated to

$$0 \to (E_{\mathcal{C}})_*/2 \to (E_{\mathcal{C}})_*/4 \to (E_{\mathcal{C}})_*/2 \to 0$$

and note that $(E_{\mathcal{C}})_* V(0) \cong (E_{\mathcal{C}})_*/2$.

Lemma 5.5.2. *All differentials $d_2 : E_2^{1,q} \to E_2^{3,q-1}$ are zero.*

Proof. Let b_n be as in Proposition 5.2.1. The set

$$B = \{h^k b_n \mid n = 0, 1, 2^{s+1}(1+4t),\ 0 \leq k \leq 3,\ 0 \leq s\}$$

generates $E_2^{1,*}$ as an $\mathbb{F}_4[v_1, k]$-module, for k as in Lemma 5.4.1. Because the differentials are $\mathbb{F}_4[v_1, k]$-linear and k acts via multiplication by h^4, it suffices to show that the d_2-differentials are zero on the elements of B. First, note that $d_2(b_n) = 0$ for all n, since the targets of these differentials are zero. Hence, it suffices to show that $d_2(h^k b_n) = 0$ for $1 \leq k \leq 3$.

The first remark is that, if $d_2(h^k b_n) = 0$, then

$$v_1 d_2(h^{k+1} b_n) = d_2(v_1 h^{k+1} b_n) = d_2(\eta h^k b_n) = \eta d_2(h^k b_n) = 0.$$

Hence, if $d_2(h^k b_n) = 0$, then $v_1 d_2(h^{k+1} b_n) = 0$. Further,

$$v_1^k d_2(h^k b_n) = d_2(\eta^k b_n) = \eta^k d_2(b_n).$$

Since $d_2(b_n) = 0$, we must have that $v_1^k d_2(h^k b_n) = 0$ for all $k \geq 0$.

Let $1 \leq k \leq 3$. Then $d_2(h^k b_0)$ is an element of internal degree $t = 0$ in $E_2^{3,k-1}$. Since $d_2(b_0) = 0$, $v_1 d_2(hb_0) = 0$. However, there is no v_1-torsion in $(E_2^{3,0})_0$, hence $d_2(hb_0) = 0$. Further, $(E_2^{3,1})_0$ and $(E_2^{3,2})_0$ are zero and $d_2(h^k b_0) = 0$ for $k = 2, 3$.

Next, consider the elements of the form $h^k b_1$ for $1 \leq k \leq 3$. Since $d_2(h^k b_1)$ is an element of internal degree $t = 6$ in $E_2^{3,k-1}$ and there is no v_1-torsion in $(E_2^{3,k-1})_6$ for $1 \leq k \leq 3$, these differentials must be zero.

The classes $h^k b_{2^{s+1}(1+4t)}$ have internal degree $3 \cdot 2^{s+2}(1+4t)$. Hence, their degree is congruent to zero modulo 3. First, consider the case when $k = 1$. The possible targets for the d_2 differentials on these classes are in $E_2^{3,0}$ and must be annihilated by v_1. Therefore, they must be of the form

$$v_1^{3(1+2^{s'+1})-1} \overline{\Delta}_{2^{s'}(1+2t')}.$$

However, such classes have internal degree congruent to 1 modulo 3, since the degree of $\overline{\Delta}_{2^{s'}(1+2t')}$ is $24 \cdot 2^s(1+2t')$ and the degree of v_1 is 2. Hence, there is no appropriate target for these differentials. Further, this implies that $d_2(h^2 b_n)$ is annihilated by v_1.

The classes which are annihilated by v_1 in $E_2^{3,1}$ are of one of the forms

$$v_1^{3(1+2^{s'+1})-2} \eta \overline{\Delta}_{2^{s'}(1+2t')},$$

$\nu\overline{\Delta}_{2^{s'}(1+2t')}$, $v_1 x \overline{\Delta}_{2^{s'}(1+2t')}$, or $y\overline{\Delta}_{2^{s'}(1+2t')}$. Here, ν has internal degree 4, x has internal degree 8 and y has internal degree 16. Again, such classes have internal degree congruent to 1 modulo 3, so there is no possible target for the differentials. This, in turn, implies that $d_2(h^3 b_n)$ is annihilated by v_1.

The classes in $E_2^{3,2}$ which are annihilated by v_1 are of one of the forms

$$v_1^{3(1+2^{s'+1})-3}\eta^2\overline{\Delta}_{2^{s'}(1+2t')},$$

$\nu^2\overline{\Delta}_{2^{s'}(1+2t')}$, $v_1\eta x\overline{\Delta}_{2^{s'}(1+2t')}$, $\eta y\overline{\Delta}_{2^{s'}(1+2t')}$ or $\nu y\overline{\Delta}_{2^{s'}(1+2t')}$. Of these classes, both $v_1\eta x\overline{\Delta}_{2^{s'}(1+2t')}$ and $\eta y\overline{\Delta}_{2^{s'}(1+2t')}$ have internal degree congruent to 0 modulo 3, so we must make a more careful analysis.

Note that $3\cdot 2^{s+2}(1+4t)\equiv 0 \mod 24$ if $s\geq 1$, and $3\cdot 2^2(1+4t)\equiv 12 \mod 24$. Since the internal degree of $\eta y\overline{\Delta}_{2^{s'}(1+2t')}$ is congruent to 18 modulo 24, it cannot be hit by a differential. The internal degree of $v_1\eta x\overline{\Delta}_{2^{s'}(1+2t')}$ is 12 modulo 24. Therefore, there are possible differentials $d_2(h^3b_{2(1+4t)})$ with targets $v_1\eta x\overline{\Delta}_{2t}$. However, by Lemma 5.5.3 and Lemma 5.5.4 below, $\beta(d_2(h^3b_{2(1+4t)}))=0$ and $\beta(v_1\eta x\overline{\Delta}_{2t})\neq 0$. Therefore, $d_2(h^3b_{2(1+4t)})\neq v_1\eta x\overline{\Delta}_{2t}$ and we must have $d_2(h^3b_{2(1+4t)})=0$. □

Lemma 5.5.3. $\beta(d_2(h^3b_{2(1+4t)}))=0$.

Proof. The maps $H^0(C_6,(E_{\mathcal{C}})_{12(1+4t)})\to H^0(C_6,(E_{\mathcal{C}})_{12(1+4t)}/2)$ are surjective and factor through $H^0(C_6,(E_{\mathcal{C}})_{12(1+4t)}/4)$. Hence, $\beta(b_{2(1+4t)})=0$. Since $\beta(h^{2t+1})=h^{2t+2}$, it follows that

$$\beta(h^3b_{2(1+4t)})=\beta(h^3)b_{2(1+4t)}+h^3\beta(b_{2(1+4t)})=h^4b_{2(1+4t)}.$$

By Lemma 4.1.2,

$$\beta(d_2(h^3b_{2(1+4t)}))=d_2(\beta(h^3b_{2(1+4t)}))=d_2(h^4b_{2(1+4t)})=kd_2(b_{2(1+4t)})=0. \quad \square$$

Lemma 5.5.4. *$\beta(v_1\eta x\overline{\Delta}_{2t})\equiv\nu^3\overline{\Delta}_{2t}$ modulo (v_1^{12}) and, hence, is non-zero.*

Proof. By Remark 5.3.3, $\overline{\Delta}_{2t}\equiv{\Delta'}^{2t}$ modulo (v_1^{12}). Since β is a derivation and we are working in characteristic two, it is zero on squares. Hence,

$$\beta(v_1\eta x\overline{\Delta}_{2t})\equiv\beta(v_1\eta x){\Delta'}^{2t} \mod (v_1^{12}).$$

So it is enough to prove that $\beta(v_1\eta x)=\nu^3$. By definition, $\beta(v_1)=\eta$, so $\beta(\eta)=0$. Further, using the fact that $\eta^2x=\nu^3$,

$$\beta(v_1\eta x)=\beta(v_1\eta)x+v_1\eta\beta(x)=\eta^2x+v_1\eta\beta(x)=\nu^3+v_1\eta\beta(x).$$

However, since $v_1^2x=0$, $\beta(x)$ is v_1-torsion and in $H^*(G'_{24},(E_{\mathcal{C}})_8V(0))$, the v_1-torsion is annihilated by (v_1). Hence, $v_1\eta\beta(x)=0$. □

The next few results will be necessary to prove that all remaining higher differentials are zero. Let $C_2=\{\pm 1\}$ in $\mathbb{S}_2$. For any group $G\subseteq\mathbb{S}_{\mathcal{C}}$ that contains C_2, let $PG=G/C_2$.

For any $\mathbb{Z}_2[\![\mathbb{S}_2]\!]$-module M on which C_2 acts trivially, the action of G descends to an action of PG. Further, there is a homomorphism

$$H^*(PG, M) \to H^*(G, M)$$

which is an isomorphism when $* = 0$.

Note that $PG_{24} \cong A_4$ and $PC_6 \cong C_3$ and that the algebraic duality resolution is a resolution of $P\mathbb{S}^1_{\mathcal{C}}$-modules, where $\mathscr{C}_0 \cong \mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}/A_4]\!]$, $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}/C_3]\!]$ and $\mathscr{C}_3 \cong \mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}/A'_4]\!]$ for $A'_4 = PG'_{24}$. Further, since C_2 acts trivially on $(E_{\mathcal{C}})_*V(0)$, the action of $\mathbb{S}^1_{\mathcal{C}}$ descends to an action of $P\mathbb{S}^1_{\mathcal{C}}$. There is a corresponding ADSS

$$F_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathscr{C}_p, (E_{\mathcal{C}})_*V(0)) \Longrightarrow H^{p+q}(P\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0)), \tag{5.5.1}$$

with $F_1^{p,q} \cong H^q(PF_p, (E_{\mathcal{C}})_*V(0))$ and a map of spectral sequences

$$\varphi : F_r^{p,q} \to E_r^{p,q}$$

induced by the projection from $\mathbb{S}^1_{\mathcal{C}}$ to $P\mathbb{S}^1_{\mathcal{C}}$.

We will relate the computation of some differentials in $E_r^{p,q}$ to the computation of differentials $F_r^{p,q}$. The advantage of this method is that the spectral sequence $F_r^{p,q}$ is sparser than $E_r^{p,q}$. Indeed, $\mathscr{C}_1$ and $\mathscr{C}_2$ are projective $\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]$-modules. Hence, for $p = 1$ or $p = 2$,

$$F_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}/C_3]\!], (E_{\mathcal{C}})_*V(0))$$

is zero when $q > 0$. Hence, $F_r^{p,q} = 0$ when $q \geq 0$ for $p = 1$ or and $p = 2$. Further, the induced maps $F_1^{p,0} \to E_1^{p,0}$ are isomorphisms as noted above. In fact, the complexes $E_1^{*,0} \cong F_1^{*,0}$ are isomorphic. Therefore, the computation of $F_2^{p,q}$ follows immediately from that of $E_2^{p,q}$ and $F_2^{p,0} \cong E_2^{p,0}$.

Let $A_4 = G_{24}/C_2$. Since C_2 acts trivially on $(E_{\mathcal{C}})_*V(0)$, the action of G_{24} descends to an action of A_4.

Lemma 5.5.5. *Let $R^\wedge = \mathbb{F}_4[\![j]\!][v_1, \Delta^{\pm 1}]/(v_1^{12} = j\Delta)$. The inclusion*

$$H^1(A_4, (E_{\mathcal{C}})_*V(0)) \to H^1(G_{24}, (E_{\mathcal{C}})_*V(0))$$

gives an isomorphism of $R^\wedge$-modules

$$H^1(A_4, (E_{\mathcal{C}})_*V(0)) \cong R^\wedge/(v_1^2)\{x\} \oplus R^\wedge/(v_1)\{\nu, y\}.$$

In particular, the module $H^1(A_4, (E_{\mathcal{C}})_*V(0))$ is annihilated by v_1^2.

Proof. Let $S_*(\rho)$ be as in Remark A.2. It suffices to prove that, for $R = \mathbb{F}_4[v_1, \Delta]$,

$$H^1(A_4, S_*(\rho)) \cong R/(v_1^2)\{x\} \oplus R/(v_1)\{\nu, y\}.$$

Consider the spectral sequence for the group extension

$$1 \to C_2 \to G_{24} \to A_4 \to 1.$$

From the associated inflation-restriction exact sequence, using the fact that $S_*(\rho)^{C_2} = S_*(\rho)$, we obtain an exact sequence

$$0 \to H^1(A_4, S_*(\rho)) \to H^1(G_{24}, S_*(\rho)) \to H^1(C_2, S_*(\rho))^{A_4}.$$

From Theorem A.14, for $R = \mathbb{F}_4[v_1, \Delta]$, we have $S_*(\rho)^{G_{24}} \cong S_*(\rho)^{A_4} \cong R$ and

$$H^1(G_{24}, S_*(\rho)) \cong R\{\eta\} \oplus R/(v_1^2)\{x\} \oplus R/(v_1)\{\nu, y\}.$$

Further, $H^1(C_2, S_*(\rho))^{A_4} \cong R\{h\}$ for h of internal degree 0. Since $R\{h\}$ is v_1-free, $R/(v_1^2)\{x\} \oplus R/(v_1)\{\nu, y\}$ maps to zero and the map $H^1(G_{24}, S_*(\rho)) \to H^1(C_2, S_*(\rho))^{A_4}$ sends η to $v_1 h$. □

Proposition 5.5.6. *The map*

$$\phi : H^*(A_4, (E_{\mathcal{C}})_*V(0)) \to H^*(G_{24}, (E_{\mathcal{C}})_*V(0))/(\eta)$$

induced by the projection $G_{24} \to G_{24}/C_2 \cong A_4$ is an isomorphism in degree $* = 0$ and is surjective in degrees $* \leq 3$.

Proof. It suffices to prove that ϕ is surjective if we replace $(E_{\mathcal{C}})_*V(0)$ by $S_*(\rho)$. It follows from Theorem A.14 that $H^*(G_{24}, S_*(\rho))/(\eta)$ is generated by the fixed points $H^0(G_{24}, S_*(\rho))$ and the elements ν, x and y in degrees $0 \leq * \leq 3$. Since C_2 has a trivial action, $S_*(\rho)^{A_4} \cong S_*(\rho)^{G_{24}}$. Hence, ϕ is an isomorphism in degree zero. By Lemma 5.5.5, ν, x and y are in the image of ϕ. The result follows from the fact that ϕ is a ring homomorphism. □

Remark 5.5.7. It follows from Proposition 5.5.6 that the map

$$\varphi : F_1^{0,*} \to E_1^{0,*}/(\eta)$$

induced by the projection $G_{24} \to G_{24}/C_2 \cong A_4$ is surjective in degrees $* \leq 3$. All classes of degree $q \geq 4$ in $E_r^{0,q}$ are multiples of k, so their differentials will be determined by differentials on classes of degree $q \leq 3$. Further, by η-linearity it suffices to show that the differentials on the classes in the image of φ of Proposition 5.5.6 are zero. It is therefore sufficient to compute some of the differentials $d_r : F_r^{0,q} \to F_r^{r,q-r+1}$ for $q \leq 3$.

The following results are generalizations of results that can be found in Henn, Karamanov and Mahowald [7, Section 6]. The first is [7, Lemma 6.1].

Lemma 5.5.8 *(Henn–Karamanov–Mahowald). Let R be a $\mathbb{Z}_2$-algebra and M be an R-module. Let*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_3} \mathscr{C}_1 \xrightarrow{\partial_2} \mathscr{C}_0 \xrightarrow{\partial_1} \mathbb{Z}_2 \to 0$$

be an exact sequence of R-modules such that $\mathscr{C}_1$ and $\mathscr{C}_2$ are projective. Define N_i recursively by $0 \to N_i \to \mathscr{C}_i \xrightarrow{\partial_i} N_{i-1} \to 0$, and let $F_r^{s,t}$ be the first quadrant spectral sequence of the exact couple

$$\begin{array}{ccc} \mathrm{Ext}_R(N_i, M) & \dashrightarrow & \mathrm{Ext}_R(N_{i-1}, M) \\ \nwarrow & & \swarrow \\ & \mathrm{Ext}_R(\mathscr{C}_i, M) & \end{array}$$

Then $E_1^{p,q} = 0$ for $0 < p < 3$ and $q > 0$. Further, there are isomorphisms

$$\mathrm{Ext}_R^q(N_0, M) \cong \begin{cases} \ker(F_1^{1,0} \to F_1^{2,0}) & q = 0 \\ F_2^{q+1,0} \cong F_3^{q+1,0} & q = 1, 2 \\ F_3^{q-2,0} & q \geq 3. \end{cases}$$

Let $j : N_0 \to \mathscr{C}_0$ be the inclusion. The only possible non-zero higher differentials are of the form $d_r : F_r^{0,q} \to F_r^{r,q-r+1}$, and they can be identified with the map $\mathrm{Ext}_R^q(\mathscr{C}_0, M) \to \mathrm{Ext}_R^q(N_0, M)$ induced by j.

Note that the algebraic duality resolution viewed as a resolution of the trivial $\mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1]\!]$-module $\mathbb{Z}_2$ and the associated spectral sequence (5.5.1) satisfy the conditions of Lemma 5.5.8.

Let $P_{-1} = \mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1/A_4]\!]$ and $P_0 = \mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1/C_3]\!]$. Let $P_0 \xrightarrow{\varepsilon} P_{-1}$ be the natural augmentation which sends the coset $[C_3]$ to $[A_4]$. Complete this to a projective resolution P_* of $\mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1/A_4]\!]$. Let P'_* be any projective resolution of $\mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1/A'_4]\!]$. Let N_0 be defined by the exact sequence

$$0 \to N_0 \to \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0.$$

Letting $Q_{-1} = N_0$, $Q_q = \mathscr{C}_{q+1} = \mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1/C_3]\!]$ for $q = 0, 1$, the complex

$$Q_2 \to Q_1 \to Q_0 \to Q_{-1}, \tag{5.5.2}$$

with maps as in the algebraic duality resolution, is the beginning of a projective resolution of N_0. The kernel of $Q_2 \to Q_1$ is isomorphic to $\mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1/A'_4]\!]$. Splicing (5.5.2) with $Q_* = P'_{*-3}$ gives a projective resolution Q_* of N_0.

Lemma 5.5.9. *There is a map $\phi : Q_* \to P_*$ such that*

$$\phi_0 : Q_0 \to P_0$$

covers the map $j : N_0 \to \mathscr{C}_0 = P_{-1}$ which sends $e_1 \to (e - \alpha)e_0$.

Proof. Note that $Q_0 \cong P_0 \cong \mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}/C_3]\!]$. So the map which sends the generator of $e \otimes 1 \in Q_0$ to $(e - \alpha) \otimes 1 \in P_0$ is well defined and covers j. Hence, it extends to a chain map ϕ. □

The following is an observation in Henn, Karamanov and Mahowald [7, Section 6]. It follows from Lemma 5.5.8.

Lemma 5.5.10. *Let $T_{*,*}$ be the double complex satisfying $T_{*,0} = P_*$ and $T_{*,1} = Q_*$ with vertical differentials δ_P and δ_Q and horizontal differentials $\phi_s : Q_s \to P_s$. Up to reindexing, the filtration of the spectral sequence of this double complex agrees with that of the ADSS.*

The following result is an adaptation of part of [7, Lemma 6.5].

Lemma 5.5.11. *Let $s > 0$. Let $z \in H^s(A_4, (E_{\mathcal{C}})_*V(0))$ be such that $v_1^r z = 0$. Let $c \in \mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(P_s, (E_{\mathcal{C}})_*V(0))$ be a cocycle which represents z. Choose an element h in $\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(P_{s-1}, (E_{\mathcal{C}})_*V(0))$ such that $\delta_P(h) = v_1^r c$. Let*

$$\phi^* : \mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(P_*,\ (E_{\mathcal{C}})_*V(0)) \to \mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(Q_*, (E_{\mathcal{C}})_*V(0))$$

be induced by ϕ. There are elements d and d' in $\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(Q_{s-1}, (E_{\mathcal{C}})_*V(0))$ and an element d'' in $\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(Q_s, (E_{\mathcal{C}})_*V(0))$ such that

$$\phi^*_{s-1}(h) = d' + v_1^r d \tag{5.5.3}$$

and $\delta_Q(d') = v_1^r d''$. For d'' as above and j as in Lemma 5.5.8,

$$j^*(z) = [d''] \in \mathrm{Ext}^s_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(N_0, (E_{\mathcal{C}})_*V(0)).$$

Proof. Note that $\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}^1_{\mathcal{C}}]\!]}(T_{*,*}, (E_{\mathcal{C}})_*V(0))$ for $T_{*,*}$ as in Lemma 5.5.10 is a double complex of $\mathbb{F}_4[v_1]$-modules which have no (v_1)-torsion. We can write

$$\phi^*_{s-1}(h) = d' + v_1^r d.$$

To prove (5.5.3), note that by v_1-linearity,

$$\delta_Q(d') + v_1^r \delta_Q(d) = \delta_Q(\phi^*_{s-1}(h)) = \phi^*_s(\delta_P(h)) = v_1^r \phi^*_s(c).$$

Hence, $\delta_Q(d') \equiv 0$ modulo (v_1^r), that is, $\delta_Q(d') = v_1^r d''$ for some d''.

Now, note that

$$v_1^r j^*(c) = \phi_s^*(v_1^r c) = \phi_s^*(\delta_P(h)) = \delta_Q(\phi_{s-1}^*(h)) = v_1^r d'' + v_1^r \delta_Q(d).$$

Since there is no v_1-torsion in $\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1]\!]}(T_{*,*}, (E_{\mathcal{C}})_*V(0))$, we must have $j^*(c) = d'' + \delta_Q(d)$. This reduces to

$$j^*(z) = [d''] \in \mathrm{Ext}^s_{\mathbb{Z}_2[\![P\mathbb{S}_{\mathcal{C}}^1]\!]}(N_0, (E_{\mathcal{C}})_*V(0)). \quad \square$$

Lemma 5.5.12. *Let z be in $F_2^{0,q}$. Then $d_2(z) = 0$.*

Proof. If $q > 1$, then $d_2(z) = 0$ since the target of the differential is zero. Suppose that $q = 1$. Then z is v_1-torsion. Let r be the smallest integer such that $v_1^r z = 0$. By Lemma 5.5.5 of Appendix A, we can choose $r = 1$ or $r = 2$. Choose h as in Lemma 5.5.11 and write

$$\phi_0(h) = (e - \phi_\alpha)(h) = d' + v_1^r d.$$

However, $\phi_\alpha \equiv id$ modulo $(2, u_1^3)$. So we must have $d' = 0$. By Lemma 5.5.8 and Lemma 5.5.11, this implies that $d_2(z) = 0$ in the ADSS for $P\mathbb{S}_{\mathcal{C}}^1$. $\square$

Corollary 5.5.13. *All differentials $d_2 : E_2^{0,q} \to E_2^{2,q-1}$ are zero.*

Proof. This follows from Remark 5.5.7 and Lemma 5.5.12. $\square$

Lemma 5.5.14. *All differentials $d_3 : E_3^{0,q} \to E_3^{3,q-1}$ are zero.*

Proof. Differentials $d_3 : E_3^{0,q} \to E_3^{3,q-2}$ are zero for degree reasons if $0 \le q < 2$. By Corollary 5.5.13, the classes $\nu\Delta^s$ survive to the E_3-term, and hence they must be permanent cycles. Thus, they represent cohomology classes in $H^*(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_*V(0))$. By Lemma 4.1.3, the differentials are $\nu\Delta^s$-linear for all $s \in \mathbb{Z}$. Using this fact and linearity with respect to η and v_1, the problem reduces to verifying the claim for $x^2\Delta^s$. However, by the same argument, x is a permanent cycle and $d_3(x^2\Delta^s) = xd_3(x\Delta^s) = 0$. $\square$

Lemma 5.5.15. *All differentials $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ are zero.*

Proof. By Lemma 5.5.12 and Lemma 5.5.14, $E_2^{*,*} \cong E_4^{*,*}$ and the spectral sequence collapses at E_4 since the targets for higher differentials are zero. $\square$

6. The action of the Morava stabilizer group

The goal of this section is to approximate the action of elements of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$. Some of our results are stronger than needed for the computations of this paper, but the better estimates are necessary for future computations. Note that the results of Section 3.3 rely on this section.

6.1. The formal group laws

Let $\mathcal{E}$ be an elliptic curve with Weierstrass equation

$$\mathcal{E} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let $F_{\mathcal{E}}(z_1, z_2)$ be the formal group law of $\mathcal{E}$, where the coordinates (z, w) at the origin are chosen so that

$$w(z) = z^3 + a_1zw(z) + a_2z^2w(z) + a_3w(z)^2 + a_4zw(z)^2 + a_6w(z)^3. \tag{6.1.1}$$

That the group $\mathbb{S}_{\mathcal{C}}$ acts on $(E_{\mathcal{C}})_*$ is a consequence of the fact that the formal group law $F_{E_{\mathcal{C}}}$ of $E_{\mathcal{C}}$ is a universal deformation of the formal group law $F_{\mathcal{C}}$ of the elliptic curve

$$\mathcal{C} : y^2 + y = x^3$$

defined over any field extension of $\mathbb{F}_2$. Further, $F_{E_{\mathcal{C}}}$ is the formal group law of an elliptic curve, namely

$$\mathcal{C}_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

defined over $(E_{\mathcal{C}})_0$. That is, $F_{E_{\mathcal{C}}} = F_{\mathcal{C}_U}$.

We start by gathering information about $F_{\mathcal{C}_U}$. We will also compute information about the formal group law of the curve

$$\mathcal{C}_{\mathbb{W}} : y^2 - y = x^3$$

defined over $\mathbb{W}$. The curve $\mathcal{C}_{\mathbb{W}}$ is a lift of $\mathcal{C}$ to $\mathbb{W}$, and $\mathcal{C}_U$ reduces to $\mathcal{C}_{\mathbb{W}}$ modulo (u_1). We will derive information about $F_{\mathcal{C}_U}$ from knowledge of $F_{\mathcal{C}_{\mathbb{W}}}$.

The following results are proved using the methods described in Silverman [14, Section 4]. We recall the key tools here. We restrict to elliptic curves $\mathcal{E}$ with homogeneous Weierstrass equation of the form

$$\mathcal{E} : y^2z + a_1xyz + a_3yz^2 = x^3.$$

Let $z = -\frac{x}{y}$ and $w = -\frac{z}{y}$, so that $(z, w(z))$ is a coordinate chart of $\mathcal{E}$ at the origin, with

$$w(z) = z^3 + a_1zw(z) + a_3w(z)^2. \tag{6.1.2}$$

This can be used to write $w(z)$ as a power series in z. Letting

$$\lambda(z_1, z_2) = \frac{w(z_2) - w(z_1)}{z_2 - z_1},$$

the line through the points $(z_1, w(z_1))$ and $(z_2, w(z_2))$ has equation

$$w(z) = \lambda(z_1, z_2)z + w(z_1) - \lambda(z_1, z_2)z_1. \tag{6.1.3}$$

(Note that there is a sign mistake in Silverman [14, Section 4.1] in the equation of the connecting line.) Substituting (6.1.3) in (6.1.1), we obtain a monic cubic polynomial whose roots are z_1, z_2 and $[-1]_{F_\mathcal{E}}(F(z_1, z_2))$. The coefficient of z^2 is $a_1\lambda(z_1, z_2) + a_3\lambda(z_1, z_2)^2$. This implies that

$$[-1]_{F_\mathcal{E}}(F(z_1, z_2)) = -z_1 - z_2 - a_1\lambda(z_1, z_2) - a_3\lambda(z_1, z_2)^2. \tag{6.1.4}$$

Noting that

$$\lambda(z, z) = \lim_{s \to z} \frac{w(s) - w(z)}{s - z} = w'(z),$$

it follows that

$$[-2]_{F_\mathcal{E}}(z) = -2z - a_1 w'(z) - a_3 w'(z)^2.$$

Finally, the series $[-1]_{F_\mathcal{E}}(z)$, which is $[-1]_{F_\mathcal{E}}(F(z, 0))$, is given by

$$[-1]_{F_\mathcal{E}}(z) = -z - a_1\frac{w(z)}{z} - a_3\frac{w^2(z)}{z^2}$$

so that $F_\mathcal{E}$ can be computed as $[-1]_{F_\mathcal{E}}([-1]_{F_\mathcal{E}}(F(z_1, z_2)))$. For example,

$$F_\mathcal{E}(x, y) \equiv x + y - a_1xy - 2a_3xy(x^2 + y^2) - 3a_3x^2y^2 \mod (x, y)^5. \tag{6.1.5}$$

The following two results give formulas for the formal group law of the curve $\mathcal{C}_U$ and of its $[-2]$-series, both integrally and modulo 2. Corollary 6.1.2 was observed computationally by the author, but was proved by Henn.

Proposition 6.1.1. *Modulo* $(x, y)^5$,

$$F_{\mathcal{C}_U}(x, y) \equiv x + y - 3u_1xy - 2(u_1^3 - 1)xy(x^2 + y^2) - 3(u_1^3 - 1)x^2y^2.$$

The formal group law $F_{\mathcal{C}_U}$ *has* $[-2]$*-series*

$$[-2]_{F_{\mathcal{C}_U}}(z) = -2z - 9z\frac{zu_1 - 2z^2u_1^2 + z^3(u_1^3 - 1)}{1 - 6zu_1 + 9z^2u_1^2 - 4z^3(u_1^3 - 1)},$$

so that

$$[-2]_{F_{\mathcal{C}_U}}(z) = -2z - 9u_1z^2 - 36u_1^2z^3 + 9z^4 - 144u_1^3z^4 + O(z^5).$$

Proof. The first claim follows directly from (6.1.5). For the curve $\mathcal{C}_U$, we have

$$w'(z) = \frac{3(z^2 + u_1 w(z))}{1 - 3u_1 z - 2(u_1^3 - 1)w(z)}.$$

Combining

$$[-2]_{F_{\mathcal{C}_U}}(z) = -2z - 3u_1 w'(z) - (u_1^3 - 1)w'(z)^2$$

and

$$(u_1^3 - 1)w(z)^2 = w(z) - z^3 - 3u_1 z w(z),$$

gives the result for $[-2]_{F_{\mathcal{C}_U}}(z)$. Its Taylor expansion is the last estimate. □

Corollary 6.1.2.

$$[-2]_{F_{\mathcal{C}_U}}(x) \equiv u_1 x^2 + \sum_{k \geq 0} u_1^{2k} x^{2k+4} \mod (2).$$

Proof. It follows from Proposition 6.1.1 that modulo 2,

$$[-2]_{F_{\mathcal{C}_\mathcal{U}}}(z) \equiv \frac{u_1 z^2 + u_1^3 z^4 + z^4}{1 + u_1^2 z^2}.$$

Therefore, modulo 2,

$$\begin{aligned}[-2]_{F_{\mathcal{C}_\mathcal{U}}}(z) &\equiv \left(u_1 z^2 + u_1^3 z^4 + z^4\right) \sum_{k \geq 0} u_1^{2k} z^{2k} \\ &\equiv u_1 z^2 + \sum_{k \geq 0} u_1^{2k} z^{2k+4}. \quad \square\end{aligned}$$

Some of the key ingredients for the proof of the next result were brought to the author's attention by Inna Zakharevich. Let $C_k = \frac{(2k)!}{k!(k+1)!}$ be the k'th Catalan number. Let

$$C(y) = \sum_{k \geq 0} C_k y^k = \frac{1 - \sqrt{1 - 4y}}{2y} \tag{6.1.6}$$

be their generating series (see, for example, Wilf [15, (2.3.9)]). Let $D(y) = yC(y)$, so that

$$D(y) = \frac{1 - \sqrt{1 - 4y}}{2}.$$

Proposition 6.1.3. *Let $\mathcal{C}_{\mathbb{W}}$ be the elliptic curve defined over $\mathbb{W}$ by the Weierstrass equation $\mathcal{C}_{\mathbb{W}} : y^2 - y = x^3$. Then*

$$[-2]_{\mathcal{C}_{\mathbb{W}}}(z) = -2z + 9z^4 \sum_{n\geq 0} (-1)^n 4^n z^{3n}.$$

For $(z, w(z))$ a coordinate chart at the origin with $w(z) = z^3 - w(z)^2$,

$$w(z) = -D((-z)^3) = \sum_{n\geq 0} (-1)^n C_n z^{3(n+1)} = \frac{\sqrt{1+4z^3} - 1}{2}.$$

Further,

$$[-1]_{\mathcal{C}_{\mathbb{W}}} F_{\mathcal{C}_{\mathbb{W}}}(z_1, z_2) = -z_1 - z_2 + \frac{(z_1^3 + z_2^3) + D(-(z_1^3 + z_2^3 + 4z_1^3 z_2^3))}{(z_2 - z_1)^2}.$$

Proof. It follows from Proposition 6.1.1 that, modulo u_1,

$$[-2]_{\mathcal{C}_{\mathbb{W}}}(z) = -2z + 9z^4 \frac{1}{1+4z^3}.$$

This proves the first claim. The second claim is equivalent to showing that $w(z) = z^3 C((-z)^3)$. It follows from (6.1.6) that $C(z) = 1 + zC(z)^2$. Therefore,

$$C((-z)^3) = 1 + (-z)^3 C((-z)^3)^2,$$

so that

$$z^3 C((-z)^3) = z^3 - (z^3 C((-z)^3))^2.$$

It also follows from (6.1.2) that, for the curve $\mathcal{C}_{\mathbb{W}}$, $w(z) = z^3 - w(z)^2$. Since $w(z)$ and $z^3 C((-z)^3)$ satisfy the same functional equation, they must be equal. Further, this implies that

$$w(z) = \frac{\sqrt{1+4z^3} - 1}{2}.$$

Finally, note that

$$\begin{aligned}\lambda(z_1, z_2) &= \frac{1}{z_2 - z_1} \left(\frac{\sqrt{1+4z_2^3} - 1}{2} - \frac{\sqrt{1+4z_1^3} - 1}{2} \right) \\ &= \frac{\sqrt{1+4z_2^3} - \sqrt{1+4z_1^3}}{2(z_2 - z_1)}.\end{aligned}$$

Using (6.1.4), it follows that

$$[-1]_{F_{\mathcal{C}_{\mathbb{W}}}}(F_{\mathcal{C}_{\mathbb{W}}}(z_1,z_2)) = -z_1 - z_2 + \lambda(z_1,z_2)^2$$
$$= -z_1 - z_2 + \frac{(z_1^3+z_2^3) + D(-(z_1^3+z_2^3+4z_1^3z_2^3))}{(z_2-z_1)^2}. \quad \square$$

Proposition 6.1.4. *Let $\mathcal{C}$ be defined over $\mathbb{F}_4$ by the Weierstrass equation $\mathcal{C} : y^2 + y = x^3$. The local uniformizer at the origin $w(z) = z^3 + w(z)^2$, satisfies $w(z) = \sum_{k\geq 0} z^{3\cdot 2^k}$. Further, $[-2]_{F_{\mathcal{C}}}(z) = z^4$ and*

$$[-1]_{F_{\mathcal{C}}}(F_{\mathcal{C}}(z_1,z_2)) = z_1 + z_2 + \sum_{k\geq 1} \sum_{n=0}^{3\cdot 2^{k-1}-1} (z_1^{2(3\cdot 2^{k-1}-1-n)} z_2^{2n}).$$

Finally, $[-1]_{F_{\mathcal{C}}}(z) = \sum_{k\geq 0} z^{3\cdot 2^k - 2}$, so that

$$F_{\mathcal{C}}(z_1,z_2) = z_1 + z_2 + z_1^2z_2^2 + z_1^6z_2^4 + z_1^4z_2^6 + z_1^8z_2^8 + z_1^{12}z_2^4 + z_1^4z_2^{12} + \dots$$

where the next term has order 22.

Proof. One can compute directly that $w(z) = \sum_{k\geq 0} z^{3\cdot 2^k}$. This implies that $C_n \neq 0$ modulo 2 if and only if $n+1 = 2^k$. Therefore, we have the following identity of power series

$$D(y) = \sum_{n\geq 0} C_n y^{n+1} = \sum_{k\geq 0} y^{2^k}.$$

Hence, using Proposition 6.1.3 modulo (2), we obtain

$$[-1]_{F_{\mathcal{C}}}(F_{\mathcal{C}}(z_1,z_2)) = z_1 + z_2 + \frac{(z_1^3+z_2^3) + D(z_1^3+z_2^3)}{z_2^2+z_1^2}$$
$$= z_1 + z_2 + \frac{1}{z_2^2+z_1^2}\left(\sum_{k\geq 1} (z_1^2)^{3\cdot 2^{k-1}} + (z_2^2)^{3\cdot 2^{k-1}}\right)$$
$$= z_1 + z_2 + \sum_{k\geq 1} \sum_{n=0}^{3\cdot 2^{k-1}-1} (z_1^{2(3\cdot 2^{k-1}-1-n)} z_2^{2n}).$$

This gives the result for $[-1]_{F_{\mathcal{C}}}(F_{\mathcal{C}}(z_1,z_2))$. Letting $z_1 = z$ and $z_2 = 0$ gives it for $[-1]_{F_{\mathcal{C}}}(z)$. A direct computation gives the estimate for $F_{\mathcal{C}}(z_1,z_2)$. $\square$

6.2. The technique for computing the action of $\mathbb{S}_{\mathcal{C}}$

The method presented here is an adaptation of the techniques used by Henn, Karamanov and Mahowald in [7]. Let γ be in $\mathbb{S}_{\mathcal{C}}$. Then $\gamma \in \mathbb{F}_4[\![x]\!]$ is a power series which satisfies

$$\gamma(F_{\mathcal{C}}(x, y)) = F_{\mathcal{C}}(\gamma(x), \gamma(y)).$$

Recall from Section 2.4 that γ gives rise to isomorphisms $\phi_\gamma : (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_*$ and $h_\gamma : \phi_\gamma^* F_{E_{\mathcal{C}}} \to F_{E_{\mathcal{C}}}$, where $h_\gamma \in (E_{\mathcal{C}})_0[\![x]\!]$. The action of γ on $(E_{\mathcal{C}})_*$ is given precisely by ϕ_γ.

The isomorphism ϕ_γ is linear over $\mathbb{W}$; hence it is sufficient to specify $\phi_\gamma(u)$ and $\phi_\gamma(u_1)$. The morphism h_γ is a power series

$$h_\gamma(x) = t_0(\gamma)x + t_1(\gamma)x^2 + t_2(\gamma)x^3 + \dots$$

where

$$t_i(\gamma) : \mathbb{S}_{\mathcal{C}} \to (E_{\mathcal{C}})_0 = \mathbb{W}[\![u_1]\!]$$

are continuous maps. By (2.4.1) $\phi_\gamma(u) = h'_\gamma(0)u = t_0(\gamma)u$, which gives the action of γ on u.

The morphism h_γ must satisfy

$$h_\gamma([-2]_{\phi_\gamma^* F_{E_{\mathcal{C}}}}(x)) = [-2]_{F_{E_{\mathcal{C}}}}(h_\gamma(x)). \tag{6.2.1}$$

This imposes a set of relations on the parameters $t_i(\gamma)$ and $\phi_\gamma(u_1)$. Further, h_γ is a lift of γ, so that $h_\gamma \equiv \gamma$ modulo $(2, u_1)$. This specifies the parameters $t_i(\gamma)$ modulo $(2, u_1)$. With this information, the relations imposed by (6.2.1) are sufficient to approximate ϕ_γ. Before executing this program, we prove a preliminary result.

Proposition 6.2.1. *If $\gamma \in \mathbb{Z}_2^\times \cap \mathbb{S}_{\mathcal{C}}$, so that $\gamma = \sum_{i\geq 0} a_i T^{2i}$, for $a_i \in \{0, 1\}$. Let $\ell = \sum_{i\geq 0} a_i(-2)^i$ in $\mathbb{Z}_2^\times \subseteq (E_{\mathcal{C}})_0$. Then $\phi_\gamma(u_1) = u_1$ and $\phi_\gamma(u) = \ell u$.*

Proof. The element γ is given by

$$\gamma(x) = a_0 x +_{F_{\mathcal{C}}} a_1[-2]_{F_{\mathcal{C}}}(x) +_{F_{\mathcal{C}}} a_2[4]_{F_{\mathcal{C}}}(x) +_{F_{\mathcal{C}}} \dots$$

Let g be the lift for γ given by

$$g(x) = a_0 x +_{F_{E_{\mathcal{C}}}} a_1[-2]_{F_{E_{\mathcal{C}}}}(x) +_{F_{E_{\mathcal{C}}}} a_2[4]_{F_{E_{\mathcal{C}}}}(x) +_{F_{E_{\mathcal{C}}}} \dots$$

Then g is an automorphism of $F_{E_{\mathcal{C}}}$, hence $\phi_\gamma : (E_{\mathcal{C}})_0 \to (E_{\mathcal{C}})_0$ is the identity and $h_\gamma(x) = g(x)$. Since $[n]_{F_{E_{\mathcal{C}}}}(x) \equiv nx$ modulo (x^2) for $n \in \mathbb{Z}$, $g(x) \equiv \ell x$ modulo (x^2). Therefore, $g'(0) = \ell$ and $\phi_\gamma(u) = \ell u$. □

Theorem 6.2.2. *Let $\gamma \in \mathbb{S}_{\mathcal{C}}$ and $t_i = t_i(\gamma)$. Then*

$$\phi_\gamma(u_1) = u_1 t_0 + \frac{2}{3}\frac{t_1}{t_0}.$$

In particular, $\phi_\gamma(u_1) \equiv t_0 u_1$ and $\phi_\gamma(u) \equiv t_0 u$ modulo (2) and $v_1 = u_1 u^{-1}$ is fixed by the action of $\mathbb{S}_{\mathcal{C}}$ modulo (2).

Proof. Recall from (6.2.1) that

$$h_\gamma([-2]_{\phi_\gamma^* F_{E_{\mathcal{C}}}}(x)) = [-2]_{F_{E_{\mathcal{C}}}}(h_\gamma(x)).$$

Using Proposition 6.1.1, one obtains the following relation on the coefficients of x^2,

$$-9\phi_\gamma(u_1)t_0 + 4t_1 = -9u_1 t_0^2 - 2t_1.$$

Because ϕ_γ is an isomorphism, t_0 is invertible. Isolating $\phi_\gamma(u_1)$ and dividing both sides by $-9t_0$ proves the claim. □

Therefore, to approximate the action of an element γ in $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$, it suffices to approximate the parameters $t_0(\gamma)$ and $t_1(\gamma)$.

6.3. Approximations for the parameters $t_i(\gamma)$

In this section, we use the technique described in Section 6.2 to approximate the parameters $t_i(\gamma)$.

Corollary 6.3.1. *Modulo $(2, u_1^6)$,*

$$\begin{aligned} t_s &\equiv t_s^4 + u_1 t_{2s+1}^2 + \binom{s+2}{2} t_0^2 t_{s+1} u_1^2 + \sum_{i=0}^{s-1} u_1^2 t_i^4 t_{2s-1-2i}^2 \\ &+ \left(\binom{s}{1} t_0^4 t_{s-1} + \binom{s}{2} t_0^4 t_{s-1} + \binom{s+3}{4} t_0^4 t_{s+2} + \binom{s+2}{1} t_{\frac{s-1}{2}}^8 \right) u_1^4. \end{aligned}$$

Proof. Let $h_\gamma(x) = \sum_{i=0}^\infty t_i x^{i+1}$. Using Corollary 6.1.2, we obtain

$$\begin{aligned} h_\gamma([-2]_{\phi_\gamma^* F_{E_{\mathcal{C}}}}(x)) &= \sum_{i=0}^\infty t_i \left(t_0 u_1 x^2 + x^4 + \sum_{i=1}^\infty (t_0 u_1)^{2i} x^{4+2i} \right)^{i+1} \\ &\equiv \sum_{i=0}^\infty t_i \left(t_0 u_1 x^2 + x^4 + t_0^2 u_1^2 x^6 + t_0^4 u_1^4 x^8 \right)^{i+1} \\ &\equiv \sum_{i=0}^\infty t_i \left(x^{4(i+1)} + \binom{i+1}{1} (t_0 u_1 x^{4i+2} + t_0^2 u_1^2 x^{4i+6} + t_0^4 u_1^4 x^{4i+8}) \right. \end{aligned}$$

$$+\binom{i+1}{2}(t_0^2u_1^2x^{4i}+t_0^4u_1^4x^{4i+8})$$
$$+\binom{i+1}{3}(t_0^3u_1^3x^{4i-2}+t_0^4u_1^4x^{4i+2}+t_0^5u_1^5x^{4i+6})$$
$$+\binom{i+1}{4}t_0^4u_1^4x^{4i-4}+\binom{i+1}{5}t_0^5u_1^5x^{4i-6}\Bigg).$$

Further,

$$[-2]_{F_{E_{\mathcal{C}}}}(h_\gamma(x)) = u_1\left(\sum_{i=0}^{\infty}t_ix^{i+1}\right)^2+\left(\sum_{i=0}^{\infty}t_ix^{i+1}\right)^4+\sum_{k=1}^{\infty}u_1^{2k}\left(\sum_{i=0}^{\infty}t_ix^{i+1}\right)^{2k+4}$$
$$\equiv\sum_{i=0}^{\infty}\left(u_1t_i^2x^{2(i+1)}+t_i^4x^{4(i+1)}+u_1^4t_i^8x^{8(i+1)}\right)+u_1^2\left(\sum_{i=0}^{\infty}t_i^2x^{2(i+1)}\right)^3$$

Next, note that $\left(\sum_{i\geq 0}a_ix^i\right)^3\equiv\sum_{k\geq 0}\sum_{2i+j=k}a_i^2a_jx^k$. Therefore,

$$u_1^2\left(\sum_{i=0}^{\infty}t_i^2x^{2(i+1)}\right)^3\equiv\sum_{k\geq 0}\sum_{2i+j=k}u_1^2t_i^4t_j^2x^{2k+6}.$$

Now, using (6.2.1), the coefficient of $x^{4(s+1)}$ gives the relation

$$t_s\equiv t_s^4+u_1t_{2s+1}^2+\binom{s+2}{2}t_0^2t_{s+1}u_1^2+\sum_{2i+j=2s-1}u_1^2t_i^4t_j^2$$
$$+\left(\binom{s}{1}+\binom{s}{2}\right)t_0^4t_{s-1}u_1^4+\binom{s+3}{4}t_0^4t_{s+2}u_1^4+\binom{s+2}{1}t^8_{\frac{s-1}{2}}u_1^4$$

(Note that the coefficient of the last term is chosen to be zero when s is even, so that when $t_{\frac{s-1}{2}}$ has a non-zero coefficient, $(s-1)/2$ is an integer.) □

Proposition 6.3.2. *For $t_i = t_i(\gamma)$ where $\gamma\in\mathbb{S}_{\mathcal{C}}$, then*

$$t_i\equiv t_i^4+u_1t_{2i+1}^2+2t_{4i+3}+2\sum_{\substack{r+s=2i\\0\leq r<s}}t_r^2t_s^2\mod (2,u_1)^2.$$

Proof. Modulo $(4,u_1)$, we have $[-2]_{F_{C_U}}(x)\equiv 2x+x^4$. This gives

$$h_\gamma([-2]_{\phi_\gamma^*F_{E_{\mathcal{C}}}}(x))\equiv\sum_{i=0}^{\infty}t_i\left(x^{4(i+1)}+2\binom{i+1}{1}x^{4i+1}\right)$$

and

$$[-2]_{F_{E_{\mathcal{C}}}}(h_\gamma(x)) \equiv \sum_{i=0}^{\infty} 2t_i x^{i+1} + \left(\sum_{i=0} t_i x^{i+1}\right)^4$$
$$\equiv \sum_{i=0}^{\infty} 2t_i x^{i+1} + \sum_{i=0} t_i^4 x^{4(i+1)} + \sum_{i=1}^{\infty} x^{4+2i} 2 \sum_{\substack{r+s=i \\ 0\leq r<s}} t_r^2 t_s^2.$$

Using (6.2.1), the coefficient of $x^{4(i+1)}$ gives the relation

$$t_i = 2t_{4i+3} + t_i^4 + 2 \sum_{\substack{r+s=2i \\ 0\leq r<s}} t_r^2 t_s^2 \mod (4, u_1).$$

The claim then follows from Corollary 6.3.1. □

Proposition 6.3.3. *Modulo* (4)

$$t_0 \equiv t_0^4 + 2t_3 + 3t_1^2 u_1 + 2t_0 t_2 u_1 + 3t_0^2 t_1 u_1^2.$$

Modulo (2)*,*

$$t_1 \equiv t_1^4 + t_3^2 u_1 + t_0^4 t_1^2 u_1^2 + t_0^2 t_2 u_1^2 + t_0^5 u_1^4 + t_0^8 u_1^4 + t_0^4 t_3 u_1^4.$$

Proof. Modulo (4), the coefficient of x^4 in $h_\gamma([-2]_{\phi_\gamma^* F_{E_{\mathcal{C}}}}(x))$ is $t_0 + \phi_\gamma(u_1)^2 t_1$ and the coefficient of $[-2]_{F_{E_{\mathcal{C}}}}(h_\gamma(x))$ is given by

$$t_0^4 + 2t_3 + 3t_1^2 u_1 + 2t_0 t_2 u_1$$

Recall from Theorem 6.2.2 that $\phi_\gamma(u_1) = u_1 t_0 + \frac{2}{3}\frac{t_1}{t_0}$. This and (6.2.1) imply that

$$t_0 + t_0^2 t_1 u_1^2 \equiv t_0^4 + 2t_3 + 3t_1^2 u_1 + 2t_0 t_2 u_1.$$

Isolating t_0 proves the first claim. A similar argument using the coefficients of x^8 give the desired relation for t_1. □

Recall that $\gamma \in \mathbb{S}_{\mathcal{C}}$ has an expansion of the form

$$\gamma = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + \ldots$$

Here the a_i are solutions to the equation $x^4 - x = 0$. Recall from Section 3 that if $\omega^s \in \mathrm{End}(F_{\mathcal{C}})$ is a solution to the equation $x^4 - x = 0$, then it corresponds the automorphism

$$\omega^s(x) = \zeta^s x,$$

where $\zeta \in \mathbb{F}_4 = (E_{\mathcal{C}})_*/(2, u_1)$. There is a copy of $\mathbb{F}_4$ in $\mathrm{End}(F_{\mathcal{C}})$ given by the ring generated by the automorphism $\omega(x)$. Further, $(E_{\mathcal{C}})_*/(2, u_1)$ is isomorphic to $\mathbb{F}_4$, with generator the image of ζ. Define a map

$$f : \mathbb{F}_4 \subseteq \mathrm{End}(F_{\mathcal{C}}) \to (E_{\mathcal{C}})_*/(2, u_1) \cong \mathbb{F}_4$$

by $f(\omega^s(x)) = \zeta^s$. If γ is as above, using the fact that $T(x) = x^2$,

$$\gamma(x) = f(a_0)x +_{F_{\mathcal{C}}} f(a_1)x^2 +_{F_{\mathcal{C}}} f(a_2)x^4 +_{F_{\mathcal{C}}} f(a_3)x^8 + \dots$$

For simplicity, we will identify a_i with $f(a_i)$ and write

$$\gamma(x) = a_0x +_{F_{\mathcal{C}}} a_1x^2 +_{F_{\mathcal{C}}} a_2x^4 +_{F_{\mathcal{C}}} a_3x^8 + \dots \tag{6.3.1}$$

Proposition 6.3.4. *For $\gamma \in S_{\mathcal{C}}$, modulo (x^{18}),*

$$\gamma(x) = x + a_1x^2 + a_2x^4 + a_1^2x^6 + a_3x^8 + a_2^2x^{10} + a_1^2a_2^2x^{12} + a_1x^{14} + (a_1^3 + a_4)x^{16}.$$

Proof. This is a direct computation using (6.3.1) and the formal group law of Proposition 6.1.4, noting that for $\gamma \in S_{\mathcal{C}}$, $a_0 = 1$. □

Corollary 6.3.5. *Let $t_i = t_i(\gamma)$ where $\gamma \in S_{\mathcal{C}}$. Modulo $(2, u_1)$, $t_0 \equiv 1$, $t_{2i} \equiv 0$ for $i \neq 0$ and*

$$\begin{array}{llll} t_1 \equiv a_1 & t_5 \equiv a_1^2 & t_9 \equiv a_2^2 & t_{13} \equiv a_1 \\ t_3 \equiv a_2 & t_7 \equiv a_3 & t_{11} \equiv a_1^2a_2^2 & t_{15} \equiv a_1^3 + a_4. \end{array}$$

Proof. This follows from Proposition 6.3.4, noting that t_i is congruent to the coefficient of x^{i+1} modulo $(2, u_1)$. □

Proposition 6.3.6. *Let $t_i = t_i(\gamma)$ for $\gamma \in S_{\mathcal{C}}$. Then modulo $(2, u_1^2)$,*

$$\begin{array}{lll} t_0 \equiv 1 + a_1^2u_1 & t_1 \equiv a_1 + a_2^2u_1 & t_2 \equiv a_1u_1 \\ t_3 \equiv a_2 + a_3^2u_1 & t_4 \equiv a_2u_1 & t_5 \equiv a_1^2 + a_1a_2u_1 \\ t_6 \equiv a_1^2u_1 & t_7 \equiv a_3 + (a_1^3 + a_4^2)u_1 & \end{array}$$

Proof. This follows from Corollary 6.3.5 and Corollary 6.3.1. □

Corollary 6.3.7. *Let $t_i = t_i(\gamma)$ where $\gamma \in S_{\mathcal{C}}$. Then*

$$t_0 \equiv 1 + a_1^2u_1 + a_1u_1^2 + (a_2 + a_2^2)u_1^3 \mod (2, u_1^4).$$

Proof. This follows from Proposition 6.3.6 and Proposition 6.3.3. □

We will need better estimates for elements which are in $F_{2/2}\mathbb{S}_{\mathcal{C}}$. Therefore, for the remainder of this section, we will always assume that $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$.

Proposition 6.3.8. *Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(2, u_1^4)$,*

$$t_3 \equiv a_2 + a_3^2 u_1 + a_4 u_1^3.$$

Modulo $(2, u_1^3)$, $t_1 \equiv a_2^2 u_1$ and $t_5 \equiv (a_2 + a_2^3)u_1^2$. Modulo $(2, u_1^6)$,

$$t_2 \equiv a_2^2 u_1^2 + a_3 u_1^4 + (a_2 + a_2^3)u_1^5.$$

Proof. It follows from Corollary 6.3.1 that, modulo $(2, u_1^4)$,

$$\begin{aligned} t_3 &\equiv t_3^4 + t_7^2 u_1 + t_1^2 t_2^4 u_1^2 + t_1^4 t_3^2 u_1^2 + t_0^4 t_5^2 u_1^2 \\ t_5 &\equiv t_5^4 + t_{11}^2 u_1 + t_3^6 u_1^2 + t_1^2 t_4^4 u_1^2 + t_2^4 t_5^2 u_1^2 + t_0^2 t_6 u_1^2 + t_1^4 t_7^2 u_1^2 + t_0^4 t_9^2 u_1^2. \end{aligned}$$

The results for t_3 and t_5 then follow from Corollary 6.3.5 and Proposition 6.3.6. It also follows from Corollary 6.3.1 that, modulo $(2, u_1^6)$,

$$t_2 \equiv t_2^4 + t_5^2 u_1 + t_1^6 u_1^2 + t_0^4 t_3^2 u_1^2 + t_0^4 t_1 u_1^4 + t_0^4 t_4 u_1^4.$$

The identity for t_2 then follows from the Corollary 6.3.5 and Proposition 6.3.6, using the identity for t_5 modulo $(2, u_1^3)$. □

Proposition 6.3.9. *Let $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(2, u_1^8)$,*

$$t_1(\gamma) \equiv a_2^2 u_1 + a_3 u_1^3 + a_3^2 u_1^5 + a_3 u_1^6 + (a_2^2 + a_2^3 + a_4 + a_4^2)u_1^7.$$

Modulo $(2, u_1^{10})$,

$$t_0(\gamma) \equiv 1 + (a_2 + a_2^2)u_1^3 + a_3 u_1^5 + a_3 u_1^8 + (a_2 + a_2^2 + a_4 + a_4^2)u_1^9.$$

Proof. Apply Proposition 6.3.3, Corollary 6.3.7 and Proposition 6.3.8 for the estimate for t_1. The result for t_0 then follows from Proposition 6.3.3. □

Proposition 6.3.10. *Let $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(4, 2u_1^2, u_1^{10})$,*

$$t_0(\gamma) \equiv 1 + 2a_2 + 2a_3^2 u_1 + (a_2 + a_2^2)u_1^3 + a_3 u_1^5 + a_3 u_1^8 + (a_2 + a_2^2 + a_4 + a_4^2)u_1^9.$$

Proof. Apply Proposition 6.3.3, Proposition 6.3.6 and Proposition 6.3.9. □

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Appendix A. The cohomology of G_{24} (by Hans-Werner Henn)

The following consists of unpublished notes by Hans-Werner Henn, which were edited by the author. She thanks him for letting her include them here.

Let $\mathcal{C}$ be the supersingular elliptic curve over $\mathbb{F}_4$ with equation $y^2 + y = x^3$. In this appendix, we calculate the cohomology of the automorphism group of this elliptic curve with coefficients in the Lubin–Tate module $(E_{\mathcal{C}})_*V(0)$ (see Section 2 for definitions).

None of the results are original. In some sense, this appendix redoes calculations by other people, for example by Bauer [1, Section 7] and by Rezk [12, Section 18]. The basic ideas go back to Hopkins. Bauer and Rezk calculate the cohomology of the Weierstrass Hopf algebroid and the calculation here can, in principle, be deduced from their calculation by inverting the discriminant Δ and passing to a suitable completion.

Our purpose is to give an independent and self-contained calculation of the group cohomology including the complete multiplicative structure. Furthermore, all elements in cohomology are defined via "Greek letter constructions" avoiding any explicit cocycles or Massey products. Everything is deduced from the knowledge of $H^*(G_{24}, (E_{\mathcal{C}})_*/(2, u_1)) \cong H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$, from the structure of the G_{24}-invariants of the symmetric algebra of a certain two dimensional representation ρ of G_{24}, and from the structure of $v_1^{-1}H^*(G_{24}, S_*(\rho)/(\Delta S_*(\rho)))$, where the discriminant Δ is an invariant class in degree 24. This knowledge is established in Lemma A.1, Corollary A.7 and Lemma A.10 below. The main computation is that of $H^*(G_{24}, S_*(\rho))$ and is given in Theorem A.14. The computation of $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$ then follows and is recorded in Theorem A.22.

Lemma A.1. *(a) There are classes $z \in H^4(Q_8, \mathbb{F}_4)$, $\widetilde{x} \in H^1(Q_8, \mathbb{F}_4)$ and $\widetilde{y} \in H^1(Q_8, \mathbb{F}_4)$ and an isomorphism of graded algebras with C_3-linear algebra action*

$$H^*(Q_8, \mathbb{F}_4) \cong \mathbb{F}_4[\widetilde{x}, \widetilde{y}, z]/(\widetilde{x}\widetilde{y}, \widetilde{x}^3 + \widetilde{y}^3),$$

where C_3 acts on Q_8 via the conjugation action of G_{24} on its normal subgroup Q_8, and C_3 acts on the right hand side by $\omega_*(\widetilde{x}) = \zeta\widetilde{x}$, $\omega_*(\widetilde{y}) = \zeta^2\widetilde{y}$ and $\omega_*(z) = z$.
(b) There are classes $k \in H^4(G_{24}, \mathbb{F}_4[u^{\pm 1}]_0)$, $a \in H^1(G_{24}, \mathbb{F}_4[u^{\pm 1}]_2)$ and $b \in H^1(G_{24}, \mathbb{F}_4[u^{\pm 1}]_4)$ and an isomorphism of graded algebras

$$H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]) \cong \mathbb{F}_4[v_2^{\pm 1}, k, a, b]/(ab, b^3 - v_2a^3).$$

(c) The subalgebra $H^*(G_{24}, \mathbb{F}_4[u^{-1}]) \subseteq H^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$ is generated as an $\mathbb{F}_4$-algebra by the elements v_2, k, a, b, $v_2^{-1}b^2$ and $v_2^{-1}a^3$.

Proof. (a) We start from the well known result that there is an isomorphism of graded algebras

$$H^*(Q_8, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, z]/(x^2 + xy + y^2, x^2y + xy^2)$$

where x and y are in dimension 1 and z is in dimension 4. The action of G_{24}/Q_8 on this algebra is trivial on z and the unique nontrivial action on $H^1 \cong (\mathbb{Z}/2)^2$. With suitable choices of x and y, we get $\omega_* x = y$, $\omega_* y = x+y$. This action has no eigenvectors over $\mathbb{F}_2$, but it has two eigenvectors over $\mathbb{F}_4$ with Galois conjugate eigenvalues equal to ζ and ζ^2. If we define $\widetilde{x} = x + \zeta y$ and $\widetilde{y} = x + \zeta^2 y$, then $\omega_*\widetilde{x} = \zeta\widetilde{x}$ and $\omega_*\widetilde{y} = \zeta^2\widetilde{y}$. Furthermore,

$$\widetilde{x}\widetilde{y} = x^2 + xy + y^2 = 0$$

and

$$\widetilde{x}^3 = x^3 + \zeta x^2 y + \zeta^2 x y^2 + y^3 = x^3 + \zeta^2 x^2 y + \zeta x y^2 + y^3 = \widetilde{y}^3.$$

The claim now follows.
(b) Let $a = u^{-1}\widetilde{x}$, $b = u^{-2}\widetilde{y}$ and $k = z$. Then we get an isomorphism of bigraded algebras

$$\mathbb{F}_4[v_2^{\pm 1}, k, a, b]/(v_2 a^3 - b^3, ab) \cong H^*(Q_8, \mathbb{F}_4[u^{\pm 1}])^{C_3}.$$

(c) This follows from the fact that $H^2(Q_8, \mathbb{F}_4[u^{-1}]_2)^{C_3} \cong \mathbb{F}_4$ is generated by $v_2^{-1}b^2$ and $H^3(Q_8, \mathbb{F}_4[u^{-1}]_0)^{C_3} \cong \mathbb{F}_4$ is generated by $v_2^{-1}a^3$. □

Consider u^{-1} and $v_1 = u_1 u^{-1}$ as elements in $(E_{\mathcal{C}})_2$. By Section 2.2, the action of G_{24} is given by

$$\begin{aligned}
\omega_*(v_1) &= v_1 & \omega_*(u^{-1}) &= \zeta^2 u^{-1} \\
i_*(v_1) &= \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta} & i_*(u^{-1}) &= \frac{v_1 - u^{-1}}{\zeta^2 - \zeta} \\
j_*(v_1) &= \frac{v_1 + 2\zeta^2 u^{-1}}{\zeta^2 - \zeta} & j_*(u^{-1}) &= \frac{\zeta v_1 - u^{-1}}{\zeta^2 - \zeta} \\
k_*(v_1) &= \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta} & k_*(u^{-1}) &= \frac{\zeta^2 v_1 - u^{-1}}{\zeta^2 - \zeta}.
\end{aligned}$$

Remark A.2. The two dimensional $\mathbb{W}$-module generated by u^{-1} and v_1 is a representation of G_{24} which we denote by ϱ. We denote its mod-2 reduction by ρ and the respective graded symmetric algebras by $S_*(\varrho)$ and $S_*(\rho)$. Because $i_*^2(u) = -u$ and $i_*^2(u_1) = u_1$, we see that for each integer $n \geq 0$ the action of G_{24} on $S_{2n}(\varrho)$ factors through the quotient $A_4 = G_{24}/(\pm 1)$. Likewise, the action of G_{24} on all of $S_*(\rho)$ factors through an action of A_4.

It follows that the element

$$\widetilde{\Delta} := \prod_{g \in V_4} g_*(u^{-1})$$

in $S_*(\rho)$ is a Q_8-invariant. One computes that $\widetilde{\Delta} \equiv u^{-1}(u^{-3} + v_1^3)$ modulo (2). It is an eigenvector for the residual action of G_{24}/Q_8; in fact,

$$\omega_*(\widetilde{\Delta}) = \zeta^2 \widetilde{\Delta}. \tag{A.3}$$

Hence, $\widetilde{\Delta}^3$ is a G_{24}-invariant and equal to the mod-2 reduction of the discriminant Δ (see Section 4.2).

The proof of the following theorem is similar to that of Goerss, Henn and Mahowald [4, Proposition 2].

Theorem A.4. *Completion at the maximal ideal $I \subseteq S_*(\varrho)[\widetilde{\Delta}^{-1}]$ induces a continuous isomorphism of $\mathbb{F}_4[G_{24}]$-algebras.*

$$S_*(\rho)[\widetilde{\Delta}^{-1}]^{\wedge}_{I} \to (E_{\mathcal{C}})_*.$$

Therefore to compute the cohomology $H^*(G_{24}, (E_2)_*V(0))$, we start by analyzing $S_*(\rho)$ and its invariants. Let $n \geq 0$ be an integer and

$$\widetilde{S}_n(\rho) = \begin{cases} S_n(\rho) & 0 \leq n \leq 3 \\ S_n(\rho)/(\widetilde{\Delta} S_{n-4}(\rho)) & 4 \leq n. \end{cases}$$

Lemma A.5. *Multiplication with $\widetilde{\Delta}$ induces a split short exact sequence of $\mathbb{F}_4[Q_8]$-modules*

$$0 \to \Sigma^8 S_n(\rho) \xrightarrow{\widetilde{\Delta}} S_{n+4}(\rho) \to \widetilde{S}_{n+4}(\rho) \to 0\ .$$

Further, for $n \geq 3$, multiplication by v_1 induces isomorphisms of $\mathbb{F}_4[Q_8]$-modules

$$v_1 : \Sigma^2 \widetilde{S}_n(\rho) \xrightarrow{\cong} \widetilde{S}_{n+1}(\rho).$$

Proof. The $n+1$ elements $\widetilde{\Delta} u^{-k} v_1^{n+1-k}$, $k = 0, \dots n$, together with the four elements $u^{-l} v_1^{n+4-l}$, $l = 0, \dots, 3$ form a basis of the $\mathbb{F}_4$-vector space $S_{n+4}(\rho)$. Therefore, the sequence is exact as a sequence of $\mathbb{F}_4[Q_8]$-modules. Furthermore, the subspace generated by $u^{-l} v_1^{n+4-l}$ with $0 \leq l \leq 3$ is Q_8-invariant. This gives the splitting.

If $n \geq 3$, the image of the elements $u^{-l} v_1^{n-l}$ for $0 \leq l \leq 3$ form a basis for $\widetilde{S}_n(\rho)$. Multiplication by v_1 sends this basis of $\widetilde{S}_n(\rho)$ to the corresponding basis of $\widetilde{S}_{n+1}(\rho)$. □

The next step is to identify the invariants.

Proposition A.6. *The Q_8-invariants of $S_*(\rho)$ are given as $\mathbb{F}_4[v_1, \widetilde{\Delta}]$.*

Proof. The split short exact sequence of Lemma A.5 gives a short exact sequence of Q_8-invariants. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^8\mathbb{F}_4[v_1, \widetilde{\Delta}] & \xrightarrow{\widetilde{\Delta}} & \mathbb{F}_4[v_1, \widetilde{\Delta}] & \longrightarrow & \mathbb{F}_4[v_1] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma^8 S_*(\rho)^{Q_8} & \xrightarrow{\widetilde{\Delta}} & S_*(\rho)^{Q_8} & \longrightarrow & (\widetilde{S}_*)^{Q_8} & \longrightarrow & 0. \end{array}$$

From part (c) of Lemma A.5 and an easy calculation, we get $(\widetilde{S}_n)^{Q_8} \cong \mathbb{F}_4\{v_1^n\}$ for each $n \geq 0$. The claim now follows from an induction over the internal degree and the five lemma. □

Corollary A.7. *The G_{24}-invariants of $S_*(\rho)$ are given as $\mathbb{F}_4[v_1, \Delta]$.*

We turn towards analyzing the cohomology algebra $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$. We begin by introducing certain classes in $H^1(G_{24}, (E_{\mathcal{C}})_*V(0))$. For this we consider the exact sequence G_{24}-modules

$$0 \to \rho \xrightarrow{2} \varrho/(4) \to \rho \to 0 \tag{A.8}$$

with associated Bockstein δ.

Lemma A.9. *(a) The class v_1 in ρ is G_{24}-invariant. The class $\eta := \delta(v_1)$ in $H^1(G_{24}, \rho)$ is nontrivial.*
(b) η is v_1-torsion free in $H^*(G_{24}, S_*(\rho))$.
(c) η is not divisible by v_1.

Proof. (a) A direct computation shows that v_1 is invariant modulo (2), but not invariant modulo (4). Hence, η is non-trivial.
(b) More generally, v_1^{2k} is invariant modulo (4) while v_1^{2k+1} is not. This shows that $\delta(v_1^{2k+1}) = v_1^{2k}\eta$ is non-trivial.
(c) If η is divisible, then there is a class $\eta' \in H^1(G_{24}, S_0(\rho))$ such that $v_1\eta' = \eta$. However, $S_0(\rho) = \mathbb{F}_4$ and $H^1(G_{24}; S_0(\rho)) = 0$ by Lemma A.1. □

Consider the graded $\mathbb{F}_4[v_1][G_{24}]$-algebras

$$\widetilde{S}_*(\rho) := S_*(\rho)/(\widetilde{\Delta} S_*(\rho))$$

and

$$\overline{S}_*(\rho) := S_*(\rho)/(\Delta S_*(\rho)).$$

By part (a) of Lemma A.5, multiplication with v_1 determines an isomorphism $\widetilde{S}_n(\rho) \to \widetilde{S}_{n+1}(\rho)$ if $n \geq 3$. Therefore, $\widetilde{S}_*(\rho)[v_1^{-1}]$ is a graded $\mathbb{F}_4[v_1^{\pm 1}][G_{24}]$-algebra with

$$(\widetilde{S}_*(\rho)[v_1^{\pm 1}])_{2k} \cong v_1^{k-3} S_3(\rho)$$

for every integer k.

Lemma A.10. *(a) There is an isomorphism of G_{24}-modules*

$$S_3(\rho) \cong \mathbb{F}_4[G_{24}] \otimes_{\mathbb{F}_4[C_6]} \mathbb{F}_4.$$

(b) The Q_8-cohomology of $\widetilde{S}_*(\rho)[v_1^{\pm 1}]$ is given by

$$H^*(Q_8, \widetilde{S}_*(\rho)[v_1^{\pm 1}]) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta].$$

It is C_3 invariant so that $H^*(G_{24}, \widetilde{S}_*(\rho)[v_1^{\pm 1}]) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta]$.
(c) The G_{24}-cohomology of $\overline{S}_*(\rho)[v_1^{\pm 1}]$ is given by

$$H^*(G_{24}, \overline{S}_*(\rho)[v_1^{\pm 1}]) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta].$$

Proof. (a) The C_6-linear map which sends 1 to u^{-3} extends to a G_{24}-linear isomorphism $\mathbb{F}_4[G_{24}] \otimes_{\mathbb{F}_4[C_6]} \mathbb{F}_4 \to S_3(\rho)$.
(b) The isomorphism of (a) restricts to an isomorphism of Q_8-modules

$$\mathbb{F}_4[Q_8] \otimes_{\mathbb{F}_4[C_2]} \mathbb{F}_4 \cong S_3(\rho).$$

The isomorphisms $(\widetilde{S}_*(\rho)[v_1^{\pm 1}])_{2k} \cong v_1^{k-3} S_3(\rho)$ and part (a) show that there is a class $h' \in H^1(Q_8, S_3(\rho)) \cong H^1(C_2, \mathbb{F}_4) \cong \mathbb{F}_4$ and, for $h = v_1^{-3} h'$, an isomorphism of graded algebras $H^*(Q_8, \widetilde{S}_*(\rho)[v_1^{\pm 1}]) \cong \mathbb{F}_4[v_1^{\pm 1}, h]$. By Lemma A.9 we know that η is v_1-torsion free in

$$H^*(G_{24}, \widetilde{S}_*(\rho)) \cong H^*(Q_8, \widetilde{S}_*(\rho))^{C_3},$$

hence it is also v_1-torsion free in $H^*(Q_8, \widetilde{S}_*(\rho))$. Therefore, $v_1^2 \eta$ must agree with h' up to a scalar. This gives the isomorphism $H^*(Q_8, \widetilde{S}_*(\rho)[v_1^{\pm 1}]) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta]$. The invariance with respect to the residual action of G_{24}/Q_8 follows from the fact that both v_1 and η are invariant.
(c) By iterated use of Lemma A.5, we obtain an isomorphism

$$S_*(\rho) \cong \widetilde{S}_*(\rho) \oplus \widetilde{\Delta} \widetilde{S}_*(\rho) \oplus \widetilde{\Delta}^2 \widetilde{S}_*(\rho) \oplus \widetilde{\Delta}^3 \widetilde{S}_*(\rho)$$

of G_{24}-modules and therefore an isomorphism

$$\overline{S}_*(\rho) \cong \widetilde{S}_*(\rho) \oplus \widetilde{\Delta} \widetilde{S}_*(\rho) \oplus \widetilde{\Delta}^2 \widetilde{S}_*(\rho).$$

Part (b) implies that

$$H^*(Q_8, \overline{S}_*(\rho)[v_1^{\pm 1}]) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta] \oplus \widetilde{\Delta} \mathbb{F}_4[v_1^{\pm 1}, \eta] \oplus \widetilde{\Delta}^2 \mathbb{F}_4[v_1^{\pm 1}, \eta].$$

The result follows by observing that, for the residual action of $C_3 \cong G_{24}/Q_8$, the first summand is invariant while the other two summands are eigenspaces for the eigenvalues ζ^2 and ζ respectively (see (A.3)). □

Next we observe that multiplication by v_1^k determines exact sequences of graded G_{24}-modules

$$0 \to \Sigma^{2k} S_n(\rho) \xrightarrow{v_1^k} S_{n+k}(\rho) \xrightarrow{p} (\mathbb{F}_4[u^{-1}, v_1]/(v_1^k))_{2(n+k)} \to 0 \qquad \text{(A.11)}$$

with associated Bockstein δ_k. In the remainder of this chapter, the Bockstein associated with the exact sequence (A.8) will not play any role, and therefore we take the liberty to simply write δ instead of δ_1.

Remark A.12. The classes v_2, v_2^2 and v_2^3 are invariant in $H^0(G_{24}, \mathbb{F}_4[u^{\pm 1}])$ but do not lift to invariants in $H^0(G_{24}, S_*(\rho))$. Further, v_2^2 is invariant in $\mathbb{F}_4[u^{-1}, v_1]/(v_1^2)$ but is not invariant in $\mathbb{F}_4[u^{-1}, v_1]/(v_1^3)$. Therefore, $\delta(v_2)$, $\delta(v_2^2)$, $\delta(v_2^3)$ and $\delta_2(v_2^2)$ are non-trivial. Define

$$\nu := \delta(v_2) \in H^1(G_{24}, S_2(\rho)),$$
$$y := \delta(v_2^3) \in H^1(G_{24}, S_8(\rho)),$$
$$x := \delta_2(v_2^2) \in H^1(G_{24}, S_4(\rho)).$$

Given a ring R and a set X, we will use the notation RX to denote the free R-module generated by the elements of X.

Lemma A.13.

(a) There are relations $v_1\nu = 0$, $v_1 y = 0$ and $v_1^2 x = 0$.
(b) There is a relation $v_1 x = \delta(v_2^2)$.
(c) The classes ν, x and y are not divisible by v_1.

Proof. (a) This follows from the definition of these classes and exactness of the long exact cohomology sequences.
(b) This is straightforward by comparing the two short exact sequences (A.11) for $k = 1$ and $k = 2$.
(c) One verifies by a direct computation that

$$H^0(G_{24}, \mathbb{F}_4[u^{-1}, v_1]/(v_1^2)) \cong \mathbb{F}_4[v_2^2, v_1]/(v_1^2)\{1\} \oplus \mathbb{F}_4[v_2^2, v_1]/(v_1)\{v_1 v_2\}$$

and that

$$H^0(G_{24}, \mathbb{F}_4[u^{-1}, v_1]/(v_1^3)) \cong \mathbb{F}_4[v_2^4, v_1]/(v_1^3)\{1\} \oplus (\mathbb{F}_4[v_2^4, v_1]/(v_1^2))\{v_1 v_2^2\}$$
$$\oplus (\mathbb{F}_4[v_2^4, v_1]/(v_1))\{v_1^2 v_2, v_1^2 v_2^3\}.$$

If ν is v_1-divisible, then there is a non-trivial class in $H^1(G_{24}, S_1(\rho))$ which is annihilated by v_1^2 and, hence, is in the image of δ_2. However, this contradicts the triviality of the group $H^0(G_{24}, \mathbb{F}_4[u^{-1}, v_1]/(v_1^2))$ in internal degree 6. Likewise, if y is v_1-divisible, then there is a non-trivial class in $H^1(G_{24}, S_7(\rho))$ which is annihilated by v_1^2. Again, this is a contradiction since $H^0(G_{24}, \mathbb{F}_4[u^{-1}, v_1]/(v_1^2))$ is trivial in internal degree 18. Finally, if x is v_1-divisible then there is a non-trivial class in $H^1(G_{24}, S_3(\rho))$ which is annihilated by v_1^3 and, hence, in the image of δ_3. However, the group $H^0(G_{24}, \mathbb{F}_4[u^{-1}, v_1]/(v_1^3))$ is trivial in internal degree 12. $\square$

Here is the main theorem giving the complete structure of the cohomology algebra $H^*(G_{24}, S_*(\rho))$.

Theorem A.14. *Let $R = \mathbb{F}_4[v_1, \Delta, k]$, where $k \in H^4(G_{24}, \mathbb{F}_4)$ is the cohomological periodicity generator.*
(a) As an R-module, $H^*(G_{24}, S_*(\rho))$ is isomorphic to

$$R\{1, \eta, \eta^2, \eta^3\} \oplus R/(v_1^2)\{x, \eta x, x^2, \eta x^2\} \oplus R/(v_1)\{\nu, y, \nu^2, \nu y, \nu^3, \Delta^{-1}\nu^2 y\},$$

where $\Delta^{-1}\nu^2 y$ is a class in $H^3(G_{24}, S_0(\rho))$ such that $\Delta(\Delta^{-1}\nu^2 y) = \nu^2 y$.
(b) The products η^2, ν^2, νy, ηx and x^2 are R-module generators and the remaining five products satisfy $\eta\nu = 0$, $xy = 0$ and

$$\eta y = v_1 x^2, \qquad \nu x = v_1 \eta x, \qquad y^2 = \nu^2 \Delta.$$

(c) The element $\Delta^{-1}\nu^2 y$ and the products η^3, ν^3 and ηx^2 are R-module generators. There are relations:

$$x^3 = \nu^2 y, \qquad \eta^2 x = \nu^3, \qquad \nu x^2 = v_1 \eta x^2,$$
$$\eta^2 y = v_1 \eta x^2, \qquad \nu y^2 = \nu^3 \Delta, \qquad y^3 = \nu^2 y \Delta$$

and the remaining possible threefold products are zero.
(d) All products of η, x and y with $\Delta^{-1}\nu^2 y$ and all fourfold products among η, ν, x and y are trivial except for $\eta^4 = v_1^4 k$.

In order to prove Theorem A.14, we calculate the cohomology of $\overline{S}_*(\rho)$ as a module over $\mathbb{F}_4[v_1, k]$. In fact we will find the following result.

Proposition A.15. *Let $\overline{R} := \mathbb{F}_4[v_1, k]$. As an $\overline{R}$-module, $H^*(G_{24}, \overline{S}_*(\rho))$ is isomorphic to*

$$\overline{R}\{1, \eta, \eta^2, \eta^3\} \oplus \overline{R}/(v_1^2)\{x, \eta x, x^2, \eta x^2\} \oplus \overline{R}/(v_1)\{\nu, y, \nu^2, \nu y, \nu^3, \Delta^{-1}\nu^2 y\}.$$

Note that, in Proposition A.15, we are using the product structure in order to describe the generators of $H^*(G_{24}, \overline{S}_*(\rho))$, but are not yet attempting to describe the cohomology as an algebra.

Proposition A.15 is deduced from the following three lemmas. We write $s \doteq t$ if $s = \ell t$ for some $\ell \in \mathbb{F}_4$.

Lemma A.16. *(a) There is an isomorphism of* $\mathbb{F}_4[v_1]$*-modules*

$$H^1(G_{24}, \overline{S}_*(\rho)) \cong \mathbb{F}_4[v_1]\{\eta\} \oplus \mathbb{F}_4[v_1]/(v_1)\{\nu, y\} \oplus \mathbb{F}_4[v_1]/(v_1^2)\{x\}.$$

(b) The reduction homomorphism

$$p_* : H^1(G_{24}, \overline{S}_*(\rho)) \to H^1(G_{24}, \overline{S}_*(\rho)/(v_1))$$

satisfies

$$p_*(\eta) \doteq a, \qquad p_*(\nu) \doteq b, \qquad p_*(x) \doteq v_2 a, \qquad p_*(y) \doteq v_2^2 b,$$

for a and b as in Lemma A.1.
(c) The connecting homomorphism associated to the exact sequence (A.11) is trivial on a, $v_2 a$, b *and* $v_2^2 b$*. It is nontrivial on the generators* $v_2^2 a$, $v_2^3 a$, $v_2 b$ *and* $v_2^3 b$*. In fact,*

$$\delta(v_2^2 a) \doteq v_1 \eta x, \qquad \delta(v_2^3 a) \doteq v_1 x^2, \qquad \delta(v_2 b) \doteq \nu^2, \qquad \delta(v_2^3 b) \doteq \nu y.$$

Further, $\nu x \doteq \eta v_1 x$ *and* $\eta y \doteq v_1 x^2$*.*

Proof. (a) Any finitely generated graded $\mathbb{F}_4[v_1]$-module is a direct sum of a free module and of cyclic torsion modules of the form $\mathbb{F}_4[v_1]/(v_1^n)$. On the other hand, we know from Lemma A.10 that the free part of $H^1(G_{24}, \overline{S}_*(\rho))$ is of rank one. By the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \overline{S}_*(\rho) \xrightarrow{v_1} \overline{S}_*(\rho) \xrightarrow{p} \overline{S}_*(\rho)/(v_1) \cong \mathbb{F}_4[u^{-1}]/(u^{-12}) \to 0,$$

we know that the submodule of $H^1(G_{24}, \overline{S}_*(\rho))$ which is annihilated by v_1 is generated by the classes ν, $v_1 x$ and y. By Lemma A.9 and Lemma A.13 the classes η, ν, x and y are not divisible by v_1, proving (a).
(b) We know from Lemma A.1 that $H^1(G_{24}, \mathbb{F}_4[u^{-1}]/(u^{-12}))$ is an $\mathbb{F}_4$-vector space on generators of the form $v_2^i a$, $v_2^i b$ for $0 \leq i \leq 3$ and the four generators of $H^1(G_{24}, \overline{S}_*(\rho))$ have to map to four of these classes. The four other generators map via δ to non-trivial elements of $H^2(G_{24}, \overline{S}_*(\rho))$. The claim in (b) then follows for degree reasons.
(c) From (a) and (b), it is clear by exactness that δ is trivial on a, b, $v_2 a$ and $v_2^2 b$ and nontrivial on the four other generators. We use that δ is p_*-linear (i.e. $\delta(p_*(t)s) = t\delta(s)$) and Lemma A.13 to conclude that

$$\delta(v_2^2 a) \doteq \delta(v_2^2 p_*(\eta)) = \eta\delta(v_2^2) = \eta v_1 x$$
$$\delta(v_2^3 a) \doteq \delta(v_2^2 p_*(x)) = x\delta(v_2^2) = v_1 x^2$$
$$\delta(v_2 b) \doteq \delta(v_2 p_*(\nu)) = \nu\delta(v_2) = \nu^2$$
$$\delta(v_2^3 b) \doteq \delta(v_2^3 p_*(\nu)) = \nu\delta(v_2^3) = \nu y.$$

Note further that

$$\delta(v_2^2 a) \doteq \delta(v_2 p_*(x)) = \nu x, \qquad \delta(v_2^3 a) \doteq \delta(v_2^3 p_*(\eta)) = \eta y$$

and hence $\nu x \doteq \eta v_1 x$ and $\eta y \doteq v_1 x^2$. □

Lemma A.17. *(a) The elements η^2, ν^2, ηx, x^2 and νy are non-trivial in $H^2(G_{24}, \overline{S}_*(\rho))$ and*

$$p_*(\eta^2) \doteq a^2, \qquad p_*(\nu^2) \doteq b^2, \qquad p_*(\eta x) \doteq v_2 a^2, \qquad p_*(x^2) \doteq v_2^2 a^2, \qquad p_*(\nu y) \doteq v_2^2 b^2.$$

(b) There is an isomorphism of $\mathbb{F}_4[v_1]$-modules

$$H^2(G_{24}, \overline{S}_*(\rho)) \cong \mathbb{F}_4[v_1]/\{\eta^2\} \oplus \mathbb{F}_4[v_1]/(v_1)\{\nu^2, \nu y\} \oplus \mathbb{F}_4[v_1]/(v_1^2)\{\eta x, x^2\} \ .$$

(c) The connecting homomorphism associated to the exact sequence (A.11) is trivial on the generators a^2, $v_2 a^2$, $v_2^2 a^2$, b^2 and $v_2^2 b^2$. It is nontrivial on the generators $v_2^3 a^2$, $v_2 b^2$ and $v_2^{-1} b^2$, and

$$\delta(v_2^3 a^2) \doteq v_1 \eta x^2, \qquad \delta(v_2 b^2) \doteq \nu^3, \qquad \delta(v_2^{-1} b^2) \doteq \Delta^{-1} \nu^2 y.$$

Further, $v_1 \eta x^2 \doteq \nu x^2 \doteq \eta^2 y$.

Proof. (a) This follows from the fact that p_* is a map of algebras and the images of the given elements are non-trivial.
(b) We claim that $H^2(G_{24}, \overline{S}_*(\rho))$ is generated as a $\mathbb{F}_4[v_1]$-module by the elements η, ν^2, ηx, x^2 and νy. In fact, by part (c) of Lemma A.16 the submodule of $H^2(G_{24}, \overline{S}_*(\rho))$ which is annihilated by v_1 is generated by the elements $v_1 \eta x$, $v_1 x^2$, ν^2 and νy. In particular, the v_1-torsion submodule is of rank four. Furthermore, we know from Lemma A.10 that the v_1-torsionfree part is of rank one and hence $H^2(G_{24}, \overline{S}_*(\rho))$ is generated as a $\mathbb{F}_4[v_1]$-module by five elements. By (a) these must be the five elements η^2, ν^2, ηx, x^2 and νy. By part (c) of Lemma A.16, the v_1-torsion submodule is as specified. This leaves η^2 which must generate a free $\mathbb{F}_4[v_1]$ submodule by Lemma A.10.
(c) By (a) and exactness, δ is trivial on a^2, b^2, $v_2 a^2$, $v_2^2 a^2$ and $v_2^2 b^2$. Finally we get from Lemma A.13 and p_*-linearity

$$\begin{aligned}
\delta(v_2^3 a^2) &\doteq x\eta\delta(v_2^2) \doteq v_1 \eta x^2 \\
\delta(v_2 b^2) &\doteq \nu^2 \delta(v_2) \doteq \nu^3 \\
\delta(v_2^{-1} b^2) = \delta(v_2^{-4} v_2^3 b^2) &\doteq \Delta^{-1} \delta(v_2^3 b^2) = \Delta^{-1} \nu^2 y.
\end{aligned}$$

(The last calculation is a calculation in the cohomology of $\Delta^{-1} S_*(\rho)$ which, in that of $\overline{S}_*(\rho)$, gives the desired relation. It uses the relation $p_*(\Delta^{-1}) = v_2^{-4}$.)

Note further that, $\delta(v_2^3 a^2) \doteq x^2 \delta(v_2) \doteq \nu x^2$ and $\delta(v_2^3 a^2) \doteq \eta^2 \delta(v_2^3) \doteq \eta^2 y$. Hence, $v_1 \eta x^2 \doteq \nu x^2 \doteq \eta^2 y$. This finishes the proof of (c). □

Lemma A.18. *(a) The elements* η^3, ν^3, ηx^2 *and* $\Delta^{-1}\nu^2 y$ *are non-trivial in* $H^3(G_{24}, \overline{S}_*(\rho))$ *and*

$$p_*(\eta^3) \doteq a^3 \qquad p_*(\nu^3) \doteq b^3 = v_2 a^3$$
$$p_*(\eta x^2) \doteq v_2^2 a^3 \qquad p_*(\Delta^{-1} y \nu^2) \doteq v_2^{-4} v_2^2 b^3 = v_2^{-1} a^3.$$

(b) There is an isomorphism of $\mathbb{F}_4[v_1]$*-modules*

$$H^3(G_{24}, \overline{S}_*(\rho)) \cong \mathbb{F}_4[v_1]\{\eta^3\} \oplus \mathbb{F}_4[v_1]/(v_1)\{\Delta^{-1}\nu^2 y, \nu^3\} \oplus \mathbb{F}_4[v_1]/(v_1^2)\{\eta x^2\}\ .$$

Proof. (a) The first part follows from the fact that p_* is a map of algebras.
(b) From (a) we see that $p_* : H^3(G_{24}, \overline{S}_*(\rho)) \to H^3(G_{24}, \overline{S}_*(\rho)/(v_1))$ is onto and hence $H^3(G_{24}, \overline{S}_*(\rho))$ is generated by η^3, ν^3, ηx^2 and $\Delta^{-1}\nu^2 y$. By part (c) of Lemma A.17, ν^3 and $\Delta^{-1}\nu^2 y$ are annihilated by v_1. Further, $v_1 \eta x^2$ is nontrivial and also annihilated by v_1. By Lemma A.10, the torsionfree part is of rank one, so this proves the claim. □

We finally turn towards the proof of Theorem A.14.

Proof of Theorem A.14. (a) Consider the short exact sequence of $\mathbb{F}_4[G_{24}]$-modules

$$0 \to \Sigma^{24} S_*(\rho) \xrightarrow{\Delta} S_{*+12}(\rho) \to \overline{S}_{*+12}(\rho) \to 0.$$

As in the proof of Lemma A.5, one can show that it is split. Therefore, the maps

$$H^*(G_{24}, S_{*+12}(\rho)) \to H^*(G_{24}, \overline{S}_{*+12}(\rho))$$

are surjective and the long exact sequence on cohomology groups gives rise to short exact sequences in each degree. The claim then follows from Proposition A.15 and the five lemma.
(b) Note that $\nu x \doteq v_1 \eta x$ and $\eta y \doteq v_1 x^2$ follow from part (c) of Lemma A.16. For the other relations, first note that $p_*(\Delta) = v_2^4$ so that

$$p_*(\eta\nu) \doteq ab = 0, \qquad p_*(y^2) \doteq v_2^4 b^2,$$
$$p_*(xy) \doteq v_2^3 ab = 0, \qquad p_*(\Delta\nu^2) \doteq v_2^4 b^2.$$

Hence, up to elements in the kernel of

$$p_* : H^2(G_{24}, S_*(\rho)) \to H^2(G_{24}, S_*(\rho)/(v_1)),$$

the relations hold in $H^2(G_{24}, S_*(\rho))$. However, in these degrees, the non-zero elements in the kernel of p_* are v_1-torsion free. Since $\eta\nu$, xy and $y^2 - \ell\Delta\nu^2$ are v_1-torsion, the relations hold as claimed.
(c) Note that $v_1 \eta x^2 \doteq \nu x^2 \doteq \eta^2 y$ follow from part (c) of Lemma A.17. Further,

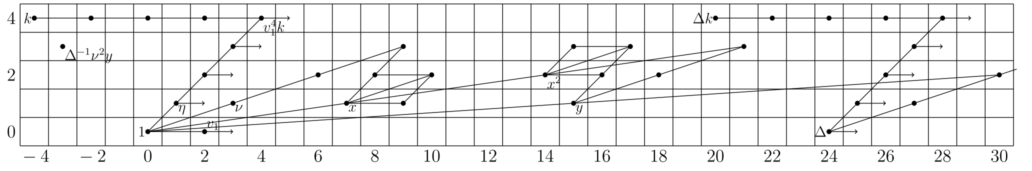

Fig. 4. The cohomology $H^*(G_{24}, S_*(\rho))$, drawn in the Adams grading $(t-s, s)$. It is periodic with respect to s with period 4 and periodicity generator k and with respect to t with period 24 and periodicity generator Δ. A $\bullet$ denotes a copy of $\mathbb{F}_4$. Lines of slope 1 denote multiplication by η and lines of slope 1/3 denote multiplication by ν. Lines of slope 1/7 denote multiplication by x and those of slope 1/15 multiplication by y. Horizontal lines denote multiplication by v_1. Classes attached to horizontal arrows are free over $\mathbb{F}_4[v_1]$.

$$p_*(\nu^3) = b^3 = v_2 a^3 \doteq p_*(x\eta^2).$$

Therefore, for an appropriate $\ell \in \mathbb{F}_4$, $\nu^3 - \ell\eta^2 x$ must be a v_1 multiple of η^3. It must be zero since it is v_1-torsion and η^3 is v_1-torsion free. Hence, $\nu^3 \doteq x\eta^2$. Finally, since

$$p_*(x^3) \doteq v_2^3 a^3 = v_2^2 b^3 \doteq p_*(\nu^2 y),$$

for an appropriate choice of ℓ in $\mathbb{F}_4$, we must have that $x^3 - \ell\nu^2 y$ is a v_1-multiple of η, and again, must therefore be zero since it is v_1-torsion and η^3 is v_1-torsion free. That the other threefold products vanish follows from part (b).
(d) With the exception of η^4 which is v_1-free, all fourfold products and all products of η, ν, x and y with $\Delta^{-1}\nu^2 y$ are v_1-torsion. However,

$$H^4(G_{24}; S_*(\rho)) \cong \mathbb{F}_4[v_1, \Delta]\{k\}$$

is v_1-torsion free. By Lemma A.10 the class η^4 is v_1-torsion free and, for degree reasons, it must be equal to $v_1^4 k$, up to a nontrivial scalar in $\mathbb{F}_4$.

Finally, note that the generators v_1, Δ and k are Galois invariant classes. Likewise, η as the mod-2 Bockstein of v_1, and ν, x and y as Bocksteins of Galois invariant classes are also Galois invariant. It follows that the multiplicative relations, which we have only proved modulo units in $\mathbb{F}_4$, do hold on the nose. □

Remark A.19. If we extend $G_{24} \cong SL_2(\mathbb{F}_3)$ by the Galois group to $G_{48} = GL_2(\mathbb{F}_3)$, then the G_{48}-cohomology of $S_*(\rho)$ is obtained from that of G_{24} by taking Galois invariants. This is the content of the following result (see Fig. 4).

Theorem A.20. *There is a ring isomorphism*

$$H^*(G_{48}, S_*(\rho)) \cong \mathbb{F}_2[v_1, \Delta, k, \eta, \nu, x, y, \Delta^{-1}\nu^2 y]/(\sim)$$

where $(\sim)$ *is the ideal generated by*

$$v_1\nu, \qquad v_1^2 x, \qquad v_1 y,$$

in cohomological degree 1,

$$\eta\nu, \qquad \nu x - v_1\eta x, \qquad \eta y - v_1 x^2, \qquad xy, \qquad y^2 - \nu^2\Delta,$$

in cohomological degree 2,

$$\eta^2 x - \nu^3, \qquad x^3 - \nu^2 y, \qquad \Delta(\Delta^{-1}\nu^2 y) - \nu^2 y$$

in cohomological degree 3 *and*

$$\eta^4 - v_1^4 k$$

in cohomological degree 4*. Further,*

$$H^*(G_{24}, S_*(\rho)) \cong \mathbb{F}_4 \otimes_{\mathbb{F}_2} H^*(G_{48}, S_*(\rho)).$$

We deduce the main result from Theorem A.20 and the following lemma, whose proof is analogous to that of Goerss, Henn and Mahowald [4, Theorem 6].

Lemma A.21. *Let* $m = 48$ *or* $m = 24$*. There is an isomorphism*

$$H^*(G_m, S_*(\rho)[\Delta^{-1}]^{\wedge}_{(j)}) \cong (H^*(G_m, S_*(\rho))[\Delta^{-1}])^{\wedge}_{(j)}.$$

Theorem A.22. *There is an isomorphism*

$$H^*(G_{48}, (E_{\mathcal{C}})_* V(0)) \cong \mathbb{F}_2[\![j]\!][v_1, \Delta^{\pm 1}, k, \eta, \nu, x, y]/(\sim)$$

where $(\sim)$ *is the ideal generated by*

$$v_1^{12} - j\Delta$$

in cohomological degree 0

$$v_1\nu, \qquad v_1^2 x, \qquad v_1 y,$$

in cohomological degree 1,

$$\eta\nu, \qquad \nu x - v_1\eta x, \qquad \eta y - v_1 x^2, \qquad xy, \qquad y^2 - \nu^2\Delta,$$

in cohomological degree 2,

$$\eta^2 x - \nu^3, \qquad x^3 - \nu^2 y,$$

in cohomological degree 3 *and*

$$\eta^4 - v_1^4 k$$

in cohomological degree 4. *Further,*

$$H^*(G_{24}, (E_{\mathcal{C}})_* V(0)) \cong \mathbb{F}_4 \otimes_{\mathbb{F}_2} H^*(G_{48}, (E_{\mathcal{C}})_* V(0))).$$

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