THE ALGEBRAIC DUALITY RESOLUTION AT $p = 2$

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Abstract. The goal of this paper is to develop some of the machinery necessary for doing $K(2)$-local computations in the stable homotopy category using duality resolutions at the prime $p = 2$. Goerss, Henn, Mahowald and Rezk have constructed an analogue of their finite resolution of the trivial $\mathbb{G}_2^1$-module $\mathbb{Z}_3$ at the prime $p = 2$. It is a finite resolution of the trivial $\mathbb{S}_2^1$-module $\mathbb{Z}_2$, which we call the algebraic duality resolution. Their construction was never published and is the main result in this paper. In the process, we give a detailed description of the structure of Morava stabilizer group $\mathbb{S}_2$ at the prime 2. We also describe the maps in the algebraic duality resolution with the precision necessary for explicit computations.

Contents

1. Introduction

The study of periodicity in the stable homotopy groups of spheres is one of the central problems in algebraic topology. It was prompted by the discovery by Adams of a periodic family of elements constructed using the j-homomorphism (see [1]). Inspired by Morava, in [23] Miller, Ravenel and Wilson set these results in a general framework based on periodic behavior in the E_2-term of the Adams-Novikov spectral sequence. Following this breakthrough, Ravenel published a series of conjectures in [25] on the structure of the stable homotopy category. Many of his conjectures were proved in the following years, in particular, by Devinatz, Hopkins and Smith in [9] and [21]. From these conjectures has arisen a framework for understanding the structure of the stable homotopy category of finite spectra called the chromatic filtration. The idea is that information about the p-local homotopy groups of a finite spectrum X can be recovered from the homotopy groups of its Morava K-theory localizations $L_{K(n)}X$. The homotopy groups of $L_{K(n)}X$ are, in theory, easier to compute than the homotopy groups of $X_{(p)}$.

1.1. **Background.** Recall that $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ for generators v_n of degree $2(p^n-1)$. Let $E(n)$ be the Johnson-Wilson spectrum. The spectrum $E(n)$ represents the Landweber exact homology theory determined by

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}],$$

together with the obvious ring homomorphism from BP_* to $E(n)_*$. The functor L_n denotes Bousfield localization with respect to $E(n)$. For each n, there is a natural transformation from L_nX to $L_{n-1}X$. Hopkins and Ravenel proved that for X a finite p-local spectrum,

$$X \simeq \operatorname{holim} L_nX.$$

This theorem is called the chromatic convergence theorem and a proof can be found in [27].

The localization L_nX can be refined as follows. Let $K(n)$ denote the n'th Morava K-theory spectrum. The spectrum $K(n)$ is the unique complex oriented ring spectrum with

$$K(n)_* = \mathbb{F}_p[v_n^{\pm}]$$

for v_n of degree $2(p^n-1)$, and whose formal group law is the Honda formal group law. This is the p-typical formal group law F_n over $K(n)_*$ defined by its p-series

$$[p]_{F_n}(x) = v_n x^{p^n}.$$

Localization with respect to $E(n)$ is equivalent to localization with respect to the wedge $K(0) \vee \ldots \vee K(n)$. It follows that the maps $L_nX \to L_{n-1}X$ fit into homotopy pull back squares

$$\begin{tikzcd} L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X. \end{tikzcd} \tag{1.1}$$

In theory, the computation of π_*L_nX can be carried out inductively by computing $\pi_*L_{n-1}X$, $\pi_*L_{K(n)}X$ and $\pi_*L_{n-1}L_{K(n)}X$, together with the maps of the homotopy pull-back (1.1). What makes this approach attractive is that $K(n)$ carries an incredible amount of geometric structure coming from the large group of automorphisms of its formal group law.

To describe this structure, we first define the Morava E-theory spectrum E_n. Let $\mathbb{W} = W(\mathbb{F}_{p^n})$ denotes the Witt vectors on $\mathbb{F}_{p^n}$. Then E_n is the spectrum which represents the Landweber exact homology theory defined by

$$(E_n)_* = \mathbb{W}[[u_1, \dots, u_{n-1}]][u^{\pm 1}],$$

with the u_i's of degree zero and u of degree -2, together with the ring homomorphism $f : BP_* \to (E_n)_*$ determined by

$$f(v_i) = \begin{cases} u_i u^{1-p^i} & 1 \le i < n \\ u^{1-p^n} & i = n \\ 0 & n < i. \end{cases}$$

The spectra $E(n)$ and E_n have the same Bousfield class. Hence, their localization functors are weakly equivalent. However, the spectrum E_n classifies deformations of the formal group law of $K(n)$ over complete local rings. Therefore, E_n inherits structure coming from deformation theory.

Indeed, let $\mathbb{S}_n$ be the group of automorphisms of F_n over $\mathbb{F}_{p^n}$. The group $\mathbb{S}_n$ is called the Morava stabilizer group. It admits an action of the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. The extended Morava stabilizer $\mathbb{G}_n$ is the extension of $\mathbb{S}_n$ by this action. By the Goerss-Hopkins-Miller theorem, E_n is an E_∞-ring spectrum and admits an action of $\mathbb{G}_n$ by maps of E_∞-ring spectra ([15]). The importance of the Morava stabilizer group and of Morava E-theory in chromatic homotopy theory is captured by the fact that, for a finite spectrum X,

$$L_{K(n)}X \simeq E_n^{h\mathbb{G}_n} \wedge X \tag{1.2}$$

Further, there is an associated descent spectral sequence computing $\pi_* L_{K(n)}X$, which can be described as follows.

For any spectrum X, the action of $\mathbb{G}_n$ on $(E_n)_*$ induces an action on

$$(E_n)_*X := \pi_* L_{K(n)}(E_n \wedge X).$$

If X is finite, this corresponds to the usual definition of E_n-homology, but this is not true for a general spectrum X. For a closed subgroup G of $\mathbb{G}_n$ and any finite spectrum X, there is a convergent descent spectral sequence

$$E_2^{s,t} := H_c^s(G, (E_n)_t X) \Longrightarrow \pi_{t-s}(E_n^{hG} \wedge X). \tag{1.3}$$

(see [5]). This spectral sequence is isomorphic to the $K(n)$-local E_n-Adams spectral described in Appendix A of [8]. Equivalences such as (1.2) and spectral sequences such as (1.3) are originally due to Devinatz and Hopkins in [8]. However, the construction of E_n^{hG} as the homotopy fixed points of a continuous action is due to Davis in [7]. The construction of (1.3) as a *descent* spectral sequence is due to Behrens and Davis in [5].

There are two important examples which are worth mentioning here. The first example is when G is the group $\mathbb{G}_n$. Then (1.3) is the descent spectral sequence computing $\pi_* L_{K(n)}X$ mentioned above,

$$H_c^s(\mathbb{G}_n, (E_n)_t X) \cong H_c^s(\mathbb{S}_n, (E_n)_t X)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} \Longrightarrow \pi_{t-s} L_{K(n)}X. \tag{1.4}$$

(Note that (1.4) was recently generalized to a wider class of spectra X by Barthel and Heard [2].)

To give the second example, we must introduce the subgroups $\mathbb{S}_n^1$ and $\mathbb{G}_n^1$. There is a norm on the group $\mathbb{S}_n$ induced by the determinant of a general linear representation. The kernel of this norm is denoted $\mathbb{S}_n^1$. One can show that

$$\mathbb{S}_n \cong \mathbb{S}_n^1 \rtimes \mathbb{Z}_p,$$

(see [11]). Similarly, the norm on $\mathbb{S}_n$ induces a norm on $\mathbb{G}_n$. The kernel is denoted $\mathbb{G}_n^1$ and

$$\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p.$$

Let π be a topological generator of $\mathbb{Z}_p$ in $\mathbb{G}_n$ and ϕ_π be its action on E_n. If X is finite, there is a fiber sequence

$$L_{K(n)}X \to E_n^{h\mathbb{G}_n^1} \wedge X \xrightarrow{\phi_\pi - \mathrm{id}} E_n^{h\mathbb{G}_n^1} \wedge X. \tag{1.5}$$

For this reason, the spectrum $E_n^{h\mathbb{G}_n^1}$ is often called the *half sphere*. One approach for computing $\pi_* L_{K(n)}X$ is to compute the spectral sequence

$$H_c^s(\mathbb{G}_n^1, (E_n)_t X) \cong H_c^s(\mathbb{S}_n^1, (E_n)_t X)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} \Longrightarrow \pi_{t-s} E_n^{h\mathbb{G}_n^1} \wedge X. \tag{1.6}$$

and then use the fiber sequence (1.5) to pass from $\pi_*(E_n^{h\mathbb{G}_n^1} \wedge X)$ to $\pi_* L_{K(n)}X$.

Because of these spectral sequences, the cohomology of closed subgroups G of $\mathbb{G}_n$ with coefficients in various $\mathbb{W}$-modules plays a central role in $K(n)$-local computations, in particular, in computing $\pi_* L_{K(n)}X$. In general, computing the E_2-terms and the differentials of descent spectral sequences of the form (1.3) is difficult and requires deep methods. The ultimate goal of this paper is to develop some machinery which can be used to compute the E_2-page of (1.6) when $n = 2$ and $p = 2$. These results will be used in [3] to compute the E_2-page of (1.6) when X is the mod 2 Moore spectrum $V(0)$.

1.2. **Existing computations.** Before stating the results of this paper, we give a brief overview of computations in the literature which are related to this problem.

The difficulty of computing the descent spectral sequence (1.4) varies with p and n. When $n = 1$, E_1 is p-complete K theory and the action of the Morava stabilizer group is given by the Adams operation. The spectral sequence (1.4) is thus well understood. When X is the sphere spectrum, it recovers the image of the j-homomorphism.

For $n \geq 3$, almost nothing is known. Further, it appears that computations will be extremely difficult, perhaps undoable. This leaves the case when $n = 2$.

Although all computations at chromatic level $n = 2$ are hard, the difficulty varies with the prime p. When $n = 2$ and $p \geq 5$, $\pi_* L_{K(2)}S$ was not computed using the descent spectral sequence (1.4). Shimumora and Yabe first computed $\pi_* L_2 S$ in [34] using the $E(2)$-Adams spectral sequence, which for any X is given by

$$\mathrm{Ext}^{s,t}_{E(2)_*E(2)}(E(2)_*, E(2)_*X) \Longrightarrow \pi_{t-s} L_2 X. \tag{1.7}$$

They computed the E_2-term using the chromatic spectral sequence (see Chapter 5 of [26]). Let S_2 denote the p-Sylow subgroup of the Morava stabilizer group $\mathbb{S}_2$. The input of the chromatic spectral sequence is the cohomology ring $H^*(S_2, \mathbb{F}_{p^2})$. This cohomology was computed by Ravenel in [24]. As opposed to the cases of $p = 2$ and $p = 3$, S_2 has finite cohomological dimension. For this reason, computations for $p \geq 5$ are relatively simpler than for $p = 2$ or $p = 3$. For $p \geq 5$, the spectral sequence (1.7) collapses and is too sparse for non-trivial extensions. Further,

$$L_{K(2)}S \simeq \mathrm{holim}_{j,k}\, L_2(M(p^j, v_1^k)),$$

for $M(p^j, v_1^k)$ generalized Moore spectra. Thus, the computation of $\pi_* L_{K(2)}S$ follows from Shimumora and Yabe's computation of $\pi_* L_2 S$. This was deduced by Behrens in his account and organization of Shimumora and Yabe's computation in

[4]. The relationship between (1.4) and (1.7) is given by the Morava change of rings theorem, which states that there is an isomorphism

$$\mathbb{W} \otimes_{\mathbb{Z}_{(p)}} \mathrm{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*, E(n)_*M(p^j, v_1^k)) \cong H^*(S_n, (E_n)_*M(p^j, v_1^k))$$

Further, for X finite, (1.4) can be realized as the following limit of spectral sequences

$$\lim_{j,k} \mathbb{W} \otimes_{\mathbb{Z}_{(p)}} \mathrm{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*, E(n)_*(M(p^j, v_1^k) \wedge X)) \Longrightarrow \mathbb{W} \otimes_{\mathbb{Z}_p} \pi_{t-s} L_{K(n)} X$$

(see [20]). However, the author does not know of a published computation of $\pi_* L_{K(2)} S$ using (1.4) directly for $p \geq 5$.

Because of the complexity of $H^*(S_2, \mathbb{F}_{p^2})$ when $p = 2$ and $p = 3$, using the spectral sequence (1.7) in these cases has not been successful. The first attempt at understanding the cohomology $H^*(S_2, \mathbb{F}_9)$ was made by Ravenel in [24]. Computations using the spectral sequence (1.7) were done by Shimomura in [31] and [33]. However, some mistakes were found in both computations. The corrected computation of $H^*(S_2, \mathbb{F}_9)$ is due to Henn in [17]. This was used to compute $\pi_* L_2 V(1)$ in [16], where $V(1)$ is the generalized Moore spectrum $M(3, v_1)$. However, in order to obtain results for the mod 3 Moore spectrum $V(0)$ and the sphere S, Goerss, Henn, Mahowald and Rezk have developed a new approach in [11]. Their idea is to use a certain finite resolution of the $K(2)$-local sphere called the *duality resolution*.

Finite resolutions of the $K(n)$-local sphere are studied extensively by Henn in [18]. The duality resolution is an example of such resolutions and is constructed in [11]. It comes in two flavors. The *algebraic* duality resolution is a finite resolution of the trivial $\mathbb{G}_2$-module $\mathbb{Z}_3$ by permutation modules induced from representations of finite subgroups G of $\mathbb{G}_2$. Its topological counterpart, the *topological* duality resolution, is a finite resolution of $E_2^{h\mathbb{G}_2}$ which realizes the algebraic duality resolution. It is composed of spectra of the form $\Sigma^k E_2^{hG}$. Using the algebraic duality resolution, Henn, Mahowald and Karamanov have computed $\pi_* L_{K(2)} V(0)$ in [19]. Although it is not in print, this method has also been used to understand (1.4) when X is the sphere spectrum S. Overall, the duality resolution approach has been extremely useful in understanding level two chromatic phenomena at the prime 3 (see [13], [12] and [14]).

The least studied and hardest problem is when $n = 2$ and $p = 2$. The literature does contain some results in this case. An associated graded for the cohomology $H^*(S_2, \mathbb{F}_4)$ appears in Ravenel in [24]. Further, Ravenel's computation contains an extensive amount of information on the multiplicative structure of this cohomology ring. Based on Ravenel's results, Shimomura and Wang have computed the E_2-page of the $E(2)$-Adams spectral sequence (1.7) for $X = V(0)$ and $X = S$ in [30] and [32] respectively. Neither paper computes the differentials in the spectral sequence (1.7). Both computations are hard to follow and verify. This beckons an alternate, more transparent approach.

The goal of this paper is to set the stage for chromatic level $n = 2$ computations at the prime $p = 2$ using (1.4) rather than (1.7). We use the duality resolution methods which were so successful at the prime $p = 3$. The central result of this paper is the construction of the algebraic duality resolution at the prime $p = 2$. The existence of such a resolution was conjectured by Mahowald using Shimomura and Wang's computations. Its construction is due to Goerss, Henn, Mahowald and Rezk, but is not in print. The complete construction is given here, together with an extensive description of the maps in this resolution. These descriptions will be used in future computations. In order to prove these results, it is necessary to describe the Morava stabilizer group $\mathbb{S}_2$ in great depth. This paper is thus a natural place to record results about the Morava stabilizer group which are interesting in themselves and may be useful in future computations. Many of these results will be used in [3] to compute $H^*(\mathbb{S}_2^1; (E_2)_*V(0))$.

1.3. **Statement of the results.** To state the main theorems, we must first define some finite subgroups of $\mathbb{S}_2$ at $p = 2$. The group $\mathbb{S}_2$ has a unique conjugacy class of maximal finite subgroups of order 24. Fix a representative and call it G_{24}. The group G_{24} isomorphic to the semi-direct product of a quaternion subgroup denoted Q_8 and a cyclic group of order 3 denoted C_3. That is

$$G_{24} \cong Q_8 \rtimes C_3.$$

Recall that $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is torsion free, any finite subgroup of $\mathbb{S}_2$ is contained in $\mathbb{S}_2^1$. However, there are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$. If π is a topological generator of $\mathbb{Z}_2$ in $\mathbb{S}_2$, the groups G_{24} and

$$G_{24}' := \pi G_{24} \pi^{-1}$$

are representatives for the distinct conjugacy classes. The group $\mathbb{S}_2$ also contains a central subgroup C_2 of order 2 generated by the automorphism $[-1](x)$ of the formal group law F_2 of $K(2)$. Therefore, $\mathbb{S}_2^1$ contains a cyclic subgroup

$$C_6 := C_2 \times C_3.$$

The group $\mathbb{S}_2^1$ is a profinite group and one can define the completed group ring

$$\mathbb{Z}_2[[\mathbb{S}_2^1]] = \lim_{i,j} \mathbb{Z}/(2^i)[\mathbb{S}_2^1/U_j],$$

where $\{U_j\}$ forms a system of open subgroups such that $\bigcap_j U_j = \{e\}$. For any finite subgroup G of $\mathbb{S}_2^1$, define

$$\mathbb{Z}_2[[\mathbb{S}_2^1/G]] := \mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[G]} \mathbb{Z}_2$$

The main result of this paper is the following theorem.

Theorem 1.8 (Goerss-Henn-Mahowald-Rezk, unpublished)**.** *Let $\mathbb{Z}_2$ be the trivial $\mathbb{S}_2^1$-module. There is an exact sequence of complete $\mathbb{S}_2^1$-modules*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

where

$$\mathscr{C}_p = \begin{cases} \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] & p = 0 \\ \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] & p = 1, 2 \\ \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']] & p = 3 \end{cases}.$$

The resolution of Theorem 1.8 is called the *algebraic duality resolution.* This name is justified by the fact that the exact sequence of Theorem 1.8 exhibits a certain twisted duality. To make this precise, let $\mathrm{Mod}(\mathbb{S}_2^1)$ denote the category of complete $\mathbb{S}_2^1$-modules. As above, let π be a topological generator of $\mathbb{Z}_2$ in $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$. For a module M in $\mathrm{Mod}(\mathbb{S}_2^1)$, let $c_\pi(M)$ denote the $\mathbb{S}_2^1$-module whose underlying $\mathbb{Z}_2$-module is equal to M, but whose $\mathbb{S}_2^1$-module structure is twisted by the element π.

Theorem 1.9 (Henn, Karamanov, Mahowald, unpublished)**.** *Let*

$$\mathscr{C}_p^* = \mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathscr{C}_p, \mathbb{Z}_2[[\mathbb{S}_2^1]]).$$

There is an isomorphism of $\mathbb{S}_2^1$-complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] & c_\pi(\mathscr{C}_3^*) \arrow[r, "\overline{\varepsilon}"] & \mathbb{Z}_2 \arrow[r] & 0.
\end{tikzcd}$$

One application of the algebraic duality resolution is given by the following theorem.

Theorem 1.10. *Let M be a finitely generated complete $\mathbb{S}_2^1$-module. There is a first quadrant spectral sequence,*

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathscr{C}_p, M) \Longrightarrow H_c^{p+q}(\mathbb{S}_2^1, M).$$

The differentials have degree

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

and

$$E_1^{p,q} \cong \begin{cases} H^q(G_{24}, M) & \text{if } p = 0; \\ H^q(C_6, M) & \text{if } p = 1, 2; \\ H^q(G'_{24}, M) & \text{if } p = 3. \end{cases} \tag{1.11}$$

The spectral sequence of Theorem 1.10 is called the algebraic duality resolution spectral sequence (ADRSS). Its computational appeal is twofold. The E_1-term of the ADRSS is composed of the cohomology of finite groups. Further, it collapses at the E_4-term.

The richest differentials are the d_1-differentials. These are induced by the maps of the exact sequence in Theorem 1.8. In order to compute the ADRSS with coefficients such as $(E_2)_*$ or $(E_2)_*V(0)$, it is necessary to have a thorough understanding of these maps. This is the content of the following theorem.

To state this result, choose a generator ω of C_3 and an element i in G_{24} such that G_{24} is generated by i and ω. That is,

$$G_{24} = \langle i, \omega \rangle .$$

Let $j = \omega i \omega^2$ and $k = \omega^2 j \omega$. The group $\mathbb{S}_2$ can be decomposed as a semi-direct product

$$\mathbb{S}_2 \cong K \rtimes G_{24}$$

for a Poincaré duality group K of dimension 4. Similarly, $\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}$ for a Poincaré duality group K^1 of dimension 3.

Theorem 1.12. *There is an element $\alpha \in K^1$ such that $\mathbb{S}_2$ is topologically generated by the elements π, α, i and ω. The group $\mathbb{S}_2^1$ is topologically generated by the elements α, i and ω.*

To state the next result, for any element τ in G_{24}, let

$$\alpha_\tau = [\tau, \alpha].$$

Recall that S_2^1 is the 2-Sylow subgroup of $\mathbb{S}_2^1$. The group S_2^1 admits a decreasing filtration, denoted $F_{n/2}S_2^1$ which will be defined in Section 2.2.

Theorem 1.13. *Let e be the canonical generator of $\mathbb{Z}_2[[\mathbb{S}_2^1]]$ and e_p be the canonical generator of $\mathscr{C}_p$. For a subgroup G of S_2^1, let IG be the kernel of the augmentation $\varepsilon : \mathbb{Z}_2[[G]] \to \mathbb{Z}_2$. The maps $\partial_p : \mathscr{C}_p \to \mathscr{C}_{p-1}$ of Theorem 1.8 can be chosen to satisfy*

$$\partial_1(e_1) = (e-\alpha)e_0$$

and

$$\partial_2(e_2) = \Theta e_1$$

for an element Θ in $\mathbb{Z}_2[[\mathbb{S}_2^1]]^{C_3}$ such that

$$\Theta \equiv e + \alpha + i + j + k - \alpha_i - \alpha_j - \alpha_k \mod \mathcal{J}$$

where

$$\mathcal{J} = (IF_{4/2}K^1, (IF_{3/2}K^1)(IS_2^1), (IK^1)^7, 2(IK^1)^3, 4IK^1, 8).$$

Further, there are isomorphisms of $\mathbb{S}_2^1$-modules $g_p : \mathscr{C}_p \to \mathscr{C}_p$ and differentials

$$\partial'_{p+1} : \mathscr{C}_{p+1} \to \mathscr{C}_p$$

such that

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "g_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "g_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "g_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "g_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial'_3"] & \mathscr{C}_2 \arrow[r, "\partial'_2"] & \mathscr{C}_1 \arrow[r, "\partial'_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_2 \arrow[r] & 0,
\end{tikzcd}$$

is an isomorphism of complexes. The map $\partial'_3 : \mathscr{C}_3 \to \mathscr{C}_2$ is given by

$$\partial'_3(e_3) = \pi(e+i+j+k)(e-\alpha^{-1})\pi^{-1}e_2.$$

Theorem 1.13 is the key to doing computations using the duality resolution spectral sequence. The most difficult part of Theorem 1.13 is giving a good estimate for $\partial_2 : \mathscr{C}_2 \to \mathscr{C}_1$. A painfully detailed description of the map ∂_2 is given in Section 3.5. Though such precision is not needed for the computations of [3], the hope is that it will be sufficient for *all* future computations using the duality resolution spectral sequence.

1.4. **Organization of the paper.** Section 2 is dedicated to the description of the Morava stabilizer group $\mathbb{S}_2$ at the prime 2. Although some of these results are true for other primes p and other chromatic level n, they are given here in the special case of $p = 2$ and $n = 2$. This allows us to make more precise statements. These results will be used in explicit computations in Section 3 and in [3]. A more general account of the structure of $\mathbb{S}_n$ can already be found in [11].

We begin by recalling the standard filtration on $\mathbb{S}_2$ and defining the norm. This allows us to define the unit norm subgroup $\mathbb{S}_2^1$. We describe what is known of the structure of the finite subgroups of $\mathbb{S}_2^1$ and their conjugacy classes. In particular, we give an explicit choice of maximal finite subgroup G_{24} in Lemma 2.18. We give an explicit set of topological generators for $\mathbb{S}_2$, proving Theorem 1.12. Finally, we introduce a subgroup K such that $\mathbb{S}_2 \cong K \rtimes G_{24}$. The computational significance of K is that it is a Poincaré duality group, hence its cohomology is finite dimensional, and almost exterior. We end this section by computing $H^*(K, \mathbb{F}_2)$ and $H^*(K, \mathbb{Z}_2)$. These results are due to Henn but are not published.

In Section 3, we introduce the finite resolution of the trivial $\mathbb{S}_2^1$-module $\mathbb{Z}_2$. This is the analogue of the finite resolution of the $\mathbb{G}_2^1$-module $\mathbb{Z}_3$ described in [11]. The construction of this resolution is due to Goerss, Henn, Mahowald and Rezk, but is not in print. We construct the algebraic duality resolution spectral sequence. We describe the duality properties of the resolution and give a proof of Theorem 1.9. We end this section by giving a detailed description of the maps in the resolution and prove Theorem 1.13.

Section 4 is an appendix which contains a brief overview of Lazard's theory of groups which are uniformly powerful. These results are used in Section 2.6 and are included here to clarify the exposition.

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2. The structure of the Morava stabilizer group

2.1. **A presentation of $\mathbb{S}_2$.** Let F_2 be the Honda formal group law of height 2 at the prime 2. It is the 2-typical formal group law over $\mathbb{F}_4$ specified by the 2-series

$$[2]_{F_2}(x) = x^4.$$

The ring of endomorphisms of F_2 is isomorphic to the maximal order $\mathcal{O}_2$ in the central division algebra $\mathbb{D}_2 = D(\mathbb{Q}_2, 1/2)$ of valuation $1/2$. We begin by describing this isomorphism. More details can be found in Chapter 1 of [6].

Let $\mathbb{W} = W(\mathbb{F}_4)$ denote the ring of Witt vectors on $\mathbb{F}_4$. The ring $\mathbb{W}$ is isomorphic to the ring of integers of the unique unramified degree 2 extension of $\mathbb{Q}_2$. It is a complete local ring with residue field $\mathbb{F}_4$. The Teichmüller character defines a group homomorphism

$$\tau : (\mathbb{F}_4)^{\times} \to \mathbb{W}^{\times}.$$

Let ω be a choice of primitive third root of unity in $\mathbb{F}_4^{\times}$, and identify ω with its Teichmüller lift $\tau(\omega)$. Given such a choice, there is an isomorphism

$$\mathbb{W} \cong \mathbb{Z}_2[\omega]/(1 + \omega + \omega^2).$$

The Galois group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ is generated by the Fröbenius σ. It is the $\mathbb{Z}_2$-linear automorphism of $\mathbb{W}$ determined by

$$\omega^{\sigma} = \omega^2.$$

The ring $\mathcal{O}_2$ is a non-commutative extension of $\mathbb{W}$

$$\mathcal{O}_2 \cong \mathbb{W}\langle S \rangle / (S^2 = 2, aS = Sa^{\sigma}),$$

for a in $\mathbb{W}$. Note that any element of $\mathcal{O}_2$ can be expressed uniquely as a linear combination $a + bS$ for a and b in $\mathbb{W}$. The division algebra $\mathbb{D}_2$ is given by

$$\mathbb{D}_2 \cong \mathcal{O}_2 \otimes_{\mathbb{Z}_2} \mathbb{Q}_2.$$

The 2-adic valuation v on $\mathbb{Q}_2$ extends uniquely to a valuation

$$v : \mathbb{D}_2 \to \frac{1}{2}\mathbb{Z} \tag{2.1}$$

in such a way that $v(S) = \frac{1}{2}$ and $\mathcal{O}_2 = \{x \in \mathbb{D} \mid v(x) \geq 0\}$. It follows that $\mathcal{O}_2^\times = \{x \in \mathbb{D} \mid v(x) > 0\}$. Therefore, any finite subgroup $G \subseteq \mathbb{D}_2^\times$ is contained in $\mathcal{O}_2^\times$.

Next, we describe the ring of endomorphisms of F_2, and give an isomorphism $\mathrm{End}(F_2) \cong \mathcal{O}_2$. A complete proof can be found in Appendix A2 of [26]. First, note that

$$\mathrm{End}(F_2) \subseteq \mathbb{F}_4[[x]].$$

To avoid confusion with the elements $\mathbb{W} \subseteq \mathcal{O}_2$, let $\zeta \in \mathbb{F}_4$ be a choice of primitive third root of unity in the field of coefficients $\mathbb{F}_4$.

Let $S(x)$ correspond to the endomorphism

$$S(x) = x^2,$$

so that

$$[2]_{F_2}(x) = x^4 = S(S(x)) = S^2(x).$$

Define

$$\omega^i(x) = \zeta^i x,$$

and $0(x) = 0$. Given an element $a \in \mathbb{W}$, one can write it uniquely as $a = \sum_{i=0}^{\infty} a_i 2^i$ where $a_i \in \mathbb{W}$ satisfies the equation

$$x^4 - x = 0$$

(that is, $a_i \in \{0, 1, \omega, \omega^2\}$.) Let $\gamma = a + bS$ be an element of $\mathcal{O}_2$. Let $a = \sum_{i \geq 0} a_{2i} 2^i$ and $b = \sum_{i \geq 0} a_{2i+1} 2^i$. Using the fact that $S^2 = 2$, the element γ can be expressed uniquely as a power series

$$\gamma = a_0 + 2a_2 + 4a_4 + \ldots + (a_1 + 2a_3 + 4a_6 + \ldots)S = \sum_{i \geq 0} a_i S^i.$$

One can show that

$$\gamma(x) = a_0(x) +_{F_2} a_1(x^2) +_{F_2} a_2(x^4) +_{F_2} \ldots +_{F_2} a_i(x^{2^i}) +_{F_2} \ldots$$

is a well-defined power series in $\mathbb{F}_4[[x]]$. This determines a ring isomorphism

$$\mathcal{O}_2 \to \mathrm{End}(F_2).$$

The Morava stabilizer group $\mathbb{S}_2$ is the group of automorphisms of F_2. Thus,

$$\mathbb{S}_2 \cong \mathcal{O}_2^\times.$$

Any element of $\mathbb{S}_2$ can be expressed uniquely as a linear combination $a + bS$ for a in $\mathbb{W}^\times$ and b in $\mathbb{W}$. One can show that the center of $\mathbb{S}_2$ is given by the Galois invariant elements in $\mathbb{W}^\times$,

$$Z(\mathbb{S}_2) \cong \mathbb{Z}_2^\times.$$

Further, the element ω in $\mathbb{W}^\times$ generates a cyclic group of order 3 in $\mathbb{S}_2$, denoted C_3. The reduction of $\mathbb{W}$ modulo 2 induces an isomorphism

$$C_3 \cong \mathbb{F}_4^\times.$$

The Galois group acts on $\mathbb{S}_2$ by

$$(a + bS)^\sigma = a^\sigma + b^\sigma S.$$

The extended Morava stabilizer group $\mathbb{G}_2$ is defined by

$$\mathbb{G}_2 := \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

2.2. **The filtration.** In what follows, we use the presentation of $\mathbb{S}_2$ induced by the isomorphism $\mathbb{S}_2 \cong \mathcal{O}_2^\times$ which was described in Section 2.1. That is,

$$\mathbb{S}_2 \cong \left(\mathbb{W}\langle S\rangle/(S^2 = 2, aS = Sa^\sigma)\right)^\times,$$

for a in $\mathbb{W}$. As described in Section 3 of [17], the group $\mathbb{S}_2$ admits the following filtration.

Recall that there is a valuation $v : \mathcal{O}_2 \to \frac{1}{2}\mathbb{Z}$ induced by (2.1), and that

$$v(S) = \frac{1}{2}.$$

Regard $\mathbb{S}_2$ as the units in $\mathcal{O}_2$. Define

$$F_{n/2}\mathbb{S}_2 = \{x \in \mathbb{S}_2 \mid v(x-1) \geq n/2\}.$$

This filtration corresponds to the filtration of $\mathbb{S}_2$ by powers of S, that is

$$F_{n/2}\mathbb{S}_2 = \{\gamma \in \mathbb{S}_2 \mid \gamma \equiv 1 \mod S^n\}. \tag{2.2}$$

Note that $\mathbb{Z}_2^\times \subseteq \mathbb{S}_2$. The motivation for indexing the filtration by half integers is that the induced filtration on $\mathbb{Z}_2^\times$ is the usual 2-adic filtration by powers of 2.

Let

$$\mathrm{gr}_{n/2}\,\mathbb{S}_2 := F_{n/2}\mathbb{S}_2/F_{(n+1)/2}\mathbb{S}_2,$$

and

$$\mathrm{gr}\,\mathbb{S}_2 = \bigoplus_{n\geq 0} \mathrm{gr}_{n/2}\,\mathbb{S}_2.$$

Define

$$S_2 := F_{1/2}\mathbb{S}_2.$$

The group S_2 is the 2-Sylow subgroup of $\mathbb{S}_2$. The map $\mathbb{S}_2 \to \mathbb{F}_4^\times$ which sends γ to a_0 has kernel S_2. It induces an isomorphism

$$\mathrm{gr}_{0/2}\,\mathbb{S}_2 \cong \mathbb{F}_4^\times.$$

Suppose that $n > 0$ and that γ is an element of $F_{n/2}\mathbb{S}_2$. Then

$$\gamma = 1 + a_n S^n + \ldots$$

for a_n in $\{0, 1, \omega, \omega^2\}$. Let $\overline{\gamma}$ denote the image of γ in $\mathrm{gr}_{n/2}\,\mathbb{S}_2$. The map defined by

$$\overline{\gamma} \mapsto a_n$$

gives a group isomorphism

$$\mathrm{gr}_{n/2}\,\mathbb{S}_2 \cong \mathbb{F}_4.$$

It follows from these observations that the subgroups $F_{n/2}\mathbb{S}_2$ form a system of open subgroups and $\mathbb{S}_2$ is a profinite topological group.

Given any subgroup G of $\mathbb{S}_2$, the filtration on $\mathbb{S}_2$ induces a filtration on G, defined by

$$F_{n/2}G = F_{n/2}\mathbb{S}_2 \cap G.$$

Let

$$\operatorname{gr} G = \bigoplus_{n \geq 0} \operatorname{gr}_{n/2} G \tag{2.3}$$

be the associated graded for this filtration.

The associated graded $\operatorname{gr} S_2$ has the structure of a restricted Lie algebra. The bracket is induced by the commutator in S_2 and the restriction is induced by squaring. In Lemma 3.1.4 of [17], Henn gives an explicit description of the structure of this Lie algebra. We record this result in the case when $p = 2$ and $n = 2$.

Lemma 2.4 (Henn)**.** *Let $[a, b]$ denote the commutator $aba^{-1}b^{-1}$ and $P(a) = a^2$. For $n, m > 0$, let $a \in F_{n/2}S_2$ and $b \in F_{m/2}S_2$. Let $\overline{a}$ be the image of a in $\operatorname{gr}_{n/2} S_2$, and $\overline{b}$ be the image of b in $\operatorname{gr}_{m/2} S_2$. Then*

$$\overline{[a, b]} \equiv \overline{a}\overline{b}^{2^n} + \overline{a}^{2^m}\overline{b} \ \in \ \operatorname{gr}_{(n+m)/2} S_2.$$

and

$$\overline{P(a)} \equiv \begin{cases} \overline{a}^3 & \in \ \operatorname{gr}_{2/2} S_2 \quad \text{if } n = 1; \\ \overline{a} + \overline{a}^2 & \in \ \operatorname{gr}_{4/2} S_2 \quad \text{if } n = 2; \\ \overline{a} & \in \ \operatorname{gr}_{n/2+1} S_2 \quad \text{if } n > 2. \end{cases}$$

2.3. **The norm and the subgroups $\mathbb{S}_2^1$ and $\mathbb{G}_2^1$.** The group $\mathbb{S}_2 \cong \mathcal{O}_2^{\times}$ acts on $\mathcal{O}_2$ by right multiplication. This gives rise to a representation $\rho : \mathbb{S}_2 \to GL_2(\mathbb{W})$, which can be described explicitly by

$$\rho(a + bS) = \begin{pmatrix} a & b \\ 2b^{\sigma} & a^{\sigma} \end{pmatrix}. \tag{2.5}$$

The restriction of the determinant to $\mathbb{S}_2$ is given by

$$\det(a + bS) = aa^{\sigma} - 2bb^{\sigma}. \tag{2.6}$$

This defines a group homomorphism $\det : \mathbb{S}_2 \to \mathbb{Z}_2^{\times}$.

Lemma 2.7. *The determinant* $\det : \mathbb{S}_2 \to \mathbb{Z}_2^{\times}$ *is split surjective.*

Before proving this lemma, we introduce elements of $\mathbb{S}_2$ that will play a key role in the remainder of this paper and in future computations. First, let

$$\pi := 1 + 2\omega. \tag{2.8}$$

By Hensel's lemma, $\mathbb{Z}_2$ contains two solutions of $f(x) = x^2 + 7$. One of them satisfies

$$\sqrt{-7} \equiv 1 + 4 \mod 8$$

This allows us to define

$$\alpha := \frac{1 - 2\omega}{\sqrt{-7}}. \tag{2.9}$$

Proof of Lemma 2.7. The group $\mathbb{Z}_2^{\times}$ is topologically generated by -1 and 3. It suffices to show that these values are in the image of the determinant. A direct computation shows that $\det(\pi) = 3$ and that $\det(\alpha) = -1$. Since α and π commute, they define a splitting. $\square$

Define the *norm* $N : \mathbb{S}_2 \to \mathbb{Z}_2^\times/\{\pm 1\}$ as the composite

$$\mathbb{S}_2 \xrightarrow{\det} \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\}.$$

For any prime p, define the group

$$U_i := \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \mod p^i\}.$$

At the prime 2, there is a canonical identification

$$\mathbb{Z}_2^\times \cong \{\pm 1\} \times U_2.$$

Therefore, the image of the norm is canonically isomorphic to the group U_2. Further, the group U_2 is non-canonically isomorphic to the additive group $\mathbb{Z}_2$.

The subgroup $\mathbb{S}_2^1$ is defined by the short exact sequence,

$$1 \to \mathbb{S}_2^1 \to \mathbb{S}_2 \xrightarrow{N} \mathbb{Z}_2^\times/\{\pm 1\} \to 1. \tag{2.10}$$

Any element γ such that $N(\gamma)$ is a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$ determines a splitting. The element π defined in (2.8) is an example. This gives a decomposition

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2. \tag{2.11}$$

The norm N extends to a homomorphism

$$N : \mathbb{G}_2 \to \mathbb{Z}_2^\times/\{\pm 1\} \times \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \to \mathbb{Z}_2^\times/\{\pm 1\},$$

where the second map is the projection. The subgroup $\mathbb{G}_2^1$ is the kernel of the extended norm and

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2. \tag{2.12}$$

However, note that there is no splitting which is equivariant under the action of the Galois group.

The filtration on $\mathbb{S}_2$ induces a filtration on $\mathbb{S}_2^1$ and

$$S_2^1 := F_{1/2}\mathbb{S}_2^1 \tag{2.13}$$

is the 2-Sylow subgroup of $\mathbb{S}_2^1$.

Remark 2.14. *Note that for odd primes p, there is a canonical isomorphism*

$$\mathbb{Z}_p^\times \cong C_{p-1} \times U_1,$$

where C_{p-1} is a cyclic group of order $p-1$. The exact sequence analogous to (2.10) is given by

$$1 \to \mathbb{S}_2^1 \to \mathbb{S}_2 \to \mathbb{Z}_p^\times/C_{p-1} \to 1.$$

Further, it has a central splitting. Therefore, when p is odd, the Morava stabilizer group is a product

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_p^\times/C_{p-1} \cong \mathbb{G}_2^1 \times \mathbb{Z}_p.$$

There is no central splitting at the prime $p = 2$ and the semi-direct products (2.11) and (2.12) are non-trivial. This is one of the reasons why the case $p = 2$ is more difficult.

2.4. **Conjugacy classes of finite subgroups.** In this section, we describe the finite subgroups of $\mathbb{S}_2$. The following result is stated as in [6], but a more extensive reference is given by [10].

Theorem 2.15 (Skolem-Noether)**.** *Let A be a finite dimensional central simple algebra over a field k. Let B be a simple k-subalgebra. If $\varphi : B \to A$ is a k-algebra homomorphism, then there exists $a \in A^{\times}$ such that $\varphi(b) = aba^{-1}$. In particular, any k-isomorphism between subalgebras of A can be realized by an inner automorphism of A, and there is an exact sequence*

$$1 \to C_A(B) \to N_A(B) \to \mathrm{Aut}(B/k) \to 1.$$

The following is a special case of Theorem 1.35 of [6].

Theorem 2.16. *The group $\mathbb{S}_2$ has a maximal finite 2-group isomorphic to a quaternion group Q_8. Any maximal finite subgroup of $\mathbb{S}_2$ is isomorphic to*

$$G_{24} := Q_8 \rtimes C_3.$$

The centralizer $C_{\mathbb{S}_2}(Q_8)$ is the center of $\mathbb{S}_2$, namely,

$$C_{\mathbb{S}_2}(Q_8) = \mathbb{Z}_2^{\times}.$$

Applying Theorem 2.15 to $A = \mathbb{D}_2$ and $B = \mathbb{Q}_2[Q_8]$ implies that the conjugacy class of Q_8 is unique in $\mathbb{D}_2$. More generally, it implies the following theorem, which is Corollary 1.30 of [6].

Theorem 2.17 (Bujard)**.** *Two finite subgroups of $\mathcal{O}_2^{\times}$ are conjugate if and only if they are isomorphic.*

It will be useful to have explicit choices of subgroups Q_8 and G_{24}. The proof of the following lemma is a direct computation.

Lemma 2.18 (Henn)**.** *Let*

$$i := \frac{1}{1+2\omega}(1-\alpha S).$$

Define $j = \omega i \omega^2$ and $k = \omega^2 i \omega = ij$. The elements i and j generate a quaternion subgroup of $\mathbb{S}_2$, denoted Q_8. Further,

$$G_{24} := Q_8 \rtimes C_3$$

is a representative for the unique conjugacy class of maximal finite subgroups.

Any automorphism of Q_8 induces a $\mathbb{Q}_2$-linear isomorphism of $\mathbb{Q}_2[Q_8]$. Therefore, $\mathrm{Aut}(Q_8)$ can be realized by inner conjugation in $\mathbb{D}_2$ (see [6] for a more detailed exposition). Proposition 2.19 describes which of these automorphisms can be realized by inner conjugation in $\mathbb{S}_2$.

Proposition 2.19 (Henn)**.** *Let Q_8 be the subgroup of $\mathbb{S}_2$ defined in Lemma 2.18. The automorphism group $\mathrm{Aut}(Q_8)$ is isomorphic to the symmetric group $\mathfrak{S}_4$ and can be realized by conjugation in $\mathbb{D}_2$. The subgroup of automorphisms which can be realized by conjugation by an element of $\mathbb{S}_2$ is isomorphic to the alternating group A_4. It is generated by conjugation by i, j and ω.*

Before proving Proposition 2.19, we describe the isomorphism $\mathrm{Aut}(Q_8) \cong \mathfrak{S}_4$. The description of this isomorphism is standard, but is included here for completeness. We do this by showing that there is a split short exact sequence

$$1 \to V \to \mathrm{Aut}(Q_8) \xrightarrow{\rho} \mathfrak{S}_3 \to 1, \tag{2.20}$$

and that the extension is isomorphic to $\mathfrak{S}_4$.

Let C_τ denote the cyclic group generated by an element τ. The map ρ is defined by the action of $\mathrm{Aut}(Q_8)$ on $\{C_i, C_j, C_{ij}\}$. An automorphism $\varphi \in \mathrm{Aut}(Q_8)$ is in $\ker(\rho)$ if and only if it satisfies $\varphi(\tau) = \pm\tau$ for $\tau \in \{i, j, ij\}$. The group of such automorphisms is generated by the automorphisms φ_i and φ_j in $\mathrm{Aut}(Q_8)$ defined by

$$\begin{aligned} &\varphi_i(i) = -i, && \varphi_i(j) = j, && \varphi_i(ij) = -ij \\ &\varphi_j(i) = i, && \varphi_j(j) = -j, && \varphi_i(ij) = -ij. \end{aligned}$$

Hence, $\ker(\rho)$ isomorphic to the Klein four group $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

Next, we define a splitting $s : \mathfrak{S}_3 \to \mathrm{Aut}(Q_8)$. The group $\mathfrak{S}_3$ is generated by the permutations σ_1 and σ_2 where

$$\begin{aligned} &\sigma_1(C_i) = C_j && \sigma_1(C_j) = C_i && \sigma_1(C_{ij}) = C_{ij} \\ &\sigma_2(C_i) = C_i && \sigma_2(C_j) = C_{ij} && \sigma_2(C_{ij}) = C_j. \end{aligned}$$

Let ϕ_1 and ϕ_2 in $\mathrm{Aut}(Q_8)$ be determined by

$$\phi_1(i) = j, \qquad \phi_1(j) = i, \qquad \phi_1(ij) = -ij \tag{2.21}$$

$$\phi_2(i) = -i, \qquad \phi_2(j) = ij \qquad \phi_2(ij) = j. \tag{2.22}$$

The splitting s is determined by $s(\sigma_1) = \phi_1$ and $s(\sigma_2) = \phi_2$. This shows that ρ is surjective, and that there is indeed a split short exact sequence (2.20). Hence,

$$\mathrm{Aut}(Q_8) \cong V \rtimes \mathfrak{S}_3. \tag{2.23}$$

A direct computation shows that

$$\begin{aligned} &\phi_1\varphi_i\phi_1^{-1} = \varphi_j && \phi_2\varphi_i\phi_2^{-1} = \varphi_i\varphi_j \\ &\phi_1\varphi_j\phi_1^{-1} = \varphi_i && \phi_2\varphi_j\phi_2^{-1} = \varphi_j. \end{aligned}$$

Let $\mathfrak{S}_4 = \mathrm{Perm}(\{1,2,3,4\})$. Let $\mathfrak{S}_3'$ be the subgroup $\mathrm{Perm}(\{1,2,3\})$ and V' be the subgroup $\{(), (12)(34), (13)(24), (14)(23)\}$. Then

$$\mathfrak{S}_4 \cong V' \rtimes \mathfrak{S}_3'. \tag{2.24}$$

Let $f : \mathfrak{S}_4 \to \mathrm{Aut}(Q_8)$ be given by

$$\begin{aligned} &f((14)(23)) = \varphi_i && f((12)) = \phi_1 \\ &f((13)(24)) = \varphi_j && f((13)) = \phi_2. \end{aligned}$$

A direct computation shows that f is an isomorphism of the extensions (2.23) and (2.24).

Proof of Proposition 2.19. By the Theorem 2.15, any element of $\mathrm{Aut}(Q_8)$ can be realized by conjugation by an element of $\mathbb{D}_2$. Conjugation by i, j and ω generate a subgroup isomorphic to the alternating group A_4. One must show that no other elements of $\mathrm{Aut}(Q_8)$ can be realized by conjugation with an element of $\mathbb{S}_2$. To prove this, it is sufficient to show that ϕ_1 and ϕ_2 defined by (2.21) and (2.22) cannot be realized by conjugation in $\mathbb{S}_2$.

Suppose that $g \in \mathbb{S}_2$ is such that $\phi_1(\tau) = g\tau g^{-1}$ for $\tau \in Q_8$. This implies that $gig^{-1} = j$. Modulo S^2, $i \equiv 1 + S$, $j \equiv 1 + \omega^2 S$ and $ij \equiv 1 + \omega S$. Let

$$g \equiv a_0 + a_1 S \mod S^2,$$

where $a_0 \in \{1, \omega, \omega^2\}$. Hence, $g^{-1} \equiv a_0^\sigma + a_1 S \mod S^2$. Then

$$gig^{-1} \equiv 1 + a_0^\sigma S$$

and

$$gijg^{-1} \equiv 1 + a_0^\sigma \omega S.$$

Since $\phi_1(i) = j$, it follows that $a_0 = \omega$. However, since $\phi_1(ij) = -ij$, it must be the case that $a_0 = 1$. This is a contradiction. Hence, ϕ_1 cannot be realized by conjugation in $\mathbb{S}_2$. The proof for ϕ_2 is similar. □

Lemma 2.25. *Let $G_{24} = Q_8 \rtimes C_3$. The normalizer of Q_8 in $\mathbb{S}_2$ is given by*

$$N_{\mathbb{S}_2}(Q_8) \cong U_2 \times G_{24}.$$

Proof. By Proposition 2.19, there is a short exact sequence

$$1 \to C_{\mathbb{S}_2}(Q_8) \to N_{\mathbb{S}_2}(Q_8) \to A_4 \to 1.$$

The centralizer is the subgroup $\mathbb{Z}_2^\times \cong C_2 \times U_2$ of $\mathbb{S}_2$. Since G_{24} is defined by the extension

$$1 \to C_2 \to G_{24} \to A_4 \to 1$$

and the elements of U_2 are in the centralizer of G_{24}, it follows that

$$N_{\mathbb{S}_2}(Q_8) \cong U_2 \times G_{24}.$$

□

Corollary 2.26. *The centralizer of Q_8 in $\mathbb{S}_2^1$ is*

$$C_{\mathbb{S}_2^1}(Q_8) \cong C_2$$

and its normalizer is

$$N_{\mathbb{S}_2^1}(Q_8) \cong G_{24}.$$

Since U_2 is torsion free and the norm is a group homomorphism, any finite subgroup $G \subseteq \mathbb{S}_2$ is contained in $\mathbb{S}_2^1$. It will be necessary to understand the conjugacy classes of maximal finite subgroups in the group $\mathbb{S}_2^1$.

Lemma 2.27. *There are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$. One is the conjugacy class of G_{24} defined in Lemma 2.18. The other is*

$$\xi G_{24} \xi^{-1},$$

where ξ is any element such that $N(\xi)$ is a topological generator of U_2.

Proof. Let $\mathbb{Z}_2^\times \subseteq \mathbb{S}_2$ be the center, and define

$$\mathbb{S}_2^0 := \mathbb{S}_2^1 \times \mathbb{Z}_2^\times.$$

The restriction of the determinant

$$\mathbb{Z}_2^\times \subseteq \mathbb{S}_2 \xrightarrow{\det} \mathbb{Z}_2^\times$$

surjects onto the $(\mathbb{Z}_2^\times)^2$. Therefore, there is an exact sequence

$$1 \to \mathbb{S}_2^0 \to \mathbb{S}_2 \to \mathbb{Z}_2^\times/(\{\pm 1\}, (\mathbb{Z}_2^\times)^2) \to 1,$$

and $\mathbb{S}_2/\mathbb{S}_2^0 \cong \mathbb{Z}/2$. If $N(\xi)$ is a topological generator for $\mathbb{Z}_2^\times/\{\pm 1\}$, then ξ is a representative for the non-trivial coset in $\mathbb{S}_2/\mathbb{S}_2^0$.

Using Theorem 2.17, there is a unique conjugacy class of subgroups isomorphic to $G_{24} \cong Q_8 \rtimes C_3$ in $\mathbb{S}_2$. Since conjugation by elements of the center $\mathbb{Z}_2^\times$ is trivial, any two conjugacy classes in $\mathbb{S}_2^1$ differ by conjugation by an element of $\mathbb{S}_2/\mathbb{S}_2^0 \cong \mathbb{Z}/2$. Therefore, there are at most 2 conjugacy classes.

Next, we show that the conjugacy classes of G_{24} and $\xi G_{24}\xi^{-1}$ are distinct in $\mathbb{S}_2^1$. Conjugation acts on the 2-Sylow subgroups; hence, it suffices to prove the claim for $Q_8 \subseteq G_{24}$. Suppose that there exists $\gamma \in \mathbb{S}_2^1$ such that

$$\xi Q_8 \xi^{-1} = \gamma Q_8 \gamma^{-1}.$$

This would imply that $\gamma^{-1}\xi \in N_{\mathbb{S}_2}(Q_8)$. By Lemma 2.25, $\gamma^{-1}\xi$ is a product $z\tau$ for $z \in U_2$ and $\tau \in G_{24}$. This implies that $\xi = \gamma z \tau$. However, $\gamma z \tau \in \mathbb{S}_2^0$. This is a contradiction, since the residue class of ξ in $\mathbb{S}_2/\mathbb{S}_2^0$ is non-trivial. Therefore, G_{24} and $\xi G_{24}\xi^{-1}$ represent distinct conjugacy classes in $\mathbb{S}_2^1$. $\square$

A choice for ξ is the element π defined in (2.8). For the remainder of this paper, G_{24}' will denote

$$G_{24}' := \pi G_{24} \pi^{-1}, \tag{2.28}$$

so that G_{24} and G_{24}' are representatives for the two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$.

2.5. **Finitely generated dense subgroups.** For the trivial modules $\mathbb{F}_2$ and $\mathbb{Z}_2$, there are isomorphisms

$$H^1(G, \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2}(G/[G,G], \mathbb{Z}_2)$$

and

$$H^1(G, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathbb{F}_2}(G/([G,G], G^2), \mathbb{F}_2).$$

Therefore, to understand the first cohomology group, it suffices to understand $[G,G]$ and G^2. For this reason, it will be important to know the decomposition of elements of $\mathbb{S}_2$ as squares and as commutators.

Figure 1 shows the decomposition of elements in the associated graded $\mathrm{gr}\, G$ as commutators. Figure 2 shows their decomposition as squares. In the column corresponding to the group G, the elements in row $n/2$ correspond to elements of $F_{n/2}G$ which project to a basis of $\mathrm{gr}_{n/2}\, G$ as an $\mathbb{F}_2$-vector space. Both figures are obtained using Lemma 2.4. The group K is the Poincaré duality subgroup and K^1 is the subgroup of elements of norm one in K. They will be described in Section 2.6.

Theorem 2.29 makes the results in Figure 1 and Figure 2 precise. For a set of elements $\{a_i\}_{i\in I} \subseteq \mathbb{S}_2$, let $\langle a_i \rangle_{i\in I}$ denote the subgroup generated by the elements a_i. Recall that G_{24} is a fixed maximal finite subgroup of $\mathbb{S}_2$, as defined in Lemma 2.18, and that

$$G_{24} = \langle i, \omega \rangle.$$

For $\gamma, \tau \in \mathbb{S}_2$, define

$$\gamma_\tau := [\tau, \gamma] = \tau\gamma\tau^{-1}\gamma^{-1}.$$

$n/2$	S_2	S_2^1	K	K^1
$1/2$	$i,\ j$	$i,\ j$		
$2/2$	$-1 = i_j,\ \varepsilon$	$-1 = i_j,\ \varepsilon$	ε	ε
$3/2$	$\varepsilon_i,\ \varepsilon_j$	$\varepsilon_i,\ \varepsilon_j$	$\varepsilon_i,\ \varepsilon_j$	$\varepsilon_i,\ \varepsilon_j$
$4/2$	$(\varepsilon_i)_j,\ \vartheta$	$(\varepsilon_i)_j$	$\varepsilon^2,\ \vartheta$	ε^2
$(2m+5)/2$	$\varepsilon_{i,m+1},\ \varepsilon_{j,m+1}$	$\varepsilon_{i,m+1},\ \varepsilon_{j,m+1}$	$\varepsilon_{i,m+1},\ \varepsilon_{j,m+1}$	$\varepsilon_{i,m+1},\ \varepsilon_{j,m+1}$
$(2m+6)/2$	$(\varepsilon_{i,m})_{\varepsilon_j},\ \vartheta^{2^{m+1}}$	$(\varepsilon_{i,m})_{\varepsilon_j}$	$(\varepsilon_{i,m})_{\varepsilon_j},\ \vartheta^{2^{m+1}}$	$(\varepsilon_{i,m})_{\varepsilon_j}$

FIGURE 1. Decomposition as commutators, where $\varepsilon_\tau = [\tau, \varepsilon]$ and $\varepsilon_{\tau,n}$ is defined inductively by $\varepsilon_{\tau,0} = \varepsilon_\tau$ and $\varepsilon_{\tau,n} = [\varepsilon, \varepsilon_{\tau,n-1}]$.

$n/2$	S_2	S_2^1	K	K^1
$1/2$	$i,\ j$	$i,\ j$		
$2/2$	$-1,\ \varepsilon$	$-1,\ \varepsilon$	ε	ε
$3/2$	$\varepsilon_i,\ \varepsilon_j$	$\varepsilon_i,\ \varepsilon_j$	$\varepsilon_i,\ \varepsilon_j$	$\varepsilon_i,\ \varepsilon_j$
$4/2$	$\varepsilon^2,\ \vartheta$	ε^2	$\varepsilon^2,\ \vartheta$	ε^2
$5/2$	$\varepsilon_i^2,\ \varepsilon_j^2$	$\varepsilon_i^2,\ \varepsilon_j^2$	$\varepsilon_i^2,\ \varepsilon_j^2$	$\varepsilon_i^2,\ \varepsilon_j^2$
$(2m)/2$	$\varepsilon^{2^{m-1}},\ \vartheta^{2^{m-2}}$	$\varepsilon^{2^{m-1}}$	$\varepsilon^{2^{m-1}},\ \vartheta^{2^{m-2}}$	$\varepsilon^{2^{m-1}}$
$(2m+1)/2$	$\varepsilon_i^{2^{m-1}},\ \varepsilon_j^{2^{m-1}}$	$\varepsilon_i^{2^{m-1}},\ \varepsilon_j^{2^{m-1}}$	$\varepsilon_i^{2^{m-1}},\ \varepsilon_j^{2^{m-1}}$	$\varepsilon_i^{2^{m-1}},\ \varepsilon_j^{2^{m-1}}$

FIGURE 2. Decomposition as squares, where $\varepsilon_\tau = [\tau, \varepsilon]$.

Theorem 2.29. *Let ε in $\mathbb{S}_2$ be such that $\det(\varepsilon) = -1$. Let $\vartheta \in F_{4/2}\mathbb{S}_2$ be such that $\det(\vartheta)$ is a topological generator of U_2. The subgroup*

$$\langle \omega, i, \varepsilon, \vartheta \rangle \tag{2.30}$$

is dense in $\mathbb{S}_2$, and the subgroup

$$\left\langle \varepsilon_i, \varepsilon_j, \varepsilon^2, \vartheta \right\rangle \tag{2.31}$$

is dense in $F_{3/2}\mathbb{S}_2$. Similarly, the subgroup

$$\langle \omega, i, \varepsilon \rangle$$

is dense in $\mathbb{S}_2^1$ and the subgroup

$$\left\langle \varepsilon_i, \varepsilon_j, \varepsilon^2 \right\rangle$$

is dense in $F_{3/2}\mathbb{S}_2^1$.

Remark 2.32. *Explicit choices of topological generators are given by $\varepsilon = \alpha$ and $\vartheta = \alpha\pi$. In what follows, we will use these elements when applying Theorem 2.29.*

Proof of Theorem 2.29. In order to prove the first statement, it suffices to show that $\langle \omega, i, \varepsilon, \vartheta \rangle$ surjects onto the finite groups

$$\mathbb{S}_2/F_{n/2}\mathbb{S}_2$$

for each n. This is equivalent to showing that it contains a subgroup which surjects onto $\mathrm{gr}_{n/2}\,\mathbb{S}_2$ for each n.

The quaternion elements i and j are in $F_{1/2}\mathbb{S}_2$. Their images in $\mathrm{gr}_{1/2}\,\mathbb{S}_2$ form an $\mathbb{F}_2$-basis. Since $\det(\varepsilon) = -1$, a direct computation shows that $\varepsilon \equiv 1 + 2\omega^s \mod S^3$ for $s = 1$ or $s = 2$. Since, $-\varepsilon = 1 + 2\omega^{2s} \mod S^3$, the elements of G_{24} together with ε generate a subgroup which surjects onto $\mathbb{S}_2/F_{3/2}\mathbb{S}_2$. The elements $\overline{\varepsilon_i}$ and $\overline{\varepsilon_j}$ are non-zero classes in $\mathrm{gr}_{3/2}\,\mathbb{S}_2$ and $\overline{\varepsilon^2} = 1$ in $\mathrm{gr}_{4/2}\,\mathbb{S}_2$. Since $\vartheta \in F_{4/2}\mathbb{S}_2$ and $\det(\vartheta)$ generates U_2, it must the case that $\vartheta \equiv 1 + 4\omega^s \mod S^5$ for $s = 1$ or $s = 2$. Under the power map P, the elements $\varepsilon_i, \varepsilon_j, \varepsilon^2$ and ϑ detect all non-zero classes in higher filtrations. Therefore, the subgroup generated by these elements is dense in $F_{3/2}\mathbb{S}_2$.

It remains to prove the statements concerning $\mathbb{S}_2^1$. The element ε has determinant -1, so it is in $\mathbb{S}_2^1$. Therefore,

$$\mathbb{S}_2/F_{3/2}\mathbb{S}_2 \cong \mathbb{S}_2^1/F_{3/2}\mathbb{S}_2^1,$$

generated by the image of G_{24} and ε. To finish the proof, it suffices to show that $\langle \varepsilon_i, \varepsilon_j, \varepsilon^2 \rangle$ is dense in $F_{3/2}\mathbb{S}_2^1$.

We define a filtration on $\mathbb{Z}_2^\times$ which makes the determinant $\det : \mathbb{S}_2 \to \mathbb{Z}_2^\times$ a filtration preserving homomorphism. Let

$$F_{0/2}\mathbb{Z}_2^\times := \mathbb{Z}_2^\times,$$

and for $n > 0$ even,

$$F_{n/2}\mathbb{Z}_2^\times = F_{(n-1)/2}\mathbb{Z}_2^\times := U_{n/2} = \{\gamma \mid \gamma \equiv 1 \mod 2^{n/2}\}.$$

With these definitions, the determinant preserves the filtration. Indeed, let γ be in $\mathbb{S}_2$. Let $n > 0$ be even and suppose that γ has an expansion of the form

$$\gamma = 1 + a_{n-1}S^{n-1} + a_n S^n \mod S^{n+1}.$$

By (2.6),

$$\det(\gamma) \equiv 1 + 2^{n/2}(a_n + a_n^\sigma) + a_{n-1}a_{n-1}^\sigma 2^{n-1} \mod 2^{n/2+1},$$

which is in $F_{n/2}\mathbb{Z}_2^\times$.

Note that there is a short exact sequence

$$1 \to F_{3/2}\mathbb{S}_2^1 \to F_{3/2}\mathbb{S}_2 \xrightarrow{\det} F_{3/2}\mathbb{Z}_2^\times \cong U_2 \to 1.$$

This induces short exact sequences of $\mathbb{F}_2$-vector spaces,

$$0 \to \mathrm{gr}_{n/2}\,\mathbb{S}_2^1 \to \mathrm{gr}_{n/2}\,\mathbb{S}_2 \to \mathrm{gr}_{n/2}\,\mathbb{Z}_2^\times \to 0.$$

Because $\mathrm{gr}_{n/2}\,\mathbb{Z}_2^\times = 0$ for n odd and $\mathbb{F}_2$ for n even and greater than zero, this implies that

$$\mathrm{gr}_{n/2}\,\mathbb{S}_2^1 = \begin{cases} \mathbb{F}_2 & \text{if } n > 0 \text{ is even;} \\ \mathbb{F}_4 & \text{if } n \text{ is odd.} \end{cases}$$

Let $G := \langle \varepsilon_i, \varepsilon_j, \varepsilon^2 \rangle$ with the induced filtration. For $n \geq 3$, the map

$$\mathrm{gr}_{n/2}\,G \to \mathrm{gr}_{n/2}\,\mathbb{S}_2^1$$

is injective. Using Lemma 2.4, one deduces that it is surjective for dimension reasons. □

The conjugation action of $\mathbb{S}_2$ on itself preserves the filtration and, for C_3, it is described by the following Lemma.

Lemma 2.33. *Let $C_3 \cong \mathbb{F}_4^\times$ act on $\mathbb{S}_2$ by conjugation. The induced action on $\mathrm{gr}_{n/2}\,\mathbb{S}_2$ is the natural right action of $\mathbb{F}_4^\times$ on $\mathbb{F}_4$ if n is odd. It is trivial if n is even.*

Proof. This is a direct computation given by

$$\omega(1+a_nS^n)\omega^{-1} = \begin{cases} 1+a_n\omega^{-1}S^n & \text{if } n \text{ is odd;} \\ 1+a_nS^n & \text{if } n \text{ is even.} \end{cases}$$

□

2.6. **The Poincaré duality subgroups K and K^1.** Let K be the closure of the subgroup generated by α and $F_{3/2}\mathbb{S}_2$. That is

$$K = \overline{\langle \alpha, F_{3/2}\mathbb{S}_2 \rangle}.$$

Let K^1 be the closure of the subgroup generated by α and $F_{3/2}\mathbb{S}_2^1$. Equivalently, K^1 is the kernel of the norm restricted to K. It follows from Theorem 2.29 that K is normal, and that there is an exact sequence

$$1 \to K \to \mathbb{S}_2 \to G_{24} \to 1.$$

Further, Lemma 2.18 gives a choice of splitting so that

$$\mathbb{S}_2 \cong K \rtimes G_{24}. \tag{2.34}$$

Similarly,

$$\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}.$$

In this section, we give the computation of the ring $H^*(K, \mathbb{F}_2)$ and $H^*(K^1, \mathbb{F}_2)$ as G_{24}-modules. The author has learned these results from Paul Goerss and Hans-Werner Henn. Some details in the proof of Theorem 2.41 given here are due to Dylan Wilson. To solve the algebraic extensions, we will use the following classical result.

Lemma 2.35. *Suppose that $H_1(G, \mathbb{Z}_2) \cong G/[G,G]$ is a finitely generated 2-group. Suppose that the residue class of an element g in $G/[G,G]$ generates a summand isomorphic to $\mathbb{Z}/2^k$. Let $x \in H^1(G, \mathbb{Z}/2) \cong \mathrm{Hom}(G, \mathbb{Z}/2)$ be the homomorphism dual to g. Then $x^2 \neq 0$ in $H^2(G, \mathbb{Z}/2)$ if and only if $k = 1$.*

Proof. This follows from the fact that $x \in H^1(G, \mathbb{Z}/2)$ has a non-zero Bockstein in the long exact sequence associated to the extension of trivial modules

$$1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$$

if and only if g generates a $\mathbb{Z}/2$ summand. □

For $0 \leq s \leq 4$, let α_s be defined by

$$\alpha_0 = \alpha, \quad \alpha_1 = \alpha_i = [i, \alpha], \quad \alpha_2 = \alpha_j = [j, \alpha], \quad \alpha_3 = \alpha^2, \quad \alpha_4 = \alpha\pi. \tag{2.36}$$

In $\mathrm{Hom}_{\mathbb{F}_2}(\mathrm{gr}\, \mathbb{S}_2, \mathbb{F}_2)$, let x_s be the function dual to the image of α_s in $\mathrm{gr}\, \mathbb{S}_2$. The conjugation action of an element τ on an element g is denote by

$$\tau_*(g).$$

It can be computed using Lemma 2.4 and the fact that

$$[\tau, \gamma]\gamma = \tau_*(\gamma).$$

Lemma 2.37. *There is an isomorphism*

$$H_1(K, \mathbb{Z}_2) \cong \mathbb{Z}/4\{\overline{\alpha}_0\} \oplus \mathbb{Z}/2\{\overline{\alpha}_1, \overline{\alpha}_2\} \oplus \mathbb{Z}_2\{\overline{\alpha}_4\},$$

where $2\alpha_0$ is the reduction of $\alpha_3 = \alpha^2$. The conjugation action of Q_8 on K factors through the quotient of Q_8 by the central subgroup C_2. The induced action on $H_1(K, \mathbb{Z}_2)$ is trivial on α_4, and is given by

$$\begin{aligned} i_*(\overline{\alpha}_0) &= \overline{\alpha}_0 + \overline{\alpha}_1 & j_*(\overline{\alpha}_0) &= \overline{\alpha}_0 + \overline{\alpha}_2 \\ i_*(\overline{\alpha}_1) &= \overline{\alpha}_1 & j_*(\overline{\alpha}_1) &= \overline{\alpha}_1 + 2\overline{\alpha}_0 \\ i_*(\overline{\alpha}_2) &= \overline{\alpha}_2 + 2\overline{\alpha}_0 & j_*(\overline{\alpha}_2) &= \overline{\alpha}_2. \end{aligned}$$

Proof. The computation of $H_1(K, \mathbb{Z}_2) = K/[K, K]$ and of the action of Q_8 are direct applications of Lemma 2.4. □

The open subgroup $F_{3/2}S_2$ has particularly nice cohomological properties. The group $F_{3/2}S_2$ is *uniformly powerful* (see Definition 4.5). This implies that $F_{3/2}S_2$ is a Poincaré duality group, and that its cohomology is an exterior algebra. This fact is the main input of the cohomological calculations which follow. The proof of Theorem 2.39 uses Lazard's theory of uniformly powerful groups, which is described in [22]. Some of Lazard's results can also be found in [35]. We have added an appendix containing a brief overview of the relevant results for completeness.

Recall the following definition from [28].

Definition 2.38. *Let G be a profinite p-group. Then G is a Poincaré duality group of dimension n if G has cohomological dimension n and*

$$H^s_c(G, \mathbb{Z}_p[[G]]) = \begin{cases} \mathbb{Z}_p & s = n \\ 0 & s \neq n. \end{cases}$$

Theorem 2.39. *The group $F_{3/2}S_2$ is a Poincaré duality group of dimension 4. The continuous group cohomology $H^*_c(F_{3/2}S_2, \mathbb{F}_2)$ is the exterior algebra generated by*

$$H^1_c(F_{3/2}S_2, \mathbb{F}_2) \cong \mathbb{F}_2\{x_1, x_2, x_3, x_4\},$$

where x_s is the function dual to the image of α_s in $\operatorname{gr} \mathbb{S}_2$ defined in (2.36). The action of α on $H^1_c(F_{3/2}S_2, \mathbb{F}_2)$ is trivial.

Similarly, $F_{3/2}S^1_2$ is a Poincaré duality group of dimension 3 and $H^*_c(F_{3/2}S^1_2, \mathbb{F}_2)$ is the exterior algebra generated by

$$H^1_c(F_{3/2}S^1_2, \mathbb{F}_2) \cong \mathbb{F}_2\{x_1, x_2, x_3\}$$

with a trivial action of α.

Proof. Let $w(x) = \max\{\frac{n}{2} \mid n \text{ odd}, \ x \in F_{n/2}S_2\}$ for $x \in S_2$. This defines a decreasing filtration on $F_{3/2}S_2$. With this filtration, $F_{3/2}S_2$ is uniformly powerful of rank 4 as defined in Definition 4.5, generated by $\operatorname{gr}_{3/2} S_2 \oplus \operatorname{gr}_{4/2} S_2$. This can be shown using Lemma 2.4. The result then follows from Theorem 5.1.5 of [35], which is also Proposition 4.6 of the appendix. The action of α and Q_8 is computed using the formulas in Lemma 2.4.

To prove the second claim note that, with the same filtration, $F_{3/2}S^1_2$ is uniformly powerful of rank 3. □

Corollary 2.40. *The group K is a Poincaré duality group of dimension 4 and the group K^1 is a Poincaré duality group of dimension 3.*

Proof. Serre proved in [29] that the cohomological dimension of a p-torsion free profinite group G is equal to the cohomological dimension of any of its open subgroups. Since K is torsion free and $F_{3/2}S_2$ is an open subgroup, the cohomological dimension of K is equal to the cohomological dimension of $F_{3/2}S_2$. Hence, K has cohomological dimension 4. Similarly, K^1 has cohomological dimension 3 since it contains $F_{3/2}S_2^1$ as an open subgroup. By Proposition 4.4.1 of [35], a profinite group G of finite cohomological dimension is a Poincaré duality group if and only if it contains an open subgroup which is a Poincaré duality group. Therefore, both K and K^1 are Poincaré duality groups. $\square$

In Section 3, we will prove that this implies that the groups K^1 and K are Poincaré duality groups.

Theorem 2.41 (Goerss-Henn, unpublished)**.** *As an $\mathbb{F}_2$-algebra,*

$$H^*(K,\mathbb{F}_2) \cong \mathbb{F}_2[x_0,x_1,x_2,x_4]/(x_0^2, x_1^2+x_0x_1, x_2^2+x_0x_2, x_4^2),$$

where x_s has degree one and is given by the function dual to the image of α_s in $\operatorname{gr}\mathbb{S}_2$ defined in (2.36). Further,

$$H^*(K^1,\mathbb{F}_2) \cong H^*(K,\mathbb{F}_2)/(x_4).$$

The conjugation action of Q_8 factors through $Q_8/C_2 \cong C_2\times C_2$. It is trivial on x_0 and x_4. On x_1 and x_2, it is described by

$$\begin{aligned} i_*(x_1) &= x_0+x_1 & \qquad j_*(x_1) &= x_1 \\ i_*(x_2) &= x_2 & \qquad j_*(x_2) &= x_0+x_2. \end{aligned}$$

so that the induced representation on $H^1(K^1,\mathbb{F}_2)$ is isomorphic to the augmentation ideal $I(Q_8/C_2)$, and $H^2(K^1,\mathbb{F}_2)$ is isomorphic to the co-augmentation ideal $I(Q_8/C_2)^*$.

Proof. The spectral sequence for the group extension

$$1 \to F_{3/2}\mathbb{S}_2 \to K \to \mathbb{Z}/2\{\overline{\alpha}_0\} \to 1$$

has E_2-page given by

$$\mathbb{F}_2[x_0]\otimes E(x_1,x_2,x_3,x_4).$$

It follows from Lemma 2.35 that $x_0^2=0$. Since x_3 is the function dual to the image of α^2 in $\operatorname{gr}\mathbb{S}_2$, we have that $d_2(x_3)=x_0^2$. Using the isomorphism $H^1(K,\mathbb{F}_2)\cong \operatorname{Hom}(K,\mathbb{F}_2)$ and Lemma 2.37, one computes that

$$H^1(K,\mathbb{F}_2)\cong \mathbb{F}_2\{x_0,x_1,x_2,x_4\}.$$

Hence, $d_r(x_i)=0$ for $i\neq 3$. All other differentials are determined by these differentials and

$$E_3 \cong E_\infty \cong E(x_0,x_1,x_2,x_4).$$

Similarly, the E_2-page for the extension

$$1 \to F_{3/2}\mathbb{S}_2^1 \to K^1 \to \mathbb{Z}/2\{\overline{\alpha}_0\} \to 1$$

is given by $\mathbb{F}_2[x_0]\otimes E(x_1,x_2,x_3)$, and

$$E_3 \cong E_\infty \cong E(x_0,x_1,x_2).$$

To determine the multiplicative extensions, note that it follows from Lemma 2.35 that $x_4^2=0$, and that x_1^2 and x_2^2 are non-zero. The only possibility is that x_1^2 and

x_2^2 are linear combinations of x_0x_1 and x_0x_2. We will show that $x_1^2 = x_0x_1$. The proof that $x_2^2 = x_0x_2$ is similar.

Let N be the closure of the normal subgroup generated by α_2. Consider the group K^1/N. One can verify using Lemma 2.4 that

$$H_1(K^1/N, \mathbb{Z}_2) \cong \mathbb{Z}/2\{\overline{\alpha}_0\} \oplus \mathbb{Z}/2\{\overline{\alpha}_1\},$$

noting in particular that $[\alpha_i, \alpha_j] \equiv \alpha^2 \mod S^5$. Therefore, x_0 and x_1 are detected in $H^*(K^1/N, \mathbb{F}_2)$. The filtration on K^1 induces a filtration on K^1/N. The group $F_{3/2}K^1/N$ is uniformly powerful. Further, $H^*(F_{3/2}K^1/N, \mathbb{F}_2) = E(x_1)$ with trivial action of α_0. The spectral sequence for the extension

$$1 \to F_{3/2}K^1/N \to K^1/N \to K^1/N/F_{3/2}K^1/N \to 1$$

has E_2-page $\mathbb{F}_2[x_0] \otimes E(x_1)$, with x_0 in degree $(1,0)$ and x_1 in degree $(0,1)$. It must collapse since x_0 and x_1 are non-trivial elements of $H^1(K^1/N, \mathbb{F}_2)$. By Lemma 2.35, there must be a non-trivial extension, $x_1^2 = b_0x_0^2 + b_1x_0x_1$. The image of x_1^2 under the natural map $H^*(K^1/N, \mathbb{F}_2) \to H^*(K^1, \mathbb{F}_2)$ must be non-zero. Since x_0^2 maps to zero, it must be the case that $b_1 \neq 0$, so that $x_1^2 = x_0x_1$ in $H^*(K^1, \mathbb{F}_2)$.

The action of Q_8 is a direct computation using Lemma 2.4. The isomorphism between $H^1(K^1, \mathbb{F}_2)$ and $I(Q_8/C_2)$, defined by

$$0 \to I(Q_8/C_2) \to \mathbb{F}_2[Q_8/C_2] \xrightarrow{\epsilon} \mathbb{F}_2 \to 0,$$

is given by sending x_0 to the invariant $e+i+j+ij$, x_1 to $e+j$ and x_2 to $e+i$. $\square$

In what follows, we will need some information about the integral homology of K^1, which is given by the following lemma.

Corollary 2.42 (Goerss-Henn, unpublished)**.** *The integral homology of K^1 is given by*

$$H_n(K^1, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & n = 0, 3; \\ \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2 & n = 1; \\ 0 & n = 2. \end{cases}$$

Further, $H_1(K^1, \mathbb{Z}_2) \cong K^1/[K^1, K^1]$ is generated by the image of α as a G_{24}-module.

Proof. The result for $n = 1$ is Lemma 2.37. The higher homology groups $H_n(K, \mathbb{F}_2)$ are forced by $H^*(K^1, \mathbb{F}_2)$. The groups $H_n(K^1, \mathbb{Z}_2)$ for $n = 2, 3$ are computed from the long exact sequence associated to

$$0 \to \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \to \mathbb{F}_2 \to 0,$$

using the fact that $H_n(K^1, \mathbb{F}_2)$ and $H_1(K^1, \mathbb{Z}_2)$ are known. $\square$

Corollary 2.43. *The integral homology of K^1 is given by*

$$H^n(K^1, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & n = 0, 3; \\ 0 & n = 1; \\ \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2 & n = 2. \end{cases}$$

Proof. This follows immediately from Corollary 2.42 and the universal coefficient theorem. $\square$

3. The algebraic duality resolution

This section is devoted to the construction of the algebraic duality resolution. Many of the results in this section are due to Goerss, Henn, Mahowald and Rezk, which we will abbreviate by *G-H-M-R*.

3.1. Preliminaries on complete modules.

In order to prove Theorem 1.8, we will need some results about complete modules, which are presented here. The results in this section were communicated to the author by G-H-M-R. In what follows, all G-modules are assumed to be finitely generated. In this section, we let p be an arbitrary prime.

Let G be a profinite group and $\{U_k\}$ be a system of open normal subgroups of G such that $\bigcap_k U_k = \{e\}$. Define,

$$\mathbb{Z}_p[[G]] := \lim_{n,k} \mathbb{Z}_p/(p^n)[G/U_k].$$

A left G-module M is *complete* if

$$M \cong \lim_{n,k} \mathbb{Z}_p/(p^n)[G/U_k] \otimes_{\mathbb{Z}_p[[G]]} M.$$

Let $\mathbb{Z}_p$ be the trivial G-module and H be a finite subgroup of G. The permutation modules

$$\mathbb{Z}_p[[G/H]] := \mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[H]} \mathbb{Z}_p,$$

are complete.

For a complete G-module M, define the $\mathbb{Z}_p[[G]]$-dual of M by

$$M^* := \mathrm{Hom}^c_{\mathbb{Z}_p[[G]]}(M, \mathbb{Z}_p[[G]]), \tag{3.1}$$

where these are the continuous homomorphisms. For $\phi \in M^*$, define

$$(g\phi)(m) = \phi(m)g^{-1}.$$

This gives M^* the structure of a complete left G-module. Let $H \subseteq G$ be a finite subgroup and let $[g]$ denote the coset gH. There is a canonical isomorphism

$$t : \mathbb{Z}_p[[G/H]] \to \mathbb{Z}_p[[G/H]]^* \tag{3.2}$$

which sends $[g]$ to the map $[g]^* : \mathbb{Z}_p[[G/H]] \to \mathbb{Z}_p[[G]]$ defined by

$$[g]^*([x]) = x \sum_{h \in H} hg^{-1}.$$

We refer the reader to Section 3.4 of [19] for a detailed discussion of $\mathbb{Z}_p[[G]]$-duals.

The following result is Lemma 4.3 of [11]. It is a profinite version of Nakayama's lemma.

Lemma 3.3 (G-H-M-R)**.** *Let G be a finitely generated profinite p-group. Let M and N be finitely generated complete G-modules and $f : M \to N$ be a map of complete G-modules. If the induced map*

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} f : \mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} M \to \mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} N$$

is surjective, then so is f. If the map

$$\mathrm{Tor}_q^{\mathbb{Z}_p[[G]]}(\mathbb{F}_p, f) : \mathrm{Tor}_q^{\mathbb{Z}_p[[G]]}(\mathbb{F}_p, M) \to \mathrm{Tor}_q^{\mathbb{Z}_p[[G]]}(\mathbb{F}_p, N)$$

is an isomorphism for $q = 0$ and surjective for $q = 1$, then f is an isomorphism.

Lemma 3.4. *Let G be a profinite p-group and let IG be the augmentation ideal. For any G-module M, the boundary map for the short exact sequence*

$$0 \to IG \to \mathbb{Z}_p[[G]] \xrightarrow{\varepsilon} \mathbb{Z}_p \to 0,$$

induces an isomorphism

$$H_{n+1}(G, M) \cong \mathrm{Tor}_n^{\mathbb{Z}_p[[G]]}(IG, M).$$

For the trivial module $M = \mathbb{Z}_p$, this isomorphism sends $g \in G/[G, G]$ to the residue class of $e - g$ in $H_0(G, IG) \cong IG/(IG)^2$. For $M = \mathbb{F}_p$, it sends $g \in G/([G, G], G^p)$ to the residue class of $e - g$ in $\mathbb{F}_p \otimes_{\mathbb{Z}_p} IG/(IG)^2$

Proof. This follows from the long exact sequence on $\mathrm{Tor}_n^{\mathbb{Z}_p[[G]]}(-, M)$, using the fact that $\mathbb{Z}_p[[G]]$ is a free G-module. □

Lemma 3.5. *Let $X = \lim_n X_n$ be a profinite G-set, where G is a finite group. Let $G\backslash X$ denote the right cosets. Then the natural map*

$$G\backslash X \to \lim_n G\backslash X_n$$

is an isomorphism. Further, there is a continuous isomorphism of continuous G-modules

$$\mathbb{Z}_p[[X]] := \lim_n \mathbb{Z}_p[X_n] \cong \prod_{Gx \in G\backslash X} \mathbb{Z}_p[G/G_x].$$

Proof. Let (Gx_n) be an element of $\lim_n G\backslash X_n$. Let $f : X_n \to X_{n-1}$ denote the structure maps. In order to prove surjectivity, it is sufficient to construct an element $y = (y_n)$ in X such that

$$Gx_n = Gy_n.$$

The proof is by induction on n. Let $x_1 = y_1$. Suppose that y_{n-1} has been defined. Then

$$Gf(x_n) = Gx_{n-1} = Gy_{n-1}.$$

Hence, there exists a $g \in G$ such that $gf(x_n) = y_{n-1}$. Define $y_n = gx_n$. This proves surjectivity.

For injectivity, suppose that $y = (y_n)$ and $z = (z_n)$ are such that $Gy_n = Gz_n$ for all n. Let G_{y_n} be the isotropy subgroup y_n. If $g \in G_{y_n}$, then

$$gy_{n-1} = f(gy_n) = f(y_n) = y_{n-1},$$

so that $G_{y_n} \subseteq G_{y_{n-1}}$. Because G is finite, $G_y = \bigcap_n G_{y_n} = G_{y_k}$ for some k. Hence, for $n \geq k$,

$$G_{y_n} = G_{y_k}.$$

For each $n \geq k$, choose g_n such that $g_n y_n = z_n$. Then

$$g_n y_k = g_n f(y_n) = f(g_n y_n) = f(z_n) = z_k.$$

This implies that $g_n \in g_k G_{y_k}$. Since $G_{y_k} = G_{y_n}$, this implies that $g_n \in g_k G_{y_n}$, so that

$$g_k y_n = z_n.$$

Further, for $n < k$

$$g_k y_n = g_k f(y_k) = f(g_k y_k) = f(z_k) = z_n.$$

Therefore, $g_k y = z$ and the map is injective.

For the second statement, note that

$$\begin{aligned}\lim_n \mathbb{Z}_p[X_n] &= \lim_n \bigoplus_{Gx\in G\backslash X_n} \mathbb{Z}_p[G/G_x]\\ &\cong \lim_n \prod_{Gx\in G\backslash X_n} \mathbb{Z}_p[G/G_x]\end{aligned}$$

as each X_n is finite. This defines a natural G-equivariant isomorphism

$$\prod_{Gx\in G\backslash X} \mathbb{Z}_p[G/G_x] \to \mathbb{Z}_p[[X]].$$

□

Lemma 3.6. *Let $\mathbb{G}$ be a profinite group, and G and H be finite subgroups. Let*

$$G_x := \{g \in G \mid gxH = xH\}.$$

Then

$$H^*(G, \mathbb{Z}_p[[\mathbb{G}/H]]) \cong \prod_{GxH} H^*(G_x, \mathbb{Z}_p).$$

Proof. By Lemma 3.5, there is an isomorphism

$$\mathbb{Z}_p[[\mathbb{G}/H]] \cong \prod_{GxH} \mathbb{Z}_p[G/G_x].$$

Hence,

$$\begin{aligned}H^*(G, \mathbb{Z}_p[[\mathbb{G}/H]]) &\cong \prod_{GxH} H^*(G, \mathbb{Z}_p[G/G_x])\\ &\cong \prod_{GxH} H^*(G_x, \mathbb{Z}_p).\end{aligned}$$

□

3.2. **The resolution.** From now on, we fix $p = 2$. The goal of this section is to prove Theorem 1.8, which was stated in Section 1.3. The proof is broken into a series of results given in Lemma 3.7, Lemma 3.9, Lemma 3.13 and Theorem 3.23.

Let G_{24} be the maximal finite subgroup of $\mathbb{S}_2$ defined in Lemma 2.18. Recall that $G_{24}' = \pi G_{24}\pi^{-1}$ for $\pi = 1 + 2\omega$ in $\mathbb{S}_2$. It was shown in Lemma 2.27 that there are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$, and that G_{24} and G_{24}' are representatives. Recall that $C_2 = \{\pm 1\}$ is the subgroup generated by $[-1](x)$ and $C_6 = C_2 \times C_3$. The group K^1 is the Poincaré duality subgroup of $\mathbb{S}_2^1$ which was defined in Section 2.6.

Lemma 3.7 (G-H-M-R, unpublished)**.** *Let $\mathscr{C}_0 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]]$ with canonical generator e_0. Let N_0 be defined by*

$$0 \to N_0 \to \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0. \tag{3.8}$$

Then N_0 is the submodule of $\mathscr{C}_0$ generated by $(e - \alpha)e_0$, where

$$\alpha = \frac{1 - 2\omega}{\sqrt{-7}}.$$

Proof. Since $\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}$, $\mathscr{C}_0 \cong \mathbb{Z}_2[[K^1]]$ as a K^1-module. Therefore, $N_0 \cong IK^1$. Lemma 3.4 implies that $H_1(K^1, \mathbb{Z}_2) \cong H_0(K^1, N_0)$, where an isomorphism sends the image of g in $K^1/[K^1, K^1]$ to the image of $e - g$ in $IK^1/(IK^1)^2$. It was shown in Corollary 2.42 that $K^1/[K^1, K^1]$ is generated by α as a G_{24}-module. This implies that, as a G_{24}-module, $H_0(K^1, N_0)$ is generated by the image of $(e - \alpha)e_0$. Therefore, the map

$$F : \mathbb{Z}_2[[\mathbb{S}_2^1]] \to N_0$$

defined by $F(\gamma) = \gamma(e - \alpha)e_0$ induces a surjective map

$$\mathbb{F}_2 \otimes_{\mathbb{Z}_p[[K^1]]} F : \mathbb{F}_2 \otimes_{\mathbb{Z}_2[[K^1]]} \mathbb{Z}_2[[\mathbb{S}_2^1]] \to \mathbb{F}_2 \otimes_{\mathbb{Z}_2[[K^1]]} \otimes N_0.$$

By Lemma 3.3, F itself is surjective, and $(e-\alpha)e_0$ generates N_0 as an $\mathbb{S}_2^1$-module. $\square$

Lemma 3.9 (G-H-M-R, unpublished)**.** *Let N_0 be as in Lemma 3.7. Let $\mathscr{C}_1 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]]$ with canonical generator e_1. There is a map*

$$\partial_1 : \mathscr{C}_1 \to N_0$$

defined by

$$\partial_1(\gamma e_1) = \gamma(e - \alpha)e_0 \tag{3.10}$$

for $\gamma \in \mathbb{Z}_2[[\mathbb{S}_2^1]]$. Further, if N_1 is defined by the exact sequence

$$0 \to N_1 \to \mathscr{C}_1 \xrightarrow{\partial_1} N_0 \to 0, \tag{3.11}$$

then any element $\Theta_0 \in \mathbb{Z}_2[[\mathbb{S}_2^1]]$ such that $\Theta_0 e_1$ is in the kernel of ∂_1 and

$$\Theta_0 e_1 \equiv (3 + i + j + k)e_1 \mod (4, IK^1)$$

generates N_1 over $\mathbb{S}_2^1$.

Proof. The element α satisfies $\tau\alpha = \alpha\tau$ for $\tau \in C_6$. Therefore, the map ∂_1 given by (3.10) is well-defined.

Let N_1 be the kernel of ∂_1. Note that $\mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \cong \mathbb{Z}_2[[K^1]]^4$ as K^1-modules, generated by e_1, ie_1, je_1 and ke_1. Therefore, there is an isomorphism of G_{24}-modules

$$H_0(K^1, \mathscr{C}_1) \cong \mathbb{Z}_2[G_{24}/C_6].$$

As K^1-modules, $H_0(K^1, \mathscr{C}_1) \cong \mathbb{Z}_2^4$ generated by the image of the classes e_1, ie_1, je_1 and ke_1. Since $N_0 \cong IK^1$, Lemma 3.4 implies that

$$H_1(K^1, N_0) \cong H_2(K^1, \mathbb{Z}_2) = 0.$$

Therefore, the long exact sequence on cohomology gives rise to a short exact sequence

$$0 \to H_0(K^1, N_1) \to H_0(K^1, \mathscr{C}_1) \to H_0(K^1, N_0) \to 0.$$

Since $H_0(K^1, N_0) \cong K^1/[K^1, K^1]$, and $K^1 \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2$ is all torsion (see Corollary 2.42), we can identify $H_0(K^1, N_1)$ with a free submodule of $H_0(K^1, \mathscr{C}_1)$. Further, it must have rank 4 over $\mathbb{Z}_2$. This can be made explicit as follows.

The map $H_0(K^1, \partial_1)$ sends the residue class of τe_1 to that of $\tau(e - \alpha)e_0$. For $\tau \in G_{24}$, $\tau^{-1}e_0 = e_0$, hence $\tau(e - \alpha)e_0 = (e - \tau_*(\alpha))e_0$, where $\tau_*(\alpha) = \tau\alpha\tau^{-1}$. Again, using the boundary isomorphism $H_1(K^1, \mathbb{Z}_2) \cong H_0(K^1, N_0)$ of Lemma 3.4, the formulas of Corollary 2.42 together with the fact that $k = ij$ can be used to compute

$$\partial_1(e_1) \equiv \overline{\alpha}, \quad \partial_1(ie_1) \equiv \overline{\alpha} + \overline{\alpha_i}, \quad \partial_1(je_1) \equiv \overline{\alpha} + \overline{\alpha_j}, \quad \partial_1(ke_1) \equiv 3\overline{\alpha} + \overline{\alpha_i} + \overline{\alpha_j}.$$

Here, $\overline{a}$ is the image of a in $H_1(K^1,\mathbb{Z}_2)$. As α generates a group isomorphic to $\mathbb{Z}/4$, and α_i and α_j both generate groups isomorphic to $\mathbb{Z}/2$, a set of $\mathbb{Z}_2$ generators for the kernel of $H_0(K^1,\partial_1)$ is given by the the elements

$$f_1 = -4e_1, \quad f_2 = 2(i-e)e_1, \quad f_3 = 2(j-e)e_1, \quad f_4 = (k-i-j-e)e_1.$$

Let

$$f = (3e+i+j+k)e_1 \in H_0(K^1, N_1).$$

Then f generates

$$H_0(K^1,N_1) \cong \mathbb{Z}_2[G_{24}/C_6],$$

as a G_{24}-module. Indeed, using the fact that $G_{24}/C_6 \cong Q_8/C_2$, one computes that

$$f_1 = 1/3(i+j+k-5)f, \quad f_2 = if - f, \quad f_3 = jf - f, \quad f_4 = -k(f+f_1).$$

(Note that $-\tau$ denotes $(-1)\cdot\tau$ for the coefficient $-1 \in \mathbb{Z}_2$, as opposed to the generator of the central C_2 in Q_8.)

Next, we show that if

$$f' \equiv f \mod (4, IK^1),$$

then f' also generates $H_0(K^1,N_1)$ as a G_{24}-module. To do this, note that $\mathbb{Z}_2[Q_8/C_2]$ is a complete local ring with maximal ideal $\mathfrak{m} = (2, IQ_8/C_2)$. Hence, any element congruent to 1 modulo $\mathfrak{m}$ is invertible. Therefore, if $f' = f + \epsilon f$ for ϵ in $\mathfrak{m}$, then f' is also a generator. However, for $a \in H_0(K_1, \mathscr{C}_1)$,

$$4ae_1 = a\frac{1}{3}((e-i)+(e-j)+(e-k)+2e)f,$$

and $a\frac{1}{3}((e-i)+(e-j)+(e-k)+2e) \in \mathfrak{m}$. Therefore, $4H_0(K_1,\mathscr{C}_1) \subseteq \mathfrak{m}f$.

Let Θ_0 in N_1 be such that

$$\Theta_0 e_1 \equiv (3+i+j+k)e_1 \mod (4, IK^1).$$

Let

$$F : \mathbb{Z}_2[[\mathbb{S}_2^1]] \to N_1$$

be the map defined by $F(\gamma) = \gamma\Theta_0 e_1$. It induces a surjective map

$$\mathbb{F}_2 \otimes_{\mathbb{Z}_p[[K^1]]} F : \mathbb{F}_2 \otimes_{\mathbb{Z}_2[[K^1]]} \mathbb{Z}_2[[\mathbb{S}_2^1]] \to \mathbb{F}_2 \otimes_{\mathbb{Z}_2[[K^1]]} \otimes N_1.$$

By Lemma 3.3, F itself is surjective, and $\Theta_0 e_1$ generates N_1 as an $\mathbb{S}_2^1$-module. □

Define $tr_{C_3} : \mathbb{Z}_2[[\mathbb{S}_2^1]] \to: \mathbb{Z}_2[[\mathbb{S}_2^1]]$ to be the $\mathbb{Z}_2$-linear map induced by

$$tr_{C_3}(g) = g + \omega g \omega^{-1} + \omega^{-1} g \omega \tag{3.12}$$

for $g \in \mathbb{S}_2^1$ and ω our chosen generator of C_3.

Lemma 3.13 (G-H-M-R, unpublished). *Let $\mathscr{C}_2 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]]$ with canonical generator e_2. Let $\Theta \in \mathbb{Z}_2[[\mathbb{S}_2^1]]$ be such that*

(1) $\tau\Theta = \Theta\tau$ for $\tau \in C_6$,
(2) Θe_1 is in the kernel of $\partial_1 : \mathscr{C}_1 \to \mathscr{C}_0$,
(3) $\Theta e_1 \equiv (3+i+j+k)e_1 \mod (4, IK^1)$.

Then the map of $\mathbb{S}_2^1$-modules $\partial_2 : \mathscr{C}_2 \to \mathscr{C}_1$

$$\partial_2(\gamma e_2) = \gamma\Theta e_2, \tag{3.14}$$

surjects onto $N_1 = \ker(\partial_1)$. Further, if N_2 is defined by the exact sequence

$$0 \to N_2 \to \mathscr{C}_2 \xrightarrow{\partial_2} N_1 \to 0, \tag{3.15}$$

then $N_2 \cong \mathbb{Z}_2[[K^1]]$ *as* K^1*-modules.*

Proof. Choose an element Θ_0 which generates N_1 as in Lemma 3.9. Recall that

$$C_6 \cong C_2 \times C_3$$

and that C_2 is in the center of $\mathbb{S}_2$. Therefore, for tr_{C_3} as defined by (3.12)

$$\Theta = \frac{1}{3} tr_{C_3}(\Theta_0)$$

satisfies properties (1), (2) and (3). The map ∂_2 given by (3.14) is well-defined and surjects onto N_1 by Lemma 3.3.

Let $N_2 \subseteq \mathscr{C}_2$ be the kernel of ∂_2 as in the statement of the result. The map ∂_2 induces an isomorphism $H_0(K^1, \mathscr{C}_2) \cong H_0(K^1, N_1)$. Hence, for all n,

$$H_n(K^1, N_2) \cong H_{n+1}(K^1, N_1) \cong H_{n+2}(K^1, N_0) \cong H_{n+3}(K^1, \mathbb{Z}_2).$$

This implies that

$$H_n(K^1, N_2) := \begin{cases} \mathbb{Z}_2 & n = 0; \\ 0 & n > 0. \end{cases}$$

Choose an element $e' \in N_2$ such that e' reduces to a generator of $\mathbb{Z}_2$ in $H_0(K^1, N_2)$. Define $\phi : \mathbb{Z}_2[[K^1]] \to N_2$ by $\phi(k) = ke'$. Then $\mathrm{Tor}_0^{\mathbb{Z}_2[[K^1]]}(\mathbb{F}_2, \phi)$ is an isomorphism, and $\mathrm{Tor}_1^{\mathbb{Z}_2[[K^1]]}(\mathbb{F}_2, \phi)$ is surjective. By Lemma 3.3, ϕ is an isomorphism of K^1-modules. □

Splicing the exact sequences (3.8), (3.11) and (3.15) gives an exact sequence

$$0 \to N_2 \to \mathscr{C}_2 \to \mathscr{C}_1 \to \mathscr{C}_0 \to \mathbb{Z}_2 \to 0 \tag{3.16}$$

which is a free resolution of $\mathbb{Z}_2$ as a trivial K^1-module. The next goal is to find an isomorphism $N_2 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]]$, where $G'_{24} = \pi G_{24} \pi^{-1}$ represents the other conjugacy class of maximal finite subgroups in $\mathbb{S}_2^1$. To prove this, we will need a few results. Before stating these, we introduce some notation.

Let G be a subgroup of $\mathbb{S}_2$ which contains the central subgroup C_2. We define

$$PG := G/C_2.$$

We let

$$A_4 := PG_{24}, \tag{3.17}$$

$$A'_4 := PG'_{24}. \tag{3.18}$$

The choice of notation is justified by the fact that both of these groups are isomorphic to the alternating group on four letters. Note also that since C_2 is central

$$PC_6 \cong C_3,$$
$$P\mathbb{S}_2^1 \cong K^1 \rtimes A_4.$$

Therefore, for any G which contains C_2,

$$\mathbb{Z}_2[[\mathbb{S}_2^1/G]] \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/PG]]$$

as $\mathbb{S}_2^1$-modules. To prove that $N_2 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]]$, it will thus be sufficient to prove that

$$N_2 \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/A'_4]]$$

as $P\mathbb{S}_2^1$-modules.

We showed in Corollary 2.40 that K^1 is a Poincaré duality group (see Definition 2.38). Further, there is an isomorphism of K^1-modules

$$\mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']] \cong \mathbb{Z}_2[[K^1]].$$

Therefore,

$$H^n(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \cong \begin{cases} \mathbb{Z}_2 & n = 3; \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

Lemma 3.20 (G-H-M-R, unpublished)**.** *The inclusion $\iota : K^1 \to P\mathbb{S}_2^1$ induces an isomorphism*

$$\iota^* : H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \to H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]).$$

Proof. The action of A_4 on $H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$ is trivial. This follows from the fact that there are no non-trivial one-dimensional representations of A_4. Indeed,

$$\mathrm{Hom}(A_4, Gl_1(\mathbb{Z}_2)) = H^1(A_4, \mathbb{Z}_2^\times)$$

and $H^1(A_4, \mathbb{Z}_2^\times) = 0$. Since $P\mathbb{S}_2^1 \cong K^1 \rtimes A_4$, there is a spectral sequence

$$H^p(A_4, H^q(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])) \Longrightarrow H^{p+q}(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]).$$

Because the action of A_4 on $H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$ is trivial, (3.19) implies that the edge homomorphism

$$H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \to H^0(A_4, H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]))$$

induced by the inclusion $\iota : K^1 \to P\mathbb{S}_2^1$ is an isomorphism. □

Lemma 3.21 (G-H-M-R, unpublished)**.** *There are surjections*

$$\eta : \mathrm{Hom}_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \to H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

and

$$\eta' : \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \to H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

making the following diagram commute:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \arrow[r, "\eta"] \arrow[d, "\iota^*"'] & H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \arrow[d, "\iota^*"] \\ \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \arrow[r, "\eta'"] & H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]), \end{tikzcd} \tag{3.22}$$

where ι^* is the map induced by the inclusion $\iota : K^1 \to P\mathbb{S}_2^1$.

Proof. Let $\mathscr{B}_p = \mathscr{C}_p$ for $0 \le p < 3$ and $\mathscr{B}_3 = N_2$. Resolving $\mathscr{B}_p$ by projective $P\mathbb{S}_2^1$-modules gives rise to spectral sequences

$$E_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(\mathscr{B}_p, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \Longrightarrow H^{p+q}(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

and

$$F_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[[K^1]]}(\mathscr{B}_p, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \Longrightarrow H^{p+q}(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]).$$

These are first quadrant cohomology spectral sequence, with differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and

$$d_r : F_r^{p,q} \to F_r^{p+r,q-r+1}.$$

Further, $\iota : K^1 \to P\mathbb{S}_2^1$ induces a map of spectral sequences

$$\iota^* : E_r^{p,q} \to F_r^{p,q}.$$

Let η be the edge homomorphism

$$\eta : E_1^{3,0} \to H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

and let η' be the edge homomorphism

$$\eta' : F_1^{3,0} \to H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]).$$

First, note that since the modules $\mathscr{B}_p$ are projective K^1-modules, $F_r^{p,q}$ collapses with $F_\infty^{p,q} = 0$ for $q > 0$ so that

$$F_\infty^{3,0} \to H^3(K^1; \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

is surjective. Hence, η' is surjective.

In order to show that η is surjective, it is sufficient to show that $E_1^{3-q,q} = 0$ for $q > 0$. For $q = 1$ and $q = 2$, this follows from the fact that $\mathbb{Z}_2[[P\mathbb{S}_2^1/C_3]]$ is a projective $P\mathbb{S}_2^1$-module. Hence, if $q > 0$, then

$$\mathrm{Ext}^q_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(\mathbb{Z}_2[[P\mathbb{S}_2^1/C_3]], \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) = 0.$$

It remains to show that

$$E_1^{0,3} = \mathrm{Ext}^3_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(\mathscr{B}_0, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

is zero, where $\mathscr{B}_0 = \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4]]$.

Let $V \cong C_2 \times C_2$ be the 2-Sylow subgroup of A_4. Then

$$\begin{aligned} E_1^{0,3} &= \mathrm{Ext}^3_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(\mathscr{B}_0, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \\ &\cong H^3(A_4, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \\ &\cong H^3(V, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])^{C_3}. \end{aligned}$$

By Lemma 3.6, there is an isomorphism

$$H^3(V, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \cong \prod_{VxA_4'} H^3(V_x, \mathbb{Z}_2).$$

Note that the inclusion $V_x \subseteq V$ cannot be an equality. Otherwise, V would be conjugate to $V' \subseteq A_4'$. This implies that V_x is either trivial, or V_x has order 2. In both cases, $H^3(V_x, \mathbb{Z}_2) = 0$ and $E_1^{0,3} = 0$. □

Theorem 3.23 (G-H-M-R, unpublished)**.** *There is an isomorphism of $\mathbb{S}_2^1$-modules*

$$\phi : \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']] \to N_2,$$

where $G_{24}' = \pi G_{24} \pi^{-1}$.

Proof. It suffices to show that $N_2 \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/PG_{24}']]$ as $P\mathbb{S}_2^1$-modules. First, note that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) &\cong \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(\mathbb{Z}_2[[K^1]], \mathbb{Z}_2[[K^1]]) \\ &\cong \mathbb{Z}_2[[K^1]]. \end{aligned}$$

Let η' be the surjection of Lemma 3.21. Choose isomorphisms f and g which make the following diagram commute

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \arrow[r, "\eta'"] \arrow[d, "f"'] & H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \arrow[d, "g"] \\ \mathbb{Z}_2[[K^1]] \arrow[r, "\varepsilon"] & \mathbb{Z}_2. \end{tikzcd} \tag{3.24}$$

A morphism $\varphi : N_2 \to \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]$ of K^1-modules is an isomorphism if and only if $\varepsilon(f(\varphi))$ is a unit in $\mathbb{Z}_2$. This is the case if and only if $g(\eta'(\varphi))$ is a unit in $\mathbb{Z}_2$. This holds if and only if the image of $\eta'(\varphi)$ in

$$H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \otimes_{\mathbb{Z}_2} \mathbb{F}_2 \tag{3.25}$$

is non-zero. Now, note that the composite

$$\iota^* \circ \eta : \mathrm{Hom}_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]) \to H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

of (3.22) is surjective. Choose an element

$$\varphi \in \mathrm{Hom}_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(N_2, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$$

such that $\iota^* \circ \eta(\varphi)$ is a generator for the group $H^3(K^1, \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']])$. Hence, the image of $\eta' \circ \iota^*(\varphi)$ in (3.25) is non-zero. Therefore,

$$\varphi : N_2 \to \mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']]$$

is an isomorphism as $P\mathbb{S}_2^1$-modules. Hence, viewed as a map of $\mathbb{S}_2^1$-modules,

$$\varphi : N_2 \to \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']]$$

is an isomorphism. Letting $\phi = \varphi^{-1}$ finishes the proof. □

Combining the previous results, we can finally prove Theorem 1.8. We restate it here for convenience.

Theorem 3.26 (G-H-M-R, unpublished)**.** *Let $\mathbb{Z}_2$ be the trivial $\mathbb{S}_2^1$-module. There is an exact sequence of complete $\mathbb{S}_2^1$-modules*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0, \tag{3.27}$$

where $\mathscr{C}_0 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]]$ and $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]]$ and $\mathscr{C}_3 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']]$. Further, (3.27) is a free resolution of the trivial K^1-module $\mathbb{Z}_2$.

Proof. Let

$$\mathscr{C}_3 := \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']].$$

Let $\phi : \mathscr{C}_3 \to N_2$ be the isomorphism of Theorem 3.23. Let $\partial_3 : \mathscr{C}_3 \to \mathscr{C}_2$ be the isomorphism ϕ followed by the inclusion of N_2 in $\mathscr{C}_2$. This gives an exact sequence

$$0 \to \mathscr{C}_3 \to \mathscr{C}_2 \xrightarrow{\partial_2} N_1 \to 0. \tag{3.28}$$

Splicing the exact sequences of (3.8), (3.11) and (3.28) finishes the proof. □

The exact sequence (3.28) is called the *algebraic duality resolution*. The duality properties it satisfies will be described in Section 3.4

3.3. **The algebraic duality resolution spectral sequence.** The algebraic duality resolution gives rise to a spectral sequence called *algebraic duality resolution spectral sequence*, which we describe here. The following result is a refinement of Theorem 1.10, which was stated in Section 1.3. We define

$$Q_8' := \pi Q_8 \pi^{-1}.$$

We also let V be the 2-Sylow subgroup of A_4 and V' be the 2-Sylow subgroup of A_4', where $A_4 \cong PG_{24}$ and $A_4' = PG_{24}'$ as defined in (3.17) and (3.18).

Theorem 3.29. *Let M be a finitely generated complete $\mathbb{S}_2^1$-module. There is a first quadrant spectral sequence,*

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathscr{C}_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M).$$

The differentials have degree

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

and

$$E_1^{p,q} \cong \begin{cases} H^q(G_{24}, M) & \text{if } p = 0; \\ H^q(C_6, M) & \text{if } p = 1, 2; \\ H^q(G_{24}', M) & \text{if } p = 3. \end{cases} \tag{3.30}$$

Similarly, there are first quadrant spectral sequences

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[G]]}(\mathscr{C}_p, M) \Longrightarrow H^{p+q}(G, M),$$

when G is S_2^1, $P\mathbb{S}_2^1$ or PS_2^1, with E_1-page given by

$$E_1^{p,q} \cong \begin{cases} H^p(Q_8; M) & \text{if } q = 0; \\ H^p(C_2; M) & \text{if } q = 1, 2; \\ H^p(Q_8'; M) & \text{if } q = 3, \end{cases}$$

for S_2^1, by

$$E_1^{p,q} \cong \begin{cases} H^p(A_4; M) & \text{if } q = 0; \\ H^p(C_3; M) & \text{if } q = 1, 2; \\ H^p(A_4'; M) & \text{if } q = 3, \end{cases}$$

for $P\mathbb{S}_2^1$, and by

$$E_1^{p,q} \cong \begin{cases} H^p(V; M) & \text{if } q = 0; \\ H^p(\{e\}; M) & \text{if } q = 1, 2; \\ H^p(V'; M) & \text{if } q = 3, \end{cases}$$

for PS_2^1.

Proof. There are two equivalent constructions. First, recall that the algebraic duality resolution is spliced from the exact sequences

$$0 \to N_i \to \mathscr{C}_i \to N_{i-1} \to 0, \tag{3.31}$$

with $\mathscr{C}_3 = N_2$ and $M_{-1} = N_{-1} = \mathbb{Z}_2$. The exact couple

$$\begin{tikzcd} \mathrm{Ext}(N_*, M) \arrow[rr, "\delta_*"] & & \mathrm{Ext}(N_{*-1}, M) \arrow[dl, "r_*"] \\ & \mathrm{Ext}(\mathscr{C}_*, M) \arrow[ul, "i^*"] & \end{tikzcd}$$

gives rise to the algebraic duality resolution spectral sequence.

Alternatively, one can resolve each $\mathscr{C}_p$ by projective $\mathbb{S}_2^1$-modules $P_{p,q}$ in order to obtain a double complex of projective modules (here, one must include $\mathscr{C}_{-1} = \mathbb{Z}_2$). The total complex $\mathrm{Tot}(P_{p,q})$ for $p \geq 0$ is a projective resolution of $\mathbb{Z}_2$ as an $\mathbb{S}_2^1$-module. The homology of $\mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathrm{Tot}(P_{p,q}), M)$ is

$$\mathrm{Ext}^{p+q}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathbb{Z}_2, M) \cong H^{p+q}(\mathbb{S}_2^1, M).$$

The identification of the E_1-term follows from Shapiro's Lemma. That is, for any finite subgroup H of $\mathbb{S}_2^1$

$$\mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[H]} \mathbb{Z}_2, M) \cong \mathrm{Ext}^q_{\mathbb{Z}_2[H]}(\mathbb{Z}_2, M) \cong H^q(H, M).$$

For the groups S_2^1, $P\mathbb{S}_2$ and PS_2^1, one applies the same construction, keeping the following isomorphisms in mind. Let $H \subseteq \mathbb{S}_2^1$ be a finite subgroup which contains C_6. Let $PH = H/C_2$ and let $Syl_2(H)$ be the 2-Sylow subgroup of H. As S_2^1-modules, there is an isomorphism

$$Z_2[[\mathbb{S}_2^1/H]] \cong \mathbb{Z}_2[[S_2^1]] \otimes_{\mathbb{Z}_2[Syl_2(H)]} \mathbb{Z}_2 \cong \mathbb{Z}_2[[S_2^1/Syl_2(H)]].$$

Similarly, as $P\mathbb{S}_2^1$ and PS_2^1-modules respectively,

$$Z_2[[\mathbb{S}_2^1/H]] \cong \mathbb{Z}_2[[P\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[PH]} \mathbb{Z}_2 \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/PH]],$$

and

$$Z_2[[\mathbb{S}_2^1/H]] \cong \mathbb{Z}_2[[PS_2^1]] \otimes_{\mathbb{Z}_2[Syl_2(PH)]} \mathbb{Z}_2 \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/Syl_2(PH)]].$$

□

3.4. **The duality.** The algebraic duality resolution (3.27) owes its name to the fact that it satisfies a certain twisted duality. This duality is crucial for computations as it allows us to understand the map $\partial_3 : \mathscr{C}_3 \to \mathscr{C}_2$ in (3.27).

Let $\mathrm{Mod}(\mathbb{S}_2^1)$ denote the category of complete $\mathbb{S}_2^1$-modules. For $M \in \mathrm{Mod}(\mathbb{S}_2^1)$, let $c_\pi(M)$ denote the $\mathbb{S}_2^1$-module whose underlying $\mathbb{Z}_2$-module is equal to M, but whose $\mathbb{S}_2^1$-module structure is twisted by the element π defined in (2.8). That is, for $\gamma \in \mathbb{S}_2^1$ and $m \in c_\pi(M)$,

$$\gamma \cdot m = \pi\gamma\pi^{-1}m.$$

If $\phi : M \to N$ is a morphism of $\mathbb{S}_2^1$-modules, let $c_\pi(\phi) : c_\pi(M) \to c_\pi(N)$ be given by

$$c_\pi(\phi)(m) = \phi(m).$$

Then $c_\pi : \mathrm{Mod}(\mathbb{S}_2^1) \to \mathrm{Mod}(\mathbb{S}_2^1)$ is a functor. In fact, c_π is an involution, since $\pi^2 = -3$ is in the center of $\mathbb{S}_2$. We can now prove Theorem 1.9, which is restated here for convenience.

Theorem 3.32 (Henn-Karamanov-Mahowald, unpublished)**.** *There exists an isomorphism of $\mathbb{S}_2^1$-complexes*

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] & c_\pi(\mathscr{C}_3^*) \arrow[r, "\overline{\varepsilon}"] & \mathbb{Z}_2 \arrow[r] & 0.
\end{tikzcd}$$

Proof. The proof is similar to the proof of Proposition 3.8 in [19]. Let $\mathscr{C}_p^* = \mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathscr{C}_p, \mathbb{Z}_2[[\mathbb{S}_2^1]])$ and $\partial_p^* = \mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\partial_p, \mathbb{Z}_2[[\mathbb{S}_2^1]])$ be the $\mathbb{S}_2^1$-duals of $\mathscr{C}_p$ and ∂_p in the sense of (3.1). The resolution (3.27) gives rise to a complex

$$0 \to \mathscr{C}_0^* \xrightarrow{\partial_1^*} \mathscr{C}_1^* \xrightarrow{\partial_2^*} \mathscr{C}_2^* \xrightarrow{\partial_3^*} \mathscr{C}_3^* \to 0. \tag{3.33}$$

Because K^1 has finite index in $\mathbb{S}_2^1$, the induced and coinduced modules of $\mathbb{Z}_2[[K^1]]$ are isomorphic (see Section 3.3 of [35]). Therefore,

$$\mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathscr{C}_p, \mathbb{Z}_2[[\mathbb{S}_2^1]]) \cong \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(\mathscr{C}_p, \mathbb{Z}_2[[K^1]]),$$

and the homology of the complex (3.33) is $H^n(K^1, \mathbb{Z}_2[[K^1]])$. By Corollary 2.40, $H^n(K^1, \mathbb{Z}_2[[K^1]])$ is 0 for $n < 3$ and $\mathbb{Z}_2$ for $n = 3$. Further, the action of G_{24} on $H^3(K^1, \mathbb{Z}_2[[K^1]]) \cong \mathbb{Z}_2$ is trivial, as there are no non-trivial one dimensional 2-adic representations of G_{24}. Hence, (3.33) is a resolution of $\mathbb{Z}_2$ as a trivial $\mathbb{S}_2^1$-module.

The module $\mathscr{C}_p^*$ is of the form $\mathbb{Z}_2[[\mathbb{S}_2^1/H]]$ via the isomorphism t defined in (3.2). Let $\overline{\varepsilon}$ be the augmentation

$$\overline{\varepsilon} : \mathscr{C}_3^* \to \mathbb{Z}_2.$$

Because the augmentation $\varepsilon : \mathbb{Z}_2[[K^1]] \to \mathbb{Z}_2$ induces an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(\mathbb{Z}_2[[K^1]], \mathbb{Z}_2),$$

one can choose an isomorphism $H^3(K, \mathbb{Z}_2[[K^1]]) \to \mathbb{Z}_2$ making the following diagram commute,

$$\begin{tikzcd} \mathscr{C}_3^* \arrow[r] \arrow[d, "\overline{\varepsilon}"] & H^3(K^1, \mathbb{Z}_2[[K^1]]) \arrow[dl, dotted] \\ \mathbb{Z}_2. & \end{tikzcd}$$

Therefore, the dual resolution is given by,

$$0 \to \mathscr{C}_0^* \xrightarrow{\partial_1^*} \mathscr{C}_1^* \xrightarrow{\partial_2^*} \mathscr{C}_2^* \xrightarrow{\partial_3^*} \mathscr{C}_3^* \xrightarrow{\overline{\varepsilon}} \mathbb{Z}_2 \to 0. \tag{3.34}$$

Take the image of this resolution in $\mathrm{Mod}(\mathbb{S}_2^1)$ under the involution c_π. Let e_3^π be the canonical generator of $c_\pi(\mathscr{C}_3^*)$. The map $f_0 : \mathscr{C}_0 \to c_\pi(\mathscr{C}_3^*)$, defined by

$$f_0(e_0) = e_3^\pi,$$

is an isomorphism of $\mathbb{S}_2^1$-modules, and the following diagram is commutative

$$\begin{tikzcd} \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0\ . \\ c_\pi(\mathscr{C}_3^*) \arrow[r, "\overline{\varepsilon}"] & \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd}$$

Therefore, f_0 induces an isomorphism $\ker \varepsilon \cong \ker \overline{\varepsilon}$. As both

$$\mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \ker \varepsilon$$

and

$$c_\pi(\mathscr{C}_2^*) \xrightarrow{c_\pi(\partial_2^*)} \mathscr{C}_3^* \xrightarrow{c_\pi(\partial_3^*)} \ker \overline{\varepsilon}$$

are the beginning of projective resolutions of $\ker\varepsilon$ and $\ker\overline{\varepsilon}$ as $P\mathbb{S}_2^1$-modules, f_0 lifts to a chain map

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \ker\varepsilon \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] & \ker\overline{\varepsilon} \arrow[r] & 0.
\end{tikzcd}$$

Let $PS_2^1 := S_2^1/C_2$, where S_2^1 denotes the 2-Sylow subgroup of $\mathbb{S}_2^1$. By construction, f_0 is an isomorphism, which implies that $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[[PS_2^1]]} f_1$ and $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[[PS_2^1]]} f_2$ are isomorphisms. As $\mathscr{C}_p$ and $c_\pi(\mathscr{C}_p^*)$ are projective $P\mathbb{S}_2^1$-modules for $p = 1, 2$, Lemma 3.3 implies that f_1 and f_2 are isomorphisms. Finally, f_3 must be an isomorphism by the five lemma. □

3.5. **A description of the maps.** This section is dedicated to proving the statements in Theorem 1.13. The first statement of Theorem 1.13 is that

$$\partial_1(e_1) = (e - \alpha)e_0.$$

This was shown in Theorem 3.26. In this section, we prove the remaining statements of that theorem.

The following result provides a description of the maps $\partial_3 : \mathscr{C}_3 \to \mathscr{C}_2$ and proves the last part of Theorem 1.13. It is a consequence of Theorem 3.32.

Theorem 3.35. *There are isomorphisms of $\mathbb{S}_2^1$-modules $g_p : \mathscr{C}_p \to \mathscr{C}_p$ and differentials*

$$\partial'_{p+1} : \mathscr{C}_{p+1} \to \mathscr{C}_p,$$

such that

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "g_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "g_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "g_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "g_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial'_3"] & \mathscr{C}_2 \arrow[r, "\partial'_2"] & \mathscr{C}_1 \arrow[r, "\partial'_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_2 \arrow[r] & 0,
\end{tikzcd} \tag{3.36}$$

is an isomorphism of complexes. The map $\partial'_3 : \mathscr{C}_3 \to \mathscr{C}_2$ is given by

$$\partial'_3(e_3) = \pi(e + i + j + k)(e - \alpha^{-1})\pi^{-1}e_2. \tag{3.37}$$

Proof. We will construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] \arrow[d, "q_3"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] \arrow[d, "q_2"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] \arrow[d, "q_1"] & c_\pi(\mathscr{C}_3^*) \arrow[r, "\varepsilon"] \arrow[d, "q_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial'_3"] & \mathscr{C}_2 \arrow[r, "\partial'_2"] & \mathscr{C}_1 \arrow[r, "\partial'_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_2 \arrow[r] & 0.
\end{tikzcd}$$

The maps g_p will be the composites of the vertical maps. First, let $e_p^\pi \in c_\pi(\mathscr{C}_p^*)$ be the canonical generator. Define isomorphisms

$$q_p : c_\pi(M_{3-p}^*) \to M_p,$$

by

$$q_p(e^{\pi}_{3-p}) = e_p.$$

Define $g_p : \mathscr{C}_p \to \mathscr{C}_p$ by

$$g_p := q_p f_p,$$

and $\partial'_{p+1} : \mathscr{C}_{p+1} \to \mathscr{C}_p$ by

$$\partial'_{p+1} := q_p c_\pi(\partial^*_{3-p})q^{-1}_{p+1}.$$

By construction, (3.36) is commutative.

In order to compute ∂'_3, it is necessary to understand ∂^*_1. By definition,

$$\partial^*_1(e^*_0)(e_1) = e^*_0((e-\alpha)e_1) = (e-\alpha)\sum_{h\in G_{24}} h.$$

But,

$$\begin{aligned}(e-\alpha)\sum_{h\in G_{24}} h &= (e-\alpha)\sum_{h\in C_6} h(e + i^{-1} + j^{-1} + k^{-1}) \\ &= \sum_{h\in C_6} h(e-\alpha)(e + i^{-1} + j^{-1} + k^{-1}) \\ &= \left((e+i+j+k)(e-\alpha^{-1})e^*_1\right)(e_1).\end{aligned}$$

Hence

$$\partial^*_1(e^*_0) = (e+i+j+k)(e-\alpha^{-1})e^*_1.$$

A diagram chase shows that ∂'_3 is given by (3.37). $\square$

The maps $\partial_1 : \mathscr{C}_1 \to \mathscr{C}_0$ and $\partial_3 : \mathscr{C}_3 \to \mathscr{C}_2$ now have explicit descriptions up to isomorphisms. The map $\partial_2 : \mathscr{C}_2 \to \mathscr{C}_1$ is harder to describe. Theorem 3.42 and Corollary 3.43 below give an estimate for this map. These are technical results which will be used for computations in [3]. Note that Theorem 3.26, Theorem 3.35 and Corollary 3.43 prove Theorem 1.13, which was stated in Section 1.3.

Recall that

$$\alpha_\tau = [\tau, \alpha] = \tau\alpha\tau^{-1}\alpha^{-1}.$$

We will need the following result to describe the element Θ of Lemma 3.13.

Lemma 3.38. *Let $n \geq 2$ and $x \in IF_{n/2}K^1$. There exist h_0, h_1 and h_2 in $\mathbb{Z}_2[[F_{n/2}K^1]]$ such that*

$$(3.39)\quad x = \begin{cases} h_0(e-\alpha^{2^{m-1}}) + h_1(e-\alpha_i^{2^{m-1}}) + h_2(e-\alpha_j^{2^{m-1}}) & \text{if } n = 2m; \\ h_0(e-\alpha^{2^{m}}) + h_1(e-\alpha_i^{2^{m-1}}) + h_2(e-\alpha_j^{2^{m-1}}) & \text{if } n = 2m+1.\end{cases}$$

Proof. Define a map of $\mathbb{Z}_2[[F_{n/2}K^1]]$-modules

$$p : \bigoplus_{i=0}^{2} \mathbb{Z}_2[[F_{n/2}K^1]]_i \to IF_{n/2}K^1,$$

by sending (h_0, h_1, h_2) to the element given by (3.39). It is sufficient to show that the map induced by p surjects onto

$$H_1(F_{n/2}K^1, \mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathbb{Z}_2[[F_{n/2}K^1]]} IF_{n/2}K^1.$$

By Lemma 2.4, $H_1(F_{n/2}K^1, \mathbb{F}_2)$ is generated by the classes $\alpha^{2^{m-1}}$, $\alpha_i^{2^{m-1}}$ and $\alpha_j^{2^{m-1}}$ if $n = 2m$, and the classes α^{2^m}, $\alpha_i^{2^{m-1}}$ and $\alpha_j^{2^{m-1}}$ if $n = 2m+1$ (see Figure 2). Therefore, $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[[F_{n/2}K^1]]} p$ is surjective, and hence so is p. $\square$

The ideal
$$\mathcal{I} = ((IK^1)^7, 2(IK^1)^3, 4(IK^1), 8)$$
will play a crucial role in the following estimates.

Corollary 3.40. *Let e_0 be the canonical generator of $\mathscr{C}_0$ and g be in $F_{8/2}K^1$. There exists h in $\mathbb{Z}_2[[\mathbb{S}_2^1]]$ such that*
$$(e-g)e_0 = h(e-\alpha)e_0$$
with
$$h \equiv 0 \mod \mathcal{I}.$$

Proof. By Lemma 3.38, there exist h_0, h_1 and h_2 in $\mathbb{Z}_2[[F_{8/2}K^1]]$ such that
$$e-g = h_0(e-\alpha^8) + h_1(e-\alpha_i^8) + h_2(e-\alpha_j^8).$$
Since
$$(e-x^8) = (e+x+x^2+x^3+x^4+x^5+x^6+x^7)(e-x)$$
this implies that
$$e-g = h_0\left(\sum_{s=0}^{7}\alpha^s\right)(e-\alpha) + h_1\left(\sum_{s=0}^{7}\alpha_i^s\right)(e-\alpha_i) + h_2\left(\sum_{s=0}^{7}\alpha_j^s\right)(e-\alpha_j).$$
Let
$$h = h_0\left(\sum_{s=0}^{7}\alpha^s\right) + h_1\left(\sum_{s=0}^{7}\alpha_i^s\right)(i-\alpha_i) + h_2\left(\sum_{s=0}^{7}\alpha_j^s\right)(j-\alpha_j).$$
If $\tau \in G_{24}$, then $\tau e_0 = e_0$. Hence
$$(\tau-\alpha_\tau)(e-\alpha)e_0 = (e-\alpha_\tau)e_0.$$
Using this fact, one verifies that $(e-g)e_0 = h(e-\alpha)e_0$. Further,
$$\sum_{s=0}^{7} x^s \equiv (1-x)^7 + 2x^4(x-1)^3 + 4x^2(x-1) \mod (8)$$

Since α, α_i and α_j are in K^1, and using the fact that K^1 is a normal subgroup, this implies that
$$h \equiv 0 \mod ((IK^1)^7, 2(IK^1)^3, 4(IK^1), 8).$$
□

We will use the following result.

Lemma 3.41. *The element $\alpha_i\alpha_j\alpha_k$ is in $F_{4/2}K^1$. The element $\alpha_i\alpha_j\alpha_k\alpha^2$ is in $F_{8/2}K^1$.*

Proof. Let $T = \alpha S$ in $\mathcal{O}_2 \cong \mathrm{End}(F_2)$. Then $T^2 = -2$, and $aT = Ta^\sigma$ for a in $\mathbb{W}$. As defined in (2.9) and Lemma 2.18, we have
$$\alpha = \frac{1}{\sqrt{-7}}(1-2\omega), \qquad i = -\frac{1}{3}(1+2\omega)(1-T),$$
$$j = -\frac{1}{3}(1+2\omega)(1-\omega^2 T), \qquad k = -\frac{1}{3}(1+2\omega)(1-\omega T).$$

Further,

$$\alpha^{-1} = -\frac{1}{\sqrt{-7}}(1-2\omega^2).$$

We use the fact that $\frac{1}{3}$ and $\frac{1}{\sqrt{-7}}$ are in $Z(\mathbb{S}_2)$. We also use the fact $\tau^{-1} = -\tau$ for $\tau = i$, j and k and the fact that $S^4 = 4$ and $S^8 = 16$.

First, note that

$$\begin{aligned} i\alpha &= -\frac{1}{3\sqrt{-7}}(1+2\omega)(1-T)(1-2\omega) \\ &= -\frac{1}{3\sqrt{-7}}(1+2\omega)((1-2\omega)-(1-2\omega^2)T) \\ &= -\frac{1}{3\sqrt{-7}}((5+4\omega)+(1-4\omega)T). \end{aligned}$$

Further,

$$\begin{aligned} i^{-1}\alpha^{-1} &= -\frac{1}{3\sqrt{-7}}(1+2\omega)(1-T)(1-2\omega^2) \\ &= -\frac{1}{3\sqrt{-7}}(1+2\omega)((1-2\omega^2)-(1-2\omega)T) \\ &= -\frac{1}{3\sqrt{-7}}((-1+4\omega)-(5+4\omega)T). \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_i &= i\alpha i^{-1}\alpha^{-1} \\ &= -\frac{1}{63}((5+4\omega)+(1-4\omega)T)((-1+4\omega)-(5+4\omega)T) \\ &= \frac{1}{63}((5+4\omega)(1-4\omega)+(1-4\omega)(1-4\omega^2)T+(5+4\omega)^2T+(1-4\omega)(5+4\omega^2)T^2) \\ &\equiv 13+(2+8\omega)T \mod S^8. \end{aligned}$$

Using the fact that $\alpha_j = \omega\alpha_i\omega^2$ and $\alpha_k = \omega^2\alpha_i\omega$, this implies that

$$\alpha_j \equiv 13+\omega^2(2+8\omega)T \mod S^8, \qquad \alpha_k \equiv 13+\omega(2+8\omega)T \mod S^8.$$

Hence,

$$\begin{aligned} \alpha_i\alpha_j &\equiv (13+(2+8\omega)T)(13+\omega^2(2+8\omega)T) \\ &\equiv \big(9+\omega^2(10+8\omega)T+(10+8\omega)T+(2+8\omega)(\omega(2+8\omega^2))T^2\big) \\ &\equiv 9+8\omega+(8+14\omega)T \mod S^8, \end{aligned}$$

so that

$$\begin{aligned} \alpha_i\alpha_j\alpha_k &\equiv (9+8\omega+(8+14\omega)T)(13+\omega(2+8\omega)T) \\ &\equiv (5+8\omega+(8+6\omega)T+(9+8\omega)\omega(2+8\omega)T+(8+14\omega)\omega^2(2+8\omega^2)T^2) \\ &\equiv 13+8\omega \mod S^8. \end{aligned}$$

This shows that $\alpha_i\alpha_j\alpha_k \equiv 1$ modulo S^4. To finish the proof, note that

$$\begin{aligned}\alpha_i\alpha_j\alpha_k\alpha^2 &\equiv (13+8\omega)\left(\frac{1}{\sqrt{-7}}(1-2\omega)\right)^2\\ &\equiv -\frac{1}{7}(13+8\omega)(1-2\omega)^2\\ &\equiv -\frac{9}{7}\\ &\equiv 1 \mod S^8.\end{aligned}$$

□

Theorem 3.42. *There is an element Θ in $\mathbb{Z}_2[[\mathbb{S}_2^1]]$ satisfying the conditions of Lemma 3.13 such that*

$$\begin{aligned}\Theta &\equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k\\ &\quad -\frac{1}{3}tr_{C_3}\Big((e-\alpha_i)(j-\alpha_j)+(e-\alpha_i\alpha_j)(k-\alpha_k)+(e-\alpha_i\alpha_j\alpha_k)(e+\alpha)\Big)\end{aligned}$$

modulo $\mathcal{I}=((IK^1)^7, 2(IK^1)^3, 4(IK^1), 8)$, where tr_{C_3} is defined by (3.12).

Proof. We will use the following facts. First, note that

$$\tau e_q = e_q,$$

for $\tau \in G_{24}$ and $q=0$, or for $\tau\in C_6$ and $q=1$. This implies that

$$\tau(e-\alpha)e_0 = (e-\alpha_\tau\alpha)e_0.$$

It also implies that

$$\omega i e_q = j e_q, \quad \omega^2 i e_q = k e_q,$$

since $j=\omega i\omega^{-1}$ and $k=\omega^{-1}i\omega$. The element $\alpha\in\mathbb{W}^\times\subseteq\mathbb{S}_2$ commutes with ω. This implies that

$$\omega\alpha_i e_q = \alpha_j e_q.$$

We will use the fact that for $\tau\in G_{24}$,

$$(\tau-\alpha_\tau)(e-\alpha)e_0 = (e-\alpha_\tau)e_0.$$

We will also use the following identity:

$$e-gh = (e-g)+(e-h)-(e-g)(e-h).$$

Let $\Theta_0 = e+i$. Then $tr_{C_3}(\Theta_0)e_1 = (3+i+j+k)e_1$ and

$$\begin{aligned}\partial_1(\Theta_0e_1) &= (e+i)(e-\alpha)e_0\\ &= (e-\alpha)e_0+(e-\alpha_i\alpha)e_0\\ &= 2(e-\alpha)e_0+(e-\alpha_i)e_0-(e-\alpha_i)(e-\alpha)e_0\\ &= (e-\alpha^2)e_0+(e-\alpha)^2e_0+(e-\alpha_i)e_0-(e-\alpha_i)(e-\alpha)e_0.\end{aligned}$$

Let

$$\Theta_1 = e+i-(e-\alpha)+(e-\alpha_i).$$

Then

$$\partial_1(\Theta_1e_1) = (e-\alpha^2)e_0+(e-\alpha_i)e_0.$$

Therefore,

$$\begin{aligned}\partial_1(tr_{C_3}(\Theta_1)e_1) &= 3(e-\alpha^2)e_0 + (e-\alpha_i)e_0 + (e-\alpha_j)e_0 + (e-\alpha_k)e_0 \\ &= 3(e-\alpha^2)e_0 + (e-\alpha_i)(e-\alpha_j)e_0 + (e-\alpha_i\alpha_j)e_0 + (e-\alpha_k)e_0 \\ &= 3(e-\alpha^2)e_0 + (e-\alpha_i)(e-\alpha_j)e_0 + (e-\alpha_i\alpha_j)(e-\alpha_k)e_0 \\ &\quad + (e-\alpha_i\alpha_j\alpha_k)e_0 \\ &= 2(e-\alpha^2)e_0 + (e-\alpha_i)(e-\alpha_j)e_0 + (e-\alpha_i\alpha_j)(e-\alpha_k)e_0 \\ &\quad + (e-\alpha_i\alpha_j\alpha_k)(e-\alpha^2)e_0 + (e-\alpha_i\alpha_j\alpha_k\alpha^2)e_0\end{aligned}$$

Let

$$\begin{aligned}\Theta_2 = {}& tr_{C_3}\big(e+i-(e-\alpha)+(e-\alpha_i)\big) - 2(e+\alpha) \\ & - (e-\alpha_i)(j-\alpha_j) - (e-\alpha_i\alpha_j)(k-\alpha_k) - (e-\alpha_i\alpha_j\alpha_k)(e+\alpha).\end{aligned}$$

Then $\Theta_2 \equiv 3+i+j+k \mod (4, IK^1)$. Further,

$$\partial_1(\Theta_2 e_1) = (e-\alpha_i\alpha_j\alpha_k\alpha^2)e_0$$

By Lemma 3.41, $\alpha_i\alpha_j\alpha_k\alpha^2 \in F_{8/2}K^1$. By Corollary 3.40, there exists h such that

$$h \equiv 0 \mod \mathcal{I},$$

where $\mathcal{I} = ((IK^1)^7, 2(IK^1)^3, 4IK^1, 8)$, and

$$\partial_0(\Theta_2 - h) = 0.$$

Define

$$\Theta = \frac{1}{3} tr_{C_3}(\Theta_2 - h).$$

Then Θ satisfies the conditions of Lemma 3.13. Further, modulo $\mathcal{I}$,

$$\begin{aligned}\Theta \equiv {}& e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \\ & - \frac{1}{3} tr_{C_3}\Big((e-\alpha_i)(j-\alpha_j) + (e-\alpha_i\alpha_j)(k-\alpha_k) + (e-\alpha_i\alpha_j\alpha_k)(e+\alpha)\Big).\end{aligned}$$

□

Corollary 3.43. *Let $\mathcal{J} = (IF_{4/2}K^1, (IF_{3/2}K^1)(IS_2^1), \mathcal{I})$. The element Θ of Theorem 3.42 satisfies*

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \mod \mathcal{J}$$

and

$$\Theta \equiv e+\alpha \mod (2, (IS_2^1)^2).$$

Proof. First, note that $\alpha_\tau \in F_{3/2}K^1$ for $\tau \in G_{24}$. Further, by Lemma 3.41, $\alpha_i\alpha_j\alpha_k$ is in $F_{4/2}K^1$. Hence, it follows from Theorem 3.42 that

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \mod \mathcal{J}.$$

For the second claim, we first prove that $\mathcal{J} \subseteq (2, (IS_2^1)^2)$. It is clear that,

$$((IF_{3/2}K^1)(IS_2^1), \mathcal{I}) \subseteq (2, (IS_2^1)^2).$$

Further, it follows from Lemma 3.38 and the fact that $(e-x^{2^k}) \equiv (e-x)^{2^k}$ modulo (2) that

$$IF_{4/2}K^1 \subseteq (2, (IS_2^1)^2).$$

Therefore, $\mathcal{J} \subseteq (2, (IS_2^1)^2)$. Hence,

$$\Theta \equiv e + \alpha + i + j + k - \alpha_i - \alpha_j - \alpha_k \mod (2, (IS_2^1)^2).$$

Since $ij = k$,

$$(e-i)(e-j) \equiv e + i + j + k \mod (2).$$

Further,

$$e - \alpha_i = i\alpha((e-\alpha^{-1})(e-i^{-1}) - (e-i^{-1})(e-\alpha^{-1})).$$

Therefore, $e+i+j+k$ and $e-\alpha_\tau$ are in $(2, (IS_2^1)^2)$. Hence, we conclude that

$$\Theta \equiv e + \alpha \mod (2, (IS_2^1)^2).$$

□

4. Appendix: Lazard's theory

The goal of the following section is to outline the work of Lazard in [22] on groups which are uniformly powerful. This theory applies to the Morava stabilizer group. In particular, it was used in Theorem 2.39 to compute $H^*(F_{3/2}\mathbb{S}_2; \mathbb{F}_2)$.

The following is Definition II.1.1.1 of [22].

Definition 4.1. *A filtration w on a group G is a function*

$$w : G \to \mathbb{R}_+^* \cup \{+\infty\},$$

satisfying

(i) $w(xy^{-1}) \geq min(w(x), w(y))$,
(ii) $w(x^{-1}y^{-1}xy) \geq w(x) + w(y)$.

Given a filtration, one can define the following distinguished subgroups (II. 1.1.2 and II.1.1.7 of [22]).

Definition 4.2. *Given a filtration w on G, let*

$$G_\nu := \{x \in G \mid w(x) \geq \nu\},$$
$$G_{\nu^+} := \{x \in G \mid w(x) > \nu\}.$$

Further, define the pieces of the associated graded by

$$\mathrm{gr}_\nu G := G_\nu / G_{\nu^+},$$

and

$$\mathrm{gr}\, G := \bigoplus \mathrm{gr}_\nu G.$$

A filtration induces a topology on G, where the groups G_ν form a system of fundamental neighborhoods of the identity. If w is such that $w(x) = +\infty$ if and only if $x = 1$, then the induced topology on G is separated. The group G is complete if

$$G = \lim_{\leftarrow} G/G_\nu.$$

One can show that $\mathrm{gr}\, G$ is graded Lie algebra, where the bracket operation on homogenous elements γ and η of $\mathrm{gr}\, G$ is defined by choosing lifts x and y and letting $[\gamma, \eta]$ be the image in $\mathrm{gr}\, G$ of the commutator $xyx^{-1}y^{-1}$. The following is Definition III.2.1.2 of [22].

Definition 4.3. *A filtered group G is* p-valuable *if, for all $x \in G$,*

(1) $w(x) < \infty$ *when* $x \neq 1$

(2) $w(x) > \dfrac{1}{p-1}$

(3) $w(x^p) = w(x) + 1$

For groups G which are p-valuable, it follows from *(3)* that the elements of $\operatorname{gr} G$ are annihilated by p. Hence, $\operatorname{gr} G$ is a Lie algebra over $\mathbb{F}_p$. One can define the restriction $P : \operatorname{gr}_\nu G \to \operatorname{gr}_{\nu+1} G$ as follows. For $\gamma \in \operatorname{gr}_\nu G$, choose a lift $x \in G$. Let $P(\gamma)$ be the image of x^p in $\operatorname{gr}_{\nu+1} G$. This is a well-defined operation. (It gives $\operatorname{gr} G$ the structure of a *mixed* Lie algebra as defined in II.1.2.5 of [22]. In fact, to obtain a mixed Lie algebra structure on $\operatorname{gr} G$, it suffices that $w(x^p) \geq \min(w(x)+1, pw(x))$. See II.1.2.10 and II.1.2.11 of [22] for a more detailed explanation.)

Let $\Gamma = \mathbb{F}_p[\Pi]$. Define the action of Π on $\operatorname{gr} G$ by

$$\Pi\gamma = P(\gamma).$$

One can show that $\operatorname{gr} G$ is a free Γ-module (see Theorem I.1.2.3 of [22]). Therefore, the following definition is sensible (see Definition III. 2.1.3 of [22]).

Definition 4.4. *Let G be p-valuable. The rank of G is the rank of* $\operatorname{gr} G$ *as a Γ-module.*

This allows us to make the following definition, which is Definition V.2.2.7 of [22].

Definition 4.5. *Let G be p-valuable. Then G is* uniformly powerful *if it is complete, of finite rank over Γ and if* $\operatorname{gr} G$ *is generated by elements of degree t for some fixed filtration t.*

Note that the terminology *uniformly powerful* is that of Symonds and Weigel in [35]. Lazard calls these groups *équi-p-valué*.

The following is Proposition V.2.5.7.1 of [22]. To state it, define

$$L = \operatorname{gr} G / \Pi \operatorname{gr} G.$$

Proposition 4.6. *Let G be a uniformly powerful group of rank r. Then $H^1_c(G, \mathbb{F}_p)$ is an r-dimensional $\mathbb{F}_p$-vector space isomorphic to the $\mathbb{F}_p$-dual L^*, and*

$$H^*_c(G, \mathbb{F}_p) \cong E(L^*).$$

Proof sketch. The Lie algebra L is abelian of rank r over $\mathbb{F}_p$, so that $H^*(L, \mathbb{F}_p) \cong E(L^*)$. To prove the proposition, one constructs an isomorphism $H^*_c(G, \mathbb{F}_p) \cong H^*(L, \mathbb{F}_p)$. □

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