A $C_2$-EQUIVARIANT ANALOG OF MAHOWALD’S THOM SPECTRUM THEOREM

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ABSTRACT. We prove that the $C_2$-equivariant Eilenberg-MacLane spectrum associated with the constant Mackey functor $F_2$ is equivalent to a Thom spectrum over $\Omega^\rho S^{\rho+1}$.

1. INTRODUCTION

Let $\mu$ be the Möbius bundle over $S^1$, regarded as a virtual bundle of dimension 0. The mod 2 Moore spectrum is the Thom spectrum

$$M(2) \cong (S^1)^\mu.$$ 

The classifying map for $\mu$ extends to a double loop map

$$\tilde{\mu} : \Omega^2 S^3 \to BO.$$ 

Mahowald proved the following theorem [Mah77]:

**Theorem 1.1** (Mahowald). There is an equivalence of spectra

$$(\Omega^2 S^3)^\mu \tilde{\cong} H F_2.$$ 

The bundle $\mu$ may also be regarded $C_2$-equivariant virtual bundle over $S^1$, by endowing both $S^1$ and the bundle with the trivial action. Since $B_{C_2} O$ is an equivariant infinite loop space [Ati68], the classifying map for $\mu$ extends to an $\Omega^\rho$-loop map

$$\tilde{\mu} : \Omega^\rho S^{\rho+1} \to B_{C_2} O.$$ 

Here, $\rho$ is the regular representation of $C_2$. The purpose of this paper is to prove the following.

**Theorem 1.2.** There is an equivalence of $C_2$-spectra

$$(\Omega^\rho S^{\rho+1})^{\mu} \tilde{\cong} H F_2.$$ 

(Here, $F_2$ denotes the constant Mackey functor with value $F_2$.)

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Conventions. Equivariant objects in this paper either live in $\text{Top}^{C_2}$, the category of $C_2$-spaces, or $\text{Sp}^{C_2}$, the category of genuine $C_2$-spectra. In both of these categories, the equivalences are those equivariant maps which induce equivalences on both $C_2$-fixed points and underlying fixed points. We let $H$ denote the Eilenberg-Maclane spectrum $H\mathbb{F}_2$, with underlying spectrum $H := H\mathbb{F}_2$. We use $H_*$ and $\pi_*^{C_2}$ to denote $RO(C_2)$-graded homology and homotopy groups (i.e. not the Mackey functors) of $C_2$-equivariant spaces and spectra, and $H_*$ and $\pi_*$ to denote the ordinary homology and homotopy groups of non-equivariant spaces and spectra. We let $\sigma$ denote the sign representation of $C_2$, and let $\rho = 1 + \sigma$ denote the regular representation. For a representation $V$, $S(V)$ denotes the unit sphere in $V$, and $S^V$ denotes its one point compactification, and $|V|$ denotes its dimension.

2. Equivariant preliminaries

Euler class. Let $a$ denote the Euler class in $\pi_{-2}^{C_2} S$, given geometrically by the inclusion

$$S^0 \hookrightarrow S^\sigma.$$ 

There is a cofiber sequence

$$C_2^+ \rightarrow S^0 \hookrightarrow S^\sigma$$

so the cofiber of $a$ is stably given by

$$Ca \cong \Sigma^{-1} C_2^+.$$ 

The transfer induces a map

$$u : S^{1-\sigma} \xrightarrow{\text{tr}} \Sigma^{-1} C_2^+ \cong Ca$$

which serves as a Thom class for the representation $\sigma$:

$$u : S^1 \rightarrow Ca \wedge S^\sigma.$$ 

For $X \in \text{Sp}^{C_2}$, we have

$$\pi_k^{C_2}(X) \cong \pi_k(X^{C_2}),$$

$$\pi_V^{C_2}(X \wedge Ca) \cong \pi_{|V|}(X^e).$$

Said differently,

$$\pi_\ast^{C_2} X \wedge Ca \cong \pi_\ast X^e[u^\pm].$$

Tate square. We will let

$$X^h := F(EC_{2^+}, X),$$

$$X^\Phi := X \wedge \widetilde{EC}_2$$

denote the homotopy completion and geometric localization of $X$, respectively. The fixed points of $X^h$ are the homotopy fixed points of $X$, and the fixed points of $X^\Phi$
are the geometric fixed points of \(X\). \(X\) is recovered from these approximations by the pullback ("Tate square") [GM95]

\[
\begin{array}{c}
X \rightarrow X^\Phi \\
\downarrow \quad \downarrow \\
X^h \rightarrow X^t 
\end{array}
\]

where the spectrum \(X^t\) is the equivariant Tate spectrum

\(X^t := (X^h)^\Phi\).

Note that a generalization of the argument establishing (2.2) yields an equivalence

\[\Sigma^{k\sigma - 1}C(a^k) \simeq S(k\sigma)_+\]

Taking a colimit, we see that we have

\[\operatorname{hocolim}_k \Sigma^{k\sigma - 1}C(a^k) \simeq EC_2^+,
\]

\[\operatorname{hocolim}_k S^{k\sigma} \simeq EC_2.\]

It follows that homotopy completion and geometric localization can be reinterpreted as \(a\)-completion and \(a\)-localization:

\[X^h \simeq X^a_\wedge,
\]

\[X^\Phi \simeq X[a^{-1}].\]

In this manner, the Tate square is equivalent to the "\(a\)-arithmetic square"

\[
\begin{array}{c}
X \rightarrow X[a^{-1}] \\
\downarrow \quad \downarrow \\
X^a_\wedge \rightarrow X^a_\wedge [a^{-1}] 
\end{array}
\]

Using (2.3), the \(a\)-Bockstein spectral sequence takes the form

\[\pi_* (X^a)[u^\pm, a] \Rightarrow \pi_*^{C_2} (X^h).\]

The \(a\)-Bockstein spectral sequence can be regarded as an \(RO(C_2)\)-graded version of the homotopy fixed point spectral sequence (see [HM17, Lem. 4.8]).

**The mod 2 Eilenberg-MacLane spectrum.** We have [HK01]

\[\pi_*^{C_2} H = \mathbb{F}_2 [a, u] \oplus \frac{\mathbb{F}_2 [a, u]}{(a^x, u^x)} \{\theta\}\]

where

\[|u| = 1 - \sigma,
\]

\[|\theta| = 2\sigma - 2.\]

The \(a\)-\(u\) divisible factor in \(\pi_* H\) is best understood from the Tate square, using

\[\pi_*^{C_2} H^h \simeq \mathbb{F}_2 [a, u^\pm],
\]

\[\pi_*^{C_2} H^\Phi \simeq \mathbb{F}_2 [a^\pm, u].\]
Actually, the second isomorphism lifts to an equivalence
\[ H^{\Phi C_2} \simeq H[a^{-1}u] := \bigvee_{i \geq 0} \Sigma^i H \]
so we have
\[ H^\Phi X \simeq H_a(X^{\Phi C_2})[a^\pm, u] \]
and, restricting the grading to trivial representations, we get
\[ (2.4) \quad H^\Phi X \simeq H_a(X^{\Phi C_2})[a^{-1}u]. \]

By applying \( \pi^C_{V^2} \) to the map
\[ H \wedge X \to H \wedge X \wedge Ca \]
we get a homomorphism
\[ (2.5) \quad \Phi^e : H_V(X) \to H_{|V|}(X^e). \]

Taking geometric fixed points of a map
\[ S^V \to H \wedge X \]
gives a map
\[ S^{V^2} \to H^{\Phi C_2} \wedge X^{\Phi C_2} \]
Using (2.4) and passing to the quotient by the ideal generated by \( a^{-1}u \), we get a homomorphism
\[ (2.6) \quad \Phi^{C_2} : H_V(X) \to H_{|V^2|}(X^{\Phi C_2}). \]

A useful lemma. Our main computational lemma is the following.

**Lemma 2.7.** Suppose that \( X \in \text{Sp}^C \) and suppose that \( \{b_i\} \) is a set of elements of \( H_*(X) \) such that

1. \( \{\Phi^e(b_i)\} \) is a basis of \( H_*(X^e) \), and
2. \( \{\Phi^{C_2}(b_i)\} \) is a basis of \( H_*(X^{\Phi C_2}) \).

Then \( H_*(X) \) is free over \( H_* \), and \( \{b_i\} \) is a basis.

**Proof.** The set \( \{b_i\} \) corresponds to a map
\[ H \wedge \bigvee S_{\{b_i\}} \to H \wedge X. \]

Assumption (1) implies this map is an equivalence upon applying \( \Phi^e \), while assumption (2) implies this map is an equivalence upon applying \( \Phi^{C_2} \). The result follows. \( \square \)

3. Homology of \( \rho \)-loop spaces

We spell out some specific algebraic structure carried by the equivariant homology of a \( \rho \)-loop space. A more detailed and general study of this algebraic structure will appear in [Hil].
**Products.** Suppose \( X = \Omega \rho Y \in \text{Top}^{C_2} \) is a \( \rho \)-loop space. Then \( X \) is in particular a 1-loop space, and is therefore an equivariant \( H \)-space with product

\[
m : X \times X \to X.
\]

However, the \( \sigma \)-loop space structure also endows \( X \) with a twisted product related to the transfer. Namely, let

\[
S^\sigma \to S^\sigma / S^0 \simeq C_2^+ \wedge S^1
\]

be the pinch map. This gives rise to a twisted product

\[
\tilde{m} : N^x \Omega Y \to \Omega^\sigma Y
\]

where

\[
N^x Z := \text{Map}(C_2, Z) = Z \times Z / C_2
\]

is the norm (with respect to Cartesian product). In particular, there is a map

(3.1)

\[
\tilde{m} : N^x \Omega^2 Y \to X.
\]

Upon applying fixed points to the map (3.1), we get an additive transfer

(3.2)

\[
t : X^e \to X^{C_2}.
\]

In homology, the \( H \)-space structure give rise to a product

\[
m : H^Y X \otimes H^W X \to H^Y + W X.
\]

Using the equivariant commutative ring spectrum structure of \( H \) [Ull13], the twisted product \( \tilde{m} \) gives rise to a “norm map” (see [BH15, Thm. 7.2])

\[
n : H_k X^e \to H_{k \rho} X.
\]

**Dyer-Lashof operations.** \( X \) has even more structure: \( X \) is an \( E_\rho \)-algebra [GM17]. Specifically, regard \( S(\rho) \) as a \( C_2 \times \Sigma_2 \)-space where \( C_2 \) acts on \( \rho \) and \( \Sigma_2 \) acts antipodally. Then the \( E_\rho \)-structure gives a map

\[
S(\rho) \times_{\Sigma_2} X^{\Sigma_2} \to X.
\]

Note that \( H \) is itself an \( E_\rho \)-ring spectrum, because it is actually an equivariant commutative ring spectrum, so \( H \wedge X_+ \) is an \( E_\rho \)-ring in \( H \)-modules. Given \( x \in H^Y(X) \), represented by a map

\[
x : S^V \to H \wedge X_+,
\]

there is an induced composite

\[
H \wedge S(\rho)_+ \wedge_{\Sigma_2} S^{2V} \xrightarrow{1 \wedge 1 \wedge x \wedge x} H \wedge S(\rho)_+ \wedge_{\Sigma_2} (H \wedge X_+)^{\Sigma_2} \xrightarrow{-} H \wedge H \wedge X_+ \xrightarrow{-} H \wedge X_+ \xrightarrow{-} H \wedge X_+
\]

(where the unlabeled maps come from the \( E_\rho \)-ring and \( H \)-module structure of \( H \wedge X_+ \)). Applying \( \pi_{C_2}^* \), we get a total power operation

\[
\mathcal{T}(x) : \tilde{H}_*(S(\rho)_+ \wedge_{\Sigma_2} S^{2V}) \to \tilde{H}_* X.
\]

For the purposes of this paper we will be only concerned with the case of \( V = k\rho - \sigma \).

We will need the following lemma.
Lemma 3.3. We have the following identification of the $C_2$-fixed point space of the extended power:

$$(S(\rho)_+ \wedge_{\Sigma_2} S^{2(kp-\sigma)})^{C_2} \approx S^{2k-1} \vee S^{2k}.$$ 

Proof. The extended power can be identified with the Thom complex of the equivariant vector bundle

$$S(\rho) \times_{\Sigma_2} \mathbb{R}^{2(kp-\sigma)} \to S(\rho)/\Sigma_2.$$ 

The fixed points is the Thom complex of the fixed point bundle. Thinking of $S(\rho)$ as the unit circle in $\mathbb{C}$, with $C_2$ acting by conjugation, the fixed points of the base are given by

$$[S(\rho)/\Sigma_2]^{C_2} = \{[1], [i]\}.$$ 

The bundle has fiber $\mathbb{R}^{2(kp-\sigma)}$ over $[1]$, and because $\Sigma_2$ acts with the antipodal action mixed with the interchange action, the fiber over $[i]$ is given by

$$\mathbb{R}^{p(kp-\sigma)} = \mathbb{R}^{(2k-1)p}.$$ 

The result follows. 

Proposition 3.4. We have

$$\widetilde{H}_* S(\rho)_+ \wedge_{\Sigma_2} S^{2(kp-\sigma)} \cong \widetilde{H}_* \{e_{2kp-\sigma-1}, e_{2kp-\sigma}\}.$$ 

Proof. Theorem 2.15 of [Wil17] implies there is a cofiber sequence

$$S^{2k-2\sigma} \to S(\rho)_+ \wedge_{\Sigma_2} S^{2(kp-\sigma)} \to S^{2kp-1}.$$ 

There are two possibilities for the long exact sequence in $\widetilde{H}_*$: either (a) the connecting homomorphism sends $\iota_{2k-1}$ to zero, or (b) the connecting homomorphism sends it to $\theta_{2kp-2\sigma}$. Only possibility (b) is compatible with Lemma 3.3 from geometric fixed point considerations. The result follows. 

Thus we get a pair of Dyer-Lashof operations

$$Q^{kp} : \widetilde{H}_{kp-\sigma} X \to \widetilde{H}_{2kp-\sigma} X,$$

$$Q^{kp-1} : \widetilde{H}_{kp-\sigma} X \to \widetilde{H}_{2kp-\sigma-1} X$$

given by the formulas

$$Q^{kp}(x) := \mathcal{P}(x)(e_{2kp-\sigma}),$$

$$Q^{kp-1}(x) := \mathcal{P}(x)(e_{2kp-\sigma-1}).$$

Remark 3.5. If $X$ is actually an equivariant infinite loop space, then $\widetilde{H}_* X$ has an action by equivariant Dyer-Lashof operations [Wil17], and these operations agree with those defined in that paper.
Compatibility with fixed points. The compatibility of all this structure with the maps $\Phi^e$ and $\Phi^{C_2}$ of (2.5) and (2.6) is summarized as follows.

**Products:** Note that $X^e$ is an $E_2$-algebra, and $X^{C_2}$ is an $E_1$-algebra. The maps $\Phi^e$ and $\Phi^{C_2}$ are algebra homomorphisms.

**Norms:** The following diagram commutes:

$$
\begin{array}{ccc}
H_k X^e & \xrightarrow{t} & H_k X^e \\
\downarrow{\scriptstyle n} & \searrow{\scriptstyle Sq} & \\
H_k X^{C_2} & \xrightarrow{\Phi^{C_2}} & H_k X^{C_2}
\end{array}
$$

Here $t$ is the transfer (3.2) and $Sq$ is the squaring map.

**Dyer-Lashof operations:** The following diagrams commute, where $\epsilon = 0, 1$:

$$
\begin{array}{ccc}
H_{k\rho-\sigma} X & \xrightarrow{\Phi^e} & H_{2k-1} X^e \\
\downarrow{\scriptstyle Q^k} & & \downarrow{\scriptstyle Q^{2k-\epsilon}} \\
H_{2k\rho-\sigma-\epsilon} X & \xrightarrow{\Phi^e} & H_{4k-2-\epsilon} X^e
\end{array}
$$

$$
\begin{array}{ccc}
H_{k\rho-\sigma} X & \xrightarrow{\Phi^{C_2}} & H_k X^{C_2} \\
\downarrow{\scriptstyle Q^k} & & \downarrow{\scriptstyle Sq} \\
H_{2k\rho-\sigma} X & \xrightarrow{\Phi^{C_2}} & H_{2k} X^{C_2}
\end{array}
$$

4. Homology of $\Omega^p S^{p+1}$

**Theorem 4.1.** There is an additive isomorphism (of $H_\ast$-modules)

$$H_\ast \Omega^p S^{p+1} \cong H_\ast \otimes E[t_0, t_1, \ldots] \otimes P[e_1, e_2, \ldots]$$

with

$$|t_i| = 2^i \rho - \sigma,$$

$$|e_i| = (2^i - 1) \rho.$$

**Proof.** Note that we have

$$H_6 \Omega^2 S^3 = \mathbb{F}_2[x_1, x_2, \ldots]$$

with

$$|x_i| = 2^i - 1.$$ 

Here $x_1$ is the fundamental class $t_1$, and

$$x_i := Q^{2^i} Q^{2^{i-1}} \cdots Q^2 x_1.$$
Define $t_0 \in H_* \Omega^p S^{p+1}$ to be the fundamental class, and define the other “generators” $e_i$ and $t_i$ by
\[
e_i := n(x_i),
\]
\[
t_i := Q^{2^i} Q^{2^{i-1}} \cdots Q^p t_0.
\]
Consider the product
\[
t^e_k := t_0^{e_0} t_1^{e_1} \cdots e_k^{e_k}, \quad t_i := Q^2 Q Q^{2i} \cdots Q^p t_0.
\]
with $e_i \in \{0, 1\}$ and $k_i \geq 0$. We compute
\[
\Phi^p(t^e_h) = x_1^{2k_1 + \epsilon_1} x_2^{2k_2 + \epsilon_1} \cdots .
\]
Mapping out of the cofiber sequence (2.1) gives a fiber sequence
\[
\Omega N \Omega S^{p+1} \to \Omega^p S^{p+1} \to \Omega S^{p+1} \xrightarrow{=} N \Omega S^{p+1}.
\]
Upon taking fixed points we get a fiber sequence
\[
\Omega^2 S^3 \to (\Omega^p S^{p+1})^C_2 \to \Omega S^2 \text{null} \to \Omega S^3.
\]
In particular there is an equivalence
\[
(\Omega^p S^{p+1})^C_2 \simeq \Omega S^2 \times \Omega^2 S^3.
\]
and we have
\[
H_*(\Omega^p S^{p+1})^C_2 \cong P[y] \oplus P[t(x_1), t(x_2), \ldots]
\]
where $y$ is the image of the fundamental class under the map
\[
S^1 \to (\Omega^p S^{p+1})^C_2.
\]
It follows that
\[
\Phi^C_2(t^e_h) = y^{e_0 + 2\epsilon_1 + 4\epsilon_2 + \cdots} t(x_1)^{k_1} t(x_2)^{k_2} \cdots .
\]
Thus the set
\[
\{t^e_h\} \subset H_* \Omega_\ast X
\]
satisfies the hypotheses of Lemma 2.7, and the result follows. \hfill \square

5. The equivariant Mahowald theorem

In order to prove Theorem 1.2 we will need to establish a Thom isomorphism
\[
H_* (\Omega^p S^{p+1}) \cong H_* \Omega^p S^{p+1}.
\]
We will do so in two steps. Recall that an $E_\rho$-algebra is just a spectrum $X$ equipped with a map $S^0 \to X$. Let Free$^*_{E_\rho} : \text{Alg}_{E_\rho}(Sp^{C_2}) \to \text{Alg}_{E_\rho}(Sp^{C_2})$ denote a homotopical left adjoint to the forgetful functor. An explicit model for this functor is the homotopy pushout of $E_\rho$-algebras:
\[
\begin{array}{c}
\text{Free}^*_{E_\rho}(S^0) \longrightarrow \text{Free}^*_{E_\rho}(X) \\
\downarrow \hspace{1cm} \downarrow \\
S^0 \longrightarrow \text{Free}^*_{E_\rho}(X)
\end{array}
\]
We will need the following theorem.

**Theorem 5.1.** Let \( f : X \to B_{C^2}O \) classify a virtual bundle of dimension zero and denote by \( \tilde{f} : \Omega^p \Sigma^p X \to B_{C^2}O \) the associated \( \Omega^p \)-map. Then there is a canonical equivalence of \( E_\rho \)-algebras in \( Sp_{C^2} \)

\[
\text{Free}^*_E (X^f) \cong (\Omega^p \Sigma^p X)^{\tilde{f}}.
\]

**Proof.** Combine the equivariant approximation theorem [GM17, RS00] with Theorem IX.7.1 and Remark X.6.4 of [LMSM86]. □

**Remark 5.2.** The non-equivariant version of Theorem 5.1 was first observed by Mark Mahowald, and then proven by Lewis. A nice modern account in the non-equivariant setting via universal properties can be found in [AB14].

**Proposition 5.3.** There is a Thom isomorphism

\[
H_*(\Omega^p S^{p+1}) \cong H_* \Omega^p S^{p+1}.
\]

**Proof.** Let \( \text{Free}^*_E, H : \text{Alg}_{E_\rho} (\text{Mod}_H) \to \text{Alg}_{E_\rho} (\text{Mod}_H) \) denote a homotopical left adjoint to the forgetful functor. Along with the previous theorem, we will need two facts:

1. \( H \wedge (-) : Sp_{C^2} \to \text{Mod}_H \) is symmetric monoidal.
2. There is a Thom isomorphism \( H \wedge (S^1)^\mu \cong H \wedge S^1_\mu \).

The proposition is now proved by the following string of equivalences:

\[
H \wedge (\Omega^p \Sigma^p S^1)^{\tilde{\mu}} \cong H \wedge \text{Free}^*_E ((S^1)^\mu) \quad \text{by Theorem 5.1}
\]
\[
\cong \text{Free}^*_E, H (H \wedge (S^1)^\mu) \quad \text{by (1)}
\]
\[
\cong \text{Free}^*_E, H (H \wedge S^1_{\mu}) \quad \text{by (2)}
\]
\[
\cong H \wedge \text{Free}^*_E, H (S^1_{\mu}) \quad \text{by (1)}
\]
\[
\cong H \wedge \Omega^p \Sigma^p S^1_{\mu}.
\]

□

**Proof of Theorem 1.2.** The Thom class is represented by a map

\[
(\Omega^p S^{p+1})^{\tilde{\mu}} \to H.
\]

We wish to show this map is an isomorphism on \( H_* \). The homology of \( H \) is the \( C_2 \)-equivariant Steenrod algebra, computed in [HK01] to be

\[
H \wedge H = H_* [\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots] / (\tau_i^2 = (u + a\tau_0) \xi_{i+1} + a\tau_{i+1})
\]

with

\[
|\tau_i| = 2^i \rho - \sigma,
\]
\[
|\xi_i| = (2^i - 1) \rho.
\]

It suffices to show it is surjective, since the two homologies are abstractly isomorphic and of finite type. Observe that the composite

\[
M(2) \cong (S^1)^\mu \to (\Omega^p S^{p+1})^{\tilde{\mu}} \to H
\]
hits $\tau_0$. Everything is hit then, by [Wil17, Thm. 5.4]. □

References


[Mah77] Mark Mahowald, A new infinite family in $\pi_{2s}^s$, Topology 16 (1977), no. 3, 249–256. MR 0445498


