# Localization for $THH(ku)$ and the topological Hochschild and cyclic homology of Waldhausen categories

Andrew J. Blumberg

Michael A. Mandell

Author address:

Department of Mathematics, The University of Texas, Austin, TX 78712
*E-mail address*: blumberg@math.utexas.edu

Department of Mathematics, Indiana University, Bloomington, IN 47405
*E-mail address*: mmandell@indiana.edu

# Contents

# Abstract

We prove a conjecture of Hesselholt and Ausoni-Rognes, establishing localization cofiber sequences of spectra

$$THH(H\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow THH(ku|KU) \longrightarrow \Sigma THH(H\mathbb{Z})$$

and

$$TC(H\mathbb{Z}) \longrightarrow TC(ku) \longrightarrow TC(ku|KU) \longrightarrow \Sigma TC(H\mathbb{Z})$$

for the topological Hochschild and cyclic homology ($THH$ and $TC$) of topological $K$-theory. These sequences support Hesselholt's view of the map $\ell \to ku$ as a "tamely ramified" extension of ring spectra, and validate the hypotheses necessary for Ausoni's simplified computation of $V(1)_*K(KU)$.

In order to make sense of the relative term $THH(ku|KU)$ and prove these results, we develop a theory of $THH$ and $TC$ of Waldhausen categories and prove the analogues of Waldhausen's theorems for $K$-theory. We resolve the longstanding confusion about localization sequences in $THH$ and $TC$, and establish a specialized dévissage theorem.

Received by the editor March 19, 2014.

2010 *Mathematics Subject Classification.* Primary 19D55; Secondary 55P43,19L41,19D10.

*Key words and phrases.* Topological Hochschild homology, topological cyclic homology, Waldhausen category, localization sequence, $K$-theory, dévissage theorem.

The first author was supported in part by an NSF postdoctoral fellowship, a Clay Liftoff fellowship, and NSF grants DMS-0111298, DMS-0906105.

The second author was supported in part by NSF grants DMS-0504069, DMS-0804272, DMS-1105255.

# Introduction

Algebraic $K$-theory provides a high-level invariant of the homotopy theory of categories with a notion of extension and equivalence. The component group, $K_0$, is the universal target for Euler characteristics, and higher algebraic $K$-theory captures subtle information intricately tied to number theory and geometry. For the algebraic $K$-theory of rings, trace methods using topological Hochschild homology ($THH$) and topological cyclic homology ($TC$) have proved remarkably successful at making $K$-theory computations tractable via the methods of equivariant stable homotopy theory.

At first glance $K$-theory and $THH$ take very different inputs and have very different formal properties. For algebraic $K$-theory, the input is typically a Waldhausen category: a category with subcategories of cofibrations and weak equivalences. For $THH$, the basic input is a spectral category: a category enriched in spectra. While $THH$ shares $K$-theory's additivity properties, $THH$ seems to lack $K$-theory's approximation and localization properties [**9**]. A specific example of this failure was studied at great length in the paper [**16**]. From the perspective of the algebraic $K$-theory of rings and connective ring spectra, where $THH$ is the stabilization of $K$-theory, this discrepancy is in some ways surprising, as one might expect $THH$ to inherit the fundamental properties of $K$-theory.

In this paper, we construct $THH$ for a general class of Waldhausen categories, and show that much of the apparent mismatch of formal properties is a consequence of the former mismatch of input data. We obtain an analogue of Waldhausen's Approximation Theorem [**35**, 1.6.7] for $THH$. On the other hand, we observe that $THH$ has two different analogues of the localization sequence in Waldhausen $K$-theory (the "Fibration Theorem" [**35**, 1.6.4]). One of the localization sequences for $THH$ was developed in our companion paper on localization in $THH$ of spectral categories [**7**, 7.1] (see Theorem 1.3.13 below); when applied to the $K$-theory of schemes, this sequence produces an analogue of the localization sequence of Thomason-Trobaugh [**32**]. The other localization sequence generalizes the localization sequence of Hesselholt-Madsen [**16**]. One of the principal contributions of this paper is to provide a conceptual explanation of the two localization sequences of $THH$ in relation to the localization sequence of $K$-theory.

As we explain in Sections 2.2 and 5.2, a Waldhausen category that admits factorizations has two spectral categories associated to it, a connective and a non-connective variant. The non-connective theory is "correct" from the perspective of abstract homotopy theory and satisfies localization for cofiber sequences of spectral categories [**7**, 7.1], but the connective theory is more closely related to $K$-theory. We show that the two theories agree under connectivity hypotheses that we make explicit in Section 3.4; in particular, for rings and connective ring spectra both spectral categories produce the expected $THH$. For exact categories, the connective version agrees with the $THH$ of exact categories defined by Dundas-McCarthy [**10**]. For categories of complexes, the non-connective version agrees with the $THH$ of the spectral derived category studied in [**7**]. Working with the non-connective theory gives the Thomason-Trobaugh style localization sequences, and working with the connective theory gives the Hesselholt-Madsen style localization sequences.

As a main application of this theory, we prove the localization sequence associated to the transfer map from $H\mathbb{Z}$ to $ku$ that was conjectured by Hesselholt and Ausoni [**1, 2**]. Specifically, we construct naturally out of the category of $ku$-modules a simplicial spectral category $W^{\Gamma}(ku|KU)$ and cofiber sequences in the stable category

$$THH(\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow THH(ku|KU) \longrightarrow \Sigma THH(\mathbb{Z})$$

and

$$TC(\mathbb{Z}) \longrightarrow TC(ku) \longrightarrow TC(ku|KU) \longrightarrow \Sigma TC(\mathbb{Z}),$$

compatible via a trace map with the localization cofiber sequence in $K$-theory established in [**5**]. Corresponding results hold for the Adams summand in the $p$-local and $p$-complete cases; see Theorem 4.2.1 below for details. These localization sequences were conjectured by Hesselholt and Ausoni-Rognes to explain the relationship of the computations of $K(\ell)$ and $K(ku)$; they support the perspective that $\ell \to ku$ should be an example of a "tamely ramified" extension of ring spectra. Furthermore, using these localization sequences, one can dramatically simplify Ausoni's computation of $K(ku)$ [**2**, 8.4] by mimicking the de Rham-Witt arguments in Hesselholt-Madsen [**16**]. These localization sequences provide the chromatic level 1 analogues of the chromatic level 0 sequence of Hesselholt and Madsen [**16**]. Another application of these localization sequences is to compute $K(KU)$. One would like to use Ausoni's computations of $K(ku)$ along with the localization cofiber sequence

$$K(\mathbb{Z}) \longrightarrow K(ku) \longrightarrow K(KU) \longrightarrow \Sigma K(\mathbb{Z})$$

to evaluate $K(KU)$. The transfer map in this sequence is controlled by the behavior of the transfer map in the associated sequences in $THH$ and $TC$, where it is easier to understand. Following Hesselholt, Ausoni [**2**, 8.3] observes that in light of his calculations, the existence of the localization cofiber sequence in $THH$ along with an algebraic fact would permit the complete identification of $V(1)_*K(KU)$.

For higher chromatic levels, Rognes has conjectured $K$-theory localization sequences of the form

$$K(BP\langle n-1\rangle^{\wedge}_{p}) \longrightarrow K(BP\langle n\rangle^{\wedge}_{p}) \longrightarrow K(E(n)^{\wedge}_{p}) \longrightarrow \Sigma K(BP\langle n-1\rangle^{\wedge}_{p})$$

as part of an ambitious program to provide a conceptual understanding of Waldhausen's $A$-theory of a point. Such sequences are attractive because they would relate the algebraic $K$-theory of the nonconnective ring spectrum $E(n)$, to which trace methods do not apply directly, to the algebraic $K$-theory of connective ring spectra $BP\langle n\rangle$, to which trace methods do apply. The corresponding conjectural localization sequences for $THH$ and $TC$ would then optimistically provide tools for organizing the trace method computations.

So far, these sequences in both algebraic $K$-theory and $TC$ remain conjectural for $n > 1$, and there is some reason to be suspicious about the existence of these sequences. However, our methods both in [**5**] and in this paper do establish the existence of the variant localization sequences

$$K(\mathbb{W}\mathbb{F}_{p^n}[\![u_1,\ldots,u_{n-1}]\!]) \longrightarrow K(BP_n) \longrightarrow K(E_n) \longrightarrow \Sigma K(\mathbb{W}\mathbb{F}_{p^n}[\![u_1,\ldots,u_{n-1}]\!])$$

and

$$TC(\mathbb{W}\mathbb{F}_{p^n}[\![u_1,\ldots,u_{n-1}]\!]) \longrightarrow TC(BP_n) \longrightarrow TC(BP_n|E_n) \longrightarrow \Sigma TC(\mathbb{W}\mathbb{F}_{p^n}[\![u_1,\ldots,u_{n-1}]\!])$$

for all $n$, where $\mathbb{W}$ denotes the $p$-typical Witt ring and $BP_n$ is the connective cover of the Lubin-Tate spectrum $E_n$. This gives a new approach to the continuation of the Rognes program, using current technology. This approach has three main advantages over the program as laid out in [**3**]:

(i) The localization sequences for $K$-theory, $TC$, and $THH$ relating the spectra $H\mathbb{W}\mathbb{F}[\![u_1,\ldots,u_{n-1}]\!]$, $BP_n$, and $E_n$ are known to exist (as mentioned above) in contrast to the sequences relating $BP\langle n-1\rangle$, $BP\langle n\rangle$, and $E(n)$, which are not (for $n>1$).
(ii) The relevant spectrum in the next step of Rognes' program for understanding $A_*$, the $K$-theory of the sphere, is $K(E_n)$ rather than $K(E(n))$.
(iii) The spectra $BP_n$ are known to be $E_\infty$ ring spectra, whereas $BP\langle n\rangle$ is currently only known to be $A_\infty$. (The Ausoni-Rognes computations require more than an $A_\infty$ structure on $BP\langle n\rangle$; the papers are written in terms of an $E_\infty$ structure, though somewhat less will suffice).

These localization sequences give the opportunity to continue the Rognes program, with attention focused on the computation of $TC(BP_n)$ and evaluation of the transfer map.

One of the interesting aspects in the construction of the localization sequences is the construction of the relative terms such as $THH(ku|KU)$ and $TC(ku|KU)$: these relative terms "mix" the weak equivalences in the category of $ku$-modules with the weak equivalences in the category of $KU$-modules, in a way which does not arise in algebraic $K$-theory. This mixing is the reason why there are two different localization sequences. In order to explain these sorts of relative terms, Rognes [**25**] has developed a theory of log ring spectra motivated by the appearance of log rings in the work of Hesselholt and Madsen [**16**]. We expect that our relative terms agree with the log $THH$ and $TC$ defined by Rognes.

Because our primary interest is the construction and explanation of the localization sequences above, we have taken a technical shortcut that drastically simplifies the theory. In Section 2.1, we introduce the concept of a simplicially enriched Waldhausen category in which the Waldhausen structure and the simplicial mapping spaces satisfy strong consistency hypotheses. The motivating example of such a category is a subcategory of the cofibrant objects in a simplicial model category with all objects fibrant; the model structure on the module categories of [**13**] satisfy this condition. For the majority of the paper we work only with simplicially enriched Waldhausen categories. In Section 5.2, we argue that simplicially enriched Waldhausen categories are not unduly restrictive by showing that a closed Waldhausen subcategory of a Waldhausen category that admits factorization is equivalent to a simplicially enriched Waldhausen category (in fact, a simplicial model category where every object is fibrant). This equivalence is functorial up to a zigzag of natural weak equivalences.

Although we have taken Waldhausen categories for the basic input to $THH$ and $K$-theory in this paper, alternatively, one could take quasi-categories as the basic input. At this stage, the quasi-category approach would require serious background treatment of the $THH$ of quasi-categories, which is not yet formalized in the literature. On the other hand, since our first step is to replace a general Waldhausen category with a stable simplicial model category, such a background treatment would be essentially independent of the main work in this paper.

In this paper, whenever we work with topological spaces, the reader should understand that we are working in the category of compactly generated weak Hausdorff spaces. We use the words "topological" or "topological space" to highlight when we are using topological spaces rather than simplicial sets; these words should not be construed to imply the use of general topological spaces rather than compactly generated weak Hausdorff spaces.

CHAPTER 1

# Review of $THH$, $TR$, and $TC$

In this chapter we review the construction and basic properties of $THH$, $TR$, and $TC$ of spectral categories. We begin in Section 1.1 by reviewing the definition of spectral categories (in symmetric spectra) and setting some conventions for the rest of the paper. In Section 1.2, we review the construction of $THH$ of spectral categories along the lines first described by Bökstedt [**8**] and the construction of $TR$ and $TC$ from $THH$. In Sections 1.3–1.4, we review the fundamental invariance properties of the $THH$ of spectral categories, including invariance under DK-equivalences, thick closure, and Morita equivalence.

None of the material in this chapter is new; it has previously appeared in substantially similar form in the authors' previous paper on $THH$, $TR$, and $TC$ of spectral categories [**7**] and is reviewed here for easy reference. Specifically, Section 1.1 streamlines and rewrites [**7**, §2] for symmetric spectra of topological spaces. Section 1.2 is based on and closely follows [**7**, §3], while Sections 1.3–1.4 review the main results of [**7**, §5–7] with most proofs omitted.

## 1.1. Review of spectral categories

This section reviews the definition of and sets conventions for spectral categories that we use throughout the remainder of the paper. Although our most common constructions naturally live in the context of symmetric spectra of simplicial sets, we occasionally need symmetric spectra of topological spaces.

Definition 1.1.1. A *spectral category* is a category enriched over symmetric spectra (of topological spaces). Specifically, a spectral category $\mathcal{C}$ consists of:

(i) A collection of objects $\operatorname{ob}\mathcal{C}$ (which need not be a small set),
(ii) A symmetric spectrum $\mathcal{C}(a,b)$ for each pair of objects $a,b \in \operatorname{ob}\mathcal{C}$,
(iii) A unit map $S \to \mathcal{C}(a,a)$ for each object $a \in \operatorname{ob}\mathcal{C}$, and
(iv) A composition map $\mathcal{C}(b,c) \wedge \mathcal{C}(a,b) \to \mathcal{C}(a,c)$ for each triple of objects $a,b,c \in \operatorname{ob}\mathcal{C}$,

satisfying the usual associativity and unit properties. We say that a spectral category is small when the objects $\operatorname{ob}\mathcal{C}$ form a set.

The previous definition makes perfect sense also in the context of symmetric spectra of simplicial sets (indeed that was the convention in [**7**]); the geometric realization/singular simplicial set adjunction that converts back and forth between symmetric spectra of simplicial sets and symmetric spectra of topological spaces is a (symmetric) monoidal functor and so converts back and forth between spectral categories in the simplicial and topological context by application to the mapping spectra.

The definition of spectral functor between spectral categories is the usual definition of an enriched functor:

Definition 1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be spectral categories. A *spectral functor* $F\colon \mathcal{C} \to \mathcal{D}$ is an enriched functor. Specifically, a spectral functor consists of:

(i) A function on objects $F\colon\ \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$, and

(ii) A map of symmetric spectra $F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa,Fb)$ for each pair of objects $a, b \in \mathrm{ob}\,\mathcal{C}$,

which is compatible with the units and the compositions in the obvious sense.

We then have the following elementary notion of weak equivalence of spectral categories. (The more useful definition of *DK-equivalence* of spectral categories is Definition 1.3.1 below.)

Definition 1.1.3. A *weak equivalence* of spectral categories is spectral functor that is a bijection on objects and a weak equivalence (stable equivalence of symmetric spectra) on all mapping spectra.

Small spectral categories generalize ring symmetric spectra and can be viewed as *rings with many objects*. From that perspective, we have the following evident concepts of modules and bimodules over spectral categories:

Definition 1.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be spectral categories. A left $\mathcal{C}$-module is a spectral functor from $\mathcal{C}$ to symmetric spectra. A right $\mathcal{D}$-module is a spectral functor from $\mathcal{D}^{\mathrm{op}}$ to symmetric spectra. A $(\mathcal{D},\mathcal{C})$-bimodule is a spectral functor from $\mathcal{D}^{\mathrm{op}} \wedge \mathcal{C}$ to symmetric spectra; a $\mathcal{C}$-bimodule is a $(\mathcal{C},\mathcal{C})$-bimodule.

Here $\mathcal{D}^{\mathrm{op}}$ denotes the spectral category with the same objects and mapping spectra as $\mathcal{D}$ but the opposite composition map. The spectral category $\mathcal{D}^{\mathrm{op}} \wedge \mathcal{C}$ has as its objects the cartesian product of the objects,

$$\mathrm{ob}(\mathcal{D}^{\mathrm{op}} \wedge \mathcal{C}) = \mathrm{ob}\,\mathcal{D}^{\mathrm{op}} \times \mathrm{ob}\,\mathcal{C} = \mathrm{ob}\,\mathcal{D} \times \mathrm{ob}\,\mathcal{C},$$

and as its mapping spectra the smash product of the mapping spectra

$$(\mathcal{D}^{\mathrm{op}} \wedge \mathcal{C})((d,c),(d',c')) = \mathcal{D}^{op}(d,d') \wedge \mathcal{C}(c,c'),$$

with unit maps the smash product of the unit maps and composition maps the smash product of the composition maps for $\mathcal{D}^{\mathrm{op}}$ and $\mathcal{C}$. Explicitly, a $(\mathcal{D},\mathcal{C})$-bimodule $\mathcal{M}$ consists of a choice of symmetric spectrum $\mathcal{M}(d,c)$ for each $d$ in $\mathrm{ob}\,\mathcal{D}$ and $c$ in $\mathrm{ob}\,\mathcal{C}$, together with maps

$$\mathcal{C}(c,c') \wedge \mathcal{M}(d,c) \wedge \mathcal{D}(d',d) \longrightarrow \mathcal{M}(d',c')$$

for each $d'$ in $\mathrm{ob}\,\mathcal{D}$ and $c'$ in $\mathrm{ob}\,\mathcal{C}$, making the obvious unit and associativity diagrams commute. In particular, for any spectral category $\mathcal{C}$, the mapping spectra $\mathcal{C}(-,-)$ define a $\mathcal{C}$-bimodule. (This example motivates the convention of listing the right module structure first.)

The work of [**28**] provides the category of $(\mathcal{D},\mathcal{C})$-bimodules with a closed model structure.

Proposition 1.1.5. ([**28**, 6.1]) *The category of $(\mathcal{D},\mathcal{C})$-bimodules forms a closed model category where the fibrations are the objectwise fibrations and the weak equivalences are the objectwise weak equivalences in the stable model structure on symmetric spectra.*

The remainder of this section records some technical observations. Because we are working with spaces rather than simplicial sets, we will often need to assume that base points are non-degenerate (include as Hurewicz cofibrations) to avoid pathologies. For spectral categories, modules, and bimodules, we use the following terminology.

DEFINITION 1.1.6. A spectral category $\mathcal{C}$ is *non-degenerately based* if each space $\mathcal{C}(a,b)(n)$ is non-degenerately based (for all objects $a$, $b$, and all $n$) and each unit map $S^0 \to \mathcal{C}(a,a)(0)$ is a Hurewicz cofibration (for all objects $a$); otherwise, we say that $\mathcal{C}$ is *degenerately based*. A $\mathcal{C}$-module or $(\mathcal{D},\mathcal{C})$-bimodule $\mathcal{M}$ is *non-degenerately based* if each space $\mathcal{M}(c)(n)$ or $\mathcal{M}(d,c)(n)$ is non-degenerately based (for all objects $c, d$ and all $n$); otherwise, we say that $\mathcal{M}$ is *degenerately based*.

The geometric realization of a spectral category or module in the simplicial context is always non-degenerately based. For an arbitrary spectral category, we can find a weakly equivalent non-degenerately based spectral category by taking the geometric realization of the singular simplicial set functor applied to its mapping spectra,

$$|\operatorname{Sing}\mathcal{C}|(a,b) := |\operatorname{Sing}\mathcal{C}(a,b)|.$$

When $\mathcal{M}$ is a $\mathcal{C}$-bimodule, $|\operatorname{Sing}\mathcal{M}|$ is a $|\operatorname{Sing}\mathcal{C}|$-bimodule. More generally, for an arbitrary bimodule over a non-degenerately based spectral category, we can find a weakly equivalent non-degenerately based replacement by applying the cofibrant replacement functor of Proposition 1.1.5.

Another technical point arises when considering the homotopy groups of symmetric spectra. In general the object in the stable category represented by a symmetric spectrum may not agree with the object represented by its underlying prespectrum. This happens for example for the desuspension spectrum $F_1S^0$. In such circumstances, the only sensible convention is to regard the underlying prespectrum as being incorrect. Thus, throughout this paper, we use the following convention.

CONVENTION 1.1.7. The *homotopy groups* of a symmetric spectrum $X$ always means the homotopy groups of $X$ as an object of the stable category, i.e., the abelian groups of maps in the stable category from $S^q$ to $X$ (for $q \in \mathbb{Z}$), and we will denote these as $\pi_q X$. A *weak equivalence* of symmetric spectra always means a weak equivalence in the stable model structure. A weak equivalence is then precisely a map that induces an isomorphism on homotopy groups.

In practice, in many cases the underlying prespectrum does represent the correct object in the stable category. We use the following terminology for this.

DEFINITION 1.1.8. A symmetric spectrum is *semistable* when fibrant approximation in the stable model structure is a weak equivalence of underlying prespectra.

When needed, we can replace an arbitrary small spectral category with a weakly equivalent spectral category that has the same objects but has mapping spectra that are $\Omega$-spectra. For example, we can do this using [**28**, §6] which constructs a cofibrantly generated Quillen model category structure on the category of small enriched categories with a fixed set of objects: The maps in this category are the spectral functors that are the identity on object sets, the fibrations are the maps $\mathcal{C} \to \mathcal{D}$ that restrict to fibrations of symmetric spectra $\mathcal{C}(x,y) \to \mathcal{D}(x,y)$ for all $x, y$ and the weak equivalences are the maps that restrict to weak equivalences $\mathcal{C}(x,y) \to \mathcal{D}(x,y)$ for all $x, y$. Following the terminology of [**28**, §6]:

DEFINITION 1.1.9. A small spectral category $\mathcal{C}$ is *pointwise fibrant* if $\mathcal{C}(x,y)$ is a fibrant symmetric spectrum (in the stable model structure) for every pair of objects

$x, y$. Likewise, $\mathcal{C}$ is said to be *pointwise cofibrant* if $\mathcal{C}(x, y)$ is a cofibrant symmetric spectrum for every pair of objects $x, y$. For a spectral functor of small spectral categories $F\colon \mathcal{C} \to \mathcal{D}$ that is the identity on the object sets, we say that $F$ is a *pointwise weak equivalence* or *pointwise level equivalence* if for every pair of objects $x, y$, the map $F\colon \mathcal{C}(x, y) \to \mathcal{D}(x, y)$ is a weak equivalence or level equivalence, respectively, of symmetric spectra.

The fibrant replacement functors of [**28**, §6], though constructed in the context of a fixed object set still behave well with respect to spectral functors that are not the identity on object sets. We then get the following proposition.

PROPOSITION 1.1.10. ([**28**, 6.3]) *Given a small spectral category $\mathcal{C}$, there exists a small spectral category $\mathcal{C}^{\Omega}$ and a spectral functor $R\colon \mathcal{C} \to \mathcal{C}^{\Omega}$ such that:*

(i) *$\mathcal{C}^{\Omega}$ has the same objects as $\mathcal{C}$ and $R$ is the identity map on objects,*
(ii) *$\mathcal{C}^{\Omega}$ is pointwise fibrant, and*
(iii) *$R$ is a pointwise weak equivalence.*

*Moreover, $(-)^{\Omega}$ and $R$ may be constructed as an endofunctor and natural transformation on the category of small spectral categories.*

Applying cofibrant replacement in the model structure of [**28**, §6], we obtain the following complementary proposition.

PROPOSITION 1.1.11. ([**28**, 6.3]) *Given a small spectral category $\mathcal{C}$, there exists a small spectral category $\mathcal{C}^{\mathrm{Cell}}$ and a spectral functor $Q\colon \mathcal{C}^{\mathrm{Cell}} \to \mathcal{C}$ such that:*

(i) *$\mathcal{C}^{\mathrm{Cell}}$ has the same objects as $\mathcal{C}$ and $Q$ is the identity map on objects,*
(ii) *$\mathcal{C}^{\mathrm{Cell}}(x, y)$ is pointwise cofibrant, and*
(iii) *$Q$ is a pointwise level equivalence.*

*Moreover, $(-)^{\mathrm{Cell}}$ and $Q$ may be constructed as an endofunctor and natural transformation on the category of small spectral categories.*

The analogous proposition in the setting of bimodules is also helpful.

PROPOSITION 1.1.12. *Assume that $\mathcal{C}$ and $\mathcal{D}$ are pointwise cofibrant small spectral categories. If $\mathcal{M}$ is a cofibrant $(\mathcal{D}, \mathcal{C})$-bimodule, then $\mathcal{M}$ is objectwise cofibrant, i.e., $\mathcal{M}(d, c)$ is a cofibrant symmetric spectrum for every $(d, c)$ in $\mathcal{D}^{\mathrm{op}} \wedge \mathcal{C}$.*

## 1.2. Review of the construction of $THH$, $TR$, and $TC$

In this section, we review the definition of $THH$, $TR$, and $TC$ of small spectral categories. We begin with a review of the cyclic bar construction for small spectral categories and the variant defined by Bökstedt [**8**] and Dundas-McCarthy [**10**] necessary for the construction of $TC$. We finish with a brief review of the definition of cyclotomic spectra and the construction of $TR$ and $TC$.

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ (including $\mathbf{0} = \{\}$), and with morphisms the injective maps. For a symmetric spectrum $A$, write $A_n$ for the $n$-th space. The association $\mathbf{n} \mapsto \Omega^n A_n$ extends to a functor from $\mathcal{I}$ to spaces. More generally, given symmetric spectra $A^0, \ldots, A^q$ and a space $X$, we obtain a functor from $\mathcal{I}^{q+1}$ to spaces that sends $\vec{\mathbf{n}} = (\mathbf{n}_0, \ldots, \mathbf{n}_q)$ to

$$\Omega^{n_0+\cdots+n_q}(A^q_{n_q} \wedge \cdots \wedge A^0_{n_0} \wedge X),$$

which is also natural in $X$. Restricting to the case when $X$ is a sphere $S^n$, we form this into a symmetric spectrum as follows.

DEFINITION 1.2.1. ([**30**, 4.2.1]) Let $D(A^q,\dots,A^0)$ be the symmetric spectrum with $n$-th space

$$D(A^q,\dots,A^0)(n) = \operatorname{hocolim}_{\vec{\mathbf{n}}\in\mathcal{I}^{q+1}} \Omega^{n_0+\cdots+n_q}(|A^q_{n_q}\wedge\cdots\wedge A^0_{n_0}|\wedge S^n),$$

and the evident structure maps.

The following is the main lemma of [**30**].

PROPOSITION 1.2.2. ([**30**, 4.2.3]) *$D(A^q,\dots,A^0)$ is canonically isomorphic in the stable category to the derived smash product of the $A^i$.*

This motivates the following definition, Dundas-McCarthy's Hochschild-Mitchell version of Bökstedt's variant of the cyclic bar construction.

DEFINITION 1.2.3. Given a small spectral category $\mathcal{C}$, a $\mathcal{C}$-bimodule $\mathcal{M}$, and a space $X$, let $\mathcal{G}(\mathcal{C};\mathcal{M};X)_{\vec{\mathbf{n}}}$ be the functor from $\mathcal{I}^{q+1}$ to spaces defined on $\vec{\mathbf{n}} = (\mathbf{n}_0,\dots,\mathbf{n}_q)$ by

$$\mathcal{G}(\mathcal{C};\mathcal{M};X)_{\vec{\mathbf{n}}} = \Omega^{n_0+\cdots+n_q}(\bigvee \mathcal{C}(c_{q-1},c_q)_{n_q}\wedge\cdots\wedge\mathcal{C}(c_0,c_1)_{n_1}\wedge\mathcal{M}(c_q,c_0)_{n_0}\wedge X),$$

where the wedge is over the $(q+1)$-tuples $(c_0,\dots,c_q)$ of objects of $\mathcal{C}$. Let

$$THH_q(\mathcal{C};\mathcal{M})(X) = \operatorname{hocolim}_{\vec{\mathbf{n}}\in\mathcal{I}^{q+1}} \mathcal{G}(\mathcal{C};\mathcal{M};X)_{\vec{\mathbf{n}}}.$$

This assembles into a simplicial space, functorially in $X$, as follows. The degeneracy maps are induced by the unit maps $S^0\to\mathcal{C}(c_i,c_i)_0$ and the functor

$$(\mathbf{n}_0,\dots,\mathbf{n}_q)\mapsto(\mathbf{n}_0,\dots,\mathbf{0},\dots,\mathbf{n}_q)$$

from $I^{q+1}$ to $I^{q+2}$. The face maps are induced by the two action maps on $\mathcal{M}$ (for $d_0$ and $d_q$) and the composition maps in $\mathcal{C}$ (for $d_1,\dots,d_{q-1}$) together with a functor $\mathcal{I}^{q+1}\to\mathcal{I}^q$ induced by the appropriate disjoint union isomorphism $(\mathbf{n}_i,\mathbf{n}_{i+1})\mapsto\mathbf{n}$ or $(\mathbf{n}_q,\mathbf{n}_0)\mapsto\mathbf{n}$ for $n=n_i+n_{i+1}$ or $n=n_q+n_0$. We write $THH(\mathcal{C};\mathcal{M})(X)$ for the geometric realization.

$THH(\mathcal{C};\mathcal{M})(X)$ is a continuous functor in the variable $X$, and so by restriction to the spheres $S^n$ specifies a symmetric spectrum which we denote $THH(\mathcal{C};\mathcal{M})$ or $THH(\mathcal{C})$ for $\mathcal{M}=\mathcal{C}$. The fact that the symmetric spectrum $THH$ is the restriction of a continuous functor implies that it is semistable [**19**, 8.7] and so the object that it represents in the stable category agrees with its underlying prespectrum. With additional hypotheses of "convergence" and "connectivity", $THH$ is often an $\Omega$-spectrum; see, for example, Proposition 2.4 of [**15**].

For most homotopical statements about $THH$, we will need to assume that $\mathcal{C}$ and $\mathcal{M}$ are non-degenerately based. When the unit maps $S^0\to\mathcal{C}(c,c)(0)$ are cofibrations, the simplicial spaces $THH_\bullet(\mathcal{C};\mathcal{M})(X)$ are "proper", meaning that the degeneracy maps are cofibrations, which is a sufficient for geometric realization to preserve level weak equivalences. The following proposition is then clear since smash products of non-degenerately based spaces preserve weak equivalences. It allows us to convert statements in [**7**] (which works with spectral categories of symmetric spectra in the context of simplicial sets) to the current context of topological spaces.

PROPOSITION 1.2.4. *If $\mathcal{C}$ is a small non-degenerately based spectral category and $\mathcal{M}$ is a non-degenerately based $\mathcal{C}$-bimodule, then the canonical map*

$$THH(|\operatorname{Sing}\mathcal{C}|,|\operatorname{Sing}\mathcal{M}|)(X)\longrightarrow THH(\mathcal{C};\mathcal{M})(X)$$

*is a weak equivalence for all $X$.*

As immediate corollaries, we obtain the following the basic properties of $THH$.

PROPOSITION 1.2.5. ([**7**, 3.6]) *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a weak equivalence of small spectral categories, $\mathcal{M}'$ a $\mathcal{C}'$-bimodule, $F^*\mathcal{M}'$ the $\mathcal{C}$-bimodule obtained by restriction of scalars, and $\mathcal{M} \to F^*\mathcal{M}'$ a weak equivalence of $\mathcal{C}$-bimodules. Then the induced map $THH(\mathcal{C};\mathcal{M}) \to THH(\mathcal{C}';\mathcal{M}')$ is a weak equivalence.*

PROPOSITION 1.2.6. ([**7**, 3.7]) *Let $\mathcal{C}$ be a small non-degenerately based spectral category.*

(i) *A weak equivalence of non-degenerately based $\mathcal{C}$-bimodules $\mathcal{M} \to \mathcal{M}'$ induces a weak equivalence $THH(\mathcal{C};\mathcal{M}) \to THH(\mathcal{C};\mathcal{M}')$.*
(ii) *A cofibration sequence of non-degenerately based $\mathcal{C}$-bimodules $\mathcal{M} \to \mathcal{M}' \to \mathcal{M}'' \to \Sigma\mathcal{M}$ induces a homotopy cofiber sequence on $THH$.*
(iii) *A fibration sequence of non-degenerately based $\mathcal{C}$-bimodules $\Omega\mathcal{M}'' \to \mathcal{M} \to \mathcal{M}' \to \mathcal{M}''$ induces a homotopy fibration sequence on $THH$.*

Additionally, we observe the following two results that are useful in arguments and applications in later chapters.

PROPOSITION 1.2.7. *Let $\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots$ be a sequence of spectrally enriched functors of non-degenerately based spectral categories and assume that either the functors are closed inclusions on mapping spectra or are induced by geometric realization from spectral functors of spectral categories enriched in symmetric spectra of simplicial sets. Let $\mathcal{C} = \operatorname{colim} \mathcal{C}_n$ and let $\mathcal{M}$ be a non-degenerately based $\mathcal{C}$ bifunctor. Then the induced map*

$$\operatorname{hocolim} THH(\mathcal{C}_n;\mathcal{M}) \longrightarrow THH(\mathcal{C};\mathcal{M})$$

*is a weak equivalence.*

PROOF. The map $\operatorname{hocolim} \mathcal{G}(\mathcal{C}_n;\mathcal{M};X)_{\vec{m}} \to \mathcal{G}(\mathcal{C};\mathcal{M};X)_{\vec{m}}$ is a weak equivalence for every $X$, $\vec{m}$. $\square$

PROPOSITION 1.2.8. *Let $\mathcal{C}_\bullet$ be a simplicial object in non-degenerately based spectral categories in which all the faces and degeneracies are the identity on objects and are Hurewicz cofibrations on each space of each mapping spectrum. Then the canonical map $|THH(\mathcal{C}_\bullet)| \to THH(|\mathcal{C}_\bullet|)$ is a weak equivalence.*

PROOF. For each $\vec{n}$ and spectral category $\mathcal{C}$, consider the symmetric spectrum

$$\mathcal{G}(\mathcal{C})_{\vec{\mathbf{n}}} = \Omega^{n_0+\cdots+n_q}\big(\bigvee \mathcal{C}(c_{q-1},c_q)_{n_q} \wedge \cdots \wedge \mathcal{C}(c_0,c_1)_{n_1} \wedge \mathcal{C}(c_q,c_0)_{n_0} \wedge S\big),$$

the symmetric spectrum obtained from assembling the spaces $\mathcal{G}(\mathcal{C},\mathcal{C},S^n)$ of Definition 1.2.3. Then

$$THH(\mathcal{C}) \cong |\operatorname{hocolim}_{\mathcal{I}^\bullet} \mathcal{G}(\mathcal{C})_{\vec{\mathbf{n}}}|.$$

For any proper simplicial non-degenerately based space $X_\bullet$, the canonical map

$$|\Omega^n(X_\bullet \wedge S)| \longrightarrow \Omega^n|X_\bullet \wedge S|$$

is a weak equivalence, indeed a level equivalence after level $n$ [**20**, 12.3], and it follows that $|THH(\mathcal{C}_\bullet)| \to THH(|\mathcal{C}_\bullet|)$ is a weak equivalence. $\square$

We now give a minimal review of the definition of $TR$ and $TC$; we refer the reader interested in more details to the excellent discussions of $TR$ and $TC$ in [**15, 16**]. For an $S^1$-space $X$, the space $THH(\mathcal{C})(X)$ has two $S^1$-actions, one coming from $X$ and the other coming from the cyclic structure. Using the diagonal action and restricting to representation spheres $S^V$ makes $THH(\mathcal{C})(-)$ into an equivariant orthogonal spectrum [**19**, §II.2]; however, $THH(\mathcal{C})$ has even more structure, that of a *cyclotomic spectrum* [**16**, §1.1], [**15**, Def. 2.2]. We refer the reader to [**7**, §4] or [**4**, §4] for a precise definition of the category of cyclotomic spectra, but in brief the structure on $THH$ derives from the fundamental fixed point map

$$(THH(\mathcal{C})(X))^H \longrightarrow THH(\mathcal{C})(X^H)$$

for $S^1$-spaces $X$ and finite subgroups $H$ of $S^1$. This induces maps in the equivariant stable category

$$r_H \colon \rho_H^{\#}\Phi^H THH(\mathcal{C}) \longrightarrow THH(\mathcal{C})$$

that are non-equivariant weak equivalences. Here $\Phi^H$ denotes the (derived) geometric fixed point spectrum, and when $H$ is the subgroup with $n$ elements, $\rho_H$ is the $n$-th root isomorphism $S^1 \cong S^1/H$; $\rho_H^{\#}$ converts the $S^1/H$-spectrum $\Phi^H THH(\mathcal{C})$ back to an $S^1$-spectrum via the isomorphism $\rho$. Essentially, a cyclotomic spectrum consists of an $S^1$-equivariant spectrum indexed on a complete universe together with weak equivalences $r_H$ of the form above, called *cyclotomic structure maps*, satisfying certain coherence properties [**15**, Def. 2.2], [**16**, §1.1]. By [**7**, 4.9] (and the obvious equivariant refinement of Proposition 1.2.4), $THH$ defines a functor from small non-degenerately based spectral categories to the point-set category of cyclotomic spectra.

For a fixed prime $p$ and each $n$, let $C_{p^n} \subset S^1$ denote the cyclic subgroup of order $p^n$. We then have maps in the (non-equivariant) stable category

$$F, R \colon THH(\mathcal{C})^{C_{p^n}} \longrightarrow THH(\mathcal{C})^{C_{p^{n-1}}}$$

where $F$ is the inclusion of the fixed points and $R$ is the map induced by the composite of the map from the fixed point spectrum to the geometric fixed point spectrum $THH(\mathcal{C})^{C_p} \to \Phi^{C_p} THH(\mathcal{C})$ and the cyclotomic structure map $r_{C_p} \colon \Phi^{C_p} THH(\mathcal{C}) \to THH(\mathcal{C})$; see [**16**, §1.1], [**15**, §2.2], or [**7**, §4]. We need functorial point-set versions of these maps to construct $TC$ as a functor on small spectral categories. In [**16**], the connectivity and convergence hypotheses used there imply that $THH(\mathcal{C})$ is an equivariant $\Omega$-spectrum relative to the family of finite subsets of $S^1$; the point-set maps $F, R$ in [**16**] are then constructed using the point-set fixed point spectra as models for the derived fixed point spectra. In our context, we need to use an $\Omega$-spectrum replacement functor in the category of cyclotomic spectra: For such a functor $Q$, we get appropriate point-set maps

$$F, R \colon Q(T)^{C_{p^n}} \longrightarrow Q(T)^{C_{p^{n-1}}}.$$

which are functorial in the cyclotomic spectrum $T$.

Definition 1.2.9. Let $Q$ be an $\Omega$-spectrum replacement functor in the category of cyclotomic spectra and write $T(\mathcal{C})$ for $Q(THH(\mathcal{C}))$. Then $TR^{\bullet}(\mathcal{C})$ is the pro-spectrum $\{T(\mathcal{C})^{C_{p^n}}\}$ under the maps $R$, and $TR(\mathcal{C})$ is the homotopy limit. $TC(\mathcal{C})$ and $TC^{\bullet}(\mathcal{C})$ are the spectrum and pro-spectrum obtained from $TR(\mathcal{C})$ and $TR^{\bullet}(\mathcal{C})$ as the homotopy equalizer of the maps $F$ and $R$.

Note that a map in the $S^1$-equivariant stable category induces a (non-equivariant) weak equivalence on fixed point spectra for all finite subgroups of $S^1$ if and only if it induces a (non-equivariant) weak equivalence on geometric fixed point spectra for all finite subgroups [**21**, XVI.6.4]. It follows that a cyclotomic map of cyclotomic spectra induces a weak equivalence of fixed point spectra for all finite subgroups of

$S^1$ if and only if it is a non-equivariant weak equivalence. In particular, we obtain the following proposition.

Proposition 1.2.10. *A spectral functor of small non-degenerately based spectral categories $\mathcal{C} \to \mathcal{D}$ that induces a weak equivalence on $THH$ induces a weak equivalence on $TR$ and $TC$.*

Likewise, using the same principle on the cofiber of a map of cyclotomic spectra, we obtain the following proposition. Applying this proposition in examples when $THH(\mathcal{C})$ is contractible, localization cofibration sequences on $TR$ and $TC$ follow from ones on $THH$.

Proposition 1.2.11. *For a strictly commuting square of small non-degenerately based spectral categories*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D}, \end{tikzcd}$$

*if the induced square on $THH$ is homotopy cocartesian, then so are the induced squares on $TR$ and $TC$.*

## 1.3. Review of the invariance properties of $THH$

In this section, we review the invariance properties of $THH$. This includes invariance under Dwyer-Kan equivalence, cofinal embeddings, and thick closure. We review the Localization Theorem of [**7**, 7.1] and the closely related theorem [**7**, 7.2] on triangulated quotients formed from “localization pairs”. We review only definitions and statements in this section and defer to [**7**] for proofs.

Definition 1.3.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a spectral functor. We say that $F$ is a *Dwyer-Kan embedding* or *DK-embedding* when for every $a, b \in \operatorname{ob}\mathcal{C}$, the map $\mathcal{C}(a,b) \to \mathcal{D}(Fa,Fb)$ is a weak equivalence.

We say that $F$ is a *Dwyer-Kan equivalence* or *DK-equivalence* when $F$ is a DK-embedding and for every $d \in \operatorname{ob}\mathcal{D}$, there exists a $c \in \operatorname{ob}\mathcal{C}$ such that $\mathcal{D}(-,d)$ and $\mathcal{D}(-,Fc)$ represent naturally isomorphic enriched functors from $\mathcal{D}^{\mathrm{op}}$ to the stable category.

We can rephrase this definition in terms of homotopy categories.

Definition 1.3.2. For a spectral category $\mathcal{C}$, the *homotopy category* $\pi_0\mathcal{C}$ is the Ab-category with the same objects, with morphism abelian groups $\pi_0\mathcal{C}(a,b)$, and with units and composition induced by the unit and composition maps of $\mathcal{C}$. The *graded homotopy category* is the $\mathrm{Ab}_*$-category with objects $\operatorname{ob}\mathcal{C}$ and morphisms $\pi_*\mathcal{C}(a,b)$.

The following proposition gives an equivalent formulation of DK-equivalence in terms of homotopy categories.

Proposition 1.3.3. *A spectral functor $\mathcal{C} \to \mathcal{D}$ is a Dwyer-Kan equivalence if and only if it induces an equivalence of graded homotopy categories $\pi_*\mathcal{C} \to \pi_*\mathcal{D}$.*

We then have the following invariance property for DK-equivalences.

Theorem 1.3.4. ([**7**, 5.9]) *A DK-equivalence of small non-degenerately based spectral categories $\mathcal{C} \to \mathcal{D}$ induces a weak equivalence $THH(\mathcal{C}) \to THH(\mathcal{D})$.*

We also have the following more general theorem for bimodule coefficients. In the statement, the $\mathcal{C}$-bimodule $F^*\mathcal{N}$ is the bimodule obtained by restriction of scalars; it is the spectral functor from $\mathcal{C}^{\mathrm{op}} \wedge \mathcal{C}$ to symmetric spectra defined by first applying $F$ to each variable and then applying $\mathcal{N}$.

THEOREM 1.3.5. ([**7**, 5.10]) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a DK-equivalence of small non-degenerately based spectral categories, $\mathcal{M}$ a $\mathcal{C}$-bimodule and $\mathcal{N}$ a $\mathcal{D}$-bimodule. A weak equivalence $\mathcal{M} \to F^*\mathcal{N}$ induces a weak equivalence $THH(\mathcal{C};\mathcal{M}) \to THH(\mathcal{D};\mathcal{N})$.*

The next theorem generalizes from DK-equivalences to cofinal DK-embeddings. For objects $a$ and $c$ of $\mathcal{D}$, say that $c$ is a *homotopy factor* of $a$ if it is a factor in the graded homotopy category $\pi_*\mathcal{D}$, i.e., if there exists an object $b$ in $\mathcal{D}$ and a natural isomorphism $\pi_*\mathcal{D}(-,c) \cong \pi_*\mathcal{D}(-,a) \times \pi_*\mathcal{D}(-,b)$ of contravariant functors from $\pi_*\mathcal{D}$ to the category of graded abelian groups. We say that a spectral functor $F\colon \mathcal{C} \to \mathcal{D}$ is *homotopy cofinal* if it induces weak equivalences on mapping spaces and each object of $\mathcal{D}$ is a homotopy factor of the image of some object in $\mathcal{C}$. The following is the most basic Morita invariance result for $THH$.

THEOREM 1.3.6. ([**7**, 5.11]) *A homotopy cofinal spectral functor $\mathcal{C} \to \mathcal{D}$ of small non-degenerately based spectral categories induces a weak equivalence $THH(\mathcal{C}) \to THH(\mathcal{D})$.*

The previous theorem generalizes further to the "thick closure". This is easiest to state and to explain in the context of pretriangulated spectral categories, which we now review.

DEFINITION 1.3.7. ([**7**, 5.4]) A spectral category $\mathcal{C}$ is *pretriangulated* means:

(i) There is an object 0 in $\mathcal{C}$ such that the right $\mathcal{C}$-module $\mathcal{C}(-,0)$ is homotopically trivial (weakly equivalent to the constant functor with value the one-point symmetric spectrum $*$).
(ii) Whenever a right $\mathcal{C}$-module $\mathcal{M}$ has the property that $\Sigma\mathcal{M}$ is weakly equivalent to a representable $\mathcal{C}$-module $\mathcal{C}(-,c)$ (for some object $c$ in $\mathcal{C}$), then $\mathcal{M}$ is weakly equivalent to a representable $\mathcal{C}$-module $\mathcal{C}(-,d)$ for some object $d$ in $\mathcal{C}$.
(iii) Whenever the right $\mathcal{C}$-modules $\mathcal{M}$ and $\mathcal{N}$ are weakly equivalent to representable $\mathcal{C}$-modules $\mathcal{C}(-,a)$ and $\mathcal{C}(-,b)$ respectively, then the homotopy cofiber of any map of right $\mathcal{C}$-modules $\mathcal{M} \to \mathcal{N}$ is weakly equivalent to a representable $\mathcal{C}$-module.

The first condition ensures the existence of a zero object in the homotopy category $\pi_0\mathcal{C}$: the usual argument shows that the left module $\mathcal{C}(0,-)$ is also homotopically trivial (in $\pi_0\mathcal{C}$, the identity map of 0 is the same as the zero map). The second condition gives a desuspension functor on $\pi_0\mathcal{C}$ and the third condition in particular produces a suspension functor on $\pi_0\mathcal{C}$: We choose $\Sigma^{-1}a$ and $\Sigma a$ representing $\Sigma^{-1}\mathcal{C}(-,a)$ and $\Sigma\mathcal{C}(-,a)$, respectively, in the derived category of right $\mathcal{C}$-modules. Then $\Sigma^{-1}a$ and $\Sigma a$ in particular represent the functors $\pi_1\mathcal{C}(-,a)$ and $\pi_{-1}\mathcal{C}(-,a)$, respectively, from $\pi_0\mathcal{C}$ to sets, and so are unique up to unique isomorphism in $\pi_0\mathcal{C}$. See [**7**, 5.4ff] for more discussion.

The terminology "pretriangulated" derives from the fact that the homotopy category is triangulated. The third condition above indicates how to form triangles.

DEFINITION 1.3.8. In a pretriangulated spectral category $\mathcal{C}$, we say that a sequence

$$a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$$

in $\pi_0\mathcal{C}$ is a *four term Puppe sequence* if there exists right $\mathcal{C}$-modules $\mathcal{M}$ and $\mathcal{N}$ and a map of right $\mathcal{C}$-modules $f\colon \mathcal{M} \to \mathcal{N}$ such that the four term Puppe sequence of $f$

$$\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow Cf \longrightarrow \Sigma\mathcal{M}$$

in the category of right $\mathcal{C}$-modules is isomorphic in the derived category of right $\mathcal{C}$-modules to the sequence

$$\mathcal{C}(-,a) \longrightarrow \mathcal{C}(-,b) \longrightarrow \mathcal{C}(-,c) \longrightarrow \mathcal{C}(-,\Sigma a)$$

such that the isomorphism $\Sigma\mathcal{M} \to \mathcal{C}(-,\Sigma a) \cong \Sigma\mathcal{C}(-,a)$ is the suspension of the isomorphism $\mathcal{M} \to \mathcal{C}(-,a)$.

THEOREM 1.3.9. ([**7**, 5.6]) *If the spectral category $\mathcal{C}$ is pretriangulated, then its homotopy category is triangulated with distinguished triangles the four term Puppe sequences. A spectral functor between pretriangulated spectral categories induces a triangulated functor on homotopy categories.*

COROLLARY 1.3.10. ([**7**, 5.7]) *A spectral functor $\mathcal{C} \to \mathcal{D}$ between pretriangulated spectral categories is a Dwyer-Kan equivalence if and only if it induces an equivalence of homotopy categories $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$.*

The following theorem indicates that there is no loss of generality in considering spectral subcategories of pretriangulated spectral categories.

THEOREM 1.3.11. ([**7**, 5.5]) *Any small spectral category $\mathcal{C}$ DK-embeds in a small pretriangulated spectral category $\tilde{\mathcal{C}}$.*

Given a set $C$ of objects in a pretriangulated spectral category $\mathcal{D}$, the *thick closure* of $C$ is the set of objects in the thick subcategory of $\pi_0\mathcal{D}$ generated by $C$. In terms of the spectral category $\mathcal{D}$, the thick closure of $C$ is the smallest set $\bar{C}$ of objects of $\mathcal{D}$ containing $C$ and satisfying:

(i) If $a$ is a homotopy factor of an object of $\bar{C}$, then $a$ is in $\bar{C}$.
(ii) If the right $\mathcal{D}$-module $\Sigma\mathcal{D}(-,a)$ is weakly equivalent to $\mathcal{D}(-,c)$ for some $c$ in $\bar{C}$, then $a$ is in $\bar{C}$.
(iii) If the right $\mathcal{D}$-module $\mathcal{D}(-,a)$ is weakly equivalent to the cofiber of a map of right $\mathcal{D}$-modules $\mathcal{M} \to \mathcal{M}'$ with $\mathcal{M}$, $\mathcal{M}'$ weakly equivalent to $\mathcal{D}(-,c)$, $\mathcal{D}(-,c')$ for $c, c'$ in $\bar{C}$, then $a$ is in $\bar{C}$.

A set is *thick* if it is its own thick closure.

THEOREM 1.3.12. ([**7**, 5.12]) *Let $\mathcal{D}$ be a pretriangulated spectral category. Let $C$ be a set of objects of $\mathcal{D}$, $\bar{C}$ its thick closure, and $C'$ a set containing $C$ and contained in $\bar{C}$. Let $\mathcal{C}$ and $\mathcal{C}'$ be the full spectral subcategories of $\mathcal{D}$ on the objects in $C$ and $C'$ respectively. If $\mathcal{C}$ and $\mathcal{C}'$ are non-degenerately based, then the inclusion $\mathcal{C} \to \mathcal{C}'$ induces a weak equivalence $THH(\mathcal{C}) \to THH(\mathcal{C}')$.*

The next theorem is the Localization Theorem of [**7**, 7.1].

THEOREM 1.3.13 (Localization Theorem [**7**, 7.1]). *Let $F\colon \mathcal{B} \to \mathcal{C}$ be a spectral functor between small pretriangulated spectral categories, and let $\mathcal{A}$ be the full spectral subcategory of $\mathcal{B}$ consisting of the objects $a$ such that $F(a)$ is isomorphic to zero in the homotopy category $\pi_0\mathcal{C}$. If the induced map from the triangulated quotient*

*$\pi_0\mathcal{B}/\pi_0\mathcal{A}$ to $\pi_0\mathcal{C}$ is cofinal, then $THH(\mathcal{C})$ is weakly equivalent the homotopy cofiber of $THH(\mathcal{A}) \to THH(\mathcal{B})$.*

There is also a version for triangulated quotients. A *localization pair* $(\mathcal{B}, \mathcal{A})$ consists of a pretriangulated spectral category $\mathcal{B}$ and a full spectral subcategory $\mathcal{A}$ such that $\pi_0\mathcal{A}$ is thick in $\pi_0\mathcal{B}$; we say that the localization pair is small when the spectral category $\mathcal{B}$ is small and non-degenerately based when $\mathcal{B}$ is non-degenerately based. (The requirement that $\mathcal{A}$ be thick is for convenience rather than necessity by Theorem 1.3.12.) The following theorem says that the cofiber of $THH$ is an invariant of a localization pair.

Theorem 1.3.14. ([**7**, 7.2]) *Let $F\colon (\mathcal{B}_1, \mathcal{A}_1) \to (\mathcal{B}_2, \mathcal{A}_2)$ be a map of small non-degenerately based localization pairs. If the induced map of triangulated quotients*

$$\pi_0\mathcal{B}_1/\pi_0\mathcal{A}_1 \longrightarrow \pi_0\mathcal{B}_2/\pi_0\mathcal{A}_2$$

*is cofinal, then the induced map*

$$C(THH(\mathcal{A}_1) \to THH(\mathcal{B}_1)) \quad \longrightarrow \quad C(THH(\mathcal{A}_2) \to THH(\mathcal{B}_2))$$

*is a weak equivalence.*

## 1.4. The Dennis-Waldhausen Morita Argument

The main tool in the proof of $THH$ invariance results is a trick due to Dennis and Waldhausen [**34**, p. 391] that we review in this section. We need it in the proof of the Sphere Theorem in Section 3.5. The argument is based on an explicit bisimplicial construction, which uses the Hochschild-Mitchell complex in place of $THH$.

Definition 1.4.1. For a small spectral category $\mathcal{C}$ and $\mathcal{C}$-bimodule $\mathcal{M}$, let

$$N^{\mathrm{cy}}_q(\mathcal{C}; \mathcal{M}) = \bigvee \mathcal{C}(c_{q-1}, c_q) \wedge \cdots \wedge \mathcal{C}(c_0, c_1) \wedge \mathcal{M}(c_q, c_0),$$

where the sum is over the $(q+1)$-tuples $(c_0, \ldots, c_q)$ of objects of $\mathcal{C}$. This becomes a simplicial object in symmetric spectra using the usual cyclic bar construction face and degeneracy maps: The unit maps of $\mathcal{C}$ induce the degeneracy maps, and the two action maps on $\mathcal{M}$ (for $d_0$ and $d_q$) and the composition maps in $\mathcal{C}$ (for $d_1, \ldots, d_{q-1}$) induce the face maps. We denote the geometric realization symmetric spectrum as $N^{\mathrm{cy}}(\mathcal{C}; \mathcal{M})$ and write $N^{\mathrm{cy}}(\mathcal{C})$ for $N^{\mathrm{cy}}(\mathcal{C}; \mathcal{C})$.

The following proposition, which is essentially the "many objects" version of [**30**, 4.2.8-9], follows from Proposition 1.2.4 and the theory developed in [**30**]. It allows us to sometimes substitute the Hochschild-Mitchell complex for $THH$

Proposition 1.4.2. ([**7**, 3.5]) *There is a natural map in the stable category from $THH(\mathcal{C}; \mathcal{M})$ to $N^{\mathrm{cy}}(\mathcal{C}; \mathcal{M})$ that is an isomorphism when $\mathcal{C}$ is pointwise cofibrant.*

In addition to the Hochschild-Mitchell complex, we also need the two-sided bar construction.

Definition 1.4.3. Let $\mathcal{C}$ be a small spectral category, $\mathcal{M}$ a right $\mathcal{C}$-module, and $\mathcal{N}$ a left $\mathcal{C}$-module. The *two-sided bar construction* $B(\mathcal{M}; \mathcal{C}; \mathcal{N})$ is the geometric realization of the simplicial symmetric spectrum $B_\bullet(\mathcal{M}; \mathcal{C}; \mathcal{N})$, where

$$B_q(\mathcal{M}; \mathcal{C}; \mathcal{N}) = \bigvee \mathcal{M}(c_q) \wedge \mathcal{C}(c_{q-1}, c_q) \wedge \cdots \wedge \mathcal{C}(c_0, c_1) \wedge \mathcal{N}(c_0),$$

where the sum is over the $(q+1)$-tuples $(c_0, \ldots, c_q)$ of objects of $\mathcal{C}$. We make this a simplicial object with the usual two-sided bar construction face and degeneracy maps: the zeroth face map is induced by the action of $\mathcal{C}$ on $\mathcal{N}$, the last face map is induced by the action of $\mathcal{C}$ on $\mathcal{M}$, and the remaining face maps are induced by the composition in $\mathcal{C}$. The degeneracy maps are induced by the unit maps $S \to \mathcal{C}(c_i, c_i)$.

The following proposition is the *Dennis-Waldhausen Morita Argument*. In the statement (and elsewhere when necessary for clarity), we write

$$B(\mathcal{M}(x); x, y \in \mathcal{C}; \mathcal{N}(y)) \qquad \text{and} \qquad N^{\mathrm{cy}}(x, y \in \mathcal{C}; \mathcal{P}(x, y))$$

for $B(\mathcal{M}; \mathcal{C}; \mathcal{N})$ and $N^{\mathrm{cy}}(\mathcal{C}; \mathcal{P})$, especially when $\mathcal{M}$, $\mathcal{N}$, and/or $\mathcal{P}$ depend on other variables.

PROPOSITION 1.4.4 (Dennis-Waldhausen Morita Argument [**7**, 6.2]). *Let $\mathcal{C}$ and $\mathcal{D}$ be small spectral categories. Let $\mathcal{P}$ be a $(\mathcal{D}, \mathcal{C})$-bimodule and $\mathcal{Q}$ a $(\mathcal{C}, \mathcal{D})$-bimodule. Then there is a natural isomorphism of symmetric spectra*

$$N^{\mathrm{cy}}(\mathcal{C}, B(\mathcal{P}, \mathcal{D}, \mathcal{Q})) \cong N^{\mathrm{cy}}(\mathcal{D}; B(\mathcal{Q}, \mathcal{C}, \mathcal{P})),$$

*that is,*

$$\begin{aligned} &N^{\mathrm{cy}}(x, y \in \mathcal{C}; B(\mathcal{P}(w, y); w, z \in \mathcal{D}; \mathcal{Q}(x, z))) \\ &\qquad\qquad \cong N^{\mathrm{cy}}(w, z \in \mathcal{D}; B(\mathcal{Q}(x, z); x, y \in \mathcal{C}; \mathcal{P}(w, y))). \end{aligned}$$

As the proof is easy, we repeat it here.

PROOF. We can identify both symmetric spectra

$$N^{\mathrm{cy}}(\mathcal{C}; B(\mathcal{P}; \mathcal{D}; \mathcal{Q})) \qquad \text{and} \qquad N^{\mathrm{cy}}(\mathcal{D}; B(\mathcal{Q}; \mathcal{C}; \mathcal{P}))$$

as the diagonal of the bisimplicial spectrum with $(q, r)$-simplices as pictured.

$$\begin{array}{ccccc} & & \mathcal{C}(c_{q-1}, x) \wedge \cdots \wedge \mathcal{C}(y, c_1) & & \\ & \wedge & & \wedge & \\ \mathcal{Q}(x, z) & & & & \mathcal{P}(w, y) \\ & \wedge & & \wedge & \\ & & \mathcal{D}(z, d_1) \wedge \cdots \wedge \mathcal{D}(d_{r-1}, w) & & \end{array}$$

These two constructions are therefore canonically isomorphic in the point-set category of symmetric spectra. $\square$

The following lemma complements Proposition 1.4.4 in the applications. Its proof is the usual simplicial contraction (see for example [**20**, 9.8]) and requires no cofibrancy or non-degenerate base point hypotheses.

LEMMA 1.4.5 (Two-Sided Bar Lemma). *Let $\mathcal{C}$ be a small spectral category, let $\mathcal{M}$ be a right $\mathcal{C}$-module, and let $\mathcal{N}$ be a left $\mathcal{C}$-module. For any object $c$ in $\mathcal{C}$, the composition maps*

$$B_\bullet(\mathcal{M}; \mathcal{C}; \mathcal{C}(c, -)) \longrightarrow \mathcal{M}(c) \qquad \textit{and} \qquad B_\bullet(\mathcal{C}(-, c); \mathcal{C}; \mathcal{N}) \longrightarrow \mathcal{N}(c)$$

*are simplicial homotopy equivalences.*

The applications we need are the following.

THEOREM 1.4.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a spectral functor between pointwise cofibrant spectral categories and let $\mathcal{L}$ be the $\mathcal{D}$-bimodule*

$$\mathcal{L}(a,b) = B(\mathcal{D}(F(-),b),\mathcal{C},\mathcal{D}(a,F(-))).$$

*Then $THH(\mathcal{D};\mathcal{L})$ is weakly equivalent to $THH(\mathcal{C};F^*\mathcal{D})$.*

PROOF. The proof is essentially the same as the proof of [**7**, 7.6]. It suffices to produce a weak equivalence between $N^{\mathrm{cy}}(\mathcal{D};\mathcal{L})$ and $N^{\mathrm{cy}}(\mathcal{C};F^*\mathcal{D})$. For this we apply Proposition 1.4.4 with $\mathcal{P} = \mathcal{C}$, and $\mathcal{Q} = \mathcal{C}$ to obtain a natural isomorphism

$$N^{\mathrm{cy}}(\mathcal{D};\mathcal{L}^{\mathcal{B}}_{\mathcal{A}}) = N^{\mathrm{cy}}(\mathcal{D};B(\mathcal{D};\mathcal{C};\mathcal{D})) \cong N^{\mathrm{cy}}(\mathcal{C};F^*B(\mathcal{D};\mathcal{D};\mathcal{D})).$$

The natural map

$$THH(\mathcal{C};F^*B(\mathcal{D};\mathcal{D};\mathcal{D})) \longrightarrow THH(\mathcal{C};F^*\mathcal{D})$$

is a weak equivalence by the Two-Sided Bar Lemma 1.4.5. □

THEOREM 1.4.7. ([**7**, 6.4]) *Let $\mathcal{C}$ and $\mathcal{D}$ be small spectral categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a spectral functor. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule, $\mathcal{N}$ a $\mathcal{D}$-bimodule and $\mathcal{M} \to F^*\mathcal{N}$ a weak equivalence. Assume that $\mathcal{C}$ and $\mathcal{D}$ are pointwise cofibrant and that $\mathcal{M}$ and $\mathcal{N}$ are non-degenerately based. If the map of symmetric spectra*

$$B(\mathcal{D}(F-,z);\mathcal{C};\mathcal{N}(w,F-)) \longrightarrow B(\mathcal{D}(-,z);\mathcal{D};\mathcal{N}(w,-))$$

*is a weak equivalence for each fixed w,z in $\mathcal{D}$. Then the map*

$$THH(\mathcal{C};\mathcal{M}) \longrightarrow THH(\mathcal{D};\mathcal{N})$$

*is a weak equivalence.*

As the proof is identical to the proof of [**7**, 6.4], we omit it here.

CHAPTER 2

# $THH$ and $TC$ of simplicially enriched Waldhausen categories

A *Waldhausen category* consists of a category $\mathcal{C}$ together with a (chosen) zero object $*$, a subcategory of *cofibrations* $co\mathcal{C}$, and a subcategory of *weak equivalences* that satisfy the following properties [**35**, §1.1–1.2]:

(i) (Cof 1, Weq 1) $co\mathcal{C}$ and $w\mathcal{C}$ contain all the isomorphisms.

(ii) (Cof 2) For every object $a$, the map $* \to a$ is a cofibration.

(iii) (Cof 3) Cofibrations admit cobase change: If $a \to b$ is a cofibration, and $a \to c$ is any map, then $b \cup_a c$ exists and $c \to b \cup_a c$ is a cofibration.

(iv) (Weq 2) *Gluing Axiom.* Given a commutative diagram

$$\begin{array}{ccccc} b & \leftarrowtail & a & \longrightarrow & c \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ b' & \leftarrowtail & a' & \longrightarrow & c' \end{array}$$

where the leftward arrows are cofibrations and the vertical arrows are weak equivalences, the induced map

$$b \cup_a c \longrightarrow b' \cup_{a'} c'$$

is a weak equivalence.

Waldhausen [**35**, §1.3] constructs the algebraic $K$-theory spectrum associated to a Waldhausen category using the $S_\bullet$ construction (which we review in Section 2.3 below). The purpose of this chapter is to construct $THH$ and $TC$ for Waldhausen categories that have an additional compatible simplicial enrichment. (We extend this definition to Waldhausen categories much more broadly in Chapter 5.)

The contents of the chapter are as follows. Section 2.1 defines simplicially enriched, enhanced simplicially enriched, and simplicially tensored Waldhausen categories, giving some examples. Section 2.2 constructs spectral categories from simplicially enriched Waldhausen categories. Section 2.3 reviews the $S_\bullet$ construction and introduces the *Moore nerve* construction, which is a version of the nerve construction that behaves better homotopically on enriched categories. Section 2.4 introduces the Moore $S'_\bullet$ construction and iterated $S'_\bullet$, which generalizes the iterated $S_\bullet$ construction and is needed for the construction of the cyclotomic trace. Section 2.5 constructs $THH$, $TR$, and $TC$ for simplicially enriched Waldhausen categories and the cyclotomic trace from $K$-theory to $TC$.

## 2.1. Simplicially enriched Waldhausen categories

In this section we introduce the structure of a simplicially enriched Waldhausen category. This structure compatibly combines a simplicial enrichment with a Waldhausen structure in a way that we make precise in Definition 2.1.1. Although this structure suffices for us to define an associated spectral category in the next section, more conditions are necessary to ensure that the homotopy theory of the enrichment matches up with the intrinsic homotopy theory of the Waldhausen category; we make these conditions precise in the definition of DK-compatible enrichment in Definition 2.1.2. In practice, and as we explain in Section 5.2, without much loss of generality, we typically have the stronger structures that we describe in Definitions 2.1.6 and 2.1.8. We begin with the most basic structure in the following definition.

Definition 2.1.1. A *simplicially enriched Waldhausen category* consists of a category $\mathcal{C} = \mathcal{C}_\bullet$ enriched in simplicial sets together with a Waldhausen category structure on $\mathcal{C}_0$ such that:

- (i) The zero object $*$ in $\mathcal{C}_0$ is a zero object for $\mathcal{C}$,
- (ii) Pushouts over cofibrations in $\mathcal{C}_0$ are pushouts in $\mathcal{C}$,
- (iii) Cofibrations $x \to y$ induce Kan fibrations $\mathcal{C}(y,z) \to \mathcal{C}(x,z)$ for all objects $z$, and
- (iv) A map $x \to y$ is a weak equivalence if and only if $\mathcal{C}(y,z) \to \mathcal{C}(x,z)$ is a weak equivalence for all objects $z$ if and only if $\mathcal{C}(z,x) \to \mathcal{C}(z,y)$ is a weak equivalence for all objects $z$.

An *enriched exact functor* between such categories is a simplicial functor $\phi\colon \mathcal{C} \to \mathcal{D}$ that restricts to an exact functor of Waldhausen categories $\mathcal{C}_0 \to \mathcal{D}_0$.

Since the initial map $* \to x$ is always a cofibration in a Waldhausen category, Definition 2.1.1 implies that all the mapping spaces $\mathcal{C}(x,y)$ are Kan complexes. The fact that weak equivalences are detected on the simplicial mapping spaces implies that weak equivalences in $\mathcal{C}_0$ are closed under retracts and satisfy the two out of three property.

As explained by Dwyer and Kan, any category with a subcategory of weak equivalences has an intrinsic homotopy theory in terms of a functorial simplicially enriched category called the Dwyer-Kan simplicial localization [**12**]. Technically, we will use exclusively the variant called the hammock localization [**11**], which we will denote by $L$. Then for a simplicial Waldhausen category $\mathcal{C}$, the Dwyer-Kan simplicial localization of the underlying category with weak equivalences, denoted $L\mathcal{C}_0$, provides a second simplicially enriched category expanding $\mathcal{C}_0$. In general, we see no reason why these two simplicial enrichments should be equivalent; we therefore introduce the following terminology.

Definition 2.1.2. Let $\mathcal{C}$ be a simplicially enriched Waldhausen category. We say that $\mathcal{C}$ is *DK-compatible* if for all objects $x, y$ in $\mathcal{C}$, the maps

$$\mathcal{C}(x,y) \longrightarrow \operatorname{diag} L\mathcal{C}_\bullet(x,y) \longleftarrow L\mathcal{C}_0(x,y)$$

are weak equivalences of simplicial sets. Here we regard $\mathcal{C}_n$ as a category with weak equivalences by declaring a map in $\mathcal{C}_n$ to be a weak equivalence if and only if some (or, equivalently, every) iterated face map takes it to a weak equivalence in $\mathcal{C}_0$.

As in Definition 1.3.1, for categories enriched in simplicial sets, spaces, or spectra, an enriched functor $\phi\colon \mathcal{C} \to \mathcal{D}$ is called a *DK-embedding* when it induces a weak equivalence $\mathcal{C}(x,y) \to \mathcal{D}(\phi(x),\phi(y))$ for all objects $x$, $y$. A DK-embedding is a *DK-equivalence* when it induces an equivalence $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ on categories of components. On the other hand, for discrete categories $\mathcal{C}_0$ and $\mathcal{D}_0$ with subcategories of weak equivalences, a functor $\mathcal{C}_0 \to \mathcal{D}_0$ that preserves weak equivalences is called a *DK-embedding* or *DK-equivalence* when it induces one on the Dwyer-Kan simplicial localizations. The main purpose of the previous definition is the following easy observation.

PROPOSITION 2.1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicially enriched Waldhausen categories and $\phi\colon \mathcal{C} \to \mathcal{D}$ a simplicial functor (not necessarily exact). Then:*

(i) *$\phi_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ preserves weak equivalences.*
(ii) *Assume furthermore that $\mathcal{C}$ and $\mathcal{D}$ are both DK-compatible. Then $\phi$ is a DK-embedding or DK-equivalence of simplicially enriched categories if and only if $\phi_0$ is a DK-embedding or DK-equivalence (respectively) of categories with weak equivalences.*

The following is an easy but important class of examples of DK-compatible simplicially enriched Waldhausen categories.

EXAMPLE 2.1.4. An exact category, or more generally, a Waldhausen category whose weak equivalences are the isomorphisms becomes a DK-compatible simplicially enriched Waldhausen category by regarding its mapping sets as discrete simplicial sets.

We also have the following less trivial examples.

EXAMPLE 2.1.5. Let $\mathcal{C}$ be a Waldhausen subcategory of cofibrant objects in simplicial closed model category $\mathcal{M}$ in which all objects are fibrant. Then $\mathcal{C}$ is a simplicially enriched Waldhausen category with its natural simplicial mapping spaces and Waldhausen structure inherited from $\mathcal{M}$. If $\mathcal{C}$ is closed under tensors with finite simplicial sets, then $\mathcal{C}$ is a DK-compatible (see Theorem 2.1.9 below). Examples of this type include:

(i) Finite cell $R$-modules for an EKMM $S$-algebra $R$, or (for $R$ connective with $\pi_0$ noetherian) cell $R$-modules that have finite stage finitely generated Postnikov towers as in [**5**].
(ii) The category of finite cell modules over a simplicial ring $A$, or the category of finite cell modules built out of finitely generated projective $A$-modules.
(iii) The category of simplicial objects on an abelian category with the "split-exact" model structure (where the cofibrations are the levelwise split monomorphisms and the weak equivalences are the simplicial homotopy equivalences).
(iv) The category of levelwise projectives in the category of simplicial objects on an abelian category with enough projectives (with the standard projective model structure). Likewise, the opposite category of the levelwise injectives in the category of cosimplicial objects on an abelian category with enough injectives (with the standard injective model structure).

In addition to being DK-compatible, the previous class of examples has an additional structure that we employ to construct non-connective spectral enrichments in the next section. We abstract this structure in the following definition.

DEFINITION 2.1.6. A *simplicially tensored Waldhausen category* is a simplicially enriched Waldhausen category in which tensors with finite simplicial sets exist and satisfy the pushout-product axiom. A tensored exact functor between simplicially tensored Waldhausen categories is a enriched exact functor that preserves tensors with finite simplicial sets.

In the previous definition, the pushout-product axiom [**29**, 2.1] asserts that given a cofibration $x \to y$ in $\mathcal{C}_0$ and a cofibration $A \to B$ of finite simplicial sets, the map

$$(x \otimes B) \cup_{x \otimes A} (y \otimes A) \longrightarrow y \otimes B$$

is a cofibration in $\mathcal{C}_0$. This axiom implies that the usual mapping cylinder construction endows $\mathcal{C}_0$ with a cylinder functor satisfying the cylinder axiom (in the sense of [**35**, §1.6]). The Kan condition on the mapping spaces combined with the tensor adjunction implies the following proposition.

PROPOSITION 2.1.7. *Let $\mathcal{C}$ be a simplicially tensored Waldhausen category.*

(i) *For any object $x$ in $\mathcal{C}$, the tensor $x \otimes (-)$ preserves weak equivalences in simplicial sets.*
(ii) *For any finite simplicial set $X$, the tensor $(-) \otimes X$ preserves weak equivalences in $\mathcal{C}$.*
(iii) *For objects $x$ and $y$ in $\mathcal{C}$, the simplicial set $\mathcal{C}(x, y)$ is canonically isomorphic to $\mathcal{C}_0(x \otimes \Delta[\cdot], y)$.*

Definition 2.1.6 provides the strongest background structure that we use; in Section 5.2, we see that Waldhausen categories quite generally admit equivalent models of this type. In our study of the $THH$ localization sequence in Chapter 4, however, we require slightly more flexibility. Using a simplicially tensored Waldhausen category as an ambient category, we will sometimes need to restrict to a subcategory.

DEFINITION 2.1.8. An *enhanced simplicially enriched Waldhausen category* is a pair $\mathcal{A} \subset \mathcal{C}$ where $\mathcal{C}$ is a simplicially tensored Waldhausen category and $\mathcal{A}$ is a full subcategory such that $\mathcal{A}_0$ is a closed Waldhausen subcategory. For $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \subset \mathcal{D}$ enhanced simplicially enriched Waldhausen categories, an *enhanced exact functor* $\mathcal{A} \to \mathcal{B}$ is a tensored exact functor of simplicially tensored Waldhausen categories $\mathcal{C} \to \mathcal{D}$ that restricts to a functor $\mathcal{A} \to \mathcal{B}$.

As in [**35**, §1.2], a *Waldhausen subcategory* $\mathcal{A}$ is a full subcategory of a Waldhausen category $\mathcal{C}$ that itself becomes a Waldhausen category by taking a weak equivalence to be a weak equivalence in $\mathcal{C}$ between objects of $\mathcal{A}$ and a cofibration to be a cofibration in $\mathcal{C}$ between objects of $\mathcal{A}$ for which the cofiber is in $\mathcal{A}$ (up to isomorphism). A *closed* Waldhausen subcategory is a Waldhausen subcategory $\mathcal{A} \subset \mathcal{C}$ that contains every object of $\mathcal{C}$ that is weakly equivalent to an object of $\mathcal{A}$. An enhanced simplicially enriched Waldhausen category inherits tensors with homotopically trivial finite simplicial sets (but not necessarily arbitrary finite simplicial sets) as well as properties (i) and (iii) of Proposition 2.1.7. We also have the following compatibility result.

THEOREM 2.1.9. *An enhanced simplicially enriched Waldhausen category $\mathcal{A} \subset \mathcal{C}$ is DK-compatible.*

PROOF. Fix objects $a, b$. Regarding $\mathcal{A}_n(a, b)$ as $\mathcal{A}_0(a \otimes \Delta[n], b)$, each category $\mathcal{A}_n$ admits a homotopy calculus of left fractions [**11**, 6.1] (see, for example, the argument for [**6**, 5.5]) and so we can replace $L\mathcal{A}_n(a, b)$ with the nerve of the category of words of the form $\mathbf{W^{-1}C}$, which we will temporarily denote as $L_n(a, b)$. An object of this category consists of a zigzag

$$a \longrightarrow x \longleftarrow b$$

of maps in $\mathcal{A}_n$, where the map $x \leftarrow b$ is a weak equivalence; a map in this category is a map in $\mathcal{A}_n$ of $x$ that is under $a$ and $b$. We check that both maps

$$L_0(a,b) \longrightarrow \operatorname{diag} L_\bullet(a,b) \longleftarrow \mathcal{A}(a,b)$$

are weak equivalences (i.e., induce weak equivalences on nerves).

For the map $L_0(a,b) \to \operatorname{diag} L_\bullet(a,b)$, we show that each iterated degeneracy $s_0^n \colon L_0(a,b) \to L_n(a,b)$ is a weak equivalence. Iterating the last face map gives a functor $\tilde{\partial}^n \colon L_n(a,b) \to L_0(a,b)$ such that the composite is the identity on $L_0(a,b)$. We need to check that that the composite $s_0^n \tilde{\partial}^n$ on $L_n(a,b)$ is a weak equivalence. Since both inclusions of $a$ in $a \otimes \Delta[1]$ and both inclusions of $b$ in $b \otimes \Delta[1]$ are weak equivalences, they induce weak equivalences

$$I_0, I_1 \colon L_n(a \otimes \Delta[1], b \otimes \Delta[1]) \longrightarrow L_n(a,b).$$

The contracting homotopy $c \colon \Delta[n] \times \Delta[1] \to \Delta[n]$ from the identity map to the inclusion of the last vertex induces a functor $C \colon \mathcal{A}_n \to \mathcal{A}_n$ sending $x$ to $x \otimes \Delta[1]$ as follows: For a map $f \colon x \to y$ in $\mathcal{A}_n$, viewed as a map $\tilde{f} \colon x \otimes \Delta[n] \to y$ in $\mathcal{A}_0$, $C(f)$ is represented by the map

$$x \otimes \Delta[1] \otimes \Delta[n] \cong x \otimes (\Delta[n] \times \Delta[1]) \longrightarrow y \otimes \Delta[1]$$

in $\mathcal{A}_0$ induced by $\tilde{f}$, $c$, and the diagonal map on $\Delta[1]$. We then get a functor

$$C \colon L_n(a,b) \longrightarrow L_n(a \otimes \Delta[1], b \otimes \Delta[1]).$$

The composite functor

$$I_0 \circ C \colon L_n(a,b) \longrightarrow L_n(a,b)$$

admits a natural transformation from the identity functor, and so induces a homotopy equivalence on nerves. It follows that $C$ is a weak equivalence. The composite functor

$$I_0 \circ C \colon L_n(a,b) \longrightarrow L_n(a,b)$$

is therefore also a weak equivalence. We have a natural transformation from $s_0^n \tilde{\partial}^n$ to $I_0 \circ C$, and so the induced maps on nerves are simplicially homotopic. This then shows that $s_0^n \tilde{\partial}^n$ is a weak equivalence.

It remains to see that the map $\mathcal{A}(a,b) \to \operatorname{diag} L_\bullet(a,b)$ is a weak equivalence. We can identify $\operatorname{diag} L_\bullet(a,b)$ as the diagonal of the bisimplicial set whose simplicial set of $q$-simplices is

$$\mathcal{A}(a,x_0) \times w\mathcal{A}(b,x_0) \times w\mathcal{A}(x_0,x_1) \times \cdots \times w\mathcal{A}(x_{q-1},x_q),$$

where $w\mathcal{A}$ denotes the components with (any, or equivalently, all) vertices in $w\mathcal{A}_0$, the subcategory of weak equivalences of the Waldhausen category $\mathcal{A}_0$. The map $\mathcal{A}(a,b) \to \operatorname{diag} L_\bullet(a,b)$ factors through a bisimplicial map from the bisimplicial set $X_{\bullet\bullet}$ whose simplicial set of $q$-simplices $X_{q\bullet}$ is

$$\mathcal{A}(a,b) \times w\mathcal{A}(b,x_0) \times w\mathcal{A}(x_0,x_1) \times \cdots \times w\mathcal{A}(x_{q-1},x_q).$$

The inclusion $\mathcal{A}(a,b) \to \operatorname{diag} X_{\bullet\bullet}$ is clearly a simplicial homotopy equivalence, and the bisimplicial map $X_{\bullet\bullet} \to L_\bullet(a,b)$ is a degreewise weak equivalence. $\square$

## 2.2. Spectral categories associated to simplicially enriched Waldhausen categories

In this section we produce for a simplicially enriched Waldhausen an associated spectral category, which is natural in enriched exact functors. The mapping spectra in this category are prolongations of Γ-spaces, and as such, are always connective. For an enhanced simplicially enriched Waldhausen category, we associate another spectral category, typically non-connective, using the suspensions in the ambient simplicially tensored Waldhausen category; it is natural in enhanced exact functors. We also explore the basic properties of these categories. We begin with the construction.

Definition 2.2.1. Let $\mathcal{C}$ be a simplicially enriched Waldhausen category. Define $\mathcal{C}^{\Gamma}$, the Γ-category associated to $\mathcal{C}$, to have objects the objects of $\mathcal{C}$ and mapping Γ-spaces

$$\mathcal{C}^{\Gamma}_q(x,y) = \mathcal{C}(x, \bigvee_q y).$$

By abuse, we will also write $\mathcal{C}^{\Gamma}$ for the enrichment in symmetric spectra obtained by prolongation. We will refer to $\mathcal{C}^{\Gamma}$ as the *connective spectral enrichment* of $\mathcal{C}$ or the connective spectral category associated to $\mathcal{C}$.

Here the composition

$$\mathcal{C}^{\Gamma}_r(y,z) \wedge \mathcal{C}^{\Gamma}_q(x,y) \longrightarrow \mathcal{C}^{\Gamma}_{qr}(x,z).$$

comes from the $\Sigma_q \wr \Sigma_r$-equivariant map

$$\mathcal{C}(y, \bigvee_r z) \longrightarrow \prod_q \mathcal{C}(y, \bigvee_r z) \longrightarrow \mathcal{C}(\bigvee_q y, \bigvee_{rq} z)$$

and composition

$$\mathcal{C}(\bigvee_q y, \bigvee_{rq} z) \wedge \mathcal{C}(x, \bigvee_q y) \longrightarrow \mathcal{C}(x, \bigvee_{rq} z).$$

This composition of Γ-spaces then induces the composition on the associated symmetric spectra. The following proposition is immediate from the construction.

Proposition 2.2.2. *For simplicially enriched Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, an enriched exact functor $\phi\colon \mathcal{C} \to \mathcal{D}$ induces a spectral functor $\phi^{\Gamma}: \mathcal{C}^{\Gamma} \to \mathcal{D}^{\Gamma}$. If $\mathcal{C}$ and $\mathcal{D}$ are DK-compatible and $\phi$ is a DK-embedding or DK-equivalence, then so is $\phi^{\Gamma}$.*

In general, we can not expect the Γ-spaces $\mathcal{C}^{\Gamma}(x,y)$ to be special or very special. On the other hand, as a prolongation of a Γ-space, the associated symmetric spectrum is semistable (Definition 1.1.8), meaning that it represents the same object in the stable category as its underlying spectrum.

Proposition 2.2.3. *The mapping symmetric spectra in $\mathcal{C}^{\Gamma}$ are semistable.*

Example 2.2.4. For $\mathfrak{E}$ be an exact category, simplicially enriched as in Example 2.1.4,

$$\mathfrak{E}^{\Gamma}_q(x,y) = \mathfrak{E}(x, \bigoplus_{i=1}^{q} y) \cong \prod_{i=1}^{q} \mathfrak{E}(x,y).$$

Prolonging to symmetric spectra, we get

$$\mathfrak{E}^{\Gamma}(x,y)(n) = \mathfrak{E}(x,y) \otimes \tilde{\mathbb{Z}}[S^n],$$

where $\tilde{\mathbb{Z}}[X] = \mathbb{Z}[X]/\mathbb{Z}[*]$. This is precisely the spectral category associated to an exact category studied by Dundas-McCarthy [**10**] and Hesselholt-Madsen [**16**].

When $\mathcal{C}$ is a simplicially tensored Waldhausen category, we can construct another enrichment in symmetric spectra using suspensions: for an object $x$ in $\mathcal{C}$, let $\Sigma x$ be the cofiber of the map

$$x \otimes \partial\Delta[1] \longrightarrow x \otimes \Delta[1].$$

Suspension defines a tensored exact functor from $\mathcal{C}$ to itself. Commuting colimits and tensors, and applying the associativity isomorphism for tensors, we can describe the iterated suspension $\Sigma^n x$ as the cofiber of the map

$$x \otimes \partial(\Delta[1]^n) \longrightarrow x \otimes \Delta[1]^n,$$

where $\Delta[1]^n = \Delta[1] \times \cdots \times \Delta[1]$. The $n$-th suspension inherits from $\Delta[1]^n$ an action of the symmetric group $\Sigma_n$.

Definition 2.2.5. Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially enriched Waldhausen category. Define $\mathcal{A}^S$ be the spectral category with objects the objects of $\mathcal{A}$ and mapping symmetric spectra

$$\mathcal{A}^S(x,y)(n) = \mathcal{C}(x, \Sigma^n y).$$

We will refer to this as the *non-connective spectral enrichment* of $\mathcal{A}$ or the non-connective spectral category associated to $\mathcal{A}$.

In the previous definition, we obtain the composition on $\mathcal{A}^S$,

$$\mathcal{A}^S(y,z) \wedge \mathcal{A}^S(x,y) \longrightarrow \mathcal{A}^S(x,z)$$

from the $\Sigma_n \times \Sigma_m$-equivariant maps

$$\mathcal{C}(y, \Sigma^m z) \wedge \mathcal{C}(x, \Sigma^n y) \longrightarrow \mathcal{C}(\Sigma^n y, \Sigma^{m+n} z) \wedge \mathcal{C}(x, \Sigma^n y) \longrightarrow \mathcal{C}(x, \Sigma^{m+n} z).$$

Note that for a enhanced simplicially enriched Waldhausen category $\mathcal{A} \subset \mathcal{C}$, the suspension of an object of $\mathcal{A}$ is an object of $\mathcal{C}$ but need not be an object in $\mathcal{A}$. As a consequence, the non-connective enrichment $\mathcal{A}^S$ depends strongly on the ambient simplicially tensored Waldhausen category $\mathcal{C}$. Recall that an enhanced exact functor has as part of its structure a tensored exact functor on the ambient simplicially tensored Waldhausen categories; the following functoriality is immediate from the construction.

Proposition 2.2.6. *An enhanced exact functor $\phi\colon \mathcal{A} \to \mathcal{B}$ between enhanced simplicially enriched Waldhausen categories induces a spectral functor $\phi^S\colon \mathcal{A}^S \to \mathcal{B}^S$. If $\phi$ is a DK-equivalence and a DK-embedding on the ambient simplicially tensored categories, then $\phi^S$ is a DK-equivalence.*

Using Proposition 2.1.7.(i) and the Kan condition, we see that the action of any even permutation on $\mathcal{A}^S(x,y)(n) = \mathcal{C}(x, \Sigma^n y)$ is homotopic to the identity. Then [**27**, 3.2] gives us the following proposition.

Proposition 2.2.7. *The mapping symmetric spectra in $\mathcal{A}^S$ are semistable.*

Example 2.2.8. Let $\mathfrak{A}$ be an abelian category with enough projectives (e.g., the opposite category of an abelian category with enough injectives), and let $\mathfrak{E} \subset \mathfrak{A}$ be an exact category (with exact sequences the sequences in $\mathfrak{E}$ that are exact in $\mathfrak{A}$). Let $\mathcal{C}$ be the simplicially tensored Waldhausen category of levelwise projectives in the category of simplicial objects of $\mathfrak{A}$, as in Example 2.1.5.(iv). Let $\mathcal{A} \subset \mathcal{C}$ be the full subcategory of $\mathcal{C}$ consisting of those objects $x$ such that $\pi_0 x$ is in $\mathfrak{E}$ and $\pi_n x = 0$ for $n > 0$. Then $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicially enriched Waldhausen category and $\pi_0$ gives a enriched exact functor $\mathcal{A} \to \mathfrak{E}$. This functor induces a DK-equivalence of the connective spectral enrichments $\mathcal{A}^\Gamma \to \mathfrak{E}^\Gamma$. On the other hand $\mathcal{A}$ has a non-connective spectral enrichment $\mathcal{A}^S$, where $\pi_n \mathcal{A}^S(x,y)$ is 0 for $n > 0$ and $\operatorname{Ext}^{-n}(\pi_0 x, \pi_0 y)$ for $n \leq 0$.

Example 2.2.9. As an example to demonstrate the significance of the ambient simplicially tensored Waldhausen category, let $\mathcal{C}$ be the Waldhausen category of countable cell EKMM $S$-modules and let $\mathcal{C}'$ the Waldhausen category of countable cell EKMM $H\mathbb{Z}$-modules (for some countable cell $S$-algebra model of $H\mathbb{Z}$). Let $\mathcal{A}$ and $\mathcal{A}'$ be the Waldhausen subcategories of Eilenberg-Mac Lane spectra with homotopy groups concentrated in degree zero in $\mathcal{C}$ and $\mathcal{C}'$ respectively. The forgetful functor $\mathcal{C}' \to \mathcal{C}$ is exact and sends $\mathcal{A}'$ into $\mathcal{A}$, inducing a DK-equivalence and hence a DK-equivalence $\mathcal{A}'^\Gamma \to \mathcal{A}^\Gamma$ but not a DK-equivalence $\mathcal{A}'^S \to \mathcal{A}^S$.

The next two propositions explore the relationship between $\mathcal{A}^\Gamma$ and $\mathcal{A}^S$.

Proposition 2.2.10. *There is a canonical spectral functor $\mathcal{A}^\Gamma \to \mathcal{A}^S$, natural in enhanced simplicially enriched Waldhausen categories $\mathcal{A} \subset \mathcal{C}$.*

Proof. The maps of simplicial sets

$$\mathcal{A}(x,y) \otimes \Delta[1]^n \longrightarrow \mathcal{A}(x, y \otimes \Delta[1]^n) \longrightarrow \mathcal{A}(x, \Sigma^n y)$$

induce equivariant maps of based simplicial sets $\Sigma^n \mathcal{A}(x,y) \to \mathcal{A}(x, \Sigma^n y)$, which assemble into the spectral functor $\mathcal{A}^\Gamma \to \mathcal{A}^S$. $\square$

In Example 2.2.9, and in fact in the examples of Example 2.1.5, the canonical map $\mathcal{A}^\Gamma \to \mathcal{A}^S$ of the previous proposition makes $\mathcal{A}^\Gamma$ a connective cover of $\mathcal{A}^S$, i.e., induces an isomorphism on the non-negative homotopy groups. The following proposition gives a sufficient general condition for this to hold.

Proposition 2.2.11. *Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially enriched Waldhausen category, and assume that for every $a, b \in \mathcal{A}$ the suspension map $\mathcal{C}(a,b) \to \mathcal{C}(\Sigma a, \Sigma b)$ is a weak equivalence. Then $\mathcal{A}^\Gamma(a,b) \to \mathcal{A}^S(a,b)$ is a connective cover.*

Proof. Fix $a, b$ and consider the functor $F(-) = \mathcal{C}(a, (b \otimes -)/(b \otimes *))$ as a functor from based finite simplicial sets to based simplicial sets; we then get $\mathcal{A}^\Gamma(a,b)$ by viewing $F$ as $\Gamma$-space and $\mathcal{A}^S(a,b)$ by viewing $\{F(S^n)\}$ as a symmetric spectrum. By the hypothesis of the proposition, the canonical map $F(-) \to \Omega F(\Sigma -)$ is a weak equivalence. The argument of [**19**, 17.9] shows that $F$ is "linear" meaning that it takes homotopy pushouts to homotopy pullbacks, and in particular, as a $\Gamma$-space $F$ is very special [**19**, 18.6]. The homotopy groups of $\mathcal{A}^\Gamma(a,b)$ are then the homotopy groups of $F(S^0) = \mathcal{A}(a,b)$. Likewise, $\{F(S^n)\}$ is an $\Omega$-spectrum, so its non-negative homotopy are also the homotopy groups of $F(S^0)$. $\square$

In the absence of the stability hypothesis of the previous proposition, $\mathcal{A}^S$ tends to better capture the stable homotopy theory of $\mathcal{A} \subset \mathcal{C}$, as indicated for example in the following proposition.

Proposition 2.2.12. *Let $\mathcal{A} \subset \mathcal{C}$ be an enhanced simplicially enriched Waldhausen category.*

(i) *For any $x, y$ in $\mathcal{A}$, the map $\mathcal{A}^S(x,y) \to \mathcal{A}^S(\Sigma x, \Sigma y)$ is a weak equivalence.*

(ii) *For a cofibration $f\colon a \to b$, $Cf$ the homotopy cofiber, and any object $z$, the sequences*

$$\Omega\mathcal{A}^S(a,z) \cong \mathcal{A}^S(\Sigma a,z) \longrightarrow \mathcal{A}^S(Cf,z) \longrightarrow \mathcal{A}^S(b,z) \longrightarrow \mathcal{A}^S(a,z)$$

$$\mathcal{A}^S(z,a) \longrightarrow \mathcal{A}^S(z,b) \longrightarrow \mathcal{A}^S(z,Cf) \longrightarrow \mathcal{A}^S(z,\Sigma a) \simeq \Sigma\mathcal{A}^S(z,a)$$

*form a fiber sequence and a cofiber sequence in the stable category, respectively.*

PROOF. Part (i) and the statement about the first sequence in part (ii) are clear. The statement about the second sequence follows from part (i) and the argument in [**17**, §III.2.1] or [**19**, 7.4.vi]. □

The proposition indicates that for a simplicially tensored Waldhausen category $\mathcal{C}$, the spectral category $\mathcal{C}^S$ is nearly pretriangulated (Definition 1.3.7). In fact, we have the following easy corollary:

COROLLARY 2.2.13. *Let $\mathcal{C}$ be a simplicially tensored Waldhausen category in which every object is weakly equivalent to a suspension. Then the category $\mathcal{C}^S$ is pretriangulated, and in particular, the category of components $\pi_0\mathcal{C}^S$ has the structure of a triangulated category with triangles coming from the Puppe sequences and translation from the suspension.*

REMARK 2.2.14. As the preceding results indicate, the construction of the mapping spectra described above provides a version of stabilization of the simplicial Waldhausen category $\mathcal{C}$, when we regard the objects of $\mathcal{C}$ as being compact. In particular, the zeroth space of (a fibrant replacement of) the mapping spectrum $\mathcal{C}^S(x,y)$ is given by

$$\operatorname{colim}_n \Omega^n\mathcal{C}(x,\Sigma^n y) \cong \operatorname{colim}_n \mathcal{C}(\Sigma^n x,\Sigma^n y).$$

It is possible to explicitly compare $\mathcal{C}^S$ to a model of the formal stabilization in terms of symmetric spectrum objects in $\mathcal{C}$. We give an example below, but general theorems of this sort are encumbered with technical hypotheses, and since we do not need such results we leave them to the interested reader.

EXAMPLE 2.2.15 (Spectral categories and stabilization in Waldhausen's algebraic $K$-theory of spaces). Let $G$ be a group-like topological monoid, let $W$ be a CW-complex on which $G$ acts, and let $R(W,G)$ denote the category of $G$-spaces which have $W$ as a retract. When restricting to objects satisfying some kind of finiteness condition, $R(W,G)$ provides Waldhausen's motivating example for a Waldhausen category and one of the models underlying the algebraic $K$-theory of spaces. We can give $R(W,G)$ the model structure in which the weak equivalences are the equivariant maps that induce underlying equivalences of spaces. The category $R(W,G)$ is in no sense stable (for example, when $G$ and $W$ are trivial, $R(W,G)$ is the category of based spaces), and the spectral category $R(W,G)^S$ is equivalent to the evident subcategory of free $\Sigma^\infty_+ G$-spectra, as expected.

## 2.3. The $S_\bullet$ and Moore nerve constructions

As part of the construction of $THH$ and $TC$ of simplicially enriched Waldhausen categories and the construction of the cyclotomic trace in the Section 2.5, we need to extend Waldhausen's $S_\bullet$ construction and the nerve category construction to the context of simplicially enriched Waldhausen categories. We begin with the $S_\bullet$ construction, where no difficulties arise.

Let $\mathrm{Ar}[n]$ denote the lexicographically ordered set of ordered pairs of integers $i, j$ where $0 \leq i \leq j \leq n$. Recall that for a Waldhausen category $\mathcal{C}_0$, $S_n\mathcal{C}_0$ is the full subcategory of the category of functors $A = a_{-,-} \colon \mathrm{Ar}[n] \to \mathcal{C}_0$ such that:

(i) $a_{i,i} = *$,
(ii) $a_{i,j} \to a_{i,k}$ is a cofibration, and
(iii) $a_{i,i} \cup_{a_{i,j}} a_{i,k} \to a_{j,k}$ is an isomorphism

for all $i \leq j \leq k$. A map in $S_\bullet\mathcal{C}_0$ is simply a natural transformation of functors $\mathrm{Ar}[n] \to \mathcal{C}_0$. This becomes a Waldhausen category with weak equivalences defined objectwise and cofibrations defined to be the objectwise cofibrations $A \to B$ such that each map $a_{i,k} \cup_{a_{i,j}} b_{i,j} \to b_{i,k}$ is a cofibration.

Definition 2.3.1. For a simplicially enriched Waldhausen category $\mathcal{C}$, let $S_n\mathcal{C}$ be the simplicially enriched category with objects the same as $S_n\mathcal{C}_0$ and with the simplicial set of maps $S_n\mathcal{C}(A,B)$ the simplicial set of natural transformations of functors $\mathrm{Ar}[n] \to \mathcal{C}$ from $A$ to $B$.

Condition (iii) in the definition of $S_\bullet$ implies that a map $A \to B$ is completely determined by the maps $a_{0,j} \to b_{0,j}$. Since the maps $a_{0,j} \to a_{0,j+1}$ are cofibrations, we can identify the simplicial set of maps $S_n\mathcal{C}(A,B)$ as a pullback over fibrations

$$S_n\mathcal{C}(A,B) \cong \mathcal{C}(a_{0,1}, b_{0,1}) \times_{\mathcal{C}(a_{0,1}, b_{0,2})} \times \cdots \times_{\mathcal{C}(a_{0,n-1}, b_{0,n})} \mathcal{C}(a_{0,n}, b_{0,n}). \tag{2.3.2}$$

That is, the simplicial set of maps computes a homotopy limit. Using this formulation of the maps, the following becomes an easy check of the definitions and standard properties of pullbacks of fibrations of Kan complexes.

Proposition 2.3.3. *Let $\mathcal{C}$ be a simplicially enriched Waldhausen category. Then:*

(i) *$S_n\mathcal{C}$ is a simplicially enriched Waldhausen category.*
(ii) *If $\mathcal{C}$ is simplicially tensored or enhanced, then so is $S_n\mathcal{C}$.*
(iii) *The face and degeneracy maps $S_m\mathcal{C} \to S_n\mathcal{C}$ are enriched exact.*
(iv) *If $\mathcal{C}$ is simplicially tensored or enhanced then the face and degeneracy maps $S_m\mathcal{C} \to S_n\mathcal{C}$ are tensored exact or enhanced exact.*

*Moreover, $S_n$ preserves enriched exact, tensored exact, and enhanced exact functors.*

Applying the spectral category constructions of the previous section, we get a simplicial spectral category $S_\bullet\mathcal{C}^\Gamma$, natural in enriched exact functors of $\mathcal{C}$. When $\mathcal{C}$ is simplicially tensored or enhanced, we get a simplicial spectral category $S_\bullet\mathcal{C}^S$, natural in tensored exact or enhanced exact functors of $\mathcal{C}$. The formula (2.3.2) for the mapping spaces then implies the following results for spectral categories.

Proposition 2.3.4. *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be an enriched exact functor between simplicially enriched Waldhausen categories that are DK-compatible. If $\phi$ is a DK-embedding, then $S_n\phi^\Gamma\colon S_n\mathcal{C}^\Gamma \to S_n\mathcal{D}^\Gamma$ is a DK-embedding.*

Proposition 2.3.5. *Let $\phi\colon (\mathcal{A} \subset \mathcal{C}) \to (\mathcal{B} \subset \mathcal{D})$ be an enhanced exact functor between enhanced simplicially enriched Waldhausen categories. If $\phi\colon \mathcal{C} \to \mathcal{D}$ is a DK-embedding, then*

$$S_n\phi^S\colon S_n\mathcal{A}^S \longrightarrow S_n\mathcal{B}^S$$

*is a DK-embedding.*

In Proposition 2.3.4, we do not necessarily get a DK-equivalence $S_n\mathcal{C}^\Gamma \to S_n\mathcal{D}^\Gamma$ from a DK-equivalence $\mathcal{C} \to \mathcal{D}$. Applying the results of [**6**], we can do slightly better in Proposition 2.3.5.

PROPOSITION 2.3.6. *Under the hypotheses of Proposition 2.3.5, if $\phi\colon \mathcal{A} \to \mathcal{B}$ and $\phi\colon \mathcal{C} \to \mathcal{D}$ are DK-equivalences, then*

$$S_n\phi^\Gamma : S_n\mathcal{A}^\Gamma \longrightarrow S_n\mathcal{B}^\Gamma \qquad \textit{and} \qquad S_n\phi^S : S_n\mathcal{A}^S \longrightarrow S_n\mathcal{B}^S$$

*are also DK-equivalences.*

PROOF. It suffices to show that for any sequence of cofibrations $b_1 \to \cdots \to b_n$ in $\mathcal{B}$, there exists a sequence of cofibrations $a_1 \to \cdots \to a_n$ in $\mathcal{A}$ and a commutative diagram

$$\begin{tikzcd} \phi(a_1) \arrow[r] \arrow[d, "\sim"'] & \phi(a_2) \arrow[r] \arrow[d, "\sim"'] & \cdots \arrow[r] & \phi(a_{n-1}) \arrow[r] \arrow[d, "\sim"'] & \phi(a_n) \arrow[d, "\sim"'] \\ b_1 \arrow[r] & b_2 \arrow[r] & \cdots \arrow[r] & b_{n-1} \arrow[r] & b_n \end{tikzcd}$$

with the vertical maps weak equivalences. We argue by induction on $n$, the base case of $n = 1$ following from the fact that $\phi$ is a DK-equivalence and all weak equivalences have homotopy inverses. Having constructed the diagram

$$\begin{tikzcd} \phi(a_1) \arrow[r] \arrow[d, "\sim"'] & \phi(a_2) \arrow[r] \arrow[d, "\sim"'] & \cdots \arrow[r] & \phi(a_{n-1}) \arrow[d, "\sim"'] & \\ b_1 \arrow[r] & b_2 \arrow[r] & \cdots \arrow[r] & b_{n-1} \arrow[r] & b_n \end{tikzcd}$$

by induction, we know from [**6**, 1.4] that the homotopy category of objects in $\mathcal{C}$ under $a_{n-1}$ is equivalent to the homotopy category of objects in $\mathcal{D}$ under $\phi(a_{n-1})$. We then get an object $a'$ a map $a_{n-1} \to a'$ in $\mathcal{C}$ and a zigzag of weak equivalences under $\phi(a_{n-1})$ in $\mathcal{D}$ from $\phi(a')$ to $b_n$. Since $b_n$ is in $\mathcal{B}$, by the embedding hypotheses, we see that $a'$ is in $\mathcal{A}$. Using an appropriate generalized interval $J$, we let $a_n = (a_{n-1} \otimes J) \cup_{a_{n-1}} a'$. The inclusion of $a_{n-1}$ in $a_n$ is a cofibration in $\mathcal{C}$, and we get a weak equivalence under $\phi(a_{n-1})$ from $\phi(a_n)$ to $b_n$. To complete the argument we need to see that $a_{n-1} \to a_n$ is a cofibration in $\mathcal{A}$, i.e., that its cofiber is in $\mathcal{A}$. This follows since $\phi(a_n/a_{n-1})$ is weakly equivalent to $b_n/b_{n-1}$, which is in $\mathcal{B}$ since by hypothesis $b_{n-1} \to b_n$ is a cofibration in $\mathcal{B}$. $\square$

Waldhausen constructed the $K$-theory spectrum $K\mathcal{C}_0$ as $w_\bullet S_\bullet^{(n)}\mathcal{C}_0$, where $S_\bullet^{(n)}$ is the iterated $S_\bullet$-construction and $w_\bullet$ is the nerve of the subcategory of weak equivalences. The previous proposition extends the iterated $S_\bullet$ construction to simplicially enriched categories. We could likewise consider the simplicially enriched categories $w_n\mathcal{C}$ with objects the sequences of weak equivalences

$$a_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} a_n$$

and simplicial sets of maps the natural transformations. Then for objects $A$ and $B$, the simplicial set of $w_n\mathcal{C}(A, B)$ becomes

$$\mathcal{C}(a_0, b_0) \times_{\mathcal{C}(a_0,b_1)} \cdots \times_{\mathcal{C}(a_{n-1},b_n)} \mathcal{C}(a_n, b_n).$$

While this works formally, it does not work well homotopically because the pullbacks are not over fibrations and so the mapping spaces are not homotopy limits.

We can sometimes resolve this problem by working with the simplicially enriched categories $\bar{w}_n\mathcal{C}$, where the objects are the sequences of maps which are weak equivalences and cofibrations; we use this construction in Section 3.3. However, this is often inconvenient and does not always produce the correct result, and so instead we describe a general technique for fixing the problem by putting choices of homotopies in the mapping spaces. As a first case, consider the following construction.

CONSTRUCTION 2.3.7. Let $\mathcal{C}$ be a simplicially enriched category and let $v\mathcal{C}_0$ be a subcategory of $\mathcal{C}_0$. We construct a topologically enriched category $v_1^M\mathcal{C}$ as follows. An object consists of a map $\alpha_0\colon a_0 \to a_1$ in $v\mathcal{C}_0$. The space of maps $v_1\mathcal{C}(A,B)$ consists of elements $f_0, f_1$ of the geometric realizations $|\mathcal{C}(a_0,b_0)|$, $|\mathcal{C}(a_1,b_1)|$ (respectively), a non-negative real number $r$, and a homotopy $f_{0,1}$ of length $r$ in $|\mathcal{C}(a_0,b_1)|$ from $\beta_0 \circ f_0$ to $f_1 \circ \alpha$; we topologize this as a subspace of

$$|\mathcal{C}(a_0,b_0)| \times |\mathcal{C}(a_1,b_1)| \times \mathbb{R} \times |\mathcal{C}(a_0,b_1)|^I.$$

Composition is induced by composition of maps and homotopies.

In the notation "$M$" stands for Moore, as this employs the Moore trick for making homotopy composition associative. In this construction, the mapping space $v_1^M\mathcal{C}(A,B)$ is homotopy equivalent to the homotopy pullback

$$|\mathcal{C}(a_0,b_0)| \times_{|\mathcal{C}(a_0,b_1)|} |\mathcal{C}(a_0,b_1)|^I \times_{|\mathcal{C}(a_0,b_1)|} |\mathcal{C}(a_1,b_1)|.$$

The Moore trick generalizes from paths to maps out of higher simplices [**24**, §2]. We understand the $n$-simplex of length $r > 0$ to be the subspace $\Delta[n]_r$ of points $(t_0, t_1, \ldots, t_n)$ of $\mathbb{R}^{n+1}$ with $t_i \geq 0$ and $\sum t_i = r$. Then given $r, s > 0$, the maps $\sigma_{r,s}^{i,n-i}\colon \Delta[i]_r \times \Delta[n-i]_s \to \Delta[n]_{r+s}$ defined by

$$\sigma_{r,s}^{i,n-i}\colon (t_0,\ldots,t_i),(u_0,\ldots,u_{n-i}) \longmapsto (t_0,\ldots,t_i+u_0,u_1,\ldots,u_{n-i})$$

decompose $\Delta[n]_{r+s}$ as a union of prisms

$$\psi_{r,s}^n\colon \Delta[n]_{r+s} \cong \bigcup_{i=0}^{n} \Delta[i]_r \times \Delta[n-i]_s.$$

(See Proof of Theorem 2.4 in [**24**, p. 162].) This decomposition clearly commutes with the simplicial face and degeneracy operations, and it is associative in that the following diagram commutes.

$$\begin{tikzcd}
\Delta[i]_q \times \Delta[j]_r \times \Delta[k]_s \arrow[r, "\sigma_{q,r}^{i,j}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\sigma_{r,s}^{j,k}"'] & \Delta[i+j]_{q+r} \times \Delta[k]_s \arrow[d, "\sigma_{q+r,s}^{i+j,k}"] \\
\Delta[i]_q \times \Delta[j+k]_{r+s} \arrow[r, "\sigma_{q,r+s}^{i,j+k}"'] & \Delta[i+j+k]_{q+r+s}
\end{tikzcd}$$

CONSTRUCTION 2.3.8 (Moore Nerve). For $\mathcal{C}$ a simplicially enriched category and $v\mathcal{C}_0$ a subcategory of $\mathcal{C}_0$, define the topologically enriched category $v_n^M\mathcal{C}$ as follows. The objects consist of the sequences of $n$ composable maps in $v\mathcal{C}_0$

$$a_0 \xrightarrow{v} \cdots \xrightarrow{v} a_n.$$

For convenience in what follows, we denote the structure map $a_i \to a_j$ as $\alpha_{i,j}$, for $i \leq j$ (and $\beta_{i,j}$, $\gamma_{i,j}$ similarly for objects $B$,$C$). An element of the space of maps from $A$ to $B$ consists of the following data:

(i) An non-negative real number $r$
(ii) For each $0 \leq m \leq n$ and each $0 \leq i_0 < \cdots < i_m < n$ a map

$$f_{i_0,\ldots,i_m} \colon \Delta[m]_r \longrightarrow |\mathcal{C}(a_{i_0}, b_{i_m})|$$

for $r > 0$, or an element of $|\mathcal{C}(a_{i_0}, b_{i_m})|$ for $r = 0$.

such that for any subset $i_{j_0}, \ldots, i_{j_\ell}$ of $i_1, \ldots, i_m$, the map

$$\beta_{i_{j_\ell}, i_m} \circ f_{i_{j_0},\ldots,i_{j_\ell}} \circ \alpha_{i_0, i_{j_0}} \colon \Delta[\ell]_r \longrightarrow |\mathcal{C}(a_{i_0}, b_{i_m})|$$

is the restriction to the face of $f_{i_0,\ldots,j_m}$ spanned by $i_{j_0}, \ldots, i_{j_\ell}$. We topologize this as a subset of the evident product. Composition is induced by the prismatic decomposition above: for $F \colon A \to B$ of length $r > 0$ and $G \colon B \to C$ of length $s > 0$, the composition $H \colon A \to C$ of length $r + s$ is defined by taking $h_{i_0,\ldots,i_m}$ to be the map

$$(g_{i_j,\ldots,i_m}(u_0, \ldots, u_{m-j}) \circ \alpha_{i_0, i_j}) \circ (\gamma_{i_j, i_m} \circ f_{i_0,\ldots,i_j}(t_0, \ldots, t_j))$$

on the $\Delta[j]_r \times \Delta[m-j]_s$ prism in the $\psi^m_{r,s}$ decomposition of $\Delta[m]_{r+s}$. For $r = 0$ or $s = 0$, composition is induced by composition in $\mathcal{C}$.

A straightforward check of the formulas verifies that this defines a topological category. Moreover, $v^M_\bullet \mathcal{C}$ assembles into a simplicial topological category with the following naturality property. (It applies in particular to the important special case $\mathcal{C} = \mathcal{D}$ with $v\mathcal{C}_0 \subsetneqq v\mathcal{D}_0$.)

Proposition 2.3.9. *Given simplicially enriched categories $\mathcal{C}$ and $\mathcal{D}$, a simplicially enriched functor $\phi \colon \mathcal{C} \to \mathcal{D}$ that takes $v\mathcal{C}_0$ into $v\mathcal{D}_0$ induces a topologically enriched simplicial functor $v^M_\bullet \mathcal{C} \to v^M_\bullet \mathcal{D}$.*

For objects $A$ and $B$, $v^M_n(A, B)$ is homotopy equivalent to the homotopy end of $\mathcal{C}(a_i, b_i)$ for $n > 0$, while $v^M_0 \mathcal{C}(a, b) = |\mathcal{C}(a, b)| \times [0, \infty)$. In particular $\mathcal{C}$ includes in $v^M_0 \mathcal{C}$ (after geometric realization) as the subcategory of maps of length zero. More generally, the nerve categories $v_n \mathcal{C}$ include (after geometric realization) as the subcategories of the Moore nerve categories $v^M_n \mathcal{C}$ of the maps of length zero. Restricting to simplicially enriched Waldhausen categories, we get the following proposition.

Proposition 2.3.10. *Let $\mathcal{C}$ be a simplicially enriched Waldhausen category and $v\mathcal{C}_0$ a subcategory of $\mathcal{C}_0$.*

(i) *If $v\mathcal{C}_0 \subset w\mathcal{C}_0$, then the inclusion of $\mathcal{C}$ in $v^M_n \mathcal{C}$ is a DK-equivalence.*
(ii) *If $v\mathcal{C}_0 \subset \mathrm{co}\,\mathcal{C}_0$, then the inclusion of $v_n \mathcal{C}$ in $v^M_n \mathcal{C}$ is a DK-equivalence.*

Finally, we use the following notation.

Definition 2.3.11. Let $\mathcal{C}$ be a simplicial enriched Waldhausen category, and let $v\mathcal{C}_0$ be a subcategory of $\mathcal{C}_0$. Define $v^M_\bullet \mathcal{C}^\Gamma$ to be the simplicial spectral category obtained from the simplicial $\Gamma$-category with

$$v^M_n \mathcal{C}^\Gamma_q(X, Y) = v^M_n \mathcal{C}(X, \bigvee_q Y).$$

For $\mathcal{A} \subset \mathcal{C}$ an enhanced simplicially enriched Waldhausen category, define $v^M_\bullet \mathcal{A}^S$ to be the simplicial spectral category with

$$v^M_n \mathcal{A}^S(X, Y)(q) = v^M_n \mathcal{C}(X, \Sigma^q Y).$$

In the formula, $\bigvee$ denotes the entry-wise coproduct; although this is not the coproduct in $v_n^M\mathcal{C}$, we can identify

$$v_n^M\mathcal{C}(\bigvee_q Y, Z) \subset \prod_q v_n^M\mathcal{C}(Y, Z)$$

as the subspace of $q$-tuples of maps, all having the same length. We then obtain $\Gamma$-category composition as in Section 2.2. Likewise, in the enhanced context, although $\Sigma^n Y$ is not a based tensor in $v_n^M\mathcal{C}$, we nevertheless have a continuous functor

$$v_n^M\mathcal{C}(Y, Z) \longrightarrow v_n^M(\Sigma^n Y, \Sigma^n Z)$$

and we obtain the spectral category composition as in Section 2.2.

## 2.4. The Moore $S'_\bullet$ construction

Although the $S_\bullet$ construction translates naturally to the enriched context, it is often useful to be able to weaken the cocartesian condition in the construction and instead work with an equivalent construction defined in terms of *homotopy cocartesian* squares called the $S'_\bullet$ construction [**5**, §2]. This flexibility plays a key role in the proof of the dévissage theorem for the localization theorem for $THH(ku)$ in Section 4.3. Such a definition also provides models of $K$-theory and $THH$ which are functorial in functors "exact up to homotopy" as explained in Section 5.1. In this section we introduce an appropriately enriched version of the $S'_\bullet$ construction, using the Moore ideas from the previous section to construct the homotopically correct enrichment.

We begin by reviewing the $S'_\bullet$ construction. For this, recall from [**6**, §2] that a *weak cofibration* is a map that is weakly equivalent (by a zigzag) to a cofibration in the category $\mathrm{Ar}\,\mathcal{C}_0$ of arrows in $\mathcal{C}_0$, and a *homotopy cocartesian square* is a square diagram that is weakly equivalent (by a zigzag) to a pushout square where one of the parallel sets of arrows consists of cofibrations.

Construction 2.4.1. Let $\mathcal{C}_0$ be a Waldhausen category. Define $S'_n\mathcal{C}_0$ to be the full subcategory of functors $A\colon \mathrm{Ar}[n] \to \mathcal{C}_0$ such that:

- The initial map $* \to a_{i,i}$ is a weak equivalence for all $i$,
- The map $a_{i,j} \to a_{i,k}$ is a weak cofibration for all $i \le j \le k$, and
- The diagram

$$\begin{array}{ccc} a_{i,j} & \longrightarrow & a_{i,k} \\ \downarrow & & \downarrow \\ a_{j,j} & \longrightarrow & a_{j,k} \end{array}$$

  is a homotopy cocartesian square for all $i \le j \le k$.

We define a map $A \to B$ to be a weak equivalence when each $a_{i,j} \to b_{i,j}$ is a weak equivalence. Clearly $S'_\bullet$ assembles into a simplicial category with the usual face and degeneracy functors.

In order to use $S'_\bullet\mathcal{C}_0$ to construct $K$-theory, we need a mild hypothesis on $\mathcal{C}_0$. We say that a Waldhausen category $\mathcal{C}_0$ admits *factorization* when any map $f\colon a \to b$ in $\mathcal{C}_0$ factors as a cofibration followed by a weak equivalence

$$\begin{tikzcd} a \arrow[r, rightarrowtail] \arrow[rr, bend right, dotted, "f"'] & Tf \arrow[r, "\sim"] & b, \end{tikzcd}$$

We say that $\mathcal{C}_0$ admits functorial factorization if this factorization may be chosen functorially in $f$ in the category $\operatorname{Ar}\mathcal{C}_0$ of arrows in $\mathcal{C}_0$. More generally, we say that $\mathcal{C}_0$ admits factorization of weak cofibrations (FWC) or functorial factorization of weak cofibrations (FFWC) when the weak cofibrations can be factored as above. Enhanced simplicially enriched Waldhausen categories always admit FFWC using the standard mapping cylinder construction.

PROPOSITION 2.4.2. *If $\mathcal{A}$ is an enhanced simplicially enriched Waldhausen category, then $\mathcal{A}$ admits FFWC.*

The significance of the hypothesis of FFWC is the following comparison result [**5**, 2.9].

PROPOSITION 2.4.3. *Let $\mathcal{C}_0$ be a Waldhausen category admitting FFWC. Then for each $n$, the inclusion $wS_n\mathcal{C}_0 \to wS'_n\mathcal{C}_0$ induces a weak equivalence on nerves.*

The previous proposition implies that $w_\bullet S'_\bullet\mathcal{C}_0$ models the $K$-theory space of $\mathcal{C}_0$. Using an iterated $S'_\bullet$ construction $S'^{(n)}_{\bullet,\dots,\bullet}\mathcal{C}_0$ as a full subcategory of functors $\operatorname{Ar}[\bullet] \times \cdots \times \operatorname{Ar}[\bullet]$ to $\mathcal{C}_0$ (see [**6**, A.5.4]) gives a model $w_\bullet S'^{(n)}_{\bullet,\dots,\bullet}\mathcal{C}_0$ for the $K$-theory spectrum.

For a simplicially enriched Waldhausen category $\mathcal{C}$, we need a version of $S'_\bullet\mathcal{C}$ (or more generally $w_\bullet S'^{(n)}_{\bullet,\dots,\bullet}\mathcal{C}$) with the correct mapping spaces. As in the construction of the Moore nerve in 2.3.8, we do this using the Moore trick, this time with the full generality of the McClure-Smith construction of the Moore Tot [**24**, §2] of a cosimplicial object.

CONSTRUCTION 2.4.4. Let $\mathcal{C}$ be a category enriched in simplicial sets, let $D$ be a small category, and let $D\mathcal{C}_0$ be the category of $D$-diagrams in $\mathcal{C}_0$. For $A = (a_d)$ and $B = (b_d)$ in $D\mathcal{C}_0$, let $D^M\mathcal{C}(A, B)$ be the McClure-Smith Moore Tot (denoted $\operatorname{Tot}'$ in [**24**, §2]) of the cosimplicial object

$$D^M\mathcal{C}(A,B)^q = \prod_{d_0 \to \cdots \to d_q} |\mathcal{C}(a_{d_q}, b_{d_0})|$$

(the cosimplicial object for the homotopy end of $|\mathcal{C}(A,B)|$). We let $D^M\mathcal{C}$ be the topologically enriched category with objects the objects of $D\mathcal{C}_0$, maps the spaces $D^M\mathcal{C}(A,B)$ above, and composition induced by the "cup-pairing" [**24**, 2.1]

$$\prod_{d_p \to \cdots \to d_0} |\mathcal{C}(b_{d_p}, c_{d_0})| \times \prod_{d'_q \to \cdots \to d'_0} |\mathcal{C}(a_{d'_q}, b_{d'_0})| \longrightarrow \prod_{d_{p+q} \to \cdots \to d_0} |\mathcal{C}(a_{d_{p+q}}, c_{d_0})|.$$

Here the map is induced on the $d_{p+q} \to \cdots \to d_0$ coordinate of the target by composition

$$\mathcal{C}(b_{d_p}, c_{d_0}) \times \mathcal{C}(a_{d_{p+q}}, b_{d_p}) \longrightarrow \mathcal{C}(a_{d_{p+q}}, c_{d_0})$$

of the maps on the $d_p \to \cdots \to d_0$ and $d_{p+q} \to \cdots \to d_p$ (i.e., $d'_i = d_{p+i}$) coordinates of the source.

As in the previous section, we obtain a connective spectral enrichment using the objectwise coproduct and (when defined) a non-connective spectral enrichment using the objectwise suspension.

We use analogous notation for the enriched categories associated to full subcategories of diagram categories, obtaining for example $S^M_n\mathcal{C}$ and $S'^M_n\mathcal{C}$ as full subcategories of the functors $\operatorname{Ar}[n] \to \mathcal{C}$. Because the Moore Tot always has the homotopy type of the homotopy end (containing it as a deformation retract), we obtain the following result as an immediate consequence.

PROPOSITION 2.4.5. *For a simplicially enriched Waldhausen category $\mathcal{C}$, the inclusion of the topologically enriched category $|S_n\mathcal{C}|$ in $S_n^M\mathcal{C}$ as the length zero part is a DK-equivalence.*

Considering more complicated diagrams, this also applies to $w_p S^{(n)}_{q_1,\dots,q_n}\mathcal{C}$. Thinking of these categories as subcategories of $w_p S'^{(n)}_{q_1,\dots,q_n}\mathcal{C}$, the more restricted homotopies in $w_p^M S^{(n)}_{q_1,\dots,q_n}\mathcal{C}$ make its mapping spaces subspaces of $(w_p S'^{(n)}_{q_1,\dots,q_n})^M\mathcal{C}$, and we get the following result.

PROPOSITION 2.4.6. *For a simplicially enriched Waldhausen category $\mathcal{C}$, the inclusion of $w_p^M S^{(n)}_{q_1,\dots,q_n}\mathcal{C}$ in $(w_p S'^{(n)}_{q_1,\dots,q_n})^M\mathcal{C}$ is a DK-embedding. If $\mathcal{C}_0$ admits FFWC, then it is a DK-equivalence.*

We write $(w_p S'^{(n)}_{q_1,\dots,q_n})^M\mathcal{C}^\Gamma$ and when appropriate $(w_p S'^{(n)}_{q_1,\dots,q_n})^M\mathcal{C}^S$ for the associated spectrally enriched categories.

## 2.5. $THH$, $TC$, and the cyclotomic trace

In this section, we apply the constructions of $THH$ and $TC$ of spectral categories in the context of the spectral enrichments associated to a simplicially enriched Waldhausen category $\mathcal{C}$. For the connective enrichments, we require Waldhausen's $S_\bullet$ construction in order to properly handle extension sequences in the Waldhausen structure for reasons first observed by McCarthy [**23**, 3.3.5] and Dundas-McCarthy [**10**, 2.3.4]; for the non-connective enrichment, the $S_\bullet$ construction turns out to be superfluous.

DEFINITION 2.5.1. For a simplicially enriched Waldhausen category $\mathcal{C}$, we define

$$WTHH^\Gamma\mathcal{C} = \Omega|THH(S_\bullet\mathcal{C}^\Gamma)|$$
$$WTR^\Gamma\mathcal{C} = \Omega|TR(S_\bullet\mathcal{C}^\Gamma)|$$
$$WTC^\Gamma\mathcal{C} = \Omega|TC(S_\bullet\mathcal{C}^\Gamma)|.$$

If $\mathcal{C}$ is a simplicially tensored Waldhausen category and $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicially enriched Waldhausen category, then we define

$$WTHH\mathcal{A} = \Omega|THH(S_\bullet\mathcal{A}^S)|$$
$$WTR\mathcal{A} = \Omega|TR(S_\bullet\mathcal{A}^S)|$$
$$WTC\mathcal{A} = \Omega|TC(S_\bullet\mathcal{A}^S)|.$$

In other words, we apply $THH$, $TR$, or $TC$ first to get simplicial (or multisimplicial) cyclotomic spectra or pro-spectra. Then we take the geometric realization in the simplicial directions, followed by loops. We have the following naturality properties.

PROPOSITION 2.5.2. *An enriched exact functor induces maps on $WTHH^\Gamma$, $WTR^\Gamma$, and $WTC^\Gamma$. A tensored exact or enhanced exact functor induces maps on $WTHH$, $WTR$, and $WTC$. Naturally weakly equivalent functors induce the same map in the stable category.*

PROOF. The only part not immediate from the construction is the last statement. We use Construction 2.3.7 for $v\mathcal{D}_0$ the subcategory of weak equivalences $w\mathcal{D}_0$. We have a pair of simplicial spectrally enriched functors $w_1^M S_\bullet \mathcal{D}^\Gamma \to S_\bullet \mathcal{D}^\Gamma$ each split by the inclusion $S_\bullet \mathcal{D}^\Gamma \to w_1^M S_\bullet \mathcal{D}^\Gamma$. Since the inclusion induces a DK-equivalence $S_\bullet \mathcal{D}^\Gamma \to w_1^M S_\bullet \mathcal{D}^\Gamma$, both maps $w_1^M S_\bullet \mathcal{D}^\Gamma \to S_\bullet \mathcal{D}^\Gamma$ induce the same map in the stable category on $THH$, $TR$, and $TC$. Now, given enriched exact functors $\phi_0, \phi_1 \colon \mathcal{C} \to \mathcal{D}$ and $h$ a natural weak equivalence between them, we get a simplicial spectrally enriched functor $S_\bullet \mathcal{C}^\Gamma \to w_1^M S_\bullet \mathcal{D}^\Gamma$ (factoring through the length zero part $w_1 S_\bullet \mathcal{D}^\Gamma$). The two composites

$$S_\bullet \mathcal{C}^\Gamma \longrightarrow w_1^M S_\bullet \mathcal{D}^\Gamma \longrightarrow S_\bullet \mathcal{D}^\Gamma$$

are the maps induced by $\phi_0$ and $\phi_1$. For tensored exact or enhanced exact functors $\phi_0$ and $\phi_1$ and a natural weak equivalence between them, the same argument applies to show that the maps on $WTHH$, $WTR$, and $WTC$ induced by $\phi_0$ and $\phi_1$ coincide in the stable category. $\square$

Applying Proposition 2.3.4 and 2.3.6 we obtain the following homotopy invariance properties.

PROPOSITION 2.5.3. *Let $\phi \colon \mathcal{C} \to \mathcal{D}$ be an enriched exact functor between simplicially enriched Waldhausen categories that are DK-compatible. Assume that $\phi$ is a DK-embedding and that every object of $S_n\mathcal{D}$ is weakly equivalent to an object in the image of $S_n\phi$ (for all $n$). Then $\phi$ induces weak equivalences on $WTHH^\Gamma$, $WTR^\Gamma$, and $WTC^\Gamma$.*

PROPOSITION 2.5.4. *Let $\phi \colon \mathcal{A} \to \mathcal{B}$ be an enhanced exact functor between enhanced simplicially enriched Waldhausen categories. If $\phi$ is a DK-equivalence of the ambient simplicially tensored categories and a DK-equivalence $\mathcal{A} \to \mathcal{B}$, then $\phi$ induces a weak equivalence on $WTHH$, $WTR$, and $WTC$.*

Implicitly in the previous propositions we passed from a level weak equivalence of simplicial spectra $X_\bullet \to Y_\bullet$ to a weak equivalence on geometric realization $|X_\bullet| \to |Y_\bullet|$. Using the standard geometric realization, we need hypotheses on $X_\bullet$ and $Y_\bullet$ for this to work (cf. 1.2.4). One sufficient hypothesis is that $X_\bullet$ and $Y_\bullet$ are *spacewise proper*: we say that a simplicial symmetric spectrum of topological spaces $X_\bullet$ is space-wise proper when the simplicial space $X_\bullet(n)$ is proper for every $n$, i.e., for each $k$, each degeneracy map $X_k(n) \to X_{k+1}(n)$ is a Hurewicz cofibration (satisfies the homotopy extension property). The following proposition applies to verify this property for the constructions in the previous propositions and the many other constructions in this paper. Its proof requires the details of the $THH$ construction in Definition 1.2.3 but is then straightforward given the standard properties of Hurewicz cofibrations.

PROPOSITION 2.5.5. *Let $\mathcal{C}_\bullet$ be a simplicial object in the category of spectral categories (in topological symmetric spectra). Assume that for all $k$ the category $\mathcal{C}_k$ is non-degenerately based and that for all objects $x, y$ of $\mathcal{C}_k$ and for each space of the mapping each degeneracy map $s^i \colon \mathcal{C}_k(x,y)(n) \to \mathcal{C}_{k+1}(s^i x, s^i y)(n)$ is a Hurewicz cofibration. Then the simplicial spectrum $THH(\mathcal{C}_\bullet)$ is spacewise proper.*

Waldhausen's approximation property provides a convenient formulation for the conditions in Propositions 2.5.3 and 2.5.4 that often holds in practice. We say that exact functor $\phi \colon \mathcal{C} \to \mathcal{D}$ has the *approximation property* when:

(i) A map $f\colon a \to b$ is a weak equivalence in $\mathcal{C}$ only if the map $\phi(f)$ in $\mathcal{D}$ is a weak equivalence.
(ii) For every map $f\colon \phi(a) \to x$ in $\mathcal{D}$, there exists a map $g\colon a \to b$ in $\mathcal{C}$ and a weak equivalence $h\colon \phi(b) \to x$ in $\mathcal{D}$ such that $g = h \circ \phi(g)$.

We then have the following $THH$ analogue of Waldhausen's Approximation Theorem. The proof is that under factorization hypotheses, the approximation property implies that $\phi$ is a DK-equivalence; see [**6**, 1.4–1.5].

THEOREM 2.5.6. *Let $\phi\colon (\mathcal{A} \subset \mathcal{C}) \to (\mathcal{B} \subset \mathcal{D})$ be an enhanced exact functor between enhanced simplicially enriched Waldhausen categories, and suppose that $\phi_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ satisfies the approximation property. If every object of $\mathcal{B}$ is weakly equivalent to the image of an object of $\mathcal{A}$, then $\phi$ induces weak equivalences on $WTHH^\Gamma$, $WTR^\Gamma$, $WTC^\Gamma$ and on $WTHH$, $WTR$, $WTC$.*

In many situations, the underlying Waldhausen category $\mathcal{C}_0$ of a simplicially enriched Waldhausen category $\mathcal{C}$ admits a second subcategory of weak equivalences $v\mathcal{C}_0$ (not necessarily related to the simplicial structure, or even satisfying the two out of three property). When $v\mathcal{C}_0$ contains all the isomorphisms and satisfies the Gluing Axiom (stated as (iv) in the introduction to this chapter), each Waldhausen category $S_n\mathcal{C}_0$ inherits a subcategory $vS_n\mathcal{C}_0$ also satisfying these properties. In this context, we have additional variants of $THH$, $TR$, and $TC$.

DEFINITION 2.5.7. Let $\mathcal{C}$ be a simplicially enriched Waldhausen category, and let $v\mathcal{C}_0$ be a subcategory of $\mathcal{C}_0$ containing all the isomorphisms and satisfying the Gluing Axiom. Then we define the connective relative $THH$, $TC$, and $TR$ of $(\mathcal{C}|v)$ as indicated below (on the left). When $\mathcal{C}$ is simplicially tensored and $\mathcal{A} \subset \mathcal{C}$ is an enhanced simplicially enriched Waldhausen category, we define the non-connective $THH$, $TC$, and $TR$ of $(\mathcal{A}|v)$ as indicated below (on the right).

$$\begin{aligned} WTHH^\Gamma(\mathcal{C}|v) &= \Omega|THH(v^M_\bullet S_\bullet \mathcal{C}^\Gamma)| & WTHH(\mathcal{A}|v) &= \Omega|THH(v^M_\bullet S_\bullet \mathcal{A}^S)| \\ WTR^\Gamma(\mathcal{C}|v) &= \Omega|TR(v^M_\bullet S_\bullet \mathcal{C}^\Gamma)| & WTR(\mathcal{A}|v) &= \Omega|TR(v^M_\bullet S_\bullet \mathcal{A}^S)| \\ WTC^\Gamma(\mathcal{C}|v) &= \Omega|TC(v^M_\bullet S_\bullet \mathcal{C}^\Gamma)| & WTC(\mathcal{A}|v) &= \Omega|TC(v^M_\bullet S_\bullet \mathcal{A}^S)| \end{aligned}$$

In the special case when $v\mathcal{C}_0$ is the category of weak equivalences $w\mathcal{C}_0$, the inclusion of each $S_n\mathcal{C}^\Gamma$ into $w^M_q S_n\mathcal{C}^\Gamma$ and (when defined) $S_n\mathcal{A}^S$ into $w^M_q S_n\mathcal{A}^S$ is a DK-equivalence. This implies the following proposition.

PROPOSITION 2.5.8. *For $\mathcal{C}$ a simplicially enriched Waldhausen category and $\mathcal{A}$ an enhanced simplicially enriched Waldhausen category, the maps*

$$\begin{aligned} WTHH^\Gamma\mathcal{C} &\longrightarrow WTHH^\Gamma(\mathcal{C}|w) & WTHH\mathcal{A} &\longrightarrow WTHH(\mathcal{A}|w) \\ WTR^\Gamma\mathcal{C} &\longrightarrow WTR^\Gamma(\mathcal{C}|w) & WTR\mathcal{A} &\longrightarrow WTR(\mathcal{A}|w) \\ WTC^\Gamma\mathcal{C} &\longrightarrow WTC^\Gamma(\mathcal{C}|w) & WTC\mathcal{A} &\longrightarrow WTC(\mathcal{A}|w) \end{aligned}$$

*are weak equivalences*

Using the model of $THH$ in the previous proposition, we have the following sharper version of Proposition 2.5.2.

PROPOSITION 2.5.9. *Let $\phi_0, \phi_1\colon \mathcal{C} \to \mathcal{D}$ be exact functors between simplicially enriched Waldhausen categories. A natural weak equivalence from $\phi_0$ to $\phi_1$ induces* *a simplicial homotopy of the induced functors*

$$WTHH^{\Gamma}(\mathcal{C}|w) \longrightarrow WTHH^{\Gamma}(\mathcal{D}|w)$$
$$WTR^{\Gamma}(\mathcal{C}|w) \longrightarrow WTR^{\Gamma}(\mathcal{D}|w)$$
$$WTC^{\Gamma}(\mathcal{C}|w) \longrightarrow WTC^{\Gamma}(\mathcal{D}|w)$$

*and similarly for the non-connective enrichments when appropriate.*

PROOF. The proof is the usual one, using the $n+2$ functors

$$w_n^M \mathcal{X} \longrightarrow w_{n+1}^M \mathcal{Y}$$

obtained by inserting the natural transformation in each position (giving it length 0 in the Moore construction). □

To construct the cyclotomic trace, we need a final variant of these constructions where we iterate the $S_\bullet$ construction.

DEFINITION 2.5.10. Let

$$\widetilde{WTHH}^{\Gamma}\mathcal{C}(n) = |THH(w_\bullet^M S_\bullet^{(n)} \mathcal{C}^{\Gamma})|.$$

The simplicial maps of spectral categories

$$\Sigma_\bullet^{n-m} w_\bullet^M S_\bullet^{(m)} \mathcal{C}^{\Gamma} \longrightarrow w_\bullet^M S_\bullet^{(n)} \mathcal{C}^{\Gamma}$$

induce maps

$$\Sigma^{n-m} \widetilde{WTHH}^{\Gamma}\mathcal{C}(m) \longrightarrow \widetilde{WTHH}^{\Gamma}\mathcal{C}(n)$$

(as in [**35**, §1.3]). These maps assemble $\widetilde{WTHH}^{\Gamma}\mathcal{C}$ into a symmetric spectrum in the category of cyclotomic spectra. We define $\widetilde{WTR}$ and $\widetilde{WTC}$ to be the $TR$ and $TC$ pro-spectra constructed from $\widetilde{WTHH}$.

As a consequence of the Additivity Theorem 3.1.1, we prove the following lemma in Section 3.1.

LEMMA 2.5.11. *The map $\Sigma^{n-m} \widetilde{WTHH}^{\Gamma}\mathcal{C}(m) \to \widetilde{WTHH}^{\Gamma}\mathcal{C}(n)$ in Definition 2.5.10 is a weak equivalence for all $n \geq m > 0$.*

We have analogous constructions and results in the relative case (using $v_\bullet^M$ in place of $w_\bullet^M$) and non-connective case (using $\mathcal{A}^S$ for $\mathcal{A}^{\Gamma}$ when $\mathcal{A}$ is enhanced). The identity $WTHH^{\Gamma}\mathcal{C} = \Omega \widetilde{WTHH}^{\Gamma}\mathcal{C}(1)$ then immediately implies the following result.

THEOREM 2.5.12. *We have natural isomorphisms in the stable category*

$$WTHH^{\Gamma}\mathcal{C} \simeq \widetilde{WTHH}^{\Gamma}\mathcal{C} \qquad WTR^{\Gamma}\mathcal{C} \simeq \widetilde{WTR}^{\Gamma}\mathcal{C} \qquad WTR^{\Gamma}\mathcal{C} \simeq \widetilde{WTC}^{\Gamma}\mathcal{C}$$

*and likewise for the relative and non-connective variants when these are defined.*

We can now define the cyclotomic trace.

DEFINITION 2.5.13. For a simplicially enriched Waldhausen category $\mathcal{C}$, the *cyclotomic trace*

$$K(\mathcal{C}_0) \longrightarrow \widetilde{WTC}^{\Gamma}\mathcal{C} \longrightarrow \widetilde{WTHH}^{\Gamma}\mathcal{C}$$

is the map induced by the inclusion of objects

$$K\mathcal{C}_0 = \mathrm{Ob}(w_\bullet S_\bullet^{(n)} \mathcal{C}_0) = \mathrm{Ob}(w_\bullet^M S_\bullet^{(n)} \mathcal{C}^{\Gamma}) \longrightarrow \widetilde{WTHH}^{\Gamma}\mathcal{C}.$$

For $v\mathcal{C}_0$ a subcategory of $\mathcal{C}_0$ containing the isomorphisms and satisfying the Gluing Axiom, the relative cyclotomic trace is the map

$$K(\mathcal{C}_0|v) \longrightarrow \widetilde{WTC}^{\Gamma}(\mathcal{C}|v) \longrightarrow \widetilde{WTHH}^{\Gamma}(\mathcal{C}|v)$$

induced by the inclusion of objects

$$K(\mathcal{C}_0|v) = \mathrm{Ob}(v_\bullet S^{(n)}_\bullet \mathcal{C}_0) = \mathrm{Ob}(v^M_\bullet S^{(n)}_\bullet \mathcal{C}^{\Gamma}) \longrightarrow \widetilde{WTHH}^{\Gamma}(\mathcal{C}|v).$$

Finally, to compare the definitions of this section with the theories used in [**7**] and Chapter 1, we state the following two theorems. The first is a consequence of the Additivity Theorem 3.1.3 and proved in Section 3.1.

Theorem 2.5.14. *Let $\mathcal{A}$ be an enhanced simplicially enriched Waldhausen category. The inclusion of $THH(\mathcal{A}^S)$ in $WTHH(\mathcal{A})$ is a weak equivalence of cyclotomic spectra.*

The second is a special case of the Sphere Theorem from Section 3.4; see Corollaries 3.4.4 and 3.4.12.

Theorem 2.5.15. *Let $R$ be a ring, a simplicial ring, or a connective ring spectrum, and let $\mathcal{A}$ be the simplicially tensored Waldhausen category of finite cell modules (built out of free or finitely generated projective modules) in Example 2.1.5.(i) or (ii) (as appropriate). Then the natural map $WTHH^{\Gamma}(\mathcal{A}) \to WTHH(\mathcal{A})$ is a weak equivalence of cyclotomic spectra.*

Thus, for a ring, simplicial ring, or connective ring spectrum, we have weak equivalences of cyclotomic spectra

$$WTHH^{\Gamma}(\mathcal{A}) \xrightarrow{\sim} WTHH(\mathcal{A}) \xleftarrow{\sim} THH(\mathcal{A}) \xleftarrow{\sim} THH(R),$$

where the last weak equivalence is a special case of Theorem 1.3.12.

CHAPTER 3

# $K$-theory theorems in $THH$ and $TC$

The purpose of this chapter is to take the standard theorems of $K$-theory as proved in [**35**] and describe versions of these for $THH$. Because weak equivalences in $THH$ and cofiber sequences in $THH$ automatically produce weak equivalences and cofiber sequences for $TC$, we typically make statements just for $THH$.

Section 3.1 reviews the *Additivity Theorem* (q.v. [**35**, 1.3.2]) for $THH$, based on work of McCarthy [**22**]. Section 3.2 proves a $THH$ version of [**35**, 1.5.5], which constructs a cofibration sequence of $THH$ spectra associated to a map. Section 3.3 proves a $THH$ version of Waldhausen's *Fibration Theorem* [**35**, 1.6.4]. Section 3.4 proves the *Sphere Theorem* (cf. [**35**, §1.7]), which in certain cases identifies the $THH$ of simplicially tensored Waldhausen category as the $THH$ of a subcategory of a subcategory of generators. Section 3.5 proves the Sphere Theorem.

## 3.1. The Additivity Theorem

In this section, we present the Additivity Theorem for the $THH$ of Waldhausen categories. The modern viewpoint, implicit in [**35**] but first written explicitly by Staffeldt [**31**], holds the Additivity Theorem as the fundamental property of $K$-theory. Following this perspective, we deduce the remaining $K$-theoretic properties of $THH$ from the Additivity Theorem in the next three sections.

To state the Additivity Theorem, we use the following notation. For $\mathcal{C}$ a simplicially enriched Waldhausen category, let $\mathcal{E}(\mathcal{C}) = S_2\mathcal{C}$ be the simplicially enriched Waldhausen category with objects the cofiber sequences $x \to y \to z$ (in $\mathcal{C}_0$). We have enriched exact functors $\alpha, \beta, \gamma$ from $\mathcal{E}(\mathcal{C})$ to $\mathcal{C}$ defined by

$$\alpha(x \to y \to z) = x, \qquad \beta(x \to y \to z) = y, \qquad \gamma(x \to y \to z) = z.$$

Theorem 3.1.1 (Additivity Theorem). *For a simplicially enriched Waldhausen category $\mathcal{C}$, the enriched exact functors $\alpha$ and $\gamma$ induce a weak equivalence of cyclotomic spectra*

$$WTHH^{\Gamma}(\mathcal{E}(\mathcal{C})) \longrightarrow WTHH^{\Gamma}(\mathcal{C}) \times WTHH^{\Gamma}(\mathcal{C}) \simeq WTHH^{\Gamma}(\mathcal{C}) \vee WTHH^{\Gamma}(\mathcal{C}).$$

McCarthy's proof of the Additivity Theorem for $K$-theory [**22**] provides a very general argument for showing that the map $(\alpha, \gamma)\colon S_\bullet\mathcal{E}(\mathcal{C}) \to S_\bullet\mathcal{C} \times S_\bullet\mathcal{C}$ induces a homotopy equivalence in various contexts. The elaboration in [**23**, §3.4-3.5] to prove the Additivity Theorem for cyclic homology of $k$-linear categories carries over essentially word for word to prove the Additivity Theorem above, just replacing "$CN$" with "$THH$" and "$k$-linear" with "spectral". (The only property of $THH$ or $CN$ needed is that it takes simplicial homotopy equivalences of simplicial (spectrally or $k$-linearly) enriched categories to weak equivalences of spectra or simplicial sets.)

The following result is both a generalization and a corollary of the Additivity Theorem above. Recall that a sequence of natural transformations of exact functors

$f \to g \to h$ from $\mathcal{C}_0$ to $\mathcal{D}_0$ forms a *cofiber sequence of exact functors*, when (taken together) they define an exact functor $\mathcal{C}$ to $\mathcal{E}(\mathcal{D})$.

COROLLARY 3.1.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicially enriched Waldhausen categories, and let $f \to g \to h$ be a sequence of enriched exact functors $\mathcal{C} \to \mathcal{D}$ that forms a cofiber sequence of exact functors. Then the maps*

$$WTHH^{\Gamma}(g) \qquad \textit{and} \qquad WTHH^{\Gamma}(f) \vee WTHH^{\Gamma}(h).$$

*from $WTHH^{\Gamma}(\mathcal{C})$ to $WTHH^{\Gamma}(\mathcal{D})$ agree in the homotopy category of cyclotomic spectra.*

PROOF. The functor $\mathcal{D} \times \mathcal{D} \to \mathcal{E}(\mathcal{D})$ sending $(a, b)$ to $a \to a \vee b \to b$ is an enriched exact functor, and the composite map

$$WTHH^{\Gamma}(\mathcal{D}) \vee WTHH^{\Gamma}(\mathcal{D}) \longrightarrow WTHH^{\Gamma}(\mathcal{D} \times \mathcal{D}) \longrightarrow WTHH^{\Gamma}(\mathcal{E}(\mathcal{D}))$$

splits the zigzag of weak equivalences in the Additivity Theorem and is therefore a weak equivalence of cyclotomic spectra. It follows that $\beta$ and $\alpha \vee \gamma$ induce the same map $\mathcal{E}(\mathcal{D}) \to \mathcal{D}$ in the homotopy category of cyclotomic spectra. Precomposing with the map $\mathcal{C} \to \mathcal{E}(\mathcal{D})$ defined by $f \to g \to h$ proves the corollary. $\square$

This corollary provides the key tool for even more general additivity statements. For example, the map $S_n\mathcal{C} \to \mathcal{C} \times S_{n-1}\mathcal{C}$ defined by sending $X = (x_{i,j})$ to $(x_{0,1}, d_0 X)$ induces a weak equivalence on $WTHH^{\Gamma}$. To see this, consider the map $\mathcal{C} \times S_{n-1}\mathcal{C} \to S_n\mathcal{C}$ sending $(x, Y)$ to $Z = (z_{i,j})$ with

$$z_{i,j} = \begin{cases} x \vee y_{0,j-1} & i = 0 \\ y_{i-1,j-1} & i > 0. \end{cases}$$

The composite map on $\mathcal{C} \times S_{n-1}\mathcal{C}$ is the identity, and the composite map on $S_n\mathcal{C}$ is $f \vee h$ for exact functors $f$ and $h$ that fit in a cofiber sequence of exact functors $f \to g \to h$ with $g$ the identity. We will use this argument many times in what follows.

When $\mathcal{A}$ is an enhanced simplicially enriched Waldhausen category, so is $\mathcal{E}(\mathcal{A})$ and the functors $\alpha, \beta, \gamma$ are enhanced exact. We have precise analogues of the previous results (with the same proof). In fact, we have the following stronger version of the Additivity Theorem for the non-connective enrichment (cf. [**7**, 10.8]).

THEOREM 3.1.3. *Let $\mathcal{A}$ be an enhanced simplicially enriched Waldhausen category. The enhanced exact functors $\alpha$ and $\gamma$ induce a weak equivalence of cyclotomic spectra*

$$THH(\mathcal{E}^S(\mathcal{A})) \longrightarrow THH(\mathcal{A}^S) \times THH(\mathcal{A}^S) \simeq THH(\mathcal{A}^S) \vee THH(\mathcal{A}^S).$$

PROOF. By Theorems 1.3.11 and 1.3.12, it suffices to consider the case when $\mathcal{A}^S$ is pretriangulated. Then the functor $a \mapsto (a \to a \to *)$ embeds $\mathcal{A}^S$ as a triangulated subcategory of $\mathcal{E}^S(\mathcal{A})$ and the functor $a \mapsto (* \to a \to a)$ induces an equivalence of $\pi_0\mathcal{A}^S$ with the triangulated quotient $\pi_0\mathcal{E}^S(\mathcal{A})/\pi_0\mathcal{A}^S$. The statement now follows from Theorem 1.3.13. $\square$

As a consequence of the Additivity Theorems 3.1.1 and 3.1.3, we can now prove Lemma 2.5.11 and Theorem 2.5.14.

Proof of Lemma 2.5.11 and Theorem 2.5.14. We will treat Lemma 2.5.11 in detail; Theorem 2.5.14 follows from the same argument using $\mathcal{A}$ in place of $S_\bullet\mathcal{C}$ and Theorem 3.1.3 in place of Theorem 3.1.1. To prove Lemma 2.5.11, it suffices to show that the map $\Sigma WTHH^\Gamma(\mathcal{C}) \to WTHH^\Gamma(S_\bullet\mathcal{C})$ is a weak equivalence. This map is induced by a simplicial map

$$\bigvee_n WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(S_n\mathcal{C}),$$

which is a levelwise weak equivalence by the Additivity Theorem and the argument following Corollary 3.1.2. □

## 3.2. The Cofiber Theorem

This section is the first of three that apply the Additivity Theorem to prove standard $K$-theory theorems in $THH$ and $TC$. This section provides a general cofibration sequence for $THH$ (and $TC$) associated to a map of Waldhausen categories by identifying the cofiber term as a version of $THH$ (cf. [**35**, §1.5]). We call this theorem the "Cofiber Theorem".

We begin with the construction of the cofiber term. For $f\colon \mathcal{C} \to \mathcal{D}$ an enriched exact functor, we define a simplicially enriched Waldhausen category $S_n f$ as follows. An object consists of an object $Y = (y_{i,j})$ of $S_n\mathcal{C}$ together with an object $X = (x_{i,j})$ of $S_{n+1}\mathcal{D}$ such that $d_0 X = f(Y)$, that is, $x_{i+1,j+1} = f(y_{i,j})$, with the structure maps for this subdiagram in $X$ identical with $f(Y)$. For objects $(X,Y), (X',Y')$, the simplicial set of maps consists of the simplicial set of natural transformations. We make this a Waldhausen category by declaring a map $(X,Y) \to (X',Y')$ to be a cofibration (resp., weak equivalence) when the restrictions $X' \to X$ (in $S_{n+1}\mathcal{D}$) and $Y' \to Y$ (in $S_n\mathcal{C}$) are both cofibrations (resp., weak equivalences). This assembles into a simplicial object in the category of simplicially enriched Waldhausen categories using the usual face and degeneracy maps on $S_n\mathcal{C}$ and the last $n+1$ face and degeneracy maps on $S_{n+1}\mathcal{D}$.

Definition 3.2.1. For $f\colon \mathcal{C} \to \mathcal{D}$ an enriched exact functor, define

$$WTHH^\Gamma(f) = |WTHH^\Gamma(S_\bullet f)|.$$

We note that when $f\colon \mathcal{A} \to \mathcal{B}$ is an enhanced exact functor between enhanced simplicially enriched Waldhausen categories, then $S_n f$ is also an enhanced simplicially enriched Waldhausen category and $S_\bullet f$ is a simplicial object in enhanced simplicially enriched Waldhausen categories. We write $WTHH(f) = |WTHH(S_\bullet f)|$.

To put this construction in perspective, we have an alternative description of $S_\bullet f$ as a pullback. For any simplicial object $Z_\bullet$, we can form the "path" object $PZ_\bullet$ precomposing with the shift operation $[n] \mapsto [n+1]$ in the category of standard simplices (or finite ordered sets). In this notation, we have a pullback square

$$\begin{tikzcd} S_\bullet f \arrow[r, "X"] \arrow[d, "Y"'] & PS_\bullet\mathcal{D} \arrow[d, "d_0"] \\ S_\bullet\mathcal{C} \arrow[r, "f"'] & S_\bullet\mathcal{D} \end{tikzcd}$$

in the category of simplicial simplicially enriched categories. The usual extra degeneracy argument produces a simplicial null homotopy on $PS_\bullet\mathcal{C}$, and applying

$WTHH^\Gamma$ and (when appropriate) $WTHH$, we get commutative squares of cyclotomic spectra

$$\begin{tikzcd} WTHH^\Gamma(f) \arrow[r] \arrow[d] & {|WTHH^\Gamma(PS_\bullet\mathcal{D})|} \arrow[d] \\ {|WTHH^\Gamma(S_\bullet\mathcal{C})|} \arrow[r] & {|WTHH^\Gamma(S_\bullet\mathcal{D})|} \end{tikzcd} \qquad \begin{tikzcd} WTHH(f) \arrow[r] \arrow[d] & {|WTHH(PS_\bullet\mathcal{B})|} \arrow[d] \\ {|WTHH(S_\bullet\mathcal{A})|} \arrow[r] & {|WTHH(S_\bullet\mathcal{B})|} \end{tikzcd}$$

where the top right entry comes with a canonical null homotopy through cyclotomic maps. We therefore get a map of cyclotomic spectra from $WTHH^\Gamma(f)$ to the homotopy fiber of the map $|WTHH^\Gamma(S_\bullet\mathcal{C})| \to |WTHH^\Gamma(S_\bullet\mathcal{D})|$, which is equivalent to the homotopy cofiber of the map $WTHH^\Gamma\mathcal{C} \to WTHH^\Gamma\mathcal{D}$. Likewise in the enhanced exact context, we get a map of cyclotomic spectra from $WTHH(f)$ to the homotopy cofiber of the map $WTHH\mathcal{A} \to WTHH\mathcal{B}$. The Cofiber Theorem asserts that these maps are weak equivalences.

THEOREM 3.2.2 (Cofiber Theorem). *For $f\colon \mathcal{C} \to \mathcal{D}$ an enriched exact functor, we have a cofiber sequence of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(\mathcal{D}) \longrightarrow WTHH^\Gamma(f) \longrightarrow |WTHH^\Gamma(S_\bullet\mathcal{C})|.$$

*For $f\colon \mathcal{A} \to \mathcal{B}$ an enhanced exact functor, we have a cofiber sequence of cyclotomic spectra*

$$WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{B}) \longrightarrow WTHH(f) \longrightarrow |WTHH(S_\bullet\mathcal{A})|.$$

PROOF. (cf. [**35**, 1.5.5]) The argument for the connective and non-connective enrichments are identical; we treat the connective case in detail. Consider the map

$$WTHH^\Gamma(\mathcal{D}) \vee \bigvee_n WTHH^\Gamma(\mathcal{C}) \longrightarrow WTHH^\Gamma(\mathcal{D} \times S_n\mathcal{C}) \longrightarrow WTHH^\Gamma(S_n f)$$

induced by sending $b, a_1, \dots, a_n$ to $(b, Y)$ and then $(X, Y)$ with $Y = (y_{i.j})$ for

$$y_{i,j} = a_{i+1} \vee \cdots \vee a_j$$

and $X = (x_{i,j})$ for

$$x_{i,j} = \begin{cases} b \vee f(y_{0,j-1}) & i = 0 \\ f(y_{i-1,j-1}) & i > 0 \end{cases}$$

with the canonical maps induced by inclusions and quotients of summands. Applying the argument following Corollary 3.1.2, we see that this map is a weak equivalence. Letting $n$ vary, these assemble into a simplicial map where we regard the domain as the simplicial cyclotomic spectrum

$$WTHH^\Gamma(\mathcal{D}) \cup_{WTHH^\Gamma(\mathcal{C})} WTHH^\Gamma(\mathcal{C}) \wedge \Delta[1].$$

On geometric realization, this induces a map from the homotopy cofiber

$$C = WTHH^\Gamma(\mathcal{D}) \cup_{WTHH^\Gamma(\mathcal{C})} WTHH^\Gamma(\mathcal{C}) \wedge I$$

to $WTHH^\Gamma(f)$ that we see is a weak equivalence. The composite map

$$C \longrightarrow WTHH^\Gamma(f) \longrightarrow WTHH^\Gamma(S_\bullet\mathcal{C})$$

factors as the connecting map $C \to \Sigma WTHH^\Gamma(\mathcal{C})$ composed with the weak equivalence $\Sigma WTHH^\Gamma(\mathcal{C}) \to |WTHH^\Gamma(S_\bullet\mathcal{C})|$. $\square$

Using the alternate models $\widetilde{WTC}$ and $\widetilde{WTHH}$ of Definition 2.5.10, we get constructions $\widetilde{WTC}^{\Gamma}(f)$ and $\widetilde{WTHH}^{\Gamma}(f)$ that admit a cyclotomic trace from $K$-theory. Because on objects, the map constructed in the proof of Theorem 3.2.2 agrees with the corresponding map in cofiber sequence on $K$-theory, we get the following theorem as an immediate consequence.

THEOREM 3.2.3. *For $f\colon \mathcal{C} \to \mathcal{D}$ an enriched exact functor, the following diagram commutes.*

$$\begin{tikzcd}
K(\mathcal{C}_0) \arrow[r] \arrow[d, "\mathrm{trc}"'] & K(\mathcal{D}_0) \arrow[r] \arrow[d, "\mathrm{trc}"'] & K(f) \arrow[r] \arrow[d, "\mathrm{trc}"'] & K(S_\bullet \mathcal{C}_0) \arrow[d, "\mathrm{trc}"'] \\
\widetilde{WTC}^{\Gamma}(\mathcal{C}) \arrow[r] \arrow[d] & \widetilde{WTC}^{\Gamma}(\mathcal{D}) \arrow[r] \arrow[d] & \widetilde{WTC}^{\Gamma}(f) \arrow[r] \arrow[d] & \widetilde{WTC}^{\Gamma}(S_\bullet \mathcal{C}) \arrow[d] \\
\widetilde{WTHH}^{\Gamma}(\mathcal{C}) \arrow[r] & \widetilde{WTHH}^{\Gamma}(\mathcal{D}) \arrow[r] & \widetilde{WTHH}^{\Gamma}(f) \arrow[r] & \widetilde{WTHH}^{\Gamma}(S_\bullet \mathcal{C})
\end{tikzcd}$$

Returning to Theorem 3.2.2, we have the following corollary that allows us to study the cofibers of exact functors in "$THH$-theoretic" terms.

COROLLARY 3.2.4. *Let $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{C} \to \mathcal{D}$ be enriched exact functors. Then the commutative square of cyclotomic spectra on the left is homotopy (co)cartesian.*

$$\begin{tikzcd}
WTHH^{\Gamma}(\mathcal{B}) \arrow[r] \arrow[d] & WTHH^{\Gamma}(f) \arrow[d] \\
WTHH^{\Gamma}(\mathcal{C}) \arrow[r] & WTHH^{\Gamma}(g \circ f)
\end{tikzcd}
\qquad
\begin{tikzcd}
WTHH(\mathcal{B}) \arrow[r] \arrow[d] & WTHH(f) \arrow[d] \\
WTHH(\mathcal{C}) \arrow[r] & WTHH(g \circ f)
\end{tikzcd}$$

*If $f$ and $g$ are enhanced exact then the commutative square of cyclotomic spectra on the right is homotopy cartesian.*

In the special case when $\mathcal{C}$ is a simplicially enriched Waldhausen subcategory of $\mathcal{D}$ and $f$ is the inclusion, $S_\bullet f$ admits an equivalent but smaller variant where we omit the choices of subquotients.

DEFINITION 3.2.5. We say that $\mathcal{C} \subset \mathcal{D}$ is a *simplicially enriched Waldhausen subcategory* when $\mathcal{C} \subset \mathcal{D}$ is full as a simplicially enriched category and $\mathcal{C}_0$ is a Waldhausen subcategory of $\mathcal{D}_0$. In this case we define $F_\bullet(\mathcal{D}, \mathcal{C})$ to be the simplicially enriched Waldhausen subcategory of the nerve of the cofibrations in $\mathcal{D}$ whose quotients lie in $\mathcal{C}$.

Concretely, $F_n(\mathcal{D}, \mathcal{C})$ has as objects the composable sequences of $n$ cofibrations

$$x_0 \rightarrowtail x_1 \rightarrowtail \cdots \rightarrowtail x_n$$

such that $x_{i+1}/x_i$ is an object of $\mathcal{C}$ for all $i$, with maps the simplicial sets of natural transformations. We have a forgetful functor from $S_\bullet(\mathcal{C} \to \mathcal{D})$ to $F_\bullet(\mathcal{D}, \mathcal{C})$ that throws away the subquotients, i.e., sending $(X, Y)$ in $S_{n+1}\mathcal{D} \times S_n \mathcal{C}$ to

$$x_{0,1} \rightarrowtail x_{0,2} \rightarrowtail \cdots \rightarrowtail x_{0,n+1}$$

in $F_n(\mathcal{D}, \mathcal{C})$, where $X = (x_{i,j})$. At each simplicial level this map is an equivalence of simplicial Waldhausen categories, and in particular induces a DK-equivalence

$$S_m S_n(\mathcal{C} \to \mathcal{D}) \longrightarrow S_m F_n(\mathcal{D}, \mathcal{C}).$$

We therefore obtain the following observation, useful in combination with Theorem 3.2.2.

PROPOSITION 3.2.6. *For $\mathcal{C} \subset \mathcal{D}$ a simplicially enriched Waldhausen subcategory, the forgetful functor from $S_\bullet(\mathcal{C} \to \mathcal{D})$ to $F_\bullet(\mathcal{D}, \mathcal{C})$ induces a weak equivalence of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{C} \to \mathcal{D}) \longrightarrow |WTHH^\Gamma(F_\bullet(\mathcal{D}, \mathcal{C}))|.$$

We have the notion of a *closed* simplicially enriched Waldhausen subcategory, which is a simplicially enriched Waldhausen subcategory $\mathcal{A} \subset \mathcal{B}$ where $\mathcal{A}_0$ is a closed Waldhausen subcategory of $\mathcal{B}_0$ (i.e., every object of $\mathcal{B}$ weakly equivalent to an object of $\mathcal{A}$ is in $\mathcal{A}$). When $\mathcal{B}$ is an enhanced simplicially enriched Waldhausen category and $\mathcal{A} \subset \mathcal{B}$ is a closed simplicially enriched Waldhausen subcategory, then $\mathcal{A}$ is also enhanced simplicially enriched. The discussion above then generalizes to show that

$$S_m S_n(\mathcal{C} \to \mathcal{D}) \longrightarrow S_m F_n(\mathcal{D}, \mathcal{C}).$$

induces an equivalence (and in particular DK-equivalence) on non-connective enrichments. It follows that

$$WTHH(\mathcal{A} \to \mathcal{B}) \longrightarrow |WTHH(F_\bullet(\mathcal{B}, \mathcal{A}))|$$

is also a weak equivalence of cyclotomic spectra.

## 3.3. The Localization Theorem

The Localization Theorem, called by Waldhausen the "Fibration Theorem", provides the most important instance of the Cofiber Theorem. Roughly speaking, this theorem states that algebraic $K$-theory takes quotient sequences of triangulated categories to cofiber sequences of spectra. In this section, we prove versions of this theorem for $THH$ and $TC$. In the case of the non-connective enrichment, we obtain a localization sequence equivalent to the one in [**7**]; in the case of the connective enrichment, we obtain a localization sequence generalizing the one in [**16**] (q.v. Chapter 4).

For the setup for the Localization Theorem, we take an enhanced simplicially enriched Waldhausen category $\mathcal{A}$ together with an additional subcategory of weak equivalences $v\mathcal{A}_0$ that contains its usual weak equivalences $w\mathcal{A}_0$. We assume that $v\mathcal{A}_0$ satisfies the *two-out-of-three property*, meaning that for composable maps $f$ and $g$, if any two of $f$, $g$, and $g \circ f$ are in $v\mathcal{A}_0$, then so is the third. We also assume that $v\mathcal{A}_0$ satisfies the *Extension Axiom* [**35**, §1.2], meaning that given a map of cofibration sequences

$$\begin{tikzcd} x \arrow[r, rightarrowtail] \arrow[d, "v"'] & y \arrow[r] \arrow[d] & y/x \arrow[d, "v"] \\ x' \arrow[r, rightarrowtail] & y' \arrow[r] & y'/x' \end{tikzcd}$$

with the outer maps $x \to x'$ and $x/y \to x'/y'$ in $v\mathcal{A}_0$, then the inner map $y \to y'$ is in $v\mathcal{A}_0$. Finally, recalling that as an enhanced simplicially enriched Waldhausen category, $\mathcal{A}$ admits tensors with contractible simplicial sets, we say that $v\mathcal{A}_0$ is *compatible with cylinders* when for any map $x \to x'$ in $v\mathcal{A}_0$, the map

$$x \longrightarrow x' \cup_x (x \otimes \Delta[1])$$

is a cofibration in $\mathcal{A}_0$, i.e., its quotient is in $\mathcal{A}$. The category of *$v$-acyclics* $\mathcal{A}_0^v$ consists of the full subcategory of objects $v$-equivalent to the trivial object $*$. Under these hypotheses, $\mathcal{A}_0^v$ forms a closed Waldhausen subcategory of $\mathcal{A}$. Moreover, $\mathcal{A}_0^v$ is *closed under extensions and cofibers* in $\mathcal{A}_0$, meaning that for a cofibration sequence in $\mathcal{A}_0$

$$x \rightarrowtail y \longrightarrow y/x,$$

if $x$ and either of $y$ or $y/x$ is in $\mathcal{A}_0^v$, then so is the other. Letting $\mathcal{A}^v$ be the full simplicially enriched subcategory of $\mathcal{A}$ consisting of the objects in $\mathcal{A}_0^v$, then $\mathcal{A}^v$ forms an enhanced simplicially enriched Waldhausen category with the inclusion functor $\mathcal{A}^v \to \mathcal{A}$ enhanced exact. We can now state the Localization Theorem.

THEOREM 3.3.1 (Localization Theorem). *With hypotheses and notation as in the previous paragraph, the following commutative squares of cyclotomic spectra are homotopy (co)cartesian.*

$$\begin{tikzcd} WTHH^{\Gamma}(\mathcal{A}^v) \arrow[r] \arrow[d] & WTHH^{\Gamma}(\mathcal{A}^v|v) \arrow[d] \\ WTHH^{\Gamma}(\mathcal{A}) \arrow[r] & WTHH^{\Gamma}(\mathcal{A}|v) \end{tikzcd} \qquad \begin{tikzcd} WTHH(\mathcal{A}^v) \arrow[r] \arrow[d] & WTHH(\mathcal{A}^v|v) \arrow[d] \\ WTHH(\mathcal{A}) \arrow[r] & WTHH(\mathcal{A}|v) \end{tikzcd}$$

*Moreover, in each square, the upper right entry is null homotopic through cyclotomic maps. Thus, we have cofiber sequences of cyclotomic spectra,*

$$WTHH^{\Gamma}(\mathcal{A}^v) \longrightarrow WTHH^{\Gamma}(\mathcal{A}) \longrightarrow WTHH^{\Gamma}(\mathcal{A}|v) \longrightarrow \Sigma WTHH^{\Gamma}(\mathcal{A}^v)$$
$$WTHH(\mathcal{A}^v) \longrightarrow WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{A}|v) \longrightarrow \Sigma WTHH(\mathcal{A}^v).$$

Although formally similar in statement and proof, the two localization sequences above are very different in practice. In the case when $\mathcal{A}$ is pretriangulated (which by Corollary 2.2.13 just means in this context that every object is weakly equivalent to a suspension), the Localization Theorem of [**7**] (Theorem 1.3.13 above) identifies the relative term $WTHH(\mathcal{A}|v)$ in the second sequence above as the $THH$ of the triangulated quotient $\pi_0\mathcal{A}^S/\pi_0(\mathcal{A}^v)^S$ (for any spectrally enriched model of this quotient).

In the special case when $\mathcal{A}$ is the category of finite cell EKMM $R$-modules for the $S$-algebra $R = HA$ for a discrete valuation ring $A$ or $R = ku$ is connective $K$-theory, we take the $v$-equivalences $v\mathcal{A}_0$ to be the $R[\beta^{-1}]$-equivalences, the maps that induce isomorphisms on homotopy groups after inverting $\beta$, where $\beta$ is a uniformizer for $A$ (when $R = HA$) or is the Bott-element (when $R = ku$). Then Theorem 2.5.15 (proved in Section 3.4) combined with Theorem 2.5.14 identify both $WTHH^{\Gamma}(\mathcal{A})$ and $WTHH(\mathcal{A})$ as $THH(R)$. In the non-connective case, we then have that $WTHH(\mathcal{A}|v)$ is equivalent to $THH(R[\beta^{-1}])$. Calculations show $WTHH(\mathcal{A}^v)$ cannot be equivalent to $THH(R/\beta)$. On the other hand, we will prove a dévissage theorem in Part 4 that identifies $WTHH^{\Gamma}(\mathcal{A}^v)$ as $THH(R/\beta)$ and calculations show that $WTHH^{\Gamma}(\mathcal{A}|v)$ cannot be equivalent to $THH(R[\beta^{-1}])$.

Returning to Theorem 3.3.1, it follows that the analogous squares in the "tilde" models $\widetilde{WTHH}^{\Gamma}$ and $\widetilde{WTC}^{\Gamma}$ are homotopy (co)cartesian as well, and we get cofiber sequences on $\widetilde{WTHH}^{\Gamma}$ and $\widetilde{WTC}^{\Gamma}$. By naturality, the maps in the squares and in the cofiber sequences commute with the cyclotomic trace. For convenient reference, we state this explicitly in the following theorem.

THEOREM 3.3.2. *Under the hypotheses of Theorem 3.3.1, the following diagram of cofiber sequences commutes.*

$$\begin{tikzcd}
K(\mathcal{A}_0^v) \arrow[r] \arrow[d, "\mathrm{trc}"'] & K(\mathcal{A}_0) \arrow[r] \arrow[d, "\mathrm{trc}"'] & K(\mathcal{A}_0|v) \arrow[r] \arrow[d, "\mathrm{trc}"'] & \Sigma K(\mathcal{A}_0^v) \arrow[d, "\mathrm{trc}"'] \\
\widetilde{WTC}^{\Gamma}(\mathcal{A}^v) \arrow[r] \arrow[d] & \widetilde{WTC}^{\Gamma}(\mathcal{A}) \arrow[r] \arrow[d] & \widetilde{WTC}^{\Gamma}(\mathcal{A}|v) \arrow[r] \arrow[d] & \Sigma\widetilde{WTC}^{\Gamma}(\mathcal{A}^v) \arrow[d] \\
\widetilde{WTHH}^{\Gamma}(\mathcal{A}^v) \arrow[r] & \widetilde{WTHH}^{\Gamma}(\mathcal{A}) \arrow[r] & \widetilde{WTHH}^{\Gamma}(\mathcal{A}|v) \arrow[r] & \Sigma\widetilde{WTHH}^{\Gamma}(\mathcal{A}^v)
\end{tikzcd}$$

We begin the proof of Theorem 3.3.1 by noting that the category of $v$-acyclics completely characterizes the $v$-equivalences $v\mathcal{A}_0$.

PROPOSITION 3.3.3. *Under the hypotheses of Theorem 3.3.1, a map $f\colon x \to y$ is in $v\mathcal{A}_0$ if and only if the homotopy cofiber*

$$Cf = y \cup_x (x \otimes \Delta[1]) \cup_x *$$

*is in $\mathcal{A}_0^v$.*

PROOF. Let $Mf = y \cup_x (x \otimes \Delta[1])$ so that $Cf = Mf/x$. The map $Mf \to y$ is a weak equivalence (and so in particular a $v$-equivalence) and the composite map $x \to Mf \to y$ is $f$ and so the inclusion of $x$ in $Mf$ is in $v\mathcal{A}_0$ if and only if $f$ is. Consider the commutative diagram of cofiber sequences

$$\begin{tikzcd}
x \arrow[r, tail, "="] \arrow[d, "="'] & x \arrow[r] \arrow[d] & * \arrow[d] \\
x \arrow[r, tail] & Mf \arrow[r] & Cf.
\end{tikzcd}$$

By the Gluing Axiom, $Cf$ is in $\mathcal{A}_0^v$ when $x \to Mf$ is in $v\mathcal{A}_0$. By the Extension Axiom $x \to Mf$ is in $v\mathcal{A}_0$ when $Cf$ is in $\mathcal{A}_0^v$. □

Let $\bar{v}\mathcal{A}_0 = v\mathcal{A}_0 \cap \mathrm{co}\,\mathcal{A}_0$ denote the subcategory of $\mathcal{A}_0$ consisting of the maps that are both cofibrations and $v$-equivalences. The previous proposition implies that $\bar{v}\mathcal{A}_0$ consists of those cofibrations whose quotients are $v$-acyclic. It follows that $F_\bullet(\mathcal{A}, \mathcal{A}^v) = \bar{v}_\bullet\mathcal{A}$, and applying Corollary 3.2.4 and Proposition 3.2.6, we get homotopy (co)cartesian squares

$$\begin{tikzcd}
WTHH^{\Gamma}(\mathcal{A}^v) \arrow[r] \arrow[d] & {|WTHH^{\Gamma}(\bar{v}_\bullet\mathcal{A}^v)|} \arrow[d] \\
WTHH^{\Gamma}(\mathcal{A}) \arrow[r] & {|WTHH^{\Gamma}(\bar{v}_\bullet\mathcal{A})|}
\end{tikzcd}
\qquad
\begin{tikzcd}
WTHH(\mathcal{A}^v) \arrow[r] \arrow[d] & {|WTHH(\bar{v}_\bullet\mathcal{A}^v)|} \arrow[d] \\
WTHH(\mathcal{A}) \arrow[r] & {|WTHH(\bar{v}_\bullet\mathcal{A})|}.
\end{tikzcd}$$

We now have what we need to prove Theorem 3.3.1.

PROOF OF THEOREM 3.3.1. To obtain the homotopy (co)cartesian squares, we just need to see that the maps

$$WTHH^{\Gamma}(\bar{v}_\bullet\mathcal{A}) \longrightarrow WTHH^{\Gamma}(v_\bullet^M\mathcal{A}) \quad \text{and} \quad WTHH(\bar{v}_\bullet\mathcal{A}) \longrightarrow WTHH(v_\bullet^M\mathcal{A})$$

are weak equivalences. The inclusion of $|\bar{v}_p S_q\mathcal{A}|$ in $|v_p^M S_q\mathcal{A}|$ is a DK-embedding and an easy mapping cylinder argument shows that it is a DK-equivalence.

It follows that $WTHH^{\Gamma}(\mathcal{A}^v|v)$ and $WTHH^{\Gamma}(\mathcal{A}^v|v)$ are weakly equivalent as cyclotomic spectra to the trivial spectrum, and to produce a null homotopy through cyclotomic maps is not much more work. The simplicial object $v_\bullet^M\mathcal{A}^v$ has an extra degeneracy which on objects inserts the trivial map at the start of the chain of maps. On maps, we use the unique (constant trivial) homotopy on any subsimplex that has the new trivial object as one of its vertices. □

## 3.4. The Sphere Theorem

In this section, we state versions of Waldhausen's "Sphere Theorem" for the $THH$ of Waldhausen categories, which we prove in the next section. These theorems allow us to deduce the important consistency result that all the different models for the $THH$ of the finite-cell modules over an EKMM $S$-algebra or a simplicial ring agree (Theorem 2.5.15 above). Before stating a precise theorem, we need two definitions,

DEFINITION 3.4.1. Let $\mathcal{C}$ be a simplicially tensored Waldhausen category. We say that $\mathcal{C}$ is *stable* when:

(i) Every object of $\mathcal{C}$ is weakly equivalent to a suspension, and
(ii) For all objects $x$ and $y$ in $\mathcal{C}$, the suspension map $\mathcal{C}(x,y) \to \mathcal{C}(\Sigma x, \Sigma y)$ is a weak equivalence.

We say that $\mathcal{C}$ is *almost stable* when it satisfies just condition (ii).

As observed in Corollary 2.2.13, the first condition implies that the non-connective spectral category $\mathcal{C}^S$ is pretriangulated, and its homotopy category $\pi_0\mathcal{C}^S$ is triangulated. The second condition implies that the homotopy category $\pi_0\mathcal{C}^S$ coincides with the homotopy category $\pi_0\mathcal{C}$ and also that the connective spectral enrichment $\mathcal{C}^\Gamma(x,y)$ is the connective cover of the non-connective spectral enrichment $\mathcal{C}^S(x,y)$ (Proposition 2.2.11). Combined with the fact that the mapping simplicial sets $\mathcal{C}(x,y)$ are Kan complexes (and that weak equivalences in $\mathcal{C}$ are homotopy equivalences in the obvious sense), this puts all the basic tools and techniques of homotopy theory and stable homotopy theory at our disposal.

In the stable case the hypotheses we need for the Sphere Theorem greatly simplify and so we will explore that case first. In addition to the stability assumptions above, we need to assume that $\mathcal{C}$ is generated by connective objects in the following sense.

DEFINITION 3.4.2. Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category. A *connective class* $Q$ in $\mathcal{C}$ is a set of objects of $\mathcal{C}$ such that for any $a, b$ in $Q$, $\mathcal{C}^S(a,b)$ is connective. If $\mathcal{C}$ is stable, then we say that $Q$ is *generating* if the smallest triangulated subcategory of the triangulated category $\pi_0\mathcal{C}^S$ that contains $Q$ is all of $\pi_0\mathcal{C}^S$.

See Definition 3.4.9 for the definition of generating when $\mathcal{C}$ is almost stable. In this terminology, we prove the following theorem, the $THH$ analogue of Waldhausen's Sphere Theorem for the stable case.

THEOREM 3.4.3 (Sphere Theorem, Stable Version). *Let $\mathcal{C}$ be a stable simplicially tensored Waldhausen category and assume that $\mathcal{C}$ has a generating connective class $Q$. Then the canonical cyclotomic maps are weak equivalences*

$$WTHH^\Gamma(\mathcal{C}) \xrightarrow{\sim} WTHH(\mathcal{C}) \xleftarrow{\sim} THH(\mathcal{C}^S) \xleftarrow{\sim} THH(Q^S).$$

*Here $Q^S$ denotes the full spectral subcategory of $\mathcal{C}^S$ on the objects of $Q$.*

We state the following corollary for ease of reference and citation; it is one case of Theorem 2.5.15.

COROLLARY 3.4.4. *Let $R$ be a connective EKMM $S$-algebra and $\mathcal{C}_R$ the category of finite cell $R$-modules. Then the canonical cyclotomic maps*

$$WTHH^{\Gamma}(\mathcal{C}_R) \xrightarrow{\sim} WTHH(\mathcal{C}_R) \xleftarrow{\sim} THH(\mathcal{C}_R^S) \xleftarrow{\sim} THH(R)$$

*are weak equivalences. Here $THH(R)$ denotes the usual Bökstedt model of $THH$ of a particular symmetric ring spectrum (or FSP) equivalent to $R$.*

The corollary follows by taking the connective class $Q$ to be the singleton set containing the one object $S_R$, the "cell zero sphere $R$-module" [**13**, III.2]. Then $THH(Q^S)$ coincides with $THH$ of the symmetric ring spectrum

$$F = Q^S(S_R, S_R) = \mathcal{C}_R^S(S_R, S_R).$$

Concretely, this has $n$-th space

$$F(n) = \mathcal{C}_R(S_R, S_R \wedge S^n),$$

and multiplication induced by composition. We can identify this as the symmetric ring spectrum (or "FSP defined on spheres") obtained from the FSP $\mathbf{F}(-) = \mathcal{C}_R(S_R, S_R \wedge -)$ by restricting to spheres $F(n) = \mathbf{F}(S^n)$.

Another symmetric ring spectrum derives from the general theory of [**26**]; writing $\mathcal{M}_S$ for the category of $S$-modules, this has spaces $\Phi(n) = \mathcal{M}_S((S_S^{-1})^{(n)}, R)$ and multiplication induced by smash product together with the multiplication on $R$. Experts know how to compare these symmetric ring spectra and therefore their $THH$, $TR$, and $TC$ spectra: Briefly, noting that $S_R = R \wedge S_S$, we construct a third symmetric ring spectrum $\Phi'$ that lies between them. $\Phi'$ has spaces

$$\begin{aligned}\Phi'(n) &= \mathcal{C}_R(S_R \wedge_S (S_S^{-1} \wedge S^1)^{(n)}, S_R \wedge S^n) \\ &\cong \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_R(S_R, S_R \wedge S^n))\end{aligned}$$

and multiplication induced both by smash product (on the $(S_S^{-1} \wedge S^1)^{(n)}$ factors) and composition (on the $F_R(S_R, S_R \wedge S^n)$ factors). We have a weak equivalence of symmetric ring spectra from $F$ to $\Phi'$ given by

$$F(n) = \mathcal{C}_R(S_R, S_R \wedge S^n) \longrightarrow \mathcal{C}_R(S_R \wedge_S (S_S^{-1} \wedge S^1)^{(n)}, S_R \wedge S^n) = \Phi'(n)$$

induced by the collapse map $S_S^{-1} \wedge S^1 \to S$; the induced map $F(n) \to \Phi'(n)$ is a weak equivalence of simplicial sets for all $n$. We have a weak equivalence of symmetric ring spectra from $\Phi$ to $\Phi'$ given by

$$\begin{aligned}\Phi(n) = \mathcal{M}_S((S_S^{-1})^{(n)}, R) &\longrightarrow \mathcal{M}_S((S_S^{-1})^{(n)}, F_R(S_R, S_R)) \\ &\longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_R(S_R, S_R \wedge S^n)) \cong \Phi'(n)\end{aligned}$$

induced by the unit map $R \to F_R(S_R, S_R)$ (which arises from the extra $R$ action on $S_R = R \wedge S_S$); again, this is a weak equivalence of simplicial sets for all $n$. For convenience, we state these remarks as a proposition.

PROPOSITION 3.4.5. *The symmetric ring spectrum in Corollary 3.4.4 is weakly equivalent to the symmetric ring spectrum obtained from the EKMM $S$-algebra $R$ by* [**26**].

For the other half of Theorem 2.5.15, we need to treat the almost stable case. This requires introducing the following subcategories of $\mathcal{C}$ associated to a connective class $Q$.

NOTATION 3.4.6. Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category and let $Q$ be a connective class. Write $\mathcal{Q}$ for the smallest closed Waldhausen category of $\mathcal{C}$ containing $Q$. For $n \geq 0$, let $\Sigma^n \mathcal{Q}$ denote the full subcategory of $\mathcal{C}$ containing all objects weakly equivalent to $\Sigma^n x$ for $x$ in $\mathcal{Q}$. Let $\Sigma^{-n}\mathcal{Q}$ denote the full subcategory of $\mathcal{C}$ containing all $x$ such that $\Sigma^n x$ is in $\mathcal{Q}$.

We note that the subcategories $\Sigma^n \mathcal{Q}$ are themselves connective classes and closed Waldhausen subcategories.

PROPOSITION 3.4.7. *Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category and let $Q$ be a connective class. Then $\Sigma^n \mathcal{Q}$ is a connective class and closed Waldhausen subcategory for all $n \in \mathbb{Z}$.*

PROOF. We begin by showing that $\mathcal{Q}$ is a connective class; stability hypothesis (ii) then shows that $\Sigma^n \mathcal{Q}$ is a connective class for all $n$. Let $Q_0$ be the collection of objects of $\mathcal{C}$ weakly equivalent to finite coproducts of objects in $Q$, and inductively let $Q_n$ be the collection of objects of $\mathcal{C}$ that are weakly equivalent to finite coproducts of homotopy pushouts $y \cup_x (x \otimes \Delta[1]) \cup_x z$ where $x, y, z \in Q_{n-1}$ and $y \cup_x (x \otimes \Delta[1]) \cup_x * \in Q_{n-1}$. If we regard $\bigcup Q_n$ as the full subcategory of $\mathcal{C}$ of objects in $Q_n$ for some $n$, it is then clear that $\mathcal{Q} = \bigcup Q_n$ is the smallest closed Waldhausen subcategory of $\mathcal{C}$ containing $Q$. To show that $\mathcal{Q}$ is a connective class, it suffices to show that for $x, y$ in $Q_n$, $\mathcal{C}^S(x, y)$ is connective, which we do by induction. We know that $x$ is weakly equivalent to a finite coproduct of homotopy pushouts of objects in $Q_{n-1}$ along maps whose homotopy cofiber is also in $Q_{n-1}$. Looking at the long exact sequence of homotopy groups from the fibration sequence in Proposition 2.2.12, we then see that $\mathcal{C}(x, z)$ is connective for all $z$ in $Q_{n-1}$. Using the same fact about $y$ and the long exact sequence of homotopy groups from the cofibration sequence in Proposition 2.2.12, we see that $\mathcal{C}(x, y)$ is connective.

By definition $\Sigma^0 \mathcal{Q} = \mathcal{Q}$ is a closed Waldhausen subcategory and it follows that $\Sigma^n \mathcal{Q}$ is a closed Waldhausen subcategory for $n < 0$ since suspension preserves homotopy pushouts. Let $n > 0$ and suppose $f \colon x \to y$ is a cofibration in $\mathcal{C}$ such that $x$, $y$, and $y/x \simeq Cf$ are all in $\Sigma^n \mathcal{Q}$. Then we can find $x'$ and $y'$ in $\mathcal{Q}$ and weak equivalences $\Sigma^n x' \to x$ and $\Sigma^n y' \to y$. By stability hypothesis (ii) and the fact that the mapping spaces in $\mathcal{C}$ are Kan complexes, we can find a map $f' \colon x' \to y'$ such that the diagram

$$\begin{tikzcd} \Sigma^n x' \arrow[r, "\Sigma^n f'"] \arrow[d, "\sim"'] & \Sigma^n y' \arrow[d, "\sim"'] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

commutes up to homotopy. Choosing a homotopy, we get a weak equivalence $C\Sigma^n f' \to Cf$. Then $y/x$ is weakly equivalent to $\Sigma^n Cf$ and it follows (again applying stability hypothesis (ii)) that $Cf$ is in $\mathcal{Q}$. For any map $x \to z$ with $z$ in $\Sigma^n \mathcal{Q}$, we can choose a compatible map $x' \to z'$ (for some $z'$ with $\Sigma^n z' \simeq z$) such that the pushout $w = z \cup_x y$ is weakly equivalent to $\Sigma^n$ of the homotopy pushout $w' = z' \cup_{x'} (x' \otimes \Delta[1]) \cup_{x'} y'$. Since $\mathcal{Q}$ is a closed Waldhausen subcategory of $\mathcal{C}$, $w'$ is in $\mathcal{Q}$, and it follows that $w$ is in $\Sigma^n \mathcal{Q}$. This shows that $\Sigma^n \mathcal{Q}$ is a closed Waldhausen subcategory of $\mathcal{C}$. □

We use the subcategories $\Sigma^n\mathcal{Q}$ to define what it means for a connective class to be generating in the almost stable case. For this, we need the following technical definitions.

DEFINITION 3.4.8. Given a class $A$ of objects of a Waldhausen category $\mathcal{C}$, we say that an object $x$ of $\mathcal{C}$ is *finitely cellularly built from* $A$ if we can find a sequence of objects $x_0, x_1, \ldots, x_n$ of $\mathcal{C}$ that fit into pushout squares

$$\begin{tikzcd} a_j \arrow[r, rightarrowtail] \arrow[d] & b_j \arrow[d] \\ x_j \arrow[r, rightarrowtail] & x_{j+1} \end{tikzcd}$$

where $x_n$ is weakly equivalent to $x$, $x_0 = *$, and for each $j$, $a_j$ is in $A$, $b_j$ is contractible (weakly equivalent to $*$), and $a_j \to b_j$ is a cofibration.

The concept of "finitely cellularly built from" above differs from other notions of "built from" in other contexts. Note in particular that an object of $A$ is not necessarily finitely cellularly built from $A$. However, suspensions of objects of $A$ are finitely cellularly built from $A$, for example.

DEFINITION 3.4.9. Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category and $Q$ a connective class. We say that $Q$ is *generating* if every object of $\mathcal{C}$ is finitely cellularly built from $\bigcup \Sigma^n\mathcal{Q}$.

The following proposition clarifies the relationship between the notions of generating given in Definitions 3.4.2 and 3.4.9.

PROPOSITION 3.4.10. *Let $\mathcal{C}$ be a stable simplicially tensored Waldhausen category and $Q$ a connective class. Then $Q$ is generating in the sense of Definition 3.4.2 if and only if it is generating in the sense of Definition 3.4.9.*

PROOF. Since the triangulated subcategory generated by $Q$ contains $\bigcup \Sigma^n\mathcal{Q}$ (cf. the proof of Proposition 3.4.7), one direction is clear. We must show that if $Q$ is generating in the sense of Definition 3.4.2, then it is generating in the sense of Definition 3.4.9. Let $C_0 = \bigcup \Sigma^n\mathcal{Q}$, and inductively let $C_n$ be the collection of objects of $\mathcal{C}$ that are weakly equivalent to the homotopy cofiber of a map between objects of $C_{n-1}$. We note that the $C_n$ are closed under suspension and desuspension and that the objects of $C_0$ are finitely cellularly built from $C_0$ (since each is equivalent to the suspension of an object of $C_0$). Generating in the sense of Definition 3.4.2 implies that $\mathcal{C} = \bigcup C_n$, so it suffices to show by induction that all objects of $C_n$ are finitely cellularly built from $C_0$. Given $f\colon x \to y$ with $x, y$ in $C_{n-1}$, we need to show that $z = Cf = y \cup_x (x \otimes \Delta[1]) \cup_x *$ is finitely cellularly built from $C_0$. Replacing $z$ with a weakly equivalent object, we can assume without loss of generality that $x$ and $y$ are isomorphic rather than just weakly equivalent to an iterated pushout. Then we build $z$ by first building $y$ and then gluing $Cb_j = (b_j \otimes \Delta[1]) \cup_{b_j} *$ along $a'_j = b_j \cup_{a_j} (a_j \otimes \Delta[1]) \cup_{a_j} *$ where $a_j \to b_j$ build $x$. Since $Cb_j$ is contractible and $a'_j$ is weakly equivalent to $\Sigma a_j$, this shows that $z$ is finitely cellularly built from $C_0$. $\square$

The following theorem now generalizes Theorem 3.4.3 to the almost stable case.

THEOREM 3.4.11 (Sphere Theorem). *Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category and assume that $\mathcal{C}$ has a generating connective class*

*$Q$. Then the canonical cyclotomic maps are weak equivalences*

$$WTHH^{\Gamma}(\mathcal{C}) \xrightarrow{\sim} WTHH(\mathcal{C}) \xleftarrow{\sim} THH(\mathcal{C}^S) \xleftarrow{\sim} THH(Q^S).$$

*where $Q^S$ denotes the full spectral subcategory of $\mathcal{C}^S$ on the objects of $Q$.*

We now have the other half of Theorem 2.5.15 as a corollary.

Corollary 3.4.12. *Let $A$ be a simplicial ring, let $\mathcal{C}_A$ be the category of finite cell $A$-modules and let $\mathcal{P}_A$ be the category of finite cell $A$-modules built out of finitely generated projective $A$-modules. Then the canonical cyclotomic maps*

$$\begin{tikzcd} WTHH^{\Gamma}(\mathcal{C}_A) \arrow[r, "\sim"] \arrow[d, "\sim"'] & WTHH(\mathcal{C}_A) \arrow[d, "\sim"'] & THH(\mathcal{C}_A^S) \arrow[l, "\sim"'] \arrow[d, "\sim"'] & THH(A) \arrow[l, "\sim"'] \\ WTHH^{\Gamma}(\mathcal{P}_A) \arrow[r, "\sim"] & WTHH(\mathcal{P}_A) & THH(\mathcal{P}_A^S) \arrow[l, "\sim"'] & \end{tikzcd}$$

*are weak equivalences.*

The vertical arrows are weak equivalences by Theorem 1.3.6 since every object of $\mathcal{P}_A$ or $S_n\mathcal{P}_A$ is a direct summand of an object of $\mathcal{C}_A$ or $S_n\mathcal{C}_A$. We get the top row from Theorem 3.4.11 taking the connective class $Q$ to be the singleton set containing the object $A$, which is clearly generating. The symmetric spectrum $Q^S(A,A) = \mathcal{C}^S(A,A)$ is just the usual symmetric ring spectrum constructed from $A$.

We have one last version of the Sphere Theorem, which is closer in spirit to Waldhausen's Sphere Theorem for $K$-theory. It also has the technical advantage of being stated purely in terms of the connective enrichments.

Theorem 3.4.13 (Sphere Theorem, Alternate Version). *Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category, let $Q$ be a generating connective class, and let $\mathcal{Q}$ be the smallest closed Waldhausen subcategory of $\mathcal{C}$ containing $Q$. The inclusion of $\mathcal{Q}$ into $\mathcal{C}$ induces a weak equivalence $WTHH^{\Gamma}(\mathcal{Q}) \to WTHH^{\Gamma}(\mathcal{C})$.*

The previous theorem is equivalent to Theorem 3.4.11, but to see this, we need more information about the categories $S_\bullet\mathcal{Q}$ implicit in the statement. The following proposition has everything we need for the comparison, plus what we need for the proofs in the next section.

Proposition 3.4.14. *Let $\mathcal{C}$ be an almost stable simplicially tensored Waldhausen category and let $Q$ be a connective class. Then $S_n\mathcal{C}$ is an almost stable simplicially tensored Waldhausen category, and $S_n\mathcal{Q}$ is a closed Waldhausen subcategory and a connective class; moreover, $S_n\Sigma^m\mathcal{Q} = \Sigma^m S_n\mathcal{Q}$. If $Q$ is generating, then so is $S_n\mathcal{Q}$.*

Proof. We saw in Proposition 2.3.3 that $S_n\mathcal{C}$ is simplicially tensored; the fact that the tensor on $S_n\mathcal{C}$ is objectwise on the diagram and the formula (2.3.2) for the mapping spaces of $S_n\mathcal{C}$ prove that $S_n\mathcal{C}$ is almost stable. Since $\mathcal{Q}$ is a closed Waldhausen subcategory of $\mathcal{C}$, $S_n\mathcal{Q}$ is a closed Waldhausen subcategory of $S_n\mathcal{C}$. Again, the formula (2.3.2) shows that the mapping spectra $S_n\mathcal{Q}^S(A,B)$ are connective. It is clear that $S_n\Sigma^m\mathcal{Q} = \Sigma^m S_n\mathcal{Q}$ since both categories are the functor categories whose objects are the sequences starting with $*$ of $n$ composable cofibrations in $\mathcal{C}$ between objects in $\Sigma^m\mathcal{Q}$ together with choices of quotients which also must be in $\Sigma^m\mathcal{Q}$.

Now assume that $Q$ is generating; it remains to show that $S_n\mathcal{Q}$ is generating. For $n = 2$, a typical object of $S_2\mathcal{C}$ is of the form $Z = [\, x \rightarrowtail y \twoheadrightarrow z\,]$ for objects $x, y, z$ in $\mathcal{C}$. Replacing $Z$ with a weakly equivalent object, we can assume without loss of generality that $x$ and $y$ are isomorphic rather than just weakly equivalent to an iterated pushout in Definition 3.4.8. Clearly the objects

$$[\, x = x \twoheadrightarrow * \,] \qquad \text{and} \qquad [\, * \rightarrowtail y = y \,]$$

can be finitely cellularly built using pushouts of objects of the same form. We can then build

$$Z' = [\, x \rightarrowtail y \cup_x (x \otimes \Delta[1]) \twoheadrightarrow y \cup_x (x \otimes \Delta[1]) \cup_x * \,]$$

by first building $[\, x \rightarrowtail x \amalg y \twoheadrightarrow y \,]$ and then using pushouts over maps of the form

$$\begin{aligned}[\, * \rightarrowtail b_j \cup_{a_j} (a_j \otimes \Delta[1]) \cup_{a_j} b_j = b_j \cup_{a_j} (a_j \otimes \Delta[1]) \cup_{a_j} b_j \,]& \\ \longrightarrow [\, * \rightarrowtail b_j \otimes \Delta[1] = b_j \otimes \Delta[1] \,]&\end{aligned}$$

where $a_j \to b_j$ are the cells building $x$. Similar observations apply for $n > 2$. $\square$

Now Theorems 3.4.11 and 3.4.13 are easily seen to be equivalent by looking at the following diagram.

$$\begin{tikzcd}
\Omega|THH(S_\bullet\mathcal{Q}^\Gamma)| \arrow[r] \arrow[d, "\sim"'] & \Omega|THH(S_\bullet\mathcal{C}^\Gamma)| \arrow[d] & & \\
\Omega|THH(S_\bullet\mathcal{Q}^S)| \arrow[r, "\sim"] & \Omega|THH(S_\bullet\mathcal{C}^S)| & THH(\mathcal{C}^S) \arrow[l, "\sim"'] & THH(Q^S). \arrow[l, "\sim"']
\end{tikzcd}$$

The lefthand vertical map is a weak equivalence since each map $S_n\mathcal{Q}^\Gamma \to S_n\mathcal{Q}^S$ is a weak equivalence (and in particular DK-equivalence) of spectral categories by Propositions 3.4.7 and 3.4.14, while the bottom horizontal maps are weak equivalences by Theorem 1.3.12 (for the first and third maps) and Theorem 2.5.14 (for the middle map). Theorem 3.4.11 then amounts to the assertion that the righthand vertical map is a weak equivalence while Theorem 3.4.13 is the assertion that the top horizontal map is a weak equivalence.

## 3.5. Proof of the Sphere Theorem

This section contains the proof of Theorem 3.4.13. We fix the almost stable simplicially tensored Waldhausen category $\mathcal{C}$ and the generating connective class $Q$, letting $\mathcal{Q}$ and $\Sigma^n\mathcal{Q}$ be as in Notation 3.4.6. Just as in Waldhausen's argument [**35**, §1.7] we need to introduce a Waldhausen category of CW complexes built out of cells based on $Q$.

Definition 3.5.1. A $Q$-CW complex is a filtered object $X$ in $\mathcal{C}$

$$\cdots \longrightarrow x_n \longrightarrow x_{n+1} \longrightarrow \cdots$$

indexed on the integers, where the arrows $x_n \to x_{n+1}$ are cofibrations for all $n$ and such that the following conditions hold for some $N \gg 0$:

(i) $x_n = *$ for $n \leq -N$,
(ii) $x_n = x_{n+1}$ for $n \geq N$, and
(iii) For all $n$, the quotient $x_{n+1}/x_n$ is an object in $\Sigma^{n+1}\mathcal{Q}$.

We call $x_N$ the *underlying object* of $X$ in $\mathcal{C}$. Let $CW_Q\mathcal{C}$ denote the category whose objects are the $Q$-CW complexes and whose maps are the maps of the underlying objects in $\mathcal{C}$. We say that a $Q$-CW complex $X$ is connective if $x_n = *$ for $n < 0$, and denote the full subcategory of connective $Q$-CW complexes by $CW_Q\mathcal{C}_{[0,\infty)}$. More generally, for $I$ an interval in $\mathbb{Z}$, write $CW_Q\mathcal{C}_I$ for the full subcategory of $Q$-CW complexes $X$ with $x_n = *$ whenever $n$ is less than the elements of $I$ and $x_n = x_{n+1}$ whenever $n+1$ is greater than the elements of $I$.

We define the mapping spectra in $CW_Q\mathcal{C}^\Gamma$ and $CW_Q\mathcal{C}^S$ as the mapping spectra of the underlying objects in $\mathcal{C}^\Gamma$ and $\mathcal{C}^S$, respectively. For the Waldhausen category structure, we use the following definition.

DEFINITION 3.5.2. A *cellular map* of $Q$-CW complexes $X \to Y$ consists of compatible maps $x_n \to y_n$ for all $n$. A cellular map is a *cellular cofibration* when each map $x_n \cup_{x_{n-1}} y_{n-1} \to y_n$ is a cofibration in $\mathcal{C}$ and the induced map $x_n/x_{n-1} \to y_n/y_{n-1}$ is a cofibration in $\Sigma^n\mathcal{Q}$.

An easy check of the definitions then proves the following proposition.

PROPOSITION 3.5.3. *The category of $Q$-CW complexes and cellular maps forms a Waldhausen category with cofibrations the cellular cofibrations of Definition 3.5.2 and weak equivalences the weak equivalences of the underlying objects in $\mathcal{C}$. For $I$ an interval in $\mathbb{Z}$, the subcategory $CW_Q\mathcal{C}_I$ forms a Waldhausen subcategory (though not a closed one).*

Since $S_n\mathcal{Q}$ is a connective class, we also have the category of $S_n\mathcal{Q}$-CW complexes in $S_n\mathcal{C}$. When we restrict to the subcategories of cellular maps, both $S_n(CW_Q\mathcal{C})$ and $CW_{S_n\mathcal{Q}}S_n\mathcal{C}$ are subcategories of the category of functors $\mathrm{Ar}[n] \times \mathbb{Z} \to \mathcal{C}$ (where the category $\mathbb{Z}$ is the ordered set of integers). An easy check of the definitions then shows that these categories coincide. More generally, for $I$ an interval in $\mathbb{Z}$, the cellular maps in $S_n(CW_Q\mathcal{C}_I)$ and $(CW_{S_n\mathcal{Q}}S_n\mathcal{C})_I$ are the same subcategory of functors $\mathrm{Ar}[n] \times I \to \mathcal{C}$. Expanding to all maps in $S_n(CW_Q\mathcal{C})$ and $CW_{S_n\mathcal{Q}}S_n\mathcal{C}$, and looking at the cofibrations and weak equivalences, we get the following proposition.

PROPOSITION 3.5.4. *The Waldhausen categories $S_n(CW_Q\mathcal{C})$ and $CW_{S_n\mathcal{Q}}S_n\mathcal{C}$ are canonically isomorphic. For any interval $I$ in $\mathbb{Z}$, the Waldhausen categories $S_n(CW_Q\mathcal{C}_I)$ and $(CW_{S_n\mathcal{Q}}S_n\mathcal{C})_I$ are canonically isomorphic.*

Because we need to restrict to cellular maps to obtain a Waldhausen category, the category $CW_Q\mathcal{C}$ does not fit into our usual framework of simplicially enriched Waldhausen categories (as the familiar example of CW complexes in spaces demonstrates). Instead, thinking of $Q$-CW complexes as objects of $\mathcal{C}$ with extra structure, we assign mapping spectra by looking at the underlying objects. We use the following notation.

NOTATION 3.5.5. Let $S_n(CW_Q\mathcal{C})^\Gamma$ denote the spectral category whose objects are the objects of $S_n(CW_Q\mathcal{C})$ and whose mapping spectra are the mapping spectra of the underlying objects in $S_n\mathcal{C}^\Gamma$. For $I$ an interval in $\mathbb{Z}$, we define $S_n(CW_Q\mathcal{C}_I)^\Gamma$ analogously.

As an alternate take on this notation, we note that under the canonical isomorphism of Proposition 3.5.4, we get the identification of spectral categories

$$S_n(CW_Q\mathcal{C}_I)^\Gamma = (CW_{S_n\mathcal{Q}}S_n\mathcal{C}_I)^\Gamma$$

As a first reduction of Theorem 3.4.13, we have the following observation. In it, the "forgetful functor" is the functor that takes a $S_n\mathcal{Q}$-CW complex to its underlying object of $S_n\mathcal{C}$.

PROPOSITION 3.5.6. *For any $n$, the forgetful functor $S_n(CW_Q\mathcal{C})^\Gamma \to S_n\mathcal{C}^\Gamma$ is a DK-equivalence.*

PROOF. Using the identification of $S_n(CW_Q\mathcal{C})$ as $CW_{S_n\mathcal{Q}}S_n\mathcal{C}$, it suffices to show that for the arbitrary almost stable simplicially tensored Waldhausen category $\mathcal{C}$ and generating connective class $Q$ the forgetful functor $CW_Q\mathcal{C}^\Gamma \to \mathcal{C}^\Gamma$ is a DK-equivalence. By definition of the mapping spectra, it is a DK-embedding, and so we just need to show that every object of $\mathcal{C}$ is weakly equivalent to the underlying object of a $Q$-CW complex. Since $Q$ is generating, and $*$ is the underlying object of a $Q$-CW complex, it suffices to show that if $y$ is the underlying object of a $Q$-CW complex $Y$, then $x = y \cup_a b$ is weakly equivalent to the underlying object of a $Q$-CW complex whenever $a$ is in $\Sigma^n\mathcal{Q}$, $b$ is contractible, and $a \to b$ is a cofibration. Using the cofibration sequence of Proposition 2.2.12 and stability hypothesis (ii), we see that we have homotopy fibration sequences

$$\mathcal{C}(a, y_m) \longrightarrow \mathcal{C}(a, y_{m+1}) \longrightarrow \mathcal{C}(a, y_{m+1}/y_m)$$

for all $m$. Since $y_{m+1}/y_m$ is in $\Sigma^{m+1}\mathcal{Q}$, for $m \geq n$ we have that $\pi_0\mathcal{C}(a, y_{m+1}/y_m) = 0$ and every map from $a$ to $y_{m+1}$ lifts up to homotopy to a map $a \to y_m$. Thus, the map $a \to y$ lifts up to homotopy to a map $a \to y_n$. Let $X$ be the $Q$-CW complex

$$X = (\cdots \longrightarrow y_{n-1} \longrightarrow y_n \cup_a b \longrightarrow y_{n+1} \cup_a b \longrightarrow \cdots),$$

Then the underlying object of $X$ is weakly equivalent to $x$. $\square$

It follows from Proposition 3.5.6 that the map

$$THH(S_n(CW_Q\mathcal{C})^\Gamma) \longrightarrow THH(S_n\mathcal{C}^\Gamma)$$

is a weak equivalence. The next step is to compare the subcategory of connective objects. The cone and suspension functor on $\mathcal{C}$ extend to cone and suspension functors of $Q$-CW complexes in the usual way: for a $Q$-CW complex $X$, let $CX$ be the $Q$-CW complex with $n$-th object $x_n \cup_{x_{n-1}} Cx_{n-1}$. The inclusion of $X$ in $CX$ is a cellular cofibration and $\Sigma X$ is its quotient. The Additivity Theorem and Corollary 3.1.2 generalize to the context of $THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)$ to show that the self-map of $|THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$ induced by $C$ coincides (in the stable category) with the sum of the identity and the map induced by $\Sigma$. Since $C$ induces the trivial map, it follows that $\Sigma$ induces the map $-\mathrm{id}$, and in particular is a weak equivalence. The analogous observations apply to $\mathcal{C}_{[0,\infty)}$, showing that suspension induces a weak equivalence on $|THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$ and on $|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)|$. Taking the homotopy colimit of the maps induced by suspension, we see that the inclusions

$$|THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)| \longrightarrow \mathrm{hocolim}_\Sigma\, |THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$$
$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| \longrightarrow \mathrm{hocolim}_\Sigma\, |THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)|$$

are weak equivalences. We use this observation in the proof of the following proposition.

PROPOSITION 3.5.7. *The inclusion*

$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$$

*is a weak equivalence.*

Proof. By the preceding observations, it suffices to prove that the map

$$\mathrm{hocolim}_\Sigma\, THH(S_n(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma) \longrightarrow \mathrm{hocolim}_\Sigma\, THH(S_n(CW_Q\mathcal{C})^\Gamma)$$

is a weak equivalence for each $n$. Again using the fact that $\mathcal{C}$ and $Q$ are arbitrary, it suffices to consider the case $n = 1$. Let $CW_Q^\Sigma\mathcal{C}^\Gamma$ be the spectrally enriched category where an object is an ordered pair $(X, m)$ where $X$ is a $Q$-CW complex and $m$ is a non-negative integer; for mapping spectra, we let

$$CW_Q^\Sigma\mathcal{C}^\Gamma((X,m),(Y,n)) = \operatorname*{colim}_{k\geq \max(m,n)} \mathcal{C}^\Gamma(\Sigma^{k-m}X, \Sigma^{k-n}Y).$$

(Composition is induced levelwise in the colimit system after taking $k$ large enough.) Let $CW_Q^\Sigma\mathcal{C}_{[0,\infty)}$ be the full subcategory of $CW_Q^\Sigma\mathcal{C}^\Gamma$ consisting of the objects $(X, m)$ with $X$ connective. By Proposition 1.2.7, the canonical maps

$$\mathrm{hocolim}_\Sigma\, THH(CW_Q\mathcal{C}^\Gamma) \longrightarrow THH(CW_Q^\Sigma\mathcal{C}^\Gamma)$$
$$\mathrm{hocolim}_\Sigma\, THH(CW_Q\mathcal{C}^\Gamma_{[0,\infty)}) \longrightarrow THH(CW_Q^\Sigma\mathcal{C}^\Gamma_{[0,\infty)})$$

are weak equivalences. The inclusion of $CW_Q^\Sigma\mathcal{C}^\Gamma_{[0,\infty)}$ in $CW_Q^\Sigma\mathcal{C}^\Gamma$ is a DK-equivalence, and so also induces a weak equivalence on $THH$. □

The previous two propositions show that the map

$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)| \longrightarrow |THH(S_\bullet\mathcal{C}^\Gamma)|$$

is a weak equivalence, reducing the proof of Theorem 3.4.13 to showing that the map

$$|THH(S_\bullet\mathcal{Q}^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)|$$

is a weak equivalence. This is an easy consequence of the following lemma.

Lemma 3.5.8. *For every $n \geq 1$, the inclusion of $CW_Q\mathcal{C}_{[0,n-1]}$ in $CW_Q\mathcal{C}_{[0,n]}$ induces a weak equivalence*

$$|THH(S_\bullet(CW_Q\mathcal{C}_{[0,n-1]})^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C}_{[0,n]})^\Gamma)|.$$

Proof of Theorem 3.4.13 from Lemma 3.5.8. The lemma implies that the maps in the homotopy colimit system

$$\mathrm{hocolim}_n\, |THH(S_\bullet(CW_Q\mathcal{C}_{[0,n]})^\Gamma)|$$

are all weak equivalences. By Proposition 1.2.7, we see that the canonical map from the homotopy colimit to $|THH(S_\bullet(CW_Q\mathcal{C}_{[0,\infty)})^\Gamma)|$ is a weak equivalence. It follows that the map

$$|THH(S_\bullet\mathcal{Q}^\Gamma)| = |THH(S_\bullet(CW_Q\mathcal{C}_{[0,0]})^\Gamma)| \longrightarrow |THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)|$$

is a weak equivalence. Composing with the weak equivalence

$$|THH(S_\bullet(CW_Q\mathcal{C})^\Gamma)| \longrightarrow |THH(S_\bullet\mathcal{C}^\Gamma)|$$

above and applying $\Omega$, we see that the map $WTHH^\Gamma(\mathcal{Q}) \to WTHH^\Gamma(\mathcal{C})$ is a weak equivalence. □

The remainder of the section is devoted to the proof of Lemma 3.5.8. The argument is somewhat roundabout, requiring the introduction of the spectral categories $S_k(CW_Q\mathcal{C}_{[0,n]})^S$, defined analogously to $S_k(CW_Q\mathcal{C}_{[0,n]})^\Gamma$ in Notation 3.5.5 but using the non-connective enrichment. The proof of Proposition 3.5.6 equally well shows that the forgetful functor $S_k(CW_Q\mathcal{C})^S \to S_k\mathcal{C}^S$ is a DK-equivalence. These non-connective enrichments are easier to understand because Proposition 2.2.12

implies that when we DK-embed $S_k\mathcal{C}^S$ in a pretriangulated spectral category, the DK-embedding takes cofiber sequences to distinguished triangles in the derived category. As a consequence, Theorem 1.3.12 tells us that the maps

$$THH(S_k\mathcal{Q}^S) \longrightarrow THH(S_k(CW_Q\mathcal{C}_{[0,n-1]})^S) \longrightarrow THH(S_k(CW_Q\mathcal{C}_{[0,n]})^S)$$

are weak equivalences. Looking at the diagram

$$\begin{tikzcd}
{|THH(S_\bullet(CW_Q\mathcal{C}_{[0,n-1]})^\Gamma)|} \arrow[r] \arrow[d] & {|THH(S_\bullet(CW_Q\mathcal{C}_{[0,n]})^\Gamma)|} \arrow[d] \\
{|THH(S_\bullet(CW_Q\mathcal{C}_{[0,n-1]})^S)|} \arrow[r, "\sim"] & {|THH(S_\bullet(CW_Q\mathcal{C}_{[0,n]})^S)|,}
\end{tikzcd}$$

we assume by induction on $n$ that the lefthand map is a weak equivalence, the base case being the already known case of $S_\bullet CW_Q\mathcal{C}_{[0,0]} = S_\bullet\mathcal{Q}$. We then prove that the top map is a weak equivalence by showing that the righthand map is a weak equivalence.

To save space and eliminate unnecessary symbols, we will now write $\mathcal{C}^n_k$ for $S_k(CW_Q\mathcal{C}_{[0,n]})$ or equivalently, $CW_{S_k\mathcal{Q}}S_k\mathcal{C}_{[0,n]}$, and $\mathcal{C}\Gamma^n_k$ and $\mathcal{C}S^n_k$ for the connective and non-connective spectral enrichments, respectively. Let $\mathcal{E}\Gamma^n_\bullet$ denote the simplicial spectral category where the objects of $\mathcal{E}\Gamma^n_k$ are the objects of $\mathcal{C}^n_k$ and for objects $X$ and $Y$, the mapping spectrum is the fiber product

$$\mathcal{E}\Gamma^n_k(X,Y) = S_k\mathcal{C}^\Gamma(x_{n-1},y_{n-1}) \times_{S_k\mathcal{C}^\Gamma(x_{n-1},y_n)} S_k\mathcal{C}^\Gamma(x_n,y_n)$$

(which is a homotopy pullback because $x_{n-1} \to x_n$ is a cofibration). We have a canonical simplicial spectral functor $\mathcal{E}\Gamma^n_\bullet \to \mathcal{C}\Gamma^n_\bullet$ sending $X$ in $\mathcal{E}\Gamma^n_k$ to $X$ viewed as an object of $\mathcal{C}\Gamma^n_k$ and using projection on the mapping spectra. We also have canonical simplicial spectral functors

$$\mathcal{E}\Gamma^n_\bullet \longrightarrow \mathcal{C}\Gamma^{n-1}_\bullet \qquad \text{and} \qquad \mathcal{E}\Gamma^n_\bullet \longrightarrow S_\bullet\Sigma^n\mathcal{Q}^\Gamma$$

sending $X$ to its $(n-1)$-skeleton $X_{n-1}$ and to $x_n/x_{n-1}$, respectively, and performing the corresponding maps on mapping spectra. Using these maps, we can identify $\mathcal{E}\Gamma^n_\bullet$ as the spectral category of extension sequences $[\ X_{n-1} \rightarrowtail X \twoheadrightarrow x_n/x_{n-1}\ ]$ in $\mathcal{C}^n_\bullet$. Although the categories $\mathcal{C}\Gamma^n_\bullet$ and $\mathcal{E}\Gamma^n$ do not exactly fit into the framework of Section 3.1, McCarthy's argument for the Additivity Theorem works quite generally and formally essentially using little more than the fact that the mapping spectra are functorial in the maps in $S_\bullet$; the Additivity Theorem generalizes to the current context, and the argument following Corollary 3.1.2 shows that the maps described above induce a weak equivalence

$$|THH(\mathcal{E}\Gamma^n_\bullet)| \xrightarrow{\sim} |THH(\mathcal{C}\Gamma^{n-1}_\bullet)| \times |THH(S_\bullet\Sigma^n\mathcal{Q}^\Gamma)|.$$

We have an analogous simplicial spectral category $\mathcal{E}S^n_\bullet$ with the analogous weak equivalence. The induction hypothesis and the weak equivalences above then imply the following proposition.

PROPOSITION 3.5.9. *The functor $\mathcal{E}\Gamma^n_\bullet \to \mathcal{E}S^n_\bullet$ induces a weak equivalence*

$$|THH(\mathcal{E}\Gamma^n_\bullet)| \xrightarrow{\sim} |THH(\mathcal{E}S^n_\bullet)|.$$

Let $(\mathcal{C}^n_k)^w$ denote the objects of $\mathcal{C}^n_k$ that are weakly equivalent to the zero object $*$, and write $(\mathcal{C}\Gamma^n_k)^w$, $(\mathcal{C}S^n_k)^w$, $(\mathcal{E}\Gamma^n_k)^w$, $(\mathcal{E}S^n_k)^w$ for the various spectral enrichments on this category, the full spectral subcategories of $\mathcal{C}\Gamma^n_k$, $\mathcal{C}S^n_k$, $\mathcal{E}\Gamma^n_k$, $\mathcal{E}S^n_k$, respectively. The mapping spectra in $(\mathcal{C}\Gamma^n_k)^w$ and $(\mathcal{C}S^n_k)^w$ are all weakly contractible so $THH$ is also weakly contractible,

$$THH((\mathcal{C}\Gamma^n_k)^w) \simeq THH((\mathcal{C}S^n_k)^w) \simeq *.$$

For the $\mathcal{E}$ categories, we have the following proposition.

PROPOSITION 3.5.10. *The canonical spectral functor $(\mathcal{E}\Gamma^n_k)^w \to (\mathcal{E}S^n_k)^w$ is a DK-equivalence.*

PROOF. Since the categories have the same object set, it suffices to show that the map is a DK-embedding, and for this it suffices to show that the mapping spectra in $(\mathcal{E}S^n_k)^w$ are connective. We note that for $X$ in $(\mathcal{C}^n_k)^w$, stability hypothesis (ii) implies that $x_{n-1}$ is an object of $S_k\Sigma^{n-1}\mathcal{Q}$ since $x_n$ is contractible and $x_n/x_{n-1}$ is an object of $S_k\Sigma^n\mathcal{Q}$. Now given $X$ and $Y$ in $(\mathcal{C}^n_k)^w$, the projection map

$$\mathcal{E}S^n_k(X,Y) = S_k\mathcal{C}^S(x_{n-1},y_{n-1}) \times_{S_k\mathcal{C}^S(x_{n-1},y_n)} S_k\mathcal{C}^S(x_n,y_n) \longrightarrow S_k\mathcal{C}^S(x_{n-1},y_{n-1})$$

is a weak equivalence. In particular, $\mathcal{E}S^n_k(X,Y)$ is connective. □

We denote by $CTHH(\mathcal{C}\Gamma^n_k, w)$ the homotopy cofiber of the inclusion

$$THH((\mathcal{C}\Gamma^n_k)^w) \longrightarrow THH(\mathcal{C}\Gamma^n_k),$$

and analogously for $CTHH(\mathcal{C}S^n_k, w)$, $CTHH(\mathcal{E}\Gamma^n_k, w)$, and $CTHH(\mathcal{E}S^n_k, w)$. In this notation, the two previous propositions then imply the following proposition.

PROPOSITION 3.5.11. *The map $|CTHH(\mathcal{E}\Gamma^n_\bullet, w)| \to |CTHH(\mathcal{E}S^n_\bullet, w)|$ is a weak equivalence.*

Since the inclusions

$$\begin{aligned} THH(\mathcal{C}\Gamma^n_k) &\longrightarrow CTHH(\mathcal{C}\Gamma^n_k, w) \\ THH(\mathcal{C}S^n_k) &\longrightarrow CTHH(\mathcal{C}S^n_k, w) \end{aligned}$$

are weak equivalences, the following lemma when combined with the previous proposition then completes the proof of Lemma 3.5.8.

LEMMA 3.5.12. *For all $k$, the maps*

$$\begin{aligned} CTHH(\mathcal{E}\Gamma^n_k, w) &\longrightarrow CTHH(\mathcal{C}\Gamma^n_k, w) \\ CTHH(\mathcal{E}S^n_k, w) &\longrightarrow CTHH(\mathcal{C}S^n_k, w) \end{aligned}$$

*are weak equivalences.*

We prove the case for the connective enrichment in detail, the case for the non-connective enrichment being similar (but slightly easier). The statement is analogous to the Localization Theorem 7.2 of [**7**] (reviewed in Chapter 1 as Theorem 1.3.14) using the Dennis-Waldhausen Morita Argument (Section 1.4) except for the fact that the subcategories above are not pretriangulated. The following proof goes roughly along the same lines as well.

For this argument $k$ is both fixed, and so replacing $\mathcal{C}\Gamma^\infty_k$ and $\mathcal{E}\Gamma^\infty_k$ by weakly equivalent spectral categories if necessary, we can assume without loss of generality that they are pointwise cofibrant and their subcategories $\mathcal{C}\Gamma^n_k$, $\mathcal{E}\Gamma^n_k$, $(\mathcal{C}\Gamma^n_k)^w$, $(\mathcal{E}\Gamma^n_k)^w$ are pointwise cofibrant.

Define the $\mathcal{E}\Gamma^n_k$-bimodule $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{C}\Gamma^n_k$-bimodule $\mathcal{L}_{\mathcal{C}}$ by

$$\mathcal{L}_{\mathcal{E}}(X,Y) = B(\mathcal{E}\Gamma^n_k(-,Y);(\mathcal{E}\Gamma^n_k)^w;\mathcal{E}\Gamma^n_k(X,-))$$
$$\mathcal{L}_{\mathcal{C}}(X,Y) = B(\mathcal{C}\Gamma^n_k(-,Y);(\mathcal{C}\Gamma^n_k)^w;\mathcal{C}\Gamma^n_k(X,-)),$$

where $B$ denotes the two-sided bar construction (Definition 1.4.3). We then have maps of $\mathcal{E}\Gamma^n_k$- and $\mathcal{C}\Gamma^n_k$-bimodules

$$\mathcal{L}_{\mathcal{E}} \longrightarrow \mathcal{E}\Gamma^n_k \qquad \text{and} \qquad \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{C}\Gamma^n_k;$$

we let $\mathcal{M}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{C}}$ be the homotopy cofibers. Then the Dennis-Waldhausen Morita Argument and specifically Theorem 1.4.6 give us weak equivalences

$$THH(\mathcal{E}\Gamma^n_k;\mathcal{L}_{\mathcal{E}}) \simeq THH((\mathcal{E}\Gamma^n_k)^w) \qquad THH(\mathcal{E}\Gamma^n_k;\mathcal{M}_{\mathcal{E}}) \simeq CTHH(\mathcal{E}\Gamma^n_k,w)$$
$$THH(\mathcal{C}\Gamma^n_k;\mathcal{L}_{\mathcal{C}}) \simeq THH((\mathcal{C}\Gamma^n_k)^w) \qquad THH(\mathcal{C}\Gamma^n_k;\mathcal{M}_{\mathcal{C}}) \simeq CTHH(\mathcal{C}\Gamma^n_k,w),$$

and we can identify the map in Lemma 3.5.12 as the map

$$THH(\mathcal{E}\Gamma^n_k;\mathcal{M}_{\mathcal{E}}) \longrightarrow THH(\mathcal{C}\Gamma^n_k;\mathcal{M}_{\mathcal{C}}). \tag{3.5.13}$$

As the mapping spectra in $(\mathcal{C}\Gamma^n_k)^w$ are weakly contractible, the spectra $\mathcal{L}_{\mathcal{C}}(X,Y)$ are weakly contractible for all $X, Y$, and it follows that the map of $\mathcal{C}\Gamma^n_k$-bimodules $\mathcal{C}\Gamma^n_k \to \mathcal{M}_{\mathcal{C}}$ is a weak equivalence. We next move towards understanding the $\mathcal{E}\Gamma^n_k$-bimodules $\mathcal{L}_{\mathcal{E}}$. We write $u$ for the canonical functor $\mathcal{E}\Gamma^n_k \to \mathcal{C}\Gamma$ and also its restriction $(\mathcal{E}\Gamma^n_k)^w \to (\mathcal{C}\Gamma)^w$. We then have a commutative diagram of $\mathcal{E}\Gamma^n_k$-bimodules

$$\begin{tikzcd}
\mathcal{L}_{\mathcal{E}} \arrow[r] \arrow[d] & u^*\mathcal{L}_{\mathcal{C}} \arrow[d] \\
\mathcal{E}\Gamma^n_k \arrow[r] & u^*\mathcal{C}\Gamma^n_k,
\end{tikzcd}$$

Letting $\mathcal{F}$ be the homotopy pullback of the deleted diagram

$$\mathcal{E}\Gamma^n_k \longrightarrow u^*\mathcal{C}\Gamma^n_k \longleftarrow u^*\mathcal{L}_{\mathcal{C}},$$

we get a map of $\mathcal{E}\Gamma^n_k$-bimodules $\mathcal{L}_{\mathcal{E}} \to \mathcal{F}$.

Proposition 3.5.14. *The map of $\mathcal{E}\Gamma^n_k$-bimodules $\mathcal{L}_{\mathcal{E}} \to \mathcal{F}$ is a weak equivalence.*

Proof. Fix $X$ and $Y$ objects in $\mathcal{E}\Gamma^n_k$; we need to show that the map $\mathcal{L}_{\mathcal{E}}(X,Y) \to \mathcal{F}(X,Y)$ is a weak equivalence. Consider the cofibration sequence

$$y_{n-1} \longrightarrow y_n \longrightarrow y_n/y_{n-1} \longrightarrow \Sigma y_{n-1}$$

obtained using a homotopy inverse weak equivalence to the collapse weak equivalence $y_n \cup_{y_{n-1}} Cy_{n-1} \to y_n/y_{n-1}$. By definition, $y_n/y_{n-1}$ is in $S_k\Sigma^n\mathcal{Q}$, and since $n \geq 1$, there exists an object $p$ in $S_k\Sigma^{n-1}\mathcal{Q}$ such that $\Sigma p$ is weakly equivalent to $y_n/y_{n-1}$. Then applying stability hypothesis (ii), we obtain from the cofibration sequence above a (homotopy class of) map $p \to y_{n-1}$ and a null homotopy $Cp \to y_n$ such that the induced map $\Sigma p \to y_n/y_{n-1}$ is homotopic to the chosen weak equivalence. Regarding $Cp$ as an object of $\mathcal{E}^n_k$, it is an object of $(\mathcal{E}^n_k)^w$ and we have constructed a cellular map $Cp \to Y$. Consider the following commutative square.

$$\begin{tikzcd} \mathcal{L}_{\mathcal{E}}(X,Y) \arrow[d] & \mathcal{L}_{\mathcal{E}}(X,Cp) \arrow[l, "a"'] \arrow[d, "c"] \\ \mathcal{F}(X,Y) & \mathcal{F}(X,Cp) \arrow[l, "b"] \end{tikzcd}$$

We complete the proof by arguing that the maps $a$, $b$, and $c$ are weak equivalences.

To analyze the map $a$, consider an object $Z$ in $(\mathcal{E}\Gamma^n_k)^w$. Since $z_n$ is weakly equivalent to $*$ in $\mathcal{C}$, $\mathcal{E}\Gamma^n_k(Z,Y)$ is weakly equivalent to the homotopy fiber of the map $S_k\mathcal{C}^\Gamma(z_{n-1},y_{n-1})$ to $S_k\mathcal{C}^\Gamma(z_{n-1},y_n)$. We can use the cofibration sequence of Proposition 2.2.12 to understand this homotopy fiber: We have that $S_k\mathcal{C}^\Gamma(z_{n-1},y_n/y_{n-1})$ is connected since $z_{n-1}$ is an object of $S_k\Sigma^{n-1}\mathcal{Q}$ and $y_n/y_{n-1}$ is an object of $S_k\Sigma^n\mathcal{Q}$. It follows that $\mathcal{E}\Gamma^n_k(Z,Y)$ is weakly equivalent to $\Omega S_k\mathcal{C}^\Gamma(z_{n-1},y_n/y_{n-1})$. The same observations apply to $Cp$. Since by construction the map $\Sigma p = Cp/p \to y_n/y_{n-1}$ is a weak equivalence, we see by naturality that the map $\mathcal{E}\Gamma^n_k(Z,Cp) \to \mathcal{E}\Gamma^n_k(Z,Y)$ is a weak equivalence. Since this holds for any $Z$ in $(\mathcal{E}\Gamma^n_k)^w$, unwinding the definition of $\mathcal{L}_{\mathcal{E}}$, we see that $a$ is a weak equivalence.

For the map $b$, we note that $\mathcal{F}(X,Y)$ being the homotopy fiber of the map $\mathcal{E}\Gamma^n_k(X,Y)$ to $\mathcal{C}\Gamma(X,Y) = S_k\mathcal{C}^\Gamma(x_n,y_n)$, it is naturally weakly equivalent to the homotopy fiber of the map $S_k\mathcal{C}^\Gamma(x_{n-1},y_{n-1})$ to $S_k\mathcal{C}^\Gamma(x_{n-1},y_n)$. As in the previous case, we can identify this up to weak equivalence as $\Omega S_k\mathcal{C}^\Gamma(x_{n-1},y_n/y_{n-1})$ since $S_k\mathcal{C}^\Gamma(x_{n-1},y_n/y_{n-1})$ is connected (which can be proved by induction up the skeletal filtration of $X$ using Proposition 2.2.12). Again, since the map $\Sigma p = Cp/p \to y_n/y_{n-1}$ is a weak equivalence, we see that $b$ is a weak equivalence.

For the map $c$, since $Cp$ is in $(\mathcal{E}\Gamma^n_k)^w$, the Two-Sided Bar Lemma 1.4.5 implies that the natural map $\mathcal{L}_{\mathcal{E}}(X,Cp) \to \mathcal{E}\Gamma^n_k(X,Cp)$ is a weak equivalence. Since $Cp$ is weakly equivalent to $*$ in $\mathcal{C}$, $\mathcal{C}\Gamma(X,Cp)$ is weakly contractible and we see that $b$ is a weak equivalence. □

The previous proposition lets us understand $\mathcal{M}_{\mathcal{E}}$.

Proposition 3.5.15. *The map of $\mathcal{E}\Gamma^n_k$-bimodules $\mathcal{M}_{\mathcal{E}} \to u^*\mathcal{M}_{\mathcal{C}}$ is a weak equivalence.*

Proof. Since homotopy fiber squares in spectra are homotopy cocartesian, the canonical map from the homotopy cofiber of $\mathcal{F} \to \mathcal{E}\Gamma^n_k$ to the homotopy cofiber of $u^*\mathcal{L}_{\mathcal{C}} \to u^*\mathcal{C}\Gamma^n_k$ is a weak equivalence. □

We now return to the map (3.5.13). We see from the previous proposition that we are in the situation where Theorem 1.4.7 applies. Thus, to see that the map

$$THH(\mathcal{E}\Gamma^n_k;\mathcal{M}_{\mathcal{E}}) \longrightarrow THH(\mathcal{C}\Gamma^n_k;\mathcal{M}_{\mathcal{C}})$$

is a weak equivalence, we just need to check that the map

$$B(\mathcal{C}\Gamma^n_k(-,Y);\mathcal{E}\Gamma^n_k;\mathcal{M}_{\mathcal{C}}(X,-)) \longrightarrow \mathcal{C}\Gamma^n_k(X,Y)$$

is a weak equivalence for all $X$, $Y$ in $\mathcal{C}\Gamma^n_k$, or equivalently in this case, for all $X$, $Y$ in $\mathcal{E}\Gamma^n_k$. Since the Two-Sided Bar Lemma 1.4.5 shows that the map

$$B(\mathcal{C}\Gamma^n_k(-,Y);\mathcal{E}\Gamma^n_k;\mathcal{E}\Gamma^n_k(X,-)) \longrightarrow \mathcal{C}\Gamma^n_k(X,Y)$$

is a weak equivalence and $\mathcal{M}_{\mathcal{C}}(X,-) \simeq \mathcal{M}_{\mathcal{E}}(X,-)$ is the homotopy cofiber of $\mathcal{L}_{\mathcal{E}}(X,-) \to \mathcal{E}\Gamma^n_k(X,-)$, it suffices to show that

$$\mathcal{G}(X,Y) = B(\mathcal{C}\Gamma^n_k(-,Y); \mathcal{E}\Gamma^n_k; \mathcal{L}_{\mathcal{E}}(X,-))$$

is weakly contractible. But we have

$$\begin{aligned}\mathcal{G}(X,Y) &= B(\mathcal{C}\Gamma^n_k(-,Y); \mathcal{E}\Gamma^n_k; B(\mathcal{E}\Gamma^n_k(-,-); (\mathcal{E}\Gamma^n_k)^w; \mathcal{E}\Gamma^n_k(X,-)))\\ &\simeq B(\mathcal{C}\Gamma^n_k(-,Y); (\mathcal{E}\Gamma^n_k)^w; \mathcal{E}\Gamma^n_k(X,-)).\end{aligned}$$

Since $\mathcal{C}\Gamma^n_k(Z,Y)$ is weakly contractible for any $Z$ in $(\mathcal{E}\Gamma^n_k)^w$, it follows that $\mathcal{G}(X,Y)$ is weakly contractible. This completes the proof that (3.5.13) is a weak equivalence and hence the proof of Lemma 3.5.12, which in turn completes the proof of Lemma 3.5.8.

CHAPTER 4

# Localization sequences for $THH$ and $TC$

In [**16**], Hesselholt and Madsen introduced a localization sequence for $THH$ and $TC$ in the context of discrete valuation rings, producing cofiber sequences

$$THH(k) \longrightarrow THH(R) \longrightarrow THH(R|F) \longrightarrow \Sigma THH(k)$$
$$TC(k) \longrightarrow TC(R) \longrightarrow TC(R|F) \longrightarrow \Sigma TC(k),$$

where $R$ denotes a discrete valuation ring, $k$ its residue field, and $F$ its field of fractions. Here $THH(R|F)$ and $TC(R|F)$ denote the $THH$ and $TC$ of a relative theory they construct. Although their constructions are slightly different than ours, we prove in Section 4.1 that this sequence arises as the Localization Theorem 3.3.1 for the connective spectral enrichment

$$WTHH^{\Gamma}(\mathcal{A}^v) \longrightarrow WTHH^{\Gamma}(\mathcal{A}) \longrightarrow WTHH^{\Gamma}(\mathcal{A}|v) \longrightarrow \Sigma WTHH^{\Gamma}(\mathcal{A}^v)$$

where $\mathcal{A}$ denotes either the category of perfect simplicial modules over the ring $R$ or the category of finite cell EKMM $HR$-modules (for the Eilenberg-Mac Lane spectrum $HR$) and $v\mathcal{A}$ denotes the subcategory of maps that induce an isomorphism on homotopy groups after inverting the action of the uniformizer (or, equivalently, tensoring over $R$ with $F$). This is in contrast to the localization sequence obtained from the non-connective spectral enrichment

$$WTHH(\mathcal{A}^v) \longrightarrow WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{A}|v) \longrightarrow \Sigma WTHH(\mathcal{A}^v)$$

which leads to the localization sequences

$$THH(R\,\mathrm{on}\,k) \longrightarrow THH(R) \longrightarrow THH(F) \longrightarrow \Sigma THH(R\,\mathrm{on}\,k)$$
$$TC(R\,\mathrm{on}\,k) \longrightarrow TC(R) \longrightarrow TC(F) \longrightarrow \Sigma TC(R\,\mathrm{on}\,k)$$

where $THH(R\,\mathrm{on}\,k)$ and $TC(R\,\mathrm{on}\,k)$ are as in Theorem 1.1 of [**7**].

Hesselholt and Ausoni [**1, 2**] conjectured that the above localization sequences generalize from the "chromatic level 0" case to "chromatic level 1" and specifically that there should be analogous cofiber sequences

$$THH(\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow THH(ku|KU) \longrightarrow \Sigma THH(\mathbb{Z})$$
$$TC(\mathbb{Z}) \longrightarrow TC(ku) \longrightarrow TC(ku|KU) \longrightarrow \Sigma TC(\mathbb{Z})$$

(as well as $p$-local and $p$-complete variants; see Theorem 4.2.1 below). Here $ku$ denotes complex connective (topological) $K$-theory, $KU$ denotes complex periodic $K$-theory. In Sections 4.2 and 4.3, we prove these sequences arise again from the Localization Theorem for the connective enrichment

$$WTHH^{\Gamma}(\mathcal{C}^v) \longrightarrow WTHH^{\Gamma}(\mathcal{C}) \longrightarrow WTHH^{\Gamma}(\mathcal{C}|v) \longrightarrow \Sigma WTHH^{\Gamma}(\mathcal{C}^v)$$

where $\mathcal{C}$ is the category of finite cell EKMM $ku$-modules and $v\mathcal{C}$ the maps that induce isomorphisms on homotopy groups after inverting the action of the Bott element. Our argument is general enough to also produce the localization sequences

$$THH(\mathbb{W}\mathbb{F}_{p^n}[\![u_1,\ldots,u_{n-1}]\!]) \longrightarrow THH(BP_n) \longrightarrow THH(BP_n|E_n) \longrightarrow \Sigma\cdots$$
$$TC(\mathbb{W}\mathbb{F}_{p^n}[\![u_1,\ldots,u_{n-1}]\!]) \longrightarrow TC(BP_n) \longrightarrow TC(BP_n|E_n) \longrightarrow \Sigma\cdots$$

for all $n$ discussed in the introduction of this paper relating the $THH$ and $TC$ of the Eilenberg-Mac Lane spectra on the Witt rings to the $THH$ and $TC$ of the connective cover $B_n$ of the Lubin-Tate spectrum $E_n$ and the corresponding relative construction.

The chapter is organized as follows. Section 4.1 compares our constructions $WTHH^{\Gamma}$ to the analogous construction of Hesselholt-Madsen [**16**]. Section 4.2 states the main theorem on localization sequences for $THH(ku)$ and reduces the proof to a dévissage theorem, Theorem 4.2.2; Section 4.3 then proves Theorem 4.2.2.

## 4.1. The localization sequence for $THH$ of a discrete valuation ring

In this section, we compare the construction of $THH$ we use here with the construction used by Hesselholt-Madsen in [**16**] to prove the localization sequences in $THH$ and $TC$ for discrete valuation rings. The main theorem of this section is then the following.

THEOREM 4.1.1. *Let $R$ be a discrete valuation ring, $k$ its quotient field and $F$ its field of fractions. Let $\mathcal{A}$ denote the category of perfect simplicial $R$-algebras and let $v\mathcal{A}$ denote the subcategory of those maps which induce isomorphisms on homotopy groups after inverting a uniformizer (i.e., after tensoring with $F$). Then the cofibration sequence*

$$WTHH^{\Gamma}(\mathcal{A}^v) \longrightarrow WTHH^{\Gamma}(\mathcal{A}) \longrightarrow WTHH^{\Gamma}(\mathcal{A}|v) \longrightarrow \Sigma WTHH^{\Gamma}(\mathcal{A}^v)$$

*of Theorem 3.3.1 on $THH$ induces on $TC$ a cofbration sequence naturally weakly equivalent to the cofibration sequence of* [**16**, 1.5.7]*, compatibly with the cyclotomic trace.*

Assuming Theorem 5.1.1 from Chapter 5, we also sketch a proof of the following theorem for the EKMM $S$-module models.

THEOREM 4.1.2. *Let $R$ be a discrete valuation ring, $k$ its quotient field and $F$ its field of fractions. Let $\mathcal{A}$ denote the category of finite cell EKMM $HR$-modules and let $v\mathcal{A}$ denote the subcategory of those maps which induce isomorphisms on homotopy groups after inverting a uniformizer (i.e., after tensoring over $R$ with $F$). Then the cofibration sequence*

$$WTHH^{\Gamma}(\mathcal{A}^v) \longrightarrow WTHH^{\Gamma}(\mathcal{A}) \longrightarrow WTHH^{\Gamma}(\mathcal{A}|v) \longrightarrow \Sigma WTHH^{\Gamma}(\mathcal{A}^v)$$

*of Theorem 3.3.1 on $THH$ induces on $TC$ a cofbration sequence naturally weakly equivalent to the cofibration sequence of* [**16**, 1.5.7]*, compatibly with the cyclotomic trace.*

We begin with a quick review of the construction used by Hesselholt-Madsen [**16**]. Let $\mathcal{C}_0$ denote the category of perfect complexes $R$-modules, i.e., the category of bounded chain complexes of finitely generated projective $R$-modules and let $v\mathcal{C}_0$ denote the subcategory of maps that induce isomorphisms on homology after inverting a uniformizer (or equivalently, tensoring over $R$ with $F$). Regarding $\mathcal{C}_0$

and $\mathcal{C}_0^v$ as exact categories, we get connective spectral enrichments $\mathcal{C}_0^\Gamma$ and $(\mathcal{C}_0^v)^\Gamma$. Hesselholt-Madsen [**16**, p. 27] then produce weak equivalences

$$THH(w_\bullet S_\bullet \mathcal{C}_0^\Gamma) \simeq THH(R), \qquad THH(w_\bullet S_\bullet (\mathcal{C}_0^v)^\Gamma) \simeq THH(k)$$

and a homotopy cartesian square

$$\begin{tikzcd} THH(w_\bullet S_\bullet (\mathcal{C}_0^v)^\Gamma) \arrow[r] \arrow[d] & THH(v_\bullet S_\bullet (\mathcal{C}_0^v)^\Gamma) \arrow[d] \\ THH(w_\bullet S_\bullet \mathcal{C}_0^\Gamma) \arrow[r] & THH(v_\bullet S_\bullet \mathcal{C}_0^\Gamma) \end{tikzcd}$$

with the upper left hand entry (canonically) contractible. Their $THH$ cofibration sequence is then

$$THH(w_\bullet S_\bullet (\mathcal{C}_0^v)^\Gamma) \longrightarrow THH(w_\bullet S_\bullet \mathcal{C}_0^\Gamma) \longrightarrow THH(v_\bullet S_\bullet \mathcal{C}_0^\Gamma) \longrightarrow \Sigma THH(w_\bullet S_\bullet (\mathcal{C}_0^v)^\Gamma).$$

Since the simplicial categories $S_n\mathcal{C}_0$ and $S_n\mathcal{C}_0^v$ are discrete, the canonical inclusions

$$w_m S_n \mathcal{C}_0^v \longrightarrow w_m^M S_n \mathcal{C}_0^v w_m S_n \mathcal{C}_0 \longrightarrow w_m^M S_n \mathcal{C}_0 v_m S_n \mathcal{C}_0 \longrightarrow v_m^M S_n \mathcal{C}_0$$

are isomorphisms, and so we can identify the $THH$ cofibration sequence of [**16**] as the cofibration sequence

$$WTHH^\Gamma(\mathcal{C}_0^v) \longrightarrow WTHH^\Gamma(\mathcal{C}_0) \longrightarrow WTHH^\Gamma(\mathcal{C}_0|v) \longrightarrow \Sigma WTHH^\Gamma(\mathcal{C}_0^v).$$

The proof of Theorem 4.1.1 then consists of essentially two parts: First reconciling the use of the category of complexes of $R$-modules ($\mathcal{C}_0$) with the use of the category of simplicial $R$-modules ($\mathcal{A}_0$), and second in the construction of connective enrichment, reconciling the use mapping spaces ($\mathcal{A}^\Gamma(x,y)$) and mapping sets ($\mathcal{A}_0^\Gamma(x,y)$).

To treat the case of $w_\bullet S_\bullet \mathcal{C}_0^v$, $w_\bullet S_\bullet \mathcal{C}_0$, and $v_\bullet S_\bullet \mathcal{C}_0$ on equal footing, we will work in the following context. Let $\mathfrak{A}$ be an abelian category, let $\mathcal{B}_0$ be a full subcategory of the category of bounded below (in the homological grading) complexes of $\mathfrak{A}$-modules, and let $v\mathcal{B}_0$ be a subcategory of $\mathcal{B}_0$ containing all the quasi-isomorphisms, satisfying the Gluing Axiom for the degreewise split monomorphisms, and satisfying the two-out-of-three property. We also assume that $\mathcal{B}_0$ contains 0, is closed under suspension and is closed under quotients and extensions by degreewise split monomorphisms, i.e., if

$$0 \longrightarrow a \longrightarrow b \longrightarrow c \longrightarrow 0$$

is a short exact sequences of chain complexes in $\mathfrak{A}$ with $a \to b$ degreewise split, if $a$ is $\mathcal{B}_0$ and either $b$ or $c$ is in $\mathcal{B}_0$ then so is the other. Let $\mathcal{A}_0$ be the subcategory of strictly connective complexes in $\mathcal{B}_0$. Then the Dold-Kan correspondence allows us to view $\mathcal{A}_0$ as a full subcategory of the category of simplicial objects in $\mathfrak{A}$, extending it to a simplicially enriched category $\mathcal{A}$. We regard $\mathcal{B}_0$ as a Waldhausen category with cofibrations the degreewise split cofibrations and weak equivalences the quasi-isomorphisms; then $\mathcal{A}_0$ is a Waldhausen subcategory (though not closed) and $\mathcal{A}$ is a simplicially tensored Waldhausen category. We prove the following lemmas.

LEMMA 4.1.3. *Under the hypotheses of the preceding paragraph, the inclusion of* $THH(v_\bullet S_\bullet \mathcal{A}_0^\Gamma)$ *in* $THH(v_\bullet S_\bullet \mathcal{B}_0^\Gamma)$ *is a weak equivalence.*

LEMMA 4.1.4. *Under the hypotheses of the preceding paragraph, the inclusion of* $THH(v_\bullet S_\bullet \mathcal{A}_0^\Gamma)$ *in* $THH(\mathcal{A}|v)$ *is a weak equivalence.*

These two lemmas then immediately imply Theorem 4.1.1.

Proof of Lemma 4.1.3. Writing $\Sigma$ for suspension and $C$ for cone, we have a cofiber sequence of enriched exact functors

$$\mathrm{Id} \longrightarrow C \longrightarrow \Sigma$$

on each $v_m\mathcal{B}_0$, and so it follows from Corollary 3.1.2 that

$$\mathrm{Id} \vee \Sigma, C \colon WTHH^{\Gamma}(v_\bullet \mathcal{B}_0) \longrightarrow WTHH^{\Gamma}(v_\bullet \mathcal{B}_0)$$

induce the same map in the stable category. On the other hand, using a simplicial contraction, it is easy to see that $C$ induces the trivial map. Thus,

$$\Sigma \colon THH(v_\bullet S_\bullet \mathcal{B}_0) \longrightarrow THH(v_\bullet S_\bullet \mathcal{B}_0)$$

is a weak equivalence. Similarly,

$$\Sigma \colon THH(v_\bullet S_\bullet \mathcal{A}_0) \longrightarrow THH(v_\bullet S_\bullet \mathcal{A}_0)$$

is a weak equivalence. Since the canonical map

$$\mathrm{colim}_\Sigma \mathcal{A}_0 \longrightarrow \mathrm{colim}_\Sigma \mathcal{B}_0$$

is an isomorphism, the lemma now follows from Proposition 1.2.7. $\square$

Proof of Lemma 4.1.4. Let $\bar{v}\mathcal{A}_0$ denote the subcategory of $v\mathcal{A}_0$ consisting of those maps that are also degreewise split monomorphisms. Then by [**16**, 1.3.9] and Proposition 2.3.10, it suffices to show that the inclusion

$$THH(\bar{v}_\bullet S_\bullet \mathcal{A}_0^\Gamma) \longrightarrow THH(\bar{v}_\bullet S_\bullet \mathcal{A}^\Gamma)$$

is a weak equivalence. Since $S_n\mathcal{B}_0$ and $vS_n\mathcal{B}_0$ satisfy the same hypotheses as $\mathcal{B}_0$, without loss of generality, it suffices to show that the inclusion

$$THH(\bar{v}_\bullet \mathcal{A}_0^\Gamma) \longrightarrow THH(\bar{v}_\bullet \mathcal{A}^\Gamma)$$

is a weak equivalence. By Proposition 1.2.8, it suffices to show that each degeneracy map

$$THH(\bar{v}_\bullet \mathcal{A}_0^\Gamma) \longrightarrow THH(\bar{v}_\bullet \mathcal{A}_n^\Gamma)$$

is a weak equivalence, which we do using an argument similar to the proof of Theorem 2.1.9.

Let $s\colon \mathcal{A}_0 \to \mathcal{A}_n$ denote the iterated degeneracy and let $d\colon \mathcal{A}_n \to \mathcal{A}_0$ denote the iterated last face map. The composite functor $d\circ s$ is the identity and so induces the identity map

$$THH(\bar{v}_\bullet \mathcal{A}_0^\Gamma) \longrightarrow THH(\bar{v}_\bullet \mathcal{A}_n^\Gamma).$$

We show that the composite $s \circ d$ is homotopic to the identity map. We have a map of simplicial sets

$$c\colon \Delta[n] \times \Delta[1] \longrightarrow \Delta[n]$$

that is a null homotopy from the identity map to the inclusion of the last vertex. We can use this to construct an exact functor $c\colon \mathcal{A}_n \to \mathcal{A}_n$ as follows. Regarding an element of $f \in \mathcal{A}_n(x,y)$ as a map $\tilde{f}\colon x \otimes \Delta[n] \to y$ in $\mathcal{A}_0$, we let $c(f) \in \mathcal{A}_n(x \otimes \Delta[1], y \otimes \Delta[1])$ be the element represented by the map

$$(x \otimes \Delta[n]) \otimes \Delta[1] \cong x \otimes (\Delta[n] \times \Delta[1]) \longrightarrow y \otimes \Delta[1]$$

in $\mathcal{A}_0$ induced by $\tilde{f}$, $c$, and the diagonal map on $\Delta[1]$. This then extends to a simplicial spectral functor

$$c_\bullet \colon \bar{v}_\bullet \mathcal{A}_n^\Gamma \longrightarrow \bar{v}_\bullet \mathcal{A}_n^\Gamma.$$

We construct two simplicial homotopies $H_0, H_1$ of simplicial spectral functors using the two inclusions $\partial_0, \partial_1$ of $\Delta[0]$ in $\Delta[1]$: On objects,

$$x_1 \longrightarrow \cdots \longrightarrow x_n$$

in $\operatorname{Ob} v_n\mathcal{A}$ is sent to

$$x_1 \longrightarrow \cdots \longrightarrow x_i \longrightarrow x_i \otimes \Delta[1] \longrightarrow \cdots \longrightarrow x_n \otimes \Delta[1]$$

in $\operatorname{Ob} v_n\mathcal{A}$ where the map $x_i \to x_i \otimes \Delta[1]$ is $\partial_0$ for $H_0$ and $\partial_1$ for $H_1$. On morphisms, $H_0$ sends

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & x_n \arrow[d, "f_n"'] \\ y_1 \arrow[r] & \cdots \arrow[r] & y_n \end{tikzcd}$$

to

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & x_i \arrow[r, "\partial_0"] \arrow[d, "f_i"'] & x_i \otimes \Delta[1] \arrow[r] \arrow[d, "c(f_i)"'] & \cdots \arrow[r] & x_n \otimes \Delta[1] \arrow[d, "c(f_n)"'] \\ y_1 \arrow[r] & \cdots \arrow[r] & y_i \arrow[r] & y_i \otimes \Delta[1] \arrow[r] & \cdots \arrow[r] & y_n \otimes \Delta[1] \end{tikzcd}$$

and $H_1$ sends it to

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d, "s(d(f_1))"'] & \cdots \arrow[r] & x_i \arrow[r, "\partial_1"] \arrow[d, "s(d(f_i))"'] & x_i \otimes \Delta[1] \arrow[r] \arrow[d, "c(f_i)"'] & \cdots \arrow[r] & x_n \otimes \Delta[1] \arrow[d, "c(f_n)"'] \\ y_1 \arrow[r] & \cdots \arrow[r] & y_i \arrow[r] & y_i \otimes \Delta[1] \arrow[r] & \cdots \arrow[r] & y_n \otimes \Delta[1]. \end{tikzcd}$$

Then $H_0$ is a simplicial homotopy of spectral functors from the identity to $c_\bullet$ and $H_1$ is a simplicial homotopy of spectral functors from $s \circ d$ to $c_\bullet$. □

We now move on to the proof of Theorem 4.1.2. Let $\mathcal{A}$ denote the category of perfect simplicial $R$-modules and now let $\mathcal{C}$ denote the category of finite cell EKMM $HR$-modules. Having proved Theorem 4.1.1, for the proof of Theorem 4.1.2, we just need to produce compatible zigzags of weak equivalences

$$WTHH^\Gamma(\mathcal{A}^v) \simeq WTHH^\Gamma(\mathcal{C}^v)$$
$$WTHH^\Gamma(\mathcal{A}) \simeq WTHH^\Gamma(\mathcal{C})$$
$$WTHH^\Gamma(\mathcal{A}|v) \simeq WTHH^\Gamma(\mathcal{C}|v).$$

Let $\mathcal{M}$ denote the full subcategory of EKMM $HR$-modules that are compact in the derived category and whose underlying spectra satisfy a cardinality bound (for any limit cardinal large enough that $\mathcal{C} \subset \mathcal{M}$). The inclusion of $\mathcal{C}$ in $\mathcal{M}$ then induces compatible weak equivalences

$$WTHH^\Gamma(\mathcal{C}^v) \xrightarrow{\simeq} WTHH^\Gamma(\mathcal{M}^v)$$
$$WTHH^\Gamma(\mathcal{C}) \xrightarrow{\simeq} WTHH^\Gamma(\mathcal{M})$$
$$WTHH^\Gamma(\mathcal{C}|v) \xrightarrow{\simeq} WTHH^\Gamma(\mathcal{M}|v),$$

We will in fact compare $THH$ of the $\mathcal{A}$ categories with $THH$ of the $\mathcal{M}$ categories. We use the functor denoted $\mathbb{M}$ in [**18**, §I.7] to construct a simplicially enriched functor $\mathcal{A} \to \mathcal{M}$ as follows.

Implicitly we are working with the standard model of $HR$ as a commutative EKMM $S$-algebra, constructed as follows. The usual Eilenberg-Mac Lane spectrum $HR$ has as its $n$-th space

$$R \otimes \tilde{\mathbb{Z}}[S^n]$$

as in Example 2.2.4; this spectrum is canonically a commutative ring orthogonal spectrum and the canonical commutative EKMM $S$-algebra $\mathbb{M}$ of this (i.e., $S \wedge_{\mathcal{L}} (-)$ applied to its Lewis-May spectrification). As $\mathbb{M}$ is a lax monoidal functor, for any (discrete) $R$-module $M$, the standard EKMM $S$-module $HM$ is $\mathbb{M}$ of the spectrum

$$M \otimes \tilde{\mathbb{Z}}[S^n]$$

and is canonically an $HA$-module. For a simplicial $R$-module $M$, geometric realization commutes with $\mathbb{M}$, and we obtain a simplicial functor $M$ from simplicial $A$-modules to EKMM $HA$-modules.

Because the construction $(-)\otimes\tilde{\mathbb{Z}}[S^{(-)}]$ does not preserve coproducts or pushouts, $M \colon \mathcal{A} \to \mathcal{M}$ is not an exact functor. But it does preserve coproducts up to weak equivalence and homotopy pushouts, so it is a *weakly exact* functor. It is also *based* in that it sends 0 to $*$ (after perhaps modifying it by an isomorphism). Theorem 5.1.1 and the work of Section 5.1 below then produces compatible zigzags of maps of cyclotomic spectra

$$\begin{gathered} WTHH^{\Gamma}(\mathcal{A}^v) \longrightarrow WTHH^{\Gamma}(\mathcal{M}^v) \\ WTHH^{\Gamma}(\mathcal{A}) \longrightarrow WTHH^{\Gamma}(\mathcal{M}) \\ WTHH^{\Gamma}(\mathcal{A}|v) \longrightarrow WTHH^{\Gamma}(\mathcal{M}|v). \end{gathered}$$

Since $M$ induces DK-equivalences

$$\begin{gathered} S_n\mathcal{A}^v \longrightarrow S_n\mathcal{M}^v \\ S_n\mathcal{A} \longrightarrow S_n\mathcal{M} \\ v_m^M S_n\mathcal{A} \longrightarrow v_m^M S_n\mathcal{M}, \end{gathered}$$

the zigzags above consist of weak equivalences. This completes the sketch proof of Theorem 4.1.2.

## 4.2. The localization sequence for $THH(ku)$

The main result of this chapter is the following theorem conjectured by Hesselholt and Ausoni-Rognes.

THEOREM 4.2.1. *The transfer maps and the canonical maps fit into cofiber sequences of cyclotomic spectra*

$$\begin{gathered} THH(\mathbb{Z}_p^{\wedge}) \longrightarrow THH(\ell_p^{\wedge}) \longrightarrow WTHH^{\Gamma}(\ell_p^{\wedge}|L_p^{\wedge}) \longrightarrow \Sigma THH(\mathbb{Z}_p^{\wedge}) \\ THH(\mathbb{Z}_{(p)}) \longrightarrow THH(\ell) \longrightarrow WTHH^{\Gamma}(\ell|L) \longrightarrow \Sigma THH(\mathbb{Z}_{(p)}) \\ THH(\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow WTHH^{\Gamma}(ku|KU) \longrightarrow \Sigma THH(\mathbb{Z}) \end{gathered}$$

*inducing cofiber sequences*

$$\begin{gathered} TC(\mathbb{Z}_p^{\wedge}) \longrightarrow TC(\ell_p^{\wedge}) \longrightarrow WTC^{\Gamma}(\ell_p^{\wedge}|L_p^{\wedge}) \longrightarrow \Sigma TC(\mathbb{Z}_p^{\wedge}) \\ TC(\mathbb{Z}_{(p)}) \longrightarrow TC(\ell) \longrightarrow WTC^{\Gamma}(\ell|L) \longrightarrow \Sigma TC(\mathbb{Z}_{(p)}) \\ TC(\mathbb{Z}) \longrightarrow TC(ku) \longrightarrow WTC^{\Gamma}(ku|KU) \longrightarrow \Sigma TC(\mathbb{Z}) \end{gathered}$$

*which are compatible via the cyclotomic trace with the corresponding cofiber sequences in algebraic $K$-theory constructed in* [**5**].

Here $WTHH^{\Gamma}(ku|KU)$ denotes the connective $THH$ of the category of finite cell $ku$-modules with the spectral enrichment induced by the canonical mapping spaces in $ku$ but weak equivalences the $KU$-equivalences. That is,

$$WTHH^{\Gamma}(ku|KU) = WTHH^{\Gamma}(\mathcal{C}_{ku}|v) = \Omega|THH(v_{\bullet}^{M} S_{\bullet}\mathcal{C}_{ku}^{\Gamma})|,$$

where $\mathcal{C}_{ku}$ is the category of finite cell EKMM $ku$-modules (as in Example 2.1.5.(i)) and $v\mathcal{C}_{ku}$ is the collection of maps $M \to N$ such that $M \wedge_{ku} KU \to N \wedge_{ku} KU$ is an equivalence, or equivalently, those maps that induce an isomorphism on homotopy groups after inverting the action of the Bott element.

The proof of this theorem follows the same general outline as the proof of the corresponding result in algebraic $K$-theory [**5**]. In particular, the localization theorem follows from a "dévissage" theorem for finitely generated finite stage Postnikov towers. We now give the definitions necessary to state this theorem. Throughout, we work with EKMM $S$-algebras and $S$-modules.

For an $S$-algebra $R$, let $\mathcal{P}_R$ denote the full subcategory of left $R$-modules that are of the homotopy type of cell $R$-modules and have only finitely many nonzero homotopy groups, all of which are finitely generated over $\pi_0 R$. We give $\mathcal{P}_R$ the structure of a simplicially tensored Waldhausen category as follows. For the simplicial structure, we use the usual simplicial enrichment obtained by regarding the category of $R$-modules as a simplicial model category. For the Waldhausen category structure, we take the weak equivalences to be the usual weak equivalences and the cofibrations to be the Hurewicz cofibrations, i.e., the maps satisfying the homotopy extension property in the category of $R$-modules. As we described in [**5**, §1], this gives $\mathcal{P}_R$ the structure of a Waldhausen category, and the pushout-product axiom on the tensors follows from [**13**, X.2.3]. (Techniques to make a version of $\mathcal{P}_R$ that is a small category are discussed in [**5**, 1.7].)

Restricting to the subcategory of the category of $S$-algebras with morphisms the maps $R \to R'$ for which $\pi_0 R'$ is finitely generated as a left $\pi_0 R$-module, we can regard $WTHH^{\Gamma}(\mathcal{P}_{(-)})$ as a contravariant functor to the homotopy category of cyclotomic spectra. We can now state the Dévissage Theorem.

THEOREM 4.2.2 (Dévissage Theorem). *Let $R$ be a connective $S$-algebra with $\pi_0 R$ left Noetherian. Then there is a natural isomorphism in the homotopy category of cyclotomic spectra $THH(\mathcal{E}^{fg}_{\pi_0 R}) \to WTHH^{\Gamma}(\mathcal{P}_R)$, where $\mathcal{E}^{fg}_{\pi_0 R}$ denotes the exact category of finitely generated left $\pi_0 R$-modules. Moreover, this isomorphism and the induced isomorphism (in the stable category) on $TC$ are compatible via the cyclotomic trace with the analogous isomorphism (in the stable category) on algebraic $K$-theory $K'(\pi_0 R) \to K'(R)$ in the Dévissage Theorem of* [**5**].

We prove Theorem 4.2.2 in the next section and use the rest of this section to prove Theorem 4.2.1 from Theorem 4.2.2. Let $R$ be one of $ku$, $\ell$, or $\ell_p^{\wedge}$, and let $\beta$ denote the appropriate Bott element in $\pi_* R$ in degree 2 or $2p-2$. Then $R[\beta^{-1}]$ is $KU$, $L$, or $L_p^{\wedge}$ respectively. For convenience, let $Z$ denote $\pi_0 R$; so $Z = \mathbb{Z}$, $\mathbb{Z}_{(p)}$, or $\mathbb{Z}_p^{\wedge}$ in the respective cases. As above we write $\mathcal{C}_A$ for the simplicially tensored Waldhausen category of finite cell $A$-modules (where $A = HZ$, $R$, or $R[\beta^{-1}]$). On $\mathcal{C}_R$ we have the additional weak equivalences $v\mathcal{C}_R$, the maps that induce an isomorphism on homotopy groups after inverting the action of the Bott element. Since $v\mathcal{C}_R$ contains the usual weak equivalences $w\mathcal{C}_R$, the hypothesis of the Localization Theorem (Theorem 3.3.1) applies and we get a cofibration sequence of cyclotomic spectra

$$WTHH^{\Gamma}(\mathcal{C}_R^v) \longrightarrow WTHH^{\Gamma}(\mathcal{C}_R) \longrightarrow WTHH^{\Gamma}(\mathcal{C}_R|v) \longrightarrow \Sigma WTHH^{\Gamma}(\mathcal{C}_R^v),$$

compatible with the analogous sequence in $K$-theory via the cyclotomic trace. Corollary 3.4.4 identifies $WTHH^{\Gamma}(\mathcal{C}_R)$ with $THH(R)$, compatibly with the cyclotomic trace. The inclusion of the $v$-acyclics $\mathcal{C}_R^v$ into the simplicially tensored Waldhausen category $\mathcal{P}_R$ described above is a tensored exact functor and a DK-equivalence. Thus, Theorem 4.2.2 identifies $\mathcal{C}_R^v$ as $THH(Z)$, compatibly with the identification of $K(\mathcal{C}_R^v)$ with $K(HZ)$.

This completes most of the proof of Theorem 4.2.1; it just remains to identify the map

$$THH(Z) \simeq WTHH^{\Gamma}(\mathcal{C}_R^v) \longrightarrow WTHH^{\Gamma}(\mathcal{C}_R) \simeq THH(R)$$

in terms of the transfer map $THH(HZ) \to THH(R)$. First, we review this transfer map. In our current context with $R = ku$, $\ell$, or $\ell_p^{\wedge}$, the Eilenberg-Mac Lane $R$-module $HZ$ is weakly equivalent to a finite cell $R$-module. If we choose a model for $HZ$ as a cofibrant associative $R$-algebra, then finite cell $HZ$-modules are cell $R$-modules and homotopy equivalent to finite cell $R$-modules. Let $\mathcal{M}_R^c$ be the simplicially tensored Waldhausen category whose objects are the $R$-modules that are homotopy equivalent to finite cell $R$-modules with the usual simplicial sets of maps, with the usual weak equivalences, and with cofibrations the Hurewicz cofibrations (using the technique of [**5**, 1.7] to make a version that is a small category). Then $\mathcal{P}_R$ is a closed Waldhausen subcategory of $\mathcal{M}_R^c$; moreover, the inclusion of $\mathcal{C}_R$ in $\mathcal{M}_R^c$ is tensored exact and a DK-equivalence, and so induces an equivalence on all versions of $THH$. We also have the analogous category $\mathcal{M}_{HZ}^c$ for $HZ$, which coincides with $\mathcal{P}_{HZ}$. The forgetful functor from $HZ$-modules to $R$-modules is a tensored exact functor $\mathcal{M}_{HZ}^c \to \mathcal{M}_R^c$. The transfer map $THH(HZ) \to THH(R)$ is by definition the map

$$\tau_{HZ}^R \colon THH(HZ) \longrightarrow THH(\mathcal{M}_R^c) \xleftarrow{\sim} THH(R),$$

where the map on the right is induced by the inclusion of $S_{HZ}$ in $\mathcal{M}_R^c$ and the map of endomorphism spectra

$$\mathcal{C}_{HZ}^S(S_{HZ}, S_{HZ}) = (\mathcal{M}_{HZ}^c)^S(S_{HZ}, S_{HZ}) \longrightarrow (\mathcal{M}_R^c)^S(S_{HZ}, S_{HZ}).$$

(We understand $THH$ of the EKMM $S$-algebra $HZ$ as $THH$ of the symmetric ring spectrum $\mathcal{C}_{HZ}^S(S_{HZ}, S_{HZ})$; cf. Corollary 3.4.4 and the remarks that follow it.)

Since the transfer map coincides with the map

$$THH(HZ) \xrightarrow{\sim} THH(\mathcal{M}_{HZ}^c) \longrightarrow THH(\mathcal{M}_R^c) \xleftarrow{\sim} THH(R),$$

applying Corollary 3.4.4 and naturality, we can also identify it as the map

$$THH(HZ) \simeq WTHH^{\Gamma}(\mathcal{M}_{HZ}^c) \longrightarrow WTHH^{\Gamma}(\mathcal{M}_R^c) \simeq THH(R).$$

Using the naturality of the isomorphism in Theorem 4.2.2, we obtain the following commutative diagram of maps in the homotopy category of cyclotomic spectra.

$$
\begin{tikzcd}
THH(Z) \arrow[r, "\sim"] \arrow[d, "="'] & WTHH^{\Gamma}(\mathcal{P}_{HZ}) \arrow[r, "="] \arrow[d] & WTHH^{\Gamma}(\mathcal{M}^{c}_{HZ}) \arrow[d] & THH(HZ) \arrow[l, "\sim"'] \arrow[d, dotted, "\tau^{R}_{HZ}"] \\
THH(Z) \arrow[r, "\sim"] & WTHH^{\Gamma}(\mathcal{P}_{R}) \arrow[r] & WTHH^{\Gamma}(\mathcal{M}^{c}_{R}) & THH(R) \arrow[l, "\sim"'] \arrow[dl, "\sim"'] \\
& WTHH^{\Gamma}(\mathcal{C}^{v}_{R}) \arrow[r] \arrow[u, "\sim"] & WTHH^{\Gamma}(\mathcal{C}_{R}) \arrow[u, "\sim"] &
\end{tikzcd}
$$

It will be obvious from the proof of Theorem 4.2.2 in the next section that the isomorphism $THH(Z) \simeq THH(HZ)$ in the top row of the diagram is the standard one, and this identifies the map $THH(Z) \to THH(R)$ as the transfer map. This completes the proof of Theorem 4.2.1.

## 4.3. Proof of the Dévissage Theorem

This section is devoted to the proof of the Dévissage Theorem, Theorem 4.2.2. The argument parallels the analogous dévissage theorem in [**5**], which we review along the way.

We fix the connective $S$-algebra $R$, writing $\mathcal{P}$ for $\mathcal{P}_R$ . Let $\mathcal{P}^n_m$ denote the full subcategory of $\mathcal{P}$ consisting of those $R$-modules whose homotopy groups $\pi_q$ are zero for $q > n$ or $q < m$. In this notation, we permit $m = -\infty$ and/or $n = \infty$, so $\mathcal{P} = \mathcal{P}^{\infty}_{-\infty}$. The categories $\mathcal{P}^n_m$ are closed Waldhausen subcategories of $\mathcal{P}_R$. The following theorem proved below parallels [**5**, 1.2].

THEOREM 4.3.1. *The inclusion $\mathcal{P}^0_0 \to \mathcal{P}$ induces a weak equivalence*

$$WTHH^{\Gamma}(\mathcal{P}^0_0) \longrightarrow WTHH^{\Gamma}(\mathcal{P}).$$

The point of the previous theorem is that $\pi_0$ provides an exact functor from $\mathcal{P}^0_0$ to the exact category of finitely generated left $\pi_0 R$-modules $\mathcal{E}^{fg}_R$. Theorem 1.3 of [**5**] proves that this functor induces a weak equivalence of $K$-theory. Since the simplicial mapping sets for $\mathcal{E}^{fg}_R$ are discrete, $\pi_0$ is also a simplicially enriched functor $\mathcal{P}^0_0 \to \mathcal{E}^{fg}_R$. It is in fact a DK-equivalence and induces a DK-equivalence $S_n\mathcal{P}^0_0 \to S_n\mathcal{E}^{fg}_R$ for all $n$. This proves the following theorem, which parallels [**5**, 1.3].

THEOREM 4.3.2. *The functor $\pi_0\colon \mathcal{P}^0_0 \to \mathcal{E}^{fg}_R$ induces a weak equivalence*

$$WTHH^{\Gamma}(\mathcal{P}^0_0) \longrightarrow WTHH^{\Gamma}(\mathcal{E}^{fg}_R) = THH(\mathcal{E}^{fg}_R).$$

Theorem 4.2.2 is an immediate consequence of the previous two theorems, with the natural isomorphism coming from the natural zigzag of weak equivalences of cyclotomic spectra

$$THH(\mathcal{E}^{fg}_R) = WTHH^{\Gamma}(\mathcal{E}^{fg}_R) \longleftarrow WTHH^{\Gamma}(\mathcal{P}^0_0) \longrightarrow WTHH^{\Gamma}(\mathcal{P}).$$

Thus, it remains to prove Theorem 4.3.1.

The proof of Theorem 4.3.1 follows the same outline as the parallel theorem [**5**, 1.2]. As in the argument there, we have the following two easy observations.

PROPOSITION 4.3.3. *The inclusion $\mathcal{P}^{\infty}_0 \to \mathcal{P}$ induces an equivalence*

$$WTHH^{\Gamma}(\mathcal{P}^{\infty}_0) \longrightarrow WTHH^{\Gamma}(\mathcal{P}).$$

Proposition 4.3.4. *The cyclotomic spectrum $WTHH^{\Gamma}(\mathcal{P}_0^{\infty})$ is weakly equivalent to the telescope of the sequence of maps*

$$WTHH^{\Gamma}(\mathcal{P}_0^{0}) \longrightarrow \cdots \longrightarrow WTHH^{\Gamma}(\mathcal{P}_0^{n}) \longrightarrow WTHH^{\Gamma}(\mathcal{P}_0^{n+1}) \longrightarrow \cdots .$$

As in [**5**], the proof of Theorem 4.3.1 will then be completed by showing that the maps

$$WTHH^{\Gamma}(\mathcal{P}_0^{n}) \longrightarrow WTHH^{\Gamma}(\mathcal{P}_0^{n+1})$$

are weak equivalences for all $n \geq 0$. Applying Proposition 3.2.6 and Theorem 3.2.2, this is equivalent to proving the following lemma.

Lemma 4.3.5. $WTHH^{\Gamma}(\mathcal{P}_0^{n} \to \mathcal{P}_0^{n+1}) \simeq \Omega|THH(S_\bullet F_\bullet(\mathcal{P}_0^{n+1}, \mathcal{P}_0^{n})^{\Gamma})|$ *is weakly contractible.*

In [**5**] the proof of the parallel (unnumbered) lemma consisted of several steps, each of which compared (multi)simplicial sets; the following diagram outlines the comparisons as stated there.

$$\begin{tikzcd}
w_\bullet S_\bullet F_\bullet(\mathcal{P}_0^{n+1}, \mathcal{P}_0^{n}) \arrow[dr] & & u_\bullet S_\bullet \mathcal{P}_0^{n+1} \arrow[dl] \arrow[r] & u_\bullet S'_\bullet \mathcal{P}_0^{n+1} \arrow[r, dotted] \arrow[d, dotted] & u_\bullet M_\bullet Z \\
& w_\bullet u_\bullet S_\bullet \mathcal{P}_0^{n+1} & & u_\bullet F^{f}_{\bullet-1} \mathcal{P}_0^{n+1} \arrow[ur, dotted] & u_\bullet F^{f}_{\bullet-1} \mathcal{P}_{n+1}^{n+1} \arrow[l, dotted] \arrow[u, dotted]
\end{tikzcd}$$

We review these constructions as needed below. Here the solid arrows are simplicial maps of diagonal simplicial sets and the dotted arrows are maps that are simplicial only in one of the simplicial directions. We correct a minor error in [**5**] below. There we claimed that the dotted arrow in the top row was a map of bisimplicial sets; it is not. The diagram for the corrected argument looks like this; it commutes up to simplicial homotopy.

$$\begin{tikzcd}
w_\bullet S_\bullet F_\bullet(\mathcal{P}_0^{n+1}, \mathcal{P}_0^{n}) \arrow[dr] & & u_\bullet S_\bullet \mathcal{P}_0^{n+1} \arrow[dl] \arrow[r] & u_\bullet S'_\bullet \mathcal{P}_0^{n+1} \arrow[r] \arrow[d, dotted] & u_\bullet S^{f} M_\bullet Z & u_\bullet M_\bullet Z \arrow[l] \\
& w_\bullet u_\bullet S_\bullet \mathcal{P}_0^{n+1} & & u_\bullet F^{f}_{\bullet-1} \mathcal{P}_0^{n+1} \arrow[urr, dotted] & & u_\bullet F^{f}_{\bullet-1} \mathcal{P}_{n+1}^{n+1} \arrow[ll, dotted] \arrow[u, dotted]
\end{tikzcd}$$

In the current context of $THH$, the line of reasoning and the diagram simplifies slightly; we use the following diagram of spectrally enriched functors, which commutes up to natural isomorphism.

(4.3.6)
$$\begin{tikzcd}
(S_p F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^{n}))^{\Gamma} \arrow[r, "\sim"] & ((u_q S'_p)^{M} \mathcal{P}_0^{n+1})^{\Gamma} \arrow[r] \arrow[d, dotted, "\sim"'] & (u_q S^{f} M_p Z)^{\Gamma} & (u_q M_p Z)^{\Gamma} \arrow[l, "\sim"'] \\
& ((u_q F^{f}_{p-1})^{M} \mathcal{P}_0^{n+1})^{\Gamma} \arrow[urr, dotted] & & ((u_q F^{f}_{p-1})^{M} \mathcal{P}_{n+1}^{n+1})^{\Gamma} \arrow[ll, dotted] \arrow[u, dotted, "\sim"']
\end{tikzcd}$$

All of the spectral categories fit into simplicial spectral categories (in the $q$ direction) and the ones on the top row fit into bisimplicial spectral categories (in $p, q$). The solid arrows are the spectrally enriched functors that respect the bisimplicial structure; the dotted arrows respect the simplicial structure in the $q$ direction. The arrows marked "$\sim$" are DK-equivalences, as shown in Propositions 4.3.8, 4.3.9, 4.3.13, and 4.3.16. The goal is to show that the composite functor $(S_pF_q(\mathcal{P}_0^{n+1},\mathcal{P}_0^n))^\Gamma \to (u_qS^fM_pZ)^\Gamma$ induces a weak equivalence

$$|THH((S_\bullet F_\bullet(\mathcal{P}_0^{n+1},\mathcal{P}_0^n))^\Gamma)| \longrightarrow |THH((u_\bullet S^f M_\bullet Z)^\Gamma)| \simeq |THH((u_\bullet M_\bullet Z)^\Gamma)|$$

and then prove Lemma 4.3.5 by showing that $|THH((u_\bullet M_\bullet Z)^\Gamma)|$ is contractible (Proposition 4.3.12).

We now begin to review the categories and maps in diagram (4.3.6). We use the following notation.

Definition 4.3.7. Let $u\mathcal{P}$ denote the subcategory of $\mathcal{P}$ consisting of those maps that induce an isomorphism on $\pi_{n+1}$ and an injection on $\pi_n$. Let $f\mathcal{P}$ denote the subcategory of $\mathcal{P}$ consisting of those maps that induce an epimorphism on $\pi_0$.

We write $u_\bullet\mathcal{P}$ for the nerve categories: An object of $u_q\mathcal{P}$ is a sequence of $q$ composable maps in $u\mathcal{P}$ and a map in $u_q\mathcal{P}$ is a commuting diagram (of maps in $\mathcal{P}$). For consistency with [**5**, 3.7], we let $F_p^f\mathcal{P}$ denote the nerve category $f_p\mathcal{P}$: An object is a sequence of $p$ composable maps in $f\mathcal{P}$ and a map is a commuting diagram (of maps in $\mathcal{P}$). We extend the definition of $u_\bullet$ in the obvious way to functor categories: In diagram (4.3.6), the category $u_qS_p\mathcal{P}_0^{n+1}$ has as objects the sequences of $q$-composable maps

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_q} A_q$$

between objects $A_i$ in $S_p\mathcal{P}_0^{n+1}$ where each $\alpha_i$ is (objectwise) in $u\mathcal{P}_0^{n+1}$; a map from $\{\alpha_i\}$ to $\{\alpha_i'\}$ consists of a map $\phi_i\colon A_i \to A_i'$ in $S_p'\mathcal{P}_0^{n+1}$ for each $i$, making the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha_1"] \arrow[d, "\phi_0"'] & A_1 \arrow[r, "\alpha_2"] \arrow[d, "\phi_1"'] & \cdots \arrow[r, "\alpha_q"] & A_q \arrow[d, "\phi_p"] \\
A_0' \arrow[r, "\alpha_1'"'] & A_1' \arrow[r, "\alpha_2'"'] & \cdots \arrow[r, "\alpha_q'"'] & A_q'
\end{tikzcd}$$

in $S_p'\mathcal{P}_0^{n+1}$ commute. We define the categories $u_qF_{p-1}^f\mathcal{P}_0^{n+1}$ and $u_qF_{p-1}^f\mathcal{P}_{n+1}^{n+1}$ analogously in terms of composable maps and diagrams in $F_{p-1}^f\mathcal{P}_0^{n+1}$ and $F_{p-1}^f\mathcal{P}_{n+1}^{n+1}$. We obtain the spectrally enriched categories $((u_qS_p')^M\mathcal{P}_0^{n+1})^\Gamma$, $((u_qF_{p-1}^f)^M\mathcal{P}_0^{n+1})^\Gamma$, and $((u_qF_{p-1}^f)^M\mathcal{P}_{n+1}^{n+1})^\Gamma$ using the Moore Tot mapping spaces (Construction 2.4.4) and the connective spectral enrichment. The usual face and degeneracy maps in the nerve construction makes $((u_\bullet S_\bullet')^M\mathcal{P}_0^{n+1})^\Gamma$ into a bisimplicial spectral category and make $((u_\bullet F_{p-1}^f)^M\mathcal{P}_0^{n+1})^\Gamma$ and $((u_\bullet F_{p-1}^f)^M\mathcal{P}_{n+1}^{n+1})^\Gamma$ into simplicial spectral categories for each $p > 0$.

Next we review the canonical inclusion

$$F_q(\mathcal{P}_0^{n+1},\mathcal{P}_0^n) \longrightarrow u_q\mathcal{P}_0^{n+1}.$$

We recall that an object of $F_q(\mathcal{P}_0^{n+1},\mathcal{P}_0^n)$ consists of a sequence of $q$ composable cofibrations in $\mathcal{P}_0^{n+1}$

$$x_0 \rightarrowtail x_1 \rightarrowtail \cdots \rightarrowtail x_q$$

such that each quotient $x_{i+1}/x_i$ is in $\mathcal{P}_0^n$. We note that for a cofibration $j\colon a \to b$ in $\mathcal{P}$ between objects of $\mathcal{P}_0^{n+1}$, the quotient $b/a$ is in $\mathcal{P}_0^{n+1}$ if and only if $j$ induces an isomorphism on $\pi_{n+1}$ and an injection on $\pi_n$, that is, if and only if $j$ is in $u\mathcal{P}$.

It follows that $F_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)$ is the full subcategory of $u_q\mathcal{P}_0^{n+1}$ consisting of those objects whose structure maps are cofibrations. We then obtain the functors

$$S_pF_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) \longrightarrow u_qS'_p\mathcal{P}_0^{n+1}$$

as the corresponding inclusions of full subcategories. When we look at mapping spaces and use the Moore enrichment, we obtain a DK-embedding

$$S_pF_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n) \longrightarrow (u_qS'_p)^M\mathcal{P}_0^{n+1}.$$

This map is a DK-equivalence since the usual cylinder argument replacing a map with a cofibration converts any diagram in $u_qS'_p\mathcal{P}_0^{n+1}$ to a weakly equivalent diagram in $S_pF_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)$. Passing to the connective spectral enrichments, we obtain the following proposition.

Proposition 4.3.8. *The spectrally enriched functor*

$$S_pF_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma \longrightarrow ((u_qS'_p)^M\mathcal{P}_0^{n+1})^\Gamma$$

*is a DK-equivalence.*

Next we review the functor $S'_p\mathcal{P}_0^{n+1} \to F^f_{p-1}\mathcal{P}_0^{n+1}$ of [**5**, 3.8]. First note that for an object $A = \{a_{i,j}\}$ in $S'_p\mathcal{P}_0^{n+1}$, the map $a_{i,p} \to a_{j,p}$ is the cofiber of the map $a_{i,j} \to a_{i,p}$ and so we have a long exact sequence of homotopy groups

$$0 \longrightarrow \pi_{n+1}a_{i,j} \longrightarrow \cdots \longrightarrow \pi_0a_{i,j} \longrightarrow \pi_0a_{i,p} \longrightarrow \pi_0a_{j,p} \longrightarrow 0.$$

In particular, the map $a_{i,p} \to a_{j,p}$ is surjective on $\pi_0$, that is, is a map in $f\mathcal{P}_0^{n+1}$. We therefore obtain a functor $S'_p\mathcal{P}_0^{n+1} \to F^f_{p-1}\mathcal{P}_0^{n+1}$ by sending each object of $S'_p\mathcal{P}_0^{n+1}$ to the object of $F^f_{p-1}\mathcal{P}_0^{n+1}$ defined by the sequence

$$a_{0,p} \longrightarrow a_{1,p} \longrightarrow \cdots \longrightarrow a_{p-1,p}.$$

In fact we have the following proposition.

Proposition 4.3.9. *The spectrally enriched functor*

$$((u_qS'_p)^M\mathcal{P}_0^{n+1})^\Gamma \longrightarrow ((u_qF^f_{p-1})^M\mathcal{P}_0^{n+1})^\Gamma$$

*is a DK-equivalence.*

Proof. Although $S'_p\mathcal{P}_0^{n+1}$ is defined in terms of homotopy cocartesian squares, it could equally well be defined in terms of homotopy cartesian squares since for EKMM $R$-modules a square is homotopy cartesian if and only if it is homotopy cocartesian. The description of the mapping space of $S_p\mathcal{P}$ in (2.3.2) has an analogue in this context: The canonical map from $S'^M_p\mathcal{P}$ to the iterated homotopy pullback

$$\mathcal{P}(a_{0,p}, b_{0,p}) \times^h_{\mathcal{P}(a_{0,p},b_{1,p})} \cdots \times^h_{\mathcal{P}(a_{p-2,p},b_{p-1,p})} \mathcal{P}(a_{p-1,p}, b_{p-1,p}).$$

is a weak equivalence. This extends to $(u_qS'_p)^M\mathcal{P}$ and from this it is easy to deduce that we have a DK-embedding. It is a DK-equivalence because every object of $u_qF^f_{p-1}\mathcal{P}_0^{n+1}$ is weakly equivalent to the image of an object in $u_qS'_p\mathcal{P}_0^{n+1}$, filling out the diagram by taking homotopy fibers. □

The inclusion of $\mathcal{P}_{n+1}^{n+1}$ as a subcategory of $\mathcal{P}_0^{n+1}$ induces a spectrally enriched functor $((u_qF^f_{p-1})^M\mathcal{P}_{n+1}^{n+1})^\Gamma \to ((u_qF^f_{p-1})^M\mathcal{P}_0^{n+1})^\Gamma$, which assembles to a simplicial spectrally enriched functor in the $q$ direction. Although not a DK-equivalence at any level, the simplicial spectrally enriched functor does induce a weak equivalence on $THH$.

PROPOSITION 4.3.10. *The inclusion* $((u_\bullet F^f_{p-1})^M \mathcal{P}^{n+1}_{n+1})^\Gamma \to ((u_\bullet F^f_{p-1})^M \mathcal{P}^{n+1}_0)^\Gamma$ *induces a weak equivalence*

$$|THH(((u_\bullet F^f_{p-1})^M \mathcal{P}^{n+1}_{n+1})^\Gamma)| \longrightarrow |THH(((u_\bullet F^f_{p-1})^M \mathcal{P}^{n+1}_0)^\Gamma)|.$$

PROOF. Consider the bisimplicial spectral category $V^\Gamma_{\bullet\bullet}$ defined as follows: in bidegree $r, s$, $V^\Gamma_{r,s}$ is the full spectral subcategory of $((u_{r+s+1} F^f_{p-1})^M \mathcal{P}^{n+1}_0)^\Gamma$ with objects the sequences of sequences of the form

$$a_0 \longrightarrow \cdots \longrightarrow a_r \longrightarrow b_0 \longrightarrow \cdots b_s$$

such that the objects $a_i$ are in $F^f_{p-1}\mathcal{P}^{n+1}_{n+1}$. Dropping the objects $\{a_i\}$ and the objects $\{b_i\}$ respectively induce bisimplicial spectrally enriched functors

$$((u_r F^f_{p-1})^M \mathcal{P}^{n+1}_{n+1})^\Gamma \longleftarrow V^\Gamma_{r,s} \longrightarrow ((u_s F^f_{p-1})^M \mathcal{P}^{n+1}_0)^\Gamma,$$

where we regard the targets as constant bisimplicial objects in the appropriate direction. Since the (connective) spectrum of maps from an object $x$ of $\mathcal{P}^{n+1}_{n+1}$ to an object $y$ of $\mathcal{P}^{n+1}_0$ is homotopy discrete with $\pi_0 = \mathrm{Hom}_{\mathbb{Z}}(\pi_{n+1}x, \pi_{n+1}y)$, we see that the map $V^\Gamma_{r,s} \to ((u_s F^f_{p-1})^M \mathcal{P}^{n+1}_0)^\Gamma$ is a DK-embedding. Furthermore, it is clear that this functor is essentially surjective (choosing an $n$-connected cover of $b_0$), and so is a DK-equivalence.

The usual arguments show that the map $V^\Gamma_{r,s} \to ((u_r F^f_{p-1})^M \mathcal{P}^{n+1}_{n+1})^\Gamma$ is a simplicial homotopy equivalence in the $s$-direction, using the homotopy inverse induced by

$$(a_0 \longrightarrow \cdots \longrightarrow a_r) \qquad \mapsto \qquad (a_0 \longrightarrow \cdots \longrightarrow a_r = a_r = \cdots = a_r).$$

Using this homotopy inverse, the composite map on (diagonal) simplicial spectral categories

$$((u_r F^f_{p-1})^M \mathcal{P}^{n+1}_{n+1})^\Gamma \longrightarrow V^\Gamma_{r,r} \longrightarrow ((u_r F^f_{p-1})^M \mathcal{P}^{n+1}_0)^\Gamma$$

is induced by

$$(a_0 \longrightarrow \cdots \longrightarrow a_r) \qquad \mapsto \qquad (a_r = a_r = \cdots = a_r),$$

and is easily seen to be simplicially homotopic to the inclusion map. □

For the categories $u_q M_p Z$, we copy the following definition from [**5**, 3.9].

DEFINITION 4.3.11. Let $Z = \pi_0 R$. Let $M_p Z$ be the category whose objects are sequences of $p-1$ composable maps of finitely generated left $Z$-modules $x_0 \to \cdots \to x_{p-1}$ and whose morphisms are commutative diagrams. Let $uM_pZ$ be the subcategory of $M_pZ$ consisting of all objects but only those maps $x \to y$ that are isomorphisms $x_i \to y_i$ for all $0 \le i \le p-1$.

We understand $M_0 Z$ to be the trivial category consisting of a single object (the empty sequence of maps) with only the identity map. As above, we let $u_q M_p Z$ denote the nerve category, which has as its objects the composable sequences of $q$ maps in $uM_pZ$ (i.e., isomorphisms in $M_pZ$) and maps the commutative diagrams of maps in $M_pZ$. We regard $u_q M_p Z$ as simplicially enriched with discrete mapping spaces and we obtain a connective spectral enrichment $(u_q M_p Z)^\Gamma$ using objectwise direct sum of finitely generated left $Z$-modules.

As above, $(u_\bullet M_p Z)^\Gamma$ assembles into a simplicial spectral category using the usual face and degeneracy maps for the nerve. We make $(u_\bullet M_\bullet Z)^\Gamma$ into a bisimplicial spectral category as follows: For $0 \le i \le p-1$, on $x_0 \to \cdots \to x_{p-1}$, the face map $\partial_i\colon u_qM_pZ \to u_qM_{p-1}Z$ is defined by dropping $x_i$ (and composing) and the degeneracy map $s_i\colon M_{p-1}Z \to M_pZ$ is defined by repeating $x_i$ (with the identity map). The face map $\partial_p\colon M_pZ \to M_{p-1}Z$ sends $x_0 \to \cdots \to x_{p-1}$ to $k_0 \to \cdots \to k_{p-2}$, where $k_i \subset x_i$ is the kernel of the composite map $x_i \to x_{p-1}$. The last degeneracy $s_{p-1}\colon M_{p-1}Z \to M_pZ$ puts 0 in as the last object in the sequence. The fundamental property of $(u_\bullet M_\bullet Z)^\Gamma$ that we need is the following.

PROPOSITION 4.3.12. *For each $q$, $|THH((u_qM_\bullet Z)^\Gamma)|$ is contractible.*

PROOF. The argument at the end of Section 3 of [**5**] constructs a simplicial contraction on the simplicial spectral category $(u_qM_\bullet Z)^\Gamma$. This simplicial contraction induces a simplicial contraction on the simplicial spectrum $THH((u_qM_\bullet Z)^\Gamma)$ and geometric realization converts this to a contraction of $|THH((u_qM_\bullet Z)^\Gamma)|$. □

Applying $\pi_{n+1}$, we get a functor $uF^f_{p-1}\mathcal{P}^{n+1}_0 \to uM_{p-1}$ and spectrally enriched functors

$$((u_qF^f_{p-1})^M\mathcal{P}^{n+1}_0)^\Gamma \longrightarrow (u_qM_pZ)^\Gamma \quad \text{and} \quad ((u_qF^f_{p-1})^M\mathcal{P}^{n+1}_{n+1})^\Gamma \longrightarrow (u_qM_pZ)^\Gamma.$$

Looking at the mapping spaces and mapping spectra, the following proposition is clear.

PROPOSITION 4.3.13. *The spectrally enriched functor*

$$((u_qF^f_{p-1})^M\mathcal{P}^{n+1}_{n+1})^\Gamma \longrightarrow (u_qM_pZ)^\Gamma$$

*is a DK-equivalence.*

In [**5**, §3], we claimed that the functors $uS'_p\mathcal{P}^{n+1}_0 \to uM_pZ$ respected the simplicial structure in the $p$ direction, which is untrue. To fix this, we introduce the category $uS^fM_\bullet Z$.

DEFINITION 4.3.14. Let $S^fM_pZ$ be the category whose objects are functors $A = a_{-,-}$ from $\mathrm{Ar}[p]$ to the category of finitely generated left $Z$-modules such that:

(i) $a_{i,i} = 0$, and
(ii) $a_{i,j} \to a_{i,k}$ is an isomorphism onto the kernel of the map $a_{i,k} \to a_{j,k}$

for all $i \leq j \leq k$. A map in $S^fM_pZ$ is a commutative diagram. The subcategory $uS^fM_pZ$ consists of those maps in $S^fM_p$ that are isomorphisms.

We make $uS^fM_\bullet Z$ a simplicial category using the usual face and degeneracy operations on $\mathrm{Ar}[\bullet]$. Basically $S^fM_\bullet Z$ is the fibration version of the $S_\bullet$ construction for the co-Waldhausen category (category with *fibrations* and weak equivalences) structure we get on the category of finitely generated left $Z$-modules by taking the fibrations to be all maps and the weak equivalences to be the isomorphisms. We have a forgetful functor $uS^fM_pZ \to uM_pZ$ which takes $A = \{a_{i,j}\}$ to the sequence

$$a_{0,p} \longrightarrow \cdots \longrightarrow a_{p-1,p}.$$

This functor is an equivalence of categories, with the inverse functor $uM_pZ \to uS^fM_pZ$ filling out the $\mathrm{Ar}[p]$ diagram from the sequence with the kernels of the maps. These functors then assemble into a simplicial functor $uM_\bullet Z \to uS^fM_\bullet Z$.

Now $\pi_{n+1}$ defines a simplicial functor $uS'_\bullet\mathcal{P}^{n+1}_0 \to uS^fM_\bullet Z$. The following theorem fixes the argument in [**5**] by replacing Theorem 3.10.

THEOREM 4.3.15. *The simplicial functors $uS'_\bullet\mathcal{P}^{n+1}_0 \to uS^fM_\bullet Z \leftarrow uM_\bullet Z$ induce weak equivalences on nerves.*

PROOF. Fix $p$. Since $uM_pZ \to uS^fM_pZ$ is an equivalence of categories, it induces a weak equivalence on nerves. The proof of Theorem 3.10 in [**5**, §4] correctly proves that the functor $uS'_p\mathcal{P}_0^{n+1} \to uM_pZ$ induces a weak equivalence on nerves, and the composite functor

$$uS'_p\mathcal{P}_0^{n+1} \longrightarrow uM_pZ \longrightarrow uS^fM_pZ$$

is naturally isomorphic to the functor $uS'_p\mathcal{P}_0^{n+1} \to uS^fM_pZ$ in the statement, so that functor also induces a weak equivalence on nerves. □

We regard the categories $u_qS^fM_p$ as simplicially enriched with discrete mapping spaces and we obtain a connective spectral enrichment $(u_qS^fM_p)^\Gamma$ using objectwise direct sum. Since the functor $u_qM_pZ \to u_qS^fM_pZ$ is an equivalence of categories, we get a DK-equivalence on the connective spectral enrichments.

PROPOSITION 4.3.16. *The spectral functor* $(u_qM_pZ)^\Gamma \to (u_qS^fM_pZ)^\Gamma$ *is a DK-equivalence.*

Finally, we have everything in place to prove Lemma 4.3.5.

PROOF OF LEMMA 4.3.5. Propositions 4.3.8, 4.3.9, 4.3.10, 4.3.13, and 4.3.16 imply that the bisimplicial map

$$THH(S_pF_q(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma) \longrightarrow THH((u_qS^fM_pZ)^\Gamma)$$

is a weak equivalence for each fixed $p, q$. Propositions 4.3.12 and 4.3.16 then imply that

$$|THH(S_\bullet F_\bullet(\mathcal{P}_0^{n+1}, \mathcal{P}_0^n)^\Gamma)| \simeq |THH((u_\bullet S^fM_\bullet Z)^\Gamma)| \simeq |THH((uM_\bullet Z)^\Gamma)|$$

is contractible. □

CHAPTER 5

# Generalization to Waldhausen categories with factorization

In previous sections, we imposed stringent hypotheses on our categories and functors. In this chapter, we relax these hypotheses and extend the theory. We begin in the first section by generalizing the maps we consider. Often, a functor between Waldhausen categories preserves the structure only "up to homotopy"; in previous work [**5**], we developed a theory of "weakly exact" functors to describe the associated functoriality of algebraic $K$-theory. In the first section (Section 5.1), we study the functoriality of $WTHH$ in weakly exact functors and show that a weakly exact functor of simplicial Waldhausen categories induces a zig-zag of spectra. We use this additional generality in the following sections to establish that our hypotheses introduced previously are generic, in the following sense. In Section 5.2, we show that any reasonable Waldhausen category (an "HCLF Waldhausen category"; see Definition 5.2.1) is connected by a weakly exact DK-equivalence to an enhanced simplicially enriched Waldhausen category. In the last section (Section 5.3), we show that if we start with a suitable spectral category $\mathcal{C}$, the two evident constructions of $THH$ (namely, $THH$ applied to the category viewed as a ring spectrum with many objects and $WTHH$ applied to the Waldhausen category of finite-cell modules) are connected by a natural zig-zag of weak equivalences.

The combination of Sections 5.1 and 5.2 allow us to regard $THH$ and $TC$ as functors from the homotopy category of HCLF Waldhausen categories (and weakly exact functors) to the homotopy categories of cyclotomic spectra and spectra, respectively; see Theorem 5.2.2.(i). Ideally, we would like to say something about coherence, perhaps showing that $THH$ and $TC$ are $\infty$-functors to spectra. The difficulty arises in Section 5.1, where a weakly exact simplicially enriched functor of Waldhausen categories only yields a zigzag of spectra instead of a map of spectra. A composable sequence of such functors yields a subdivided simplex of maps, a multi-dimensional zigzag. While it is clear that there should be a clean and straightforward $\infty$-category interpretation of this kind of generalized functor, we know of no good theory to plug it into, and this monograph does not seem like the appropriate place to develop one.

The work in Section 5.2, on the other hand, does have a relatively straightforward to interpretation as an $\infty$-functor in the context of quasi-categories (using the homotopy coherent nerve). The functor $\mathcal{C} \mapsto \tilde{\mathcal{C}}$ from HCLF Waldhausen categories to simplicially enriched Waldhausen categories is easily seen to be the composite of a lax pseudofunctor with an op-lax pseudofunctor, where we regard both categories of Waldhausen categories as strict 2-categories with 2-morphisms the natural weak equivalences. A (strictly unital) lax or op-lax pseudo-functor of strict 2-categories then induces a map on homotopy coherent nerves of the topologically enriched categories obtained by geometric realization of the nerve of the morphism categories. After strictifying the units, we get a zigzag of $\infty$-functors. We leave the details to a future paper.

## 5.1. Weakly exact functors

In this section, we still consider functors that preserve the simplicial enrichment, but now we drop the hypothesis that the functor is exact, and substitute the up to weak equivalence version of this hypothesis that the functor is "weakly exact" [**6**, §2]. For Waldhausen categories that admit functorial factorization of weak cofibrations (FFWC), a weakly exact functor is the minimum structure necessary to induce a map on $K$-theory. The purpose of this section is to explain the proof of the following theorem, which provides the corresponding result in our setting.

THEOREM 5.1.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicially enriched Waldhausen categories and assume that the underlying Waldhausen category of $\mathcal{D}$ admits FFWC. Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a simplicially enriched functor that restricts to a based weakly exact functor on the underlying Waldhausen categories, then it induces a map*

$$WTHH^{\Gamma}(\mathcal{C}) \longrightarrow WTHH^{\Gamma}(\mathcal{D})$$

*in the homotopy category of cyclotomic spectra. This map is compatible with the cyclotomic trace in that the following diagram commutes in the stable category.*

$$\begin{tikzcd}
K\mathcal{C} \arrow[r, "\mathrm{trc}"] \arrow[d] & WTC^{\Gamma}(\mathcal{C}) \arrow[r] \arrow[d] & WTHH(\mathcal{C}) \arrow[d] \\
K\mathcal{D} \arrow[r, "\mathrm{trc}"'] & WTC^{\Gamma}\mathcal{D} \arrow[r] & WTHH^{\Gamma}(\mathcal{D})
\end{tikzcd}$$

In the case of enhanced simplicially enriched Waldhausen categories, we have the following version of the previous theorem.

THEOREM 5.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be enhanced simplicially enriched Waldhausen categories with ambient simplicially tensored Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$ respectively. If $\phi\colon \mathcal{C} \to \mathcal{D}$ is a simplicially enriched functor that sends $\mathcal{A}$ into $\mathcal{B}$ and restricts to a based weakly exact functor on the underlying Waldhausen categories, then it induces a map in the homotopy category of cyclotomic spectra*

$$WTHH(\mathcal{A}) \longrightarrow WTHH(\mathcal{B})$$

*making the following diagram commute in the homotopy category of cyclotomic spectra.*

$$\begin{tikzcd}
WTHH^{\Gamma}(\mathcal{A}) \arrow[r] \arrow[d] & WTHH^{\Gamma}(\mathcal{B}) \arrow[d] \\
WTHH(\mathcal{A}) \arrow[r] & WTHH(\mathcal{B})
\end{tikzcd}$$

We also have the following theorem for natural weak equivalences between enriched weakly exact functors.

THEOREM 5.1.3. *Let $\phi$ and $\phi'$ be as in Theorem 5.1.1 or Theorem 5.1.2 above. If there is a natural weak equivalence from $\phi$ to $\phi'$, then the induced maps from*

*$WTHH^\Gamma(\mathcal{C})$ to $WTHH^\Gamma(\mathcal{D})$ agree in the homotopy category of cyclotomic spectra and (for Theorem 5.1.2) the induced maps from $WTHH(\mathcal{A})$ to $WTHH(\mathcal{B})$ agree in the homotopy category of cyclotomic spectra.*

The proof of these theorems requires the $S'^{M}_{\bullet}$ construction from Section 2.4; a weakly exact functor is precisely a functor that is compatible with that construction. We begin with the definition of weakly exact functor.

DEFINITION 5.1.4 ([**6**, 2.1]). Let $\mathcal{C}_0$ and $\mathcal{D}_0$ be Waldhausen categories. A functor $\phi\colon \mathcal{C}_0 \to \mathcal{D}_0$ is *weakly exact* if the initial map $* \to \phi(*)$ in $\mathcal{D}_0$ is a weak equivalence and $\phi$ preserves weak equivalences, weak cofibrations, and homotopy cocartesian squares. We say that a weakly exact functor $\phi$ is *based* if the initial map $* \to \phi(*)$ is the identity.

It follows that a functor that preserves weak equivalences will preserve weak cofibrations and homotopy cocartesian squares if and only if it takes cofibrations to weak cofibrations and takes pushouts along cofibrations to homotopy cocartesian squares.

Let

$$W'THH^\Gamma\mathcal{C} = \Omega|THH(S'^{M}_{\bullet}\mathcal{C}^\Gamma)|$$
$$\widetilde{W'THH}^\Gamma\mathcal{C}(n) = |THH((w_\bullet S'^{(n)}_{\bullet,\dots,\bullet})^M\mathcal{C}^\Gamma)|.$$

If $\mathcal{A}$ in an enhanced simplicially enriched Waldhausen category, let

$$W'THH\mathcal{A} = \Omega|THH(S'^{M}_{\bullet}\mathcal{A}^S)|$$
$$\widetilde{W'THH}\mathcal{A}(n) = |THH((w_\bullet S'^{(n)}_{\bullet,\dots,\bullet})^M\mathcal{A}^S)|.$$

Proposition 2.4.6 now implies the following theorem.

THEOREM 5.1.5. *Let $\mathcal{C}$ be a simplicially enriched Waldhausen category that admits FFWC. The maps of cyclotomic spectra*

$$WTHH^\Gamma(\mathcal{C}) \longrightarrow W'THH^\Gamma(\mathcal{C}) \quad \text{and} \quad \widetilde{WTHH}^\Gamma(\mathcal{C}) \longrightarrow \widetilde{W'THH}^\Gamma(\mathcal{C})$$

*are weak equivalences. If $\mathcal{A}$ is an enhanced simplicially enriched Waldhausen category, then the maps of cyclotomic spectra*

$$WTHH(\mathcal{A}) \longrightarrow W'THH(\mathcal{A}) \quad \text{and} \quad \widetilde{WTHH}(\mathcal{A}) \longrightarrow \widetilde{W'THH}(\mathcal{A})$$

*are weak equivalences.*

Functoriality of $THH$ in weakly exact functors requires one more twist. Because an exact functor $\mathcal{C}_0 \to \mathcal{D}_0$ preserves coproducts, an enriched exact functor induces a functor on spectral enrichments. For a weakly exact functor $\phi$, the map

$$\sigma_{(n)}\colon \phi(c_1) \vee \cdots \vee \phi(c_n) \longrightarrow \phi(c_1 \vee \cdots \vee c_n)$$

is generally not an isomorphism, though it is always a weak equivalence. To fix this problem, we use a zigzag with the following construction.

CONSTRUCTION 5.1.6. For simplicially enriched Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $\phi\colon \mathcal{C} \to \mathcal{D}$ that is simplicially enriched and based weakly exact, let $\phi^*(S'^{M}_{\bullet}\mathcal{C})^\Gamma$ be the (simplicial) topological $\Gamma$-category whose objects are the objects of $S'^{M}_{\bullet}\mathcal{C}$ and whose $\Gamma$-space of maps $\phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}_{q}(a,b)$ (for each fixed $\bullet = 0,1,2,\ldots$) consists of maps

$$f_0 \in S'^{M}_{\bullet}\mathcal{C}(a, \bigvee_q b), \qquad f_1 \in S'^{M}_{\bullet}\mathcal{D}(\phi(a), \bigvee_q \phi(b)),$$

a non-negative real number $s$, and a homotopy $f_{0,1}$ of length $s$ from $\phi(f_0)$ to $\sigma_{(q)} \circ f_1$, which we topologize as a subset of

$$S'^{M}_{\bullet}\mathcal{C}(a, \bigvee_q b) \times S'^{M}_{\bullet}\mathcal{D}(\phi(a), \bigvee_q \phi(b)) \times \mathbb{R} \times S'^{M}_{\bullet}\mathcal{D}(\phi(a), \phi(\bigvee_q b))^{I}$$

as in Construction 2.3.7. Composition works as follows: Given $(f_0, f_1, s, f_{0,1})\colon a \to b$ in level $q$ and $(g_0, g_1, t, g_{0,1})\colon b \to c$ in level $r$, the composition is $(g_0 \circ f_0, g_1 \circ f_1, s + t, h_{0,1})$ where $g_0 \circ f_0$ and $g_1 \circ f_1$ are the compositions in $(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}$ and $(S'^{M}_{\bullet}\mathcal{D})^{\Gamma}$, and $h_{0,1}$ is the homotopy that does

$$\phi(\bigvee_q g_0) \circ f_{0,1}$$

on $[0,s] \subset [0, s+t]$ (composing as in Definition 2.3.11) and does

$$\sigma_{(r)} \circ (\bigvee_q g_{0,1}) \circ f_1$$

on the length $t$ part $[s, s+t]$ of $[0, s+t]$. We define $\phi_*(w_\bullet S'^{(n)\,M}_{\bullet,\ldots,\bullet}\mathcal{C})^{\Gamma}$ similarly.

We have canonical (simplicial) spectral functors

$$\phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma} \longrightarrow S'^{M}_{\bullet}\mathcal{C}^{\Gamma} \qquad \text{and} \qquad \phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma} \longrightarrow S'^{M}_{\bullet}\mathcal{D}^{\Gamma}$$

induced by projecting on to the relevant factors in the product

$$\phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}_{q}(a,b) \subset S'^{M}_{\bullet}\mathcal{C}^{\Gamma}_{q}(a,b) \times S'^{M}_{\bullet}\mathcal{D}^{\Gamma}_{q}(\phi(a), \phi(b)) \times \mathbb{R} \times S'^{M}_{\bullet}\mathcal{D}(\phi(a), \phi(\bigvee_q b))^{I}.$$

Because the Moore construction for $\phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}_{q}(a,b)$ is homotopy equivalent to the homotopy pullback and the maps $\sigma_{(n)}$ above are weak equivalences, the projection

$$\phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}_{q}(a,b) \longrightarrow S'^{M}_{\bullet}\mathcal{C}^{\Gamma}_{q}(a,b)$$

is always a weak homotopy equivalence. We now have the following theorem.

THEOREM 5.1.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicially enriched Waldhausen categories and let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a functor that is simplicially enriched and based weakly exact. Then we have zigzags of (simplicial or multisimplicial) spectrally enriched functors, with the leftward arrow a DK-equivalence.*

$$S'^{M}_{\bullet}\mathcal{C}^{\Gamma} \xleftarrow{\sim} \phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}_{\mathcal{C}} \longrightarrow S'^{M}_{\bullet}\mathcal{D}^{\Gamma}$$

$$w_\bullet S'^{(n)\,M}_{\bullet,\ldots,\bullet}\mathcal{C}^{\Gamma} \xleftarrow{\sim} \phi_*(w_\bullet S'^{(n)\,M}_{\bullet,\ldots,\bullet}\mathcal{C})^{\Gamma}_{\mathcal{C}} \longrightarrow w_\bullet S'^{(n)\,M}_{\bullet,\ldots,\bullet}\mathcal{D}^{\Gamma}$$

Returning to $THH$, we define

$$W'THH^{\Gamma}(\phi_*\mathcal{C}) = \Omega|THH(\phi_*(S'^{M}_{\bullet}\mathcal{C})^{\Gamma}|$$

$$\widetilde{W'THH}^{\Gamma}(\phi_*\mathcal{C})(n) = |THH(\phi_*(S'^{(n)\,M}_{\bullet,\ldots,\bullet}\mathcal{C})^{\Gamma}|.$$

The fact that $\phi$ is based gives simplicial suspension maps, imparting the structure of a symmetric spectrum (in the category of cyclotomic spectra). The symmetric spectrum structure is compatible with the functors in the previous theorem, as summarized in the next result.

THEOREM 5.1.8. *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a simplicially enriched functor that restricts to a based weakly exact functor $\mathcal{C}_0 \to \mathcal{D}_0$. Then we have the following maps of cyclotomic spectra*

$$\begin{array}{ccc}
WTHH^{\Gamma}(\mathcal{C}) & \qquad & \widetilde{WTHH}^{\Gamma}(\mathcal{C}) \\
\downarrow & & \downarrow \\
W'THH^{\Gamma}(\mathcal{C}) & & \widetilde{W'THH}^{\Gamma}(\mathcal{C}) \\
\sim\uparrow & & \sim\uparrow \\
W'THH^{\Gamma}(\phi_*\mathcal{C}) & & \widetilde{W'THH}^{\Gamma}(\phi_*\mathcal{C}) \\
\downarrow & & \downarrow \\
W'THH^{\Gamma}(\mathcal{D}) & & \widetilde{W'THH}^{\Gamma}(\mathcal{D}) \\
\uparrow & & \uparrow \\
WTHH^{\Gamma}(\mathcal{D}) & & \widetilde{WTHH}^{\Gamma}(\mathcal{D})
\end{array}$$

*and the upward maps marked "$\sim$" are weak equivalences. If $\mathcal{D}_0$ admits FFWC, then all upward maps are weak equivalences.*

For Theorem 5.1.1, we have the cyclotomic trace induced by the inclusion of objects, producing the commutative diagram

$$\begin{tikzcd}
K(\mathcal{C}) \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddd] & \widetilde{WTC}^{\Gamma}(\mathcal{C}) \arrow[r] \arrow[d] & \widetilde{WTHH}^{\Gamma}(\mathcal{C}) \arrow[d] \\
 & \widetilde{W'TC}^{\Gamma}(\mathcal{C}) \arrow[r] & \widetilde{W'THH}^{\Gamma}(\mathcal{C}) \\
 & \widetilde{W'TC}^{\Gamma}(\phi_*\mathcal{C}) \arrow[r] \arrow[u, "\sim"] \arrow[d] & \widetilde{W'THH}^{\Gamma}(\phi_*\mathcal{C}) \arrow[u, "\sim"] \arrow[d] \\
 & & \\
K'(\mathcal{D}) \arrow[r] & \widetilde{W'THH}^{\Gamma}(\mathcal{D}) \arrow[r] & \widetilde{W'THH}^{\Gamma}(\mathcal{D}) \\
K(\mathcal{D}) \arrow[r] \arrow[u, "\sim"] & \widetilde{WTC}^{\Gamma}(\mathcal{D}) \arrow[r] \arrow[u, "\sim"] & \widetilde{WTHH}^{\Gamma}(\mathcal{D}) \arrow[u, "\sim"]
\end{tikzcd}$$

where here $K'(\mathcal{D})$ denotes $K$-theory constructed from the $S'_\bullet$ construction. This completes the proof the Theorem 5.1.1.

Theorem 5.1.2 is entirely similar, using the map

$$\tau_n\colon \Sigma^n\phi(c) \longrightarrow \phi(\Sigma^n c)$$

in place of $\sigma_{(n)}$ above. We can see that $\tau_1$ is a weak equivalence by writing the suspension as a homotopy pushout, and $\tau_n$ is a weak equivalence since $\tau_n = \tau_1 \circ \cdots \circ \tau_1$. Given enhanced simplicially enriched Waldhausen categories $\mathcal{A}$ and $\mathcal{B}$ with ambient simplicially tensored Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, respectively, and $\phi\colon \mathcal{C} \to \mathcal{D}$ a functor that is simplicially enriched, based weakly exact, and restricts to a functor $\mathcal{A} \to \mathcal{B}$, we then define $\phi_*\mathcal{A}^S$ as the spectrally enriched category whose set of objects is the same as $\mathcal{A}$ and whose spectrum of maps $\phi_*\mathcal{A}^S(a,b)$ is defined by letting

$$\phi_*\mathcal{A}^S(a,b)(n) \subset |\mathcal{A}^S(a,b)(n)| \times |\mathcal{B}^S(\phi(a),\phi(b))(n)| \times \mathbb{R} \times |\mathcal{B}(\phi(a),\phi(\Sigma^n b))|^I$$

be the subspace of elements $(f_0, f_1, s, f_{0,1})$ where $f_{0,1}$ is a length $s \geq 0$ homotopy from $\phi(f_0)$ to $\tau_n \circ f_1$. For a diagram $D$, we define $\phi_*(D^M\mathcal{A})^S$ likewise. This gives us the non-connective analogue of Theorem 5.1.7.

THEOREM 5.1.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be enhanced simplicially enriched Waldhausen categories with ambient simplicially tensored Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, respectively, and let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a functor that is simplicially enriched, based weakly exact, and restricts to a functor $\mathcal{A} \to \mathcal{B}$. Then we have a zigzag of spectrally enriched functors, with the leftward arrow a DK-equivalence.*

$$\mathcal{A}^S \xleftarrow{\sim} \phi_*\mathcal{A}^S \longrightarrow \mathcal{B}^S$$

Writing

$$W'THH(\phi_*\mathcal{A}) = \Omega|THH(\phi_* S'^{M}_{\bullet}\mathcal{A}^S)|$$
$$\widetilde{W'TH}H(\phi_*\mathcal{A})(n) = |THH(\phi_*(wS'^{(n)}_{\bullet,\dots,\bullet})^M\mathcal{A}^S)|$$

we obtain non-connective analogue of Theorem 5.1.8.

THEOREM 5.1.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be enhanced simplicially enriched Waldhausen categories with ambient simplicially tensored Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, respectively, and let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a functor that is simplicially enriched, based weakly exact, and restricts to a functor $\mathcal{A} \to \mathcal{B}$. Then we have the following maps of cyclotomic spectra*

$$\begin{array}{ccc}
WTHH(\mathcal{A}) & \qquad & \widetilde{WTH}H(\mathcal{A}) \\
\downarrow & & \downarrow \\
W'THH(\mathcal{A}) & & \widetilde{W'TH}H(\mathcal{A}) \\
\sim\uparrow & & \sim\uparrow \\
W'THH(\phi_*\mathcal{A}) & & \widetilde{W'TH}H(\phi_*\mathcal{A}) \\
\downarrow & & \downarrow \\
W'THH(\mathcal{B}) & & \widetilde{W'TH}H(\mathcal{B}) \\
\sim\uparrow & & \sim\uparrow \\
WTHH(\mathcal{B}) & & \widetilde{WTH}H(\mathcal{B})
\end{array}$$

*and the upward maps are weak equivalences.*

Finally for Theorem 5.1.3, choosing a natural weak equivalence from $\phi$ to $\phi'$, we obtain a simplicially enriched and weakly exact functor $\Phi$ from $\mathcal{C}$ to $w_1\mathcal{D}$. We obtain the zigzag

$$THH(S'^{M}_{\bullet}\mathcal{C}^\Gamma) \xleftarrow{\sim} THH(\Phi_*(S'^{M}_{\bullet}\mathcal{C})^\Gamma) \longrightarrow THH(((S'_{\bullet}w_1)^M\mathcal{D})^\Gamma) \xleftarrow{\sim} THH(S'^{M}_{\bullet}\mathcal{D}^\Gamma),$$

and a similar zigzag in the non-connective case (when it applies).

## 5.2. Embedding in simplicially tensored Waldhausen categories

In previous sections we worked under the stringent compatibility hypotheses in our definition of a simplicially enriched Waldhausen category. In this section, we show how to produce a DK-compatible simplicially enriched Waldhausen category from a Waldhausen category satisfying a certain technical hypothesis.

Definition 5.2.1. An *HCLF Waldhausen category* is a Waldhausen category that admits a homotopy calculus of left fractions (HCLF) as defined in [**11**, 6.1.(ii)].

We prove the following theorem.

Theorem 5.2.2. *Let $\mathcal{C}$ be an HCLF Waldhausen category. Then there exists a DK-compatible simplicially enriched Waldhausen category $\widetilde{\mathcal{C}}$ and a based weakly exact functor $\tilde{\imath}\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ that is a DK-equivalence (on simplicial localizations). Moreover:*

(i) *$WTHH^{\Gamma}(\widetilde{\mathcal{C}})$ is a functor from the category of HCLF Waldhausen categories and weakly exact maps to the homotopy category of cyclotomic spectra.*
(ii) *As a map in the stable category, $K(\mathcal{C}) \to K(\widetilde{\mathcal{C}})$ is natural in exact functors of $\mathcal{C}$.*
(iii) *As a map in the stable category, the cyclotomic trace $K(\widetilde{\mathcal{C}}) \to WTHH^{\Gamma}(\widetilde{\mathcal{C}})$ is natural in weakly exact functors of $\mathcal{C}$.*
(iv) *$\widetilde{\mathcal{C}}$ admits FFWC.*
(v) *If $\mathcal{C}$ is a DK-compatible simplicially enriched Waldhausen category then $\tilde{\imath}$ is naturally weakly equivalent to a simplicially enriched functor $\tilde{\imath}'$, which is also based weakly exact.*
(vi) *If $\mathcal{C}$ can be given the structure of an enhanced simplicially enriched Waldhausen category, then $\tilde{\imath}'$ induces DK-equivalence $S_n\mathcal{C} \to S'^{M}_{n}\widetilde{\mathcal{C}}$ for all $n$ and so induces a weak equivalence on $WTHH^{\Gamma}$.*

In the context of part (v), Theorem 5.1.7 gives a zigzag of spectrally enriched functors relating $\mathcal{C}^{\Gamma}$ and $\widetilde{\mathcal{C}}^{\Gamma}$, all of which are DK-equivalences in this case.

As we showed in [**6**, §5, App. A], a Waldhausen category that admits factorization (every map factors as a cofibration followed by a weak equivalence) and any closed Waldhausen subcategory of such a category in particular admits a homotopy calculus of left fractions. In this context, we can also produce an enhanced exact Waldhausen category.

Theorem 5.2.3. *Let $\mathcal{C}$ be a Waldhausen category that admits factorization and let $\mathcal{A}$ be a closed Waldhausen subcategory. Let $\breve{\mathcal{A}}$ be the full subcategory of $\widetilde{\mathcal{C}}$ of objects weakly equivalent to objects from $\mathcal{A}$. Then $\widetilde{\mathcal{C}}$ is a simplicially tensored Waldhausen category, $\breve{\mathcal{A}}$ is a closed Waldhausen subcategory, and the induced based weakly exact functor $\breve{\imath}\colon \mathcal{A} \to \breve{\mathcal{A}}$ is a DK-equivalence. Moreover:*

(i) *$WTHH(\breve{\mathcal{A}})$ is a functor from the category of pairs (Waldhausen category, closed Waldhausen subcategory) and weakly exact maps to the homotopy category of cyclotomic spectra.*
(ii) *There is a based weakly exact functor $\breve{\jmath}\colon \breve{\mathcal{A}} \to \widetilde{\mathcal{A}}$ such that $\breve{\jmath}\circ\breve{\imath}$ is naturally weakly equivalent to $\tilde{\imath}$. (In particular, $\breve{\jmath}$ is a DK-equivalence.)*
(iii) *The map of cyclotomic spectra $WTHH^{\Gamma}(\breve{\mathcal{A}}) \to WTHH^{\Gamma}(\widetilde{\mathcal{A}})$ induced by $\breve{\jmath}$ is a weak equivalence and natural in the homotopy category of cyclotomic spectra.*

In the context of the previous theorem, when $\mathcal{C}$ is a simplicially tensored Waldhausen category, $\mathcal{A}$ is an enhanced simplicially enriched Waldhausen category, and part (v) of Theorem 5.2.2 gives us a based weakly exact simplicially enriched functor $\breve{\imath}'\colon \mathcal{A} \to \breve{\mathcal{A}}$, weakly equivalent to $\breve{\imath}$; namely, $\breve{\imath}'$ is the restriction to $\mathcal{A}$ of $\tilde{\imath}'\colon \mathcal{C} \to \widetilde{\mathcal{C}}$.

Theorem 5.1.9 then produces a zigzag of spectrally enriched functors between $\mathcal{A}^S$ and $\breve{\mathcal{A}}^S$, all of which are DK-equivalences in this case.

The proof of the previous theorems works by embedding $\mathcal{C}$ in a simplicial model category in which all objects are fibrant. We do this using a variant of a presheaf construction in Toën and Vezzosi [**33**] to define the $K$-theory of a simplicial category. In the following discussion, let $L\mathcal{C}$ denote the simplicial category obtained as the Dwyer-Kan hammock simplicial localization of $\mathcal{C}$ with respect to the weak equivalences in the given Waldhausen structure.

Definition 5.2.4. Let $SF(L\mathcal{C})$ denote the category of simplicial functors from $L\mathcal{C}$ to based simplicial sets taking values in a fixed but sufficiently large cardinal depending on $\mathcal{C}$. We regard $SF(L\mathcal{C})$ as a simplicial model category using the injective model structure [**14**], where cofibrations and weak equivalences are defined objectwise and fibrations are defined by the right-lifting property with respect to the acyclic cofibrations; in this model structure, all objects are cofibrant. The opposite category $(SF(L\mathcal{C}))^{\mathrm{op}}$ then has the opposite simplicial model structure and all objects are fibrant.

Since the cofibrations in $SF(L\mathcal{C})$ are the injections, it is clear that $SF(L\mathcal{C})$ satisfies the pushout-product axiom, which is one of the equivalent forms of Quillen's SM7; in other words, $SF(L\mathcal{C})$ is a simplicial model category. It follows that $(SF(L\mathcal{C}))^{\mathrm{op}}$ is likewise a simplicial model category. Heller [**14**, §4] shows that the injective model structure has functorial factorizations, and in particular, we have a fibrant replacement functor in $SF(L\mathcal{C})$. In $(SF(L\mathcal{C}))^{\mathrm{op}}$, this gives functorial factorization and a cofibrant approximation functor. It will be useful for us to have these as simplicial functors and to preserve the zero object $*$. We prove the following lemma at the end of the section.

Lemma 5.2.5. *The category $SF(L\mathcal{C})$ admits simplicial endo-functors $P^c$ and $I^f$ such that $P^c$ is a cofibrant approximation functor for the projective model structure, $I^f$ is a fibrant approximation functor for the injective model structure, and $P^c(*) = * = I^f(*)$.*

The full subcategory of cofibrant objects in $(SF(L\mathcal{C}))^{\mathrm{op}}$ inherits the structure of a Waldhausen category.

Definition 5.2.6. Let $\widetilde{\mathcal{C}}$ be the full subcategory of $(SF(L\mathcal{C}))^{\mathrm{op}}$ consisting of cofibrant objects weakly equivalent to the opposite of a corepresentable in the image of $\mathcal{C}$, i.e., weakly equivalent to a functor of the form $L\mathcal{C}(x, -)$, where $x$ is an object of $\mathcal{C}$. When $\mathcal{A}$ is a closed Waldhausen subcategory of $\mathcal{C}$, let $\breve{\mathcal{A}}$ be the full subcategory of $(SF(L\mathcal{C}))^{\mathrm{op}}$ consisting of cofibrant objects weakly equivalent to the opposite of a corepresentable of an object $\mathcal{A}$.

As observed in Example 2.1.5, $\widetilde{\mathcal{C}}$ becomes a DK-compatible simplicially enriched Waldhausen category when given the Waldhausen structure induced by the model structure. The Yoneda embedding

$$Y_{\mathcal{C}} \colon x \mapsto L\mathcal{C}(x, -)$$

gives us a functor $Y_{\mathcal{C}}$ from $\mathcal{C}$ to $(SF(L\mathcal{C}))^{\mathrm{op}}$ that we can compose with $I^f$ to obtain a functor $\mathcal{C} \to \widetilde{\mathcal{C}}$. We showed in [**6**, 6.2] that under the hypothesis of homotopy calculus of left fractions, the simplicial localization mapping spaces take homotopy cocartesian squares to homotopy cartesian squares, and hence to homotopy cocartesian squares in $(SF(L\mathcal{C}))^{\mathrm{op}}$. It follows that $I^f Y_{\mathcal{C}}$ is a weakly exact functor $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ and a DK-equivalence. It is not, however, a based weakly exact functor as the zero object of $\mathcal{C}$ is generally not a zero object in $L\mathcal{C}$. On the other hand, $L\mathcal{C}(*,-) \to L\mathcal{C}(x,-)$ is an objectwise injection (as it is split by the map $L\mathcal{C}(x,-) \to L\mathcal{C}(*,-)$), and so the based functor

$$Y'_{\mathcal{C}} \colon x \mapsto L\mathcal{C}(x,-)/L\mathcal{C}(*,-)$$

is weakly equivalent to $Y_{\mathcal{C}}$ and hence is a based weakly exact functor and DK-equivalence. This proves the following proposition.

PROPOSITION 5.2.7. *Let $\mathcal{C}$ be a Waldhausen category that admits a homotopy calculus of left fractions. Then the functor $\tilde{\imath} = I^f Y'_{\mathcal{C}} \colon \mathcal{C} \to \widetilde{\mathcal{C}}$ is a based weakly exact functor and a DK-equivalence.*

When $\mathcal{C}$ is a simplicially enriched Waldhausen category, $\tilde{\imath}$ is a simplicial functor $L\mathcal{C} \to \widetilde{\mathcal{C}}$ but generally not a simplicial functor $\mathcal{C}_\bullet \to \widetilde{\mathcal{C}}$. We can regard the functor

$$x \mapsto \operatorname{diag} L\mathcal{C}_\bullet(x,-)/L\mathcal{C}_\bullet(*,-)$$

as a simplicial functor from $\mathcal{C}$ to $(SF(L\mathcal{C}))^{\mathrm{op}}$. Composing with $I^f$, we get a simplicial functor $\tilde{\imath}' \colon \mathcal{C} \to \widetilde{\mathcal{C}}$. The inclusion of $L\mathcal{C}_0$ in $L\mathcal{C}_\bullet$ induces a natural transformation $\tilde{\imath} \to \tilde{\imath}'$, which is a natural weak equivalence when $\mathcal{C}$ is DK-compatible (by definition). This proves the following proposition.

PROPOSITION 5.2.8. *If $\mathcal{C}$ is a DK-compatible simplicially enriched Waldhausen category, then $\tilde{\imath}$ is weakly equivalent to a simplicial functor, which is also a based weakly exact DK-equivalence.*

When $\mathcal{C}$ is a DK-compatible simplicially enriched Waldhausen category, just as in Proposition 2.3.4, looking at the formula for mapping spectra in $S_n\mathcal{C}$ and $S'^{M}_n\widetilde{\mathcal{C}}$, we see that $\tilde{\imath}'$ induces a DK-embedding $S_n\mathcal{C} \to S'^{M}_n\widetilde{\mathcal{C}}$. If we assume the hypothesis of part (vi), then $\mathcal{C}$ admits tensors with $\Delta[1]$, and for weak cofibration $x \to y$, the map $x \vee y \to (x \otimes \Delta[1]) \cup_{x \otimes \{1\}} y$ is a cofibration, i.e., $\mathcal{C}$ has functorial mapping cylinders for weak cofibrations in the terminology of [**6**, 2.6]. Since in any simplicially enriched Waldhausen category, weak equivalences are closed under retracts, we can apply [**6**, 6.1] to characterize the weak cofibrations in $\mathcal{C}$ as precisely those maps whose images in $\widetilde{\mathcal{C}}$ are weak cofibrations. Moreover, tensors with generalized intervals exist in $\mathcal{C}$, and arguing as in the proof of Proposition 2.3.6, we see that every object of $S'_n\widetilde{\mathcal{C}}$ is weakly equivalent to the image of an object of $S'_n\mathcal{C}$, i.e., that the DK-embedding is a DK-equivalence. The induced map (from Theorem 5.1.1)

$$WTHH^{\Gamma}(\mathcal{C}) \longrightarrow WTHH^{\Gamma}(\widetilde{\mathcal{C}})$$

is then a weak equivalence.

Now drop the assumption that $\mathcal{C}$ is simplicially enriched, and assume instead that $\mathcal{C}$ admits factorization. Then Waldhausen [**35**, p. 357] shows that we can form homotopy colimits in $\mathcal{C}$ over diagrams in finite partially ordered sets as iterated pushouts over cofibrations. Since any finite simplicial set is weakly equivalent to the nerve of a finite partially ordered set, it follows that for any weakly corepresentable $C$ and any finite simplicial set $X$, the simplicial functor $C^X$ is also weakly corepresentable. This proves the following proposition.

PROPOSITION 5.2.9. *If $\mathcal{C}$ admits factorization then $\widetilde{\mathcal{C}}$ is a simplicially tensored Waldhausen category.*

We also have the corresponding proposition for closed Waldhausen subcategories.

PROPOSITION 5.2.10. *If $\mathcal{A}$ is a closed Waldhausen subcategory of $\mathcal{C}$, then $\breve{\mathcal{A}} \subset \widetilde{\mathcal{C}}$ is an enhanced simplicial Waldhausen category and $\breve{\imath}\colon \mathcal{A} \to \breve{\mathcal{A}}$ is a based weakly exact functor and a DK-equivalence on simplicial localizations.*

We obtain the functor $\breve{\jmath}\colon \breve{\mathcal{A}} \to \widetilde{\mathcal{A}}$ as the restriction to $\breve{\mathcal{A}}$ of the functor $I^f_{\mathcal{A}} \circ R^{\mathcal{A}}_{\mathcal{C}}$, where $R^{\mathcal{A}}_{\mathcal{C}}$ denotes the functor $(SF(L\mathcal{C}))^{\mathrm{op}} \to (SF(L\mathcal{A}))^{\mathrm{op}}$ obtained by restricting an $L\mathcal{C}$ diagram to $L\mathcal{A}$ and $I^f_{\mathcal{A}}$ denotes the endo-functor $I^f$ in $\widetilde{\mathcal{A}}$. Writing $Y'_{\mathcal{A}}$ and $Y'_{\mathcal{C}}$ for the modified Yoneda embeddings as above, then

$$\breve{\jmath} \circ \breve{\imath} = I^f_{\mathcal{A}} R^{\mathcal{A}}_{\mathcal{C}} I^f_{\mathcal{C}} Y'_{\mathcal{C}} \qquad \text{and} \qquad \tilde{\imath} = I^f_{\mathcal{A}} Y'_{\mathcal{A}}.$$

Under the hypothesis of homotopy calculus of left fractions, the natural map $Y'_{\mathcal{A}} \to R^{\mathcal{A}}_{\mathcal{C}} Y'_{\mathcal{C}}$ in $SF(L\mathcal{A})$ is a weak equivalence; combining this with the canonical weak equivalence $\mathrm{Id} \to I^f_{\mathcal{C}}$ in $SF(L\mathcal{C})$ and reversing arrows to work in $(SF(L\mathcal{A}))^{\mathrm{op}}$ gives natural weak equivalences

$$\breve{\jmath} \circ \breve{\imath} = I^f_{\mathcal{A}} R^{\mathcal{A}}_{\mathcal{C}} I^f_{\mathcal{C}} Y'_{\mathcal{C}} \longrightarrow I^f_{\mathcal{A}} R^{\mathcal{A}}_{\mathcal{C}} Y'_{\mathcal{C}} \longrightarrow I^f_{\mathcal{A}} Y'_{\mathcal{A}} = \tilde{\imath}$$

in $\widetilde{\mathcal{A}}$.

The previous observations, propositions, and definitions cover all of the statements in Theorems 5.2.2 and 5.2.3 except for the naturality statements. The next result begins the study of naturality.

THEOREM 5.2.11. *Let $\mathcal{C}$ and $\mathcal{C}'$ be Waldhausen categories that admit homotopy calculi of left fractions, and let $\phi\colon \mathcal{C} \to \mathcal{C}'$ be a weakly exact functor. Then there exists a simplicial functor $\widetilde{\phi}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}'$ that restricts to a based weakly exact functor of the underlying Waldhausen categories and makes the diagram of functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\phi"'] & \widetilde{\mathcal{C}} \arrow[d, "\widetilde{\phi}"] \\ \mathcal{C}' \arrow[r] & \widetilde{\mathcal{C}}' \end{tikzcd}$$

*commute up to a zigzag of natural weak equivalences.*

*If $\mathcal{A}$ and $\mathcal{A}'$ are closed Waldhausen subcategories of $\mathcal{C}$ and $\mathcal{C}'$ (respectively) and $\phi$ restricts to a functor from $\mathcal{A}$ to $\mathcal{A}'$, then the functor $\widetilde{\phi}$ restricts to a functor $\breve{\phi}\colon \breve{\mathcal{A}} \to \breve{\mathcal{A}}'$ making the diagram of functors*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "\phi"'] & \breve{\mathcal{A}} \arrow[d, "\breve{\phi}"] \\ \mathcal{A}' \arrow[r] & \breve{\mathcal{A}}' \end{tikzcd}$$

*commute up to a zigzag of natural weak equivalences.*

We prove this theorem below, but first state the following corollary.

COROLLARY 5.2.12. *Let $\mathcal{C}$ and $\mathcal{C}'$ be Waldhausen categories that admit homotopy calculi of left fractions, and let $\phi\colon \mathcal{C} \to \mathcal{C}'$ be a weakly exact functor. If $\phi$ induces a DK-equivalence on passage to simplicial localizations, then the functor $\widetilde{\phi}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}'$ is a DK-equivalence. Moreover, $\widetilde{\phi}$ and (when appropriate) $\breve{\phi}$ induce an equivalence of cyclotomic spectra on $WTHH^{\Gamma}$ and $WTHH$, respectively.*

The proof of Theorem 5.2.11 combines the simplicially enriched cofibrant and fibrant approximation functors with left Kan extension. Fix the functor $\phi\colon \mathcal{C} \to \mathcal{C}'$. Left Kan extension gives rise to a functor $\mathrm{Lan}_{\phi}\colon SF(L\mathcal{C}) \to SF(L\mathcal{C}')$ and we let $\widetilde{\phi}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}'$ be the composite functor

$$SF(L\mathcal{C}) \xrightarrow{P^c} SF(L\mathcal{C}) \xrightarrow{\mathrm{Lan}_{\phi}} SF(L\mathcal{C}') \xrightarrow{I^f} SF(L\mathcal{C}').$$

By construction $\widetilde{\phi}$ preserves weak equivalences and is equipped with a zig-zag of natural weak equivalences connecting $\tilde{\imath} \circ \widetilde{\phi}$ to $\phi \circ \tilde{\imath}$. This completes the proof of Theorem 5.2.11 .

Most of Corollary 5.2.12 follows immediately from Theorem 5.2.11. To see that $\widetilde{\phi}$ induces a weak equivalence on $WTHH^{\Gamma}$, we need to see that the induced functor $S_n\widetilde{\mathcal{C}} \to {S'_n}^M\widetilde{\mathcal{C}}'$ is a DK-equivalence. The argument for Proposition 2.3.6 adapts to the current context to complete the proof.

The proof of the naturality statements in Theorems 5.2.2 and 5.2.3 now follow from an easy check that functors $\widetilde{\phi}$ compose as expected up to a zigzag of natural weak equivalences. Somewhat more work shows that this construction actually preserves composition up to coherent homotopy; we defer this to a future paper.

Finally, we need to prove Lemma 5.2.5. The specifics of the simplicial category $L\mathcal{C}$ play no role: the lemma holds for the category of simplicial functors from any small simplicial category $\mathcal{D}$ to based simplicial sets, and we argue in this context. We prove the following lemma, of which Lemma 5.2.5 is a special case.

LEMMA 5.2.13. *Let $\mathcal{D}$ be a small simplicial category and let $\mathfrak{S}_*^{\mathcal{D}}$ denote the category of simplicial functors from $\mathcal{D}$ to based simplicial sets. Then the projective and injective model structures both admit factorization functors that are simplicial functors and that send the identity on $*$ to the factorization $* = * = *$.*

The most basic case is when $\mathcal{D}$ is the trivial category and $\mathfrak{S}_*^{\mathcal{D}}$ is the category of based simplicial sets. Let $C$ denote the set of generating cofibrations ($\partial\Delta[n] \to \Delta[n]$, $n = 0, 1, 2, \ldots$) and let $A$ denote the set of generating acyclic cofibrations ($\Lambda_j[n] \to \Delta[n]$, $n = 0, 1, 2, \ldots$). Then the usual construction of the factorization functors uses the small objects argument as follows. Given $f\colon x \to y$, the factorization of $f$ as an acyclic cofibration $x \to x'$ followed by a fibration $x' \to y$ is constructed as $x' = \operatorname{colim} x'_n$, where $x'_0 = x$ and inductively $x'_{n+1}$ is constructed as the pushout

$$x'_{n+1} = x'_n \cup_{\coprod a} (\coprod b)$$

where the coproduct is over commutative diagrams

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ x'_n \arrow[r] & y \end{tikzcd}$$

with $i\colon a \to b$ ranging over the elements of $A$. The version we need for Lemma 5.2.13 instead uses the based simplicial set of maps in place of the set of maps above: We construct $x'_n$ inductively as the pushout

$$x'_{n+1} = x'_n \cup_{\coprod a \wedge D_i} (\coprod b \wedge D_i)$$

where the coproduct is over the elements $i\colon a \to b$ in $A$ and

$$D_i = \mathfrak{S}_*(a, x'_n) \times_{\mathfrak{S}_*(a,y)} \mathfrak{S}_*(b, y)$$

is the based simplicial set of commutative diagrams of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ x'_n \arrow[r] & y. \end{tikzcd}$$

The induced map $x'_n \to x'_{n+1}$ and the colimit map $x \to x'$ is an injection and weak equivalence and the map $x' \to y$ is a fibration. Moreover, this functor is clearly a simplicial functor into the appropriate diagram category. The analogous construction using $C$ instead of $A$ constructs the other factorization. When applied to the identity map on the trivial based simplicial set $*$, each $D_i$ is the trivial based simplicial set $*$, and so we get that each map $* \cong x_n \to x'_{n+1}$ and $x_{n+1} \to y = *$ is an isomorphism. Thus, (replacing the factorization functors with naturally isomorphic functors if necessary), we have that the factorization of $* = *$ is $* = * = *$.

A slight modification of the factorization functors in Heller [**14**] constructs the factorizations in the general case. Let $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$ denote the simplicial category $\prod_{\mathrm{Ob}\,\mathcal{D}} \mathfrak{S}_*$, and (following the notation in [**14**]), let $J^*$ denote the forgetful functor from $\mathfrak{S}_*^{\mathcal{D}}$ to $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$ that remembers just the objects in the diagram (values of the functor) and forgets the maps. Let $J_P$ be its left adjoint; since we are working in based simplicial sets, $J_P X$ is the simplicial functor

$$J_P X = \bigvee_{c \in \mathrm{Ob}\,\mathcal{D}} X(c) \wedge \mathcal{D}(c, -)_+.$$

Likewise, let $J^I$ be the right adjoint of $J^*$,

$$J^I X = \prod_{c \in \mathrm{Ob}\,\mathcal{D}} X(c)^{\mathcal{D}(-,c)},$$

where $X(c)^{\mathcal{D}(-,c)}$ denotes the based simplicial set of unbased simplicial maps from $\mathcal{D}(-, c)$ to $X(c)$. We note that for any $X$, $J_P X$ is cofibrant in the projective model structure and more generally, $J_P$ sends (objectwise) cofibrations and acyclic cofibrations in $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$ to cofibrations and acyclic cofibrations in the projective model structure on $\mathfrak{S}_*^{\mathcal{D}}$. Likewise $J^F$ sends (objectwise) fibrations and acyclic fibrations to fibrations and acyclic fibrations in the injective model structure on $\mathfrak{S}_*^{\mathcal{D}}$.

The factorization functors for the projective model structure are constructed as follows. For $f\colon X \to Y$, let $Z_0 = X$ and construct $Z_{n+1}$ inductively as follows. First factor $J^* Z_n \to J^* Y$ objectwise

$$J^* Z_n \longrightarrow W_n \longrightarrow J^* Y$$

using the simplicial factorization functor (for the appropriate factorization) on based simplicial sets constructed above, and let $Z_{n+1}$ be the pushout

$$Z_{n+1} = Z_n \cup_{J_P J^* Z_n} J_P W_n,$$

with the factorization $Z_{n+1} \to Y$ induced by the map $J_P W_n \to Y$. Letting $Z = \operatorname{colim} Z_n$, we get a factorization $X \to Z \to Y$, with the map $X \to Z$ a cofibration or acyclic cofibration (as appropriate) in the projective model structure. We note that the underlying map in $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$ from $J^*Z_n$ to $J^*Z_{n+1}$ factors through $W_n$. It follows that we can identify $J^*Z$ as $\operatorname{colim} W_n$ and the underlying map $J^*Z \to J^*Y$ in $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$ as the colimit of the maps $W_n \to J^*Y$. Since by construction, these maps are objectwise acyclic fibrations or fibrations of simplicial sets, the map $j^*Z \to J^*Y$ is an objectwise acyclic fibration or fibration as required. We note that when $X = * = Y$, by construction each $W_n$ is $*$ and $J_P W_n$ is isomorphic to $*$, and so we end up with both factorizations of $* = *$ as $* = * = *$.

The factorization functors on the injective model structure are precisely dual. We start with $Z_0 = Y$, and inductively construct $Z_{n+1}$ as follows. Using the appropriate objectwise factorization functor, we factor $J^*X \to J^*Z_n$ in $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$ as

$$J^*X \longrightarrow W_n \longrightarrow J^*Z_n,$$

and we define $Z_{n+1}$ as the pullback

$$Z_{n+1} = Z_n \times_{J^I J^* Z_n} J^I W_n.$$

We let $Z = \lim Z_n$ and get a factorization $X \to Z \to Y$ with $Z \to Y$ by construction a fibration or acyclic fibration (as appropriate) in the injective model structure. Again looking at the underlying map in $\mathfrak{S}_*^{\mathrm{Ob}\,\mathcal{D}}$, we see that the map $X \to Z$ is an objectwise acyclic cofibration or cofibration as appropriate. Again, the factorization of $* = *$ becomes $* = * = *$. This completes the proof of Lemma 5.2.13.

## 5.3. Spectral categories and Waldhausen categories

The work of the previous section showed how to associate a spectral category to any well-behaved Waldhausen category. On the other hand, given a spectral category $\mathcal{C}$, we can produce a simplicially tensored Waldhausen category by passage to the Waldhausen category $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ of "finite cell right $\mathcal{C}$-modules" described below. In this section we show that when $\mathcal{C}$ is pretriangulated (Definition 1.3.7), the spectral category associated to $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ in Definition 5.2.6 recovers the original spectral category $\mathcal{C}$ up to DK-equivalence.

As a general principal, it does not matter which modern category of spectra we use as a model when discussing small spectral categories. The monoidal Quillen equivalences relating the various categories of diagram spectra and EKMM $S$-modules [**18, 19, 26**] allow us to convert a spectral category on any of these models to one on any other. In particular, the following theorem is an easy consequence of the work of [**28**] (extended by the techniques of [**13**] for dealing with non-cofibrant units that arise there).

THEOREM 5.3.1. *Fix a set of objects $O$. For $\mathcal{S}$ a modern category of spectra from* [**19**] *or* [**13**]*, let $\mathcal{S}O$-Cat denote the category of $\mathcal{S}$-enriched categories with object set $O$ and functors that are the identity on the object set $O$. Then:*

(i) *The category $\mathcal{S}O$-Cat forms a closed model category where the weak equivalences and fibrations are the functors that induce a weak equivalence or positive fibration, respectively, on mapping spectra.*
(ii) *The monoidal Quillen equivalences from* [**18, 19, 26**] *induce Quillen equivalences between the categories $\mathcal{S}O$-Cat for the various $\mathcal{S}$.*

Because of this theorem, without loss of generality, we can assume that our spectral category $\mathcal{C}$ comes enriched in EKMM $S$-modules, which have the technical advantage that every object is fibrant. On the other hand, since our goal is to compare with the non-connective enrichment of a simplicially tensored Waldhausen category, our comparison must be between spectral categories enriched in symmetric spectra. Again, we use the previous theorem. Spectral categories enriched in EKMM $S$-modules are always fibrant in the model structure of the previous theorem, so the associated spectral category enriched in symmetric spectra has (the same object set and) mapping spectra $\Phi\mathcal{C}(x,y)$, where $\Phi$ is the lax symmetric monoidal right adjoint functor from EKMM $S$-modules to symmetric spectra defined in [**26**]. Specifically, for an EKMM $S$-module $X$,

$$\Phi X(n) = \mathcal{M}_S((S_S^{-1})^{(n)}, X).$$

Here $\mathcal{M}_S$ denotes the mapping spaces (in simplicial sets) for the category of EKMM $S$-modules and $S_S^{-1}$ denotes the canonical cell $(-1)$-sphere $S$-module [**13**, III.2]; $\Phi X$ is always a positive $\Omega$-spectrum and when $X$ is a mapping spectrum, $\Phi X$ often turns out to be an $\Omega$-spectrum (for example, this happens for $X = \mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,y)$ where $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is the spectral category defined below). The lax monoidal natural transformation is induced by

$$\begin{aligned}\Phi X(m) \wedge \Phi Y(n) = \mathcal{M}_S((S_S^{-1})^{(m)}, X) \wedge \mathcal{M}_S((S_S^{-1})^{(n)}, Y)\\ \longrightarrow \mathcal{M}_S((S_S^{-1})^{(m+n)}, X \wedge Y) = \Phi(X \wedge Y)(m+n)\end{aligned}$$

and the map $S \to \Phi S$ is induced by the map $S^0 \to \mathcal{M}_S(S,S)$ sending the non-base point to the identity element.

NOTATION 5.3.2. For $\mathcal{C}$ a spectral category in EKMM $S$-modules, write $\Phi\mathcal{C}$ for the associated spectral category in symmetric spectra described above.

Now given $\mathcal{C}$ a spectral category in EKMM $S$-modules we associate a Waldhausen category to $\mathcal{C}$ as follows. Let $\mathcal{M}_{\mathcal{C}^{\mathrm{op}}}$ denote the category of (right) $\mathcal{C}$-modules, the category of enriched functors from $\mathcal{C}^{\mathrm{op}}$ to the category of EKMM $S$-modules. We make $\mathcal{M}_{\mathcal{C}^{\mathrm{op}}}$ into a model category with the projective model structure: The weak equivalences and fibrations are the objectwise weak equivalences and fibrations. The cofibrations in this model structure are the retracts of relative cell inclusions, where a cell is of the form

$$\mathcal{C}(-,x) \wedge S_S^q \wedge S_+^{n-1} \longrightarrow \mathcal{C}(-,x) \wedge S_S^q \wedge D_+^n$$

for some object $x$ in $\mathcal{C}$, $q \in \mathbb{Z}$, $n \geq 0$, where $S^{n-1} \to D^n$ is the standard $n$-cell in topology. We then have a subcategory of finite cell $\mathcal{C}$-modules, having objects the $\mathcal{C}$-modules built from $*$ by attaching finitely many cells. If we insist on using canonical pushouts in building these complexes (or restrict to a skeleton), then the resulting subcategory we get is small.

NOTATION 5.3.3. For $\mathcal{C}$ a small spectral category in EKMM $S$-modules, let $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ be the small subcategory of $\mathcal{M}_{\mathcal{C}^{\mathrm{op}}}$ of finite cell $\mathcal{C}$-modules.

We have a spectrally enriched functor $\mathcal{C} \to \mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ sending $x$ to $\mathcal{C}(-,x) \wedge S_S^0$. By the Yoneda lemma

$$\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(-,x) \wedge S_S^0, \mathcal{C}(-,y) \wedge S_S^0) \cong F_S(S_S^0, \mathcal{C}(x,y) \wedge S_S^0)$$

(where $F_S$ denotes the function $S$-module) and the map

$$\mathcal{C}(x,y) \longrightarrow F_S(S_S^0, \mathcal{C}(x,y) \wedge S_S^0)$$

is a weak equivalence. The following theorem is now clear from the construction of $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$.

THEOREM 5.3.4. *For $\mathcal{C}$ a small spectral category in EKMM $S$-modules, the spectrally enriched functor $\mathcal{C} \to \mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is a DK-embedding, and $\pi_0\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is the thick subcategory of $\pi_0\mathcal{M}_{\mathcal{C}^{\mathrm{op}}}$ generated by the image of $\mathcal{C}$. In particular, $\mathcal{C} \to \mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is a DK-equivalence if and only if $\mathcal{C}$ is pretriangulated.*

Since $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is a subcategory of cofibrant objects in a simplicial model category with all objects fibrant, it fits into the context of Example 2.1.5, and is canonically a simplicially enriched Waldhausen category. In fact, it is easy to see that the tensor in $\mathcal{M}_{\mathcal{C}^{\mathrm{op}}}$ of an object of $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ with a finite simplicial set is isomorphic to an object of $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$, so $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is a simplicially tensored Waldhausen category. The following is the main theorem of this section; combined with the previous theorem, it gives the zigzag of DK-equivalence of spectral categories $\Phi\mathcal{C} \simeq \widetilde{\mathcal{F}}^S_{\mathcal{C}^{\mathrm{op}}}$ when $\mathcal{C}$ is pretriangulated.

THEOREM 5.3.5. *For $\mathcal{C}$ a small spectral category in EKMM $S$-modules, there are zigzags of DK-equivalences of spectral categories (in symmetric spectra in simplicial sets)*

$$\Phi\mathcal{F}_{\mathcal{C}^{\mathrm{op}}} \simeq \mathcal{F}^S_{\mathcal{C}^{\mathrm{op}}} \simeq \widetilde{\mathcal{F}}^S_{\mathcal{C}^{\mathrm{op}}},$$

*where $\widetilde{\mathcal{F}}_{\mathcal{C}^{\mathrm{op}}}$ denotes the simplicially tensored Waldhausen category constructed from $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ by Definition 5.2.6.*

The zigzag of DK-equivalences $\mathcal{F}^S_{\mathcal{C}^{\mathrm{op}}} \simeq \widetilde{\mathcal{F}}^S_{\mathcal{C}^{\mathrm{op}}}$ is the one obtained from applying Theorem 5.1.9 to the simplicially enriched based weakly exact functor $\tilde{\imath}' \colon \mathcal{F}_{\mathcal{C}^{\mathrm{op}}} \to \widetilde{\mathcal{F}}_{\mathcal{C}^{\mathrm{op}}}$ in part (v) of Theorem 5.2.2. That leaves us with constructing the zigzag of DK-equivalences $\Phi\mathcal{F}_{\mathcal{C}^{\mathrm{op}}} \simeq \mathcal{F}^S_{\mathcal{C}^{\mathrm{op}}}$, which is just the generalization of Proposition 3.4.5 to rings with many objects. The proof is essentially identical: Let $\Phi'\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ denote the spectral category (in symmetric spectra in simplicial sets) with the same objects as $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$, but with mapping spectra $\Phi'\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,y)$ defined by

$$\Phi'\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,y)(n) = \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y)),$$

where we have written $F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}$ for the mapping spectrum in $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ to avoid confusion with the mapping space (simplicial set) $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,y)$. For $y = x$, we have the unit $S \to \Phi'\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,x)$ induced by the unit for $\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,x)$ and the canonical isomorphism

$$\mathcal{M}_S((S_S^{-1} \wedge S^1)^{(0)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^0 y)) = \mathcal{M}_S(S, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x,y)) \cong \mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x,y).$$

Composition is induced by the smash product map

$$\begin{aligned}\mathcal{M}_S((S_S^{-1} \wedge S^1)^{(m)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(y, \Sigma^m z)) \wedge \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y))\\ \longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(m+n)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(y, \Sigma^m z) \wedge F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y))\end{aligned}$$

and the composition map

$$\begin{aligned}F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(y, \Sigma^m z) \wedge F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y) \longrightarrow F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(\Sigma^n y, \Sigma^{m+n} z) \wedge F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y)\\ \longrightarrow F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^{m+n} z)\end{aligned}$$

analogous to the one in Definition 2.2.5. We then have spectral functors

$$\Phi\mathcal{F}_{\mathcal{C}^{\mathrm{op}}} \longrightarrow \Phi'\mathcal{F}_{\mathcal{C}^{\mathrm{op}}} \longleftarrow \mathcal{F}^S_{\mathcal{C}^{\mathrm{op}}}$$

defined as follows. The functor $\Phi\mathcal{F}_{\mathcal{C}^{\mathrm{op}}} \to \Phi'\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}$ is the map

$$\mathcal{M}_S((S_S^{-1})^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x,y)) \longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y))$$

induced by $n$-fold suspension

$$F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x,y) \longrightarrow F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(\Sigma^n x, \Sigma^n y) \cong \Omega^n F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y)$$

and the adjunction

$$\mathcal{M}_S((S_S^{-1})^{(n)}, \Omega^n F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y)) \cong \mathcal{M}_S((S_S^{-1})^{(n)} \wedge S^n, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y)).$$

The functor $\mathcal{F}^S_{\mathcal{C}^{\mathrm{op}}} \to \Phi'$ is induced by the map

$$\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}(x, \Sigma^n y) = \mathcal{M}_S(S, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y)) \longrightarrow \mathcal{M}_S((S_S^{-1} \wedge S^1)^{(n)}, F_{\mathcal{F}_{\mathcal{C}^{\mathrm{op}}}}(x, \Sigma^n y))$$

induced by the canonical collapse map $S_S^{-1} \wedge S^1 \to S$. On mapping spaces, both these functors are weak equivalences (in fact, level equivalences) of symmetric spectra, and so the functors are DK-equivalences. This completes the proof of Theorem 5.3.5.

# Bibliography


1. Christian Ausoni, *Topological Hochschild homology of complex $K$-theory*, Amer. J. Math. **127** (2005), no. 6, 1261–1313.
2. ______, *On the algebraic $K$-theory of the complex $K$-theory spectrum*, Invent. Math. **180** (2010), no. 3, 611–668.
3. Christian Ausoni and John Rognes, *Algebraic $K$-theory of topological $K$-theory*, Acta Math. **188** (2002), no. 1, 1–39. MR MR1947457 (2004f:19007)
4. A. J. Blumberg and M. A. Mandell, *The homotopy theory of cyclotomic spectra*, arXiv:1303.1694 [math.KT], preprint 2013.
5. Andrew J. Blumberg and Michael A. Mandell, *The localization sequence for the algebraic $K$-theory of topological $K$-theory*, Acta Math. **200** (2008), no. 2, 155–179. MR MR2413133 (2009f:19003)
6. ______, *Algebraic $K$-theory and abstract homotopy theory*, Adv. Math. **226** (2011), no. 4, 3760–3812. MR 2764905
7. ______, *Localization theorems in topological Hochschild homology and topological cyclic homology*, Geom. Topol. **16** (2012), no. 2, 1053–1120. MR 2928988
8. M. Bökstedt, *Topological Hochschild homology*, Preprint, 1990.
9. Bjørn Ian Dundas, *$K$-theory theorems in topological cyclic homology*, J. Pure Appl. Algebra **129** (1998), no. 1, 23–33. MR MR1626647 (99h:19004)
10. Bjørn Ian Dundas and Randy McCarthy, *Topological Hochschild homology of ring functors and exact categories*, J. Pure Appl. Algebra **109** (1996), no. 3, 231–294. MR MR1388700 (97i:19001)
11. W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35. MR MR578563 (81h:55019)
12. ______, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284. MR MR579087 (81h:55018)
13. A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR MR1417719 (97h:55006)
14. Alex Heller, *Homotopy theories*, Mem. Amer. Math. Soc. **71** (1988), no. 383, vi+78. MR MR920963 (89b:55013)
15. Lars Hesselholt and Ib Madsen, *On the $K$-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), no. 1, 29–101. MR MR1410465 (97i:19002)
16. ______, *On the $K$-theory of local fields*, Ann. of Math. (2) **158** (2003), no. 1, 1–113. MR MR1998478 (2004k:19003)
17. L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986, With contributions by J. E. McClure. MR MR866482 (88e:55002)
18. M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and $S$-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR 1922205 (2003i:55012)
19. M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR MR1806878 (2001k:55025)
20. J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271. MR MR0420610 (54 #8623b)
21. ______, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D.

Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR MR1413302 (97k:55016)

22. Randy McCarthy, *On fundamental theorems of algebraic $K$-theory*, Topology **32** (1993), no. 2, 325–328. MR 1217072 (94e:19002)
23. ———, *The cyclic homology of an exact category*, J. Pure Appl. Algebra **93** (1994), no. 3, 251–296. MR MR1275967 (95b:19002)
24. James E. McClure and Jeffrey H. Smith, *Operads and cosimplicial objects: an introduction*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 133–171. MR 2061854 (2005b:55018)
25. John Rognes, *Topological logarithmic structures*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 401–544. MR 2544395 (2010h:14029)
26. Stefan Schwede, *$S$-modules and symmetric spectra*, Math. Ann. **319** (2001), no. 3, 517–532. MR 2001m:55050
27. ———, *On the homotopy groups of symmetric spectra*, Geom. Topol. **12** (2008), no. 3, 1313–1344. MR MR2421129 (2009c:55006)
28. Stefan Schwede and Brooke Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), 287–334 (electronic). MR MR1997322 (2004i:55026)
29. Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511. MR MR1734325 (2001c:18006)
30. Brooke Shipley, *Symmetric spectra and topological Hochschild homology*, $K$-Theory **19** (2000), no. 2, 155–183. MR MR1740756 (2001h:55010)
31. Ross E. Staffeldt, *On fundamental theorems of algebraic $K$-theory*, $K$-Theory **2** (1989), no. 4, 511–532. MR 990574 (90g:18009)
32. R. W. Thomason and Thomas Trobaugh, *Higher algebraic $K$-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR MR1106918 (92f:19001)
33. Bertrand Toën and Gabriele Vezzosi, *A remark on $K$-theory and $S$-categories*, Topology **43** (2004), no. 4, 765–791. MR MR2061207 (2005e:19001)
34. Friedhelm Waldhausen, *Algebraic $K$-theory of topological spaces. II*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer, Berlin, 1979, pp. 356–394. MR MR561230 (81i:18014b)
35. ———, *Algebraic $K$-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR MR802796 (86m:18011)

# Index