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Derived Koszul duality and involutions in the algebraic K-theory of spaces

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Abstract

We interpret different constructions of the algebraic K-theory of spaces as an instance of derived Koszul (or bar) duality and also as an instance of Morita equivalence. We relate the interplay between these two descriptions to the homotopy involution. We define a geometric analog of the Swan theory $G^{\mathbb{Z}}(\mathbb{Z}[\pi])$ in terms of $\Sigma^\infty_+\Omega X$ and show that it is the algebraic K-theory of the E_∞ ring spectrum $DX = S^{X_+}$.

1. *Introduction*

Associated to a space X are two ring spectra: $\Sigma^\infty_+\Omega X$, the free suspension spectrum on the based loop space of X, and $DX = S^{X_+}$, the Spanier–Whitehead dual of X. Waldhausen [**31**] defined the algebraic K-theory of X, $A(X)$, as the K-theory of the ring spectrum $\Sigma^\infty_+\Omega X$. This theory has deep geometric content: when X is a manifold, $A(X)$ contains the stable pseudo-isotopy theory of X, and when X is a finite complex, $A(X)$ is a receptacle for 'higher torsion invariants' [**10**] and closely related to transfers [**18**]. On the other hand, the unstable homotopy theory of X is encoded in the E_∞ ring spectrum DX (see [**20**]). Recent work by Morava [**24**] conjectures the structure and properties for a category of homotopy theoretic motives in terms of the stabilization of a category of correspondences; one candidate construction put forward is built from algebraic K-theory of ring spectra of the form DX.

Based on computations in THH motivated by string topology, Cohen conjectured a duality between $K(DX)$ and $A(X)$ as modules over $K(S)$ (see [**6**]). Although the non-connectivity of DX means that trace methods fail to apply to $K(DX)$, in this paper we construct such a duality in terms of derived Koszul duality when X is a simply connected finite CW complex.

In differential graded algebra, derived Koszul duality (or bar duality) concerns the contravariant adjunction between the category of augmented differential graded algebras and itself [**16**, **23**] (named for the special case of Koszul algebras [**1**, **2**, **25**]). The dual of an augmented differential graded k-algebra A is an augmented differential graded k-algebra E that models the A-module endomorphisms of k, $\mathrm{End}_A(k,k)$. Under mild hypotheses, $A \simeq \mathrm{End}_E(k,k)$; the contravariant functors $\mathrm{Hom}_A(-,k)$ and $\mathrm{Hom}_E(-,k)$ form an adjunction on the module categories and an equivalence between various thick subcategories of the derived categories.

In our context, $\Sigma^\infty_+\Omega X$ forms an augmented S-algebra, and we can identify the augmented S-algebra of $\Sigma^\infty_+\Omega X$-endomorphisms of S as the augmented S-algebra DX (see [**8**, Section 4.22]). In fact, the coherent $\Sigma^\infty_+\Omega X$-module equivalence $S \wedge S \simeq S$ makes the endomorphism ring spectrum naturally commutative, compatible with the natural commutative S-algebra structure on DX. Interpreting $\mathrm{Ext}_{\Sigma^\infty_+\Omega X}(-,S)$ and $\mathrm{Ext}_{DX}(-,S)$ as contravariant adjoints on

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derived categories

$$\mathcal{D}_{\Sigma^\infty_+\Omega X} \rightleftarrows \mathcal{D}_{DX},$$

we get equivalences upon restricting to certain subcategories. For example, the subcategory $\mathcal{D}^c_{\Sigma^\infty_+\Omega X}$ of compact objects of $\mathcal{D}_{\Sigma^\infty_+\Omega X}$ is the thick subcategory generated by $\Sigma^\infty_+\Omega X$ and is equivalent under this adjunction to the thick subcategory $\mathcal{T}_{DX}(S)$ of $\mathcal{D}_{DX}$ generated by S. This is reminiscent of Waldhausen's comparison of the stable category of ΩX-spaces with the stable category of retractive spaces over X. Writing $\mathcal{M}_{DX}(S)$ for the subcategory of the model category of DX-modules that are isomorphic in $\mathcal{D}_{DX}$ to objects in $\mathcal{T}_{DX}(S)$, we prove the following theorem. In the following theorem and all theorems in this section, we understand X to be a simply connected finite CW complex.

THEOREM 1.1. *In the notation above, $K(\mathcal{M}_{DX}(S))$ is weakly equivalent to $A(X) = K(\Sigma^\infty_+\Omega X)$.*

On the other hand, the subcategory $\mathcal{D}^c_{DX}$ of compact objects of $\mathcal{D}_{DX}$ is the thick subcategory generated by DX and is equivalent under the adjunction above to the thick subcategory $\mathcal{T}_{\Sigma^\infty_+\Omega X}(S)$ of $\mathcal{D}_{\Sigma^\infty_+\Omega X}$ generated by S. The category $\mathcal{T}_{\Sigma^\infty_+\Omega X}(S)$ is a geometric analog of the category of finite rank projective $\pi_1 X$-modules, whose K-theory $G^{\mathbb{Z}}_i(\mathbb{Z}[\pi])$ was studied by Swan [**28**]. We call the K-theory of the category $\mathcal{M}_{\Sigma^\infty_+\Omega X}(S)$ the geometric Swan theory of the space X and denote it as $G(X)$. We prove the following theorem.

THEOREM 1.2. *In the notation above, $G(X) = K(\mathcal{M}_{\Sigma^\infty_+\Omega X}(S))$ is weakly equivalent to $K(DX)$.*

In fact, both $G(X)$ and $K(DX)$ are commutative ring spectra, and the equivalence is a weak equivalence of ring spectra. Likewise, $A(X)$ is a module spectrum over $G(X)$ and $K(\mathcal{M}_{DX}(S))$ is a module spectrum over $K(DX)$; the equivalence in Theorem 1.1 is a weak equivalence of module spectra. In fact, we have the following more precise result.

THEOREM 1.3. *The weak equivalence $G(X) \to K(DX)$ is a map of E_∞ ring spectra. The weak equivalence $K(\mathcal{M}_{DX}(S)) \to A(X)$ is a map of $G(X)$-modules.*

Because X is a finite CW complex, the $\Sigma^\infty_+\Omega X$-module S is compact, and so we can interpret the map on K-theory induced by inclusion of the thick subcategory generated by S into the thick subcategory of compact $\Sigma^\infty_+\Omega X$-modules in terms of Waldhausen's fibration theorem. We obtain a localization sequence of K-theory spectra

$$G(X) \longrightarrow A(X) \longrightarrow K(\mathcal{C}_{\Sigma^\infty_+\Omega X}/\epsilon), \tag{1.4}$$

where $\mathcal{C}_{\Sigma^\infty_+\Omega X}/\epsilon$ is the Waldhausen category of compact $\Sigma^\infty_+\Omega X$-modules but with weak equivalences the maps whose cofiber is in $\mathcal{T}_{\Sigma^\infty_+\Omega X}(S)$. (We typically do not have a corresponding transfer $A(X) \to G(X)$ because S is not usually a compact DX-module when X is a finite complex.) We intend to study this sequence further in a future paper.

Derived Koszul duality between categories of modules over differential graded algebras is a contravariant phenomenon, but there is also an associated covariant Morita adjunction switching chirality from left modules to right modules [**8**]. (For a survey on Morita theory in stable homotopy theory, see [**27**].) In the presence of an anti-involution (for example, commutativity), we can use the anti-involution to obtain a Morita adjunction between categories of left modules. In the context of DX and $\Sigma^\infty_+\Omega X$, we get two covariant adjunctions

$$\mathcal{D}_{\Sigma^\infty_+\Omega X} \rightleftarrows \mathcal{D}_{DX}$$

given by the adjoint pairs

$$\mathrm{Ext}_{\Sigma^\infty_+\Omega X}(S,-), \mathrm{Tor}^{DX}(-,S) \quad \text{and} \quad \mathrm{Tor}^{\Sigma^\infty_+\Omega X}(-,S), \mathrm{Ext}_{DX}(S,-).$$

The first restricts to an equivalence

$$\mathcal{T}_{\Sigma^\infty_+\Omega X}(S) \simeq \mathcal{D}^c_{DX}$$

(in fact, as S is compact as an $\Sigma^\infty_+\Omega X$ module, the adjunction restricts to embed $\mathcal{D}_{DX}$ as the localizing subcategory of S in $\mathcal{D}_{\Sigma^\infty_+\Omega X}$). The second restricts to an equivalence

$$\mathcal{D}^c_{\Sigma^\infty_+\Omega X} \simeq \mathcal{T}_{DX}(S).$$

These equivalences give rise to equivalences on algebraic K-theory, akin to the equivalences of Theorems 1.1 and 1.2. The composites are self-maps on $A(X)$ and $G(X)$. In Section 5, we identify these self-maps as the standard homotopy involutions.

Experts will recognize that Theorems 1.1 and 1.2 fit into the framework of [**8**, **29**]; the benefit of the approach here is the description in terms of concrete models, which allow more direct comparisons than in the abstract approach, and more precise results such as Theorem 1.3.

Readers may also wonder about the connection to the work of Goresky, Kottwitz, and MacPherson on Koszul duality [**14**]. From our perspective, they study the 'dual' setting in which $G = \Omega BG$ is compact and BG is infinite. Our techniques apply to recover (integral) liftings of their equivalences of derived categories; in fact, this case was studied in [**16**]. Because this example is not connected as closely to A-theory, we have chosen to omit a detailed discussion.

The paper is organized as follows. In Section 2, we review and slightly extend the passage from algebraic structures on Waldhausen categories to algebraic structures on K-theory spectra, using the technology developed in [**12**]. In Section 3, we introduce the concrete models for $\Sigma^\infty_+\Omega X$ and the endomorphism ring spectra, which allow a good point-set model for the adjunctions in the remaining sections. Section 4 studies the contravariant adjunction and proves Theorems 1.1–1.3. Finally, Section 5 studies the point-set model of the covariant adjunctions of [**8**] and identifies the composite homotopy endomorphisms on $A(X)$ and $G(X)$ as the standard homotopy involutions.

2. *Algebraic structures on Waldhausen K-theory*

Since even before the advent of the theory of symmetric spectra [**15**], experts have understood that any algebraic structure on a Waldhausen category induces an analogous structure on Waldhausen K-theory. Sources for results of this type in the literature include [**12**, **13**, Appendix A, **31**, p. 342]. We briefly review the current state of the theory here.

We refer the reader to [**31**, Section 1.2] for the definition of a Waldhausen category (called there a 'category with cofibrations and weak equivalences'). Recall that Waldhausen's $S_\bullet$ construction [**31**, Section 1.3] produces a simplicial Waldhausen category $S_\bullet\mathcal{C}$ from a Waldhausen category $\mathcal{C}$ and is defined as follows. Let $\mathrm{Ar}[n]$ denote the category with objects (i,j) for $0 \leqslant i \leqslant j \leqslant n$ and a unique map $(i,j) \to (i',j')$ for $i \leqslant i'$ and $j \leqslant j'$. $S_n\mathcal{C}$ is defined to be the full subcategory of the category of functors $A : \mathrm{Ar}[n] \to \mathcal{C}$ such that:

(i) for all i, $A_{i,i} = *$;
(ii) the map $A_{i,j} \to A_{i,k}$ is a cofibration for all $i \leqslant j \leqslant k$;
(iii) the diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a pushout square for all $i \leqslant j \leqslant k$;

where we write $A_{i,j}$ for $A(i,j)$. The last two conditions can be simplified to the hypothesis that each map $A_{0,j} \to A_{0,j+1}$ is a cofibration and the induced maps $A_{0,j}/A_{0,i} \to A_{i,j}$ are isomorphisms. This becomes a Waldhausen category by defining a map $A \to B$ to be a weak equivalence when each $A_{i,j} \to B_{i,j}$ is a weak equivalence in $\mathcal{C}$, and to be a cofibration when each $A_{i,j} \to B_{i,j}$ and each induced map $A_{i,k} \cup_{A_{i,j}} B_{i,j} \to B_{i,k}$ is a cofibration in $\mathcal{C}$.

As $S_\bullet\mathcal{C}$ forms a simplicial Waldhausen category, the construction can be iterated to form $S_\bullet S_\bullet \ldots S_\bullet\mathcal{C}$. For our purposes, it is convenient to have an 'all at once' construction of the qth iterate $S^{(q)}_{\bullet,\ldots,\bullet}\mathcal{C}$. For this construction, we need the following terminology (see also [**26**, Section 2]).

Definition 2.1. Let $[n]$ denote the ordered set $0 \leqslant 1 \leqslant \ldots \leqslant n$. For a Waldhausen category $\mathcal{C}$, a functor $C : [n_1] \times \ldots \times [n_q] \to \mathcal{C}$ is *cubically cofibrant* means that the following conditions hold.

(i) Every map $C(i_1, \ldots, i_q) \to C(j_1, \ldots, j_q)$ is a cofibration.

(ii) In every subsquare $(1 \leqslant r < s \leqslant q)$

$$\begin{array}{ccc} C(i_1,\ldots,i_q) & \longrightarrow & C(i_1,\ldots,i_r+1,\ldots,i_q) \\ \downarrow & & \downarrow \\ C(i_1,\ldots,i_s+1,\ldots,i_q) & \longrightarrow & C(i_1,\ldots,i_r+1,\ldots,i_s+1,\ldots,i_q) \end{array}$$

the induced map from the pushout to the lower-right entry

$$\begin{aligned} &C(i_1,\ldots,i_r+1,\ldots,i_q) \cup_{C(i_1,\ldots,i_q)} C(i_1,\ldots,i_s+1,\ldots,i_q) \\ &\quad\longrightarrow C(i_1,\ldots,i_r+1,\ldots,i_s+1,\ldots,i_q) \end{aligned}$$

is a cofibration.

(iii) In general, in every m-dimensional subcube specified by choosing m distinct coordinates $1 \leqslant r_1 < r_2 < \ldots < r_m \leqslant n$, the induced map from the colimit over the diagram obtained by deleting $Q = C(i_1, \ldots, i_{r_1}+1, \ldots, i_{r_2}+1, \ldots, i_{r_m}+1, \ldots, i_n)$ to Q is a cofibration.

Construction 2.2 (Iterated $S_\bullet$ construction). Let $\mathrm{Ar}[n_1, \ldots, n_q]$ denote the category $\mathrm{Ar}[n_1] \times \ldots \times \mathrm{Ar}[n_q]$. For a functor

$$A : \mathrm{Ar}[n_1,\ldots,n_q] = \mathrm{Ar}[n_1] \times \ldots \times \mathrm{Ar}[n_q] \longrightarrow \mathcal{C},$$

we write $A_{i_1,j_1;\ldots;i_q,j_q}$ for the value of A on the object $((i_1,j_1),\ldots,(i_q,j_q))$. For a Waldhausen category $\mathcal{C}$, let $S^{(q)}_{n_1,\ldots,n_q}\mathcal{C}$ be the full subcategory of functors A (as above) such that:

(i) whenever $i_k = j_k$ for some k, $A_{i_1,j_1;\ldots;i_q,j_q} = *$;

(ii) the subfunctor

$$C(j_1,\ldots,j_q) = A_{0,j_1;\ldots;0,j_q} : [n_1] \times \ldots \times [n_q] \longrightarrow \mathcal{C}$$

is cubically cofibrant;

(iii) for every object $(i_1, j_1; \ldots; i_q, j_q)$ in $\mathrm{Ar}[n_1] \times \ldots \times \mathrm{Ar}[n_q]$, every $1 \leqslant r \leqslant q$, and every $j_r \leqslant k \leqslant n_r$, the square

$$\begin{array}{ccc} A_{i_1,j_1;\ldots;i_q,j_q} & \longrightarrow & A_{i_1,j_1;\ldots;i_r,k;\ldots;i_q,j_q} \\ \downarrow & & \downarrow \\ A_{i_1,j_1;\ldots;j_r,j_r;\ldots;i_q,i_q} & \longrightarrow & A_{i_1,j_1;\ldots;j_r,k;\ldots;i_q,i_q} \end{array}$$

is a pushout square.

The subcategory $\mathrm{w}S^{(q)}_{n_1,\ldots,n_q}\mathcal{C}$ consists of the maps in $S_{n_1,\ldots,n_q}\mathcal{C}$ that are objectwise weak equivalences. We understand $S^{(0)}\mathcal{C}$ to be $\mathcal{C}$ and we see that $S^{(1)}_n\mathcal{C}$ is $S_n\mathcal{C}$.

Following Waldhausen [31, p. 330], we define the K-theory spectrum of a Waldhausen category $\mathcal{C}$ to be the spectrum with qth space

$$K\mathcal{C}(q) = N(\mathrm{w}S^{(q)}_{\bullet,\ldots,\bullet}\mathcal{C}) = |N_\bullet(\mathrm{w}S^{(q)}_{\bullet,\ldots,\bullet}\mathcal{C})|,$$

the geometric realization of the nerve of the multisimplicial category $\mathrm{w}S^{(q)}_{\bullet,\ldots,\bullet}\mathcal{C}$. The suspension maps $\Sigma K\mathcal{C}(q) \to K(q+1)$ are induced on diagrams by the projection map

$$\mathrm{Ar}[n_1] \times \ldots \times \mathrm{Ar}[n_q] \times \mathrm{Ar}[n_{q+1}] \longrightarrow \mathrm{Ar}[n_1] \times \ldots \times \mathrm{Ar}[n_q].$$

Defining an action of Σ_q on $K\mathcal{C}(q)$ by permuting the simplicial directions, we see from the explicit description of $S^{(q)}_{\bullet,\ldots,\bullet}\mathcal{C}$ above that $K\mathcal{C}$ forms a symmetric spectrum.

We can encode an algebraic structure on a set of symmetric spectra using a *symmetric multicategory* (also called *colored operad*). A symmetric multicategory $\mathcal{M}$ enriched in (small) categories consists of:

(i) a set of *objects* $\mathrm{Ob}\,\mathcal{M}$;

(ii) a (small) category of *k-morphisms* $\mathcal{M}_k(x_1,\ldots,x_k;y)$ for all $k = 0,1,2,\ldots$ and all $x_1,\ldots,x_k,y \in \mathrm{Ob}\,\mathcal{M}$;

(iii) a unit object 1_x in $\mathcal{M}_1(x;x)$ for each $x \in \mathrm{Ob}\,\mathcal{M}$;

(iv) for every permutation $\sigma \in \Sigma_k$, an isomorphism

$$\sigma^* : \mathcal{M}_k(x_1,\ldots,x_k;y) \longrightarrow \mathcal{M}_k(x_{\sigma 1},\ldots,x_{\sigma k}),$$

compatibly assembling to an action of Σ_k on $\coprod \mathcal{M}_k(x_1,\ldots,x_k;y)$;

(v) composition maps

$$\begin{aligned}&\mathcal{M}_n(y_1,\ldots,y_n;z) \times (\mathcal{M}_{j_1}(x_{1,1},\ldots,x_{1,j_1};y_1) \times \ldots \times \mathcal{M}_{j_n}(x_{n,1},\ldots,x_{n,j_n};y_n))\\ &\quad\longrightarrow \mathcal{M}_j(x_{1,1},\ldots,x_{n,j_n};z)\end{aligned}$$

satisfying the analog of the usual conditions for an operad [22, pp. 1–2]; these are written out in [12, Section 2]. The following definition is standard.

Definition 2.3. Let $\mathcal{M}$ be a symmetric multicategory enriched in small categories. An $\mathcal{M}$-algebra A in symmetric spectra consists of a symmetric spectrum $A(x)$ for each $x \in \mathrm{Ob}\,\mathcal{M}$ and maps of symmetric spectra

$$N(\mathcal{M}_k(x_1,\ldots,x_k;y)) \wedge A(x_1) \wedge \ldots \wedge A(x_k) \longrightarrow A(y),$$

for all k, $x_1,\ldots,x_k,y$, which are compatible with the composition maps and identity objects of $\mathcal{M}$. Here (as above), $N(-)$ denotes the geometric realization of the nerve of the category. When $k=0$, we understand the map pictured above as $N(\mathcal{M}(;y)) \wedge S \to A(y)$.

To define an $\mathcal{M}$-algebra in Waldhausen categories, we first need to describe the kinds of functors to which objects of $\mathcal{M}_k$ should map.

Definition 2.4. Let $\mathcal{C}_1,\ldots,\mathcal{C}_n$ and $\mathcal{D}$ be Waldhausen categories. A functor

$$F : \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

is *multiexact* if it satisfies the following conditions:

(i) $F(X_1,\ldots,X_n) = *$ if any of $X_1,\ldots,X_n$ is $*$;

(ii) F is exact in each variable (preserves weak equivalences, cofibrations, and pushouts over cofibrations in each variable, keeping the other variables fixed);

(iii) given cofibrations $X_{k,0} \to X_{k,1}$ in $\mathcal{C}_k$ for all k, the diagram

$$A(i_1,\ldots,i_n) = F(X_{1,i_1},\ldots,X_{n,i_n}) : [1] \times \ldots \times [1] \longrightarrow \mathcal{D}$$

is cubically cofibrant.

We define the category of multiexact functors

$$\mathrm{Mult}_n(\mathcal{C}_1,\ldots,\mathcal{C}_n;\mathcal{D}),$$

to have objects the multiexact functors and maps the natural weak equivalences. For $n = 0$, we define $\mathrm{Mult}_0(;D)$ to be $\mathrm{w}\mathcal{D}$, the subcategory of weak equivalences in $\mathcal{D}$.

Because multiexact functors compose into multiexact functors, the definition above makes the category of small Waldhausen categories into a symmetric multicategory enriched in categories. Following [**12**], we define an $\mathcal{M}$-algebra in Waldhausen categories as a map of symmetric multicategories enriched in categories.

Definition 2.5. Let $\mathcal{M}$ be a symmetric multicategory enriched in small categories. An $\mathcal{M}$-algebra C in Waldhausen categories consists of a Waldhausen category $C(x)$ for each $x \in \mathrm{Ob}\,\mathcal{M}$ and functors

$$\mathcal{M}_k(x_1,\ldots,x_k;y) \longrightarrow \mathrm{Mult}_k(\mathcal{C}(x_1),\ldots,\mathcal{C}(x_k);\mathcal{D}),$$

for all k, $x_1,\ldots,x_k,y$, which are compatible with the permutations, composition maps, and identity objects of $\mathcal{M}$.

Recalling the universal property of the smash product of symmetric spectra [**15**, 2.1.4], the following theorem is immediate from inspection of the definitions above.

Theorem 2.6. *Waldhausen's algebraic K-theory functor naturally takes $\mathcal{M}$-algebras in Waldhausen categories to $\mathcal{M}$-algebras in symmetric spectra.*

In particular, as explained in [**12**, Section 9], the preceding theorem applies to describe the algebraic structures on K-theory spectra induced by pairings on the level of Waldhausen categories. Suppose that $\mathcal{C}$ is a Waldhausen category which is also a permutative category, where the product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a biexact functor; we will refer to $\mathcal{C}$ as a *permutative Waldhausen category*. Recall that a permutative category is a rigidified form of a symmetric monoidal category: a permutative category is a symmetric monoidal category where the product satisfies strict associativity and unit relations (the associativity and unit isomorphisms are the identity). If $\mathcal{C}$ is a permutative Waldhausen category, then a *strict Waldhausen module* over $\mathcal{C}$ consists of a Waldhausen category $\mathcal{Q}$ and a biexact functor $\mathcal{C} \times \mathcal{Q} \to \mathcal{Q}$ satisfying the evident strict associativity and unit relations.

The structure of a permutative Waldhausen category on $\mathcal{C}$ is equivalent to an algebra in Waldhausen categories for the symmetric multicategory $E\Sigma_*$ (see [**12**, Section 3]), where the unique object of $E\Sigma_*$ is taken to $\mathcal{C}$. Then $K\mathcal{C}$ becomes an $E\Sigma_*$-algebra in symmetric spectra; this is a particular type of E_∞-algebra symmetric spectrum, which is an associative ring symmetric spectra by neglect of structure (the symmetric multicategory of objects of $E\Sigma_*$ is the operad Σ_* of sets). Similarly, the structure of a strict Waldhausen module over $\mathcal{C}$ on $\mathcal{Q}$ is equivalent to specifying an algebra in Waldhausen categories for the symmetric multicategory associated to $E\Sigma_*$ parameterizing modules, called $E\ell\mathbf{M}^{\Sigma_*}$ in [**12**, Section 9.1], such that the 'ring object' is taken to $\mathcal{C}$ and the 'module object' to $\mathcal{Q}$. Then $K\mathcal{Q}$ becomes a $K\mathcal{C}$-module in symmetric spectra.

Corollary 2.7. *Let $\mathcal{C}$ be a permutative Waldhausen category. Then $K\mathcal{C}$ is naturally an $E\Sigma_*$-algebra symmetric spectrum, and in particular an associative ring symmetric spectrum. Moreover, if $\mathcal{D}$ is a strict Waldhausen $\mathcal{C}$-module, then $K\mathcal{D}$ is naturally a $K\mathcal{C}$-module.*

Working with a permutative product has the appealing consequence that the multicategory that arises is a familiar one, namely, the categorical Barratt–Eccles operad $E\Sigma_*$. However, the categories that we work with in this paper (and that tend to arise in practice) are symmetric monoidal categories rather than permutative categories. This is no real limitation, as a standard construction [**17**] rectifies any symmetric monoidal category into an equivalent permutative category: the rectification of $\mathcal{C}$ is a category $\mathcal{C}'$ with objects the 'words' in the objects of $\mathcal{C}$, where a word $(X_1, X_2, \ldots, X_r)$ corresponds to the product

$$\lambda(X_1, \ldots, X_r) = (\ldots(X_1 \otimes X_2) \otimes \ldots) \otimes X_r$$

in $\mathcal{C}$; we associate the empty word in $\mathcal{C}'$ to the unit of the monoidal product. The morphisms in $\mathcal{C}'$ are precisely the morphisms in $\mathcal{C}$ between the associated products

$$\mathcal{C}'((X_1, \ldots, X_r), (Y_1, \ldots, Y_s)) = \mathcal{C}(\lambda(X_1, \ldots, X_r), \lambda(Y_1, \ldots, Y_s)).$$

Concatenation provides the permutative structure. Sending a word to the associated product λ defines a strong symmetric monoidal functor $\mathcal{C}' \to \mathcal{C}$. The inclusion of $\mathcal{C}$ in $\mathcal{C}'$ as the singleton words is also a strong symmetric monoidal functor; the composite functor $\mathcal{C} \to \mathcal{C}$ is the identity, whereas the composite functor $\mathcal{C}' \to \mathcal{C}'$ is naturally isomorphic to the identity via the map corresponding to the identity map on the associated product. When $\mathcal{C}$ is a Waldhausen category and $\otimes$ is biexact, we use the variant where we look at words in objects that are not $*$ together with a distinguished zero object $*$, and force a concatenation in $\mathcal{C}'$ with $*$ to result in $*$. The resulting category $\mathcal{C}''$ becomes a Waldhausen category when we define the cofibrations and weak equivalences to be those maps that correspond to weak equivalences and cofibrations in $\mathcal{C}$. The functors above remain strong symmetric monoidal equivalences, but now are exact functors as well.

Alternatively, at the cost of complicating the multicategory in Corollary 2.7, we can work directly with symmetric monoidal Waldhausen categories (that is, Waldhausen categories that are symmetric monoidal under a biexact product). Specifying such a structure on $\mathcal{C}$ is equivalent to specifying the structure of an algebra over a certain symmetric multicategory $\mathcal{B}$ enriched in small categories, defined as follows: $\mathrm{Ob}\,\mathcal{B}$ is a single element. For $k = 1$, $\mathcal{B}_1$ is the category with one object and the identity morphism. For $k > 1$, $\mathcal{B}$ is the category with objects the labeled planar binary trees with k leaves, having a unique morphism between any two objects. The permutation action permutes the labels. As above, there is a symmetric multicategory parameterizing modules in this setting; an action of $\mathcal{C}$ on a Waldhausen category $\mathcal{D}$ through a biexact functor endows $(\mathcal{C}, \mathcal{D})$ with the structure of an algebra over this module multicategory. We have the following consequence.

Corollary 2.8. *Let $\mathcal{C}$ be a symmetric monoidal Waldhausen category. Then $K\mathcal{C}$ is naturally a $\mathcal{B}$-algebra symmetric spectrum. Moreover, if $\mathcal{D}$ is a symmetric monoidal Waldhausen $\mathcal{C}$-module, then $K\mathcal{D}$ is naturally a $K\mathcal{C}$-module (parameterized by the multicategory of E_∞-modules associated to $\mathcal{B}$).*

3. *Models for endomorphism S-algebras and the double centralizer condition*

Classically, for a k-algebra R and an R-module M, the double centralizer condition for M is the requirement that the natural map

$$R \longrightarrow \mathrm{End}_{\mathrm{End}_R(M,M)}(M, M)$$

be an isomorphism. Dwyer, Greenlees, and Iyengar [8] studied the derived form of this condition. They study the example of $R = \Sigma^{\infty}_{+}\Omega X$ and $DX \simeq \mathrm{End}_R(S, S)$ in [8, Section 4.22]. We review this example in this section in terms of specific models we use in the remainder of the paper.

In our context, we are interested in the case when X is a finite CW complex. As we shall see below, Dwyer's results on convergence of the Eilenberg–Moore spectral sequence [7] imply that the double centralizer map cannot be a weak equivalence unless X is simply connected (as this is the only case in which $\pi_1 X$ acts nilpotently on $H_0(\Omega X)$). Once we restrict to this context, we can assume without loss of generality that X is the geometric realization of a reduced finite simplicial set. Then we have a topological group model G for ΩX (given by the geometric realization of the Kan loop group), and a free G-CW complex P whose quotient by G is X (the twisted cartesian product $G_\bullet \times_\tau X_\bullet$ for the universal twisting function τ; see, for example, [21, Chapter VI]).

Notation 3.1. Let X, P, and G be as above. Let $R = \Sigma^{\infty}_{+}G$, regarded as an EKMM S-algebra [11, IV.7.8]. Let $SP = \Sigma^{\infty}_{+}P$ and let $E = F_R(SP, SP)$.

We regard S as an R-algebra via the augmentation $R \to S$ (induced by the map $G \to *$). The map $SP \to S$ (induced by the map $P \to *$) is a weak equivalence of R-modules. Although SP is not cofibrant, it is semi-cofibrant [19, 1.2], meaning that the functor $SP \wedge_S (-) = P_+ \wedge (-)$ from S-modules to R-modules preserves cofibrations and acyclic cofibrations [19, 1.3(a)]. As in EKMM S-module categories all objects are fibrant, E represents the correct endomorphism algebra $\mathrm{Ext}_R(S, S)$ (see [19, 6.3]).

These particular models show the strong parallel between the double centralizer condition for $\Sigma^{\infty}_{+}\Omega X$ and the bar duality theory of [16]. The diagonal map $P \to P \times P \to X \times P$ induces an X-comodule structure on SP

$$SP = \Sigma^{\infty}_{+} P \longrightarrow \Sigma^{\infty}_{+}(X \times P) \cong X_+ \wedge \Sigma^{\infty}_{+} P = X_+ \wedge SP.$$

This in turn endows SP with a left DX-module structure

$$DX \wedge_S SP \longrightarrow DX \wedge_S (X_+ \wedge SP) \cong (DX \wedge X_+) \wedge_S SP \longrightarrow S \wedge_S SP \cong SP.$$

This left DX-module structure commutes with the left R-module structure, and so defines a map of S-algebras

$$DX \longrightarrow F_R(SP, SP) = E.$$

To see that this map is a weak equivalence, consider the following diagram:

$$\begin{tikzcd} DX \arrow[r] \arrow[dr, "\cong"'] & F_R(SP, SP) \arrow[d] \\ & F_R(SP, S), \end{tikzcd}$$

where the right-hand map is induced by the map $SP \to S$ (induced by the map $P \to *$), and the slanted map is the isomorphism induced by the isomorphism $P/G = X$. This diagram commutes as the top-right composite is adjoint to the map $DX \wedge SP \to S$ induced by the diagonal $P \to X \times P$ followed by evaluation of DX on X and the trivial map $P \to *$, whereas the slanted map is induced by the map $P \to X$ followed by evaluation of DX on X:

$$\begin{tikzcd} DX \wedge P_+ \arrow[r] \arrow[d] & DX \wedge X_+ \wedge P_+ \arrow[d] \\ DX \wedge X_+ \arrow[r] & S. \end{tikzcd}$$

As the map $F_R(SP, SP) \to F_R(SP, S)$ is a weak equivalence, the S-algebra map $DX \to E$ is a weak equivalence.

We can obtain a model for the map $R \to \mathrm{Ext}_E(S, S)$ as follows. First, it is convenient to choose a cofibrant S-algebra approximation $E' \to DX$. Then the two-sided bar construction $SP' = B(DX, E', SP)$ is a semi-cofibrant DX-module approximation of SP. Furthermore, E'-maps $SP \to SP$ induce DX-maps $SP' \to SP'$. By construction, the (left) action of R on SP commutes with the (left) action of DX, making SP' an R-module in the category of DX-modules, or equivalently, producing a map of S-algebras $R \to F_{DX}(SP', SP')$.

This constructs the S-algebra map; we need to show that this map is a weak equivalence. Consider the cobar construction $C^\bullet(*, X, P)$,

$$C^n(*, X, P) = \underbrace{X \times \ldots \times X}_{n \text{ factors}} \times P,$$

with cosimplicial maps induced from the diagonal, the inclusion of the basepoint, and the map $P \to X$. The inclusion of G as the fiber of the fibration $P \to X$ induces a weak equivalence $G \to \mathrm{Tot}\, C^\bullet(*, X, P)$. Likewise, we get a map

$$R \longrightarrow \Sigma^\infty_+ \mathrm{Tot}\, C^\bullet(*, X, P) \longrightarrow \mathrm{Tot}\, \Sigma^\infty_+ C^\bullet(*, X, P).$$

Results of Dwyer [7] and Bousfield [5] (for $D_* = \pi^S_*$) show that this map is a weak equivalence, as X is simply connected. Moreover, when X is not simply connected, the 'only if' part of Dwyer's results shows that no model of this map will be a weak equivalence. The map $E' \to DX$ induces weak equivalences

$$E' \wedge_S \ldots \wedge_S E' \longrightarrow D(X \times \ldots \times X).$$

Together with the weak equivalence of E'-modules $SP \to S$, these induce weak equivalences

$$\begin{aligned} \Sigma^\infty_+(X \times \ldots \times X \times P) \cong X_+ \wedge \ldots \wedge X_+ \wedge SP &\longrightarrow F_S(E' \wedge_S \ldots \wedge_S E' \wedge S, SP) \\ &\longrightarrow F_S(E' \wedge_S \ldots \wedge_S E' \wedge SP, SP) \\ &\cong F_{DX}(DX \wedge_S E' \wedge_S \ldots \wedge_S E' \wedge_S SP, SP). \end{aligned}$$

These maps are compatible with the cosimplicial structure on the cobar construction and the maps induced by the simplicial structure on the bar construction $B(DX, E', SP)$, and induce a weak equivalence on Tot. Finally, the weak equivalence of E'-modules $SP' \to SP$ induces a weak equivalence $F_{DX}(SP', SP') \to F_{DX}(SP', SP)$. This describes the maps in the following diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[dd] & F_{DX}(SP', SP') \arrow[d] \\ & F_{DX}(B(DX, E', SP), SP) \\ \mathrm{Tot}\, \Sigma^\infty_+ C^\bullet(*, X, P) \arrow[r] & \mathrm{Tot}\, F_S(E' \wedge_S \ldots \wedge_S E', SP). \arrow[u] \end{tikzcd}$$

We have shown all maps but the top one to be weak equivalences, and so it suffices to observe that the diagram commutes up to homotopy. At each cosimplicial level, the right-down composite is adjoint to the map

$$R \wedge_S E' \wedge_S \ldots \wedge_S E' \wedge_S SP \longrightarrow SP$$

induced by the action of E' and R on SP. The down-right-up composite is adjoint to the composite map

$$R \wedge_S E' \wedge_S \ldots \wedge_S E' \wedge_S SP \longrightarrow R \wedge_S S \longrightarrow SP$$

induced by the augmentation $E' \to S$, the weak equivalence $SP \to S$, and the inclusion of G in P. A contraction $P \times I \to P$ onto the basepoint of P induces a homotopy from the former map to the latter map.

4. *Contravariant equivalences in algebraic K-theory and geometric Swan theory of spaces*

We now turn to the adjoint functors

$$\mathrm{Ext}_R(-, S)\colon \mathcal{D}_R \rightleftarrows \mathcal{D}_{DX} : \mathrm{Ext}_{DX}(-, S)$$

and describe our point-set model for the Quillen adjunction on the model categories $\mathcal{M}_R$ and $\mathcal{M}_{DX}$. The easiest and most obvious point-set model for these functors would be to use the adjunction $F_R(-, SP) : \mathcal{M}_R \to \mathcal{M}_{DX}$ as in the previous section, but instead we use an equivalent functor with better multiplicative properties.

The diagonal map $G \to G \times G$ induces a diagonal map $R \to R \wedge_S R$, which is clearly a map of S-algebras. This endows the category $\mathcal{M}_R$ of R-modules with a symmetric monoidal product, given by $\wedge_S$ on the underlying S-modules. As DX is a commutative S-algebra, the category $\mathcal{M}_{DX}$ has a symmetric monoidal product $\wedge_{DX}$. The diagonal map $SP \to SP \wedge_S SP$ on SP makes SP a cocommutative coalgebra in the category of R-modules and the diagonal map $SP \to SP \wedge_{DX} SP$ makes SP a cocommutative coalgebra in the category of DX-modules. For our adjunctions, we need a version of SP that is a commutative algebra in both categories.

Notation 4.1. Let $SP^\vee = F_S(SP, S) \cong S^{P_+}$, a left $(R \wedge_S DX)$-module.

Here F_S (and more generally F_R and F_{DX}, which we use below) denotes the function module construction of [**11**, Section III.6.1]. The commuting left R-module and DX-module structures on SP make $F_S(SP, S)$ naturally a right $(R \wedge_S DX)$-module, and we turn it into a left $(R \wedge_S DX)$-module using commutativity of DX and the anti-involution $R \to R$ induced by the inverse map $G \to G$. Using the diagonal map on SP, we get now a map of left $(R \wedge_S DX)$-modules

$$SP^\vee \wedge_{DX} SP^\vee \longrightarrow SP^\vee,$$

which is easily seen to be associative and commutative in the appropriate sense.

Moreover, we have a zigzag of weak equivalences of left $(R \wedge_S DX)$-modules relating SP and $SP^\vee$:

$$SP^\vee \longleftarrow SP^\vee \wedge_S SP \longrightarrow SP,$$

where we make $SP^\vee \wedge_S SP$ a left $(R \wedge DX)$-module using the diagonal R-module structure and the DX-module structure on $SP^\vee$. The leftward map is induced by the map of R-modules $SP \to S$, and the rightward map is induced by the diagonal on SP and evaluation:

$$SP^\vee \wedge_S SP = F_S(SP, S) \wedge_S SP \longrightarrow F_S(SP, S) \wedge_S SP \wedge_S SP \longrightarrow S \wedge_S SP \cong SP.$$

This is clearly a map of R-modules as each map in the composite is, and it is a map of DX-modules as the DX-module structure on $SP^\vee$ is adjoint to the map

$$DX \wedge_S SP^\vee \wedge_S SP \longrightarrow DX \wedge_S SP^\vee \wedge_S (X_+ \wedge SP) \cong (DX \wedge X_+) \wedge_S (SP^\vee \wedge_S SP) \longrightarrow S$$

induced by the diagonal on SP and evaluation.

Using the commuting left R-module and DX-module structures on $SP^\vee$, we get adjoint functors

$$F_R(-,SP^\vee)\colon \mathcal{M}_R \rightleftarrows \mathcal{M}_{DX} : F_{DX}(-,SP^\vee)$$

between the (point-set) categories of R-modules and DX-modules modeling the $\mathrm{Ext}_R(-,S)$ and $\mathrm{Ext}_{DX}(-,S)$ adjunction on derived categories. The unit maps of this adjunction are the maps

$$X \longrightarrow F_{DX}(F_R(X,SP^\vee),SP^\vee) \quad \text{and} \quad Y \longrightarrow F_R(F_{DX}(Y,SP^\vee),SP^\vee)$$

adjoint to the (R-module and DX-module) maps

$$M \wedge_S F_{DX}(M,SP^\vee) \longrightarrow SP^\vee \quad \text{and} \quad N \wedge_S F_R(N,SP^\vee) \longrightarrow SP^\vee$$

induced by evaluation. As fibrations and weak equivalences in EKMM module categories are detected on the underlying S-modules, the functors $F_R(-,SP^\vee)$ and $F_{DX}(-,SP^\vee)$ convert cofibrations and acyclic cofibrations to fibrations and acyclic fibrations. The adjunction above is therefore a Quillen adjunction.

We note that these functors are lax symmetric monoidal. We have the natural transformations

$$\begin{aligned} F_R(M_1,SP^\vee) \wedge_{DX} F_R(M_2,SP^\vee) &\longrightarrow F_R(M_1 \wedge_S M_2, SP^\vee \wedge_{DX} SP^\vee) \\ &\longrightarrow F_R(M_1 \wedge_S M_2, SP^\vee) \end{aligned}$$

and

$$\begin{aligned} F_{DX}(N_1,SP^\vee) \wedge_S F_{DX}(N_2,SP^\vee) &\longrightarrow F_{DX}(N_1 \wedge_{DX} N_2, SP^\vee \wedge_{DX} SP^\vee) \\ &\longrightarrow F_{DX}(N_1 \wedge_{DX} N_2, SP^\vee) \end{aligned}$$

induced by the multiplication $SP^\vee \wedge_{DX} SP^\vee \to SP^\vee$, which is both a map of R-modules and of DX-modules. Note that because G is a CW complex, $M_1 \wedge_S M_2$ is in fact a cofibrant R-module when M_1 and M_2 are cofibrant R-modules. The lax unit natural transformations

$$DX \longrightarrow F_R(S,SP^\vee) \cong F_S(SP \wedge_R S, S) \quad \text{and} \quad S \longrightarrow F_{DX}(DX,SP^\vee) \cong SP^\vee$$

are induced by the identification $P/G = X$ (for the first map) and the map of R-modules $SP \to S$ (for the second map).

When M is a cofibrant R-module approximation to SP (for example, $X = SP \wedge S_c$ for a cofibrant S-module approximation of S), we have a weak equivalence of DX-modules

$$DX \longrightarrow E = F_R(SP,SP) \longrightarrow F_R(M,SP) \simeq F_R(M,SP^\vee).$$

It follows that the left derived functors of $F_R(-,SP^\vee)$ and $F_{DX}(-,SP^\vee)$ induce an equivalence between the thick subcategories of the homotopy categories generated by S in $\mathcal{D}_R$ and by DX in $\mathcal{D}_{DX}$. The latter is the category of compact objects $\mathcal{D}^c_{DX}$. As in Section 1, we denote the former subcategory by $\mathcal{T}_R(S)$.

Proposition 4.2. *The derived functors* $\mathrm{Ext}_R(-,S)$ *and* $\mathrm{Ext}_{DX}(-,S)$ *induce inverse equivalences between* $\mathcal{T}_R(S)$ *and* $\mathcal{D}^c_{DX}$.

Likewise, when N is a cofibrant DX-module approximation to SP', we have a weak equivalence of R-modules

$$R \longrightarrow F_{DX}(SP',SP') \longrightarrow F_{DX}(SP',SP) \simeq F_{DX}(SP',SP^\vee) \longrightarrow F_{DX}(N,SP^\vee).$$

It follows that the left derived functors of $F_R(-,SP^\vee)$ and $F_{DX}(-,SP^\vee)$ induce an equivalence between the thick subcategories of the homotopy categories generated by S in $\mathcal{D}_{DX}$ and by R

in $\mathcal{D}_R$. The latter is the category of compact objects $\mathcal{D}_R^c$. As in Section 1, we denote the former subcategory by $\mathcal{T}_{DX}(S)$.

PROPOSITION 4.3. *The derived functors* $\mathrm{Ext}_R(-,S)$ *and* $\mathrm{Ext}_{DX}(-,S)$ *induce inverse equivalences between* $\mathcal{D}_R^c$ *and* $\mathcal{T}_{DX}(S)$.

We obtain Waldhausen category structures modeling each of the subcategories $\mathcal{D}_R^c$, $\mathcal{T}_R(S)$ in $\mathcal{D}_R$ and $\mathcal{D}_{DX}^c$, $\mathcal{T}_{DX}(S)$ in $\mathcal{D}_R$ as follows. We consider the full subcategory of cofibrant objects in the model category of R-modules or DX-modules whose images in the homotopy category lie in the subcategory in question. (To make these categories small, we can fix a set $\mathfrak{X}$ of sufficiently large cardinality and restrict to objects whose point sets are subsets of $\mathfrak{X}$ as in [**4**, 1.7].) We denote these Waldhausen categories as $\mathcal{M}_R^c$, $\mathcal{M}_R(S)$, $\mathcal{M}_{DX}^c$, and $\mathcal{M}_{DX}(S)$, respectively. We then get associated K-theory spectra, including Waldhausen's algebraic K-theory of X and the geometric Swan theory of X.

DEFINITION 4.4. In the notation above, $A(X)=K(R)=K(\mathcal{M}_R^c)$, $G(X)=K(\mathcal{M}_R(S))$, $K(DX)=K(\mathcal{M}_{DX}^c)$.

The biexact smash product $\wedge_S$ makes $G(X)$ into an E_∞ ring symmetric spectrum and the biexact smash product $\wedge_{DX}$ makes $K(DX)$ into an E_∞ ring symmetric spectrum by Theorem 2.6. Likewise, the biexact functors $\wedge_S$ and $\wedge_{DX}$ make $A(X)$ into a module over $G(X)$ and $K(\mathcal{M}_{DX}(S))$ into a module over $K(DX)$. We next explain how the functors $F_R(-,SP^\vee)$ and $F_{DX}(-,SP^\vee)$ induce weak equivalences of these E_∞ ring symmetric spectra and modules.

Although the functors $F_R(-,SP^\vee)$ and $F_{DX}(-,SP^\vee)$ are not exact (and do not land in the model Waldhausen categories), we do immediately obtain weak equivalences of spectra $K(DX)\to G(X)$ and $K(\mathcal{M}_{DX}(S))\to A(X)$, using the $S'_\bullet$ construction, a homotopical variant of the $S_\bullet$ construction introduced in [**4**]. Rather than working with pushouts over cofibrations, the $S'_\bullet$ construction depends on a theory of 'homotopy cocartesian' squares. The $S'_\bullet$ construction replaces the cofibrations and pushouts in $S_\bullet$ with homotopy cocartesian squares. Under mild hypotheses [**3**, Appendix A], the natural inclusion $S_\bullet\mathcal{C}\to S'_\bullet\mathcal{C}$ is a weak equivalence. To study the products and pairings, we take a different approach that allows us to continue working only with objects that are cofibrant.

First consider the categories $\mathcal{M}_R(S)$ and $\mathcal{M}_{DX}^c$. For each $q,n_1,\ldots,n_q$, consider the category whose objects consist of an element A of $S^{(q)}_{n_1,\ldots,n_q}\mathcal{M}_R(S)$, an element B of $S^{(q)}_{n_1,\ldots,n_q}\mathcal{M}_{DX}^c$, and weak equivalences

$$\phi_{i_1,j_1;\ldots;i_q,j_q}\colon A_{i_1,j_1;\ldots;i_q,j_q}\longrightarrow F_{DX}(B_{n_1-j_1,n_1-i_1;\ldots;n_q-j_q,n_q-i_q},SP^\vee),$$

making the $\mathrm{Ar}[n_1]\times\ldots\times\mathrm{Ar}[n_q]$ diagram commute. A map (A,B,ϕ) to (A',B',ϕ') consists of weak equivalences $A\to A'$, $B\to B'$ such that the composite

$$A_*\longrightarrow A'_*\xrightarrow{\phi'}F_{DX}(B'_*,SP^\vee)\longrightarrow F_{DX}(B_*,SP^\vee)$$

is ϕ_*. This forms a multisimplicial category, where we use the opposite ordering in each simplicial direction on $S^{(q)}_{\bullet,\ldots,\bullet}\mathcal{M}_{DX}^c$. Taking the classifying space, we obtain a sequence of spaces $T(q)$ with the structure of a symmetric spectrum.

The smash products on $\mathcal{M}_R$ and $\mathcal{M}_{DX}$ and the lax symmetric monoidal transformations above induce maps

$$T(p)\wedge T(q)\longrightarrow T(p+q)$$

and a multiplication $T\wedge T\to T$. We obtain a map $T\to G(X)$ dropping the $\mathcal{M}_{DX}^c$ data; we also obtain a map $T\to K(DX)$ by dropping the $\mathcal{M}_R(S)$ data and using the canonical homeomorphism between the geometric realization of a simplicial set and its opposite. Both maps preserve the E_∞ structures; Lemmas 4.6 and 4.7 complete the proof of Theorem 1.1 and the first part of Theorem 1.3 by showing that these maps are weak equivalences.

The analogous construction, with $\mathcal{M}^c_R$ and $\mathcal{M}_{DX}(S)$ in place of $\mathcal{M}_R(S)$ and $\mathcal{M}^c_{DX}$, respectively, produces a symmetric spectrum U that is a module over T. The analogous maps $U \to A(X)$ and $U \to K(\mathcal{M}_{DX}(S))$ are T-module maps; again Lemmas 4.6 and 4.7 show that these maps are weak equivalences and complete the proof of Theorem 1.2 and the remaining part of Theorem 1.3.

Before stating Lemma 4.6, we abstract the construction used to build the pieces of T and U. Consider the following construction.

Construction 4.5. Let $\mathcal{C}$ be a Waldhausen category, $\overline{\mathcal{M}}$ be a pointed closed model category, and $\mathcal{M}$ be a closed Waldhausen subcategory of cofibrant objects in $\overline{\mathcal{M}}$, that is, a Waldhausen category under the cofibrations and weak equivalences from $\overline{\mathcal{M}}$, which is closed under weak equivalences in $\overline{\mathcal{M}}$. Let $F : \mathcal{C} \to \overline{\mathcal{M}}$ be a contravariant functor that takes $*$ to $*$, cofibrations to fibrations, and weak equivalences to weak equivalences. Define MF to be the following category. An object of MF consists of an object A of $\mathcal{M}$, an object B of $\mathcal{C}$, and a weak equivalence $\phi : A \to FB$. A map in MF from (A, B, ϕ) to (A', B', ϕ') consists of weak equivalences $A \to A'$ and $B \to B'$ such that the composite map

$$A \longrightarrow A' \xrightarrow{\phi'} FB' \longrightarrow FB$$

is ϕ. We have canonical functors $MF \to \mathrm{w}\mathcal{C}$ and $MF \to \mathrm{w}\mathcal{M}$ obtained by dropping the $\mathcal{M}$ and $\mathcal{C}$ data, respectively.

Lemma 4.6. *With notation as above*:

(i) *if for every object A of $\mathcal{C}$, FA is weakly equivalent in $\overline{\mathcal{M}}$ to an object of $\mathcal{M}$, then the functor $MF \to \mathrm{w}\mathcal{C}$ induces a weak equivalence on nerves*;

(ii) *if $\mathcal{C}$ is a closed Waldhausen subcategory of cofibrant objects in a closed model category $\overline{\mathcal{C}}$ and F is a left Quillen adjoint that induces an equivalence between the full subcategories of $\mathrm{Ho}\,\overline{\mathcal{C}}$ and $\mathrm{Ho}\,\overline{\mathcal{M}}$ generated by $\mathcal{C}$ and $\mathcal{M}$, respectively, then $MF \to \mathrm{w}\mathcal{M}$ also induces a weak equivalence on nerves.*

Proof. For the first statement, we apply Quillen's Theorem A. For an object B of $\mathcal{C}$, the relevant category $FM \downarrow B$ has objects the maps $\phi : A \to FC$, $\gamma : C \to B$, where A is a cofibrant object in $\overline{\mathcal{M}}$, C is an object in $\mathcal{C}$, and ϕ and γ are weak equivalences. The nerve of this category is equivalent to the nerve of the subcategory where $C = B$ and γ is the identity. This is the category of cofibrant approximations of the fibrant object FC; work of Dwyer–Kan (cf. [**9**, 6.12]) shows that the nerve of this category is contractible.

For the second statement, let G denote the contravariant left adjoint of F. Then, under the hypotheses of the second statement, a map $A \to FB$ is a weak equivalence if and only if the adjoint map $B \to GA$ is a weak equivalence. The second statement now follows from the first. $\square$

In the case considered above, we are looking at functors F of the form

$$S^{(q)}_{\bullet,\dots,\bullet}\mathcal{M}_{DX}(S) \longrightarrow \mathrm{Ar}[\bullet,\dots,\bullet](\mathcal{M}_R) \quad \text{or} \quad S^{(q)}_{\bullet,\dots,\bullet}\mathcal{M}^c_{DX} \longrightarrow \mathrm{Ar}[\bullet,\dots,\bullet](\mathcal{M}_R),$$

where we have written $\mathrm{Ar}[\bullet,\dots,\bullet](\mathcal{C})$ for the category of functors from $\mathrm{Ar}[\bullet,\dots,\bullet]$ to $\mathcal{C}$ (where $\mathrm{Ar}[n_1,\dots,n_q]$ is as in Construction 2.2). Both $S^{(q)}_{\bullet,\dots,\bullet}\mathcal{M}_{DX}(S)$ and $S^{(q)}_{\bullet,\dots,\bullet}\mathcal{M}^c_{DX}$ are closed Waldhausen subcategories of the cofibrant objects in $\mathrm{Ar}[\bullet,\dots,\bullet](\mathcal{M}_{DX})$. As every map in $\mathrm{Ar}[\bullet,\dots,\bullet](\mathcal{M}_R)$ is weakly equivalent to a cofibration, and a commuting square in $\mathcal{M}_R$ is a homotopy pushout square if and only if it is a homotopy pullback square if and only if it is weakly equivalent to a pullback square of fibrations, an easy inductive argument proves the following lemma.

LEMMA 4.7. *The functor $F_{DX}(-, SP^{\vee})$ induces equivalences between*

(i) *the full subcategory of the homotopy category of* $\mathrm{Ar}[n_1, \ldots, n_q](\mathcal{M}_{DX})$ *generated by objects of* $S^{(q)}_{n_1,\ldots,n_q}\mathcal{M}^c_{DX}$*; and*

(ii) *the full subcategory of the homotopy category of* $\mathrm{Ar}[n_1, \ldots, n_q](\mathcal{M}_R)$ *generated by objects of* $S^{(q)}_{n_1,\ldots,n_q}\mathcal{M}_R(S)$.

It also induces equivalences between

(i) *the full subcategory of the homotopy category of* $\mathrm{Ar}[n_1, \ldots, n_q](\mathcal{M}_{DX})$ *generated by objects of* $S^{(q)}_{n_1,\ldots,n_q}\mathcal{M}_{DX}(S)$*; and*

(ii) *the full subcategory of the homotopy category of* $\mathrm{Ar}[n_1, \ldots, n_q](\mathcal{M}_R)$ *generated by objects of* $S^{(q)}_{n_1,\ldots,n_q}\mathcal{M}^c_R$.

5. *Covariant equivalences in algebraic K-theory and geometric Swan theory of spaces*

Using the models described in Section 3, the generalized Morita theory of [**8**] admits a point-set refinement into adjoint pairs of covariant functors

$$F_R(SP, -)\colon \mathcal{M}_R \leftrightarrows \mathcal{M}_{DX} :(-) \wedge_{DX} SP$$

and

$$(-) \wedge_R SP'\colon \mathcal{M}_R \rightleftarrows \mathcal{M}_{DX} :F_{DX}(SP', -),$$

forming Quillen adjunctions. Here we switch between left and right modules at will using the commutativity of DX and the anti-involution on R (induced by the inverse map on the topological group G).

As S is compact in $\mathcal{D}_R$, the first adjunction induces an equivalence between the localizing subcategory of $\mathcal{D}_R$ generated by S and $\mathcal{D}_{DX}$, and, in particular, it restricts to an equivalence between $\mathcal{T}_R(S)$ and $\mathcal{D}^c_{DX}$. In general, S is not compact in $\mathcal{D}_{DX}$, but nonetheless the second adjoint pair yields an equivalence between $\mathcal{T}_{DX}(S)$ and $\mathcal{D}^c_R$. In this case, one of the functors in each pair is exact, and so Waldhausen's approximation theorem (or the more general formulations of [**3** or **29**]) implies that these equivalences induce the equivalences

$$K(DX) \longrightarrow G(X) \quad \text{and} \quad A(X) \longrightarrow K(\mathcal{M}_{DX}(S)).$$

Combining these equivalences with the equivalences of the previous section, we obtain self-homotopy equivalences on $A(X)$ and $G(X)$. We complete our analysis by identifying these as the standard Spanier–Whitehead duality involution on $A(X)$ and an analogous involution on $G(X)$. (Note that when X is a smooth manifold, this involution is generally not compatible with the involution on pseudo-isotopy theory unless X is parallelizable [**30**].) Roughly, the involution on $A(X)$ is given by the functor that takes a left R-module M to the right R-module $\mathrm{Ext}_R(M, R)$, which we transform into a left R-module via the anti-involution $R \to R^{\mathrm{op}}$. For $G(X)$, the involution is similar but with $\mathrm{Ext}_S(M, S)$ instead.

Because the duality maps are contravariant, it is convenient to work with Waldhausen categories, modeling the opposite categories of $\mathcal{D}^c_R$ and $\mathcal{T}_R(S)$. As observed in [**4**, Section 1], $\mathcal{M}^{\mathrm{op}}_R$ has the structure of a Waldhausen category with weak equivalences the maps opposite to the usual weak equivalences and cofibrations the maps opposite to the Hurewicz fibrations. Let $\mathcal{M}^{\mathrm{op},c}_R$ be the full subcategory of objects that are opposite to compact objects in $\mathcal{D}^c_R$, and let $\mathcal{M}^{\mathrm{op}}_R(S)$ be the full subcategory of $\mathcal{M}^{\mathrm{op}}_R$ opposite to objects in $\mathcal{T}_R(S)$ (again, we can make these latter two Waldhausen categories small by restricting to subsets of a set with high cardinality). The argument for [**4**, 1.1] (see the discussion following [**4**, 2.9]) provides the weak equivalences

$$A(X) = K(\mathcal{M}_R^c) \simeq K(\mathcal{M}_R^{\mathrm{op},c}) \quad \text{and} \quad G(X) = K(\mathcal{M}_R(S)) \simeq K(\mathcal{M}_R^{\mathrm{op}}(S)).$$

Essentially, the map on S_n sends $A = \{A_{i,j}\}$ to $A' = \{A'_{i,j}\}$, where $A'_{i,j} \simeq A_{n-j,n-i}$ and the pushouts over cofibrations have been replaced by equivalent pullbacks over fibrations.

The functors $F_R(-,R) : \mathcal{M}_R^c \to \mathcal{M}_R^{\mathrm{op},c}$ and $F_S(-,S) : \mathcal{M}_R(S) \to \mathcal{M}_R^{\mathrm{op}}(S)$ are then exact. Under the equivalences above, the induced maps on K-theory represent the canonical involution. Thus, it now suffices to compare our composite functors to these functors.

In the case of $A(X)$, the composite of our equivalences is the functor $\mathcal{M}_R^c \to \mathcal{M}_R^{\mathrm{op},c}$ defined as

$$M \longmapsto F_{DX}(M \wedge_R SP', SP^\vee).$$

By adjunction, this is naturally isomorphic to $F_R(-, F_{DX}(SP', SP^\vee))$. The weak equivalence $R \to F_{DX}(SP', SP^\vee)$ then induces a natural weak equivalence from the duality functor $F_R(-,R)$.

For $G(X)$, the argument above shows that the composite map on $K(DX) \to K(\mathcal{M}_{DX}^{\mathrm{op},c})$ is the functor $F_R(- \wedge_{DX} SP, SP^\vee)$ and is naturally weakly equivalent to the duality map $F_{DX}(-,DX)$. On the other hand, $F_R(SP,-) : \mathcal{M}_R^{\mathrm{op}}(S) \to \mathcal{M}_{DX}^{\mathrm{op},c}$ is exact and the following solid arrow diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathcal{M}_R(S) \arrow[r, dotted, "F_S(-,S)"] & \mathcal{M}_R^{\mathrm{op}}(S) \arrow[d, "F_R(SP,-)"] \\
\mathcal{M}_{DX}^c \arrow[u, dotted, "(-)\wedge_{DX} SP"] \arrow[ur, "F_{DX}(-,SP^\vee)"] \arrow[r, "F_R(-\wedge_{DX} SP, SP^\vee)"'] & \mathcal{M}_{DX}^{\mathrm{op},c}.
\end{tikzcd}$$

The composite of the dotted arrows is the functor $F_S((-) \wedge_{DX} SP, S)$. By the smash-function adjunction, we see that this functor is naturally isomorphic to $F_{DX}(-, F_S(SP,S))$, which is the diagonal arrow as $SP^\vee = F_S(SP,S)$.

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