THE LOCAL COHOMOLOGY SPECTRAL SEQUENCE
FOR TOPOLOGICAL MODULAR FORMS

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ABSTRACT. We discuss proofs of local cohomology theorems for topological modular forms, based on Mahowald–Rezk duality and on Gorenstein duality, and then make the associated local cohomology spectral sequences explicit, including their differential patterns and hidden extensions.

1. INTRODUCTION

Several interesting ring spectra satisfy duality theorems relating local cohomology to Anderson or Brown–Comenetz duals. The algebraic precursor of these results is due to Grothendieck [Har67], and is a local analogue of Serre's projective duality theorem. In each case there is a covariant local cohomology spectral sequence converging to the homotopy of the local cohomology spectrum, and a contravariant Ext spectral sequence computing the homotopy of a functionally dual spectrum. As a consequence of self-dualities intrinsic to the ring spectra in question, the results of the two calculations agree up to a shift in grading, in spite of their opposite variances. It is the purpose of this paper to make these self-dualities explicit for the connective topological modular forms spectrum. Figures 8.1 and 8.9 depict the 2- and 3-complete dualities, respectively. A reader wondering if this paper is of interest might glance at these figures; they do require explanation (given below), but the structural patterns are immediately and strikingly apparent in the pictures. A reader new to tmf might prefer to start with the simpler charts for $p = 3$, as preparation for the case of $p = 2$. We also treat the much simpler case of the connective real K-theory spectrum.

We work at one prime p at a time, write $ko = ko_p^\wedge$ and $ku = ku_p^\wedge$ for the p-completed real and complex connective topological K-theory spectra, and write $tmf = tmf_p^\wedge$ for the p-completed connective topological modular forms spectrum. We also consider a 2-complete spectrum $tmf_1(3) = tmf_1(3)_2^\wedge$ and a 3-complete spectrum $tmf_0(2) = tmf_0(2)_3^\wedge$ related to elliptic curves with $\Gamma_1(3)$ and $\Gamma_0(2)$ level structures, respectively. These are all commutative S_p-algebras, where $S_p = S_p^\wedge$ denotes the p-completed sphere spectrum. See [DFHH14] and [BR21] for theoretical and computational background regarding topological modular forms. For $p = 2$ there are Bott and Mahowald classes $B \in \pi_8(tmf)$ and $M \in \pi_{192}(tmf)$ detected by the modular forms c_4 and Δ^8, respectively. The homotopy groups $\pi_*(tmf)$ for $0 \le * \le 192$ are shown in Figure 8.2. The red dots indicate B-power torsion classes, and the entire picture repeats M-periodically.

For any commutative S_p-algebra R and a choice of finitely generated ideal $J = (x_1, \dots, x_d) \subset \pi_*(R)$, the local cohomology spectrum $\Gamma_J R$ encapsulates the J-power torsion of $\pi_*(R)$, together with its right derived functors. There is a local cohomology spectral sequence

$$E_2^{s,t} = H_J^s(\pi_*(R))_t \Longrightarrow_s \pi_{t-s}(\Gamma_J R)$$

(in Adams grading), which can be used to compute its homotopy. For $R = tmf$ at $p = 2$ and $J = (B, M)$ the spectral sequence collapses to a short exact sequence

$$0 \to H^2_{(B,M)}(\pi_*(tmf))_{n+2} \longrightarrow \pi_n(\Gamma_{(B,M)}tmf) \longrightarrow H^1_{(B,M)}(\pi_*(tmf))_{n+1} \to 0$$

in each topological degree n, cf. Figures 8.3 and 8.4, while for $J = (2, B, M)$ its E_2-term is concentrated in filtration degrees $1 \leq s \leq 3$ and contains nonzero d_2-differentials, cf. Figures 8.5 through 8.8.

The S_p-module Anderson and Brown–Comenetz duals $I_{\mathbb{Z}_p} R$ and IR are defined as function spectra $F_{S_p}(R, I_{\mathbb{Z}_p})$ and $F_{S_p}(R, I)$, where $I_{\mathbb{Z}_p}$ and I are so designed that the associated homotopy spectral sequences collapse to a short exact sequence

$$0 \to \mathrm{Ext}_{\mathbb{Z}_p}(\pi_{m-1}(R), \mathbb{Z}_p) \longrightarrow \pi_{-m}(I_{\mathbb{Z}_p} R) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\pi_m(R), \mathbb{Z}_p) \to 0$$

and an isomorphism $\pi_{-m}(IR) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\pi_m(R), \mathbb{Q}_p/\mathbb{Z}_p)$, respectively. The local cohomology duality theorems for tmf at $p = 2$ establish equivalences

$$(1.1) \qquad\qquad \Gamma_{(B,M)}tmf \simeq \Sigma^{-22} I_{\mathbb{Z}_2}(tmf)$$

$$(1.2) \qquad\qquad \Gamma_{(2,B,M)}tmf \simeq \Sigma^{-23} I(tmf),$$

which in particular imply that the covariantly defined $\pi_n(\Gamma_{(B,M)}tmf)$ and the contravariantly defined $\pi_{-m}(I_{\mathbb{Z}_2}tmf)$ are isomorphic for $n + m = -22$, and similarly that $\pi_n(\Gamma_{(2,B,M)}tmf)$ and $\mathrm{Hom}_{\mathbb{Z}_2}(\pi_m(tmf), \mathbb{Q}_2/\mathbb{Z}_2)$ are isomorphic for $n+m = -23$.

Figure 8.1 illustrates $\pi_*(tmf)$, $\pi_*(\Gamma_{(B,M)}tmf)$, $\pi_*(I_{\mathbb{Z}_2}(tmf))$ and the duality isomorphism, up to a degree shift, between the latter two graded abelian groups. More precisely, $\pi_*(tmf)$ is isomorphic to the 'basic block' $\pi_*(N)$ shown in the first part of the figure, tensored with $\mathbb{Z}[M]$. The local cohomology $\pi_*(\Gamma_{(B,M)}tmf)$ and the Anderson dual $\pi_*(I_{\mathbb{Z}_2}tmf)$ are isomorphic to $\pi_*(\Gamma_B N)$ and $\pi_*(I_{\mathbb{Z}_2}N)$ tensored with $\mathbb{Z}[M]/M^\infty$, respectively, up to appropriate degree shifts. The second part of the figure shows the covariantly defined $\pi_*(\Gamma_B N)$, while the third part shows the contravariantly defined $\pi_*(\Sigma^{171} I_{\mathbb{Z}_2}N)$. The nearly mirror symmetric isomorphism between the latter two graded abelian groups thus exhibits the duality isomorphism (1.1), in its 'basic block' form $\Gamma_B N \simeq \Sigma^{171} I_{\mathbb{Z}_2}N$. The same structure is presented in greater detail in Figures 8.2, 8.3 and 8.4, where the additive generators are named and the module action by η, ν (and B) is shown by lines increasing the topological degree by 1, 3 (and 8), respectively, but the distinctive symmetry implied by the duality theorem is most easily seen in the first figure.

Several different approaches lead to proofs of such local cohomology duality theorems. For fp-spectra X, i.e., bounded below and p-complete spectra whose mod p cohomology is finitely presented as a module over the Steenrod algebra, Mahowald and Rezk [MR99] determined the cohomology of the Brown–Comenetz dual of the finite $E(n)$-acyclisation $C_n^f X$. In many cases $C_n^f R$ is a local cohomology spectrum, and we show in Theorem 4.8 how this leads to duality theorems for $R = ko$ at all primes, and for $R = tmf$ at $p = 2$ and $p = 3$. This strategy ties nicely in with chromatic homotopy theory.

Next, Dwyer, Greenlees and Iyengar [DGI06] showed that for augmented ring spectra $R \to k$ such that $\pi_*(R) \to k$ is algebraically Gorenstein, the k-cellularisation $\mathrm{Cell}_k R$ is often equivalent to a suspension of IR or $I_{\mathbb{Z}_p} R$, for $k = \mathbb{F}_p$ or $k = \mathbb{Z}_p$, respectively. We use descent methods to extend this to ring spectra with a good map to an augmented ring spectrum $T \to k$ satisfying the algebraic Gorenstein property, e.g., with $\pi_*(T) = k[x_1, \ldots, x_d]$ polynomial over k. Moreover, $\mathrm{Cell}_k R$ is in many cases a local cohomology spectrum, and we show in Theorem 5.19 how this leads to duality theorems for $R = ko$ and $R = tmf$, at all primes p. This strategy emphasises commutative algebra inspired by algebraic geometry.

There is a growing list [Gre93], [Gre95], [BG97], [BG97b], [BG03], [DGI06], [BG08], [BG10], [Gre16], [GM17], [GS18] of examples known to enjoy Gorenstein duality. Many of them are of equivariant origin, or have $R = C^*(X)$ for a manifold X, or arise from Serre duality in derived algebraic geometry. For instance, Stojanoska [Sto12], [Sto14] used Galois descent and homotopy fixed point spectral sequences to deduce Anderson self-duality for Tmf from its covers $Tmf(2)$ (at $p = 3$) and $Tmf(3)$ (at $p = 2$). More recently, Bruner and Rognes [BR21] used a variant of the descent arguments above to directly deduce local cohomology duality theorems for tmf at $p = 2$ and $p = 3$ from similar theorems for $tmf_1(3)$ and $tmf_0(2)$, respectively. We summarise these results in Theorems 6.1 and 6.2.

The main goal of this paper is to draw on the Hopkins–Mahowald calculation of $\pi_*(tmf)$, as presented in [BR21], to make the local cohomology spectral sequences for $R = tmf$ at $p = 2$ and at $p = 3$, with $J = (B, M)$ and $J = (p, B, M)$, completely explicit. In order to determine the differential patterns and some of the hidden (filtration-shifting) multiplicative extensions in these spectral sequences, we rely on the local cohomology duality theorems to identify the abutments with shifts of the Anderson and Brown–Comenetz duals of tmf. This is carried out in Subsections 8.A and 8.B for $p = 2$, and in Subsections 8.C and 8.D for $p = 3$. See also the explanations in Subsection 8.E of the graphical conventions used in the charts. As a warm-up we first go through the corresponding, but far simpler, calculations for $R = ko$ at $p = 2$ in Section 7.

2. Colocalisations

2.A. **Small and proxy-small.** The stable homotopy category of spectra and, more generally, the homotopy category of R-modules for any fixed S-algebra R, are prototypical triangulated categories. We keep the terminology from [DGI06, 3.15, 4.6]: A full subcategory of a triangulated category is *thick* if it is closed under equivalences, integer suspensions, cofibres and retracts, and it is *localising* if it is furthermore closed under coproducts. An object A *finitely builds* an object X if X lies in the thick subcategory generated by A, and more generally A *builds* X if X lies in the localising subcategory generated by A. An R-module A is *small* if it is finitely built from R, and more generally it is *proxy-small* if there is a small R-module K that both builds A and is finitely built by A.

2.B. **Acyclisation.** We recall three related colocalisations. First, for any spectrum X and integer $n \geq 0$ let $C_n^f X \to X$ denote its *finite $E(n)$-acycl(ic)isation*, as defined by Miller [Mil92, §2]. Here $E(n)$ denotes the n-th p-local Johnson–Wilson spectrum, with coefficient ring $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$. The map $F(A, C_n^f X) \to F(A, X)$ is an equivalence for each finite $E(n)$-acyclic A, and $C_n^f X$ is built from finite $E(n)$-acyclic spectra. There is a natural equivalence $C_n^f X \simeq$

$X \wedge C_n^f S$, so for any R-module M the spectrum $C_n^f M$ admits a natural R-module structure. A p-local finite spectrum has *type $n + 1$* if it is $E(n)$-acyclic but not $E(n + 1)$-acyclic. If X is p-local, then by Hopkins–Smith [HS98, Thm. 7] any one choice of a finite spectrum A of type $n+1$ suffices to build $C_n^f X$. Inductively for each $n \geq 0$, Hovey and Strickland [HS99, Prop. 4.22] build a cofinal tower of *generalised Moore spectra S/I* of type $n + 1$, for suitable ideals $I = (p^{a_0}, v_1^{a_1}, \ldots, v_{n-1}^{a_{n-1}}, v_n^{a_n})$, such that there are homotopy cofibre sequences

$$\Sigma^{2(p^n - 1)a_n} S/I' \xrightarrow{v_n^{a_n}} S/I' \longrightarrow S/I$$

with $I' = (p^{a_0}, v_1^{a_1}, \ldots, v_{n-1}^{a_{n-1}})$. Here $S/() = S$ and $v_0 = p$. By [HS99, Prop. 7.10(a)] there is a natural equivalence $\mathrm{hocolim}_I F(S/I, X) \simeq C_n^f X$, where S/I ranges over this tower.

2.C. **Cellularisation.** Second, let k and M be R-modules. The *k-cellularisation* of M is the R-module map $\mathrm{Cell}_k M \to M$ such that $F_R(k, \mathrm{Cell}_k M) \to F_R(k, M)$ is an equivalence, and such that $\mathrm{Cell}_k M$ is built from k in R-modules. It can be realised as the cofibrant replacement in a right Bousfield localisation of the stable model structure on R-modules in symmetric spectra, cf. [Hir03, §5.1, §4.1], hence always exists.

Lemma 2.1. *If two R-modules k and ℓ mutually build one another, then $\mathrm{Cell}_k M \simeq \mathrm{Cell}_\ell M$ for all R-modules M. Conversely, if $\mathrm{Cell}_k M \simeq \mathrm{Cell}_\ell M$ for all M, then k and ℓ mutually build one another.*

Proof. If k builds ℓ, then $F_R(\ell, \mathrm{Cell}_k M) \to F_R(\ell, M)$ is an equivalence. If ℓ builds k, then $\mathrm{Cell}_k M$ is built from ℓ. If both conditions hold, then $M \to \mathrm{Cell}_k M$ is the ℓ-cellularisation of M.

If $\mathrm{Cell}_k \ell \simeq \mathrm{Cell}_\ell \ell = \ell$, then k builds ℓ, and if $k = \mathrm{Cell}_k k \simeq \mathrm{Cell}_\ell k$ then ℓ builds k, so if both hold then k and ℓ build one another. $\square$

Lemma 2.2. *If A is a p-local finite spectrum of type $n + 1$, and M is a p-local R-module, then $C_n^f M \simeq \mathrm{Cell}_{R \wedge A} M$ as R-modules.*

Proof. We know that R builds M in R-modules, and A builds $C_n^f S_{(p)}$ in S-modules, so $R \wedge A$ builds $M \wedge C_n^f S_{(p)} \simeq C_n^f M$ in R-modules. Moreover, $F_R(R \wedge A, C_n^f M) \to F_R(R \wedge A, M)$ is an equivalence, since this the same map as $F(A, C_n^f M) \to F(A, M)$. Hence $C_n^f M$ is the $R \wedge A$-cellularisation of M. $\square$

Let $\mathcal{E} = F_R(k, k)$ be the endomorphism S-algebra of the R-module k. An R-module M is *effectively constructible* from k if the natural map

$$F_R(k, M) \wedge_{\mathcal{E}} k \longrightarrow M$$

is an equivalence. It is proved in [DGI06, Thm. 4.10] that, if k is proxy-small, then this map always realises the k-cellularisation of M. Hence $\mathrm{Cell}_k M$ is determined by the right $\mathcal{E}$-module structure on $F_R(k, M)$, for proxy-small k.

2.D. **Local cohomology.** Third, suppose that R is a commutative S-algebra, and let $J = (x_1, \ldots, x_d)$ be a finitely generated ideal in the graded ring $\pi_*(R)$. For each $x \in \pi_*(R)$ define the *x-power torsion spectrum $\Gamma_x R$* by the homotopy (co-)fibre sequence

$$\Sigma^{-1} R[\tfrac{1}{x}] \xrightarrow{\alpha} \Gamma_x R \xrightarrow{\beta} R \xrightarrow{\gamma} R[\tfrac{1}{x}].$$

For any R-module M let

$$\Gamma_J M = \Gamma_{x_1} R \wedge_R \cdots \wedge_R \Gamma_{x_d} R \wedge_R M$$

be the *local cohomology spectrum*. By [GM95, §1, §3], this R-module only depends on the radical $\sqrt{J}$ of the ideal J. The convolution product of the short filtrations $\alpha \colon \Sigma^{-1} R[1/x_i] \to \Gamma_{x_i} R$ for $1 \le i \le d$ leads to a length d decreasing filtration of $\Gamma_J M$, with subquotients

$$F^s/F^{s+1} \simeq \bigvee_{1 \le i_1 < \cdots < i_s \le d} \Sigma^{-1} R[\frac{1}{x_{i_1}}] \wedge_R \cdots \wedge_R \Sigma^{-1} R[\frac{1}{x_{i_s}}] \wedge_R M \,.$$

In Adams indexing, the associated spectral sequence has E_1-term

$$E_1^{s,t} = \pi_{t-s}(F^s/F^{s+1}) \cong \bigoplus_{1 \le i_1 < \cdots < i_s \le d} \pi_t(M[\frac{1}{x_{i_1}}, \ldots, \frac{1}{x_{i_s}}])$$

for $0 \le s \le d$, with differentials $d_r \colon E_r^{s,t} \to E^{s+r,t+r-1}$. The d_1-differentials are induced by the various localisation maps $\gamma \colon R \to R[1/x_i]$, and the cohomology of (E_1, d_1) defines the *local cohomology groups* of the $\pi_*(R)$-module $\pi_*(M)$, in the sense of Grothendieck [Har67]. This construction defines the *local cohomology spectral sequence*

$$(2.1) \qquad\qquad E_2^{s,t} = H_J^s(\pi_*(M))_t \Longrightarrow_s \pi_{t-s}(\Gamma_J M) \,,$$

which is a strongly convergent $\pi_*(R)$-module spectral sequence, cf. [GM95, (3.2)]. As in the topological case, the local cohomology groups $H_J^*(\pi_*(M))$ only depend on J through its radical in $\pi_*(R)$, not on the explicit generators $x_1, \ldots, x_d$. We emphasise that the local cohomology spectrum $\Gamma_J M$, and the associated spectral sequence, are covariantly functorial in M.

Definition 2.3. Given a finite sequence $J = (x_1, \ldots, x_d)$ of elements in $\pi_*(R)$, we let

$$R/J = R/x_1 \wedge_R \cdots \wedge_R R/x_d \,,$$

where each R/x is defined by a homotopy cofibre sequence

$$\Sigma^{|x|} R \xrightarrow{x} R \longrightarrow R/x \longrightarrow \Sigma^{|x|+1} R \,.$$

We shall also write R/J for this R-module in contexts where J is interpreted as the ideal in $\pi_*(R)$ generated by the given sequence of elements. This is, however, an abuse of notation, since R/J depends upon the chosen generators for the ideal, not just on the ideal J itself. We may refer to R/J and $\Gamma_J R$ as the *Koszul complex* and the *stable Koszul complex*, respectively.

Lemma 2.4. *If $J = (x_1, \ldots, x_d)$ is a finitely generated ideal in $\pi_*(R)$, and M is any R-module, then $\mathrm{Cell}_{R/J} M \simeq \Gamma_J M$ as R-modules.*

Proof. We show that $\Gamma_J M$ is the R/J-cellularisation of M. An inductive argument, as in the proof of [DGI06, Prop. 9.3], shows that R/J finitely builds $R/x_1^m \wedge_R \cdots \wedge_R R/x_d^m$ for each $m \ge 1$. Passing to the colimit over m, it follows that R/J builds $\Gamma_J R$. Since R builds M, it also follows that R/J builds $\Gamma_J M$. Finally, $F_R(R/J, \Gamma_J M) \to F_R(R/J, M)$ is an equivalence, because $F_R(R/J, N[1/x_i]) \simeq *$ for each R-module N and any $1 \le i \le d$. $\square$

With notation as above, if $R \wedge A$ and R/J mutually build one another, then $C_n^f M \simeq \mathrm{Cell}_{R \wedge A} M \simeq \mathrm{Cell}_{R/J} M \simeq \Gamma_J M$ by Lemmas 2.2 and 2.4. Under slightly different hypotheses we can close the cycle and obtain this conclusion directly.

Lemma 2.5. *Let $I = (p^{a_0}, \ldots, v_n^{a_n})$ and $J = (x_1, \ldots, x_d)$. If*

(1) *each x_i acts nilpotently on each $\pi_* F(S/I, R)$, and*
(2) *$v_s^{a_s}$ acts nilpotently on $\pi_* F(S/(p^{a_0}, \ldots, v_{s-1}^{a_{s-1}}), R/J)$ for each $0 \leq s \leq n$,*

then $C_n^f M \simeq \Gamma_J M$ as R-modules.

Proof. Item (1) ensures that $F(S/I, \Gamma_J R) \simeq \Gamma_J F(S/I, R)$ is equivalent to $F(S/I, R)$ for each I in the cofinal system, and passage to homotopy colimits implies that

$$C_n^f \Gamma_J R \xrightarrow{\simeq} C_n^f R$$

is an equivalence. Item (2) ensures that $C_n^f R/J \simeq \mathrm{hocolim}_I F(S/I, R/J)$ is equivalent to R/J, which implies that

$$C_n^f \Gamma_J R \xrightarrow{\simeq} \Gamma_J R$$

is an equivalence, since R/J builds $\Gamma_J R$. Hence $C_n^f R \simeq \Gamma_J R$, and more generally $C_n^f M = C_n^f R \wedge_R M \simeq \Gamma_J R \wedge_R M = \Gamma_J M$. $\qquad\square$

2.E. **A composite functor spectral sequence.** Let $I, J \subset R_*$ be finitely generated ideals in a graded commutative ring, and let M_* be an R_*-module. If $I = (x)$ we write $\Gamma_x M_* = H_I^0(M_*)$ and $M_*/x^\infty = H_I^1(M_*)$ for the kernel and the cokernel of the localisation homomorphism γ below.

$$0 \to \Gamma_x M_* \longrightarrow M_* \xrightarrow{\gamma} M_*[1/x] \longrightarrow M_*/x^\infty \to 0.$$

More generally, let $\Gamma_I M_* = H_I^0(M_*)$ denote the I-power torsion submodule of M_*. The identity $\Gamma_I(\Gamma_J M_*) = \Gamma_{I+J} M_*$ leads to a composite functor spectral sequence

$$E_2^{i,j} = H_I^i(H_J^j(M_*)) \Longrightarrow_i H_{I+J}^{i+j}(M_*).$$

This is a case of the double complex spectral sequence of [CE56, §XV.6]. When $I = (x)$ and $J = (y)$, it arises by horizontally filtering the condensation of the central commutative square below, leading to an E_1-term given by the inner modules in the upper and lower rows, and an E_2-term given by the modules at the four corners.

$$
\begin{array}{ccccccc}
\Gamma_x(M_*/y^\infty) & \rightarrowtail & M_*/y^\infty & \longrightarrow & M_*[1/x]/y^\infty & \twoheadrightarrow & (M_*/y^\infty)/x^\infty \\
& & \uparrow & & \uparrow & & \\
& & M_*[1/y] & \xrightarrow{\gamma} & M_*[1/x, 1/y] & & \\
& & \gamma\uparrow & & \gamma\uparrow & & \\
& & M_* & \xrightarrow{\ \gamma\ } & M_*[1/x] & & \\
& & \uparrow & & \uparrow & & \\
\Gamma_x(\Gamma_y M_*) & \rightarrowtail & \Gamma_y M_* & \longrightarrow & \Gamma_y M_*[1/x] & \twoheadrightarrow & (\Gamma_y M_*)/x^\infty
\end{array}
$$

For bidegree reasons, the spectral sequence collapses at this stage, so that

$$E_2^{i,j} = E_\infty^{i,j} = \begin{cases} \Gamma_x(\Gamma_y M_*) & \text{for } (i,j) = (0,0), \\ (\Gamma_y M_*)/x^\infty & \text{for } (i,j) = (1,0), \\ \Gamma_x(M_*/y^\infty) & \text{for } (i,j) = (0,1), \\ (M_*/y^\infty)/x^\infty & \text{for } (i,j) = (1,1). \end{cases}$$

It follows that we have identities

$$\Gamma_x(\Gamma_y M_*) = \Gamma_{(x,y)} M_* = H^0_{(x,y)}(M_*)$$

$$(M_*/y^\infty)/x^\infty = M_*/(x^\infty, y^\infty) = H^2_{(x,y)}(M_*),$$

and a natural short exact sequence

$$0 \to (\Gamma_y M_*)/x^\infty \longrightarrow H^1_{(x,y)}(M_*) \longrightarrow \Gamma_x(M_*/y^\infty) \to 0.$$

If $\Gamma_y M_* \subset \Gamma_x M_*$, so that the y-power torsion is entirely x-power torsion, then $(\Gamma_y M_*)/x^\infty = 0$ and $H^1_{(x,y)}(M_*) \cong \Gamma_x(M_*/y^\infty)$.

3. Dualities

3.A. **Artinian and Noetherian S_p-modules.** Let S_p denote the p-completed sphere spectrum. The category of S_p-modules contains a subcategory of *p-power torsion modules* satisfying $\Gamma_p M \simeq M$, and a subcategory of *p-complete modules* for which $M \simeq M_p^\wedge$. The (covariant) functors Γ_p and $(-)_p^\wedge$ give mutually inverse equivalences between these full subcategories, cf. [HPS97, Thm. 3.3.5].

We say that a p-power torsion module M is *Artinian* if each homotopy group $\pi_t(M)$ is an Artinian $\mathbb{Z}_p$-module, i.e., a finite direct sum of modules of the form $\mathbb{Q}_p/\mathbb{Z}_p$ or $\mathbb{Z}/p^a$ for $a \geq 1$. Dually, we say that a p-complete module M is *Noetherian* if each homotopy group $\pi_t(M)$ is a Noetherian $\mathbb{Z}_p$-module, i.e., a finite direct sum of modules of the form $\mathbb{Z}_p$ or $\mathbb{Z}/p^a$ for $a \geq 1$. The latter are the same as the finitely generated $\mathbb{Z}_p$-modules. The simultaneously Artinian and Noetherian S_p-modules M are those for which each $\pi_t(M)$ is finite.

3.B. **Brown–Comenetz duality.** We recall two related dualities. First, working in S_p-modules, the *Brown–Comenetz duality* spectrum I represents the cohomology theory

$$I^t(M) = \text{Hom}_{\mathbb{Z}_p}(\pi_t(M), \mathbb{Q}_p/\mathbb{Z}_p),$$

cf. [BC76]. This makes sense because $\mathbb{Q}_p/\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module. Letting $IM = F_{S_p}(M, I)$, we obtain a contravariant endofunctor I of S_p-modules, with $\pi_{-t}(IM) = I^t(M)$. It maps p-power torsion modules to p-complete modules, and restricts to a functor from Artinian S_p-modules to Noetherian S_p-modules, since

$$\text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}_p \quad \text{and} \quad \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}/p^a, \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}/p^a.$$

In general, it does not map p-complete modules to p-power torsion modules, but it does restrict to a functor from Noetherian S_p-modules to Artinian S_p-modules. Moreover, the natural map

$$\rho \colon M \longrightarrow I(IM)$$

is an equivalence for M that are Artinian or Noetherian. Hence the two restrictions of I are mutually inverse contravariant equivalences between Artinian S_p-modules and Noetherian S_p-modules. If the S_p-module action on M extends to a (left or right) R-module structure, then IM is naturally a (right or left) R-module.

3.C. **Anderson duality.** Second, the Eilenberg–MacLane spectrum $I_{\mathbb{Q}_p} = H\mathbb{Q}_p$ represents the ordinary rational cohomology theory

$$I_{\mathbb{Q}_p}^t(M) = \mathrm{Hom}_{\mathbb{Z}_p}(\pi_t(M), \mathbb{Q}_p)$$

in S_p-modules. The canonical surjection $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ induces a map of cohomology theories, and a map $I_{\mathbb{Q}_p} \to I$ of representing spectra, whose homotopy fibre defines the *Anderson duality* spectrum $I_{\mathbb{Z}_p}$, cf. [And69] and [Kai71]. Letting $I_{\mathbb{Z}_p}M = F_{S_p}(M, I_{\mathbb{Z}_p})$ and $I_{\mathbb{Q}_p}M = F_{S_p}(M, I_{\mathbb{Q}_p})$, we obtain a natural homotopy fibre sequence

$$\Sigma^{-1}IM \longrightarrow I_{\mathbb{Z}_p}M \longrightarrow I_{\mathbb{Q}_p}M \longrightarrow IM\,,$$

which lifts the injective resolution $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$. The associated long exact sequence in homotopy splits into short exact sequences

$$(3.1) \qquad 0 \to \mathrm{Ext}_{\mathbb{Z}_p}(\pi_{t-1}(M), \mathbb{Z}_p) \longrightarrow \pi_{-t}(I_{\mathbb{Z}_p}M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\pi_t(M), \mathbb{Z}_p) \to 0\,.$$

If the S_p-module action on M extends to a (left or right) R-module structure, then $I_{\mathbb{Z}_p}M$ is naturally a (right or left) R-module, and the short exact sequence above is one of $\pi_*(R)$-modules.

The contravariant endofunctor $I_{\mathbb{Z}_p}$ on S_p-modules is equivalent to $\Sigma^{-1}I$ on the subcategory of p-power torsion modules, since $I_{\mathbb{Q}_p}$ is trivial on these objects. More relevant to us is the fact that it maps Noetherian S_p-modules to Noetherian S_p-modules, since

$$\mathrm{Ext}_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \qquad\qquad \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$$
$$\mathrm{Ext}_{\mathbb{Z}_p}(\mathbb{Z}/p^a, \mathbb{Z}_p) \cong \mathbb{Z}/p^a \qquad\qquad \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}/p^a, \mathbb{Z}_p) = 0\,.$$

Moreover, the natural map

$$(3.2) \qquad\qquad\qquad \rho\colon M \longrightarrow I_{\mathbb{Z}_p}(I_{\mathbb{Z}_p}M)$$

is an equivalence for Noetherian M, cf. [Yos75, Thm. 2] and [Kna99, Cor. 2.8]. Hence $I_{\mathbb{Z}_p}$ restricts to a contravariant self-equivalence of Noetherian S_p-modules, being its own inverse equivalence.

We emphasise that the Brown–Comenetz and Anderson dual spectra, IM and $I_{\mathbb{Z}_p}M$, and the algebraic expressions for their homotopy groups, are contravariantly functorial in the S_p- or R-module M.

Lemma 3.1. *There are natural equivalences*

$$I(\Gamma_pM) \simeq (IM)_p^{\wedge} \simeq \Sigma(I_{\mathbb{Z}_p}M)_p^{\wedge}$$

Proof. For S_p-modules M and N we have $\Gamma_pS \wedge_{S_p} M = \Gamma_pM$ and $F_{S_p}(\Gamma_pS, N) = N_p^{\wedge}$. The first equivalence then follows from the adjunction $F_{S_p}(\Gamma_pS \wedge_{S_p} M, I) \simeq F_{S_p}(\Gamma_pS, F_{S_p}(M, I))$. The second equivalence follows from the homotopy fibre sequence defining the Anderson dual, since $(I_{\mathbb{Q}_p}M)_p^{\wedge}$ is trivial. If the S_p-module structure on M extends to an R-module structure, then this is respected by all of these equivalences. $\square$

4. Mahowald–Rezk duality

4.A. **Spectra with finitely presented cohomology.** Let A denote the mod p Steenrod algebra, where p is a prime. We write $H^*(X)$ for the mod p cohomology of a spectrum X, with its natural left A-module structure. For $n \geq 0$ let $A(n)$ be the finite sub (Hopf) algebra of A that is generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$ for $p = 2$, and by $\beta, P^1, \ldots, P^{p^{n-1}}$ for p odd. Also let $E(n)$ be the exterior sub (Hopf) algebra

of $A(n)$ generated by $Q_0, Q_1, \ldots, Q_n$, where $Q_0 = Sq^1$ and $Q_i = [Sq^{2^i}, Q_{i-1}]$ for $i \geq 1$ and $p = 2$, and $Q_0 = \beta$ and $Q_{i+1} = [P^{p^i}, Q_i]$ for $i \geq 0$ and p odd.

Let X be a spectrum that is p-complete and bounded below. Following Mahowald and Rezk [MR99, §3] we say that X is an *fp-spectrum* if $H^*(X)$ is finitely presented as an A-module. This is equivalent to asking that $H^*(X) \cong A \otimes_{A(n)} M$ is induced up from a finite $A(n)$-module M, for some n. We say that a graded abelian group π_* is *finite* if the direct sum $\bigoplus_t \pi_t$ is finite. The class of p-local finite spectra V such that $\pi_*(V \wedge X)$ is finite generates a thick subcategory of the stable homotopy category, and is therefore equal to the class of p-local finite spectra of type $\geq m+1$ for some well-defined integer $m \geq 0$. We then say that X has *fp-type* equal to m. In each case $n \geq m$, sometimes with strict inequality, cf. [BR, Prop. 3.9].

Theorem 4.1 ([MR99, Prop. 4.10, Thm. 8.2]). *Let X be p-complete and bounded below, with $H^*(X) \cong A \otimes_{A(n)} M$ for some finite $A(n)$-module M. Then $IC_n^f X$ is p-complete and bounded below, with $H^*(IC_n^f X) \cong A \otimes_{A(n)} \mathrm{Hom}_{\mathbb{F}_p}(M, \Sigma^{a(n)} \mathbb{F}_p)$, where $a(n)$ is the top degree of a nonzero class in $A(n)$.*

Recall that we write $ko = ko_p^\wedge$ and $tmf = tmf_p^\wedge$ for the p-completed connective real K-theory and topological modular forms spectra, respectively, and that these are commutative S-algebras.

Proposition 4.2 ([MR99, Cor. 9.3]). *There is an equivalence of ko-modules*
$$\Sigma^6 ko \simeq IC_1^f ko$$
(at all primes p), and an equivalence of tmf-modules
$$\Sigma^{23} tmf \simeq IC_2^f tmf$$
(at $p = 2$ and at $p = 3$). The underlying S_p-modules are Noetherian and bounded below.

Proof. For $X = ko$ completed at $p = 2$, we have $H^*(ko) \cong A//A(1) = A \otimes_{A(1)} \mathbb{F}_2$ by Stong [Sto63], so
$$H^*(IC_1^f ko) \cong A \otimes_{A(1)} \Sigma^6 \mathbb{F}_2 = \Sigma^6 A//A(1) \,,$$
since $a(1) = 6$. Choosing a map $S^6 \to IC_1^f ko$ generating the lowest homotopy (and homology) group, and using the natural ko-module structure on the target, we obtain a ko-module map $\phi: \Sigma^6 ko \to IC_1^f ko$. The induced A-module homomorphism $\phi^*: H^*(IC_1^f ko) \to H^*(\Sigma^6 ko)$ has the form $\Sigma^6 A//A(1) \to \Sigma^6 A//A(1)$, and is an isomorphism in degree 6, hence is an isomorphism in all degrees. It follows that ϕ is an equivalence of 2-complete ko-modules.

For $X = \ell = BP\langle 1 \rangle$ completed at any prime p we have $H^*(\ell) \cong A//E(1)$, essentially by [Sin68], so $H^*(IC_1^f \ell) \cong \Sigma^{2p} A//E(1)$ and $IC_1^f \ell \simeq \Sigma^{2p} \ell$. For p odd this implies the claim for $ko \simeq \bigvee_{i=0}^{(p-3)/2} \ell$.

For $X = tmf$ completed at $p = 2$, we have $H^*(tmf) \cong A//A(2) = A \otimes_{A(2)} \mathbb{F}_2$ by Hopkins–Mahowald and Mathew [Mat16, Thm. 1.1]. Hence
$$H^*(IC_2^f tmf) \cong A \otimes_{A(2)} \Sigma^{23} \mathbb{F}_2 = \Sigma^{23} A//A(2) \,,$$
since $a(2) = 23$. Choosing a map $S^{23} \to IC_2^f tmf$ generating the lowest homotopy group, and using the natural tmf-module structure on the target, we obtain a tmf-module map $\phi: \Sigma^{23} tmf \to IC_2^f tmf$. The induced A-module homomorphism

$\phi^* \colon H^*(IC_2^f tmf) \to H^*(\Sigma^{23} tmf)$ has the form $\Sigma^{23} A//A(2) \to \Sigma^{23} A//A(2)$, and is an isomorphism in degree 23, hence is an isomorphism in all degrees. It follows that ϕ is an equivalence of 2-complete tmf-modules.

For $X = tmf$ completed at $p = 3$, we have $H^*(tmf) \cong A \otimes_{A(2)} M$ for a finite $A(2)$-module M with $\mathrm{Hom}_{\mathbb{F}_3}(M, \mathbb{F}_3) \cong \Sigma^{-64} M$, by Proposition 4.3 below. It follows that

$$H^*(IC_2^f tmf) \cong A \otimes_{A(2)} \Sigma^{23} M \cong \Sigma^{23} H^*(tmf)$$

as A-modules, since $a(2) = 87$ and $87 - 64 = 23$. Choosing a map $S^{23} \to IC_2^f tmf$ generating the lowest homotopy group, and using the natural tmf-module structure on the target, we obtain a tmf-module map $\phi \colon \Sigma^{23} tmf \to IC_2^f tmf$. The induced A-module homomorphism $\phi^* \colon H^*(IC_2^f tmf) \to H^*(\Sigma^{23} tmf)$ has the form $\Sigma^{23} A \otimes_{A(2)} M \to \Sigma^{23} A \otimes_{A(2)} M$, and is an isomorphism in degree 23. It follows from the relation $P^3 g_0 = P^1 g_8$ that ϕ^* is also an isomorphism in degree $23 + 8 = 31$, hence in all degrees, and that ϕ is an equivalence of 3-complete tmf-modules.   □

Proposition 4.3. *At $p = 3$ there is an isomorphism $H^*(tmf) \cong A \otimes_{A(2)} M$ of A-modules, where*

$$M = \frac{A(2)//E(2)\{g_0, g_8\}}{(P^1 g_0, P^3 g_0 = P^1 g_8)}$$

is a finite $A(2)$-module of dimension 18 satisfying $\mathrm{Hom}_{\mathbb{F}_3}(M, \mathbb{F}_3) \cong \Sigma^{-64} M$.

Proof. Let all spectra be implicitly completed at $p = 3$. Let $\Psi = S \cup_\nu e^4 \cup_\nu e^8$, where $\nu = \alpha_1$ is detected by P^1. According to [Mat16, Thm. 4.16], there is an equivalence $tmf \wedge \Psi \simeq tmf_0(2)$ of tmf-module spectra, where $\pi_*(tmf_0(2)) = \mathbb{Z}_p[a_2, a_4]$ with $|a_2| = 4$ and $|a_4| = 8$. We take as known that $H^*(tmf_0(2)) \cong A//E(2)\{g_0, g_8\}$, where $|g_0| = 0$ and $|g_8| = 8$. The homotopy cofibre sequences

$$S \longrightarrow \Psi \longrightarrow \Sigma^4 C\nu$$
$$\Sigma^4 C\nu \longrightarrow \Sigma^4 \Psi \longrightarrow S^{12}$$

induce homotopy cofibre sequences

$$tmf \longrightarrow tmf_0(2) \longrightarrow \Sigma^4 tmf \wedge C\nu$$
$$\Sigma^4 tmf \wedge C\nu \longrightarrow \Sigma^4 tmf_0(2) \longrightarrow \Sigma^{12} tmf$$

of tmf-modules. Passing to cohomology, we get two short exact sequences, which we splice together to an exact complex

$$0 \to \Sigma^{12} H^*(tmf) \to \Sigma^4 A//E(2)\{g_0, g_8\} \xrightarrow{\partial} A//E(2)\{g_0, g_8\} \to H^*(tmf) \to 0$$

of A-modules. Here $\partial(\Sigma^4 g_0) \in \mathbb{F}_3\{P^1 g_0\}$ and $\partial(\Sigma^4 g_8) \in \mathbb{F}_3\{P^3 g_0, P^1 g_8\}$. Hence this complex is induced up from an exact complex

$$0 \to \Sigma^{12} M \to \Sigma^4 A(2)//E(2)\{g_0, g_8\} \xrightarrow{\partial} A(2)//E(2)\{g_0, g_8\} \to M \to 0$$

of $A(2)$-modules, where the rank of ∂ is twice the dimension of M. The dimension of $A(2)//E(2)$ is 27, so the dimension of M is 18. By exactness, $\partial(\Sigma^4 g_0)$ and $\partial(\Sigma^4 g_8)$ are nonzero. From [Cul21, Cor. 6.7] it follows that we can choose the signs of the generators so that $\partial(\Sigma^4 g_0) = P^1 g_0$ and $\partial(\Sigma^4 g_8) = P^3 g_0 - P^1 g_8$. This gives the stated presentation of M.

TABLE 4.1. Algebra generators x_k for $\pi_*(tmf)$ at $p = 2$

x	k	$\lvert x \rvert$
η	$0, 1, 4$	1
ν	$0, 1, 2, 4, 5, 6$	3
ϵ	$0, 1, 4, 5$	8
κ	$0, 4$	14
$\bar{\kappa}$	0	20
B	$0, 1, 2, 3, 4, 5, 6, 7$	8
C	$0, 1, 2, 3, 4, 5, 6, 7$	12
D	$1, 2, 3, 4, 5, 6, 7$	0
M	0	192

TABLE 4.2. Algebra generators x_k for $\pi_*(tmf)$ at $p = 3$

x	k	$\lvert x \rvert$
ν	$0, 1$	3
β	0	10
B	$0, 1, 2$	8
C	$0, 1, 2$	12
D	$1, 2$	0
H	0	72

Applying $D = \mathrm{Hom}_{\mathbb{F}_3}(-, \mathbb{F}_3)$ we find that $D(A(2)//E(2)) \cong \Sigma^{-64} A(2)//E(2)$, and can calculate that $D\partial$ has the same form as ∂, so that the dual of the exact $A(2)$-module complex above presents $D(\Sigma^{12} M) = \Sigma^{-12} DM$ as $\Sigma^{-76} M$, which implies that M is concentrated in degrees $0 \le * \le 64$, and is self-dual. $\square$

4.B. **Local cohomology theorems by Mahowald–Rezk duality.**

Notation 4.4. The graded ring structure of $\pi_*(ko)$ is well known [Bot59]. We use the notation
$$\pi_*(ko) = \mathbb{Z}_p[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$
where $\lvert \eta \rvert = 1$, $\lvert A \rvert = 4$ and $\lvert B \rvert = 8$, cf. [BR21, Ex. 2.30]. If p is odd this simplifies to $\pi_*(ko) = \mathbb{Z}_p[A]$. We call B the *Bott element*.

Notation 4.5. The graded ring structure of $\pi_*(tmf)$ is also known [DFHH14, Ch. 13], [BR21, Ch. 9, Ch. 13], up to a couple of finer points.

For $p = 2$, the graded commutative $\mathbb{Z}_2$-algebra $\pi_*(tmf)$ is generated by forty classes x_k, where $x \in \{\eta, \nu, \epsilon, \kappa, \bar{\kappa}, B, C, D, M\}$ and $0 \le k \le 7$. The indices k that occur are shown in Table 4.1. We abbreviate x_0 to x, and note that $\lvert x_k \rvert = \lvert x \rvert + 24k$ is positive in each case. We call $B \in \pi_8(tmf)$ and $M \in \pi_{192}(tmf)$ the *Bott element* and the *Mahowald element*, respectively. See Figure 8.2 for the mod 2 Adams E_∞-term for tmf in the range $0 \le t - s \le 192$.

For $p = 3$, the graded commutative $\mathbb{Z}_3$-algebra $\pi_*(tmf)$ is generated by twelve classes x_k, where $x \in \{\nu, \beta, B, C, D, H\}$ and $0 \le k \le 2$. The values of k that occur are shown in Table 4.2. We abbreviate x_0 to x, and again note that $|x_k| = |x| + 24k$ is positive in each case. We call $B \in \pi_8(tmf)$ and $H \in \pi_{72}(tmf)$ the *Bott element* and the *Hopkins–Miller element*, respectively. See Figure 8.10 for the mod 3 (*tmf*-module) Adams E_∞-term for tmf in the range $0 \le t - s \le 72$.

To avoid repetitive case distinctions we will sometimes write $\mathbb{Z}_p[B, M]$, (p, B, M) or (B, M), both for $p = 2$ and for $p = 3$, in spite of the fact that the correct notations for $p = 3$ would be $\mathbb{Z}_3[B, H]$, $(3, B, H)$ or (B, H). In effect, the element 'M' should be read as 'H' for $p = 3$.

Definition 4.6. For any p-complete connective S-algebra R with $\pi_0(R) = \mathbb{Z}_p$ let

$$\mathfrak{n}_0 = \ker(\pi_*(R) \to \mathbb{Z}_p)$$

denote the ideal in $\pi_*(R)$ given by the classes in positive degrees, and let

$$\mathfrak{n}_p = \ker(\pi_*(R) \to \mathbb{F}_p)$$

denote the maximal ideal generated by $\mathfrak{n}_0$ and p.

We shall review the precise structure of $\pi_*(tmf)$ as a $\mathbb{Z}_p[B, M]$-module in Section 8, but for now we will only need the following, more qualitative, properties. Their analogues for ko are straightforward.

Lemma 4.7. *The following hold for $p = 2$ and for $p = 3$.*
  (1) *The graded group $\pi_*(tmf)$ is finitely generated as a $\mathbb{Z}_p[B, M]$-module.*
  (2) *The radical of $(B, M) \subset \pi_*(tmf)$ is $\mathfrak{n}_0$, and the radical of (p, B, M) is $\mathfrak{n}_p$.*
  (3) *The graded group $\pi_*(tmf/(p, B, M))$ is finite.*
  (4) *Each $\pi_t(\Gamma_{(B,M)}tmf)$ is a finitely generated $\mathbb{Z}_p$-module.*

Proof. We use the notation and results of Subsections 8.A and 8.C. In particular, $N = tmf/M$, with $N_* \cong \pi_*(N)$.

(1) Since $\pi_*(tmf) \cong N_* \otimes \mathbb{Z}[M]$, it suffices to check that N_* is finitely generated as a $\mathbb{Z}_p[B]$-module, which is clear from the explicit expressions given in Theorems 8.4 and 8.15.

(2) Because $\pi_*(tmf)/(B, M)$ is finitely generated over $\mathbb{Z}_p$, each positive degree class is nilpotent, and therefore each class in $\mathfrak{n}_0$ lies in the radical of (B, M). The other conclusions follow.

(3) Since $\Gamma_B N_*$ is finite, each $ko[k]$ is B-torsion free, and each $ko[k]/B$ is a finitely generated $\mathbb{Z}_p$-module, it follows that $\pi_*(N/B)$ is a finitely generated $\mathbb{Z}_p$-module. Hence $\pi_*(N/(p, B)) \cong \pi_*(tmf/(p, B, M))$ is finite.

(4) Likewise, since $\Gamma_B N_*$ is finite and each $ko[k]/B^\infty$ is bounded above and finitely generated over $\mathbb{Z}_p$ in each degree, it follows that $\pi_*(\Gamma_B N)$ is bounded above and finitely generated over $\mathbb{Z}_p$ in each degree. Hence $\pi_*(\Gamma_{(B,M)}tmf) \cong \pi_*(\Gamma_B N) \otimes \mathbb{Z}[M]/M^\infty$ is also bounded above and finitely generated over $\mathbb{Z}_p$ in each degree. $\square$

Theorem 4.8. *There are equivalences of ko-modules*

$$\Gamma_{(p,B)}ko = \Gamma_{\mathfrak{n}_p}ko \simeq C_1^f ko \simeq \Sigma^{-6}I(ko)$$

(at all primes p), and equivalences of tmf-modules

$$\Gamma_{(p,B,M)}tmf = \Gamma_{\mathfrak{n}_p}tmf \simeq C_2^f tmf \simeq \Sigma^{-23}I(tmf)$$

(at $p = 2$ and at $p = 3$). The underlying S_p-modules are Artinian and bounded above.

Proof. For any S_p-module M the natural homomorphism

$$\rho \colon \pi_t(M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Hom}_{\mathbb{Z}_p}(\pi_t(M), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$$

is injective. Hence, if $\pi_{-t}(IM) = \mathrm{Hom}_{\mathbb{Z}_p}(\pi_t(M), \mathbb{Q}_p/\mathbb{Z}_p)$ is a Noetherian (= finitely generated) $\mathbb{Z}_p$-module, then ρ exhibits $\pi_t(M)$ as a submodule of an Artinian $\mathbb{Z}_p$-module, which must itself be Artinian. This applies with $M = C_1^f ko$, since we know from Proposition 4.2 that $IM \simeq \Sigma^6 ko$, and ko is a Noetherian S_p-module. It also applies with $M = C_2^f tmf$, since $IM \simeq \Sigma^{23} tmf$ and tmf is a Noetherian S_p-module by Lemma 4.7(1). Hence $\rho \colon M \to I(IM)$ is in fact an equivalence in these cases, so that

$$C_1^f ko \simeq I(IC_1^f ko) \simeq I(\Sigma^6 ko) \simeq \Sigma^{-6} I(ko)$$
$$C_2^f tmf \simeq I(IC_2^f tmf) \simeq I(\Sigma^{23} tmf) \simeq \Sigma^{-23} I(tmf) \,.$$

We use Lemma 2.5 with $n = 1$, $R = M = ko$ and $J = (p, B)$ to see that

$$(4.1) \qquad\qquad C_1^f ko \simeq \Gamma_{(p,B)} ko \,.$$

The finite spectra S/I with $I = (p^{a_0}, v_1^{a_1})$, and their Spanier–Whitehead duals, have type 2, so $\pi_* F(S/I, ko)$ is finite, since ko has fp-type 1. Hence both p and B act nilpotently on this graded $\mathbb{Z}_p$-module. This confirms the first condition. For the second condition, note that $R/J = ko/(p, B)$ and $F(S/p^{a_0}, R/J)$ have finite graded homotopy groups, hence p^{a_0} acts nilpotently on the former and $v_1^{a_1}$ acts nilpotently on the latter. The radical in $\pi_*(ko)$ of $J = (p, B)$ equals $\sqrt{J} = \mathfrak{n}_p$, and as reviewed in Subsection 2.D this implies the equivalence

$$\Gamma_{(p,B)} ko \simeq \Gamma_{\mathfrak{n}_p} ko \,.$$

Similarly, we use Lemma 2.5 with $n = 2$, $R = M = tmf$ and $J = (p, B, M)$ to see that

$$(4.2) \qquad\qquad C_2^f tmf \simeq \Gamma_{(p,B,M)} tmf \,.$$

The finite spectra S/I with $I = (p^{a_0}, v_1^{a_1}, v_2^{a_2})$, and their Spanier–Whitehead duals, have type 3, so $\pi_* F(S/I, tmf)$ is finite, since tmf has fp-type 2. Hence p, B and M act nilpotently on this graded $\mathbb{Z}_p$-module. Next, note that $R/J = tmf/(p, B, M)$, $F(S/p^{a_0}, R/J)$ and $F(S/(p^{a_0}, v_1^{a_1}), R/J)$ have finite homotopy, by Lemma 4.7(3), so that p^{a_0} acts nilpotently on the first, $v_1^{a_1}$ acts nilpotently on the second, and $v_2^{a_2}$ acts nilpotently on the third. The radical in $\pi_*(tmf)$ of $J = (p, B, M)$ equals $\sqrt{J} = \mathfrak{n}_p$, by Lemma 4.7(2), which implies the equivalence

$$\Gamma_{(p,B,M)} tmf \simeq \Gamma_{\mathfrak{n}_p} tmf \,. \qquad\qquad \square$$

Remark 4.9. As an alternative to Lemma 2.5, we could use Lemmas 2.2 and 2.4 to establish (4.1). For $p = 2$ the Adams complex $A = S/(2, v_1^4)$ from [Ada66, §12] has type 2 and satisfies $ko \wedge S/(2, v_1^4) \simeq ko/(2, B)$. Hence $C_1^f ko \simeq \mathrm{Cell}_{ko \wedge A} ko \simeq \mathrm{Cell}_{ko/(2,B)} ko \simeq \Gamma_{(2,B)} ko$. For p odd the complex $S/(p, v_1)$ has type 2 and $ko \wedge S/(p, v_1) \simeq ko/(p, A^m)$ with $m = (p-1)/2$. Hence $C_1^f ko \simeq \Gamma_{(p,A^m)} ko \simeq \Gamma_{(p,B)} ko$, since (p, A^m) and (p, B) have the same radical.

Likewise, we could use the existence [BHHM08] of a 2-local finite complex $A = S/(2, v_1^4, v_2^{32})$ of type 3 satisfying $tmf \wedge A \simeq tmf/(2, B, M)$ to deduce that $C_2^f tmf \simeq$

$\mathrm{Cell}_{tmf\wedge A} \simeq \mathrm{Cell}_{tmf/(2,B,M)}\, tmf \simeq \Gamma_{(2,B,M)}tmf$. Similar remarks apply at $p = 3$, using the type 3 complex $A = S/(3, v_1, v_2^9)$ constructed in [BP04]. In these cases the argument for (4.2) using Lemma 2.5 is dramatically simpler, as is to be expected, since asking for $R\wedge A$ and R/J to be equivalent is a much more restrictive condition than asking that they build one another.

Theorem 4.10. *There are equivalences of ko-modules*

$$\Gamma_B ko = \Gamma_{\mathfrak{n}_0} ko \simeq \Sigma^{-5} I_{\mathbb{Z}_p}(ko)$$

(at all primes p), and equivalences of tmf-modules

$$\Gamma_{(B,M)}tmf = \Gamma_{\mathfrak{n}_0}tmf \simeq \Sigma^{-22} I_{\mathbb{Z}_p}(tmf)$$

(at $p = 2$ and at $p = 3$). The underlying S_p-modules are Noetherian and bounded above.

Proof. Proposition 4.2, equivalence (4.1), and Lemma 3.1 applied to the ko-module $M = \Gamma_B ko$, give equivalences

$$\Sigma^6 ko \simeq IC_1^f ko \simeq I(\Gamma_{(p,B)}ko) \simeq \Sigma(I_{\mathbb{Z}_p}\Gamma_B ko)_p^\wedge \,.$$

Here $\Gamma_B ko$ is a Noetherian S_p-module, which implies that $I_{\mathbb{Z}_p}\Gamma_B ko$ is also Noetherian and (in particular) p-complete. Hence $\Sigma^5 ko \simeq I_{\mathbb{Z}_p}\Gamma_B ko$, which implies that

$$\Gamma_B ko \simeq I_{\mathbb{Z}_p}(I_{\mathbb{Z}_p}\Gamma_B ko) \simeq I_{\mathbb{Z}_p}(\Sigma^5 ko) \simeq \Sigma^{-5} I_{\mathbb{Z}_p}(ko)\,.$$

Furthermore, $\Gamma_B ko = \Gamma_{\mathfrak{n}_0} ko$, since the radical of (B) in $\pi_*(ko)$ equals $\mathfrak{n}_0$.

Similarly, Proposition 4.2, equivalence (4.2), and Lemma 3.1 applied to the tmf-module $M = \Gamma_{(B,M)}tmf$, give equivalences

$$\Sigma^{23}tmf \simeq IC_2^f tmf \simeq I(\Gamma_{(p,B,M)}tmf) \simeq \Sigma(I_{\mathbb{Z}_p}\Gamma_{(B,M)}tmf)_p^\wedge \,.$$

Part (4) of Lemma 4.7 asserts that $\Gamma_{(B,M)}tmf$ is a Noetherian S_p-module, which implies that $I_{\mathbb{Z}_p}\Gamma_{(B,M)}tmf$ is also Noetherian and p-complete. Hence $\Sigma^{22}tmf \simeq I_{\mathbb{Z}_p}\Gamma_{(B,M)}tmf$, which implies that

$$\Gamma_{(B,M)}tmf \simeq I_{\mathbb{Z}_p}(I_{\mathbb{Z}_p}\Gamma_{(B,M)}tmf) \simeq I_{\mathbb{Z}_p}(\Sigma^{22}tmf) \simeq \Sigma^{-22} I_{\mathbb{Z}_p}(tmf)\,.$$

Finally, $\Gamma_{(B,M)}tmf = \Gamma_{\mathfrak{n}_0}tmf$, since the radical of (B,M) in $\pi_*(tmf)$ equals $\mathfrak{n}_0$ by Lemma 4.7(2). □

5. Gorenstein duality

5.A. **Gorenstein maps of S-algebras.** The original version [DGI06, Def. 8.1] of the following definition was slightly more restrictive, but by [DGI06, Prop. 8.4] there is no difference when k is proxy-small as an R-module.

Definition 5.1. Let $R \to k$ be a map of S-algebras. We say that $R \to k$ is *Gorenstein of shift a* if there is an equivalence of left k-modules

$$F_R(k, R) \simeq \Sigma^a k\,.$$

Our next aim is to prove Proposition 5.3. We write $\mathbb{F}_p$ and $\mathbb{Z}_p$ for the mod p and p-adic integral Eilenberg–MacLane spectra, respectively, with their unique (commutative) S_p-algebra structures.

Suppose that R is an S_p-algebra and that $k = \mathbb{F}_p$. There is then an equivalence

$$k \simeq Ik = F_{S_p}(k, I) \cong F_R(k, IR)$$

of left k-modules, where Ik and $IR = F_{S_p}(R, I)$ are the Brown–Comenetz duals of k and R, and where k acts from the right on the domains of the two mapping spectra. Hence, if $R \to k$ is Gorenstein of shift k, then there is a k-module equivalence

$$F_R(k, R) \simeq \Sigma^a k \simeq F_R(k, \Sigma^a IR).$$

Recall the notation $\mathcal{E} = F_R(k, k)$ from Subsection 2.C. Restriction along $R \to k$ defines an S-algebra map $k^{op} \to \mathcal{E}$, and the left $\mathcal{E}$-action on k induces right $\mathcal{E}$-actions on $F_R(k, R)$ and $F_R(k, \Sigma^a IR)$. If R is connective with $\pi_0(R) = \mathbb{Z}_p$, then $\mathcal{E}$ is coconnective with $\pi_0(\mathcal{E}) \cong k^{op}$ a field. According to [DGI06, Prop. 3.9] the k-module equivalence above then extends to an $\mathcal{E}$-module equivalence, so that

$$F_R(k, R) \wedge_{\mathcal{E}} k \simeq F_R(k, \Sigma^a IR) \wedge_{\mathcal{E}} k.$$

Moreover, if k is proxy-small, so that k-cellularisation is effectively constructible by [DGI06, Thm. 4.10], then we can rewrite this as an equivalence

$$\text{Cell}_k R \simeq \text{Cell}_k(\Sigma^a IR).$$

Finally, if $\pi_*(IR)$ is p-power torsion, then k builds IR as the homotopy colimit of a refined Whitehead tower in R-modules, so that $\text{Cell}_k(\Sigma^a IR) \simeq \Sigma^a IR$. Hence these hypotheses ensure that $R \to k$ has Gorenstein duality of shift a, in the following sense.

Definition 5.2. A map $R \to k = \mathbb{F}_p$ of S_p-algebras has *Gorenstein duality of shift a* if there is an equivalence

$$\text{Cell}_k R \simeq \Sigma^a IR.$$

Similarly, a map $R \to k = \mathbb{Z}_p$ of S_p-algebras has *Gorenstein duality of shift a* if there is an equivalence

$$\text{Cell}_k R \simeq \Sigma^a I_{\mathbb{Z}_p} R.$$

Proposition 5.3. *Let $k = \mathbb{F}_p$ or $\mathbb{Z}_p$, let $R \to k$ be a map of connective S_p-algebras with $\pi_0(R) = \mathbb{Z}_p$, and let $\mathcal{E} = F_R(k, k)$. Suppose that*

    (1) *$R \to k$ is Gorenstein of shift a, and*

    (2) *k is proxy-small as an R-module.*

For $k = \mathbb{F}_p$ also assume that

    (3) *$\text{Hom}_{\mathbb{Z}_p}(\pi_*(R), \mathbb{Q}_p/\mathbb{Z}_p)$ is p-power torsion.*

Then $R \to k$ has Gorenstein duality of shift a.

Proof. The case $k = \mathbb{F}_p$ was discussed above. When $k = \mathbb{Z}_p$, we have an equivalence

$$k \simeq F_{S_p}(k, I_{\mathbb{Z}_p}) \cong F_R(k, I_{\mathbb{Z}_p} R)$$

of k-modules, where $I_{\mathbb{Z}_p} R = F_{S_p}(R, I_{\mathbb{Z}_p})$ is the Anderson dual of R. Hence, if $R \to k$ is Gorenstein of shift a then there is a k-module equivalence

$$F_R(k, R) \simeq \Sigma^a k \simeq F_R(k, \Sigma^a I_{\mathbb{Z}_p} R).$$

Since R is connective with $\pi_0(R) = \mathbb{Z}_p$, it follows that $\mathcal{E}$ is coconnective with $\pi_0(\mathcal{E}) \cong k^{op}$ and $\pi_{-1}(\mathcal{E}) = 0$. We prove in Proposition 5.4 below that this implies that the k-module equivalence above extends to an $\mathcal{E}$-module equivalence, so that

$$F_R(k, R) \wedge_{\mathcal{E}} k \simeq F_R(k, \Sigma^a I_{\mathbb{Z}_p} R) \wedge_{\mathcal{E}} k.$$

If k is proxy-small, then we can rewrite this as

$$\text{Cell}_k R \simeq \text{Cell}_k(\Sigma^a I_{\mathbb{Z}_p} R).$$

Since $\pi_*(I_{\mathbb{Z}_p} R)$ is a bounded above graded $\mathbb{Z}_p$-module, it follows that $I_{\mathbb{Z}_p} R$ is built from k, so that $\mathrm{Cell}_k R \simeq \Sigma^a I_{\mathbb{Z}_p} R$. $\qquad \square$

5.B. Uniqueness of $\mathcal{E}$-module structures.

Proposition 5.4. *Let $k = \mathbb{F}_p$ or $\mathbb{Z}_p$, and let $k^{op} \to \mathcal{E}$ be a map of coconnective S-algebras inducing an isomorphism on π_0. For $k = \mathbb{Z}_p$ also assume that $\pi_{-1}(\mathcal{E}) = 0$. Then any two right $\mathcal{E}$-module structures on k are equivalent.*

Proof. When $k = \mathbb{F}_p$, this is a special case of [DGI06, Prop. 3.9]. When $k = \mathbb{Z}_p$, we refine the proof of that proposition. Let k_1 and k_2 be right $\mathcal{E}$-modules whose restricted k-module structures are given by the usual left action on k. Let $M = \mathcal{E}$ as an $\mathcal{E}$-module, with the usual right action, and choose $\mathcal{E}$-module maps $f_1 \colon M \to k_1$ and $f_2 \colon M \to k_2$ inducing isomorphisms on π_0. We shall extend M to a cellular $\mathcal{E}$-module N such that f_1 extends to an $\mathcal{E}$-module equivalence $g_1 \colon N \to k_1$, and such that f_2 extends to an $\mathcal{E}$-module map $g_2 \colon N \to k_2$. It will then follow that g_2 is also an equivalence, and k_1 and k_2 are equivalent as $\mathcal{E}$-modules.

Note that $\pi_{-1}(M) = 0$. As a first approximation to N we build a cellular $\mathcal{E}$-module M' by attaching $\mathcal{E}$-cells of dimension ≤ 0 to M, so that $\pi_{-1}(M') = 0$ and $\pi_t(M) \to \pi_t(M')$ is trivial for each $t \leq -2$. More precisely, for each $t \leq -2$ choose an $\mathcal{E}$-module map

$$\bigvee_\alpha \Sigma^t \mathcal{E} \xrightarrow{\phi_t} \bigvee_\beta \Sigma^t \mathcal{E}$$

such that

$$0 \to \bigoplus_\alpha \mathbb{Z}_p \xrightarrow{\pi_t(\phi_t)} \bigoplus_\beta \mathbb{Z}_p \longrightarrow \pi_t(M) \to 0$$

is a free $\mathbb{Z}_p$-resolution of $\pi_t(M)$. There is then a map $C\phi_t \to M$ from the homotopy cofibre of ϕ_t, inducing an isomorphism on π_t. The composite $C\phi_t \to M \to k_1$ is null-homotopic, since $C\phi_t$ has cells in dimensions t and $t+1 \leq -1$ only, while k_1 is connective. Let M' be the mapping cone of the sum over t of the maps $C\phi_t \to M$, and let $f_1' \colon M' \to k_1$ extend f_1. Then M' has the stated properties.

Iterating the process infinitely often, and letting N be the (homotopy) colimit of the sequence $M \subset M' \subset \ldots$, we calculate that $\pi_t(N) = 0$ for $t \neq 0$, while $g_1 \colon N \to k_1$ is a π_0-isomorphism, and therefore an equivalence.

We obtained N from M by attaching cells of dimensions ≤ 0, so the obstructions to extending $f_2 \colon M \to k_2$ lie in the negative homotopy groups of k_2, which are trivial. Hence an extension $g_2 \colon N \to k_2$ exists. It must be a π_0-isomorphism, since f_1, f_2 and g_1 have this property, and is therefore an equivalence, as claimed. $\qquad \square$

Remark 5.5. The hypothesis on $\pi_{-1}(\mathcal{E})$ can in general not be omitted; see [DGI06, Rem. 3.11].

5.C. **Gorenstein descent.** Suppose that we are given maps $R \to T \to k$ of S-algebras, and that T is somehow easier to work with than R. A *descent theorem* for a property P gives hypotheses under which P for $T \to k$ implies P for $R \to k$. We first apply this idea in the case of the Gorenstein property.

Lemma 5.6. *Let $T \to k$ be a map of S-algebras, and suppose that the homomorphism of coefficient rings $\pi_*(T) \to \pi_*(k)$ is (algebraically) Gorenstein in the sense that $\mathrm{Ext}^{*,*}_{\pi_*(T)}(\pi_*(k), \pi_*(T))$ is a free $\pi_*(k)$-module of rank 1, on a generator in bidegree (s,t). Then $T \to k$ is Gorenstein of shift $a = t - s$.*

Grothendieck's definition given in [Har67, p. 63] is more restrictive, but this condition suffices for our purposes.

Proof. The conditionally convergent Ext spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(T)}(\pi_*(k), \pi_*(T)) \Longrightarrow_s \pi_{t-s} F_T(k, T)$$

of [EKMM97, Thm. IV.4.1] is a $\pi_*(k)$-module spectral sequence that collapses at the E_2-term, hence is strongly convergent. It follows that $\pi_* F_T(k, T) \cong \Sigma^a \pi_*(k)$ as $\pi_*(k)$-modules, so that $F_T(k, T) \simeq \Sigma^a k$ as k-modules. $\square$

Example 5.7. $\mathbb{Z}_p \to \mathbb{F}_p$ is Gorenstein of algebraic shift $(s,t) = (1,0)$ and of topological shift $a = -1$.

Lemma 5.8. *Let $R \to T \to k$ be maps of S-algebras, and suppose that $R \to T$ is Gorenstein of shift b. Then $T \to k$ is Gorenstein of shift a if and only if $R \to k$ is Gorenstein of shift $a + b$.*

Proof. By hypothesis, $F_R(T, R) \simeq \Sigma^b T$ as left T-modules. It follows that

$$F_R(k, R) \cong F_T(k, F_R(T, R)) \simeq F_T(k, \Sigma^b T) \cong \Sigma^b F_T(k, T)$$

as left k-modules. Hence $F_T(k, T) \simeq \Sigma^a k$ if and only if $F_R(k, R) \simeq \Sigma^{a+b} k$. $\square$

Example 5.9. If $\pi_*(T) \cong \mathbb{Z}_p[y_1, \ldots, y_d]$ is polynomial on finitely many generators, and $k = \mathbb{Z}_p$, then the ring homomorphism $\pi_*(T) \to \mathbb{Z}_p$ is Gorenstein of shift $(s,t) = (d, -\sum_{i=1}^d |y_i|)$. Hence the S-algebra map $T \to \mathbb{Z}_p$ is Gorenstein of shift

$$a = -d - \sum_{i=1}^d |y_i| = -\sum_{i=1}^d (|y_i| + 1).$$

Moreover, $\pi_*(T) \to \mathbb{F}_p$ is Gorenstein of shift $(d + 1, -\sum_{i=1}^d |y_i|)$ and $T \to \mathbb{F}_p$ is Gorenstein of shift $-d - 1 - \sum_{i=1}^d |y_i|$.

Proposition 5.10. *The S-algebra maps*

$$ko \longrightarrow ku \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{F}_p$$

are Gorenstein of shift -2, -3 and -1, respectively. Hence $ko \to \mathbb{Z}_p$ is Gorenstein of shift -5 and $ko \to \mathbb{F}_p$ is Gorenstein of shift -6.

At $p = 2$ the S-algebra maps

$$tmf \longrightarrow tmf_1(3) \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{F}_2$$

are Gorenstein of shift -12, -10 and -1, respectively. At $p = 3$ the S-algebra maps

$$tmf \longrightarrow tmf_0(2) \longrightarrow \mathbb{Z}_3 \longrightarrow \mathbb{F}_3$$

are Gorenstein of shift -8, -14 *and* -1, *respectively. At* $p \geq 5$ *the* S-*algebra maps*

$$tmf \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{F}_p$$

are Gorenstein of shift -22 *and* -1, *respectively. Hence* $tmf \to \mathbb{Z}_p$ *is Gorenstein of shift* -22, *and* $tmf \to \mathbb{F}_p$ *is Gorenstein of shift* -23, *uniformly at all primes.*

Proof. The homotopy rings

$$\pi_*(ku) \cong \mathbb{Z}_p[u]$$
$$\pi_*(tmf_1(3)) \cong \mathbb{Z}_2[a_1, a_3]$$
$$\pi_*(tmf_0(2)) \cong \mathbb{Z}_3[a_2, a_4]$$
$$\pi_*(tmf) \cong \mathbb{Z}_p[c_4, c_6]\,,$$

where $p \geq 5$ in the last case, are all polynomial, with $|u| = 2$, $|a_i| = 2i$ and $|c_i| = 2i$. See [Bot59], [MR09, Prop. 3.2], [Beh06, §1.3], [Del75, Prop. 6.1] or [BR21, §9.3, Thm. 13.4]. This accounts for the Gorenstein shifts by -3, -10, -14 and -22, as in Example 5.9.

The shifts by -1 are covered by Example 5.7.

By Wood's theorem [BG10, Lem. 4.1.2], and its parallels [Mat16, Thm. 4.12, Thm. 4.15] for topological modular forms, there are equivalences

$$ko \wedge C\eta \simeq ku$$
(5.1) $$tmf \wedge \Phi \simeq tmf_1(3)$$
$$tmf \wedge \Psi \simeq tmf_0(2)$$

of ko- or tmf-modules, according to the case. Here $C\eta = S \cup_\eta e^2$ is a 2-cell, 2-dimensional Spanier–Whitehead self-dual spectrum, Φ is an 8-cell, 12-dimensional Spanier–Whitehead self-dual 2-local spectrum [BR21, Lem. 1.42] with mod 2 cohomology $H^*(\Phi) \cong A(2)//E(2) \cong \Phi A(1)$ realising the double of $A(1)$, and $\Psi = S \cup_\nu e^4 \cup_\nu e^8$ is a 3-cell 8-dimensional Spanier–Whitehead self-dual 3-local spectrum [BR21, Def. 13.3] with mod 3 cohomology $H^*(\Psi) \cong P(0) = \langle P^1 \rangle$. The duality equivalences $D(C\eta) \simeq \Sigma^{-2} C\eta$, $D\Phi \simeq \Sigma^{-12}\Phi$ and $D\Psi \simeq \Sigma^{-8}\Psi$ account for the Gorenstein shifts by -2, -12 and -8, respectively. For example, in the case of tmf at $p = 2$ we have equivalences

$$F_{tmf}(tmf_1(3), tmf) \simeq F_{tmf}(tmf \wedge \Phi, tmf) \simeq F(\Phi, tmf)$$
$$\simeq tmf \wedge D\Phi \simeq tmf \wedge \Sigma^{-12}\Phi \simeq \Sigma^{-12} tmf_1(3)$$

of tmf-modules. $\qquad\qquad\square$

5.D. **Small descent.** We can use descent to verify that k is a (proxy-)small R-module in the cases relevant for Sections 7 and 8.

Lemma 5.11. *Let* $R \to T \to k$ *be maps of* S-*algebras, such that* T *is small as an* R-*module and* k *is small as a* T-*module. Then* k *is small as an* R-*module.*

Proof. Since T finitely builds k as a T-module, this remains true as R-modules. Hence R finitely builds k. $\qquad\qquad\square$

Lemma 5.12. *Let* T *be a commutative* S-*algebra with* $\pi_*(T) \cong \mathbb{Z}_p[y_1, \ldots, y_d]$. *Then* $\mathbb{Z}_p$ *and* $\mathbb{F}_p$ *are small as* T-*modules.*

Proof. $T/y_1 \wedge_T \cdots \wedge_T T/y_d \simeq \mathbb{Z}_p$ and $\mathbb{Z}_p/p \simeq \mathbb{F}_p$ are finitely built from T. $\qquad\square$

Remark 5.13. More generally, if $\pi_*(k)$ is a perfect $\pi_*(T)$-module, meaning that it admits a finite length resolution by finitely generated projective $\pi_*(T)$-modules, then k is small as a T-module.

Corollary 5.14. $\mathbb{Z}_p$ *and* $\mathbb{F}_p$ *are small as ko-modules and as tmf-modules, at all primes* p.

Proof. Lemma 5.12 applies to the commutative S_p-algebras ko for $p \geq 3$, ku for all p, $tmf_1(3)$ for $p = 2$, $tmf_0(2)$ for $p = 3$, and tmf for $p \geq 5$. Lemma 5.11 then covers the cases of ko at $p = 2$, and tmf at $p \in \{2, 3\}$, in view of (5.1). $\square$

5.E. **Descent of algebraic cellularisation.**

Definition 5.15. Let R be a commutative S-algebra and k an R-module. We say that R has *algebraic k-cellularisation by J* if $J = (x_1, \ldots, x_d) \subset \pi_*(R)$ is a finitely generated ideal with

$$\mathrm{Cell}_k M \simeq \Gamma_J M$$

for all R-modules M.

This condition only depends on the radical $\sqrt{J}$ of J, and by Lemmas 2.1 and 2.4 it is equivalent to asking that the R-modules k and R/J mutually build one another.

Lemma 5.16. *Let T be a commutative S-algebra with $\pi_*(T) \cong \mathbb{Z}_p[y_1, \ldots, y_d]$. Then T has algebraic $\mathbb{Z}_p$-cellularisation by $(y_1, \ldots, y_d)$, and algebraic $\mathbb{F}_p$-cellularisation by $(p, y_1, \ldots, y_d)$.*

Proof. Letting $J' = (y_1, \ldots, y_d)$ or $J' = (p, y_1, \ldots, y_d)$ we have $T/J' \simeq k = \mathbb{Z}_p$ or $T/J' \simeq k = \mathbb{F}_p$, according to the case. Hence $\mathrm{Cell}_k M \simeq \mathrm{Cell}_{T/J'} M \simeq \Gamma_{J'} M$. $\square$

Lemma 5.17. *Let $\phi \colon R \to T$ be a map of commutative S-algebras, where R is connective with $\pi_0(R) = \mathbb{Z}_p$. Let $J = (x_1, \ldots, x_d) \subset \pi_*(R)$ be such that $\pi_*(R/J)$ is bounded above, and suppose that T has algebraic $\mathbb{Z}_p$-cellularisation by*

$$J' = (\phi(x_1), \ldots, \phi(x_d)) \subset \pi_*(T).$$

Then R has algebraic $\mathbb{Z}_p$-cellularisation by J.

Similarly, if $\pi_*(R/J)$ is p-power torsion and bounded above, and T has algebraic $\mathbb{F}_p$-cellularisation by J', then R has algebraic $\mathbb{F}_p$-cellularisation by J.

Proof. In the case $k = \mathbb{Z}_p$, the R-module $\mathbb{Z}_p$ builds R/J since $\pi_*(R/J)$ is bounded above. Conversely, R builds T so R/J builds $T \wedge_R R/J = T/J'$. By hypothesis, T/J' builds $\mathbb{Z}_p$ in T-modules, hence also in R-modules. Thus R/J builds $\mathbb{Z}_p$ in R-modules.

Similarly, for $k = \mathbb{F}_p$ the R-module $\mathbb{F}_p$ builds R/J since $\pi_*(R/J)$ is p-power torsion and bounded above. Conversely, R/J builds T/J' as before. By hypothesis, T/J' builds $\mathbb{F}_p$ in T-modules, hence also in R-modules. Thus R/J builds $\mathbb{F}_p$ in R-modules. $\square$

Recall that $B \in \pi_8(tmf)$ (together with $B + \epsilon$) is detected by the modular form c_4, while we write M for $M \in \pi_{192}(tmf)$ detected by Δ^8 when $p = 2$, and for $H \in \pi_{72}(tmf)$ detected by Δ^3 when $p = 3$. For uniformity of notation, let us also write M for the class in $\pi_{24}(tmf)$ detected by Δ when $p \geq 5$.

Proposition 5.18. *The commutative S_p-algebra ko has algebraic $\mathbb{Z}_p$-cellularisation by (B), and algebraic $\mathbb{F}_p$-cellularisation by (p, B), for all primes p.*

$$\operatorname{Cell}_{\mathbb{Z}_p} ko \simeq \Gamma_{(B)} ko$$
$$\operatorname{Cell}_{\mathbb{F}_p} ko \simeq \Gamma_{(p,B)} ko$$

Likewise, tmf has algebraic $\mathbb{Z}_p$-cellularisation by (B, M), and algebraic $\mathbb{F}_p$-cellularisation by (p, B, M), for all primes p.

$$\operatorname{Cell}_{\mathbb{Z}_p} tmf \simeq \Gamma_{(B,M)} tmf$$
$$\operatorname{Cell}_{\mathbb{F}_p} tmf \simeq \Gamma_{(p,B,M)} tmf$$

Proof. For ko, we apply Lemma 5.17 to the complexification map $\phi\colon ko \to ku$ with $J = (B)$, where $\phi(B) = u^4$. Then $\pi_*(ko/B) \cong \mathbb{Z}_p\{1, \eta, \eta^2, A\}/(2\eta, 2\eta^2)$ is finitely generated over $\mathbb{Z}_p$. Moreover, $J' = (u^4)$ has radical $(u) \subset \pi_*(ku)$. According to Lemma 5.16, ku has algebraic $\mathbb{Z}_p$-cellularisation by (u), hence it also has algebraic $\mathbb{Z}_p$-cellularisation by J'.

Similarly, with $J = (p, B)$ we see that $\pi_*(ko/(p, B))$ is finite and $J' = (p, u^4)$ has radical $(p, u) \subset \pi_*(ku)$, so ku has algebraic $\mathbb{F}_p$-cellularisation by (p, u) and by J'.

For tmf at $p = 2$ we apply Lemma 5.17 to the map $\phi\colon tmf \to tmf_1(3)$ with $J = (B, M)$, where

$$\phi(B) = c_4 \qquad c_4 = a_1(a_1^3 - 24a_3)$$
$$\phi(M) = \Delta^8 \qquad \Delta = a_3^3(a_1^3 - 27a_3)\,,$$

according to the formulas for $\Gamma_1(3)$-modular forms. See [BR21, §9.3] and the more detailed references therein. It is clear from Theorem 8.4 that $\pi_*(tmf/(B, M)) \cong \pi_*(N/B)$ is finitely generated over $\mathbb{Z}_2$. Moreover, $J' = (c_4, \Delta^8)$ has radical $(a_1, a_3) \subset \pi_*(tmf_1(3))$, so $tmf_1(3)$ has algebraic $\mathbb{Z}_2$-cellularisation by (a_1, a_3) according to Lemma 5.16, hence also by J'.

Similarly, with $J = (2, B, M)$ we see that $\pi_*(tmf/(2, B, M)) \cong \pi_*(N/(2, B))$ is finite and $J' = (2, c_4, \Delta^8)$ has radical $(2, a_1, a_3)$, so tmf has algebraic $\mathbb{F}_2$-cellularisation by $(2, a_1, a_3)$ and by J'.

For tmf at $p = 3$ we apply Lemma 5.17 to the map $\phi\colon tmf \to tmf_0(2)$ with $J = (B, H)$, where

$$\phi(B) = c_4 \qquad c_4 = 16(a_2^2 - 3a_4)$$
$$\phi(H) = \Delta^3 \qquad \Delta = 16a_4^2(a_2^2 - 4a_4)$$

according to the formulas for $\Gamma_0(2)$-modular forms. See [BR21, §13.1] and the more detailed references therein. It is clear from Theorem 8.15 that $\pi_*(tmf/(B, H)) \cong \pi_*(N/B)$ is finitely generated over $\mathbb{Z}_3$. Moreover, $J' = (c_4, \Delta^3)$ has radical $(a_2, a_4) \subset \pi_*(tmf_0(2))$, so $tmf_0(2)$ has algebraic $\mathbb{Z}_3$-cellularisation by (a_2, a_4) according to Lemma 5.16, hence also by J'.

Similarly, with $J = (3, B, H)$ we see that $\pi_*(tmf/(3, B, H)) \cong \pi_*(N/(3, B))$ is finite and $J' = (3, c_4, \Delta^3)$ has radical $(3, a_2, a_4)$, so tmf has algebraic $\mathbb{F}_3$-cellularisation by $(3, a_2, a_4)$ and by J'.

For tmf at $p \geq 5$, the ideal $J' = (B, M) = (c_4, \Delta)$, with $\Delta = (c_4^3 - c_6^2)/1728$, has radical (c_4, c_6). Hence tmf has algebraic $\mathbb{Z}_p$-cellularisation by (c_4, c_6) and by J'.

Similarly, the ideal $J' = (p, B, M) = (p, c_4, \Delta)$ has radical (p, c_4, c_6), so tmf has algebraic $\mathbb{F}_p$-cellularisation by (p, c_4, c_6) and by J'. $\square$

5.F. **Local cohomology theorems by Gorenstein duality.**

Theorem 5.19. *There are equivalences of ko-modules*

$$\Gamma_B ko = \Gamma_{\mathfrak{n}_0} ko \simeq \Sigma^{-5} I_{\mathbb{Z}_p}(ko)$$

and equivalences of tmf-modules

$$\Gamma_{(B,M)} tmf = \Gamma_{\mathfrak{n}_0} tmf \simeq \Sigma^{-22} I_{\mathbb{Z}_p}(tmf)$$

at all primes p.

Proof. We apply Proposition 5.3 to $R \to k$ with $R = ko$ or $R = tmf$ and $k = \mathbb{Z}_p$. Then $R \to k$ is Gorenstein of shift $a = -5$ or $a = -22$ by Proposition 5.10, and k is small, hence proxy-small, as an R-module by Corollary 5.14. Hence $\text{Cell}_k R \simeq \Sigma^a I_{\mathbb{Z}_p} R$ in each case. Moreover, $\text{Cell}_k R \simeq \Gamma_J R$ for $J = (B) \subset \pi_*(ko)$ or $J = (B, M) \subset \pi_*(tmf)$, by Proposition 5.18. Finally, $\Gamma_J R \simeq \Gamma_{\mathfrak{n}_0} R$ since J has radical $\mathfrak{n}_0$ in each case, cf. Lemma 4.7(2). $\qquad\square$

Theorem 5.20. *There are equivalences of ko-modules*

$$\Gamma_{(p,B)} ko = \Gamma_{\mathfrak{n}_p} ko \simeq \Sigma^{-6} I(ko)$$

and equivalences of tmf-modules

$$\Gamma_{(p,B,M)} tmf = \Gamma_{\mathfrak{n}_p} tmf \simeq \Sigma^{-23} I(tmf)$$

at all primes p.

Proof. We apply Proposition 5.3 to $R \to k$ with $R = ko$ or $R = tmf$ and $k = \mathbb{F}_p$. Then $R \to k$ is Gorenstein of shift $a = -6$ or $a = -23$ by Proposition 5.10, and k is small, hence proxy-small, as an R-module by Corollary 5.14. Furthermore, $\pi_t(R)$ is a finitely generated $\mathbb{Z}_p$-module for each t, as is clear from Theorems 8.2 and 8.4 below, so $\text{Hom}_{\mathbb{Z}_p}(\pi_*(R), \mathbb{Q}_p/\mathbb{Z}_p)$ is p-power torsion in each degree. Hence $\text{Cell}_k R \simeq \Sigma^a I R$ in each case. Moreover, $\text{Cell}_k R \simeq \Gamma_J R$ for $J = (p, B) \subset \pi_*(ko)$ or $J = (p, B, M) \subset \pi_*(tmf)$, by Proposition 5.18. Finally, $\Gamma_J R \simeq \Gamma_{\mathfrak{n}_p} R$ since J has radical $\mathfrak{n}_p$ in each case, cf. Lemma 4.7(2). $\qquad\square$

5.G. **ko- and tmf-module Steenrod algebras.** For completeness, we record the structure of $\pi_*(\mathcal{E})$ in our main cases of interest, where $\mathcal{E} = F_R(k, k)$, $R = ko$ or tmf, and $k = \mathbb{F}_p$.

Proposition 5.21 ([Hil07], [DFHH14], [BR21]). *For $p = 2$ there are algebra isomorphisms*

$$\pi_* F_{ko}(\mathbb{F}_2, \mathbb{F}_2) \cong A(1)$$
$$\pi_* F_{tmf}(\mathbb{F}_2, \mathbb{F}_2) \cong A(2) \,.$$

For $p = 3$ there is a square-zero quadratic extension

$$\pi_* F_{tmf}(\mathbb{F}_3, \mathbb{F}_3) = A_{tmf} \longrightarrow A(1) \,,$$

where A_{tmf} is generated by classes β and P^1 in cohomological degrees 1 and 4, subject to $\beta^2 = 0$, $\beta(P^1)^2\beta = (\beta P^1)^2 + (P^1\beta)^2$ and $(P^1)^3 = 0$. In each case, classes in homotopical degree $-m$ correspond to classes in cohomological degree m.

Proof. Restriction along $S \to tmf$ induces an S-algebra map

$$\mathcal{E} = F_{tmf}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow F_S(\mathbb{F}_2, \mathbb{F}_2)$$

and an algebra homomorphism $\pi_*(\mathcal{E}) \to A$ to the mod 2 Steenrod algebra. Base change along $S \to \mathbb{F}_2$ lets us rewrite the S-algebra map as

$$\mathcal{E} \cong F_{\mathbb{F}_2 \wedge tmf}(\mathbb{F}_2 \wedge \mathbb{F}_2, \mathbb{F}_2) \longrightarrow F_{\mathbb{F}_2}(\mathbb{F}_2 \wedge \mathbb{F}_2, \mathbb{F}_2) \,.$$

Since the dual Steenrod algebra $A_* = \pi_*(\mathbb{F}_2 \wedge \mathbb{F}_2)$ is free as an $H_*(tmf) = \pi_*(\mathbb{F}_2 \wedge tmf)$-module, the Ext spectral sequences for these two function spectra collapse, and let us rewrite the algebra homomorphism as the monomorphism

$$\pi_*(\mathcal{E}) \cong \operatorname{Hom}_{H_*(tmf)}(A_*, \mathbb{F}_2) \longrightarrow \operatorname{Hom}_{\mathbb{F}_2}(A_*, \mathbb{F}_2) \cong A \,.$$

By duality, this identifies $\pi_*(\mathcal{E})$ with the $H^*(tmf) = A//A(2)$-comodule primitives in A, which is precisely the subalgebra $A(2)$.

The proof for ko is the same, replacing $A(2)$ with $A(1)$.

The result for $p = 3$ is due to Henriques and Hill [Hil07, Thm. 2.2], [DFHH14, §13.3], except for the comment that the extension is square-zero, which appears in [BR21, §13.1]. □

6. Thera duality

A third line of proof is discussed in [BR21, §10.3, §10.4, §13.5], yielding the following theorems.

Theorem 6.1 ([BR21, Thm. 10.6, Prop. 10.12]). *There are equivalences of 2-complete tmf-modules*

$$\Sigma^{23} tmf \simeq I(\Gamma_{(2,B,M)} tmf)$$
$$\Sigma^{22} tmf \simeq I_{\mathbb{Z}_2}(\Gamma_{(B,M)} tmf) \,.$$

Theorem 6.2 ([BR21, Thm. 13.20, Prop. 13.21]). *There are equivalences of 3-complete tmf-modules*

$$\Sigma^{23} tmf \simeq I(\Gamma_{(3,B,H)} tmf)$$
$$\Sigma^{22} tmf \simeq I_{\mathbb{Z}_3}(\Gamma_{(B,H)} tmf) \,.$$

This approach combines descent with a strengthening of the Cohen–Macaulay property, equivalent to the Gorenstein property. One first observes that

$$\Sigma^{11} tmf_1(3) \simeq I(\Gamma_{(2,a_1,a_3)} tmf_1(3)) \,,$$

because the local cohomology of $\pi_*(tmf_1(3)) = \mathbb{Z}_2[a_1, a_3]$ at the maximal ideal $\mathfrak{n}_2 = (2, a_1, a_3)$ is concentrated in a single cohomological degree, and, moreover, its $\mathbb{Z}_2$-module Pontryagin dual is a free $\pi_*(tmf_1(3))$-module on one generator. The conclusion for tmf follows by faithful descent along $tmf \to tmf_1(3) \simeq tmf \wedge \Phi$, since Φ is Spanier–Whitehead self-dual.
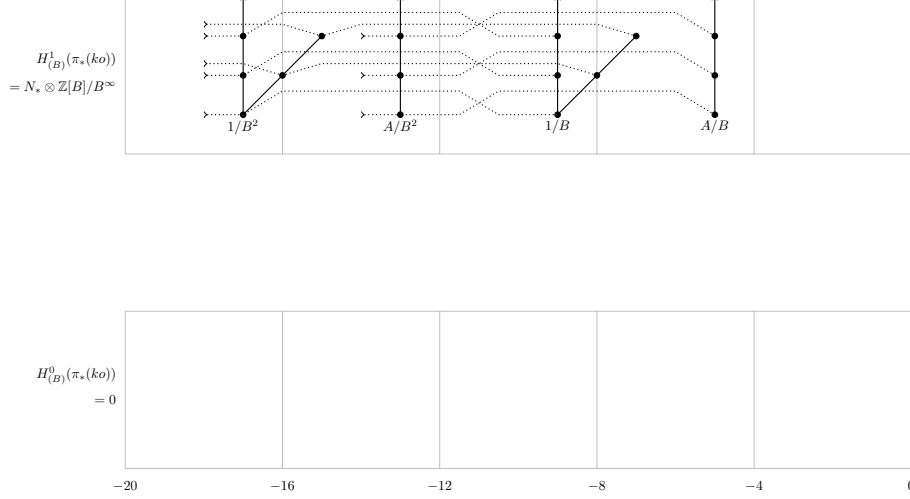
7. Topological K-theory

As a warm-up to Section 8, we spell out the structure of the local cohomology spectral sequences

$$E_2^{s,t} = H_{(B)}^s(\pi_*(ko))_t \Longrightarrow_s \pi_{t-s}(\Gamma_B ko) \cong \pi_{t-s}(\Sigma^{-5} I_{\mathbb{Z}_2}(ko))$$

and

$$E_2^{s,t} = H_{(2,B)}^s(\pi_*(ko))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)} ko) \cong \pi_{t-s}(\Sigma^{-6} I(ko)) \,.$$

FIGURE 7.1. $E_2^{s,t} = H_{(B)}^s(\pi_*(ko))_t \Longrightarrow_s \pi_{t-s}(\Gamma_B ko)$

Multiplication by B acts injectively on the depth 1 graded commutative ring

$$\pi_*(ko) = \mathbb{Z}_2[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 - 4B)$$

and we let N_* denote a basic block for this action.

Definition 7.1. In this section only, let $N_* \subset \pi_*(ko)$ be the $\mathbb{Z}_2$-submodule of classes in degrees $0 \leq * < 8$, and let $N = ko/B$.

Lemma 7.2. *The composite $N_* \otimes \mathbb{Z}[B] \to \pi_*(ko) \otimes \mathbb{Z}[B] \dot\to \pi_*(ko)$ is an isomorphism. As a $\mathbb{Z}_2$-module, $N_* = \mathbb{Z}_2\{1, \eta, \eta^2, A\}/(2\eta, 2\eta^2)$ is a split extension by the 2-torsion submodule $\Gamma_2 N_* = \mathbb{Z}/2\{\eta, \eta^2\}$ of the 2-torsion free quotient $N_*/\Gamma_2 N_* = \mathbb{Z}_2\{1, A\}$.*

Lemma 7.3. $H_{(B)}^0(\pi_*(ko)) = 0$ *and* $H_{(B)}^1(\pi_*(ko)) \cong N_* \otimes \mathbb{Z}[B]/B^\infty$.

Proof. These are the cohomology groups of the complex

$$0 \to \pi_*(ko) \xrightarrow{\gamma} \pi_*(ko)[1/B] \to 0\,,$$

which we may rewrite as $0 \to N_* \otimes \mathbb{Z}[B] \xrightarrow{\gamma} N_* \otimes \mathbb{Z}[B^{\pm 1}] \to 0$. $\square$

Proposition 7.4. *The local cohomology spectral sequence*

$$E_2^{s,t} = H_{(B)}^s(\pi_*(ko))_t \Longrightarrow_s \pi_{t-s}(\Gamma_B ko) \cong \pi_{t-s}(\Sigma^{-5} I_{\mathbb{Z}_2} ko)$$

has E_2-term concentrated on the $s = 1$ line, with $E_2^{1,*} = N_* \otimes \mathbb{Z}[B]/B^\infty$. There is no room for differentials or hidden extensions, so $E_2 = E_\infty$. Hence there are isomorphisms

$$\Sigma^{-1} \mathbb{Z}_2\{1, A\} \otimes \mathbb{Z}[B]/B^\infty \cong \Sigma^{-5} \operatorname{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2\{1, A\} \otimes \mathbb{Z}[B], \mathbb{Z}_2)$$

and

$$\Sigma^{-1} \mathbb{Z}/2\{\eta, \eta^2\} \otimes \mathbb{Z}[B]/B^\infty \cong \Sigma^{-6} \operatorname{Ext}_{\mathbb{Z}_2}(\mathbb{Z}/2\{\eta, \eta^2\} \otimes \mathbb{Z}[B], \mathbb{Z}_2)\,.$$
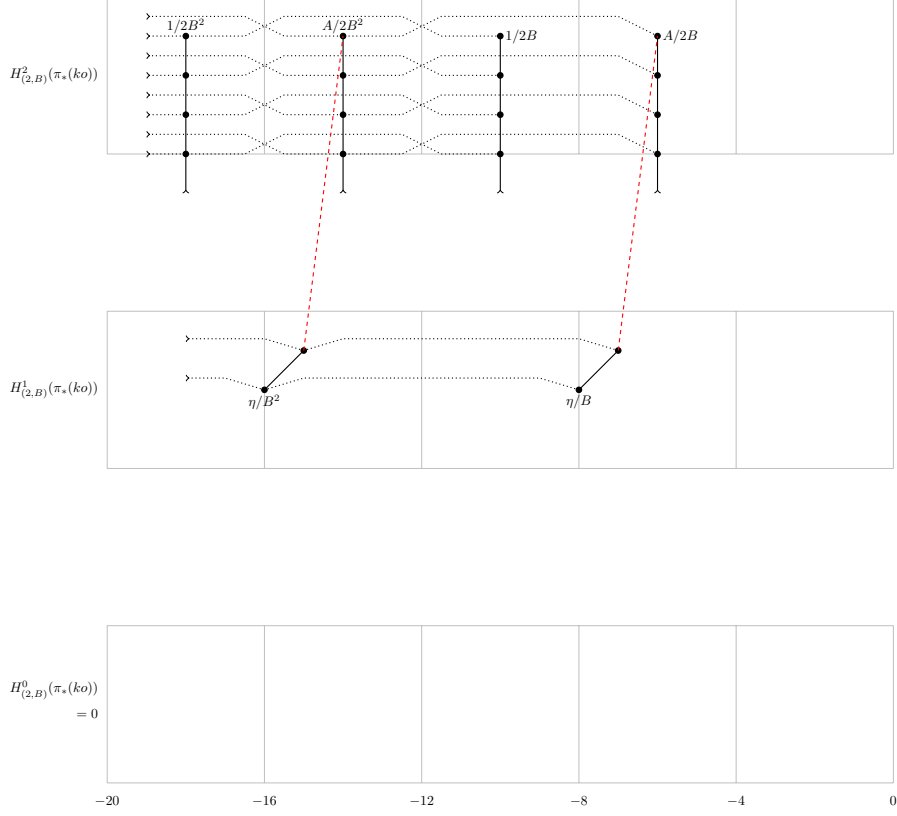
FIGURE 7.2. $E_2^{s,t} = H_{(2,B)}^s(\pi_*(ko))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)}ko)$

Proof. See Figure 7.1 and recall the short exact sequence (3.1). $\qquad\square$

Lemma 7.5. $H_{(2,B)}^s(\pi_*(ko)) \cong H_{(2)}^{s-1}(N_*) \otimes \mathbb{Z}[B]/B^\infty$ *where*

$$H_{(2)}^0(N_*) = \mathbb{Z}/2\{\eta, \eta^2\}$$
$$H_{(2)}^1(N_*) = \mathbb{Q}_2/\mathbb{Z}_2\{1, A\}\,.$$

Proof. See Lemma 8.6 for the proof of the first isomorphism. The $H_{(2)}^*(N_*)$ are the cohomology groups of the complex

$$0 \to N_* \xrightarrow{\gamma} N_*[1/2] \to 0\,. \qquad\square$$

Proposition 7.6. *The local cohomology spectral sequence*

$$E_2^{s,t} = H_{(2,B)}^s(\pi_*(ko))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)}ko) \cong \pi_{t-s}(\Sigma^{-6}I(ko))$$

has E_2-term

$$E_2^{s,t} \cong H_{(2)}^{s-1}(N_*) \otimes \mathbb{Z}[B]/B^\infty\,.$$

There is no room for differentials, so $E_2 = E_\infty$. Hence there are isomorphisms

$$\Sigma^{-2}\mathbb{Q}_2/\mathbb{Z}_2\{1, A\} \otimes \mathbb{Z}[B]/B^\infty \cong \Sigma^{-6}\operatorname{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2\{1, A\} \otimes \mathbb{Z}[B], \mathbb{Q}_2/\mathbb{Z}_2)$$

and

$$\Sigma^{-1}\mathbb{Z}/2\{\eta,\eta^2\} \otimes \mathbb{Z}[B]/B^\infty \cong \Sigma^{-6}\operatorname{Hom}_{\mathbb{Z}_2}(\mathbb{Z}/2\{\eta,\eta^2\} \otimes \mathbb{Z}[B], \mathbb{Q}_2/\mathbb{Z}_2)\,.$$

Moreover, there are hidden η-extensions as shown by sloping dashed red lines.

Proof. See Figure 7.2. □

8. Topological modular forms

We can now work out the structure of the local cohomology spectral sequences

$$E_2^{s,t} = H_{(B,M)}^s(\pi_*(tmf))_t$$
$$\Longrightarrow_s \pi_{t-s}(\Gamma_{(B,M)}tmf) \cong \pi_{t-s}(\Sigma^{-22}I_{\mathbb{Z}_p}(tmf))$$

and

$$E_2^{s,t} = H_{(p,B,M)}^s(\pi_*(tmf))_t$$
$$\Longrightarrow_s \pi_{t-s}(\Gamma_{(p,B,M)}tmf) \cong \pi_{t-s}(\Sigma^{-23}I(tmf))$$

for $p = 2$ and for $p = 3$. Recall the algebra generators for $\pi_*(tmf)$ listed in Table 4.1 for $p = 2$ and in Table 4.2 for $p = 3$. In each case multiplication by M acts injectively on the depth 1 graded commutative ring $\pi_*(tmf)$, and we let N_* denote a basic block for this action. (The notation BB is used for a similar object in [GM17].) To begin, we review the $\mathbb{Z}_p[B, M]$-module structure on $\pi_*(tmf)$ and the $\mathbb{Z}_p[B]$-module structure on N_*, in the notation from [BR21, Ch. 9].

8.A. **(B, M)-local cohomology of tmf.** Let $p = 2$ in this subsection and the next. See Figure 8.2 for the mod 2 Adams E_∞-term for tmf in the range $0 \leq t - s \leq 192$, with all hidden 2-, η- and ν-extensions shown. There are no hidden B- or M-extensions in this spectral sequence.

Definition 8.1. Let $N_* \subset \pi_*(tmf)$ be the $\mathbb{Z}_2[B]$-submodule generated by all classes in degrees $0 \leq * < 192$, and let $N = tmf/M$.

Theorem 8.2 ([BR21, Thm. 9.27]). *The composite homomorphisms*

$$N_* \otimes \mathbb{Z}[M] \longrightarrow \pi_*(tmf) \otimes \pi_*(tmf) \xrightarrow{\cdot} \pi_*(tmf)$$
$$N_* \subset \pi_*(tmf) \longrightarrow \pi_*(N)$$

are isomorphisms. Hence $\pi_*(tmf)$ is a (split) extension of $\mathbb{Z}_2[B, M]$-modules

$$0 \to \Gamma_B N_* \otimes \mathbb{Z}[M] \longrightarrow \pi_*(tmf) \longrightarrow \frac{N_*}{\Gamma_B N_*} \otimes \mathbb{Z}[M] \to 0\,.$$

Definition 8.3. Let $\nu_3 = \eta_1^3$ and $\nu_7 = 0$, and set $d_k = 8/\gcd(k, 8)$, so that $d_0 = 1$, $d_4 = 2$, $d_2 = d_6 = 4$, $d_1 = d_3 = d_5 = d_7 = 8$ and $d_{7-k}\nu_k = 0$ for $0 \leq k \leq 7$.

Theorem 8.4 ([BR21, Thm. 9.26]). *As a $\mathbb{Z}_2[B]$-module, N_* is a split extension*

$$0 \to \Gamma_B N_* \longrightarrow N_* \longrightarrow \frac{N_*}{\Gamma_B N_*} \to 0\,.$$

The B-power torsion submodule $\Gamma_B N_*$ is given in Table 8.1. It is concentrated in degrees $3 \leq * \leq 164$, and is finite in each degree. The action of B is as indicated in the table, together with $B \cdot \epsilon_1 = 2\bar{\kappa}^2$, $B \cdot \eta\nu_2 = 2\bar{\kappa}^3$ and $B \cdot \epsilon_5\kappa = 4\nu\nu_6$.

The B-torsion free quotient of N_* is the direct sum

$$\frac{N_*}{\Gamma_B N_*} = \bigoplus_{k=0}^{7} ko[k]$$

of the following eight $\mathbb{Z}_2[B]$-modules, with $ko[k]$ concentrated in degrees $* \geq 24k$:

$$ko[0] = \mathbb{Z}_2[B]\{1, C\} \oplus \mathbb{Z}/2[B]\{\eta, \eta^2\}$$
$$ko[1] = \mathbb{Z}_2\{D_1\} \oplus \mathbb{Z}_2[B]\{B_1, C_1\} \oplus \mathbb{Z}/2[B]\{\eta_1, \eta\eta_1\}$$
$$ko[2] = \mathbb{Z}_2\{D_2\} \oplus \mathbb{Z}_2[B]\{B_2, C_2\} \oplus \mathbb{Z}/2[B]\{\eta B_2, \eta_1^2\}$$
$$ko[3] = \mathbb{Z}_2\{D_3\} \oplus \mathbb{Z}_2[B]\{B_3, C_3\} \oplus \mathbb{Z}/2[B]\{\eta B_3, \eta^2 B_3\}$$
$$ko[4] = \mathbb{Z}_2\{D_4\} \oplus \mathbb{Z}_2[B]\{B_4, C_4\} \oplus \mathbb{Z}/2[B]\{\eta_4, \eta\eta_4\}$$
$$ko[5] = \mathbb{Z}_2\{D_5\} \oplus \mathbb{Z}_2[B]\{B_5, C_5\} \oplus \mathbb{Z}/2[B]\{\eta B_5, \eta_1\eta_4\}$$
$$ko[6] = \mathbb{Z}_2\{D_6\} \oplus \mathbb{Z}_2[B]\{B_6, C_6\} \oplus \mathbb{Z}/2[B]\{\eta B_6, \eta^2 B_6\}$$
$$ko[7] = \mathbb{Z}_2\{D_7\} \oplus \mathbb{Z}_2[B]\{B_7, C_7\} \oplus \mathbb{Z}/2[B]\{\eta B_7, \eta^2 B_7\}\,.$$

The $\mathbb{Z}_2[B]$-module structures are specified by $B \cdot D_k = d_k B_k$ for each $1 \leq k \leq 7$.

Remark 8.5. *The submodule $N_* \subset \pi_*(tmf)$ is preserved by the action of η, ν, ϵ, κ and $\bar{\kappa}$. To check this, note that the B^2-torsion classes κC_7, $\bar{\kappa} B_7$ and $\bar{\kappa} C_7$ are zero. It follows that the isomorphism $N_* \otimes \mathbb{Z}[M] \cong \pi_*(tmf)$ also respects the action by these elements.*

Lemma 8.6.
$$H^s_{(B,M)}(\pi_*(tmf)) \cong H^{s-1}_{(B)}(N_*) \otimes \mathbb{Z}[M]/M^\infty\,.$$

Proof. The spectral sequence
$$E_2^{i,j} = H^i_{(B)}(H^j_{(M)}(N_* \otimes \mathbb{Z}[M])) \Longrightarrow_i H^{i+j}_{(B,M)}(N_* \otimes \mathbb{Z}[M])$$

collapses at the $j = 1$ line, where $H^1_{(M)}(\mathbb{Z}[M]) = \mathbb{Z}[M]/M^\infty = \mathbb{Z}[M^{-1}]\{1/M\}$. □

Lemma 8.7.
$$H^0_{(B)}(N_*) = \Gamma_B N_*$$

and

$$H^1_{(B)}(N_*) = N_*/B^\infty = \bigoplus_{k=0}^{7} ko[k]/B^\infty$$

is the direct sum of the following eight $\mathbb{Z}_2[B]$-modules, with $ko[k]/B^\infty$ concentrated in degrees $* \leq 24k + 4$:

$$ko[0]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{1, C\} \oplus \mathbb{Z}/2[B]/B^\infty\{\eta, \eta^2\}$$
$$ko[1]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_1, C_1\}/(8B_1/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta_1, \eta\eta_1\}$$
$$ko[2]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_2, C_2\}/(4B_2/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta B_2, \eta_1^2\}$$
$$ko[3]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_3, C_3\}/(8B_3/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta B_3, \eta^2 B_3\}$$
$$ko[4]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_4, C_4\}/(2B_4/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta_4, \eta\eta_4\}$$
$$ko[5]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_5, C_5\}/(8B_5/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta B_5, \eta_1\eta_4\}$$
$$ko[6]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_6, C_6\}/(4B_6/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta B_6, \eta^2 B_6\}$$
$$ko[7]/B^\infty = \mathbb{Z}_2[B]/B^\infty\{B_7, C_7\}/(8B_7/B) \oplus \mathbb{Z}/2[B]/B^\infty\{\eta B_7, \eta^2 B_7\}\,.$$

TABLE 8.1. B-power torsion in N_* at $p = 2$

n	$\Gamma_B N_n$	generator	n	$\Gamma_B N_n$	generator
3	$\mathbb{Z}/8$	ν	85	$\mathbb{Z}/2$	$\eta_1 \bar{\kappa}^3$
6	$\mathbb{Z}/2$	ν^2	90	$\mathbb{Z}/2$	$\eta_1^2 \bar{\kappa}^2$
8	$\mathbb{Z}/2$	ϵ	99	$\mathbb{Z}/8$	ν_4
9	$\mathbb{Z}/2$	$\eta\epsilon$	100	$\mathbb{Z}/2$	$\eta\nu_4$
14	$\mathbb{Z}/2$	κ	102	$\mathbb{Z}/2$	$\nu\nu_4$
15	$\mathbb{Z}/2$	$\eta\kappa$	104	$\mathbb{Z}/2$	ϵ_4
17	$\mathbb{Z}/2$	$\nu\kappa$	105	$(\mathbb{Z}/2)^2$	$\eta\epsilon_4$
20	$\mathbb{Z}/8$	$\bar{\kappa}$	$-$	$-$	$\eta_1 \bar{\kappa}^4$
21	$\mathbb{Z}/2$	$\eta\bar{\kappa}$	110	$\mathbb{Z}/4$	κ_4
22	$\mathbb{Z}/2$	$\eta^2\bar{\kappa} = B \cdot \kappa$	111	$\mathbb{Z}/2$	$\eta\kappa_4$
27	$\mathbb{Z}/4$	ν_1	113	$\mathbb{Z}/2$	$\nu\kappa_4$
28	$\mathbb{Z}/2$	$\eta\nu_1 = B \cdot \bar{\kappa}$	116	$\mathbb{Z}/4$	$\bar{\kappa}D_4$
32	$\mathbb{Z}/2$	ϵ_1	117	$\mathbb{Z}/2$	$\eta_4\bar{\kappa}$
33	$\mathbb{Z}/2$	$\eta\epsilon_1$	118	$\mathbb{Z}/2$	$\eta\eta_4\bar{\kappa} = B \cdot \kappa_4$
34	$\mathbb{Z}/2$	$\kappa\bar{\kappa}$	123	$\mathbb{Z}/4$	ν_5
35	$\mathbb{Z}/2$	$\eta\kappa\bar{\kappa} = B \cdot \nu_1$	124	$\mathbb{Z}/2$	$\eta\nu_5$
39	$\mathbb{Z}/2$	$\eta_1\kappa$	125	$\mathbb{Z}/2$	$\eta^2\nu_5 = B \cdot \eta_4\bar{\kappa}$
40	$\mathbb{Z}/4$	$\bar{\kappa}^2$	128	$\mathbb{Z}/2$	ϵ_5
41	$\mathbb{Z}/2$	$\eta\bar{\kappa}^2$	129	$\mathbb{Z}/2$	$\eta\epsilon_5$
42	$\mathbb{Z}/2$	$\eta^2\bar{\kappa}^2 = B \cdot \kappa\bar{\kappa}$	130	$\mathbb{Z}/4$	$\kappa_4\bar{\kappa}$
45	$\mathbb{Z}/2$	$\eta_1\bar{\kappa}$	131	$\mathbb{Z}/2$	$\eta\kappa_4\bar{\kappa} = B \cdot \nu_5$
46	$\mathbb{Z}/2$	$\eta\eta_1\bar{\kappa}$	135	$\mathbb{Z}/2$	$\eta_1\kappa_4$
51	$\mathbb{Z}/8$	ν_2	136	$\mathbb{Z}/2$	$\eta\eta_1\kappa_4 = B \cdot \epsilon_5$
52	$\mathbb{Z}/2$	$\eta\nu_2$	137	$\mathbb{Z}/2$	$\nu_5\kappa$
53	$\mathbb{Z}/2$	$\eta^2\nu_2 = B \cdot \eta_1\bar{\kappa}$	138	$\mathbb{Z}/2$	$\eta\nu_5\kappa = B \cdot \kappa_4\bar{\kappa}$
54	$\mathbb{Z}/4$	$\nu\nu_2$	142	$\mathbb{Z}/2$	$\epsilon_5\kappa$
57	$\mathbb{Z}/2$	$\nu^2\nu_2$	147	$\mathbb{Z}/8$	ν_6
59	$\mathbb{Z}/2$	$B\nu_2$	148	$\mathbb{Z}/2$	$\eta\nu_6$
60	$\mathbb{Z}/4$	$\bar{\kappa}^3$	149	$\mathbb{Z}/2$	$\eta^2\nu_6$
65	$(\mathbb{Z}/2)^2$	$\eta_1\bar{\kappa}^2$	150	$\mathbb{Z}/8$	$\nu\nu_6$
$-$	$-$	$\nu_2\kappa$	153	$\mathbb{Z}/2$	$\nu^2\nu_6$
66	$\mathbb{Z}/2$	$\eta\nu_2\kappa$	155	$\mathbb{Z}/2$	$B\nu_6$
68	$\mathbb{Z}/2$	$\nu\nu_2\kappa$	156	$\mathbb{Z}/2$	$B\eta\nu_6$
70	$\mathbb{Z}/2$	$\eta_1^2\bar{\kappa}$	161	$\mathbb{Z}/2$	$\nu_6\kappa$
75	$\mathbb{Z}/2$	η_1^3	162	$\mathbb{Z}/2$	$\eta\nu_6\kappa$
80	$\mathbb{Z}/2$	$\bar{\kappa}^4$	164	$\mathbb{Z}/2$	$\nu\nu_6\kappa$

Here $\mathbb{Z}_2[B]/B^\infty = \mathbb{Z}_2[B^{-1}]\{1/B\}$ *and* $\mathbb{Z}/2[B]/B^\infty = \mathbb{Z}/2[B^{-1}]\{1/B\}$.

Proof. The relations $B \cdot \eta_k = \eta B_k$ from [BR21, Def. 7.22(7)] ensure that

$$ko[k][1/B] = \mathbb{Z}_2[B^{\pm1}]\{B_k, C_k\} \oplus \mathbb{Z}/2[B^{\pm1}]\{\eta B_k, \eta^2 B_k\}$$

for each $0 \le k \le 7$, from which the formulas for $ko[k]/B^\infty$ follow. Note that $B \cdot D_k = d_k B_k$ in $ko[k]$ implies the relation $d_k \cdot B_k/B = 0$ in $ko[k]/B^\infty$. $\square$

Theorem 8.8. *At $p = 2$, the local cohomology spectral sequence*

$$E_2^{s,t} = H_{(B,M)}^s(\pi_*(tmf))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(B,M)}tmf) \cong \pi_{t-s}(\Sigma^{-22}I_{\mathbb{Z}_2}(tmf))$$

has E_2-term

$$H_{(B,M)}^s(\pi_*(tmf))_* \cong H_{(B)}^{s-1}(N_*) \otimes \mathbb{Z}[M]/M^\infty$$

where $H_{(B)}^*(N_*)$ is displayed in Figures 8.3 and 8.4. There is no room for differentials, so $E_2 = E_\infty$. There are hidden additive extensions

$$d_{7-k} \cdot \nu_k \doteq C_k/B$$

(multiplied by all negative powers of M) for $0 \le k \le 6$, indicated by vertical dashed red lines in the figures. Moreover, there are hidden η- and ν-extensions as shown by sloping dashed and dotted red lines in these figures.

Proof. See [BR21, §9.2] for the η- and ν-multiplications in $\Gamma_B N_*$ that are not evident from the notation. We note in particular the relation $\nu^2\nu_4 = \eta\epsilon_4 + \eta_1\bar{\kappa}^4$ in degree 105. The dotted black lines show B-multiplications. The homotopy cofibre (and fibre) sequences

$$\Sigma^{192}\Gamma_{(B,M)}tmf \xrightarrow{M} \Gamma_{(B,M)}tmf \longrightarrow \Gamma_B N$$

$$I_{\mathbb{Z}_2}N \longrightarrow I_{\mathbb{Z}_2}(tmf) \xrightarrow{M} I_{\mathbb{Z}_2}(\Sigma^{192}tmf)$$

and the equivalence $\Gamma_{(B,M)}tmf \simeq \Sigma^{-22}I_{\mathbb{Z}_2}(tmf)$ imply an equivalence

$$\Gamma_B N \simeq \Sigma^{171}I_{\mathbb{Z}_2}N$$

of *tmf*-modules. For each $0 \le k \le 6$ the group $\pi_{24k+3}(\Gamma_B N) \cong \pi_{-24(7-k)}(I_{\mathbb{Z}_2}N)$ sits in a short exact sequence

$$0 \to \mathrm{Ext}_{\mathbb{Z}_2}(\pi_{24(7-k)-1}(N), \mathbb{Z}_2) \longrightarrow \pi_{-24(7-k)}(I_{\mathbb{Z}_2}N)$$
$$\longrightarrow \mathrm{Hom}_{\mathbb{Z}_2}(\pi_{24(7-k)}(N), \mathbb{Z}_2) \to 0\,,$$

cf. (3.1). Here $\pi_{24(7-k)-1}(N) = 0$ and

$$\pi_{24(7-k)}(N) \cong \mathbb{Z}_2\{B^{3(7-k)}, \dots, B^3 D_{6-k}, D_{7-k}\} \cong \mathbb{Z}_2^{8-k}\,,$$

so $\pi_{24k+3}(\Gamma_B N) \cong \mathbb{Z}_2^{8-k}$ is 2-torsion free. In each case this implies that ν_k, which generates a cyclic group $\langle\nu_k\rangle$ of order d_{7-k} in $\Gamma_B N_*$, lifts to a class of infinite order in $\pi_{24k+3}(\Gamma_B N)$. Since ν_k is (B- or) B^2-torsion in $\Gamma_B N_*$, its lift must also be (B- or) B^2-torsion, and the only possibility is that d_{7-k} times the lift of ν_k is a 2-adic unit times the image of $C_k/B \in N_*/B^\infty$. Hence there is a hidden 2-extension from $\frac{1}{2}d_{7-k}\nu_k$ in Adams bidegree $(t - s, s) = (24k + 3, 0)$ to C_k/B in bidegree $(t - s, s) = (24k + 3, 1)$, in the local cohomology spectral sequence

$$E_2^{s,t} = H_{(B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_B N) \cong \pi_{t-s}(\Sigma^{171}I_{\mathbb{Z}_2}N)\,.$$

This translates to a hidden 2-extension from $\frac{1}{2}d_{7-k}\nu_k/M$ in bidegree $(t-s,s) = (24k-190,1)$ to C_k/BM in bidegree $(t-s,s) = (24k-190,2)$ in the local cohomology spectral sequence for $\Gamma_{(B,M)}tmf$, together with its multiples by all negative powers of M.

There is no room for further hidden 2-extensions, by elementary η-, ν- and B-linearity considerations. The hidden η- and ν-extensions are present in $\pi_*(I_{\mathbb{Z}_2}N)$, hence also in $\pi_*(\Gamma_B N)$ and in $\pi_*(\Gamma_{(B,M)}tmf)$, with the appropriate degree shifts. $\qquad\square$

8.B. $(2, B, M)$-**local cohomology of** tmf.

Lemma 8.9.

$$H^s_{(2,B,M)}(\pi_*(tmf)) \cong H^{s-1}_{(2,B)}(N_*) \otimes \mathbb{Z}[M]/M^\infty .$$

Proof. Replace (B) by $(2,B)$ in the proof of Lemma 8.6. $\qquad\square$

Proposition 8.10. *All B-power torsion in N_* is 2-power torsion, so*

$$H^0_{(2,B)}(N_*) = \Gamma_B N_*$$
$$H^1_{(2,B)}(N_*) = \Gamma_2(N_*/B^\infty)$$
$$H^2_{(2,B)}(N_*) = N_*/(2^\infty, B^\infty)$$

with a short exact sequence

$$0 \to (\Gamma_2 N_*)/B^\infty \longrightarrow \Gamma_2(N_*/B^\infty) \longrightarrow \Gamma_B(N_*/2^\infty) \to 0 .$$

Here

$$(\Gamma_2 N_*)/B^\infty = \mathbb{Z}/2[B]/B^\infty\{\eta, \eta^2, \eta_1, \eta\eta_1, \eta B_2, \eta_1^2, \eta B_3, \eta^2 B_3,$$
$$\eta_4, \eta\eta_4, \eta B_5, \eta_1\eta_4, \eta B_6, \eta^2 B_6, \eta B_7, \eta^2 B_7\} ,$$

and

$$\Gamma_B(N_*/2^\infty) = \bigoplus_{k=1}^{7} \mathbb{Z}/d_k\{B_k/B\} ,$$

while

$$N_*/(2^\infty, B^\infty) = \bigoplus_{k=0}^{7} \mathbb{Z}_2[B]/(2^\infty, B^\infty)\{B_k/B, C_k\} ,$$

where $\mathbb{Z}_2[B]/(2^\infty, B^\infty) = \mathbb{Q}_2/\mathbb{Z}_2[B^{-1}]\{1/B\}$.

Proof. This follows from the composite functor spectral sequence of Subsection 2.E with $R_* = \pi_*(tmf)/M$, first applied with $x = B$ and $y = 2$, and thereafter with $x = 2$ and $y = B$. The formulas for $(\Gamma_2 N_*)/B^\infty$ and $N_*/(2^\infty, B^\infty)$ follow from the expressions for N_* and N_*/B^∞ in Theorem 8.4 and Lemma 8.7. Only the summands $\mathbb{Z}_2[D_k] \oplus \mathbb{Z}_2[B]\{B_k\} \subset ko[k]$ of N_* contribute to $\Gamma_B(N_*/2^\infty)$, where $B \cdot D_k = d_k B_k$. The B-power torsion in $ko[k]/2^\infty$ equals $\mathbb{Z}/d_k\{D_k/d_k\} \subset \mathbb{Q}_2/\mathbb{Z}_2\{D_k\}$, which we can rewrite as $\mathbb{Z}/d_k\{B_k/B\}$. $\qquad\square$

Theorem 8.11. *The local cohomology spectral sequence*

$$E_2^{s,t} = H^s_{(2,B,M)}(\pi_*(tmf))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B,M)}tmf) \cong \pi_{t-s}(\Sigma^{-23}I(tmf))$$

has E_2-term

$$H^s_{(2,B,M)}(\pi_*(tmf))_* \cong H^{s-1}_{(2,B)}(N_*) \otimes \mathbb{Z}[M]/M^\infty$$

where $H^*_{(2,B)}(N_*)$ is displayed in Figures 8.5 through 8.8. There are d_2-differentials

$$d_2(\nu_k) \doteq C_k/d_{7-k}B$$

(multiplied by all negative powers of M) for $0 \le k \le 6$, indicated by the green zigzag arrows increasing the filtration by 2. There are no hidden additive extensions, but several hidden η- and ν-extensions, as shown by sloping dashed and dotted red lines in these figures.

Proof. The homotopy (co-)fibre sequences

$$\Sigma^{192}\Gamma_{(2,B,M)}tmf \xrightarrow{M} \Gamma_{(2,B,M)}tmf \longrightarrow \Gamma_{(2,B)}N$$

$$IN \longrightarrow I(tmf) \xrightarrow{M} I(\Sigma^{192}tmf)$$

and the equivalence $\Gamma_{(2,B,M)}tmf \simeq \Sigma^{-23}I(tmf)$ imply an equivalence

$$\Gamma_{(2,B)}N \simeq \Sigma^{170}IN$$

of tmf-modules. For each $0 \le k \le 6$ the group

$$\pi_{24k+3}(\Gamma_{(2,B)}N) \cong \pi_{-24(7-k)+1}(IN) \cong \mathrm{Hom}_{\mathbb{Z}_2}(\pi_{24(7-k)-1}(N), \mathbb{Q}_2/\mathbb{Z}_2)$$

is trivial, since $\pi_{24(7-k)-1}(N) = 0$. Hence the group $\langle \nu_k \rangle = \mathbb{Z}/d_{7-k}\{\nu_k\}$ in degree $24k + 3$ of $\Gamma_{(2,B)}N_* = \Gamma_B N_*$ cannot survive to E_∞ in the local cohomology spectral sequence

$$E_2^{s,t} = H^s_{(2,B)}(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)}N) \cong \pi_{t-s}(\Sigma^{170}IN).$$

This means that d_2 must act injectively on $\langle \nu_k \rangle$. Since ν_k is (B- or) B^2-torsion, the only possible target in bidegree $(t - s, s) = (24k + 2, 2)$ is $\mathbb{Q}_2/\mathbb{Z}_2\{C_k/B\}$, and therefore d_2 maps $\langle \nu_k \rangle$ isomorphically to the subgroup of this target that is generated by $C_k/d_{7-k}B$.

This translates to a d_2-differential in the local cohomology spectral sequence for $\Gamma_{(2,B,M)}tmf$ from ν_k/M in bidegree $(t - s, s) = (24k - 190, 1)$ to $C_k/d_{7-k}BM$ in bidegree $(t - s, s) = (24k - 191, 3)$ together with its multiples by all negative powers of M. The 2-, η- and ν-extensions in $\pi_*(N)$ and $\pi_*(IN)$ are also present in $\pi_*(\Gamma_{(2,B)}N)$ and in $\pi_*(\Gamma_{(2,B,M)}tmf)$, with the appropriate degree shifts, and those that increase the local cohomology filtration degree are displayed with red lines. $\square$

Remark 8.12. Let $\Theta N_* \subset \Gamma_B N_*$ be the part of the B-power torsion in N_* that is not in degrees $* \equiv 3 \mod 24$, omitting the subgroups $\langle \nu_k \rangle$ for $0 \le k \le 6$ from Table 8.1. This equals the kernel of the d_2-differential in the $(2, B)$-local cohomology spectral sequence, which is also the image of the edge homomorphism $\pi_*(\Gamma_{(2,B)}N) \to \Gamma_B N_*$. Furthermore, let $\Theta \pi_*(tmf)$ be the part of $\Gamma_B \pi_*(tmf)$ that is not in degrees $* \equiv 3 \mod 24$, which equals the image of the edge homomorphism $\pi_*(\Gamma_{(2,B)}tmf) \to \Gamma_B \pi_*(tmf)$.

The image of the 2-complete tmf-Hurewicz homomorphism $\pi_*(S) \to \pi_*(tmf)$ is the direct sum of $\mathbb{Z}$ in degree 0, the 8-periodic groups $\mathbb{Z}/2\{\eta B^k\}$ and $\mathbb{Z}/2\{\eta^2 B^k\}$ for $k \ge 0$, the group $\mathbb{Z}/8\{\nu\}$ in degree 3, and the 192-periodic groups $\Theta \pi_*(tmf) \cong \Theta N_* \otimes \mathbb{Z}[M]$. This was conjectured by Mahowald, was proved for degrees $n \le 101$ and $n = 125$ in [BR21, Thm. 11.89], and has now been proved in all degrees by Behrens, Mahowald and Quigley [BMQ]. The three first summands of the tmf-Hurewicz image are also detected by the Adams d- and e-invariants. To see that

the fourth summand is contained in the image from $\pi_*(\Gamma_{(2,B)}tmf)$, one can use the commutative diagram

$$
\begin{array}{ccc}
C_1^f S & \longrightarrow & S \\
\downarrow & & \downarrow \\
C_1^f tmf & \longrightarrow & tmf
\end{array}
$$

and the equivalence $C_1^f tmf \simeq \Gamma_{(2,B)}tmf$ from Lemmas 2.2 and 2.4.

8.C. (B, H)-**local cohomology of** tmf. Let $p = 3$ in this subsection and the next. See Figure 8.10 for the mod 3 (tmf-module) Adams E_∞-term for tmf in the range $0 \leq t - s \leq 72$, with all hidden ν-extensions shown. There are no hidden B- or H-extensions in this spectral sequence.

Definition 8.13. Let $N_* \subset \pi_*(tmf)$ be the $\mathbb{Z}_3[B]$-submodule generated by all classes in degrees $0 \leq * < 72$, and let $N = tmf/H$.

Theorem 8.14 ([BR21, Lem. 13.16]). *The composite homomorphisms*

$$
N_* \otimes \mathbb{Z}[H] \longrightarrow \pi_*(tmf) \otimes \pi_*(tmf) \xrightarrow{\ \cdot\ } \pi_*(tmf)
$$
$$
N_* \subset \pi_*(tmf) \longrightarrow \pi_*(N)
$$

are isomorphisms. Hence $\pi_*(tmf)$ is a (split) extension of $\mathbb{Z}_3[B, H]$-modules

$$
0 \to \Gamma_B N_* \otimes \mathbb{Z}[H] \longrightarrow \pi_*(tmf) \longrightarrow \frac{N_*}{\Gamma_B N_*} \otimes \mathbb{Z}[H] \to 0 \,.
$$

Theorem 8.15 ([BR21, Thm. 13.18]). *As a $\mathbb{Z}_3[B]$-module, N_* is a split extension*

$$
0 \to \Gamma_B N_* \longrightarrow N_* \longrightarrow \frac{N_*}{\Gamma_B N_*} \to 0 \,.
$$

The B-power torsion submodule $\Gamma_B N_*$ is given in Table 8.2. It is concentrated in degrees $3 \leq * \leq 40$, and is annihilated by $(3, B)$.

The B-torsion free quotient of N_* is the direct sum

$$
\frac{N_*}{\Gamma_B N_*} = \bigoplus_{k=0}^{2} ko[k]
$$

of the following three $\mathbb{Z}_3[B]$-modules, with $ko[k]$ concentrated in degrees $* \geq 24k$:

$$
ko[0] = \mathbb{Z}_3[B]\{1, C\}
$$
$$
ko[1] = \mathbb{Z}_3\{D_1\} \oplus \mathbb{Z}_3[B]\{B_1, C_1\}
$$
$$
ko[2] = \mathbb{Z}_3\{D_2\} \oplus \mathbb{Z}_3[B]\{B_2, C_2\} \,.
$$

The $\mathbb{Z}_3[B]$-module structures are specified by $B \cdot D_1 = 3B_1$ and $B \cdot D_2 = 3B_2$.

Lemma 8.16.

$$
H_{(B,H)}^s(\pi_*(tmf)) \cong H_{(B)}^{s-1}(N_*) \otimes \mathbb{Z}[H]/H^\infty \,.
$$

Proof. Replace M by H in the proof of Lemma 8.6. □

TABLE 8.2. B-power torsion in N_* at $p = 3$

n	$\Gamma_B N_n$	generator
3	$\mathbb{Z}/3$	ν
10	$\mathbb{Z}/3$	β
13	$\mathbb{Z}/3$	$\nu\beta$
20	$\mathbb{Z}/3$	β^2
27	$\mathbb{Z}/3$	ν_1
30	$\mathbb{Z}/3$	β^3
37	$\mathbb{Z}/3$	$\nu_1\beta$
40	$\mathbb{Z}/3$	β^4

Lemma 8.17.

$$H^0_{(B)}(N_*) = \Gamma_B N_*$$

and

$$H^1_{(B)}(N_*) = N_*/B^\infty = \bigoplus_{k=0}^{2} ko[k]/B^\infty$$

is the direct sum of the following three $\mathbb{Z}_3[B]$-modules, with $ko[k]/B^\infty$ concentrated in degrees $* \leq 24k + 4$:

$$ko[0]/B^\infty = \mathbb{Z}_3[B]/B^\infty\{1, C\}$$
$$ko[1]/B^\infty = \mathbb{Z}_3[B]/B^\infty\{B_1, C_1\}/(3B_1/B)$$
$$ko[2]/B^\infty = \mathbb{Z}_3[B]/B^\infty\{B_2, C_2\}/(3B_2/B).$$

Proof. For $k \in \{1, 2\}$, the relation $B \cdot D_k = 3B_k$ in $ko[k]$ implies the relation $3 \cdot B_k/B = 0$ in $ko[k]/B^\infty$. □

Theorem 8.18. *At $p = 3$, the local cohomology spectral sequence*

$$E_2^{s,t} = H^s_{(B,H)}(\pi_*(tmf))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(B,H)}tmf) \cong \pi_{t-s}(\Sigma^{-22} I_{\mathbb{Z}_3}(tmf))$$

has E_2-term

$$H^s_{(B,H)}(\pi_*(tmf))_* \cong H^{s-1}_{(B)}(N_*) \otimes \mathbb{Z}[H]/H^\infty$$

where $H^*_{(B)}(N_*)$ is displayed in Figure 8.11. There is no room for differentials, so $E_2 = E_\infty$. There are hidden additive extensions

$$3 \cdot \nu \doteq C/B \quad and \quad 3 \cdot \nu_1 \doteq C_1/B$$

(multiplied by all negative powers of H), indicated by vertical dashed red lines in the figure. Moreover, there is a hidden ν-extension from β^2 to B_1/B, shown by a sloping dotted red line.

Proof. We refer to [BR21, Prop. 13.14] for the relation $\nu\nu_1 \doteq \beta^3$. The equivalence $\Gamma_{(B,H)}tmf \simeq \Sigma^{-22} I_{\mathbb{Z}_3}(tmf)$ implies an equivalence $\Gamma_B N \simeq \Sigma^{51} I_{\mathbb{Z}_3} N$ of tmf-modules. For $k \in \{0, 1\}$ the group $\pi_{24k+3}(\Gamma_B N) \cong \pi_{-24(2-k)}(I_{\mathbb{Z}_3} N)$ sits in an extension

$$0 \to \mathrm{Ext}_{\mathbb{Z}_3}(\pi_{24(2-k)-1}(N), \mathbb{Z}_3) \to \pi_{-24(2-k)}(I_{\mathbb{Z}_3} N) \to \mathrm{Hom}_{\mathbb{Z}_3}(\pi_{24(2-k)}(N), \mathbb{Z}_3) \to 0.$$

Here $\pi_{24(2-k)-1}(N) = 0$ and

$$\pi_{24(2-k)}(N) \cong \mathbb{Z}_3\{B^{3(2-k)}, \ldots, D_{2-k}\} \cong \mathbb{Z}_3^{3-k},$$

so $\pi_{24k+3}(\Gamma_B N) \cong \mathbb{Z}_3^{3-k}$. Since ν_k is B-torsion, there must be a 3-extension in $\pi_{24k+3}(\Gamma_B N)$ from ν_k to a 3-adic unit times C_k/B. The ν-extension from ν_1 to β^3 in $\pi_*(N)$ appears in dual form in $\pi_*(I_{\mathbb{Z}_3}N)$, $\pi_*(\Gamma_B N)$ and $\pi_*(\Gamma_{(B,H)}tmf)$, and appears as a hidden ν-extension from β^2 to B_1/B in the second of these.   □

8.D. $(3, B, H)$-**local cohomology of** tmf.

Lemma 8.19.

$$H^s_{(3,B,H)}(\pi_*(tmf)) \cong H^{s-1}_{(3,B)}(N_*) \otimes \mathbb{Z}[H]/H^\infty.$$

Proof. Replace (B) by $(3, B)$ in the proof of Lemma 8.16.   □

Proposition 8.20.

$$H^0_{(3,B)}(N_*) = \Gamma_B N_* = \mathbb{Z}/3\{\nu, \beta, \nu\beta, \beta^2, \nu_1, \beta^3, \nu_1\beta, \beta^4\}$$

$$H^1_{(3,B)}(N_*) = \Gamma_3(N_*/B^\infty) = \Gamma_B(N_*/3^\infty) = \mathbb{Z}/3\{B_1/B, B_2/B\}$$

$$H^2_{(3,B)}(N_*) = N_*/(3^\infty, B^\infty) = \bigoplus_{k=0}^{2} \mathbb{Z}_3[B]/(3^\infty, B^\infty)\{B_k/B, C_k\}.$$

Proof. This follows from the composite functor spectral sequence of Subsection 2.E with $R_* = \pi_*(tmf)/H$, first applied with $x = 3$ and $y = B$, and thereafter with $x = B$ and $y = 3$. The groups $(\Gamma_B N_*)/3^\infty$ and $(\Gamma_3 N_*)/B^\infty$ vanish. The 3-power torsion in $ko[k]/B^\infty$ is trivial for $k = 0$, and equals $\mathbb{Z}/3\{B_k/B\}$ for $k \in \{1, 2\}$.   □

Theorem 8.21. *The local cohomology spectral sequence*

$$E_2^{s,t} = H^s_{(3,B,H)}(\pi_*(tmf))_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(3,B,H)}tmf) \cong \pi_{t-s}(\Sigma^{-23}I(tmf))$$

has E_2-term

$$H^s_{(3,B,H)}(\pi_*(tmf))_* \cong H^{s-1}_{(3,B)}(N_*) \otimes \mathbb{Z}[H]/H^\infty$$

where $H^*_{(3,B)}(N_*)$ is displayed in Figure 8.12. There are d_2-differentials

$$d_2(\nu_k) \doteq C_k/3B$$

(multiplied by all negative powers of M) for $k \in \{0, 1\}$, indicated by the green zigzag arrows increasing the filtration by 2. There are no hidden additive extensions, but hidden ν-extensions from β^2 to B_1/B and from B_2/B to $C_2/3B$, as shown by sloping dashed red lines in this figure.

Proof. The equivalence $\Gamma_{(3,B,M)}tmf \simeq \Sigma^{-23}I(tmf)$ implies an equivalence

$$\Gamma_{(3,B)}N \simeq \Sigma^{50}IN$$

of tmf-modules. For each $k \in \{0, 1\}$ the group

$$\pi_{24k+3}(\Gamma_{(3,B)}N) \cong \pi_{-24(2-k)+1}(IN) \cong \mathrm{Hom}_{\mathbb{Z}_3}(\pi_{24(2-k)-1}(N), \mathbb{Q}_3/\mathbb{Z}_3)$$

is trivial, since $\pi_{24(2-k)-1}(N) = 0$. Hence the group $\langle \nu_k \rangle \cong \mathbb{Z}/3$ in degree $24k+3$ of $\Gamma_{(3,B)}N_* = \Gamma_B N_*$ cannot survive to E_∞ in the local cohomology spectral sequence

$$E_2^{s,t} = H^s_{(3,B)}(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(3,B)}N) \cong \pi_{t-s}(\Sigma^{50}IN).$$

Since ν_k is B-torsion, it follows that d_2 maps $\langle \nu_k \rangle$ isomorphically to the subgroup of $\mathbb{Q}_3/\mathbb{Z}_3\{C_k/B\}$ that is generated by $C_k/3B$. This translates to a d_2-differential in the local cohomology spectral sequence for $\Gamma_{(3,B,H)}tmf$ from ν_k/H in bidegree $(t-s, s) = (24k-70, 1)$ to $C_k/3BH$ in bidegree $(t-s, s) = (24k-71, 3)$ together with its multiples by all negative powers of H. The ν-extensions in $\pi_*(N)$ and $\pi_*(IN)$ are also present in $\pi_*(\Gamma_{(3,B)}N)$ and in $\pi_*(\Gamma_{(3,B,H)}tmf)$, with the appropriate degree shifts, and those that increase the local cohomology filtration degree are displayed in red. $\square$

Remark 8.22. Let $\Theta N_* \subset \Gamma_B N_*$ be the part of the B-power torsion in N_* that is not in degrees $* \equiv 3 \mod 24$, omitting $\mathbb{Z}/3\{\nu\}$ and $\mathbb{Z}/3\{\nu_1\}$ from Table 8.2. Likewise, let $\Theta \pi_*(tmf)$ be the part of $\Gamma_B \pi_*(tmf)$ that is not in degrees $* \equiv 3 \mod 24$, which equals the image of the $(3, B)$-local cohomology spectral sequence edge homomorphism $\pi_*(\Gamma_{(3,B)}tmf) \to \Gamma_B \pi_*(tmf)$.

Mahowald conjectured that the image of the 3-complete tmf-Hurewicz homomorphism $\pi_*(S) \to \pi_*(tmf)$ is the direct sum of $\mathbb{Z}$ in degree 0, the group $\mathbb{Z}/3\{\nu\}$ in degree 3, and the 72-periodic groups $\Theta \pi_*(tmf) \cong \Theta N_* \otimes \mathbb{Z}[H]$. This was proved for degrees $n < 154$ in [BR21, Prop. 13.29].

8.E. **Charts.** Figure 8.1 shows $N_* \cong \pi_*(N)$, $\pi_*(\Gamma_B N)$ and $\pi_*(\Sigma^{171} I_{\mathbb{Z}_2} N)$ in the range $-9 \leq * \leq 72$, visible as three horizontal wedges. The vertical direction has no intrinsic meaning. Circled numbers represent finite cyclic groups of that order, squares represent infinite cyclic groups, and each ellipse containing '22' represents a Klein Vierergruppe. Horizontal dashed lines show multiplication by B, which extends indefinitely to the right in the upper wedge, and indefinitely to the left in the middle and lower wedges. Thick vertical lines indicate additive extensions, by which a square and a circle combine to an infinite cyclic group. The passage from the upper to the middle wedge is given by taking the homotopy fibre of the localisation map $\gamma \colon N \to N[1/B]$, leaving the B-power torsion (shown in red) in place and replacing copies of $\mathbb{Z}[B]$ or $\mathbb{Z}/2[B]$ (shown in blue) by desuspended copies of $\mathbb{Z}[B]/B^\infty$ or $\mathbb{Z}/2[B]/B^\infty$ (shown in green), respectively. The passage from the upper to the lower wedge takes the torsion-free part of $\pi_*(N)$ to its linear dual in degree $171 - *$, and takes the torsion in $\pi_*(N)$ to its Pontryagin dual in degree $170 - *$. The local cohomology theorem asserts that the middle and lower wedges are isomorphic. Note in particular how this is achieved in degrees $* \equiv -1, 3 \mod 24$.

Figure 8.2 shows the E_∞-term of the mod 2 Adams spectral sequence

$$E_2^{s,t} = \text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s}(tmf)$$

for $0 \leq t - s \leq 192$, together with all hidden 2-, η- and ν-extensions in this range. There is also a more subtle multiplicative relation in degree 105, cf. the proof of Theorem 8.8. The vertical coordinate gives the Adams filtration s. The B-power torsion classes are shown in red, and selected product factorisations in terms of the algebra indecomposables in $\pi_*(tmf)$ are shown. The B-periodic classes are shown in black, and usually only the $\mathbb{Z}[B]$-module generators are labelled. The $\mathbb{Z}[B]$-submodule generated by the classes in degrees $0 \leq * < 192$ defines the basic block N_*, which repeats M-periodically, so that $\pi_*(tmf) \cong N_* \otimes \mathbb{Z}[M]$. Note how the additive structure of N_* also appears in the upper wedge of Figure 8.1.

Figures 8.3 and 8.4 show the collapsing local cohomology spectral sequence for $\Gamma_B N$, in the range $-20 \leq t - s \leq 172$, broken into four sections. In each section

the lower row shows $H^0_{(B)}(N_*) = \Gamma_B N_*$, while the upper row shows $H^1_{(B)}(N_*) = N_*/B^\infty$ shifted one unit to the left. Multiplication by 2, η, ν and B is shown by lines increasing the topological degree by 0, 1, 3 and 8, respectively. The dotted arrows coming from the left indicate classes that are infinitely divisible by B. Multiplications that connect the lower and upper rows increase the local cohomology filtration, hence are hidden, and are shown in red. The additive extensions in degrees $* \equiv 3 \mod 24$ are also carried over to the central wedge of Figure 8.1.

It may be easiest to study these charts by starting in high degrees and descending from there. The top terms in N_* that are not B-divisible are $\mathbb{Z}_2\{C_7\}$, $\mathbb{Z}/2\{\eta^2 B_7\}$, $\mathbb{Z}/2\{\eta B_7\}$ and $\mathbb{Z}/8\{B_7\}$ in degrees 180 and 178 to 176, while the topmost B-power torsion in N_* is $\mathbb{Z}/2\{\nu\nu_6\kappa\}$ in degree 164. These contribute $\mathbb{Z}_2\{C_7/B\}$, $\mathbb{Z}/2\{\eta^2 B_7/B\}$, $\mathbb{Z}/2\{\eta B_7/B\}$ and $\mathbb{Z}/8\{B_7/B\}$ to N_*/B^∞ in internal degrees 172 and 170 to 168, shifted to topological degrees 171 and 169 to 167 in $\pi_*(\Gamma_B N)$, together with $\mathbb{Z}/2\{\nu\nu_6\kappa\}$ in degree 164 of $\Gamma_B N_*$ and $\pi_*(\Gamma_B N)$. In the Anderson dual, the bottom term $\mathbb{Z}_2\{1\}$ of $\pi_*(N)$ contributes a copy of $\mathbb{Z}_2$ in degree 171 of $\pi_*(\Sigma^{171} I_{\mathbb{Z}_2} N)$, while the terms $\mathbb{Z}/2\{\eta\}$, $\mathbb{Z}/2\{\eta^2\}$, $\mathbb{Z}/8\{\nu\}$ and $\mathbb{Z}/2\{\nu^2\}$ contribute the groups $\mathbb{Z}/2$, $\mathbb{Z}/2$, $\mathbb{Z}/8$ and $\mathbb{Z}/2$ in degrees 169 to 167 and 164. The duality theorem matches these groups isomorphically.

Figures 8.5 through 8.8 show the local cohomology spectral sequence for $\Gamma_{(2,B)}N$, in the range $-20 \le t - s \le 172$. In each figure the lower row shows $H^0_{(2,B)}(N_*) = \Gamma_B N_*$, the middle row shows $H^1_{(2,B)}(N_*) = \Gamma_2(N_*/B^\infty)$ shifted one unit to the left, and the upper row shows $H^2_{(2,B)}(N_*) = N_*/(2^\infty, B^\infty)$ shifted two units to the left. There are nonzero d_2-differentials from topological degrees $* \equiv 3 \mod 24$, leaving $E_3 = E_\infty$. Multiplications by 2, η, ν and B, infinitely B-divisible towers, and hidden extensions, are shown as in the previous figures. Note how the abutment $\pi_*(\Gamma_{(2,B)}N)$ is Pontryagin dual to $\pi_{170-*}(N)$.

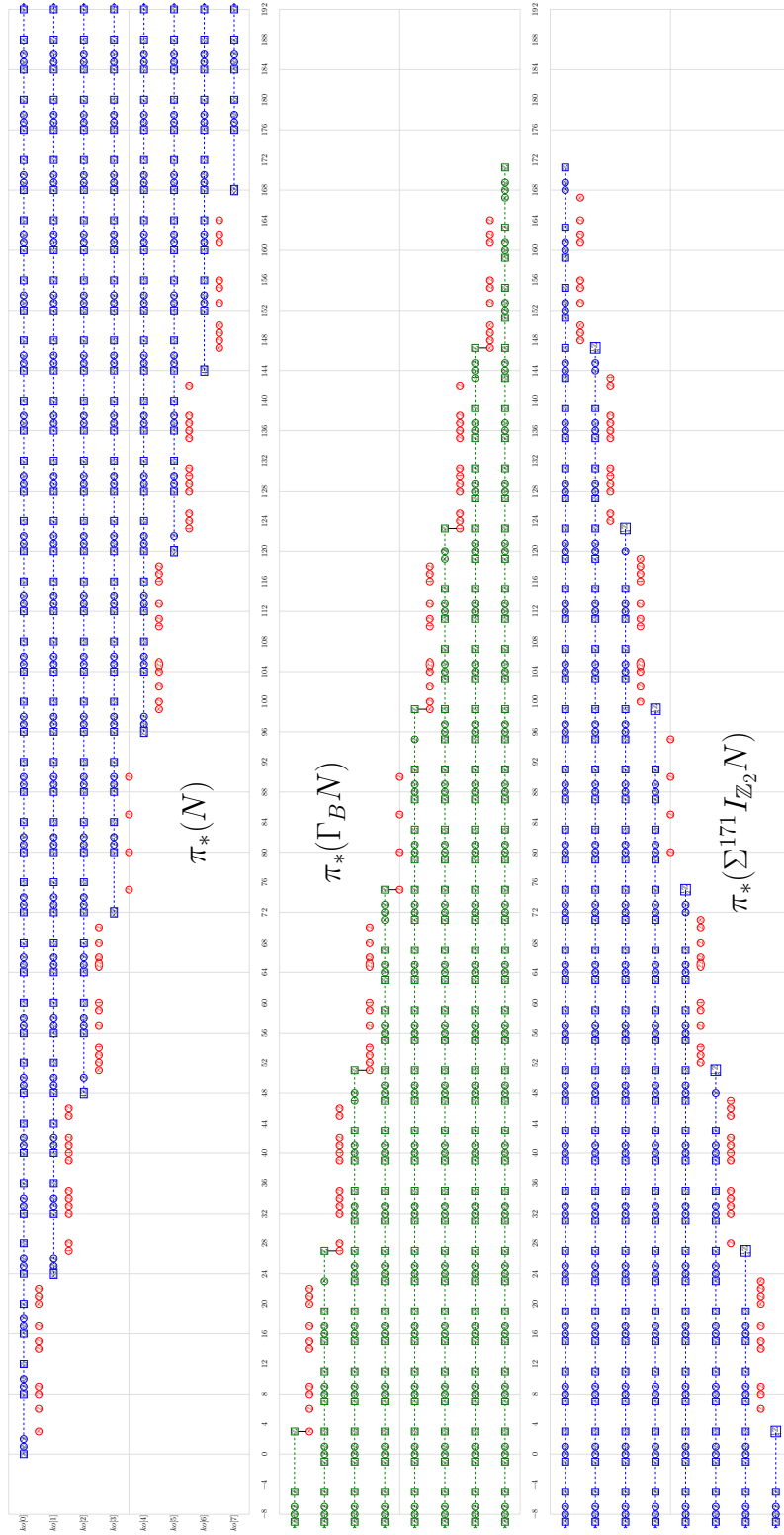The charts for $p = 3$ follow the same conventions as for $p = 2$.

FIGURE 8.1. Homotopy of the basic block N of tmf at $p = 2$, of its B-local cohomology, and of its shifted Anderson dual
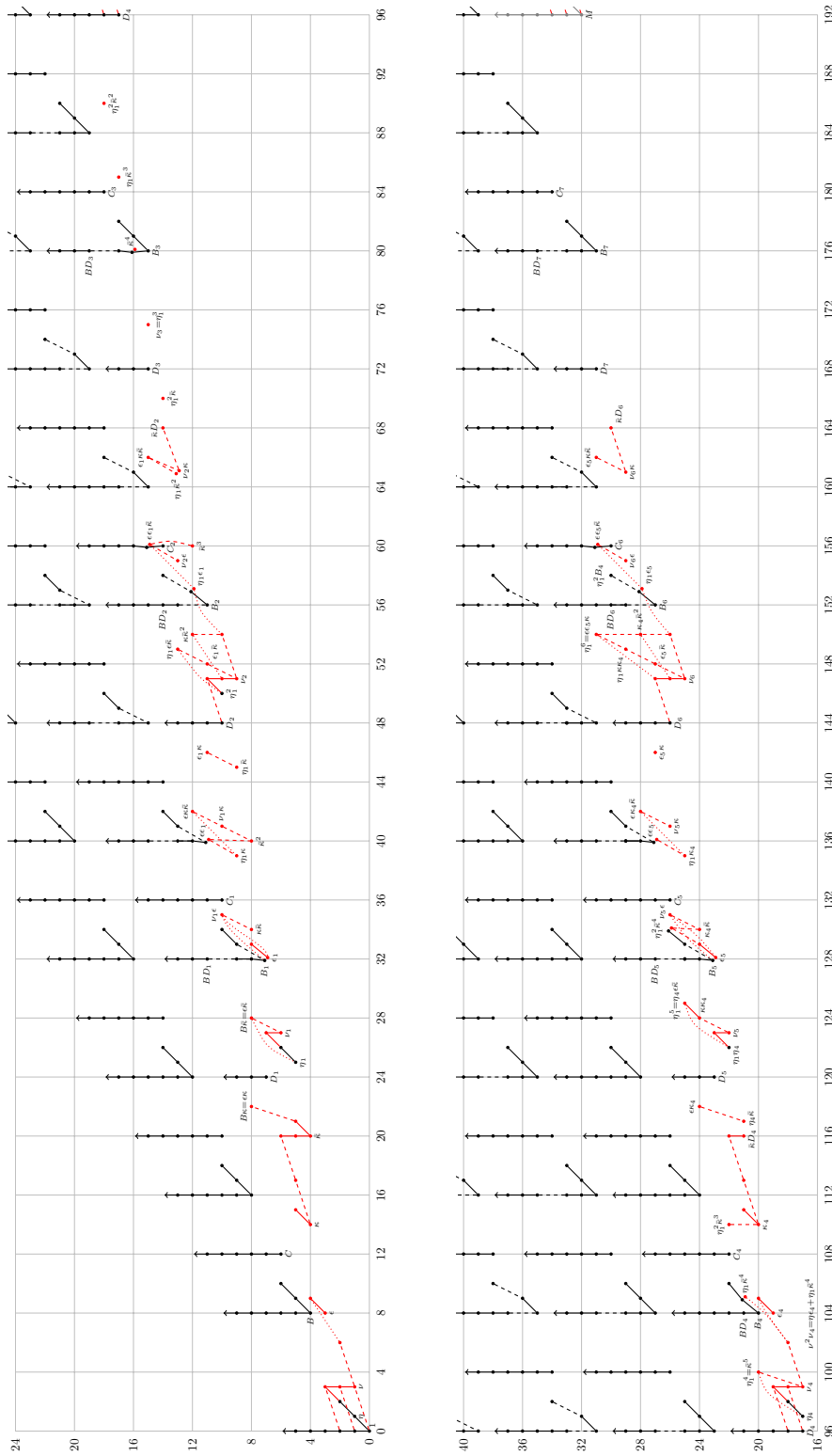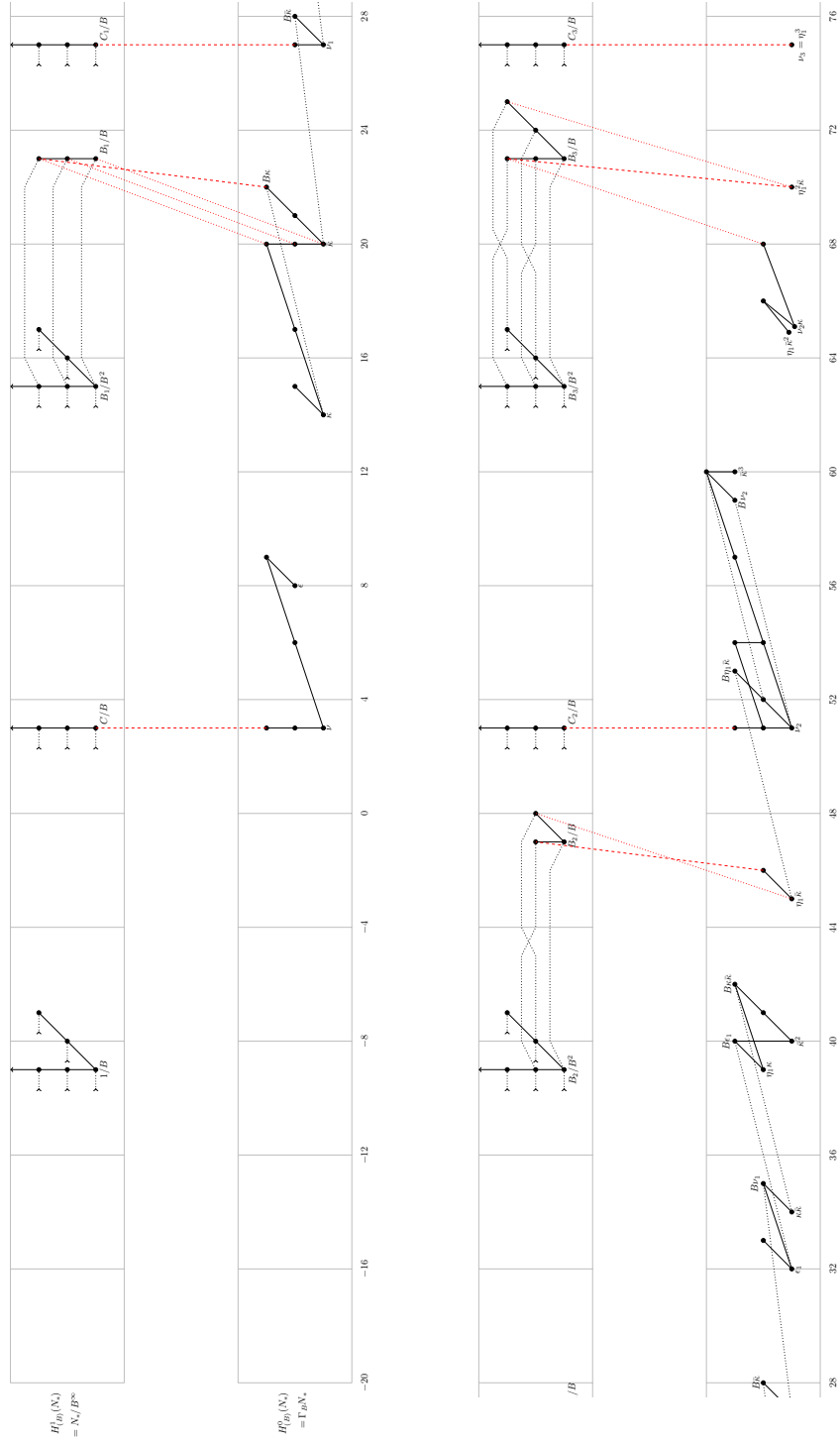
FIGURE 8.2. $\pi_*(tmf)$ at $p = 2$ for $0 \leq * \leq 192$

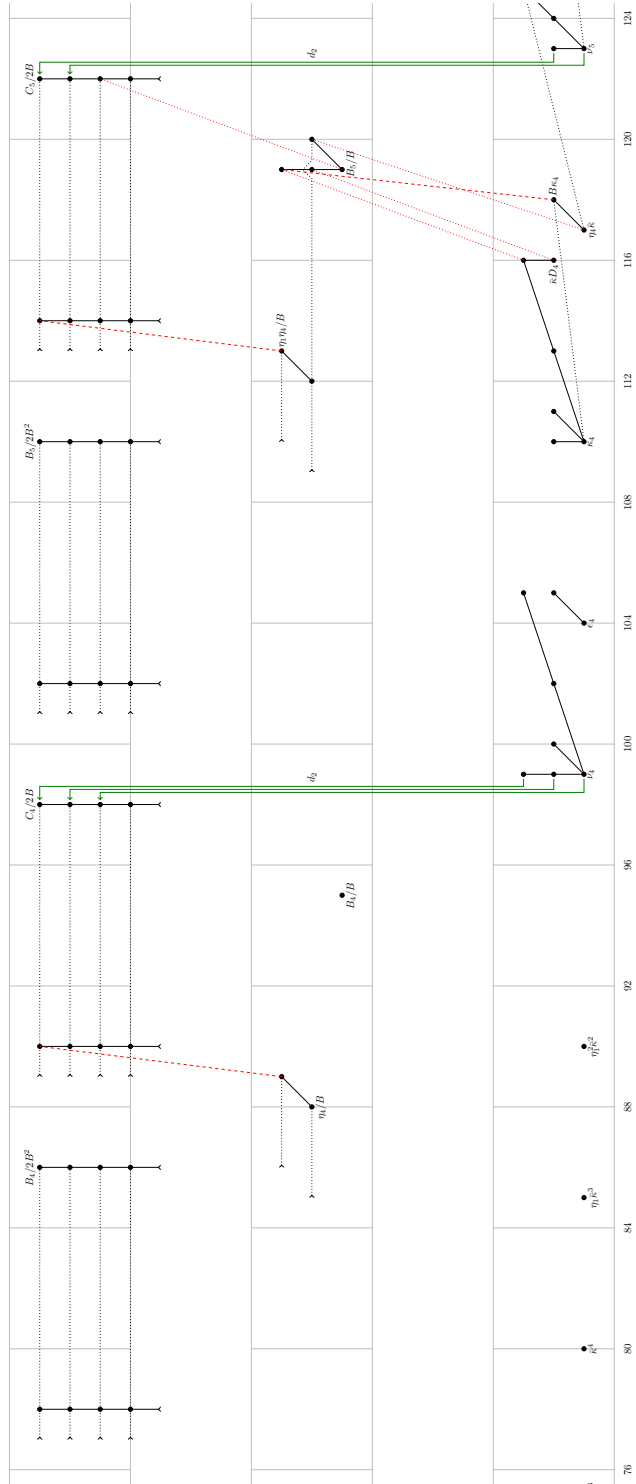FIGURE 8.3. $E_2^{s,t} = H_{(B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_B N)$ at $p = 2$

FIGURE 8.4. $E_2^{s,t} = H_{(B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_B N)$ at $p = 2$

FIGURE 8.5. $E_2^{s,t} = H_{(2,B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)}N)$

FIGURE 8.6. $E_2^{s,t} = H_{(2,B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)}N)$

FIGURE 8.7. $E_2^{s,t} = H_{\langle 2,B\rangle}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{\langle 2,B\rangle})N$
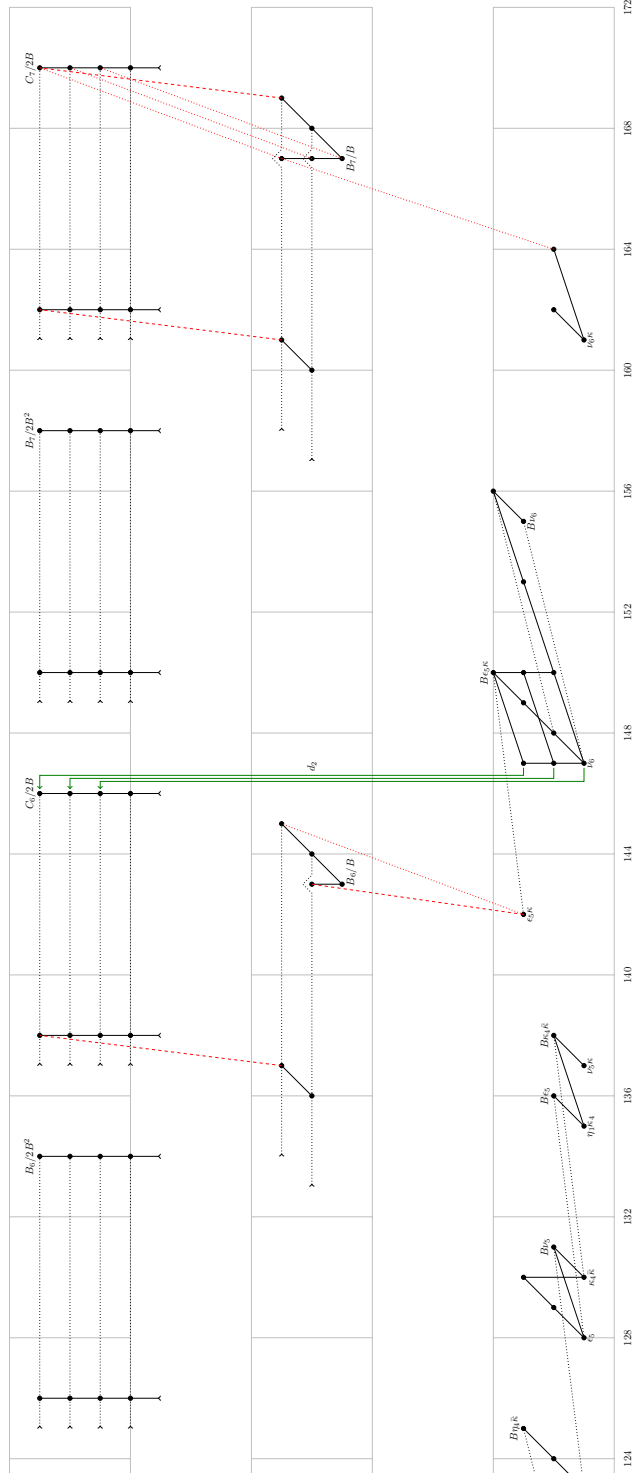
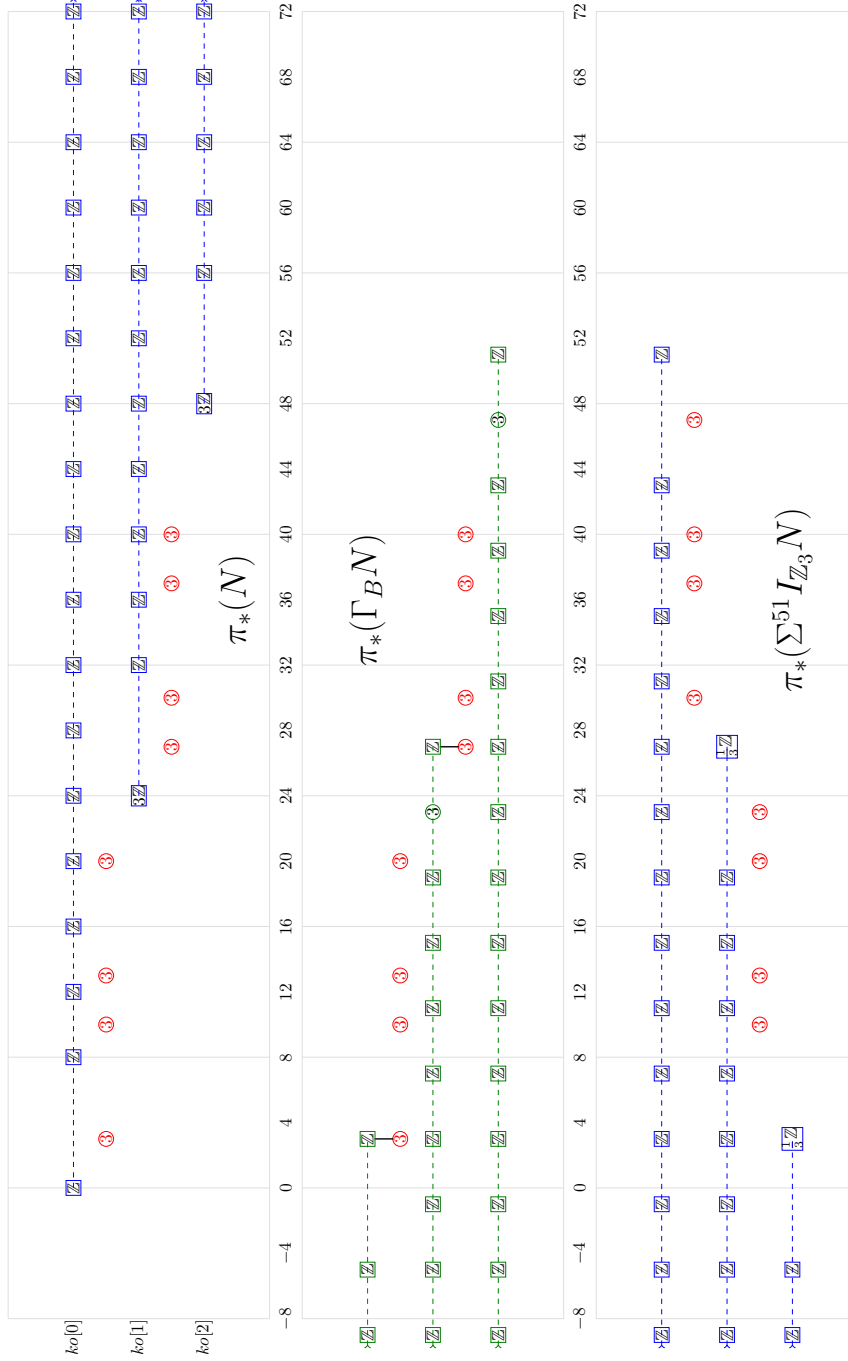FIGURE 8.8. $E_2^{s,t} = H_{(2,B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(2,B)}N)$

FIGURE 8.9. Homotopy of the basic block N of tmf at $p = 3$, of its B-local cohomology, and of its shifted Anderson dual
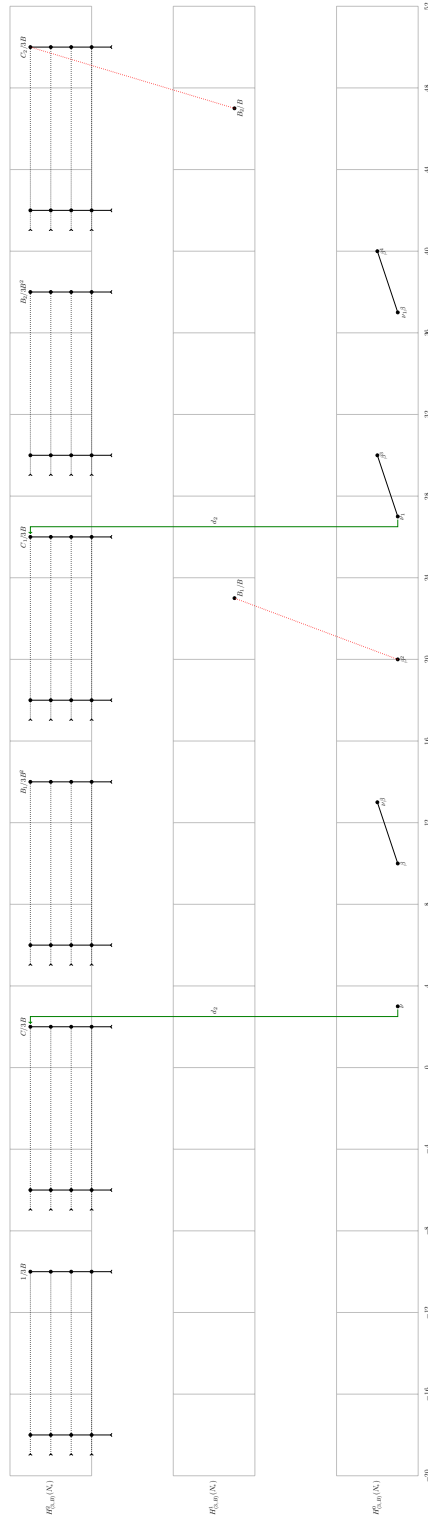
FIGURE 8.10. $\pi_*(tmf)$ at $p = 3$ for $0 \leq * \leq 72$

FIGURE 8.11. $E_2^{s,t} = H_{(B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_B N)$ at $p = 3$

FIGURE 8.12. $E_2^{s,t} = H_{(3,B)}^s(N_*)_t \Longrightarrow_s \pi_{t-s}(\Gamma_{(3,B)}N)$

References

[Ada66] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71, DOI 10.1016/0040-9383(66)90004-8. MR198470

[And69] D. W. Anderson, *Universal coefficient theorems for K-theory* (1969). Mimeographed notes.

[Beh06] Mark Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), no. 2, 343–402, DOI 10.1016/j.top.2005.08.005. MR2193339

[BHHM08] M. Behrens, M. Hill, M. J. Hopkins, and M. Mahowald, *On the existence of a v_2^{32}-self map on $M(1, 4)$ at the prime 2*, Homology Homotopy Appl. **10** (2008), no. 3, 45–84. MR2475617

[BMQ] Mark Behrens, Mark Mahowald, and J. D. Quigley, *The 2-primary Hurewicz image of tmf*. arXiv:2011.08956.

[BP04] Mark Behrens and Satya Pemmaraju, *On the existence of the self map v_2^9 on the Smith-Toda complex $V(1)$ at the prime 3*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 9–49, DOI 10.1090/conm/346/06284. MR2066495

[BG97a] D. J. Benson and J. P. C. Greenlees, *Commutative algebra for cohomology rings of virtual duality groups*, J. Algebra **192** (1997), no. 2, 678–700, DOI 10.1006/jabr.1996.6944. MR1452682

[BG97b] D. J. Benson and J. P. C. Greenlees, *Commutative algebra for cohomology rings of classifying spaces of compact Lie groups*, J. Pure Appl. Algebra **122** (1997), no. 1-2, 41–53, DOI 10.1016/S0022-4049(96)00078-3. MR1479347

[BG08] David J. Benson and J. P. C. Greenlees, *Localization and duality in topology and modular representation theory*, J. Pure Appl. Algebra **212** (2008), no. 7, 1716–1743, DOI 10.1016/j.jpaa.2007.12.001. MR2400738

[Bot59] Raoul Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337, DOI 10.2307/1970106. MR110104

[BC76] Edgar H. Brown Jr. and Michael Comenetz, *Pontrjagin duality for generalized homology and cohomology theories*, Amer. J. Math. **98** (1976), no. 1, 1–27, DOI 10.2307/2373610. MR405403

[BG03] R. R. Bruner and J. P. C. Greenlees, *The connective K-theory of finite groups*, Mem. Amer. Math. Soc. **165** (2003), no. 785, viii+127, DOI 10.1090/memo/0785. MR1997161

[BG10] Robert R. Bruner and J. P. C. Greenlees, *Connective real K-theory of finite groups*, Mathematical Surveys and Monographs, vol. 169, American Mathematical Society, Providence, RI, 2010. MR2723113

[BR21] Robert R. Bruner and John Rognes, *The Adams spectral sequence for topological modular forms*, Mathematical Surveys and Monographs, vol. 253, American Mathematical Society, Providence, RI, 2021.

[BR] Robert R. Bruner and John Rognes, *The Adams spectral sequence for the image-of-J spectrum*. arXiv:2105.02601.

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR0077480

[Cul21] D. Culver, *The Adams spectral sequence for 3-local tmf*, J. Homotopy Relat. Struct. **16** (2021), no. 1, 1–40, DOI 10.1007/s40062-020-00271-3. MR4225505

[Del75] P. Deligne, *Courbes elliptiques: formulaire d'après J. Tate*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1975, pp. 53–73. Lecture Notes in Math., Vol. 476 (French). MR0387292

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014. MR3223024

[DGI06] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar, *Duality in algebra and topology*, Adv. Math. **200** (2006), no. 2, 357–402, DOI 10.1016/j.aim.2005.11.004. MR2200850

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. MR1417719

[Gre93] J. P. C. Greenlees, *K-homology of universal spaces and local cohomology of the representation ring*, Topology **32** (1993), no. 2, 295–308, DOI 10.1016/0040-9383(93)90021-M. MR1217070

[Gre95] J. P. C. Greenlees, *Commutative algebra in group cohomology*, J. Pure Appl. Algebra **98** (1995), no. 2, 151–162, DOI 10.1016/0022-4049(94)00040-P. MR1319967

[Gre16] J. P. C. Greenlees, *Ausoni-Bökstedt duality for topological Hochschild homology*, J. Pure Appl. Algebra **220** (2016), no. 4, 1382–1402, DOI 10.1016/j.jpaa.2015.09.007. MR3423453

[GM95] J. P. C. Greenlees and J. P. May, *Completions in algebra and topology*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 255–276, DOI 10.1016/B978-044481779-2/50008-0. MR1361892

[GM17] J. P. C. Greenlees and Lennart Meier, *Gorenstein duality for real spectra*, Algebr. Geom. Topol. **17** (2017), no. 6, 3547–3619, DOI 10.2140/agt.2017.17.3547. MR3709655

[GS18] J. P. C. Greenlees and V. Stojanoska, *Anderson and Gorenstein duality*, Geometric and topological aspects of the representation theory of finite groups, Springer Proc. Math. Stat., vol. 242, Springer, Cham, 2018, pp. 105–130. MR3901158

[Har67] Robin Hartshorne, *Local cohomology*, Lecture Notes in Mathematics, No. 41, Springer-Verlag, Berlin-New York, 1967. A seminar given by A. Grothendieck, Harvard University, Fall, 1961. MR0224620

[Hil07] Michael A. Hill, *The 3-local* tmf*-homology of* $B\Sigma_3$, Proc. Amer. Math. Soc. **135** (2007), no. 12, 4075–4086, DOI 10.1090/S0002-9939-07-08937-X. MR2341960

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041

[HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49, DOI 10.2307/120991. MR1652975

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114, DOI 10.1090/memo/0610. MR1388895

[HS99] Mark Hovey and Neil P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100, DOI 10.1090/memo/0666. MR1601906

[Kai71] Paul C. Kainen, *Universal coefficient theorems for generalized homology and stable cohomotopy*, Pacific J. Math. **37** (1971), 397–407. MR319198

[Kna99] Karlheinz Knapp, *Anderson duality in K-theory and* Im(J)*-theory*, K-Theory **18** (1999), no. 2, 137–159, DOI 10.1023/A:1007763715735. MR1711716

[MR99] Mark Mahowald and Charles Rezk, *Brown-Comenetz duality and the Adams spectral sequence*, Amer. J. Math. **121** (1999), no. 6, 1153–1177. MR1719751

[MR09] Mark Mahowald and Charles Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. **5** (2009), no. 2, Special Issue: In honor of Friedrich Hirzebruch., 853–872, DOI 10.4310/PAMQ.2009.v5.n2.a9. MR2508904

[Mat16] Akhil Mathew, *The homology of tmf*, Homology Homotopy Appl. **18** (2016), no. 2, 1–29, DOI 10.4310/HHA.2016.v18.n2.a1. MR3515195

[Mil92] Haynes Miller, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) **37** (1992), no. 1-2, 383–389. Papers in honor of José Adem (Spanish). MR1317588

[Sin68] William M. Singer, *Connective fiberings over* BU *and* U, Topology **7** (1968), 271–303, DOI 10.1016/0040-9383(68)90006-2. MR232392

[Sto12] Vesna Stojanoska, *Duality for topological modular forms*, Doc. Math. **17** (2012), 271–311. MR2946825

[Sto14] Vesna Stojanoska, *Calculating descent for 2-primary topological modular forms*, An alpine expedition through algebraic topology, Contemp. Math., vol. 617, Amer. Math. Soc., Providence, RI, 2014, pp. 241–258, DOI 10.1090/conm/617/12286. MR3243402

[Sto63] Robert E. Stong, *Determination of* $H^*(BO(k, \cdots, \infty), Z_2)$ *and* $H^*(BU(k, \cdots, \infty), Z_2)$, Trans. Amer. Math. Soc. **107** (1963), 526–544, DOI 10.2307/1993817. MR151963

[Yos75] Zen-ichi Yosimura, *Universal coefficient sequences for cohomology theories of* CW-*spectra*, Osaka Math. J. **12** (1975), no. 2, 305–323. MR388375

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