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THE ADAMS DIFFERENTIALS ON THE CLASSES h_j^3

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Abstract. In filtration 1 of the Adams spectral sequence, using secondary cohomology operations, Adams [Ada60] computed the differentials on the classes h_j, resolving the Hopf invariant one problem. In Adams filtration 2, using equivariant and chromatic homotopy theory, Hill–Hopkins–Ravenel [HHR16] proved that the classes h_j^2 support non-trivial differentials for $j \geq 7$, resolving the celebrated Kervaire invariant one problem. The precise differentials on the classes h_j^2 for $j \geq 7$ and the fate of h_6^2 remains unknown.

In this paper, in Adams filtration 3, we prove an infinite family of non-trivial d_4-differentials on the classes h_j^3 for $j \geq 6$, confirming a conjecture of Mahowald. Our proof uses two different deformations of stable homotopy theory—$\mathbb{C}$-motivic stable homotopy theory and $\mathbb{F}_2$-synthetic homotopy theory—both in an essential way. Along the way, we also show that h_j^2 survives to the Adams E_5-page and that h_6^2 survives to the Adams E_9-page.

Contents

1. Introduction

One of the basic goals of stable homotopy theory is to obtain a systematic understanding of the differentials in the Adams spectral sequence. Perhaps surprisingly, many questions in differential topology also require and reduce to a core Adams spectral sequence computation. As an example, let us consider the following question: When is the n-sphere parallelizable? Classical constructions using division algebras provide parallelizations of S^1, S^3 and S^7 and the remaining problem is to show that all other spheres are not parallelizable. For this it suffices to show that there does not exist a class with Hopf invariant one in the n-stem.[1] In order to approach this problem Adams began by reformulating earlier work of Adem into the following theorem.

Theorem 1.1 (Adem, Adams [Ada58])**.** *Hopf invariant one classes can only exist in dimensions of the form $2^j - 1$. Moreover, there exists a Hopf invariant one class in the $(2^j - 1)$-stem if and only if the class h_j on the 1-line of the Adams spectral sequence is a permanent cycle.*

[1] In fact, one only needs to show that there is no Hopf invariant one class in the image of J, which is easier than the full Hopf invariant one problem (see [BM58]).

Then, using secondary cohomology operations, Adams computed the first infinite family of nonzero differentials in the Adams spectral sequence, resolving the Hopf invariant one problem.

Theorem 1.2 (Adams [Ada60]). *$d_2(h_j) = h_0 h_{j-1}^2 \neq 0$ for all $j \geq 4$.*

On the Adams 2-line we have classes h_j^2 and Browder showed that they too are intimately connected with a fundamental problem in differential topology.

Theorem 1.3 (Browder [Bro69]). *A smooth framed manifold with Kervaire invariant one can only exist in dimensions of the form $2(2^j - 1)$. Moreover, the following statements are equivalent.*

- *The class h_j^2 is a permanent cycle in the Adams spectral sequence.*
- *There exists a smooth framed manifold of dimension $2(2^j - 1)$ with Kervaire invariant one.*

Using equivariant and chromatic methods, Hill, Hopkins and Ravenel proved the following celebrated theorem, resolving the Kervaire invariant problem in large dimensions.

Theorem 1.4 (Hill–Hopkins–Ravenel [HHR16]). *The classes h_j^2 support nonzero Adams differentials for $j \geq 7$.*[2]

Computations of the homotopy groups of spheres show that the Kervaire classes h_j^2 are permanent cycles for $0 \leq j \leq 5$ [BMT70, BJM84, Xu16]. The final case, the fate of h_6^2 in dimension 126, remains open.

Continuing in this manner we are led to consider the classes h_j^3 on the Adams 3-line. We begin by asking whether they too lie at the heart of some geometric problem.

Question 1.5. Do the classes h_j^3 also have an interpretation in differential topology?

Barratt, Mahowald and Tangora [BMT70] proved that the classes h_j^3 are permanent cycles for $j \leq 4$. At Toda's 60^{th} birthday conference in 1988 Mahowald then made the following conjecture[3] regarding the h_j^3 family:

Conjecture 1.6 (Mahowald). *The classes h_j^3 are not permanent cycles for j large.*

Recently, Isaksen, Wang and the second author proved that h_5^3 supports a nonzero Adams differential using motivic stable homotopy theory. Let g_j be the generator of the group $\mathrm{Ext}_{\mathcal{A}}^{4,3\cdot 2^{j+2}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$.

Theorem 1.7 (Isaksen–Wang–Xu [IWX20b]). *$d_4(h_5^3) = h_0^3 g_3 \neq 0$.*

Previously, Wu and Lin had, respectively, shown that h_6^3 and h_7^3 support nonzero Adams differentials.

Theorem 1.8 (Lin [Lin98], Wu [Wu13]). *$d_4(h_6^3) = h_0^3 g_4 \neq 0$ and $d_4(h_7^3) \neq 0$.*

Lin and Wu analyze h_6^3 and h_7^3 via a study of the Kahn–Priddy transfer map. Lin also went on to conjecture that $d_4(h_j^3) = h_0^3 g_{j-2}$ for $j \geq 6$ [Lin98] and Wu verified this conjecture in the case $j = 6$.

In this article we prove Lin's refinement of Mahowald's Conjecture, thereby determining the fate the h_j^3 family.

Theorem 1.9. *$d_4(h_j^3) = h_0^3 g_{j-2} \neq 0$ for $j \geq 6$.*

[2] Note that the targets of the Adams differentials on the Kervaire classes h_j^2 have not been identified.

[3] See [Min95, Remark 3.1].

The proof of Theorem 1.9 uses two different deformations of the category of spectra—$\mathbb{C}$-motivic spectra and $\mathbb{F}_2$-synthetic spectra—in an essential way. We prove Theorem 1.9 as the Betti realization of a corresponding differential in the motivic Adams spectral sequence. This differential in turn is lifted from the the motivic Adams spectral sequence of a certain 2 cell complex—the cofiber of τ. By a theorem of Gheorghe, Wang and the second author [GWX21, Theorem 1.3], the motivic Adams spectral sequence for the cofiber of τ is isomorphic to the algebraic Novikov spectral sequence. The crucial step in our proof is then to establish a certain non-trivial differential in the algebraic Novikov spectral sequence.

The most delicate part of our argument is in lifting the algebraic Novikov differential proved in Section 6 to the motivic Adams spectral sequence and it is at this point where we need several auxiliary inputs from $\mathbb{F}_2$-synthetic spectra. In Section 7 we analyze the differentials on the classes h_j^2 using a synthetic refinement of the "inductive approach to Kervaire invariant one" from [BJM83]. As a consequence of this we obtain the following theorem.

Theorem 1.10. [4] *Fix an $r \geq 2$ and suppose that θ_j is a lift of h_j^2 from $\mathbb{S}/\lambda$ to $\mathbb{S}/\lambda^r$. If $2\theta_j = 0$ and $\lambda^2\theta_j^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^r)$, then there exists a class θ_{j+1} lifting h_{j+1}^2 to $\mathbb{S}/\lambda^r$ such that $2\theta_{j+1} = 0$ in $\pi_{**}(\mathbb{S}/\lambda^r)$.*

As a corollary of Theorem 1.10 we show that

Corollary 1.11. *The class h_j^2 survives to the Adams E_5-page for all $j \geq 0$. In other words,*

$$d_r(h_j^2) = 0 \quad \textit{for} \quad 2 \leq r \leq 4.$$

Specializing to the case $j = 6$ we are able to refine our arguments to obtain partial progress towards understanding the fate of h_6^2.

Theorem 1.12. *The class h_6^2 survives to the Adams E_9-page. In other words,*

$$d_r(h_6^2) = 0 \quad \textit{for} \quad 2 \leq r \leq 8.$$

Remark 1.13. Adams proved that $h_j^4 = 0$ for all $j \geq 1$ and h_0^n is a nonzero permanent cycles detecting 2^n, therefore as a consequence of Theorem 1.9 the only remaining class of the form h_j^n whose fate we do not know is h_6^2.

The New Doomsday Conjecture.

The infinite families of Adams differentials in Theorems 1.2 and 1.9 share a key structural feature: their sources and targets naturally fit into Sq^0-families. We expect that this is not a coincidence.

Recollection 1.14. The commutative $\mathbb{F}_2$-algebra structure on the stack of additive formal groups provides us with algebraic Steenrod operations[5] acting on the cohomology of the Steenrod algebra. In particular, there is an operation

$$\mathrm{Sq}^0 : \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2).$$

Given a class x in the cohomology of the Steenrod algebra we obtain an infinite Sq^0-family of classes $\{x, \mathrm{Sq}^0(x), \mathrm{Sq}^0(\mathrm{Sq}^0(x)), \dots\}$ by iterating Sq^0. We say that a Sq^0-family is non-trivial if all members of the family are non-zero.

Example 1.15. On the Adams 1-line the Hopf classes $\{h_j\}_{j\geq 0}$ form a Sq^0-family as

$$\mathrm{Sq}^0(h_j) = h_{j+1}.$$

Similarly, because Sq^0 is a ring map, we have Sq^0-families $\{h_j^2\}_{j\geq 0}$ and $\{h_j^3\}_{j\geq 0}$ as well.

[4] See Section 7 for our conventions regarding synthetic spectra.

[5] See Chapters IV–VI of [BMMS86].

Example 1.16. The classes $g_j \in \mathrm{Ext}_{\mathcal{A}}^{4,2^{j+3}+2^{j+2}}(\mathbb{F}_2,\mathbb{F}_2)$ in the statement of Theorem 1.9 also form a Sq^0-family.[6]

In 1995, Minami [Min95] made the following conjecture regarding Sq^0-families:

Conjecture 1.17 (New Doomsday Conjecture). *For any Sq^0-family $\{x_j\}_{j\geq 0}$ on the Adams E_2-page only finitely many classes survive to the E_∞-page.*

Adams's solution of the Hopf invariant one problem (Theorem 1.2) shows that the New Doomsday Conjecture is true on the Adams 1-line. Similarly, Hill–Hopkins–Ravenel's solution of the Kervaire invariant one problem (Theorem 1.4) was the last, hardest, case of the New Doomsday Conjecture on the Adams 2-line.

Remark 1.18. On the Adams 3-line and above, Conjecture 1.17 remains open. With our Theorem 1.9, the remaining open cases are the Sq^0-families $\{h_j^2 h_{j+k+1}+h_{j+1}h_{j+k}^2\}_{j\geq 0}$ for $k \geq 2$.

Based on the uniformity of the differentials in Theorems 1.2 and 1.9 we propose the following refinement of Minami's conjecture:

Conjecture 1.19 (Uniform Doomsday Conjecture). *Let $\{a_j\}_{j\geq 0}$ be a non-trivial Sq^0-family on the Adams E_2-page. Then, there exists another Sq^0-family $\{b_j\}_{j\geq 0}$, an $r \geq 2$ and a class c such that*

$$d_r(a_j) = c \cdot b_j \neq 0$$

for $j \gg 0$.

Example 1.20. Bruner's formulas for power operations on Adams differentials naturally produces families of differentials of the form predicted by Conjecture 1.19. For example one obtains families of differentials

(1) $d_2(h_{j+1}) = h_0 \cdot \mathrm{Sq}^1(h_j) = h_0 h_j^2$,
(2) $d_2(h_{j+1}^2) = h_0 \cdot \mathrm{Sq}^1(h_j^2) = 0$,
(3) $d_2(c_{j+1}) = h_0 \cdot \mathrm{Sq}^1(c_j) = h_0 f_j$,
(4) $d_2(d_{j+1}) = h_0 \cdot \mathrm{Sq}^1(d_j) = 0$,
(5) $d_2(e_{j+1}) = h_0 \cdot \mathrm{Sq}^1(e_j) = h_0 x_j$,
(6) $d_2(f_{j+1}) = h_0 \cdot \mathrm{Sq}^1(f_j) = 0$,
(7) $d_2(g_{j+1}) = h_0 \cdot \mathrm{Sq}^1(g_j) = 0$,
(8) $d_2(p_{j+1}) = h_0 \cdot \mathrm{Sq}^1(p_j) = h_0 h_j p_j'$,
(9) $d_2(D_3(j+1)) = h_0 \cdot \mathrm{Sq}^1(D_3(j)) = h_0 K_j$,
(10) $d_2(p_{j+1}') = h_0 \cdot \mathrm{Sq}^1(p_j') = h_0 T_j$,

for $j \geq 1$.

Prior to the present work, all known cases of Conjecture 1.19 followed as corollaries of Bruner's power operation formulas and the authors' core motivation in undertaking this project was to provide a more substantiative test of this conjecture. The differentials of Theorem 1.9 cannot be obtained from Bruner's formulas. In fact, reversing the flow of information, the differentials of Theorem 1.9 can be interpreted as the first computation of an infinite family of *hidden* power operations in the Adams spectral sequence for the sphere.

1.1. An outline of the paper.

In Section 2 we review the Miller square, motivic homotopy theory and set up notation for many of the spectral sequences we will use throughout the paper. In Section 3 we reduce the proof of Theorem 1.9 to a collection of propositions which we will verify across the remaining sections of the paper. In Section 4 we study the E_2-page of the Cartan–Eilenberg spectral sequence near h_j^3. In Section 5 we show that there are no Cartan–Eilenberg differentials near h_j^3 and in particular it is at this point that we show that the target of the Adams differential in Theorem 1.9 is non-trivial. In Section 6

[6] In fact, the g_j are defined to be the classes in the Sq^0-family generated by $g_1 := g$.

we prove the key family of algebraic Novikov differentials at the heart of our proof. In Section 7 we use $\mathbb{F}_2$-synthetic homotopy theory to analyze the classes h_j^2, proving Theorem 1.10, Theorem 1.12 and the final input necessary for the proof of Theorem 1.9.

1.2. Notations and conventions.

(1) We write S^n for the n-sphere in spectra,
$S^{s,w}$ for the $\mathbb{C}$-motivic (s, w)-sphere $\Sigma^{s-w}(\mathbb{G}_m)^{\otimes w}$ and
$\mathbb{S}^{k,s}$ for the $\mathbb{F}_2$-synthetic (k, s)-sphere $\Sigma^{-s}\nu(S^{k+s})$.
(2) The indices we use in the various spectral sequences we consider are as follows: In the classical Adams E_2-page $\mathrm{Ext}_{\mathcal{A}}^{a,t}(\mathbb{F}_2, \mathbb{F}_2)$, we use a for the Adams filtration and t for the internal degree. In the motivic Adams E_2-page $\mathrm{Ext}_{\mathcal{A}^{\mathrm{mot}}}^{a,t,w}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])$, we use a for the Adams filtration and t for the internal degree and w for weight. Moreover, from the motivic Cartan–Eilenberg spectral sequence, every element in $\mathrm{Ext}_{\mathcal{A}^{\mathrm{mot}}}^{a,t,w}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])$ has a Cartan–Eilenberg filtration, denoted by k, and an Adams–Novikov filtration s, satisfying that $a = s + k$.
(3) In order to avoid notational collisions we write τ for the usual $\mathbb{C}$-motivic map τ and λ for the $\mathbb{F}_2$-synthetic map usually denoted τ.
(4) In the final pair of sections it becomes important to distinguish between several different classes connected with Kervaire invariant one. We use h_j^2 for the classes on the Adams E_2-page. We use ϑ_j for certain class on the Adams–Novikov E_2-page which map to h_j^2 under the Thom reduction map (see Definition 6.6). We use Θ_j for the Kervaire invariant one classes in the stable homotopy groups of spheres. We use θ_j for a choice of class in the synthetic homotopy groups of $\mathbb{S}/\lambda^k$ (which exists when h_j^2 survives to the E_{k+1}-page of the Adams spectral sequence).
(5) All objects are p-complete unless otherwise noted.
(6) Outside of Section 2 we work entirely at the prime 2.

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2. The Miller square and motivic homotopy theory

In this section we review necessary background for the proof of our main theorem—Theorem 1.9. We begin by recalling the Miller square, introduced in [Mil81], which captures the interplay between the Adams spectral sequence and the Adams–Novikov spectral sequence. We then discuss the recent cofiber of *tau* method developed by Gheorghe, Isaksen, Wang and the second author [GWX21, IWX20b, IWX20a] which uses the motivic stable homotopy category over $\mathbb{C}$ and the motivic Adams spectral sequence to categorify the Miller square.

2.1. The Miller square.

The Adams spectral sequence and the Adams–Novikov spectral sequence are two of the most effective methods of computing the homotopy groups of the p-completed sphere spectrum, S^0. They are spectral sequences of the form:

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \cong E_2^{s,t} \Longrightarrow \pi_{t-s}S^0, \qquad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$
$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}^{s,t}(\mathrm{BP}_*, \mathrm{BP}_*) \cong E_2^{s,t} \Longrightarrow \pi_{t-s}S^0, \qquad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

where $\mathcal{A}$ is the mod p dual Steenrod algebra and BP is the Brown–Peterson spectrum at the prime p. For degrees, s is the homological degree and is referred as the Adams filtration (resp. the Adams–Novikov filtration), and t is the internal degree.

It is important to understand connections between them. A first connection is given by the Thom reduction map $\mathrm{BP} \to \mathbb{F}_p$, which induces a map of spectral sequences

$$\mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*,\, \mathrm{BP}_*) \longrightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_p,\, \mathbb{F}_p)$$

that preserves the (s,t)-degrees. However, a general homotopy class in π_*S^0 doesn't usually have the same Adams filtration as the Adams–Novikov filtration. So this map is not very useful for comparison of the Adams filtration and the Adams–Novikov filtration of a surviving homotopy class—it only tells us the latter is less or equal to the former.

A fundamental connection is the Miller square. We have an algebraic Novikov spectral sequence converging to the Adams–Novikov E_2-page, and a Cartan–Eilenberg spectral sequence converging to the Adams E_2-page. It turns out the E_2-pages of these two algebraic spectral sequences are isomorphic.

The algebraic Novikov spectral sequence comes the filtration of powers of the augmentation ideal $I = (p, v_1, v_2, \cdots) \subset \mathrm{BP}_*$. It has the form:

$$\mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I,\, I^k/I^{k+1}) \cong E_2^{s,k,t'} \Longrightarrow \mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*,\, \mathrm{BP}_*)$$

$$d_r : E_r^{s,k,t'} \longrightarrow E_r^{s+1,k+r-1,t'}$$

where s and k are homological degrees, and t' is internal degree. In particular, in the Adams–Novikov gradings, all differentials in the algebraic Novikov spectral sequence look like Adams–Novikov d_1-differentials.

Let $\mathcal{P}$ be the sub-Hopf algebra of squares inside $\mathcal{A}$

$$\mathcal{P} \cong \mathbb{F}_2[\xi_1^2, \xi_2^2, \cdots] \subseteq \mathbb{F}_2[\xi_1, \xi_2, \cdots] \cong \mathcal{A}$$

and let $\mathcal{Q}$ be the quotient Hopf algebra, $\mathcal{Q} \cong \mathcal{A} \otimes_{\mathcal{P}} \mathbb{F}_2 \cong \Lambda_{\mathbb{F}_2}[\xi_1, \xi_2, \cdots]$. The Cartan-Eilenberg spectral sequence comes from the extension of Hopf algebras $\mathcal{P} \to \mathcal{A} \to \mathcal{Q}$. It has the form:

$$E_2^{s,k,t} = \mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_p, \mathrm{Ext}^k_{\mathcal{Q}}(\mathbb{F}_p, \mathbb{F}_p)) \Longrightarrow \mathrm{Ext}^{s+k,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p)$$

$$d_r : E_r^{s,k,t} \longrightarrow E_r^{s+r,k-r-1,t}$$

where s and k are homological degrees, and t is internal degree. In particular, in the Adams gradings, all differentials in the Cartan–Eilenberg spectral sequence look like Adams d_1-differentials. The Hopf algebra $\mathcal{Q}$ is cocommutative and primitively generated by the exterior classes ξ_{i+1}, therefore we have

$$\mathrm{Ext}^{*,*}_{\mathcal{Q}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[q_0, q_1, \cdots],$$

where q_i corresponds to $[\xi_{i+1}]$ and has (k,t) bidegree $(1, 2^{i+1} - 1)$.

We identify the E_2-pages of the Cartan–Eilenberg spectral sequence and the algebraic Novikov spectral sequence by using the isomorphism of Hopf algebroids

$$(\mathrm{BP}_*/I,\ \mathrm{BP}_*\mathrm{BP}/I) \cong (\mathbb{F}_p,\ \mathcal{P})$$

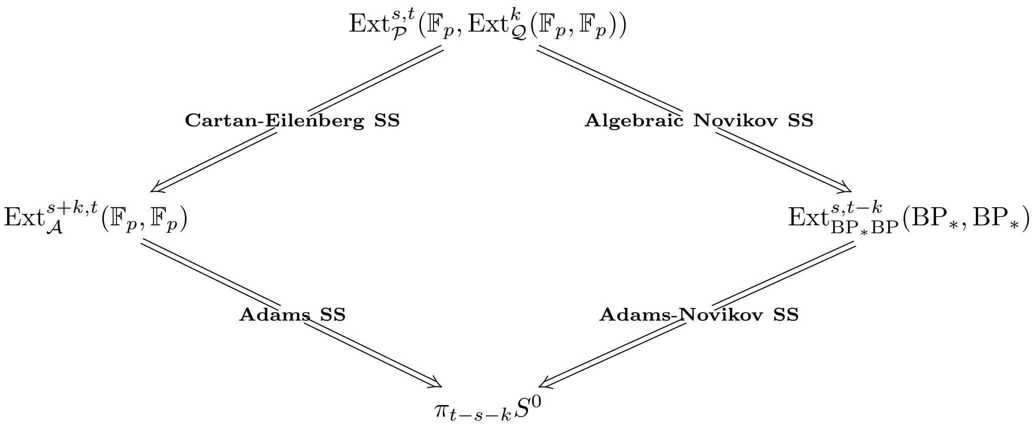
and the associated isomorphism of $\mathcal{P}$-comodule algebras

$$\mathrm{Ext}^*_{\mathcal{Q}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p[q_0, q_1, \cdots] \cong \mathbb{F}_p[v_0, v_1, \cdots] \cong \oplus_* I^*/I^{*+1}$$

where the middle isomorphisms identifies q_i with v_i. Taking into account that q_i has (s,k,t)-degree $(0, 1, 2^{i+1}-1)$ and v_i has (s,k,t')-degree $(0, 1, 2^{i+1}-2)$ the isomorphisms above provide an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_p, \mathrm{Ext}^k_{\mathcal{Q}}(\mathbb{F}_p, \mathbb{F}_p)) \xrightarrow{\quad\cong\quad} \mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^k/I^{k+1})$$

by sending s to s, k to k, and t to $t' + k$. Altogether, we have introduced the Miller square:



Remark 2.1. Miller's original motivation for studying his square was in order to deduce d_2-differentials in the Adams spectral sequence from d_2-differentials in the algebraic Novikov spectral sequence (see [Mil81]). Using motivic homotopy theory, one can generalize this method to obtain information about d_r-differentials for $r \geq 2$. We will explain this in the next subsection.

Remark 2.2. At odd primes, the dual Steenrod algebra admits an additional *Cartan grading* which places ξ_i in degree 0 and τ_i in degree 1. As the differentials in the Cartan–Eilenberg spectral sequence must respect the Cartan grading, this spectral sequence collapses at the E_2-page.

2.2. Motivic homotopy theory.

We work with the motivic stable homotopy category over $\mathbb{C}$. For the gradings, we denote by $S^{1,0}$ the simplicial sphere, and by $S^{1,1}$ the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 - 0$. We use the same notation for their suspension spectra. We denote by $\mathbb{F}_p^{\mathrm{mot}}$ the mod p motivic Eilenberg-Mac Lane spectrum that represents the mod p motivic cohomology, and by BPGL the motivic Brown-Peterson spectrum at the prime p. When the context is clear, we abuse notation and also write $S^{n,w}$ for the $\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum in bidegree (n, w).

There is a map

$$\tau : S^{0,-1} \longrightarrow S^{0,0}$$

that induces a nonzero map on mod p motivic homology. We denote by $S^{0,0}/\tau$ the cofiber of τ.

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow S^{0,0}/\tau \longrightarrow S^{1,-1}.$$

There is a Betti Realization functor $\mathbf{Re}$ from the motivic stable homotopy category over $\mathbb{C}$ to the classical stable homotopy category. We have

$$\mathbf{Re}(S^{n,w}) \simeq S^n,$$
$$\mathbf{Re}(\mathbb{F}_p^{\mathrm{mot}}) \simeq \mathbb{F}_p,$$
$$\mathbf{Re}(\mathrm{BPGL}) \simeq \mathrm{BP}.$$

The motivic dual Steenrod algebra over $\mathcal{C}$ is a Hopf algebra over $\pi_{*,*}(\mathbb{F}_p^{\mathrm{mot}}) = \mathbb{F}_p[\tau]$ which takes the following form (see [Voe03, Sec. 12])

$$\mathcal{A}^{\mathrm{mot}} \cong \mathbb{F}_2[\tau][\tau_1,\ \tau_2, \cdots][\xi_1^2,\ \xi_2^2, \cdots]/\tau_i^2 = \tau\xi_i^2,$$
$$\mathbf{Re}(\tau) = 1,\ \mathbf{Re}(\tau_i) = \xi_i,\ \mathbf{Re}(\xi_i^2) = \xi_i^2.$$

We then have the motivic Adams spectral sequence ([DI10]) of the form

$$\mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \cong E_2^{a,t,w} \Longrightarrow \pi_{t-a,w} S^{0,0}$$
$$d_r : E_r^{a,t,w} \to E_r^{a+r,t+r-1,w}$$

The quotient map $S^{0,0} \to S^{0,0}/\tau$ induces a map of the motivic Adams spectral sequences.

$$\mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \longrightarrow \mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p).$$

The following theorem is crucial in the computation of classical and motivic stable homotopy groups of spheres over $\mathbb{C}$.

Theorem 2.3 ([GWX21, Theorem 1.17])**.** *There is an isomorphism of tri-graded spectral sequences between the motivic Adams spectral sequence for $S^{0,0}/\tau$ and the algebraic Novikov spectral sequence.*

$$\begin{array}{ccc}
\mathrm{Ext}^{s+k,t'-k,\frac{t'}{2}}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p) & \xrightarrow{\cong} & \mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathbb{F}_p, I^k/I^{k+1}) \\
\Big\Downarrow \textbf{Motivic Adams SS} & & \Big\Downarrow \textbf{Algebraic Novikov SS} \\
\pi_{t'-s,\frac{t'}{2}}(S^{0,0}/\tau) & \xrightarrow{\cong} & \mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).
\end{array}$$

Remark 2.4. Note that the Ext-groups in the left column in Theorem 2.3 are only defined when t' is odd, meanwhile in the right column the Ext-groups vanish for t' odd for sparsity reasons.

As explained in [GWX21, Subsection 1.3], the motivic deformation and the naturality of the Adams spectral sequences give us a zig-zag diagram.

$$\begin{array}{ccccc}
\mathrm{Ext}^{a,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) & \xleftarrow{\mathbf{Re}} & \mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) & \longrightarrow & \mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \\
\Big\Downarrow \textbf{Adams SS} & & \Big\Downarrow \textbf{Motivic Adams SS} & & \Big\Downarrow \textbf{Motivic Adams SS} \\
\pi_{t-a} S^0 & \xleftarrow{\mathbf{Re}} & \pi_{t-a,w} S^{0,0} & \longrightarrow & \pi_{t-a,w} S^{0,0}/\tau
\end{array}$$

The right-hand horizontal maps are induced by the quotient map $S^{0,0} \to S^{0,0}/\tau$, and the left-hand horizontal maps are given by the Betti realization functor. The diagram of spectral sequences allow us to build up connections between the differentials in the classical Adams spectral sequence and the algebraic Novikov spectral sequence (by Theorem 2.3) through the motivic world.

Remark 2.5. It is often useful to combine the isomorphisms in the Miller square and in Theorem 2.3 and obtain the following isomorphism between the E_2-pages of the Cartan-Eilenberg spectral sequence and the motivic Adams spectral sequence for $S^{0,0}/\tau$:

$$\mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_p, \mathrm{Ext}^k_{\mathcal{Q}}(\mathbb{F}_p, \mathbb{F}_p)) \xrightarrow{\cong} \mathrm{Ext}^{s+k,t,\frac{t-k}{2}}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p).$$

We also have the motivic Adams-Novikov spectral sequence ([HKO11, Isa19]) of the form.

$$\mathrm{Ext}^{s,t',w}_{\mathrm{BPGL}_{*,*}\mathrm{BPGL}}(\mathrm{BPGL}_{*,*}, \mathrm{BPGL}_{*,*}) \cong E_2^{s,t',w} \Longrightarrow \pi_{t'-s,w} S^{0,0}$$
$$d_r : E_r^{s,t',w} \to E_r^{s+r,t'+r-1,w}$$

In [Isa19, Sections 6.1, 6.2], Isaksen proves the following rigidity theorem.

Theorem 2.6. *After a re-grading, the motivic Adams-Novikov spectral sequence for $\pi_{*,*}S^{0,0}$ is isomorphic to a τ-Bockstein spectral sequence.*

Finally, for some of our later arguments we need a motivic version of the Cartan–Eilenberg spectral sequence. Notably, this spectral sequence satisfies a rigidity theorem analogous to Theorem 2.6.

Construction 2.7. We define

$$\begin{aligned}\mathcal{P}^{\mathrm{mot}} &:= \mathbb{F}_2[\tau][\xi_1^2,\ \xi_2^2, \cdots] \cong \mathcal{P}[\tau],\\ \mathcal{Q}^{\mathrm{mot}} &:= \mathcal{A}^{\mathrm{mot}} \otimes_{\mathcal{P}^{\mathrm{mot}}} \mathbb{F}_2[\tau] \cong \mathbb{F}_2[\tau][\tau_1,\ \tau_2, \cdots]\end{aligned}$$

The associated extension of Hopf algebroids $\mathcal{P}^{\mathrm{mot}} \to \mathcal{A}^{\mathrm{mot}} \to \mathcal{Q}^{\mathrm{mot}}$ gives us a motivic version of the classical Cartan-Eilenberg spectral sequence. It has the form

$$\mathrm{Ext}^{s,t,w}_{\mathcal{P}^{\mathrm{mot}}}(\mathbb{F}_2[\tau],\ \mathrm{Ext}^{k}_{\mathcal{Q}^{\mathrm{mot}}}(\mathbb{F}_2[\tau],\ \mathbb{F}_2[\tau])) \cong E_2^{s,k,t,w} \Rightarrow \mathrm{Ext}^{s+k,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_2[\tau],\ \mathbb{F}_2[\tau]),$$

$$d_r : E_r^{s,k,t,w} \to E_r^{s+r,k-r+1,t,w}.$$

Note that in this spectral sequence all d_r-differentials look like motivic Adams d_1-differential in the tri-gradings.

Theorem 2.8. *After a re-grading, the motivic Cartan-Eilenberg spectral sequence for $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau])$ is isomorphic to a τ-Bockstein spectral sequence.*

Proof. We have

$$\begin{aligned}\mathrm{Ext}^{s,t,w}_{\mathcal{P}^{\mathrm{mot}}}(\mathbb{F}_2[\tau], \mathrm{Ext}^{k}_{\mathcal{Q}^{\mathrm{mot}}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])) &\cong \mathrm{Ext}^{s,t,w}_{\mathcal{P}^{\mathrm{mot}}}(\mathbb{F}_2[\tau], \mathrm{Ext}^{k}_{\mathcal{Q}}(\mathbb{F}_2, \mathbb{F}_2))[\tau])\\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_2, \mathrm{Ext}^{k}_{\mathcal{Q}}(\mathbb{F}_2, \mathbb{F}_2))[\tau].\end{aligned}$$

Here τ has (s,k,t,w)-degrees $(0,0,0,-1)$, and every element in $\mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_2, \mathrm{Ext}^{k}_{\mathcal{Q}}(\mathbb{F}_2, \mathbb{F}_2))$ satisfies $w = \frac{t-k}{2}$ by Remark 2.5. Note that by sparseness this group is nonzero only if $t-k$ is even.

Consider $S^{0,0}/\tau$, we have the E_2-page of its MCESS isomorphic to $\mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_2, \mathrm{Ext}^{k}_{\mathcal{Q}}(\mathbb{F}_2, \mathbb{F}_2))$. Since all differentials in MCESS preserve the w and t-degrees, it must preserve the k-degree in the case for $S^{0,0}/\tau$, so its MCESS collapses at the E_2-page.

Back to $S^{0,0}$, again due to degree reasons, all d_r-differentials have the form

$$d_{2n+1}x = \tau^n y, \quad x, y \in \mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_2, \mathrm{Ext}^{k}_{\mathcal{Q}}(\mathbb{F}_2, \mathbb{F}_2)),$$

and $d_{2n} = 0$. One can check that there are no τ-extensions in MCESS. This is precisely saying that after a regrading, the MCESS for $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])$ is isomorphic to a τ-Bockstein spectral sequence and completes the proof. □

Remark 2.9. The important consequence of Theorem 2.6 is that, the differentials in the classical Adams-Novikov spectral sequence completely determine the differentials in the motivic Adams-Novikov spectral sequence, and vice versa. Similarly, our Theorem 2.8 tells us that the differentials in the classical and motivic Cartan-Eilenberg spectral sequences determine each other.

Corollary 2.10. *Suppose there are no non-trivial Cartan–Eilenberg differentials entering tridegrees $(s,k,t) = (s,*,t)$. Let $\{\overline{x}_i\}_{i\in I}$ be a collection of generators of the permanent cycles on the Cartan–Eilenberg E_2-page in tridegree $(s,k,t) = (s,*,t)$. Then, we can lift each $\overline{x}_i$ to a class x_i on the E_2-page of the motivic Adams spectral sequence for $S^{0,0}$ and any choice of lifts $\{x_i\}_{i\in I}$ provides a basis for $E_2^{s,*,t}$ as a free $\mathbb{F}_2[\tau]$-module.*

3. Proof of the main theorem

In this section, we give the proof of our main theorem, Theorem 1.9, modulo three inputs from the later sections of the paper. Our core strategy is to use the motivic zig-zag to gain information about the classical Adams d_4 differential on h_j^3 from the associated algebraic Novikov d_4 differential.

Recollection 3.1. As in the classical Steenrod algebra, $\xi_1^{2^j}$ is a primitive element in the motivic dual Steenrod algebra. We let h_j denote the associated class on the E_2-page of the motivic Adams spectral sequence. h_j lives in (a, s, t, w)-degree $(1,\ 1,\ 2^j,\ 2^{j-1})$. Similarly, we also denote the induced classes on the E_2-pages of the classical Adams sseq and motivic Adams sseq for $S^{0,0}/\tau$ by h_j.

In fact, the main result of this section is a computation of $d_4(h_j^3)$ in the motivic Adams spectral sequence (with Theorem 1.9 being obtained from this by Betti realization). Before proceeding let us summarize what we need from later sections.

- In Theorem 3.2 we describe the E_2-pages of all three spectral sequences in the motivic zig-zag (classical Adams, motivic Adams, motivic Adams for $S^{0,0}/\tau$) in a neighborhood of the classes h_j^3. This theorem is proved in Sections 4 and 5 and summarized in Figure 1.
- In Theorem 3.4 we compute the differentials on h_j^3 in the motivic Adams sseq for $S^{0,0}/\tau$. This theorem is proved in Section 6.
- In Theorem 3.5 we give partial information on the classical Adams differentials on h_j^3. This theorem is proved in Section 7 using $\mathbb{F}_2$-synthetic homotopy theory.

Theorem 3.2. *Let $j \geq 6$.*

(1) The E_2-page of the classical Adams spectral sequence for S^0,

$$\mathrm{Ext}_{\mathcal{A}}^{a,t}(\mathbb{F}_2, \mathbb{F}_2) \cong E_2^{a,t} \Rightarrow \pi_{t-a} S^0,$$

takes the following form near h_j^3:

$(a,\quad t-a)$	$\mathrm{Ext}_{\mathcal{A}}^{a,\,t}(\mathbb{F}_2,\,\mathbb{F}_2)$
$(4,\quad 3\cdot 2^j - 4)$	$\mathbb{F}_2\{g_{j-2}\}$
$(5,\quad 3\cdot 2^j - 4)$	$\mathbb{F}_2\{h_0 g_{j-2}\}$
$(6,\quad 3\cdot 2^j - 4)$	*contains* $\mathbb{F}_2\{h_0^2 g_{j-2}\}$
$(7,\quad 3\cdot 2^j - 4)$	*contains* $\mathbb{F}_2\{h_0^3 g_{j-2}\}$
$(3,\quad 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_j^3\}$
$(4,\quad 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_0 h_j^3\}$
$(5,\quad 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_0^2 h_j^3, h_1 g_{j-2}\}$

In the case $j = 6$ degree $(a, t-s) = (5, 3\cdot 2^6 - 4)$ contains the additional class $h_7 D_3(0)$. In the case $j = 7$ degree $(a, t-s) = (5, 3\cdot 2^7 - 3)$ contains the additional class $h_8 D_3(1)$.

(2) The E_2-page of the motivic Adams spectral sequence for $S^{0,0}$,

$$\mathrm{Ext}_{\mathcal{A}^{\mathrm{mot}}}^{a,t,w}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau]) \cong E_2^{a,t,w} \Rightarrow \pi_{t-a,w} S^{0,0},$$

takes the following form near h_j^3:

$(a,\quad t-a)$	$\mathrm{Ext}^{a,\,t,\,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_2[\tau],\,\mathbb{F}_2[\tau])$
$(4,\quad 3\cdot 2^j-4)$	$\mathbb{F}_2[\tau]\{g_{j-2}\}$
$(5,\quad 3\cdot 2^j-4)$	$\mathbb{F}_2[\tau]\{h_0g_{j-2}\}$
$(6,\quad 3\cdot 2^j-4)$	*contains* $\mathbb{F}_2[\tau]\{h_0^2g_{j-2}\}$, τ*-torsion free*
$(7,\quad 3\cdot 2^j-4)$	*contains* $\mathbb{F}_2[\tau]\{h_0^3g_{j-2}\}$, τ*-torsion free*
$(3,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2[\tau]\{h_j^3\}$
$(4,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2[\tau]\{h_0h_j^3\}$
$(5,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2[\tau]\{h_0^2h_j^3, h_1g_{j-2}\}$

where each of the generators $g_{j-2}, h_0g_{j-2}, h_0^2g_{j-2}, h_0^3g_{j-2}, h_j^3, h_0h_j^3, h_0^2h_j^3$ *has weight* $w=3\cdot 2^{j-1}$ *and* h_1g_{j-2} *has weight* $w=3\cdot 2^{j-1}+1$*. In the case* $j=6$ *degree* $(a,t-a)=(5,3\cdot 2^6-4)$ *contains an additional* $\mathbb{F}_2[\tau]$ *summand with generator* $h_7D_3(0)$*. In the case* $j=7$ *degree* $(a,t-a)=(5,3\cdot 2^7-3)$ *contains an additional* $\mathbb{F}_2[\tau]$ *summand with generator* $h_8D_3(1)$ *of weight* $w=3\cdot 2^6+1$*.*

(3) *The* E_2*-page of the motivic Adams spectral sequence for* $S^{0,0}/\tau$*,*

$$\mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_2[\tau],\mathbb{F}_2)\cong E_2^{a,t,w}\Rightarrow \pi_{t-a,w}S^{0,0}/\tau,$$

takes the following form near h_j^3*:*

$(a,\quad 2w-t+a,\quad t-a)$	$\mathrm{Ext}^{a,\,t,\,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_2[\tau],\,\mathbb{F}_2)$
$(4,\quad 4,\quad 3\cdot 2^j-4)$	$\mathbb{F}_2\{g_{j-2}\}$
$(5,\quad 4,\quad 3\cdot 2^j-4)$	$\mathbb{F}_2\{h_0g_{j-2}\}$
$(6,\quad 4,\quad 3\cdot 2^j-4)$	*contains* $\mathbb{F}_2\{h_0^2g_{j-2}\}$
$(7,\quad 4,\quad 3\cdot 2^j-4)$	*contains* $\mathbb{F}_2\{h_0^3g_{j-2}\}$
$(3,\quad 1,\quad 3\cdot 2^j-3)$	0
$(4,\quad 1,\quad 3\cdot 2^j-3)$	0
$(5,\quad 1,\quad 3\cdot 2^j-3)$	0
$(6,\quad 1,\quad 3\cdot 2^j-3)$	0
$(3,\quad 3,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_j^3\}$
$(4,\quad 3,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_0h_j^3\}$
$(5,\quad 3,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_0^2h_j^3\}$
$(6,\quad 3,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_0^3h_j^3\}$ *or* 0
$(5,\quad 5,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_1g_{j-2}\}$

where the generators have the same weights as in (2). In the case $j=6$ *degree* $(a,2w-t+a,t-a)=(5,4,3\cdot 2^6-4)$ *possibly contains an additional class* $h_7D_3(0)$*. In the case* $j=7$ *degree* $(a,2w-t+a,t-a)=(5,5,3\cdot 2^7-3)$ *contains an additional class* $h_8D_3(1)$*.*

(4) *Under the Betti realization and reduction mod* τ *maps each generator maps to the generator of the same name.*

As an amplification of the first four lines of the table in Theorem 3.2(2) we have the following corollary.

Corollary 3.3. *The Betti realization map from the* E_2*-page of the motivic Adams sseq for* $S^{0,0}$ *to the* E_2*-page of the Adams sseq for* S^0 *is injective in the tridegrees of* h_0g_{j-2}*,* $h_0^2g_{j-2}$*,* $h_0^3g_{j-2}$ *and their* τ*-multiples.*

Theorem 3.4. *Let* $j\geq 6$*. In the motivic Adams sseq for* $S^{0,0}/\tau$*,*

(1) $d_2(h_j^3)=0$*,*

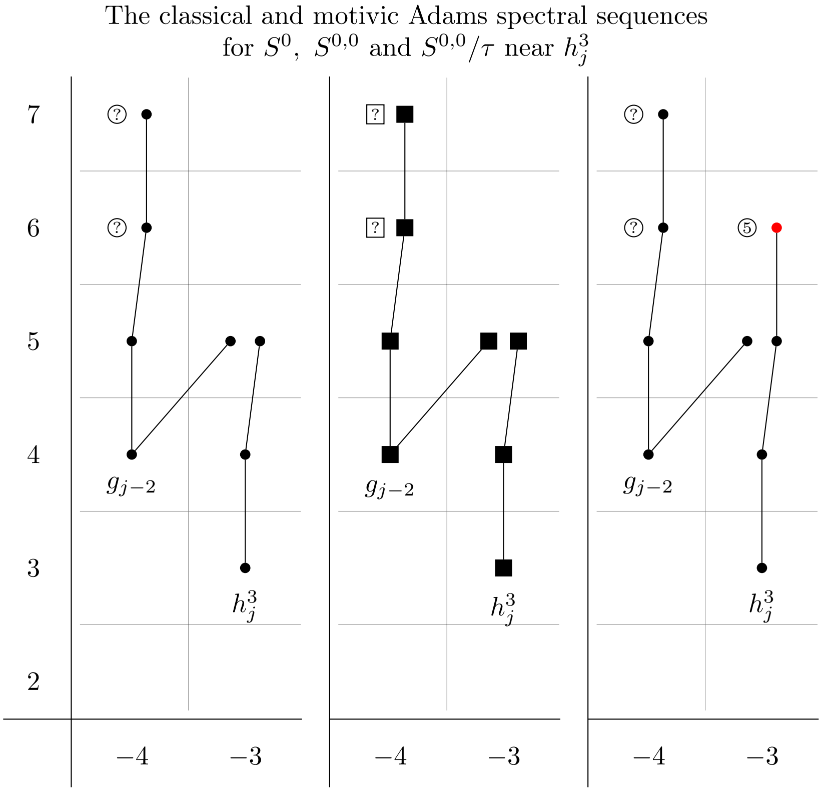


FIGURE 1. The horizontal degree is the topological stem $t-a$, shifted by $3\cdot 2^j$. The vertical degree is the Adams filtration a. Left and Right: Each dot • denotes a copy of $\mathbb{F}_2$. Circles ∘ denote possible copies of $\mathbb{F}_2$ and numbers inside the circle indicate the s-degree. The red dot indicates a class that is possibly zero. Middle: Each solid square ■ denotes a copy of $\mathbb{F}_2[\tau]$. Hollowed squares □ denote possible copies of $\mathbb{F}_2[\tau]$.

(2) $d_3(h_j^3) = 0$ and
(3) $d_4(h_j^3) = h_0^3 g_{j-2}$.

Theorem 3.5. *Let $j \geq 6$. In the classical Adams sseq for S^0,*

(1) $d_2(h_j^3) = 0$,
(2) $d_3(h_j^3)$ is either 0 or $h_0^2 g_{j-2}$ and
(3) $d_4(h_j^3)$ is either 0 or $h_0^3 g_{j-2}$ (if defined).

With all our inputs ready we now begin proving our main theorem.

Lemma 3.6. *Let $j \geq 6$. In the motivic Adams sseq for $S^{0,0}/\tau$, there are no non-zero d_2 or d_3-differentials entering the tridegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$ or $h_0^3 g_{j-2}$.*

Proof. Using Theorem 3.2(3) we can read off that the only potential sources for a d_2 or d_3-differential entering the tridegree of one of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$ or $h_0^3 g_{j-2}$ are h_j^3, $h_0 h_j^3$ and $h_0^2 h_j^3$. The lemma now follows as a corollary of Theorem 3.4(1,2). □

Lemma 3.7. *Let $j \geq 6$.*

(1) In the motivic Adams sseq for $S^{0,0}$, $d_2(h_j^3) = 0$.
(2) For $j \neq 7$, in the Adams sseq for S^0, there are no non-zero d_2-differentials entering the bidegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$ and $h_0^3 g_{j-2}$.

(3) *For $j \neq 7$, in the motivic Adams sseq for $S^{0,0}$, there are no non-zero d_2-differentials entering the tridegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$, $h_0^3 g_{j-2}$ or their τ-multiples.*
(4) *The Betti realization map from the E_3-page of the motivic Adams sseq for $S^{0,0}$ to the E_3-page of the Adams sseq for S^0 is injective in the tridegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$, $h_0^3 g_{j-2}$ and their τ-multiples.*

Proof. Using Corollary 3.3 we can deduce (1) from Theorem 3.5(1). For $j \neq 7$ we can read off from Theorem 3.2(1) that the only potential sources for Adams differential entering the bidegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$ and $h_0^3 g_{j-2}$ are the classes h_j^3, $h_0 h_j^3$, $h_0^2 h_j^3$ and $h_1 g_{j-2}$. Thus, (2) follows from Theorem 3.5(1) and Example 1.20 (which tells us that $d_2(g_{j-2}) = 0$). The injectivity from Corollary 3.3 implies that any entering motivic d_2-differential would induce an entering classical d_2-differential, therefore (2) implies (3). For $j \neq 7$, (4) is obtained by combining (2), (3) and Corollary 3.3.

For the $j = 7$ case of (4) we observe that the additional generator $h_8 D_3(1)$ has degree $(a, t, w) = (5, 3 \cdot 2^7 + 2, 3 \cdot 2^6 + 1)$ and that in order for this differential to create τ-torsion on the motivic Adams E_3-page its target must be τ-divisible. This would force us to have a non-trivial class in degree $(a, t, w) = (7, 3 \cdot 2^7 + 3, 3 \cdot 2^6 + 2)$. On the other hand, such a class would contradict the fact that the motivic Adams E_2-page vanishes when $t < 2w$. □

Lemma 3.8. *Let $j \geq 6$.*

(1) *In the classical Adams sseq for S^0, $d_3(h_j^3) = 0$.*
(2) *In the motivic Adams sseq for $S^{0,0}$, $d_3(h_j^3) = 0$.*
(3) *In the classical Adams sseq for S^0, there are no non-zero d_3-differentials entering the bidegrees of $h_0^2 g_{j-2}$ and $h_0^3 g_{j-2}$.*
(4) *In the motivic Adams sseq for $S^{0,0}$, there are no non-zero d_3-differentials entering the tridegrees of $h_0^2 g_{j-2}$, $h_0^3 g_{j-2}$ or their τ-multiples.*
(5) *The Betti realization map from the E_4-page of the motivic Adams sseq for $S^{0,0}$ to the E_4-page of the Adams sseq for S^0 is injective in the tridegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$, $h_0^3 g_{j-2}$ and their τ-multiples.*

Proof. We begin with (2). The injectivity statement from Lemma 3.7(4) allows us to upgrade Theorem 3.5 to the claim that in the motivic Adams sseq for $S^{0,0}$, $d_3(h_j^3)$ is either 0 or $h_0^2 g_{j-2}$. Next we examine the reduction mod τ map to the motivic Adams sseq for $S^{0,0}/\tau$. From Theorem 3.2(3) and Lemma 3.6 we know that $h_0^2 g_{j-2}$ is non-zero on the E_3-page of the motivic Adams sseq for $S^{0,0}/\tau$. Thus, (2) follows from Theorem 3.4(2) which tells us that $d_3(h_j^3) = 0$ in the motivic Adams sseq for $S^{0,0}/\tau$. (1) follows from (2) by Betti realization.

From Theorem 3.2(1) we can read off that the only potential sources for Adams differential entering the bidegrees of $h_0 g_{j-2}$, $h_0^2 g_{j-2}$ and $h_0^3 g_{j-2}$ are the classes h_j^3 and $h_0 h_j^3$. Thus, (3) follows from (1). The injectivity from Lemma 3.7(4) implies that any entering motivic d_3-differential would induce an entering classical d_3-differential, therefore (3) implies (4). (5) is obtained by combining (3), (4) and Lemma 3.7(4). □

Theorem 3.9. *Let $j \geq 6$. In the motivic Adams sseq for $S^{0,0}$ we have*

$$d_4(h_j^3) = h_0^3 g_{j-2} \neq 0$$

which Betti realizes to the non-trivial Adams differential of Theorem 1.9

Proof. From Lemmas 3.7(1) and 3.8(2) we know that the class h_j^3 survives to the E_4-page of the motivic Adams sseq. Under the reduction mod τ map to E_4-page of the motivic Adams sseq for $S^{0,0}/\tau$ the class $d_4(h_j^3)$ is sent to a non-zero class by Theorem 3.4(3) and Lemma 3.6, therefore $d_4(h_j^3) \neq 0$. On the other hand, the injectivity statement from

Lemma 3.8(4) allows us to upgrade Theorem 3.5(3) to the claim that in the motivic Adams sseq for the sphere $d_4(h_j^3)$ is either 0 or $h_0^3 g_{j-2}$. It follows that

$$d_4(h_j^3) = h_0^3 g_{j-2} \neq 0.$$

Using the injectivity from Lemma 3.8(4) again we obtain the desired non-trivial classical Adams d_4-differential from the motivic one. □

Remark 3.10. One may prove directly that the element $h_0^3 g_{j-2}$ survives to the classical Adams E_4-page, but it is not logically necessary for the proof of Theorem 3.9—Theorem 3.9 in fact implies that $h_0^3 g_{j-2}$ does not support a nonzero d_2 or d_3-differential.

4. The Cartan–Eilenberg E_2-page

Our goal in the next pair of sections is to prove Theorem 3.2. In this section we prove a weak form of Theorem 3.2(3). In fact, as explained in Section 2 the motivic Adams sseq for $S^{0,0}/\tau$ is isomorphic to the Cartan–Eilenberg sseq, therefore the main object of this section is the Cartan–Eilenberg E_2-page. In order to gain access to this E_2-page we construct an algebraic Atiyah–Hirzebruch sseq, converging to the Cartan-Eilenberg E_2-page. Then, in Section 5 we study the differentials in the Cartan–Eilenberg sseq and use this information to complete the proof of Theorem 3.2.

In order to motivate our strategy, let us focus on a particular claim from Theorem 3.2: the class $h_0^3 g_{n+1}$ on the Adams 7-line is non-trivial. On the E_2-page of the Cartan–Eilenberg sseq this class is detected by $q_0^3 \cdot g_n$ and therefore it suffices for us to do two things (1) show that $q_0^3 \cdot g_n$ is non-trivial on the Cartan–Eilenberg E_2-page and (2) show that $q_0^3 \cdot g_n$ is not the target of a Cartan–Eilenberg differential. The class $q_0^3 \cdot q_n$ is detected by a non-trivial class of the same name on the algebraic Atiyah–Hirzebruch E_1-page and we prove (1) by showing that this class is not the target of an algebraic Atiyah–Hirzebruch differential. We depict this strategy in the diagram of spectral sequences below.

$$\begin{array}{ll} q_0^3 \cdot g_n \in & \mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathbb{F}_2[q_0, q_1, \cdots] \\ & \Big\Downarrow \text{algebraic Atiyah-Hirzebruch SS} \\ q_0^3 \cdot g_n \in & \mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2[q_0, q_1, \cdots]) \\ & \Big\Downarrow \text{Cartan-Eilenberg SS} \\ h_0^3 g_{n+1} \in & \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \end{array}$$

Notation 4.1. In order to give names to elements on the E_2-page we use the injective map

$$\mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2[q_0, q_1, \ldots])$$

together with the Frobenius isomorphism $\mathcal{P} \cong \mathcal{A}$ so that we can use the familiar names from $\mathrm{Ext}_{\mathcal{A}}$. Note that this use of the Frobenius in naming classes means that the class a on Cartan–Eilenberg E_2-page will detect the class $\mathrm{Sq}^0(a)$ in $\mathrm{Ext}_{\mathcal{A}}$. In particular, this means that the class h_j^3 on the motivic Adams E_2-page corresponds to the class h_{j-1}^3 on the Cartan–Eilenberg E_2-page.

Proposition 4.2. *Let $j \geq 5$. The E_2-page of the Cartan–Eilenberg spectral sequence takes the following form near h_j^3:*

$(s,\ k,\ t)$	$\mathrm{Ext}_{\mathcal{P}}^{s,t}(\mathbb{F}_2, \mathrm{Ext}_{\mathcal{Q}}^{k}(\mathbb{F}_2,\mathbb{F}_2))$
$(4,\ 1,\ 3\cdot 2^j+1)$	*contains* $\mathbb{F}_2\{q_0 g_{j-2}\}$
$(4,\ 2,\ 3\cdot 2^j+2)$	*contains* $\mathbb{F}_2\{q_0^2 g_{j-2}\}$
$(4,\ 3,\ 3\cdot 2^j+3)$	*contains* $\mathbb{F}_2\{q_0^3 g_{j-2}\}$
$(1,\ 2,\ 3\cdot 2^j)$	0
$(1,\ 3,\ 3\cdot 2^j+1)$	0
$(1,\ 4,\ 3\cdot 2^j+2)$	0
$(1,\ 5,\ 3\cdot 2^j+3)$	0
$(3,\ 1,\ 3\cdot 2^j+1)$	$\mathbb{F}_2\{q_0 h_j^3\}$ *or* 0
$(3,\ 2,\ 3\cdot 2^j+2)$	$\mathbb{F}_2\{q_0^2 h_j^3\}$ *or* 0
$(3,\ 3,\ 3\cdot 2^j+3)$	$\mathbb{F}_2\{q_0^3 h_j^3\}$ *or* 0

As discussed in Section 2 we have an isomorphism $\mathrm{Ext}_{\mathcal{Q}}(\mathbb{F}_2,\mathbb{F}_2)\cong\mathbb{F}_2[q_0,q_1,\dots]$ where the polynomial generator q_i has degree $(k,t)=(1,2^{i+1}-1)$. The $\mathcal{P}$-comodule structure on this polynomial algebra is given by

$$\psi:\mathbb{F}_2[q_0,q_1,\cdots]\to\mathcal{P}\otimes\mathbb{F}_2[q_0,q_1,\cdots] \tag{1}$$

$$\psi(q_n)=\sum_{i=0}^{n}\xi_{n-i}^{2^{i+1}}\otimes q_i,\ \text{ where } \xi_0=1.$$

(see [Rav86, Theorem 4.3.3]). For further discussion of the Cartan–Eilenberg sseq see [AM17, Mil81, Rav86].

4.1. An algebraic Atiyah-Hirzebruch spectral sequence.

In this subsection we construct an algebraic Atiyah–Hirzebruch spectral sequence which converges to the Cartan–Eilenberg E_2-page. The algAH sseq is the main tool we use in our proof of Proposition 4.2.

Construction 4.3. The $\mathcal{P}$-comodule $\mathbb{F}_2[q_0,\dots]$ can be described as the free polynomial algebra on the $\mathcal{P}$-comodule $\mathbb{F}_2\{q_i\}_{i\geq 0}$. We place an increasing filtration on $\mathbb{F}_2\{q_i\}_{i\geq 0}$ where q_i is in filtration i. Passing to polynomial algebras we obtain a filtered commutative algebra in $\mathcal{P}$-comodules whose underlying object is $\mathbb{F}_2[q_0,\dots]$. Associated to this filtration is a multiplicative spectral sequence computing $\mathrm{Ext}_{\mathcal{P}}(\mathbb{F}_2,\mathbb{F}_2[q_0,\dots])$ which we will call the algebraic Atiyah–Hirzebruch spectral sequence.

Lemma 4.4. *The associated graded of the filtration on $\mathbb{F}_2[q_0,q_1,\cdots]$ from Construction 4.3 is given by the graded $\mathcal{P}$-comodule algebra $\mathbb{F}_2[q_0,q_1,\cdots]$ with trivial $\mathcal{P}$-comodule structure where the monomial $q_{i_1}q_{i_2}\cdots q_{i_k}$ lives in filtration $i_1+i_2+\cdots+i_k$.*

Proof. The associated graded of a polynomial algebra on a filtered P-comodule M is the polynomial algebra on the associated graded of M. Therefore, it suffices for us to observe that since we used the cellular filtration on $\mathbb{F}_2\{q_i\}_{i\geq 0}$ the associated graded has trivial P-comodule structure. □

As a consequence of Lemma 4.4 the algAH sseq of Construction 4.3 has signature

$$\mathrm{Ext}_{\mathcal{P}}(\mathbb{F}_2,\mathbb{F}_2)\otimes\mathbb{F}_2[q_0,q_1,\cdots]\cong E_1\Longrightarrow\mathrm{Ext}_P(\mathbb{F}_2,\mathbb{F}_2[q_0,q_1,\cdots])$$
$$d_r: E_r^{s,k,t,i}\to E_r^{s+1,k,t,i-r}$$

where we give classes in $\mathrm{Ext}_{\mathcal{P}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$ degree $(s,k,t,i)=(s,0,t,0)$ and q_n has degree $(s,k,t,i)=(0,1,2^{n+1}-1,n)$.

Remark 4.5. The multiplicative structure on the algAH sseq coming from its construction via a filtered commutative algebra includes compatibility of the product structure on the E_1-page with products in $\mathrm{Ext}_P(\mathbb{F}_2, \mathbb{F}_2[q_0, q_1, \dots])$ and a Liebniz rule for differentials.

Remark 4.6. The E_1-page of the algAH sseq has a basis of elements of the form $q_{i_1} q_{i_2} \cdots q_{i_k} \cdot a$, where $a \in \mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. Furthermore, since the k-degree just records the number of q's, algAH differentials preserve the number of q's.

In general, the differentials in algAH sseq can be computed by embedding into the cobar complex that computes the E_2-page of Cartan–Eilenberg sseq. So in particular, the primary algAH differentials can be computed by using the $\mathcal{P}$-comodule structure map ψ.

Example 4.7. We have
$$\psi(q_1) = \xi_1^2 \otimes q_0 + 1 \otimes q_1.$$
Since ξ_1 detects h_0 in $\mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$, we have ξ_1^2 detects h_0 in $\mathrm{Ext}_{\mathcal{P}}^{1,2}(\mathbb{F}_2, \mathbb{F}_2)$ (in the naming convention of Notation 4.1). This gives us algAH differentials
$$d_1(q_1 \cdot a) = q_0 \cdot h_0 a,$$
for $a \in \mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Example 4.8. We have
$$\psi(q_2) = \xi_2^2 \otimes q_0 + \xi_1^4 \otimes q_1 + 1 \otimes q_0.$$
Examining the top AH filtration terms we obtain
$$d_1(q_2 \cdot a) = q_1 \cdot h_1 a.$$
If $h_1 a = 0$ in $\mathrm{Ext}_{P}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, then we have $d_1(q_2 \cdot a) = 0$. In this case, we have
$$d_2(q_2 \cdot a) = q_0 \cdot \langle h_0, h_1, a \rangle.$$
This is due to the fact that in cobar complex for $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, the nullhomotopy of $h_0 h_1$ is ξ_2. Note that the indeterminacy of the set $q_0 \cdot \langle h_0, h_1, a \rangle$ is killed by d_1-differentials.

Example 4.9.
$$\psi(q_1^2) = \psi(q_1)^2 = \xi_1^4 \otimes q_0^2 + 1 \otimes q_1^2,$$
therefore we have
$$d_2(q_1^2 \cdot a) = q_0^2 \cdot h_1 a.$$

Generalizing, these examples we have the following lemma.

Lemma 4.10. *We have the following differentials in* algAHSS *for* $n \geq 0$.

(1) $d_1(q_{n+1} \cdot a) = q_n \cdot h_n a.$
(2) $d_2(q_{n+1}^2 \cdot a) = q_n^2 \cdot h_{n+1} a.$
(3) $d_2(q_{n+2} \cdot a) = q_n \cdot \langle h_n, h_{n+1}, a \rangle$, *if* $h_{n+1} a = 0$.
(4) $d_4(q_{n+2}^2 \cdot a) = q_n^2 \cdot \langle h_{n+1}, h_{n+2}, a \rangle$, *if* $h_{n+2} a = 0$.
(5) $d_3(q_{n+3} \cdot a) = q_n \cdot \langle h_n, h_{n+1}, h_{n+2}, a \rangle$, *if* $h_{n+2} a = 0$ *and* $0 \in \langle h_{n+1}, h_{n+2}, a \rangle$.
(6) $d_6(q_{n+3}^2 \cdot a) = q_n^2 \cdot \langle h_{n+1}, h_{n+2}, h_{n+3}, a \rangle$, *if* $h_{n+3} a = 0$ *and* $0 \in \langle h_{n+2}, h_{n+3}, a \rangle$.

Proof. The differentials in (1) and (2) follow from the comodule structure map ψ modulo elements in lower AH filtration
$$\psi(q_{n+1}) \equiv \xi_1^{2^{n+1}} \otimes q_n + 1 \otimes q_{n+1},$$
$$\psi(q_{n+1}^2) = \psi(q_{n+1})^2 \equiv \xi_1^{2^{n+2}} \otimes q_n^2 + 1 \otimes q_{n+1}^2,$$
and the facts that $\xi_1^{2^{n+1}}$ and $\xi_1^{2^{n+2}}$ detects h_n and h_{n+1} in the cobar complex that computes $\mathrm{Ext}_{P}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

The differentials in (3) and (4) follow from the comodule structure map ψ modulo elements in lower AH filtration

$$\psi(q_{n+2}) \equiv \xi_2^{2^{n+1}} \otimes q_n + \xi_1^{2^{n+2}} \otimes q_{n+1} + 1 \otimes q_{n+2},$$
$$\psi(q_{n+2}^2) = \psi(q_{n+2})^2 \equiv \xi_2^{2^{n+2}} \otimes q_n^2 + \xi_1^{2^{n+3}} \otimes q_{n+1}^2 + 1 \otimes q_{n+2}^2,$$

and the facts that $\xi_2^{2^{n+1}}$ and $\xi_2^{2^{n+2}}$ detects the nullhomotopy of $h_n h_{n+1}$ and $h_{n+1}h_{n+2}$ in the cobar complex that computes $\mathrm{Ext}_{\mathcal{P}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Similarly, the elements $\xi_3^{2^{n+1}}$ and $\xi_3^{2^{n+2}}$ detects the null homotopy of the Massey products $\langle h_n, h_{n+1}, h_{n+2}\rangle$ and $\langle h_{n+1}, h_{n+2}, h_{n+3}\rangle$, and the differentials in (5) and (6) follow.

Note that indeterminacies of the expressions $q_n \cdot \langle h_n, h_{n+1}, a\rangle$, $q_n^2 \cdot \langle h_{n+1}, h_{n+2}, a\rangle$, $q_n \cdot \langle h_n, h_{n+1}, h_{n+2}, a\rangle$, and $q_n^2 \cdot \langle h_{n+1}, h_{n+2}, h_{n+3}, a\rangle$ are killed by shorter differentials. □

4.2. **The algebraic Atiyah–Hirzebruch E_1-page.**

In this subsection we compute the E_1-page of the algAH sseq in the degrees we will need for proving Proposition 4.2.

Recollection 4.11. Recall that we have the following generators in the cohomology of the Steenrod algebra

$$h_n \in \mathrm{Ext}_{\mathcal{A}}^{1,2^n}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{P}}^{1,2^{n+1}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2,$$
$$c_n \in \mathrm{Ext}_{\mathcal{A}}^{3,11\cdot 2^n}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{P}}^{3,11\cdot 2^{n+1}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2,$$
$$g_n \in \mathrm{Ext}_{\mathcal{A}}^{4,3\cdot 2^{n+2}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{P}}^{4,3\cdot 2^{n+3}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\,.$$

Lemma 4.12. *In degree $(s,k,t) = (3,3,3\cdot 2^{n+3}+3)$ the E_1-page of the algAH sseq consists of the following classes for $n \geq 4$,*

(1) $q_0q_1q_n \cdot c_n$,
(2) $q_0^3 \cdot h_{n+1}^2 h_{n+3} = q_0^3 \cdot h_{n+2}^3$,
(3) $q_0^2 q_{n+1} \cdot h_0 h_{n+1} h_{n+3}$,
(4) $q_0^2 q_{n+3} \cdot h_0 h_{n+1}^2$,
(5) $q_0 q_{n+1}^2 \cdot h_0^2 h_{n+3}$,
(6) $q_0 q_{n+1} q_{n+3} \cdot h_0^2 h_{n+1}$,
(7) $q_{n+1}^2 q_{n+3} \cdot h_0^3$,
(8) $q_0^2 q_{n+2} \cdot h_0 h_{n+2}^2$,
(9) $q_0 q_{n+2}^2 \cdot h_0^2 h_{n+2}$,
(10) $q_{n+2}^3 \cdot h_0^3$,
(11) $q_0q_1q_{n+1} \cdot h_n^2 h_{n+3}$,
(12) $q_0 q_n^2 \cdot h_1 h_{n+1} h_{n+3}$,
(13) $q_0 q_n q_{n+1} \cdot h_1 h_n h_{n+3}$,
(14) $q_0 q_{n+1} q_{n+3} \cdot h_1 h_n^2$,
(15) $q_1 q_n^2 \cdot h_0 h_{n+1} h_{n+3}$,
(16) $q_1 q_n q_{n+1} \cdot h_0 h_n h_{n+3}$,
(17) $q_1 q_{n+1} q_{n+3} \cdot h_0 h_n^2$,
(18) $q_0 q_{n+1}^2 \cdot h_1 h_{n+2}^2$,
(19) $q_0 q_{n+2}^2 \cdot h_1 h_{n+1}^2$,
(20) $q_1 q_{n+1}^2 \cdot h_0 h_{n+2}^2$,
(21) $q_1 q_{n+2}^2 \cdot h_0 h_{n+1}^2$.

Proof. As discussed in subsection 4.1, the E_1-page of the algAH sseq in degree $(s,k,t) = (3,3,3\cdot 2^{n+3}+3)$ has a basis of elements of the form $q_iq_jq_k \cdot a$, where

$$a \in \mathrm{Ext}_{\mathcal{P}}^{3,2*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}}^{3,*}(\mathbb{F}_2, \mathbb{F}_2).$$

The classical Adams 3-line $\mathrm{Ext}_{\mathcal{A}}^{3,*}$ is generated by the elements c_ℓ and $h_ah_bh_c$. So the candidates are of the following two forms

- $q_iq_jq_k \cdot c_\ell$,
- $q_iq_jq_k \cdot h_ah_bh_c$.

We may also assume that $i \leq j \leq k$ and $a \leq b \leq c$.

We start by considering classes of the form $q_iq_jq_k \cdot c_\ell$. The class $q_iq_jq_k \cdot c_\ell$ has t-degree

$$(2^{i+1}-1) + (2^{j+1}-1) + (2^{k+1}-1) + 11 \cdot 2^{\ell+1}$$

from this we obtain the equation

$$3 + 3 \cdot 2^{n+3} = (2^{i+1} - 1) + (2^{j+1} - 1) + (2^{k+1} - 1) + 11 \cdot 2^{\ell+1},$$

which can be simplified to the equation

$$1 + 2 + 2^{n+2} + 2^{n+3} = 2^i + 2^j + 2^k + 2^l + 2^{l+1} + 2^{l+3}. \qquad (*)$$

Since we are considering $n \geq 4$, the right hand side of $(*)$ is 3 mod 64, and is at least 195.

When $\ell = 0$, the equation $(*)$ becomes

$$2^{n+2} + 2^{n+3} = 2^i + 2^j + 2^k + 8,$$

Not possible for $n \geq 4$.

When $\ell = 1$, we must have $i = 0$, then equation $(*)$ becomes

$$2^{n+2} + 2^{n+3} = 2^j + 2^k + 4 + 16.$$

Not possible for $n \geq 4$.

When $\ell = 2$, we must have $i = 0, j = 1$, then equation $(*)$ becomes

$$2^{n+2} + 2^{n+3} = 2^k + 4 + 8 + 32.$$

Not possible for $n \geq 4$.

When $\ell \geq 3$, we must have $i = 0, j = 1$, then equation $(*)$ becomes

$$2^{n+2} + 2^{n+3} = 2^k + 2^\ell + 2^{\ell+1} + 2^{\ell+3}.$$

We must have $k = \ell = n$, which is at least 4. This is case $(1) : q_0 q_1 q_n \cdot c_n$.

Next we consider classes of the form $q_i q_j q_k \cdot h_a h_b h_c$. The class $q_i q_j q_k \cdot h_a h_b h_c$ has t-degree

$$(2^{i+1} - 1) + (2^{j+1} - 1) + (2^{k+1} - 1) + 2^{a+1} + 2^{b+1} + 2^{c+1}.$$

from this we obtain the equation

$$3 + 3 \cdot 2^{n+3} = (2^{i+1} - 1) + (2^{j+1} - 1) + (2^{k+1} - 1) + 2^{a+1} + 2^{b+1} + 2^{c+1},$$

which can be simplified to the equation

$$1 + 2 + 2^{n+2} + 2^{n+3} = 2^i + 2^j + 2^k + 2^a + 2^b + 2^c. \qquad (**)$$

Either two or three terms on the right hand side of $(**)$ contribute to $1+2$. So we only have the following 4 possibilities for the unordered set $\{2^i,\ 2^j,\ 2^k,\ 2^a,\ 2^b,\ 2^c\}$.

- $\{1,\ 1,\ 1,\ 2^{n+1},\ 2^{n+1},\ 2^{n+3}\}$,
- $\{1,\ 1,\ 1,\ 2^{n+2},\ 2^{n+2},\ 2^{n+2}\}$,
- $\{1,\ 2,\ 2^n,\ 2^n,\ 2^{n+1},\ 2^{n+3}\}$,
- $\{1,\ 2,\ 2^{n+1},\ 2^{n+1},\ 2^{n+2},\ 2^{n+2}\}$.

For the set $\{1,\ 1,\ 1,\ 2^{n+1},\ 2^{n+1},\ 2^{n+3}\}$, we have candidates

$$q_0^3 \cdot h_{n+1}^2 h_{n+3},\ q_0^2 q_{n+1} \cdot h_0 h_{n+1} h_{n+3},\ q_0^2 q_{n+3} \cdot h_0 h_{n+1}^2,$$

$$q_0 q_{n+1}^2 \cdot h_0^2 h_{n+3},\ q_0 q_{n+1} q_{n+3} \cdot h_0^2 h_{n+1},\ q_{n+1}^2 q_{n+3} \cdot h_0^3.$$

These are the cases $(2) - (7)$.

For the set $\{1,\ 1,\ 1,\ 2^{n+2},\ 2^{n+2},\ 2^{n+2}\}$, we have candidates

$$q_0^3 \cdot h_{n+2}^3,\ q_0^2 q_{n+2} \cdot h_0 h_{n+2}^2,\ q_0 q_{n+2}^2 \cdot h_0^2 h_{n+2},\ q_{n+2}^3 \cdot h_0^3.$$

Note that we have a relation $h_{n+2}^3 = h_{n+1}^2 h_{n+3}$ in $\mathrm{Ext}_A^{3,*}$ and hence in $\mathrm{Ext}_P^{3,2*}$. These are the cases (2) and $(8) - (10)$.

For the set $\{1,\ 2,\ 2^n,\ 2^n,\ 2^{n+1},\ 2^{n+3}\}$, we have candidates

$$q_0 q_1 q_{n+1} \cdot h_n^2 h_{n+3},\ q_0 q_n^2 \cdot h_1 h_{n+1} h_{n+3},\ q_0 q_n q_{n+1} \cdot h_1 h_n h_{n+3},\ q_0 q_{n+1} q_{n+3} \cdot h_1 h_n^2,$$

$$q_1 q_n^2 \cdot h_0 h_{n+1} h_{n+3},\ q_1 q_n q_{n+1} \cdot h_0 h_n h_{n+3},\ q_1 q_{n+1} q_{n+3} \cdot h_0 h_n^2.$$

Note that we have a relation $h_n h_{n+1} = 0$ in $\mathrm{Ext}_A^{2,*}$ and hence in $\mathrm{Ext}_P^{2,2*}$. So certain candidates are already zero (e.g. $q_0q_1q_n \cdot h_nh_{n+1}h_{n+3}$). These are the cases (11) – (17).

For the set $\{1,\ 2,\ 2^{n+1},\ 2^{n+1},\ 2^{n+2},\ 2^{n+2}\}$, we have candidates

$$q_0q_{n+1}^2 \cdot h_1h_{n+2}^2,\ q_0q_{n+2}^2 \cdot h_1h_{n+1}^2,\ q_1q_{n+1}^2 \cdot h_0h_{n+2}^2,\ q_1q_{n+2}^2 \cdot h_0h_{n+1}^2.$$

Again, due the relation $h_nh_{n+1} = 0$, certain candidates are already zero so we don't list them. These are the cases (18) – (21).

This completes the discussion of elements of the form $q_iq_jq_k \cdot h_ah_bh_c$ and therefore the proof of this lemma. □

Lemma 4.13. *In degree $(s, k, t) = (1, 5, 3\cdot 2^{n+3}+3)$ the E_1-page of the algAHSS consists of the following classes for $n \geq 4$,*

(1) $q_0^3q_{n+2}q_{n+3} \cdot h_0$,
(2) $q_0^4q_{n+3} \cdot h_{n+2}$,
(3) $q_0^4q_{n+2} \cdot h_{n+3}$,
(4) $q_0q_1q_{n+1}^2q_{n+3} \cdot h_0$,
(5) $q_0^2q_{n+1}^2q_{n+3} \cdot h_1$,
(6) $q_0^2q_1q_{n+1}q_{n+3} \cdot h_{n+1}$,
(7) $q_0^2q_1q_{n+1}^2 \cdot h_{n+3}$,
(8) $q_0q_1q_{n+2}^3 \cdot h_0$,
(9) $q_0^2q_{n+2}^3 \cdot h_1$,
(10) $q_0^2q_1q_{n+2}^2 \cdot h_{n+2}$,
(11) $q_1q_n^2q_{n+1}q_{n+3} \cdot h_1$,
(12) $q_1^2q_nq_{n+1}q_{n+3} \cdot h_n$,
(13) $q_1^2q_n^2q_{n+3} \cdot h_{n+1}$,
(14) $q_1^2q_n^2q_{n+1} \cdot h_{n+3}$,
(15) $q_1q_{n+1}^2q_{n+2}^2 \cdot h_1$,
(16) $q_1^2q_{n+1}q_{n+2}^2 \cdot h_{n+1}$,
(17) $q_1^2q_{n+1}^2q_{n+2} \cdot h_{n+2}$,
(18) $q_{n-1}^2q_nq_{n+1}q_{n+3} \cdot h_2$,
(19) $q_2q_{n-1}q_nq_{n+1}q_{n+3} \cdot h_{n-1}$,
(20) $q_2q_{n-1}^2q_{n+1}q_{n+3} \cdot h_n$,
(21) $q_2q_{n-1}^2q_nq_{n+3} \cdot h_{n+1}$,
(22) $q_2q_{n-1}^2q_nq_{n+1} \cdot h_{n+3}$,
(23) $q_n^4q_{n+3} \cdot h_2$,
(24) $q_2q_n^3q_{n+3} \cdot h_n$,
(25) $q_2q_n^4 \cdot h_{n+3}$,
(26) $q_n^2q_{n+1}q_{n+2}^2 \cdot h_2$,
(27) $q_2q_nq_{n+1}q_{n+2}^2 \cdot h_n$,
(28) $q_2q_n^2q_{n+2}^2 \cdot h_{n+1}$,
(29) $q_2q_n^2q_{n+1}q_{n+2} \cdot h_{n+2}$,
(30) $q_{n+1}^4q_{n+2} \cdot h_2$,
(31) $q_2q_{n+1}^3q_{n+2} \cdot h_{n+1}$,
(32) $q_2q_{n+1}^4 \cdot h_{n+2}$.

Proof. The E_1-page of the algAH sseq in degree $(s, k, t) = (1, 5, 3 \cdot 2^{n+3} + 3)$ has a basis of elements of the form

$$q_{i_1}q_{i_2}q_{i_3}q_{i_4}q_{i_5} \cdot h_m$$

where $i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5$.

Considering the t-degrees, we have an equation

$$3 + 3 \cdot 2^{n+3} = 2^{m+1} + (2^{i_1+1} - 1) + (2^{i_2+1} - 1) + (2^{i_3+1} - 1) + (2^{i_4+1} - 1) + (2^{i_5+1} - 1),$$

which can be simplified to the equation

$$4 + 2^{n+2} + 2^{n+3} = 2^m + 2^{i_1} + 2^{i_2} + 2^{i_3} + 2^{i_4} + 2^{i_5}.$$

Up to four terms on the right hand side contribute to 4. So we have the following 9 possibilities for the unordered set $\{2^m,\ 2^{i_1},\ 2^{i_2},\ 2^{i_3},\ 2^{i_4},\ 2^{i_5}\}$.

- $\{1,\ 1,\ 1,\ 1,\ 2^{n+2},\ 2^{n+3}\}$,
- $\{1,\ 1,\ 2,\ 2^{n+1},\ 2^{n+1},\ 2^{n+3}\}$,
- $\{1,\ 1,\ 2,\ 2^{n+2},\ 2^{n+2},\ 2^{n+2}\}$,
- $\{2,\ 2,\ 2^n,\ 2^n,\ 2^{n+1},\ 2^{n+3}\}$,
- $\{2,\ 2,\ 2^{n+1},\ 2^{n+1},\ 2^{n+2},\ 2^{n+2}\}$,
- $\{4,\ 2^{n-1},\ 2^{n-1},\ 2^n,\ 2^{n+1},\ 2^{n+3}\}$,
- $\{4,\ 2^n,\ 2^n,\ 2^n,\ 2^n,\ 2^{n+3}\}$,
- $\{4,\ 2^n,\ 2^n,\ 2^{n+1},\ 2^{n+2},\ 2^{n+2}\}$,
- $\{4,\ 2^{n+1},\ 2^{n+1},\ 2^{n+1},\ 2^{n+1},\ 2^{n+2}\}$.

The rest of proof works similarly to the proof of Lemma 4.12. □

Remark 4.14. Note that since q_0 acts injectively on the algAH E_1-page we can read off the collection of elements on the E_1-page in degree $(s, k, t) = (3, 2, 3 \cdot 2^{n+3} + 2)$ by extracting the subset of elements divisible by q_0 from Lemma 4.12.

4.3. algebraic Atiyah–Hirzebruch differentials.

In this subsection we complete the proof of Proposition 4.2 by computing the relevant degrees of the algAH sseq. Our computation of the algAH differentials uses multiplicative and Massey product structures in $\mathrm{Ext}_{\mathcal{P}}$ together with descriptions of differentials in the algAH sseq from Lemma 4.10.

We begin with the following lemma which we will need in order to conclude that the targets of various algAH differentials are non-trivial.

Lemma 4.15. *The following elements are nonzero in* $\mathrm{Ext}^{4,*}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$.

- h_0c_n, *for* $n\geq 2$,
- $h_0^3h_n$, *for* $n\geq 3$,
- $h_0h_n^2h_{n+3}$, *for* $n\geq 3$,
- $h_1h_n^2h_{n+3}$, *for* $n\geq 4$,
- $h_1h_n^3$, *for* $n\geq 5$,
- $h_0^2h_{n+1}h_{n+3}$, *for* $n\geq 1$,
- $h_0^2h_nh_{n+3}$, *for* $n\geq 2$,
- $h_0^2h_n^2$, *for* $n\geq 4$,
- h_1c_n, *for* $n\geq 3$.

Proof. The complete description of $\mathrm{Ext}^{\leq 4,*}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is given as Theorems 1.2 and 1.3 in [Lin08]. In particular, all relations that are satisfied among the elements h_n and c_n up to the Adams filtration 4 are the following:

$$h_nh_{n+1}=0,\ h_nh_{n+2}^2=0,\ h_n^2h_{n+2}=h_{n+1}^3,\ h_n^2h_{n+3}^2=0,$$
$$h_jc_n=0 \text{ for } j=n-1,n,n+2 \text{ and } n+3.$$

One checks that this lemma is true. □

Lemma 4.16.

- $\mathrm{Ext}_{\mathcal{P}}^{3,\ 3\cdot 2^{n+3}+3}(\mathbb{F}_2,\mathrm{Ext}^3_{\mathcal{Q}}(\mathbb{F}_2,\mathbb{F}_2))$ *is either* $\mathbb{F}_2\{q_0^3\cdot h_{n+2}^3\}$ *or* 0.
- $\mathrm{Ext}_{\mathcal{P}}^{4,\ 3\cdot 2^{n+3}+3}(\mathbb{F}_2,\mathrm{Ext}^3_{\mathcal{Q}}(\mathbb{F}_2,\mathbb{F}_2))$ *contains* $q_0^3\cdot g_n\neq 0$.
- $\mathrm{Ext}_{\mathcal{P}}^{4,\ 3\cdot 2^{n+3}+2}(\mathbb{F}_2,\mathrm{Ext}^2_{\mathcal{Q}}(\mathbb{F}_2,\mathbb{F}_2))$ *contains* $q_0^2\cdot g_n\neq 0$.
- $\mathrm{Ext}_{\mathcal{P}}^{4,\ 3\cdot 2^{n+3}+1}(\mathbb{F}_2,\mathrm{Ext}^1_{\mathcal{Q}}(\mathbb{F}_2,\mathbb{F}_2))$ *contains* $q_0\cdot g_n\neq 0$.

Proof. We begin by noting that $q_0^3\cdot g_n$, $q_0^2\cdot g_n$ and $q_0^1\cdot g_n$ are permanent cycles in the algAH sseq (since they are each a product of permanent cycles). In order to prove the second bullet point, we will show that $q_0^3\cdot g_n$ is not hit by an algAH differential. Note that if $q_0^3\cdot g_n$ is non-zero on the E_∞-page of the algAH sseq, then so are $q_0^2\cdot g_n$ and $q_0\cdot g_n$. Therefore, the third and fourth bullet points will follow. We prove the first bullet point by showing that in the algAH sseq this degree has only $q_0^3\cdot h_{n+2}^3$ by the E_7-page.

In Lemma 4.12 we determined the E_1-page of the algAH sseq in degree $(s,\ k,\ t)=(3,\ 2,\ 3\cdot 2^j+2)$. What we must do now is show that for all 21 elements in Lemma 4.12, other than case (2), each element either supports or is killed by a short algAH differential, so none of them can kill $q_0^3\cdot g_n$.

It is clear that being an element in $\mathrm{Ext}^{*,*}_{\mathcal{P}}(\mathbb{F}_2,\mathbb{F}_2)\{q_0^3\}$, $q_0^3\cdot h_{n+2}^3$ is a permanent cycle so it cannot kill $q_0^3\cdot g_n$. This accounts for the case (2).

Using Lemma 4.10(1) and the Liebniz rule, we obtain the algAH d_1-differentials displayed in Figure 2.

For the cases (1), (7), (10), (11), (13) – (17), (20), (21), by Lemma 4.15, the targets are all nonzero for $n\geq 4$. Note that in the case (14) for $n=4$, the first term of the target $q_0q_4q_7\cdot h_1h_4^3$ is zero, while the second term $q_0q_5q_6\cdot h_1h_4^2h_6$ is nonzero. For $n\geq 5$, both terms are nonzero.

Using Lemma 4.10(2) and the Liebniz rule, we have the following algAH d_2-differentials.

$$(12):\quad d_2(q_0q_{n+1}^2\cdot h_1h_{n+3})=q_0q_n^2\cdot h_1h_{n+1}h_{n+3},$$
$$(18):\quad d_2(q_0q_{n+2}^2\cdot h_1h_{n+2})=q_0q_{n+1}^2\cdot h_1h_{n+2}^2.$$

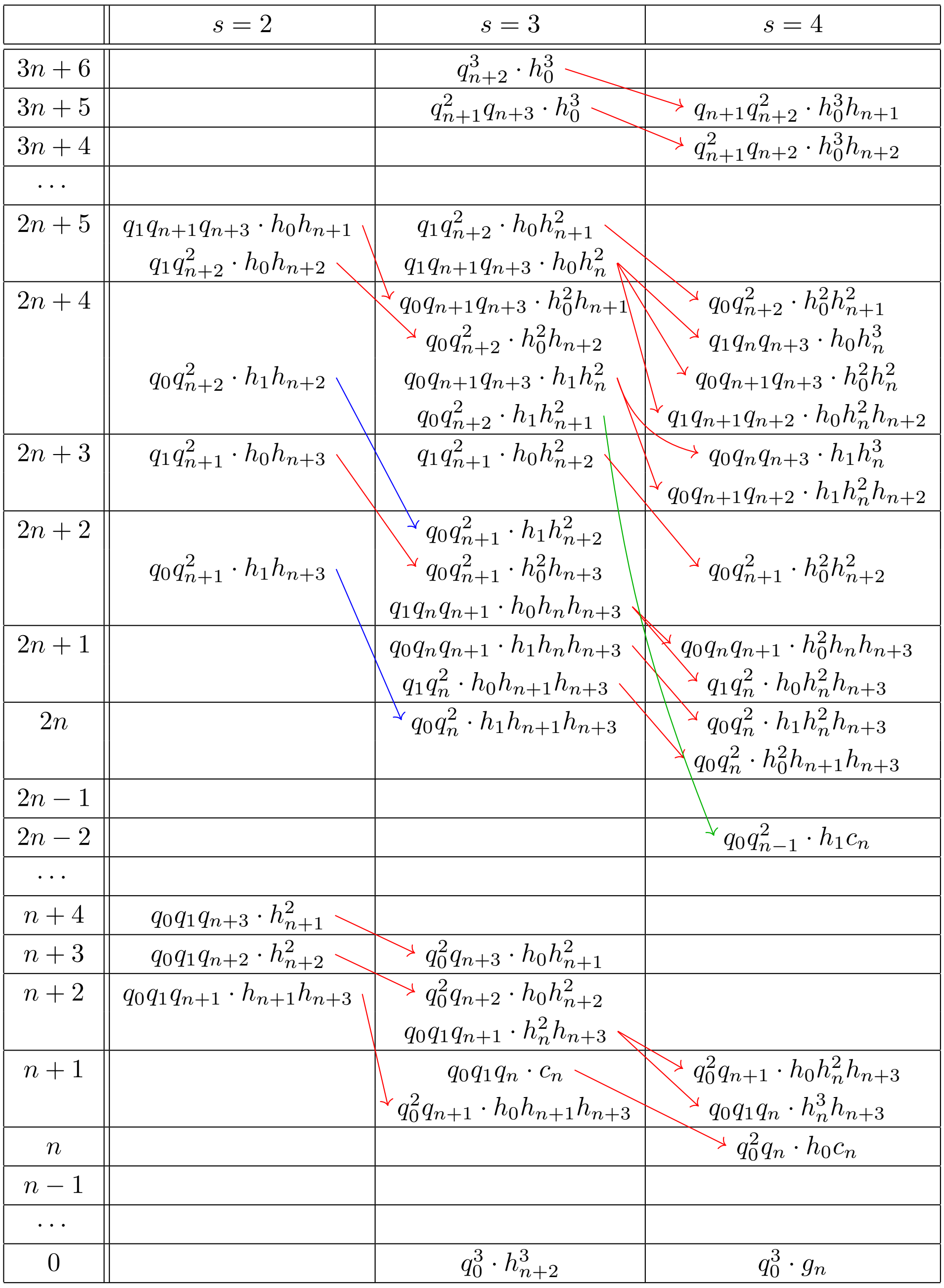


FIGURE 2. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t) = (3,3,3\cdot 2^{n+3}+3)$. In this chart the vertical axis is the algebraic Atiyah–Hirzebruch filtration and the horizontal axis is the s-degree. d_1-differentials are red, d_2-differentials are blue and d_6-differentials are green.

One observes that the sources and targets of the above two d_2-differentials survives to the E_2-page of the algAH sseq. In fact, for the case (18), we have a d_1-differential.

$$d_1(q_0 q_{n+2} q_{n+3} \cdot h_1) = q_0 q_{n+2}^2 \cdot h_1 h_{n+2} + q_0 q_{n+1} q_{n+3} \cdot h_1 h_{n+1}.$$

So in the E_2-page, we have a relation that $q_0q_{n+2}^2 \cdot h_1h_{n+2} = q_0q_{n+1}q_{n+3} \cdot h_1h_{n+1}$. One may also use Lemma 4.10(3) to prove a d_2-differential

$$\begin{aligned}(18): \quad d_2(q_0q_{n+1}q_{n+3} \cdot h_1h_{n+1}) &= q_0q_{n+1}^2 \cdot h_1\langle h_{n+1}, h_{n+2}, h_{n+1}\rangle \\ &= q_0q_{n+1}^2 \cdot h_1h_{n+2}^2,\end{aligned}$$

which is equivalent to the above one.

Using Lemma 4.10(6), we have the following algAH d_6-differential

$$\begin{aligned}(19): \quad d_6(q_0q_{n+2}^2 \cdot h_1h_{n+1}^2) &= q_0q_{n-1}^2 \cdot h_1\langle h_n, h_{n+1}, h_{n+2}, h_{n+1}^2\rangle \\ &= q_0q_{n-1}^2 \cdot h_1c_n.\end{aligned}$$

One observes that all targets of the above algAH differentials are linearly independent. Since most of the targets only have one term, this is not hard to check. This completes the proof. □

Lemma 4.17. $\mathrm{Ext}_{\mathcal{P}}^{3,\ 3\cdot 2^{n+3}+2}(\mathbb{F}_2, \mathrm{Ext}_{\mathcal{Q}}^2(\mathbb{F}_2,\mathbb{F}_2))$ *is either* $\mathbb{F}_2\{q_0^2 \cdot h_{n+2}^3\}$ *or* 0.

Proof. We consider all elements in the E_1-page of algAH sseq in degree $(s,\ k,\ t) = (3,\ 2,\ 3\cdot 2^j + 2)$. In Remark 4.14 we described how each such element can be obtained by dividing one of the elements from Lemma 4.12 by q_0. From this we obtain the following list of classes in degree $(s,\ k,\ t) = (3,\ 2,\ 3\cdot 2^j + 2)$.

(1) $q_1q_n \cdot c_n$,
(2) $q_0^2 \cdot h_{n+1}^2h_{n+3} = q_0^2 \cdot h_{n+2}^3$,
(3) $q_0q_{n+1} \cdot h_0h_{n+1}h_{n+3}$,
(4) $q_0q_{n+3} \cdot h_0h_{n+1}^2$,
(5) $q_{n+1}^2 \cdot h_0^2h_{n+3}$,
(6) $q_{n+1}q_{n+3} \cdot h_0^2h_{n+1}$,
(8) $q_0q_{n+2} \cdot h_0h_{n+2}^2$,
(9) $q_{n+2}^2 \cdot h_0^2h_{n+2}$,
(11) $q_1q_{n+1} \cdot h_n^2h_{n+3}$,
(12) $q_n^2 \cdot h_1h_{n+1}h_{n+3}$,
(13) $q_nq_{n+1} \cdot h_1h_nh_{n+3}$,
(14) $q_{n+1}q_{n+3} \cdot h_1h_n^2$,
(18) $q_{n+1}^2 \cdot h_1h_{n+2}^2$,
(19) $q_{n+2}^2 \cdot h_1h_{n+1}^2$.

Other than the cases (2), (5), (6) and (9), the differentials in the proof of Lemma 4.16 are q_0-divisible, therefore these candidates do not survive the algAH sseq. For the cases (6) and (9), we have an algAH d_1-differential

$$(6,9): \quad d_1(q_{n+2}q_{n+3} \cdot h_0^2) = q_{n+1}q_{n+3} \cdot h_0^2h_{n+1} + q_{n+2}^2 \cdot h_0^2h_{n+2}$$

and by Lemma 4.10(2), we have the following algAH d_2-differential.

$$(9): \quad d_2(q_{n+2}^2 \cdot h_0^2h_{n+2}) = q_{n+1}^2 \cdot h_0.$$

For case (5), we have the following algAH d_2-differential by Lemma 4.10(2),

$$(5): \quad d_2(q_{n+2}^2 \cdot h_1h_{n+2}) = q_{n+1}^2 \cdot h_1h_{n+2}^2.$$

This completes the proof. □

Lemma 4.18. $\mathrm{Ext}_{\mathcal{P}}^{3,\ 3\cdot 2^{n+3}+1}(\mathbb{F}_2, \mathrm{Ext}_{\mathcal{Q}}^1(\mathbb{F}_2,\mathbb{F}_2))$ *is either* $\mathbb{F}_2\{q_0 \cdot h_{n+2}^3\}$ *or* 0.

Proof. We consider all elements in the E_1-page of algAH sseq in degree $(s,\ k,\ t) = (3,\ 1,\ 3\cdot 2^j + 2)$. As in Remark 4.14 we can determine all such elements by beginning with the list from Lemma 4.12 and dividing by q_0^2.

(2) $q_0 \cdot h_{n+1}^2h_{n+3} = q_0 \cdot h_{n+2}^3$,
(3) $q_{n+1} \cdot h_0h_{n+1}h_{n+3}$,
(4) $q_{n+3} \cdot h_0h_{n+1}^2$,
(8) $q_{n+2} \cdot h_0h_{n+2}^2$,

	$s=2$	$s=3$	$s=4$
$2n+5$	$q_{n+2}q_{n+3}\cdot h_0^2$		
$2n+4$	$q_{n+2}^2\cdot h_1h_{n+2}$	$q_{n+1}q_{n+3}\cdot h_1h_n^2$ $q_{n+1}q_{n+3}\cdot h_0^2h_{n+1}$ $q_{n+2}^2\cdot h_0^2h_{n+2}$ $q_{n+2}^2\cdot h_1h_{n+1}^2$	
$2n+3$			$q_nq_{n+3}\cdot h_1h_n^3$ $q_{n+1}q_{n+2}\cdot h_1h_n^2h_{n+2}$
$2n+2$	$q_{n+1}^2\cdot h_1h_{n+3}$	$q_{n+1}^2\cdot h_1h_{n+2}^2$ $q_{n+1}^2\cdot h_0^2h_{n+3}$	$q_{n+1}^2\cdot h_0^2h_{n+2}^2$
$2n+1$		$q_nq_{n+1}\cdot h_1h_nh_{n+3}$	$q_nq_{n+1}\cdot h_0^2h_nh_{n+3}$
$2n$		$q_n^2\cdot h_1h_{n+1}h_{n+3}$	$q_n^2\cdot h_0^2h_{n+1}h_{n+3}$ $q_n^2\cdot h_1h_n^2h_{n+3}$
$2n-1$			
$2n-2$			$q_{n-1}^2\cdot h_1c_n$
$\cdots$			
$n+4$	$q_1q_{n+3}\cdot h_{n+1}^2$		
$n+3$	$q_1q_{n+2}\cdot h_{n+2}^2$	$q_0q_{n+3}\cdot h_0h_{n+1}^2$	
$n+2$	$q_1q_{n+1}\cdot h_{n+1}h_{n+3}$	$q_0q_{n+2}\cdot h_0h_{n+2}^2$ $q_1q_{n+1}\cdot h_n^2h_{n+3}$	
$n+1$		$q_1q_n\cdot c_n$ $q_0q_{n+1}\cdot h_0h_{n+1}h_{n+3}$	$q_0q_{n+1}\cdot h_0h_n^2h_{n+3}$ $q_1q_n\cdot h_n^3h_{n+3}$
n			$q_0q_n\cdot h_0c_n$
$n-1$			
$\cdots$			
0		$q_0^2\cdot h_{n+2}^3$	$q_0^2\cdot g_n$

FIGURE 3. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t)=(3,2,3\cdot 2^{n+3}+2)$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red, d_2-differentials are blue and d_6-differentials are green.

For the cases (3) and (8), we have the following Atiyah-Hirzebruch d_1-differentials:

$$(3): \quad d_1(q_{n+2}\cdot h_0h_{n+3}) = q_{n+1}\cdot h_0h_{n+1}h_{n+3},$$
$$(8): \quad d_1(q_{n+3}\cdot h_0h_{n+2}) = q_{n+2}\cdot h_0h_{n+2}^2.$$

For the case (4), by Lemma 4.10(5), we have the following algAH d_3-differential

$$\begin{aligned}(4): \quad d_3(q_{n+3}\cdot h_0h_{n+1}^2) &= q_n\cdot h_0\langle h_n,h_{n+1},h_{n+2},h_{n+1}^2\rangle\\ &= q_n\cdot h_0c_n.\end{aligned}$$

This completes the proof. □

Lemma 4.19. $\mathrm{Ext}_{\mathcal{P}}^{1,\,3\cdot 2^{n+3}+3}(\mathbb{F}_2,\mathrm{Ext}_{\mathcal{Q}}^{5}(\mathbb{F}_2,\mathbb{F}_2))\cong 0$.

Proof. From Lemma 4.13 we know the E_1-page of the algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t)=(1,5,3\cdot 2^{n+3}+3)$. Using Lemma 4.10(1) we

	$s=2$	$s=3$	$s=4$
$n+3$	$q_{n+3} \cdot h_0 h_{n+2}$	$q_{n+3} \cdot h_0 h_{n+1}^2$	
$n+2$	$q_{n+2} \cdot h_0 h_{n+3}$	$q_{n+2} \cdot h_0 h_{n+2}^2$	
$n+1$		$q_{n+1} \cdot h_0 h_{n+1} h_{n+3}$	
n			$q_n \cdot h_0 c_n$
$\cdots$			
0		$q_0 \cdot h_{n+2}^3$	$q_0 \cdot g_n$

FIGURE 4. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t) = (3, 2, 3 \cdot 2^{n+3} + 2)$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red and d_3-differentials are green.

	$s=0$	$s=1$	$s=2$
$5n+6$		$q_{n+1}^4 q_{n+2} \cdot h_2$	
$5n+5$		$q_n^2 q_{n+1} q_{n+2}^2 \cdot h_2$	$q_{n+1}^5 \cdot h_2 h_{n+1}$
$5n+4$			$q_n^3 q_{n+2}^2 \cdot h_2 h_n$
$5n+3$		$q_n^4 q_{n+3} \cdot h_2$	
$5n+2$		$q_{n-1}^2 q_n q_{n+1} q_{n+3} \cdot h_2$	$q_n^4 q_{n+2} \cdot h_2 h_{n+2}$
$5n+1$			$q_{n-1}^3 q_{n+1} q_{n+3} \cdot h_2 h_{n-1} + q_{n-1}^2 q_n^2 q_{n+3} \cdot h_2 h_n$ $+ q_{n-1}^2 q_n q_{n+1} q_{n+2} \cdot h_2 h_{n+2}$
$\cdots$			
$4n+8$	$q_2 q_{n+1}^2 q_{n+2}^2 \cdot 1$		
$4n+7$		$q_1 q_{n+1}^2 q_{n+2}^2 \cdot h_1$ $q_2 q_{n+1}^3 q_{n+2} \cdot h_{n+1}$ $q_2 q_n q_{n+1} q_{n+2}^2 \cdot h_n$	
$4n+6$	$q_2 q_n^2 q_{n+1} q_{n+3} \cdot 1$	$q_2 q_{n+1}^4 \cdot h_{n+2}$ $q_2 q_n^2 q_{n+2}^2 \cdot h_{n+1}$	$q_1 q_{n+1}^3 q_{n+2} \cdot h_1 h_{n+1} + q_2 q_{n+1}^4 \cdot h_{n+1}^2$ $q_1 q_n q_{n+1} q_{n+2}^2 \cdot h_1 h_n + q_2 q_n^2 q_{n+2}^2 \cdot h_n^2$
$4n+5$		$q_1 q_n^2 q_{n+1} q_{n+3} \cdot h_1$ $q_2 q_n^3 q_{n+3} \cdot h_n$ $q_2 q_n^2 q_{n+1} q_{n+2} \cdot h_{n+2}$ $q_2 q_{n-1} q_n q_{n+1} q_{n+3} \cdot h_{n-1}$	$q_1 q_{n+1}^4 \cdot h_1 h_{n+2}$ $q_1 q_n^2 q_{n+2}^2 \cdot h_1 h_{n+1}$
$4n+4$		$q_2 q_{n-1}^2 q_{n+1} q_{n+3} \cdot h_n$	$q_1 q_n^3 q_{n+3} \cdot h_1 h_n$ $q_1 q_n^2 q_{n+1} q_{n+2} \cdot h_1 h_{n+2}$ $q_2 q_n^3 q_{n+2} \cdot h_n h_{n+2}$ $q_1 q_{n-1} q_n q_{n+1} q_{n+3} \cdot h_1 h_{n-1} + q_2 q_{n-1}^2 q_{n+1} q_{n+3} \cdot h_{n-1}^2$ $+ q_2 q_{n-1} q_n q_{n+1} q_{n+2} \cdot h_{n-1} h_{n+2}$
$4n+3$		$q_2 q_{n-1}^2 q_n q_{n+3} \cdot h_{n+1}$	$q_1 q_{n-1}^2 q_{n+1} q_{n+3} \cdot h_1 h_n + q_2 q_{n-1}^2 q_{n+1} q_{n+2} \cdot h_n h_{n+2}$
$4n+2$		$q_2 q_n^4 \cdot h_{n+3}$	$q_1 q_{n-1}^2 q_n q_{n+3} \cdot h_1 h_{n+1} + q_2 q_{n-1}^3 q_{n+3} \cdot h_{n-1} h_{n+1}$
$4n+1$		$q_2 q_{n-1}^2 q_n q_{n+1} \cdot h_{n+3}$	$q_1 q_n^4 \cdot h_1 h_{n+3}$
$4n$			$q_1 q_{n-1}^2 q_n q_{n+1} \cdot h_1 h_{n+3} + q_2 q_{n-1}^3 q_{n+1} \cdot h_{n-1} h_{n+3}$ $+ q_2 q_{n-1}^2 q_n^2 \cdot h_n h_{n+3}$

FIGURE 5. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t) = (1, 5, 3 \cdot 2^{n+3} + 3)$ and algAH filtration $\geq 4n$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red.

compute the relevant algAH d_1-differentials. These differentials are displayed in Figures 5 and 6. In order to pass the E_2-page we must also verify that the targets of these d_1-differentials are all linearly independent. For this we note that two classes $q_{i_1} q_{i_2} q_{i_3} q_{i_4} q_{i_5} \cdot a$ and $q_{j_1} q_{j_2} q_{j_3} q_{j_4} q_{j_5} \cdot b$ (with the i's and j's in non-decreasing order) on the E_1-page can only satisfy a relation if $i_k = j_k$ for $k = 1, \ldots, 5$. Examining Figures 5 and 6 we see that no pair of classes satisfy this condition.

	$s=0$	$s=1$	$s=2$
$3n+8$	$q_1^2q_{n+2}^3\cdot 1$		
$3n+7$	$q_1^2q_{n+1}^2q_{n+3}\cdot 1$	$q_1^2q_{n+1}q_{n+2}^2\cdot h_{n+1}$ $q_0q_1q_{n+2}^3\cdot h_0$	
$3n+6$		$q_1^2q_{n+1}q_{n+2}^2\cdot h_{n+1}$ $q_0q_1q_{n+1}^2q_{n+3}\cdot h_0$ $q_1^2q_nq_{n+1}q_{n+3}\cdot h_n$ $q_0^2q_{n+2}^3\cdot h_1$	$q_0^2q_{n+2}^3\cdot h_0^2+q_0q_1q_{n+1}q_{n+2}^2\cdot h_0h_{n+1}$
$3n+5$		$q_0^2q_{n+1}^2q_{n+3}\cdot h_1$ $q_1^2q_n^2q_{n+3}\cdot h_{n+1}$	$q_0^2q_{n+1}^2q_{n+3}\cdot h_0^2+q_0q_1q_{n+1}^2q_{n+2}\cdot h_0h_{n+2}$ $q_1^2q_n^2q_{n+3}\cdot h_n^2+q_1^2q_nq_{n+1}q_{n+2}\cdot h_nh_{n+2}$ $q_0^2q_{n+1}q_{n+2}^2\cdot h_1h_{n+1}$
$3n+4$			$q_0^2q_{n+1}^2q_{n+2}\cdot h_1h_{n+2}$
$3n+3$		$q_1^2q_n^2q_{n+1}\cdot h_{n+3}$	$q_0^2q_n^2q_{n+3}\cdot h_1h_{n+1}+q_1^2q_n^2q_{n+1}\cdot h_{n+2}^2$
$3n+2$			$q_1^2q_n^3\cdot h_nh_{n+3}$
$\cdots$			
$2n+6$	$q_0^2q_1q_{n+2}q_{n+3}\cdot 1$		
$2n+5$		$q_0^2q_1q_{n+1}q_{n+3}\cdot h_{n+1}$ $q_0^2q_1q_{n+2}^2\cdot h_{n+2}$ $q_0^3q_{n+2}q_{n+3}\cdot h_0$	
$2n+4$			$q_0^3q_{n+1}q_{n+3}\cdot h_0h_{n+1}$ $q_0^3q_{n+2}^2\cdot h_0h_{n+2}$
$2n+3$		$q_0^2q_1q_{n+1}^2\cdot h_{n+3}$	
$2n+2$			$q_0^3q_{n+1}^2\cdot h_0h_{n+3}$
$\cdots$			
$n+3$		$q_0^4q_{n+3}\cdot h_{n+2}$	
$n+2$		$q_0^4q_{n+2}\cdot h_{n+3}$	$q_0^4q_{n+2}\cdot h_{n+2}^2$
$n+1$			$q_0^4q_{n+1}\cdot h_{n+1}h_{n+3}$

FIGURE 6. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t)=(1,5,3\cdot 2^{n+3}+3)$ and algAH filtration $\leq 3n+8$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red and d_2-differentials are blue.

On the E_2-page of the algAH sseq only one of the 32 candidates from Lemma 4.13 is still present: (13). Using Lemma 4.10(2,3), we obtain the following d_2-differential.

$$(13):\quad d_2(q_1^2q_n^2q_{n+3}\cdot h_{n+1}) = q_0^2q_n^2q_{n+3}\cdot h_1h_{n+1} + q_1^2q_n^2q_{n+1}\cdot h_{n+2}^2.$$

As the target is non-zero on the E_2-page of the algAH sseq this d_2-differential is non-zero. In particular, the algAH sseq is empty in degree $(s,k,t)=(1,5,3\cdot 2^{n+3}+3)$ starting from the E_3-page. □

Lemma 4.20. $\mathrm{Ext}_{\mathcal{P}}^{1,\,3\cdot 2^{n+3}+2}(\mathbb{F}_2,\mathrm{Ext}_{\mathcal{Q}}^4(\mathbb{F}_2,\mathbb{F}_2))\cong 0.$

Proof. Dividing by q_0 as in Remark 4.14 we can determine the E_1-page of the algAH sseq in degree $(s,k,t)=(1,4,3\cdot 2^{n+3}+2)$ from Lemma 4.13. It contains the following classes:

(1) $q_0^2q_jq_{j+1}\cdot h_0$,
(2) $q_0^3q_{j+1}\cdot h_j$,
(3) $q_0^3q_j\cdot h_{j+1}$,
(4) $q_1q_{j-1}^2q_{j+1}\cdot h_0$,
(5) $q_0q_{j-1}^2q_{j+1}\cdot h_1$,
(6) $q_0q_1q_{j-1}q_{j+1}\cdot h_{j-1}$,
(7) $q_0q_1q_{j-1}^2\cdot h_{j+1}$,
(8) $q_1q_j^3\cdot h_0$,
(9) $q_0q_j^3\cdot h_1$,
(10) $q_0q_1q_j^2\cdot h_j$.

	$s=0$	$s=1$	$s=2$
$3n+7$		$q_1q_{n+2}^3 \cdot h_0$	
$3n+6$		$q_0q_{n+2}^3 \cdot h_1$ $q_1q_{n+1}^2q_{n+3} \cdot h_0$	$q_0q_{n+2}^3 \cdot h_0^2 + q_1q_{n+1}q_{n+2}^2 \cdot h_0h_{n+1}$
$3n+5$		$q_0q_{n+1}^2q_{n+3} \cdot h_1$	$q_0q_{n+1}q_{n+2}^2 \cdot h_1h_{n+1}$ $q_0q_{n+1}^2q_{n+3} \cdot h_0^2 + q_1q_{n+1}^2q_{n+2} \cdot h_0h_{n+2}$
$3n+4$			$q_0q_{n+1}^2q_{n+2} \cdot h_1h_{n+2}$
$\cdots$			
$2n+6$	$q_0q_1q_{n+2}q_{n+3} \cdot 1$		
$2n+5$		$q_0q_1q_{n+1}q_{n+3} \cdot h_{n+1}$ $q_0q_1q_{n+2}^2 \cdot h_{n+2}$ $q_0^2q_{n+2}q_{n+3} \cdot h_0$	
$2n+4$			$q_0^2q_{n+1}q_{n+3} \cdot h_0h_{n+1}$ $q_0^2q_{n+2}^2 \cdot h_0h_{n+2}$
$2n+3$		$q_0q_1q_{n+1}^2 \cdot h_{n+3}$	
$2n+2$			$q_0^2q_{n+1}^2 \cdot h_0h_{n+3}$
$\cdots$			
$n+3$		$q_0^3q_{n+3} \cdot h_{n+2}$	
$n+2$		$q_0^3q_{n+2} \cdot h_{n+3}$	$q_0^3q_{n+2} \cdot h_{n+2}^2$
$n+1$			$q_0^3q_{n+1} \cdot h_{n+1}h_{n+3}$

FIGURE 7. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t) = (1,4,3\cdot 2^{n+3}+3)$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red.

Using Lemma 4.10(1) we compute the relevant algAH d_1-differentials. These differentials are displayed in Figure 7. In particular, the algAH sseq is empty in degree $(s,k,t) = (1,4,3\cdot 2^{n+3}+2)$ starting from the E_2-page. □

Lemma 4.21. $\mathrm{Ext}_{\mathcal{P}}^{1,\,3\cdot 2^{n+3}+1}(\mathbb{F}_2, \mathrm{Ext}_{\mathcal{Q}}^{3}(\mathbb{F}_2,\mathbb{F}_2)) \cong 0$.

Proof. Dividing by q_0 as in Remark 4.14 again we can determine the E_1-page of the algAH sseq in degree $(s,k,t) = (1,3,3\cdot 2^{n+3}+1)$. It contains the following classes:

(1) $q_0q_jq_{j+1} \cdot h_0$,
(2) $q_0^2q_{j+1} \cdot h_j$,
(3) $q_0^2q_j \cdot h_{j+1}$,
(4) $q_{j-1}^2q_{j+1} \cdot h_1$,
(5) $q_1q_{j-1}q_{j+1} \cdot h_{j-1}$,
(6) $q_1q_{j-1}^2 \cdot h_{j+1}$,
(7) $q_j^3 \cdot h_1$,
(8) $q_1q_j^2 \cdot h_j$.

Using Lemma 4.10(1) we compute the relevant algAH d_1-differentials. These differentials are displayed in Figure 8. In particular, the algAH sseq is empty in degree $(s,k,t) = (1,3,3\cdot 2^{n+3}+1)$ starting from the E_2-page. □

Lemma 4.22. $\mathrm{Ext}_{\mathcal{P}}^{1,\,3\cdot 2^{n+3}}(\mathbb{F}_2, \mathrm{Ext}_{\mathcal{Q}}^{2}(\mathbb{F}_2,\mathbb{F}_2)) \cong 0$.

Proof. Dividing by q_0 as in Remark 4.14 again we can determine the E_1-page of the algAH sseq in degree $(s,k,t) = (1,2,3\cdot 2^{n+3})$. It contains the following classes:

(1) $q_jq_{j+1} \cdot h_0$,
(2) $q_0q_{j+1} \cdot h_j$,
(3) $q_0q_j \cdot h_{j+1}$,

	$s=0$	$s=1$	$s=2$
$3n+6$		$q_{n+2}^3 \cdot h_1$	
$3n+5$		$q_{n+1}^2 q_{n+3} \cdot h_1$	$q_{n+1} q_{n+2}^2 \cdot h_1 h_{n+1}$
$3n+4$			$q_{n+1}^2 q_{n+2} \cdot h_1 h_{n+2}$
$\cdots$			
$2n+6$	$q_1 q_{n+2} q_{n+3} \cdot 1$		
$2n+5$		$q_1 q_{n+1} q_{n+3} \cdot h_{n+1}$ $q_1 q_{n+2}^2 \cdot h_{n+2}$ $q_0 q_{n+2} q_{n+3} \cdot h_0$	
$2n+4$			$q_0 q_{n+1} q_{n+3} \cdot h_0 h_{n+1}$ $q_0 q_{n+2}^2 \cdot h_0 h_{n+2}$
$2n+3$		$q_1 q_{n+1}^2 \cdot h_{n+3}$	
$2n+2$			$q_0 q_{n+1}^2 \cdot h_0 h_{n+3}$
$\cdots$			
$n+3$		$q_0^2 q_{n+3} \cdot h_{n+2}$	
$n+2$		$q_0^2 q_{n+2} \cdot h_{n+3}$	$q_0^2 q_{n+2} \cdot h_{n+2}^2$
$n+1$			$q_0^2 q_{n+1} \cdot h_{n+1} h_{n+3}$

FIGURE 8. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t) = (1,3,3\cdot 2^{n+3}+3)$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red.

	$s=0$	$s=1$	$s=2$
$2n+5$		$q_{n+2} q_{n+3} \cdot h_0$	
$2n+4$			$q_{n+1} q_{n+3} \cdot h_0 h_{n+1} + q_{n+2}^2 \cdot h_0 h_{n+2}$
$\cdots$			
$n+3$		$q_0 q_{n+3} \cdot h_{n+2}$	
$n+2$		$q_0 q_{n+2} \cdot h_{n+3}$	$q_0 q_{n+2} \cdot h_{n+2}^2$
$n+1$			$q_0 q_{n+1} \cdot h_{n+1} h_{n+3}$

FIGURE 9. The algebraic Atiyah–Hirzebruch spectral sequence in degree $(s,k,t) = (1,2,3\cdot 2^{n+3}+3)$. In this chart the vertical axis is the algAH filtration and the horizontal axis is the s-degree. d_1-differentials are red.

Using Lemma 4.10(1) we compute the relevant algAH d_1-differentials. These differentials are displayed in Figure 9. In particular, the algAH sseq is empty in degree $(s,k,t) = (1,2,3\cdot 2^{n+3})$ starting from the E_2-page. □

Proof (of Proposition 4.2). The various components of this proposition appeared in the preceding lemmas in this subsection. □

5. THE CARTAN–EILENBERG SPECTRAL SEQUENCE

In this section we show that the Cartan–Eilenberg sseq has no differentials in a neighborhood around h_j^3 and use this to complete the proof of Theorem 3.2.

Lemma 5.1. *Let $j \geq 5$. There are no CE differentials entering the following $(a, t-a)$ degrees*[7] $(4, 3\cdot 2^j - 4)$, $(5, 3\cdot 2^j - 4)$, $(6, 3\cdot 2^j - 4)$ *or* $(7, 3\cdot 2^j - 4)$.

[7]Recall that $a = s + k$.

Proof. Recall that Cartan-Eilenberg d_r-differentials changes the tri-degrees in the following way

$$d_r : E_r^{s,k,t} \to E_r^{s+r,k-r+1,t}.$$

Rewriting this in the $(a, s, t-a)$ basis we obtain

$$d_r : E_r^{a,s,t-a} \to E_r^{a+1,s+r,t-a-1}.$$

From the sparsity of the CE sseq we know that all differentials have odd length. The following table contains a list of each differential we must rule out, and why each differential doesn't occur.

$d_r : E_r^{a,s,t-a} \to E_r^{a+1,s+r,t-a-1}$	argument
$d_3 : E_3^{3,1,3\cdot 2^j-3} \to E_3^{4,4,3\cdot 2^j-4}$	zero source
$d_3 : E_3^{4,1,3\cdot 2^j-3} \to E_3^{5,4,3\cdot 2^j-4}$	zero source
$d_3 : E_3^{5,1,3\cdot 2^j-3} \to E_3^{6,4,3\cdot 2^j-4}$	zero source
$d_3 : E_3^{5,3,3\cdot 2^j-3} \to E_3^{6,6,3\cdot 2^j-4}$	source all permanent cycles
$d_5 : E_5^{5,1,3\cdot 2^j-3} \to E_5^{6,6,3\cdot 2^j-4}$	zero source
$d_3 : E_3^{6,1,3\cdot 2^j-3} \to E_3^{7,4,3\cdot 2^j-4}$	zero source
$d_3 : E_3^{6,3,3\cdot 2^j-3} \to E_3^{7,6,3\cdot 2^j-4}$	source all permanent cycles
$d_5 : E_5^{6,1,3\cdot 2^j-3} \to E_5^{7,6,3\cdot 2^j-4}$	zero source

In each case Proposition 4.2 provides the required information about the source group. □

Recollection 5.2. Building on Lin's work on the 4-line, in [Che11, Theorem 1.2], Chen gives a complete description of $\mathrm{Ext}_{\mathcal{A}}$ up to Adams filtration 5. In particular, Chen's work provides the following information about $\mathrm{Ext}_{\mathcal{A}}$ in a neighborhood of h_j^3:

$(a,\quad t-a)$	$\mathrm{Ext}_{\mathcal{A}}^{a,\,t}(\mathbb{F}_2,\,\mathbb{F}_2)$
$(4,\quad 3\cdot 2^j-4)$	$\mathbb{F}_2\{g_{j-2}\}$
$(5,\quad 3\cdot 2^j-4)$	$\mathbb{F}_2\{h_0 g_{j-2}\}$
$(3,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_j^3\}$
$(4,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_0 h_j^3\}$
$(5,\quad 3\cdot 2^j-3)$	$\mathbb{F}_2\{h_0^2 h_j^3, h_1 g_{j-2}\}$

for $j \geq 8$. For $j = 7$ degree $(a, t-s) = (5, 3\cdot 2^7-3)$ contains the additional class $h_8 D_3(1)$. For $j = 6$ degree $(a, t-s) = (5, 3\cdot 2^6-4)$ contains the additional class $h_7 D_3(0)$.

Proposition 5.3. *Let $j \geq 5$. On the E_2-page of the classical Adams spectral sequence for S^0 the class $h_0^3 g_{j-2}$ is non-zero.*

Proof. In the cases $j = 5, 6$, this is known from stemwise calculation of the Ext-groups. See for example [Bru, IWX20a] for $j = 5$ and [Nas] for $j = 6$. We will prove the proposition for $j \geq 7$.

The class q_0^3 survives the algAH sseq and CE sseq, and detects the class h_0^3 in the Adams E_2-page. From the Frobenius isomorphism between $\mathrm{Ext}_{\mathcal{P}}^{*,2*}(\mathbb{F}_2,\mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2,\mathbb{F}_2)$, the element g_n in $\mathrm{Ext}_{\mathcal{P}}^{4,3\cdot 2^{n+3}}(\mathbb{F}_2,\mathbb{F}_2)$ detects g_{n+1} in $\mathrm{Ext}_{\mathcal{A}}^{4,3\cdot 2^{n+3}}(\mathbb{F}_2,\mathbb{F}_2)$. Therefore, the element $q_0^3 \cdot g_n$ detects the element $h_0^3 g_{n+1}$ in the Adams E_2-page.

For this we observe that $h_0^3 g_{j-2}$ is detected on the Cartan–Eilenberg E_2-page by $q_0^3 g_{j-3}$ and that this class is not hit by a CE differential since there are no differential which enter this tridegree by Lemma 5.1. □

Together Recollection 5.2 and Proposition 5.3 complete the proof of Theorem 3.2(1).

Lemma 5.4. *Let $j \geq 6$. The E_2-page of the Cartan–Eilenberg spectral sequence takes the following form near h_j^3:*

$(a,\ s,\ t-a)$	$\mathrm{Ext}^{a,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbb{F}_2[\tau], \mathbb{F}_2)$
$(4,\ 4,\ 3\cdot 2^j - 4)$	$\mathbb{F}_2\{g_{j-2}\}$
$(5,\ 4,\ 3\cdot 2^j - 4)$	$\mathbb{F}_2\{h_0 g_{j-2}\}$
$(3,\ 3,\ 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_j^3\}$
$(4,\ 3,\ 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_0 h_j^3\}$
$(5,\ 3,\ 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_0^2 h_j^3\}$
$(5,\ 5,\ 3\cdot 2^j - 3)$	$\mathbb{F}_2\{h_1 g_{j-2}\}$

For $j = 7$ there is an additional class, $h_8 D_3(1)$, in degree $(a, s, t-a) = (5, 5, 3\cdot 2^7 - 3)$. For $j = 6$ there is possibly an additional class, $h_7 D_3(0)$, in degree $(a, s, t-a) = (5, 4, 3\cdot 2^6 - 4)$.

Proof. Using the Frobenius isomorphism $\mathrm{Ext}^{s,2t}_{\mathcal{P}} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}$ the information on lines 1, 3 and 6 can be extracted from Lin's computation of $\mathrm{Ext}_{\mathcal{A}}$ through Adams filtration 5 (see [Che11, Theorem 1.2]).

Recall that we proved that the CE E_2-page contains a non-trivial class $h_0 g_{j-2}$ in Proposition 4.2. Suppose the CE E_2-page had another element x besides $h_0 g_{j-2}$ in degree $(a, s, t-a) = (5, 4, 3\cdot 2^j - 4)$. This element would be a permanent cycle for degree reasons and would survive to the E_∞-page since no differentials enter this tridegree by Lemma 5.1. This would imply that the rank of degree $(a, t-a) = (5, 3\cdot 2^j - 4)$ in the Adams E_2-page is at least 2. This contradicts Lin's calculations of $\mathrm{Ext}_{\mathcal{A}}$, therefore no such x can exist and we obtain the conclusion in line 2. In the $j = 6$ case there the rank of $\mathrm{Ext}_{\mathcal{A}}$ is 2 so we may have another class detecting $h_7 D_3(0)$.

Recall that Proposition 4.2 lets us reduce proving the claims on lines 4 and 5 to showing that $q_0^2 h_j^3 \neq 0$. From Lin's computations we know that $h_0^2 h_{j+1}^3 \neq 0$ on the Adams E_2-page. $h_0^2 h_{j+1}^3$ would be detected on the CE E_2-page by $q_0^2 h_j^3$ if this class was non-zero and if this class is zero, then it must be detected in $(a, s, t-a) = (5, 5, 3\cdot 2^j - 3)$. On the other hand in $(a, s, t-a) = (5, 5, 3\cdot 2^j - 3)$ the only class is $h_1 g_{j-2}$ which is already detecting $h_1 g_{j-2}$ (in the $j = 7$ case there is also $h_8 D_3(1)$, but this is already detecting $h_8 D_3(1)$). The conclusion follows. □

Together Proposition 4.2 and Lemma 5.4 complete the proof of Theorem 3.2(3). We end the section by using the motivic CE sseq (and Corollary 2.10 in particular) to prove Theorem 3.2(2,4).

Lemma 5.5. *Let $j \geq 6$. There are no CE differentials entering the following $(a, t-a)$ degrees $(3, 3\cdot 2^j - 3)$, $(4, 3\cdot 2^j - 3)$ and $(5, 3\cdot 2^j - 3)$.*

Proof. Convergence of the CE sseq implies that the rank of the CE E_2-page is an upper bound on the rank of the Adams E_2-page and that this bound is sharp in degree $(a, t-a)$ exactly when there are no differentials entering or leaving degree (a, t). Comparing the ranks from Theorem 3.2(3) with the ranks from Theorem 3.2(1) we see the bound is sharp and obtain the desired conclusion. □

Proof (of Theorem 3.2(2,4)). Using Corollary 2.10 we combine the statements about Cartan–Eilenberg differentials from Lemmas 5.1 and 5.5 with our knowledge of the Cartan–Eilenberg E_2-page to prove Theorem 3.2(2).

In order to prove Theorem 3.2(4) we begin by observing that Corollary 2.10 lets us match up names of classes with names of lifts, so it suffices to determine where each class goes under Betti realization. In $(a, t-a)$ degrees $(4, 3\cdot 2^j - 4)$, $(5, 3\cdot 2^j - 4)$, $(3, 3\cdot 2^j - 3)$ and $(4, 3\cdot 2^j - 3)$ the target of Betti realization is a single $\mathbb{F}_2$ so the conclusion is automatic. In $(a, t-a)$ degrees $(6, 3\cdot 2^j - 4)$, $(7, 3\cdot 2^j - 4)$ and $(5, 3\cdot 2^j - 3)$ we can choose our lifts of Cartan–Eilenberg permanent cycles to be the products $h_0^2 \cdot g_{j-2}$, $h_0^3 \cdot g_{j-2}$, $h_0 \cdot h_j^3$, $h_0^2 \cdot h_j^3$ and $h_1 \cdot g_{j-2}$ and the compatibility of Betti realization with products lets us conclude. □

6. The key algebraic Novikov differential

In this section we prove the key motivic Adams differentials for $S^{0,0}/\tau$ appearing in Theorem 3.4. Through [GWX21, Theorem 1.17] this is equivalent to the following family of algebraic Novikov differentials:

Proposition 6.1. *In the algebraic Novikov spectral sequence we have differentials*

$$d_2(h_j^3) = 0, \quad d_3(h_j^3) = 0, \quad d_4(h_j^3) = q_0^3 g_{j-2}$$

for $j \geq 5$.

Remark 6.2. Note that in Proposition 6.1 we are using Cartan–Eilenberg names rather than motivic Adams names. In particular, the class $q_0^3 g_{j-2}$ corresponds to the class $h_0^3 g_{j-1}$ in Theorem 3.4 and the condition $j \geq 5$ corresponds to the condition $j \geq 6$.[8]

Speaking broadly, the proof of Proposition 6.1 has two parts. In the first part, we relate the differential on h_j^3 to a product $\vartheta_j \vartheta_{j+1}$ on the Adams–Novikov E_2-page. Where ϑ_j refers to a particular choice of class that maps to h_j^2 under the Thom reduction map

$$\mathrm{Ext}^{2,2^{j+1}}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \to \mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

whose precise definition will be given in Definition 6.6. In the second part, we use explicit cocycle representatives to identify $\vartheta_j \vartheta_{j+1}$ in the algebraic Novikov spectral sequence and thereby prove Proposition 6.1.

Remark 6.3. In Section 7, we will explain how the product $\vartheta_j \vartheta_{j+1}$ is related to the Adams differentials supported by the Kervaire invariant class h_j^2 and is therefore of independent interest.

Speaking concretely, the section is organized as follows. In Section 6.1 we explain how algebraic Novikov differentials can be computed in terms of cocycles and reduce Proposition 6.1 to understanding the product $\vartheta_j \vartheta_{j+1}$. In Section 6.2 we identify the class detecting the product $\vartheta_j \vartheta_{j+1}$ in the algebraic Novikov spectral sequence. In Section 6.3, we provide material which, although not strictly necessary for the proof of Proposition 6.1, is useful for the reader looking to extend our results and methods.

Remark 6.4. The family of differentials we prove in Proposition 6.1 appears as though it might be obtainable inductively via algebraic Steenrod operations. Unfortunately, as the moduli of formal groups is an integral object, the operation we would like to use is not definable without a lift of Frobenius. This does, however, suggest an alternative approach to proving Proposition 6.1 for $j \gg 0$.

Instead of studying $\mathcal{M}_{\mathrm{fg}}$ we can study another closely related object,

$$\mathbb{W}\left(\left(\mathcal{M}_{\mathrm{fg}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)\right)^{\mathrm{perf}}\right),$$

which *does* admit a lift of Frobenius. This is the Witt vectors of the perfection of the reduction mod p of the moduli of formal groups. Although there is no comparison map between these two stacks, it seems that computations on the Witt vector side can be lifted back to the moduli of formal groups in an asymptotic sense (i.e. for $j \gg 0$). On the Witt vector side the lift of Frobenius would let us reduce the proof of the analog of Proposition 6.1 to a single calculation.

[8]See Section 5 for more on this translations of names.

6.1. **Algebraic Novikov differentials in terms of cocycles.**

In Section 2 we introduced the algebraic Novikov spectral sequence by filtering the cobar complex for $\mathrm{BP}_*\mathrm{BP}$ by powers of the augmentation ideal I of BP_*. This spectral sequence takes the form

$$\mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I,\, I^k/I^{k+1}) \cong E_2^{s,k,t'} \Longrightarrow \mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*,\, \mathrm{BP}_*)$$
$$d_r : E_r^{s,k,t'} \longrightarrow E_r^{s+1,k+r-1,t'}.$$

In this subsection we explain how algebraic Novikov differentials can be calculated using cocycles and prove Proposition 6.1 modulo a lemma which we defer to Section 6.2.

Notation 6.5. We write $\mathrm{cb}(-)$ for the cobar complex of a Hopf algebroid and use bar notation for elements of this complex. For example this means that we write $[t_1|t_2^2]$ for the class $t_1 \otimes t_2^2$ in $(\mathrm{BP}_*\mathrm{BP}) \otimes_{\mathrm{BP}_*} (\mathrm{BP}_*\mathrm{BP})$. Note that we are also identifying $\mathrm{BP}_*\mathrm{BP}$ with $\mathrm{BP}_*[t_1, t_2, \dots]$ in the usual way.

We let $I \cdot \mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$ denote the (levelwise) augmentation ideal of this cosimplicial ring. The quotient by this ideal is $\mathrm{cb}(\mathrm{BP}_*\mathrm{BP}/I)$.

Suppose we have a class ϵ on the E_2-page of the algebraic Novikov spectral sequence. For simplicity we will assume that the k-degree of ϵ is zero. The differentials $d_*(\epsilon)$ can be computed using the following procedure:

(1) Pick a cocycle e in $\mathrm{cb}(\mathrm{BP}_*\mathrm{BP}/I)$ which represents ϵ.
(2) Pick a lift e_1 of e to the cobar complex $\mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$. Note that although e is a cocyle, e_1 may not be a cocycle.[9]
(3) Compute $d(e_1)$ in $\mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$ and denote it by c_1. Note that $c_1 \in I \cdot \mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$ since e_1 is a cocycle modulo the augmentation ideal. Now proceed to step (4.1).
(4.r) We currently have e_r with $c_r = d(e_r)$ such that $c_r \in I^r \cdot \mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$. In order to proceed we break into two cases:
- Suppose there exists a $w_r \in I \cdot \mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$ such that
$$d(w_r) \equiv c_r \pmod{I^{r+1}}.$$
Pick such a w_r and proceed to step $(4.r + 1)$ with
$$e_{r+1} = e_r + w_r.$$
(Note that $c_{r+1} = d(e_{r+1}) \in I^{r+1} \cdot \mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$.) This corresponds to the statement that $d_r(\epsilon) = 0$.[10]
- If no such w_r exists, then we may conclude that there is a class
$$\zeta \in \mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^r/I^{r+1})$$
which is detected by c_r and survives to the E_r-page of the algebraic Novikov spectral sequence where it is hit by the differential $d_r(\epsilon) = \zeta$.

In order to apply this procedure to the classes h_j^3 there is one more wrinkle we need to iron out. In the statement of Proposition 6.1 both the source and target of the differential we wish to prove were stated in terms of the names for classes coming from the *Cartan–Eilenberg* E_2*-page*. This means we will have to provide a precise procedure for translating between these naming schemes.

From Section 2 we recall that the isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{F}_2, \mathbb{F}_2[q_0, q_1, \dots]) \xrightarrow{\quad\cong\quad} \bigoplus_{t'+k=t} \mathrm{Ext}^{s,t'}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^k/I^{k+1})$$

---

[9] This is the mechanism by which algebraic Novikov differentials arise.

[10] More specifically, the class w_r can be viewed as a class which supports a differential pre-empting the one on ϵ. Note that this implies that in order to compute the d_r-differential one needs knowledge of all shorter length differentials.

sends a q_i to v_i and uses the fact that $\mathrm{BP}_*\mathrm{BP}/I \cong \mathcal{P}$ in order to provide an isomorphism at the level of cobar complexes. The most subtle point here is that this isomorphism sends t_i to $\chi(\xi_i^2)$ where χ is the antipode on $\mathcal{P}$.

Definition 6.6. Let T_j denote the following class in the cobar complex $\mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$:

$$T_j \coloneqq \frac{1}{2}d(t_1^{2^j}) = \frac{1}{2}\left(\Delta(t_1^{2^j}) - t_1^{2^j} \otimes 1 - 1 \otimes t_1^{2^j}\right).$$

The class T_j is a cocycle since $d^2 = 0$ and the $\mathrm{BP}_*\mathrm{BP}$-cobar complex is torsion free. We will let ϑ_j denote the class on the Adams–Novikov E_2-page detected by T_j. In Example 6.10 we will check that ϑ_j maps to h_{j-1}^2 under the quotient map to the cohomology of $\mathcal{P}$.

For Proposition 6.1 we will need the following lemma whose proof we defer to the next subsection.

Lemma 6.7. *For $j \geq 5$, the product $\vartheta_j\vartheta_{j+1}$ is detected by $q_0^2 g_{j-2}$ in the algebraic Novikov spectral sequence.*

Now we prove Proposition 6.1.

Proof of Proposition 6.1. We follow the procedure outlined above. The first step is picking a lift of h_j^3 to the the cobar complex $\mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$. We pick the class:

$$t_1^{2^j}|T_{j+1}$$

where T_{j+1} is the cocycle given in Definition 6.6 which detects ϑ_{j+1}. We can compute the algebraic Novikov differential on h_j^3 by applying the cobar differential to this cocycle. In particular, we have

$$d(t_1^{2^j}|T_{j+1}) = d(t_1^{2^j})|T_{j+1} + t_1^{2^j}|d(T_{j+1}) = 2T_j|T_{j+1}.$$

Now we need to produce the correction terms for step (4). From Lemma 6.7 we know for $j \geq 5$, $\vartheta_j\vartheta_{j+1}$ is detected by $q_0^2 g_{j-2}$ in the algebraic Novikov spectral sequence. In terms of cocycles this means that there exists a correction term c such that

$$T_j|T_{j+1} + c \in I^3 \cdot \mathrm{cb}(\mathrm{BP}_*\mathrm{BP})$$

and the image of this cocycle in the cohomology of I^3/I^4 is detected by $q_0^3 g_{j-2}$.

Therefore, we have

$$d(t_1^{2^j}|T_{j+1} + 2c) = d(t_1^{2^j})|T_{j+1} + t_1^{2^j}|d(T_{j+1}) + 2d(c) = 2T_j|T_{j+1} + 2d(c)$$

which is detected by $q_0^3 g_{j-2}$. This implies that for $j \geq 5$,

$$d_2(h_j^3) = 0, \quad d_3(h_j^3) = 0, \quad d_4(h_j^3) = q_0^3 g_{j-2}.$$

□

6.2. **A key product relation.**

In this subsection we prove Lemma 6.7, identifying the class detecting the product $\vartheta_j\vartheta_{j+1}$ in the algebraic Novikov spectral sequence in the following refined form.

Lemma 6.8. *For $j \geq 5$, there exists a correction term c_j such that $T_jT_{j+1} + d(c_j)$ is zero modulo $(4, v_1^4)$. Moreover, the image of this cocycle in I^2/I^3 is detected by $q_0^2 g_{j-2}$.*

Remark 6.9. This lemma is sufficient to conclude that $q_0^2 g_{j-2}$ is a permanent cycle in the algebraic Novikov spectral sequence, but does not guarantee that this class is non-zero (even on the E_2-page). For this we need the material from Section 5.

The correction term c_j we use is

$$
\begin{aligned}
c_j &:= [t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^j}] + 7[t_2^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^j}] \\
&+ 2[t_2^{2^{j-2}}|t_2^{2^{j-2}}|t_2^{2^{j-1}}] \\
&+ 2[t_1^{2^{j-2}}|t_1^{2^{j-1}}|t_2^{2^{j-1}}] \cdot \left([t_2^{2^{j-2}}|1|1] + [1|t_2^{2^{j-2}}|1]\right) \\
&+ 2\left([t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^{j-1}}] + [t_1^{2^{j-2}}|t_1^{2^{j-2}}|t_1^{2^j}]\right) \cdot \left([t_2^{2^{j-1}}|1|1] + [1|t_2^{2^{j-1}}|1] + [1|1|t_2^{2^{j-1}}]\right) \\
&+ 2[t_1^{3\cdot 2^{j-2}}|t_1^{5\cdot 2^{j-2}}|t_1^{2^j}] \\
&+ 2[t_1^{2^j}|t_1^{3\cdot 2^{j-1}}|t_1^{2^{j-1}}] + 2[t_1^{3\cdot 2^{j-1}}|t_1^{2^j}|t_1^{2^{j-1}}] \\
&+ 2[t_1^{2^{j-2}}|t_2^{2^{j-2}}|t_1^{2^{j+1}}] + 2[t_1^{2^{j-1}}|t_2^{2^{j-1}}|t_1^{2^j}].
\end{aligned}
$$

We will return to the issue of how we produced this correction term in the next subsection and at that point the reason for our somewhat unnatural choice of line breaks will become clear. The remainder of the proof involves computing $T_j|T_{j+1} + d(c_j)$. We will do this by showing that this cocycle stabilizes for $j \gg 0$ and then using a small computer program to verify the desired conclusion holds in a single (now universal) case directly.

We will need to give formulas for the cobar differential on certain powers of t_1 and t_2 modulo $(8, v_1^4)$. The key observation here is the following congruence of binomial coefficients

$$
\binom{a\cdot 2^{n+k}}{b\cdot 2^n + c} \equiv \begin{cases} \binom{a\cdot 2^k}{b} & c = 0 \\ 0 & c = 1, \dots, 2^n - 1 \end{cases} \pmod{2^{k+1}}.
$$

This congruence allows us to replace cocycles which, a priori, would have a number of terms depending on j with cocycles that are independent of j in a certain sense. As an example to illustrate the point:

$$
\begin{aligned}
d(t_1^{2^{n+3}}) &= \sum_{i=1}^{2^{n+3}-1} \binom{2^{n+3}}{i} [t_1^i|t_1^{2^{n+3}-i}] \\
&\equiv \binom{8}{1}[t_1^{1\cdot 2^n}|t_1^{7\cdot 2^n}] + \binom{8}{2}[t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}] + \binom{8}{3}[t_1^{3\cdot 2^n}|t_1^{5\cdot 2^n}] + \binom{8}{4}[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&\quad + \binom{8}{5}[t_1^{5\cdot 2^n}|t_1^{3\cdot 2^n}] + \binom{8}{6}[t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}] + \binom{8}{7}[t_1^{7\cdot 2^n}|t_1^{1\cdot 2^n}] \\
&\equiv 8[t_1^{1\cdot 2^n}|t_1^{7\cdot 2^n}] + 12[t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}] + 8[t_1^{3\cdot 2^n}|t_1^{5\cdot 2^n}] + 6[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&\quad + 8[t_1^{5\cdot 2^n}|t_1^{3\cdot 2^n}] + 12[t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}] + 8[t_1^{7\cdot 2^n}|t_1^{1\cdot 2^n}] \pmod{16}
\end{aligned}
$$

Example 6.10. Using the mod 4 version of these congruences we can determine the image of ϑ_j under the quotient map to the cohomology of $\mathcal{P}$.

$$
T_j = \frac{1}{2} d(t_1^{2^j}) \equiv [t_1^{2^{j-1}}|t_1^{2^{j-1}}] \pmod{2}
$$

Under the isomorphism between $\mathrm{BP}_*\mathrm{BP}/I$ and $\mathcal{P}$ followed by the isomorphism with $\mathcal{A}$ the class $[t_1^{2^{j-1}}]$ is detected by h_{j-1} and therefore the image of ϑ_j maps to h_{j-1}^2.

This same formula also allows us to rewrite $T_j|T_{j+1}$ modulo 8 in terms of powers of t_1 for $j \geq 3$

$$
\begin{aligned}
T_j|T_{j+1} \equiv\ & 4[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{1\cdot 2^{j-2}}|t_1^{7\cdot 2^{j-2}}] + 4[t_1^{2^{j-2}}|t_1^{3\cdot 2^{j-2}}|t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}] + 2[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}] \\
&+ 4[t_1^{3\cdot 2^{j-2}}|t_1^{2^{j-2}}|t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}] + 4[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-2}}|t_1^{5\cdot 2^{j-2}}] + 4[t_1^{1\cdot 2^{j-3}}|t_1^{7\cdot 2^{j-3}}|t_1^{2^j}|t_1^{2^j}]
\end{aligned}
$$

$$
\begin{aligned}
&+2[t_1^{2^{j-2}}|t_1^{3\cdot 2^{j-2}}|t_1^{2^j}|t_1^{2^j}] \quad + 4[t_1^{3\cdot 2^{j-3}}|t_1^{5\cdot 2^{j-3}}|t_1^{2^j}|t_1^{2^j}] \quad + 1[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^j}|t_1^{2^j}] \\
&+4[t_1^{5\cdot 2^{j-3}}|t_1^{3\cdot 2^{j-3}}|t_1^{2^j}|t_1^{2^j}] \quad + 2[t_1^{3\cdot 2^{j-2}}|t_1^{2^{j-2}}|t_1^{2^j}|t_1^{2^j}] \quad + 4[t_1^{7\cdot 2^{j-3}}|t_1^{1\cdot 2^{j-3}}|t_1^{2^j}|t_1^{2^j}] \\
&+4[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{5\cdot 2^{j-2}}|t_1^{3\cdot 2^{j-2}}] + 4[t_1^{2^{j-2}}|t_1^{3\cdot 2^{j-2}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^{j-1}}] + 2[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^{j-1}}] \\
&+4[t_1^{3\cdot 2^{j-2}}|t_1^{2^{j-2}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^{j-1}}] + 4[t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{7\cdot 2^{j-2}}|t_1^{1\cdot 2^{j-2}} \qquad \pmod 8.
\end{aligned}
$$

Some further formulas we will need are

$$
\begin{aligned}
d(t_1^{3\cdot 2^{n+2}}) &= \sum_{i=1}^{3\cdot 2^{n+2}-1} \binom{3\cdot 2^{n+2}}{i} [t_1^i|t_1^{3\cdot 2^{n+2}-i}] \equiv \sum_{i=1}^{11} \binom{12}{i} [t_1^{i\cdot 2^n}|t_1^{(12-i)\cdot 2^n}] \\
&\equiv 4[t_1^{1\cdot 2^n}|t_1^{11\cdot 2^n}] + 2[t_1^{2\cdot 2^n}|t_1^{10\cdot 2^n}] + 4[t_1^{3\cdot 2^n}|t_1^{9\cdot 2^n}] + 7[t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] + 4[t_1^{6\cdot 2^n}|t_1^{6\cdot 2^n}] \\
&\quad + 7[t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}] + 4[t_1^{9\cdot 2^n}|t_1^{3\cdot 2^n}] + 2[t_1^{10\cdot 2^n}|t_1^{2\cdot 2^n}] + 4[t_1^{11\cdot 2^n}|t_1^{1\cdot 2^n}] \pmod 8,
\end{aligned}
$$

$$
\begin{aligned}
d(t_1^{5\cdot 2^{n+1}}) &= \sum_{i=1}^{5\cdot 2^{n+1}-1} \binom{5\cdot 2^{n+1}}{i} [t_1^i|t_1^{5\cdot 2^{n+1}-i}] \equiv \sum_{i=1}^{9} \binom{10}{i} [t_1^{i\cdot 2^n}|t_1^{(10-i)\cdot 2^n}] \\
&\equiv 2[t_1^{1\cdot 2^n}|t_1^{9\cdot 2^n}] + [t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] + 2[t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}] + 2[t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&\quad + [t_1^{8\cdot 2^n}|t_1^{2\cdot 2^n}] + 2[t_1^{9\cdot 2^n}|t_1^{1\cdot 2^n}] \pmod 4,
\end{aligned}
$$

$$
\begin{aligned}
\Delta(t_2^{2^{n+4}}) &= \left([t_2|1] - [t_1|t_1^2] + v_1[t_1|t_1] + [1|t_2]\right)^{2^{n+4}} \\
&= \sum_{|I|=4,\ \deg(I)=2^{n+4}} (-1)^{i_2} \binom{2^{n+4}}{I} v_1^{i_3} [t_1^{i_2+i_3} t_2^{i_1}|t_1^{2i_2+i_3} t_2^{i_4}] \\
&\equiv \sum_{|I|=4,\ \deg(I)=4} (-1)^{i_2\cdot 2^{n+2}} \binom{4}{I} v_1^{i_3\cdot 2^{n+2}} [t_1^{(i_2+i_3)\cdot 2^{n+2}} t_2^{i_1\cdot 2^{n+2}}|t_1^{(2i_2+i_3)\cdot 2^{n+2}} t_2^{i_4\cdot 2^{n+2}}] \\
&\equiv \sum_{|I|=4,\ \deg(I)=4,\ i_3=0} \binom{4}{I} [t_1^{i_2\cdot 2^{n+2}} t_2^{i_1\cdot 2^{n+2}}|t_1^{i_2\cdot 2^{n+3}} t_2^{i_4\cdot 2^{n+2}}] \\
&\equiv 1[t_2^{16\cdot 2^n}|1] + 4[t_1^{4\cdot 2^n} t_2^{12\cdot 2^n}|t_1^{8\cdot 2^n}] + 4[t_2^{12\cdot 2^n}|t_2^{4\cdot 2^n}] + 6[t_1^{8\cdot 2^n} t_2^{8\cdot 2^n}|t_1^{16\cdot 2^n}] \\
&\quad + 4[t_1^{4\cdot 2^n} t_2^{8\cdot 2^n}|t_1^{8\cdot 2^n} t_2^{4\cdot 2^n}] + 6[t_2^{8\cdot 2^n}|t_2^{8\cdot 2^n}] + 4[t_1^{12\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{24\cdot 2^n}] + 4[t_1^{8\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{16\cdot 2^n} t_2^{4\cdot 2^n}] \\
&\quad + 4[t_1^{4\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{8\cdot 2^n} t_2^{8\cdot 2^n}] + 4[t_2^{4\cdot 2^n}|t_2^{12\cdot 2^n}] + 1[t_1^{16\cdot 2^n}|t_1^{32\cdot 2^n}] + 4[t_1^{12\cdot 2^n}|t_1^{24\cdot 2^n} t_2^{4\cdot 2^n}] \\
&\quad + 6[t_1^{8\cdot 2^n}|t_1^{16\cdot 2^n} t_2^{8\cdot 2^n}] + 4[t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n} t_2^{12\cdot 2^n}] + 1[1|t_2^{16\cdot 2^n}] \pmod{8, v_1^4}.
\end{aligned}
$$

In each case we have indexed the given formula so that it is valid for each $n \geq 0$.

Lemma 6.11. *The value of $T_j|T_{j+1} + d(c_j)$ modulo $(8, v_1^4)$ stabilizes for $j \geq 5$ in the sense that after expanding this cocycle as a sum of monomials in t_1 and t_2 we can increase j by taking a termwise square.*

Proof. We've already expanded $T_j|T_{j+1}$ using the formula for $\Delta(t_1^{2^{n+3}})$. From this we can read off that the conclusion of the lemma is valid for this part of the cocycle once $j \geq 3$. Examining the terms which appear in c_j we see that they are all produced from products and bars of the following terms

$$\underline{t_1^{1\cdot 2^{j-2}}},\ t_1^{2\cdot 2^{j-2}},\ \underline{t_1^{3\cdot 2^{j-2}}},\ t_1^{4\cdot 2^{j-2}},\ \underline{t_1^{5\cdot 2^{j-2}}},\ t_1^{6\cdot 2^{j-2}},\ \underline{t_1^{8\cdot 2^{j-2}}},\ \underline{t_2^{1\cdot 2^{j-2}}},\ t_2^{2\cdot 2^{j-2}}$$

where the underlined terms only appear with coefficient 2. The formulas provided above then give us the desired stabilization once $j \geq 5$. □

Proof (of Lemma 6.8). Using either the program provided in Appendix A or pencil, paper and patience one may give an explicit expansion of the cocycle $T_5|T_6 + d(c_5)$ modulo $(8, v_1^4)$. Applying Lemma 6.11 we can then covert this into an explicit expansion of $T_j|T_{j+1} + d(c_j)$ valid for $j \geq 5$.[11] We reproduce this expansion below (with $n = j-3$ for notational compactness):

$$
\begin{aligned}
&4[t_1^{12\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_1^{8\cdot 2^n}|t_1^{12\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{12\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{14\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_1^{8\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{12\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{10\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{8\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{12\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{12\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{12\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{6\cdot 2^n}] && + 4[t_1^{6\cdot 2^n}|t_1^{9\cdot 2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{11\cdot 2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{5\cdot 2^n}|t_1^{9\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{9\cdot 2^n}|t_1^{3\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{6\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{7\cdot 2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{5\cdot 2^n}|t_1^{3\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{3\cdot 2^n}|t_1^{5\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2^n}|t_1^{7\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{6\cdot 2^n}|t_1^{2^n}|t_1^{9\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{3\cdot 2^n}|t_1^{9\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{5\cdot 2^n}|t_1^{9\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{5\cdot 2^n}|t_1^{2^n}|t_1^{10\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2^n}|t_1^{5\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{2^n}|t_1^{11\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{3\cdot 2^n}|t_1^{2^n}|t_1^{12\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2^n}|t_1^{3\cdot 2^n}|t_1^{12\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{12\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{12\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{12\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{14\cdot 2^n}] && + 4[t_1^{3\cdot 2^n} t_2^{2^n}|t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{3\cdot 2^n}|t_1^{6\cdot 2^n} t_2^{2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{2^n} t_2^{2^n}|t_1^{2\cdot 2^n}|t_1^{64}] && + 4[t_1^{2\cdot 2^n}|t_1^{2^n}|t_1^{2\cdot 2^n} t_2^{2^n}|t_1^{64}] && + 4[t_1^{6\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n} t_2^{2^n}|t_1^{4\cdot 2^n} t_2^{2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_2^{2^n}|t_2^{2^n}|t_1^{64}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{10\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{12\cdot 2^n}] \\
&+ 4[t_1^{2^n}|t_1^{2^n}|t_2^{2\cdot 2^n}|t_1^{64}] && + 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{10\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n} t_2^{2\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{12\cdot 2^n} t_2^{2\cdot 2^n}] && + 4[t_1^{2^n} t_2^{3\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2^n} t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2^n} t_2^{2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2^n}|t_1^{2\cdot 2^n} t_2^{3\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_2^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_1^{4\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_2^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_2^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{6\cdot 2^n}] && + 4[t_1^{2\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_2^{4\cdot 2^n}|t_1^{3\cdot 2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2^n} t_2^{4\cdot 2^n}|t_1^{2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_2^{4\cdot 2^n}|t_1^{2^n}|t_1^{3\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_2^{3\cdot 2^n}|t_2^{2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_2^{2^n}|t_2^{3\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2^n} t_2^{4\cdot 2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{2^n}|t_1^{2^n} t_2^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{4\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{2\cdot 2^n}] && + 4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n} t_2^{4\cdot 2^n}|t_1^{4\cdot 2^n}] \\
&+ 4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_2^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n}|t_1^{2^n}|t_1^{2^n} t_2^{4\cdot 2^n}|t_1^{8\cdot 2^n}] && + 4[t_1^{2\cdot 2^n} t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n} t_2^{2\cdot 2^n}]
\end{aligned}
$$

[11] Note that if one does this expansion by hand it is actually easier to skip over Lemma 6.11 and just use the formulas given prior to that lemma directly.

$$\begin{array}{lll}
+\,4[t_1^{2^n}|t_1^{2^n}|t_1^{2\cdot 2^n}t_2^{4\cdot 2^n}|t_1^{8\cdot 2^n}]x & +\,4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{6\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}] \\
+\,4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}] & +\,4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{8\cdot 2^n}t_2^{2\cdot 2^n}] \\
+\,4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{3\cdot 2^n}|t_1^{5\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}] \\
+\,4[t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}t_2^{4\cdot 2^n}] & +\,4[t_1^{4\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{2^n}|t_1^{2^n}|t_1^{8\cdot 2^n}t_2^{4\cdot 2^n}] \\
+\,4[t_1^{2^n}|t_1^{2^n}|t_1^{2\cdot 2^n}|t_1^{8\cdot 2^n}t_2^{4\cdot 2^n}] & +\,4[t_1^{3\cdot 2^n}t_2^{2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2^n}t_2^{2^n}|t_1^{4\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_1^{2\cdot 2^n}|t_1^{5\cdot 2^n}t_2^{2^n}|t_1^{2\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{3\cdot 2^n}|t_1^{2\cdot 2^n}t_2^{2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2^n}|t_1^{4\cdot 2^n}t_2^{2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_1^{2\cdot 2^n}|t_1^{2^n}t_2^{2^n}|t_1^{6\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{5\cdot 2^n}|t_1^{2\cdot 2^n}t_2^{2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{2^n}|t_1^{6\cdot 2^n}t_2^{2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}] & +\,4[t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}] \\
+\,4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2^n}t_2^{2\cdot 2^n}|t_1^{2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_1^{2\cdot 2^n}t_2^{2^n}|t_2^{2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_2^{2^n}|t_1^{2\cdot 2^n}t_2^{2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}t_2^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_1^{2^n}|t_1^{2^n}t_2^{2\cdot 2^n}|t_1^{4\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_1^{4\cdot 2^n}t_2^{2^n}|t_2^{2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2\cdot 2^n}|t_2^{2^n}|t_1^{4\cdot 2^n}t_2^{2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}|t_1^{2\cdot 2^n}t_2^{2\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2^n}|t_1^{2^n}|t_1^{4\cdot 2^n}t_2^{2\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_2^{2\cdot 2^n}|t_1^{2^n}t_2^{2^n}|t_1^{2\cdot 2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_2^{2\cdot 2^n}|t_1^{2^n}|t_1^{2\cdot 2^n}t_2^{2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2^n}t_2^{2^n}|t_1^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_2^{4\cdot 2^n}] & +\,4[t_1^{2^n}|t_1^{2\cdot 2^n}t_2^{2^n}|t_2^{2\cdot 2^n}|t_2^{4\cdot 2^n}] \\
+\,4[t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}|t_2^{2\cdot 2^n}] & +\,4[t_2^{2\cdot 2^n}|t_2^{2^n}|t_2^{2^n}|t_2^{4\cdot 2^n}] & +\,4[t_2^{2^n}|t_2^{2^n}|t_2^{2\cdot 2^n}|t_2^{4\cdot 2^n}]
\end{array}$$

Since every term is divisible by 4 we learn that $T_j|T_{j+1} + d(c_j)$ is zero modulo $(4, v_1^4)$ as desired. In order to determine which class detects this cocycle in I^2/I^3 we first note that no v_n's appear other than $v_0 = 2$. This means that if we divide this class by 4, view it as an element of the cobar complex of $\mathcal{P}$, and at that point it detects a class H, then the original cocycle was detected by $q_0^2 H$ in the algebraic Novikov spectral sequence.

In order to determine the relevant class in the cohomology of $\mathcal{P}$ we make two observations. First, as a consequence of the results of [Lin08] the only nontrivial element of the cohomology of $\mathcal{P}$ in this bidegree is g_{j-2}, therefore it suffices to show that this class is detected by something nontrivial. Second, in the May spectral sequence g_{j-2} is detected by $h_{2,j-2}^4$ (see [Tan70]). In order to identify our class we would like to pass to the associated graded of the May filtration. However, the t_i's don't directly correspond to the ξ_i's in $\mathcal{P}$—they differ by an application of the anti-involution. Thankfully, this map is the identity on the May E_0-page so we can safely ignore this issue. After filtering out by terms of May filtration 11 and lower we obtain

$$[t_2^{2^{j-2}}|t_2^{2^{j-2}}|t_2^{2^{j-2}}|t_2^{2^{j-2}}] + [t_2^{2^{j-2}}|t_2^{2^{j-3}}|t_2^{2^{j-3}}|t_2^{2^{j-1}}] + [t_2^{2^{j-3}}|t_2^{2^{j-3}}|t_2^{2^{j-2}}|t_2^{2^{j-1}}].$$

Passing to the May E_1-page we can rewrite this as

$$h_{2,j-1}h_{2,j-3}^2 h_{2,j-2} + h_{2,j-2}^4 + h_{2,j-1}h_{2,j-2}h_{2,j-3}^2$$

which is equal to $h_{2,j-2}^4$, finishing the proof. □

Corollary 6.12. *The product $\vartheta_j\vartheta_{j+1}$ is nontrivial on the Adams–Novikov E_2-page.*

Proof. From Lemma 6.7 we know that $\vartheta_j\vartheta_{j+1}$ is detected by $q_0^2 g_{j-2}$ in the algebraic Novikov spectral sequence. This means that it will suffice to show this class isn't hit by a differential.

Examining the information about the E_2 page of the algebraic Novikov spectral sequence from Theorem 3.2(3) the only potential differentials which could hit $q_0^2 g_{j-2}$ are a d_2 differential on $q_0 h_j^3$ or a d_3 differential on h_j^3. We showed that $d_2(h_j^3)$ and $d_3(h_j^3)$ are zero in the Proposition 6.1. □

6.3. Constructing the correction term.

In this short subsection we digress and discuss the choice of the correction term c_j. Although c_j was originally produced by inspection we offer the following as evidence that it is perhaps not so difficult to produce such a term. We hope that this serves as a useful guide to the reader trying to make similar calculations in the future.

First note that we have rigged things so that at each step no v_i appears (other than $v_0 = 2$). What this means is that the associated graded of the algebraic Novikov filtration becomes, as far as we see, the same as the cobar complex for $\mathcal{P}$. What gain does this provide us? Well, we can now filter things by the May filtration and successively add terms to work our way up the May filtration. As an example: The cocycle we started with $T_j|T_{j+1}$ is quite complicated, but mod 2 things are manageable

$$T_j|T_{j+1} \equiv [t_1^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^j}|t_1^{2^j}] \pmod 2$$

We use the term $[t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^j}]$ to swap the middle pair of powers of t_1. Next we add the term $[t_2^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^j}]$ in order to use the May d_1 differential killing $h_{j-1}h_j$.

At this point, after adding $d(-)$ of these two correction terms we get something which is zero mod 2. This means we get to move up one stage in the algebraic Novikov filtration. Now we look at things mod 4, since we get a cocycle mod 4 which is divisible by 2 we consider it again as a cocycle in $\mathcal{P}$. We also have to make a choice of lift of our coefficients on the correction term and we use the ones that appear in c_j (i.e. one and seven). The reasoning behind this choice is related to how we initially wrote down this cocycle (by inspection) and it likely makes little difference in the end what coefficients were chosen.

The next stage involves examining the cocycle

$$T_j|T_{j+1} + d\left([t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^j}] + 7[t_2^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^j}]\right) \pmod 4.$$

The leading term in the May filtration of this cocycle is $2[t_2^{2^{j-2}}|t_2^{2^{j-2}}|t_1^{2^{j-1}}|t_1^{2^j}]$ in May filtration 8. In the May E_1 page this cocycle is detected by $h_{2,j-2}^2 h_{1,j-1}h_{1,j}$ and this class is hit by a May d_1 differential coming off of $h_{2,j-2}^2 h_{2,j-1}$. Using this we conclude that the next correction term we want to add is $2[t_2^{2^{j-2}}|t_2^{2^{j-2}}|t_2^{2^{j-1}}]$.

The leading May filtration of the cocycle

$$T_j|T_{j+1} + d\left([t_1^{2^{j-1}}|t_1^{3\cdot 2^{j-1}}|t_1^{2^j}] + 7[t_2^{2^{j-1}}|t_1^{2^{j-1}}|t_1^{2^j}] + 2[t_2^{2^{j-2}}|t_2^{2^{j-2}}|t_2^{2^{j-1}}]\right) \pmod 4$$

is $2[t_2^{2^{j-2}}|t_1^{2^{j-2}}|t_1^{2^{j-1}}|t_2^{2^{j-1}}] + 2[t_1^{2^{j-2}}|t_1^{2^{j-1}}|t_2^{2^{j-2}}|t_2^{2^{j-1}}]$ in May filtration 8. In the May E_1 page this cocycle is already zero so we add a correction term that swaps the t_1's and t_2's so that they cancel (see the third line of the formula for c_j). After adding this correction the top May filtration is now reduced from 8 to 6.

Repeating this process we eventually eliminate the entire thing—proving that $\vartheta_j\vartheta_{j+1}$ is divisible by 4 (with the desired correction term c_j providing a witness to divisibility in the cobar complex). In Appendix A we have annotated the corrections terms we add with the May lengths of the corresponding May differentials we are using at each step. One of the reasons this process was relatively straightforward was that we used only May d_0 and d_1 differentials.

7. The inductive approach revisited

In this section, which can for the most part be read independently of the rest of the paper, we investigate the fate of the Kervaire invariant one classes in the Adams spectral sequence. The celebrated Hill–Hopkins–Ravenel theorem on Kervaire invariant one tells us that h_j^2 supports a non-trivial Adams differential for every $j \geq 7$ [HHR16]. However, their work provides no further identifying information about these differentials.[12] As a corollary of our study of these classes we provide a new lower bound on the length of the HHR differentials, showing that $d_4(h_j^2) = 0$.

[12]For example, it remains possible that the lengths of these differentials grows without bound as j increases.

We obtain our lower bound by revisiting one of the most promising proposals for constructing Θ_j, the inductive approach of Barratt–Jones–Mahowald [BJM83]. The crux of this approach is a construction which proceeds

$$\begin{pmatrix} \Theta_j \text{ exists} \\ 2\Theta_j = 0 \\ \Theta_j^2 = 0 \end{pmatrix} \implies \begin{pmatrix} \Theta_{j+1} \text{ exists} \\ 2\Theta_{j+1} = 0 \end{pmatrix}.$$

In light of the HHR theorem this approach must break down at some point, meaning at least one of Θ_5^2 or Θ_6^2 is non-zero. The idea we pursue in this section is that it should be possible to run the inductive approach internal to the Adams spectral sequence on a fixed page. The main result of this section, Theorem 7.16, can be informally summarized as saying that,

$$\begin{pmatrix} \theta_j \text{ survives to the } E_r\text{-page} \\ 2\theta_j = 0 \text{ on the } E_r\text{-page} \\ \theta_j^2 = 0 \text{ on the } E_{r-2}\text{-page} \end{pmatrix} \implies \begin{pmatrix} \theta_j \text{ survives to the } E_r\text{-page} \\ 2\theta_j = 0 \text{ on the } E_r\text{-page} \end{pmatrix}.$$

The power of this result lies in the fact that the inductive hypothesis we must verify lies 2 pages prior to the conclusion. This means that if we proceed by induction on r (rather than induction on j) the classes θ_j^2 are already defined for all j at the outset. This opens the possibility that θ_j^2 might be identified on the Adams E_r-page simultaneously for all j and that being non-zero we might, then read off the HHR differentials. In order to make the expected outcome as concrete and precise as possible we make the following conjecture, which is a special case of Conjecture 1.19.

Conjecture 7.1 (Uniform Kervaire differentials conjecture)**.** *There exists a Sq^0-family of classes* $(\mathrm{HHR})_j$ *on the Adams E_2-page (defined for $j \gg 0$) and another class x on the E_2-page such that*

$$d_r(h_j^2) = x \cdot (\mathrm{HHR})_j \neq 0.$$

In order to make the idea of "working on a fixed page of the Adams spectral sequence" rigorous we will work in the category of $\mathbb{F}_2$-synthetic spectra introduced in [Pst18]. In the first subsection of this section we provide a lightning introduction to this category and its connection to the Adams spectral sequence. In the second subsection we run the Barratt–Jones–Mahowald inductive argument internal to the category of $\mathbb{F}_2$-synthetic spectra. In the final subsection we investigate a certain product related to the differential on h_j^3.

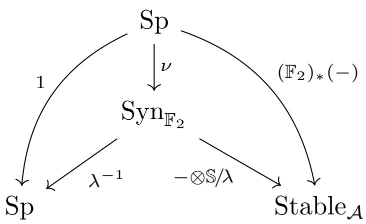
Remark 7.2. It is in this section that the distinction between Θ_j, θ_j and h_j^2 becomes important and for this reason we pause to remind the reader of our conventions. We use Θ_j for the Kervaire invariant one classes in the stable homotopy groups of spheres. We use h_j^2 for the class on the Adams E_2-page. We use θ_j for a choice of class in the synthetic homotopy groups of $\mathbb{S}/\lambda^k$ (which exists when h_j^2 survives to the E_{k+1}-page of the Adams spectral sequence).

7.1. **A lightning introduction to synthetic spectra.**

In this subsection we provide a minimal introduction to the category of $\mathbb{F}_2$-synthetic spectra. For a more complete introduction focusing on the construction of this category see [Pst18]. For a short introduction tailored to using synthetic spectra for computational purposes see [BHS19, Sections 9 and A]. For a more comprehensive account of synthetic homotopy theory see the (forthcoming) book [Bur23]. Since our main application is only in the case of $\mathbb{F}_2$ we will stick to this case throughout our exposition.

7.1.1. *Synthetic spectra.*

Construction 7.3 (Pstrągowski)**.** There is a stable, presentably symmetric monoidal ∞-category $\mathrm{Syn}_{\mathbb{F}_2}$ which fits into a diagram of symmetric monoidal functors



with the following properties:

(1) ν commutes with filtered colimits and sends a cofiber sequence to a cofiber sequence if and only if it induces a short exact sequence on $\mathbb{F}_2$-homology.[13]
(2) The functors $- \otimes \mathbb{S}/\lambda$ and λ^{-1} each commute with colimits.
(3) The functor λ^{-1} is a localization.

For each spectrum X the object νX records all information present in the Adams spectral sequence for X. In fact, one can reasonably think about the object νX as *being* the Adams spectral sequence for X.

7.1.2. *Synthetic homotopy groups.*
Before we can justify our claim that νX records the data present in the Adams spectral sequence for X we will need to introduce a way to measure a synthetic spectrum. For this our preferred method is via the *bigraded synthetic homotopy groups* and the action of the canonical bigraded homotopy element λ on them. For the reader familiar with motivic spectra over $\mathbb{C}$ this pattern should be relatively familiar.

Definition 7.4 ([Pst18, Definitions 4.6 and 4.9])**.** The *bigraded synthetic sphere* $\mathbb{S}^{k,s}$ is defined[14] to be $\Sigma^{-s}\nu S^{k+s}$. As is standard we will omit the superscripts in the case $(0,0)$, using the symbol $\mathbb{S}$ for the monoidal unit of $\mathrm{Syn}_{\mathbb{F}_2}$. For any synthetic spectrum X, the *bigraded homotopy group* $\pi_{k,s}(X)$ is defined to be the abelian group of homotopy classes of maps with source $\mathbb{S}^{k,s}$ and target X.

Definition 7.5 ([Pst18, Definition 4.27])**.** For each spectrum X we have an assembly map $\Sigma\nu X \to \nu\Sigma X$. In the case of S^{-1} this provides us with a canonical map[15]

$$\lambda : \mathbb{S}^{0,-1} \simeq \Sigma\nu S^{-1} \longrightarrow \nu\Sigma S^{-1} \simeq \mathbb{S}^{0,0}\,.$$

The class $\lambda \in \pi_{0,-1}\,\mathbb{S}$ and its behavior is the most important feature of the category of synthetic spectra. As an example say that X is λ-*invertible* if the map

$$\lambda : \Sigma^{0,-1}X \to X$$

is an equivalence. Then, the symmetric monoidal localization functor associated to the map λ is the functor λ^{-1} given above. In particular, the full subcategory of λ-invertible objects is equivalent to the category of spectra. At the opposite extreme, let the symbol $\mathbb{S}/\lambda$ denotes the cofiber of λ, then we have a simple description of the homotopy groups of $\mathbb{S}/\lambda \otimes X$.

Lemma 7.6 ([Pst18, Lemma 4.56])**.** *For any spectrum X, there is a natural isomorphism of bigraded abelian groups*

$$\pi_{t-s,s}(\mathbb{S}/\lambda \otimes \nu X) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2, (\mathbb{F}_2)_*X)$$

between homotopy groups mod λ and the E_2-page of the Adams spectral sequence for X.

[13]This condition on cofiber sequences is exactly the minimal one so that we have a long exact sequence at the level of the Adams E_2-page.

[14]We warn the reader that our conventions differ from those in some previous references. With the conventions used here the two indices of $\mathbb{S}^{k,s}$ are the x and y coordinates in an Adams chart. We have found that in practice these are the most user-friendly conventions.

[15]Typically this class is referred to as τ. Since we make use of both the synthetic and motivic categories, we denote it by λ instead.

In fact, this lemma is only a prelude to [BHS19, Theorem 9.19] which gives a precise translation[16] between Adams spectral sequence data for X and bigraded synthetic homotopy groups for νX. Again for the readers familiar with motivic spectra over $\mathbb{C}$ this pattern should be relatively familiar.

7.1.3. *The λ-Bockstein tower.*

With the class λ in hand we can refine our understanding of the diagram in Construction 7.3.

Theorem 7.7 (Pstrągowski)**.**

(1) The cofiber of λ, which we denote $\mathbb{S}/\lambda$, admits the structure of a commutative algebra in $\mathrm{Syn}_{\mathbb{F}_p}$.

(2) There is a natural symmetric monoidal equivalence

$$\mathrm{Mod}(\mathrm{Syn}_{\mathbb{F}_p};\mathbb{S}/\lambda) \to \mathrm{Stable}_{\mathcal{A}},$$

from the stable ∞-category of modules over $\mathbb{S}/\lambda$ to Hovey's stable ∞-category of comodules over the Steenrod algebra $\mathcal{A}$. The composite of ν with $\mathbb{S}/\lambda \otimes -$ and this equivalence is naturally equivalent to the $\mathbb{F}_p$-homology functor.

In fact, we can go beyond just $\mathbb{S}/\lambda$ and study $\mathbb{S}/\lambda^n$ for varying n. To do this in a coherent way we will need [BHS20, Example C.15] and the surrounding material. As a consequence of this $\mathrm{Syn}_{\mathbb{F}_p}$ has the structure of a locally filtered category in the sense of [Lur15] where the canonical shift map is λ. Through this we make the following construction.

Construction 7.8. The locally filtered structure on $\mathrm{Syn}_{\mathbb{F}_p}$ comes from a symmetric monoidal left adjoint

$$i : \mathrm{Sp}^{\mathbf{Fil}} \to \mathrm{Syn}_{\mathbb{F}_p}$$

which sends $S(1)$ to $\mathbb{S}^{0,1}$. This provides us with a tower of commutative algebras

$$\mathbb{S} \to \cdots \to \mathbb{S}/\lambda^3 \to \mathbb{S}/\lambda^2 \to \mathbb{S}/\lambda.$$

Using $r_{n,m}$ to denote these restriction maps, we have cofiber sequences

$$\Sigma^{0,m-n}\,\mathbb{S}/\lambda^{n-m} \xrightarrow{\lambda^m} \mathbb{S}/\lambda^n \xrightarrow{r_{n,m}} \mathbb{S}/\lambda^m \xrightarrow{\delta_{n,m}} \Sigma^{1,m-n-1}\,\mathbb{S}/\lambda^{n-m}.$$

The commutative algebra structure on $\mathbb{S}/\lambda^n$ provides us with a symmetric monoidal category of modules over this object. In view of the connection between the λ-Bockstein and Adams differentials we think of the category of $\mathbb{S}/\lambda^n$-modules as providing a way to work at the level of the Adams E_{n+1}-page.

Warning 7.9. Outside the $n = 1$ case, the bigraded homotopy groups of $\mathbb{S}/\lambda^n$ are not literally given by the Adams E_{n+1}-page. Instead, thinking through the mechanics of the λ-bockstein spectral sequence one finds that there is a surjective map

$$\pi_{k,s}(\mathbb{S}/\lambda^n) \longrightarrow E_{n+1}^{s,k+s}$$

with kernel generated by the image of $\underline{\lambda}$ together with the λ^{n-1}-torsion classes.

7.1.4. *Examples of synthetic homotopy groups.*

In order to prepare for later subsections we work through several examples of synthetic homotopy groups. Each of our examples will correspond to understanding a small region of the Adams spectral sequence for the sphere near h_j^2. Before proceeding, we suggest the reader new to synthetic spectra look at [BHS19, Section A.2] which works through an example chosen for its instructiveness in full detail.

[16]Again we warn the reader that our grading differs by a shearing from those used in loc. cit. That said, k and s are the same.

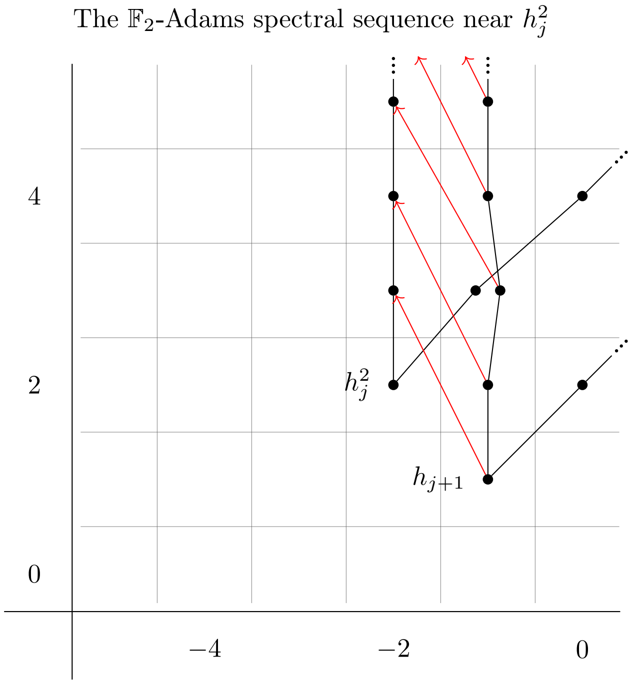


FIGURE 10. The Adams spectral sequence for the sphere near h_j^2 for $j \geq 10$. We have indexed the chart so that 0 on the x-axis corresponds to topological degree 2^{j+1}.

In Figure 10 we display the Adams spectral sequence for the sphere near the class h_j^2 for[17] $j \geq 10$. The structure of the E_2-page in this range can be obtained from [Che11] and the indicated differentials are the Hopf invariant one differentials proved by Adams [Ada60]. The class $h_1 h_{j+1}$ is a permanent cycle detecting the class η_{j+1} in the homotopy groups of spheres constructed by Mahowald [Mah77].

Lemma 7.10. *Near h_j^2 for $j \geq 9$ the synthetic homotopy groups satisfy*

$$\pi_{2^{j+1}-4,4}(\mathbb{S}/\lambda^2) = 0, \qquad \pi_{2^{j+1}-3,4}(\mathbb{S}/\lambda^2) = 0, \qquad \pi_{2^{j+1}-2,4}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\widetilde{2^2\theta_j}\},$$
$$\pi_{2^{j+1}-4,3}(\mathbb{S}/\lambda^3) = 0, \qquad \pi_{2^{j+1}-3,3}(\mathbb{S}/\lambda^3) = 0, \qquad \pi_{2^{j+1}-2,3}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\widetilde{2\theta_j}\},$$
$$\pi_{2^{j+1}-2,2}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\theta_j\},$$

where θ_j is a choice of lift of h_j^2 to $\mathbb{S}/\lambda^3$.

In Figure 11 we display the Adams spectral sequence for the sphere around the classes h_7^2 and h_8^2. As above, the structure of the E_2-page is obtained from [Che11] and the indicated differentials are the Hopf invariant one differentials. Additionally, we will show in Remark 7.17 that $d_3(h_8^2) = 0$.

Lemma 7.11. *Near h_7^2 and h_8^2 the synthetic homotopy groups satisfy*

$$\pi_{252,4}(\mathbb{S}/\lambda^2) \cong \mathbb{F}_2\{\underline{\lambda} V_0'\}, \qquad \pi_{253,4}(\mathbb{S}/\lambda^2) = 0, \qquad \pi_{254,4}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\widetilde{2^2\theta_7}\},$$
$$\pi_{252,3}(\mathbb{S}/\lambda^2) = 0, \qquad \pi_{253,3}(\mathbb{S}/\lambda^3) = 0, \qquad \pi_{254,3}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\widetilde{2\theta_7}\},$$
$$\pi_{252,3}(\mathbb{S}/\lambda^3) \cong \mathbb{F}_2\{\underline{\lambda^2} V_0'\}, \qquad \pi_{254,2}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\theta_7\},$$

[17]In the $j = 9$ case which is not pictured there is a single extra dot, V_2', located at $(-1, 5)$.

The $\mathbb{F}_2$-Adams spectral sequence near h_7^2 and h_8^2

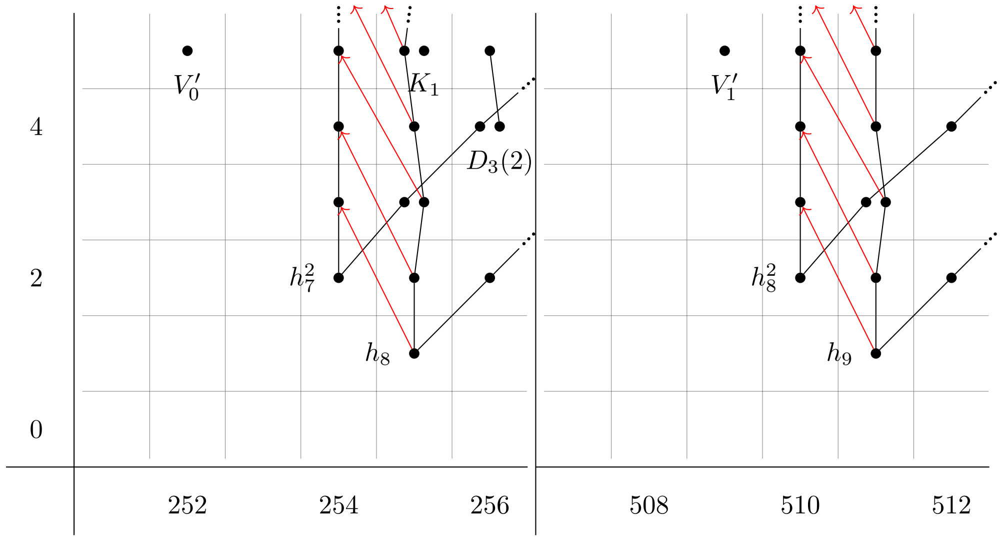


FIGURE 11. Left: The $\mathbb{F}_2$-Adams spectral sequence for the sphere near h_7^2. Right: Near h_8^2. Note that $d_3(h_8^2) = 0$ by Remark 7.17.

where θ_7 is a choice of lift of h_7^2 to $\mathbb{S}/\lambda^3$

$$\pi_{508,4}(\mathbb{S}/\lambda^2) = 0, \quad \pi_{509,4}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\underline{\lambda} V_1'\}, \quad \pi_{510,4}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\widetilde{2}^2\theta_8\},$$
$$\pi_{508,3}(\mathbb{S}/\lambda^3) = 0, \quad \pi_{509,3}(\mathbb{S}/\lambda^2) = 0, \quad \pi_{510,3}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\widetilde{2}\theta_8\},$$
$$\pi_{510,2}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\theta_8\},$$

and θ_8 is a choice of lift of h_8^2 to $\mathbb{S}/\lambda^3$.

We leave it as an instructive exercise for the reader to work out the proofs of Lemmas 7.10, 7.11 using [BHS19, Theorem 9.19].

In Figure 12 we display the Adams spectral sequence for the sphere near the class h_6^2. In this case our knowledge of the E_2-page comes from [Bru] and the d_2 differentials were computed in [Chu21b] using an implementation of an algorithm of Nassau refining Baues' work on algorithmic computation of Adams d_2 differentials. Applying [BHS19, Theorem 9.19] we obtain the following lemma.

Lemma 7.12. *Near h_6^2 the synthetic homotopy groups satisfy*

$$\pi_{124,4}(\mathbb{S}/\lambda^2) = 0, \quad \pi_{125,4}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\underline{\lambda} K_0\}, \quad \pi_{126,4}(\mathbb{S}/\lambda^2) = \mathbb{F}_2\{\widetilde{2}^2\theta_j,\ [D_3(1)]\},$$
$$\pi_{124,3}(\mathbb{S}/\lambda^3) = 0, \quad \pi_{125,3}(\mathbb{S}/\lambda^2) = 0, \quad \pi_{126,3}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\widetilde{2}\theta_j,\ \underline{\lambda}[D_3(1)]\},$$
$$\pi_{125,3}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\underline{\lambda}^2 K_0\}, \quad \pi_{126,2}(\mathbb{S}/\lambda^3) = \mathbb{F}_2\{\theta_j,\ \underline{\lambda}^2 D_3(1)\},$$

where θ_6 is a choice of lift of h_6^2 to $\mathbb{S}/\lambda^3$, $[D_3(1)]$ is a choice of lift of $D_3(1)$ to $\mathbb{S}/\lambda^2$.

Remark 7.13. The reader may have noticed that, somewhat counter-intuitively, we have chosen to present our examples in the opposite of the usual order. Our reason for doing this is to emphasize that the complexity of E_2 page is minimal for the generic element in a Sq^0-family. We expect that this phenomenon is robust.

7.1.5. *Power operations.*

The study of power operations in the Adams spectral sequence began with work of Adams, Barratt and Mahowald on the quadratic construction. These ideas were developed further by several others over the next two decades attaining a relatively stable

The E_2-page of the $\mathbb{F}_2$-Adams spectral sequence near h_6^2

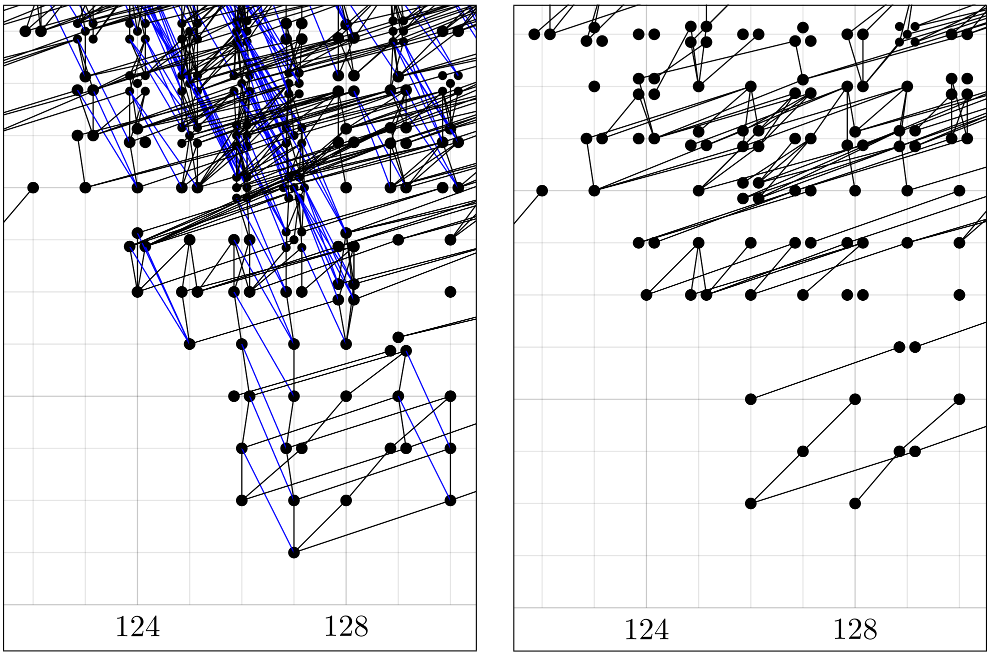


FIGURE 12. In this figure, reproduced from [Chu21a], we display the E_2-page of the Adams spectral sequence near h_6^2 with all d_2 differentials and the corresponding E_3-page.

form with Bruner's treatment in [BMMS86]. In order to give our synthetic inductive argument we will need to introduce the synthetic incarnation of this material.[18]

For the purposes of this paper we only need the simplest example of a power operation: the quadratic construction. This construction, which can be performed in any symmetric monoidal category, takes a map f with target the unit and sends it to the composite

$$X^{\otimes 2}_{hC_2} \xrightarrow{f^{\otimes 2}_{hC_2}} \mathbb{1}^{\otimes 2}_{hC_2} \xrightarrow{\mu} \mathbb{1}$$

where μ is the multiplication map provided by the symmetric monoidal structure.

The quadratic construction uses three things: colimits, the symmetric monoidal structure and the commutative algebra structure on the unit. Since a symmetric monoidal left adjoint preserves each of these we learn that the quadratic construction is compatible with such functors. Now consider the following span of symmetric monoidal left adjoints.

$$\begin{array}{ccccc} & \mathrm{Syn}_{\mathbb{F}_p} & \xrightarrow{\mathbb{S}/\lambda^n \otimes -} & \mathrm{Mod}(\mathrm{Syn}_{\mathbb{F}_p}; \mathbb{S}/\lambda^n) & \\ \swarrow^{\lambda^{-1}} & & & & \searrow^{\mathbb{S}/\lambda \otimes_{\mathbb{S}/\lambda^n} -} \\ \mathrm{Sp} & & & & \mathrm{Stable}(\mathcal{A}) \end{array}$$

Although not immediately obvious, the existence of this span essentially encodes (nearly) all known compatibilites between power operations and the Adams spectral

[18] For a more relaxed introduction see the corresponding chapter in [Bur23]

sequence. As a demonstration of this we consider the example of a map of synthetic spheres.

Example 7.14. Suppose we have a map $\alpha : \mathbb{S}^{k,s} \to \mathbb{S}^{0,0}$. Applying the quadratic construction we obtain an associated map

$$\mathrm{Sq}(\alpha) : (\mathbb{S}^{k,s})^{\otimes 2}_{hC_2} \to \mathbb{S}^{0,0}\,.$$

The usual filtration on the C_2 orbits provides us with a filtration on the source whose associated graded is given by $\mathbb{S}^{2k+i,2s-i}$ in its i^{th} piece. The bottom piece of this filtration is a copy of $\mathbb{S}^{2k,2s}$ and composing with the associated inclusion gives α^2. Inverting λ on $(\mathbb{S}^{k,s})^{\otimes 2}_{hC_2}$ gives us $(S^k)^{\otimes 2}_{hC_2} \simeq \Sigma^k \mathbb{R}\mathrm{P}^\infty_k$ which allows us to work out the attaching maps.[19] If we think about α in terms of the class it represents on inverting λ, then the (changing) values of the s-degree of the cells of the quadratic construction are recording lower bounds on the Adams filtrations of power operations.

Now let's tensor down to $\mathbb{S}/\lambda$. The map α now becomes some class a on the Adams E_2-page. The induced filtration on the quadratic construction now splits because there are no attaching maps of the appropriate degree on the E_2-page, so

$$\mathbb{S}/\lambda \otimes (\mathbb{S}^{k,s})^{\otimes 2}_{hC_2} \simeq \oplus_{i\geq 0}\Sigma^{2k+i,2s-i}\,\mathbb{S}/\lambda.$$

This provides a family of power operations Q_i for $i \geq 0$ where $Q_0(a) = a^2$. Examining [May70] we learn that $Q_i(a) = \mathrm{Sq}^{s-i}(a)$ and $Q_i(a) = 0$ for $i > s$. This tells us that the quadratic construction in the synthetic category unifies the algebraic Steenrod operations on the Adams E_2-page with the quadratic construction in the category of spectra.

Bruner's formulas for Adams differentials on algebraic Steenrod squares can now be read off by examining the λ-Bockstein spectral sequence for $(\mathbb{S}^{k,s})^{\otimes 2}_{hC_2}$ and pushing these differentials forward using $\mathrm{Sq}(\alpha)$.

In general, if we think about X in terms of its bigraded cells, then $\pi_{**}(\nu\,\mathbb{F}_p\,/\lambda \otimes X)$ is a bigraded vector space which records the locations of these cells. Then using the fact that $\nu\,\mathbb{F}_p\,/\lambda \otimes -$ is symmetric monoidal we can work out the cells needed for $X^{\otimes 2}_{hC_2}$.

Example 7.15. Suppose that X has two cells e_1 and e_2 where e_j is in degree (k_j, s_j). Then, $X^{\otimes 2}_{hC_2}$ has cells $Q_i(e_j)$ for $i \geq 0$ in degree $(2k_j + i, 2s_j - i)$ and a single cell e_1e_2 in degree $(k_1 + k_2, s_1 + s_2)$.

7.2. **The inductive approach.**

We now come to the main task of this section: giving a synthetic refinement of the inductive approach to constructing Θ_j. In order to orient the reader we give a brief outline of the main points of the original version of this argument.

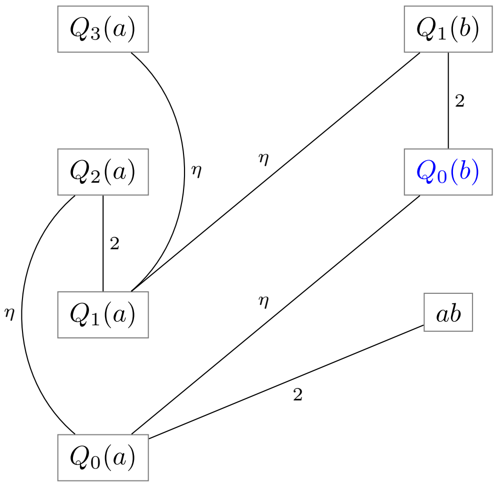
(1) Using the hypothesis that $2\Theta_j = 0$ construct a map

$$\widehat{\Theta}_j : \Sigma^{2^{j+1}-2}\mathbb{S}^0/2 \to \mathbb{S}^0$$

which is Θ_j on the bottom cell.

(2) Apply the quadratic construction to $\widehat{\Theta}_j$. A cell diagram of the source of the resulting map in shown below.

[19] Since all the attaching maps decrease s they are uniquely determined by what happens on inverting λ, i.e. by the attaching maps in a stunted projective space.



Each cell in this diagram is labelled by its associated Dyer–Lashof operation where a is the bottom cell of the copy of $\mathbb{S}^0/2$ and b is the top cell.

(3) Composing $D_2(\widehat{\Theta}_j)$ with the inclusion of the bottom cell gives Θ_j^2 and assuming we have a nullhomotopy of this map, the induced map out of the cell indicated in blue will be Θ_{j+1} and the difference of the pair of cells above it will record a nullhomotopy of $2\Theta_{j+1}$.

Before proceeding further we remark that since inverting λ is symmetric monoidal and commutes with colimits any lift of this procedure to the synthetic category (using maps between spheres) will produce something lying over Θ_j. In essence, all that this would do is record an Adams filtration bound on Θ_{j+1}. The next theorem shows that if we linearize with respect to $\mathbb{S}/\lambda^r$, then the same argument works though with a weaker output and an inductive hypothesis which is easier to check (a substantial victory)!

Theorem 7.16. *Fix an $r \geq 2$ and suppose that θ_j is a lift of h_j^2 from $\mathbb{S}/\lambda$ to $\mathbb{S}/\lambda^r$. If $2\theta_j = 0$ and $\lambda^2\theta_j^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^r)$, then there exists a class θ_{j+1} lifting h_{j+1}^2 to $\mathbb{S}/\lambda^r$ such that $2\theta_{j+1} = 0$ in $\pi_{**}(\mathbb{S}/\lambda^r)$.*

Throughout the proof of Theorem 7.16 we will work in the symmetric monoidal category of $\mathbb{S}/\lambda^r$-modules. In order to simplify notation we will use $\mathbb{1}$ to denote $\mathbb{S}/\lambda^r$ (since this is the monoidal unit of the category) and $\mathbb{1}^{k,s}$ to denote $\Sigma^{k,s}\mathbb{1}$. Before proceeding we pause to point out the key ideas in this proof:

- Although we work in $\mathbb{S}/\lambda^r$-modules, we can often maintain control over maps at the level of their image under the functor $\mathbb{S}/\lambda \otimes_{\mathbb{S}/\lambda^r} -$.
- In $\mathbb{S}/\lambda$-modules we have good control over the quadratic construction since it induces the action of algebraic Steenrod squares on the cohomology of the Steenrod algebra.

Proof. Using a choice of nullhomotopy of $2\theta_j$ we can write down a map

$$\widehat{\theta}_j : \mathbb{1}^{2^{j+1}-2,2}/2 \to \mathbb{1}$$

which is θ_j on the bottom cell. For later use we record the following property (which all choices of $\widehat{\theta}_j$ share):

(∗) Upon tensoring down to $\mathbb{S}/\lambda$ the source of $\widehat{\theta}_j$ splits into two cells. The bottom cell maps to $\mathbb{1}$ by h_j^2 while the top cell maps to $\mathbb{1}$ by h_{j+1} (as long as $r \geq 2$ and $j \geq 3$).

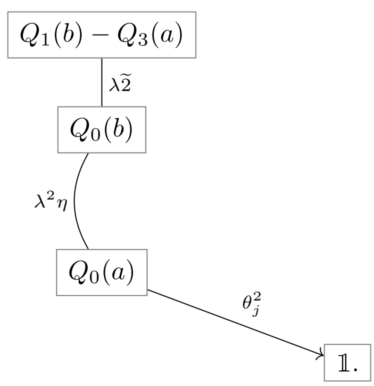
To verify $(*)$ consider the map of λ-Bockstein spectral spectral sequences induced by $\widehat{\theta}_j$. Label the bottom cell of the source by a and the top cell by b. Then, we have λ-Bockstein differential $d_1(b) = h_0 a$. By naturality we learn $d_1(\widehat{\theta}_j(b)) = h_0 h_j^2 \neq 0$, which implies $\widehat{\theta}_j(b)$ is the unique non-zero class in $\mathbb{S}/\lambda$ in this degree—h_{j+1}[20].

Applying the quadratic construction (i.e. the functor $(-)^{\otimes 2}_{hC_2}$) and postcomposing with the multiplication map on $\mathbb{1}$ we obtain a map

$$\mathrm{Sq}(\widehat{\theta}_j) : (\mathbb{1}^{2^{j+1}-2,2}/2)^{\otimes 2}_{hC_2} \to \mathbb{1}$$

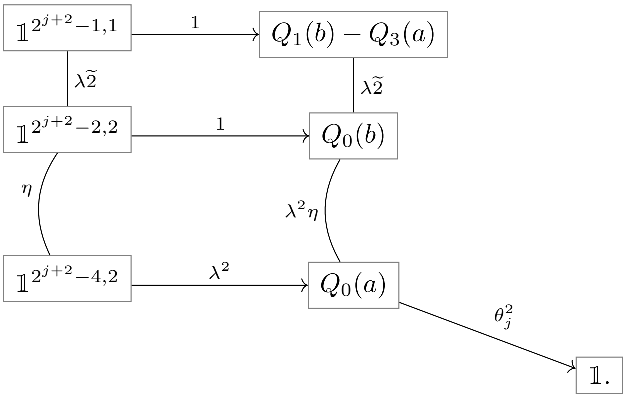
which is θ_j^2 on the bottom cell. Now we make a couple of observations. First, we can read off from Example 7.14 that the cell diagram for $(\mathbb{S}^{2^{j+1}-2}/2)^{\otimes 2}_{hC_2}$ given above is also a cell structure for $(\mathbb{1}^{2^{j+1}-2,2}/2)^{\otimes 2}_{hC_2}$ where the bigradings of the cells have constant weight (the difference of the two coordinates). Second, after tensoring down to $\mathbb{S}/\lambda$ the source splits as a sum of (shifts of) copies of the unit and the Dyer–Lashof operations the cells were labelled by now correspond to the Dyer-Lashof operations (i.e. Steenrod square) they record in $\mathcal{A}$-comodules.

As above, we will label the bottom and top cell of $\mathbb{1}^{2^{j+1}-2,2}/2$ as a and b respectively. Using our identification of what b does on tensoring down to $\mathbb{S}/\lambda$ from $(*)$ we can read off that the cell $Q_0(b)$ in $(\mathbb{1}^{2^{j+1}-2,2}/2)^{\otimes 2}_{hC_2}$ will map to $\mathbb{1}$ via h_{j+1}^2. This means that if we can remove the cells below this one, then we will have a construction of θ_{j+1}. After including a subcomplex of $(\mathbb{1}^{2^{j+1}-2,2}/2)^{\otimes 2}_{hC_2}$ we obtain a map



Note the powers of λ which appear in the attaching maps of this cell structure due to the bigradings of the cells. Next we compose this map with the map that pushes that the λ^2 off the bottom as indicated,

[20] Note that in this argument we have *proved* the Hopf invariant differentials rather than citing them. The reader should compare this argument with the one which can be made using Bruner's power operations formulas.



Using our assumption that $\lambda^2\theta_j^2$ is nullhomotopic we can factor this composite through $\mathbb{1}^{2^{j+2}-2,2}/2$ via a map which is h_j^2 on the bottom cell if we tensor down to $\mathbb{S}/\lambda$. This map is a good choice of $\widehat{\theta}_{j+1}$ and its construction completes the proof. □

Remark 7.17. The condition that $\lambda^2\theta_j^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^r)$ is implied by asking that $\theta_j^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^{r-2})$. This means that as a corollary of the fact that $h_j^4 = 0$ for $j \geq 1$ we learn that h_j^2 lifts to $\mathbb{S}/\lambda^3$. In particular, $d_2(h_j^2) = 0$ and $d_3(h_j^2) = 0$.

Proposition 7.18. *There exist lifts θ_j of the classes h_j^2 to $\mathbb{S}/\lambda^4$ with the property that $2\theta_j = 0$ and $\lambda^2\theta_j^2 = 0$.*[21] *In more classical language the first claim is equivalent to saying that $d_4(h_j^2) = 0$.*

Proof. We proceed by induction on j using Theorem 7.16. The inductive step requires that we check at each step that $\lambda^2\theta_j^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^4)$. In fact we will check the stronger statement that $\theta_j^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^2)$.

For $j \leq 4$ this follows from the fact that $\Theta_j^2 = 0$. If $j \neq 7$, then the conclusion follows from the fact that $\mathrm{Ext}_{\mathcal{A}}^{5,2^{j+2}+1} = 0$ (see Figures 10, 11 and 12).

In the case $j = 7$ we must rule out the possibility that $\theta_6^2 = \underline{\lambda} V_0'$ in $\pi_{252,4}(\mathbb{S}/\lambda^2)$. In Lemma 7.22, in the next subsection, we will give a Toda bracket expression $\theta_6 \in \langle 2, \theta_5, \lambda\theta_5, \widetilde{2}\rangle$ in $\pi_{126,2}(\mathbb{S}/\lambda^2)$. Multiplying by θ_6 and shuffling we have

$$\theta_6^2 \in \theta_6 \cdot \langle 2, \theta_5, \lambda\theta_5, \widetilde{2}\rangle \subseteq \langle\langle\theta_6, 2, \theta_5\rangle, \lambda\theta_5, \widetilde{2}\rangle.$$

To proceed we would like to determine the value of the 3-fold $\langle\theta_6, 2, \theta_5\rangle \in \pi_{189,3}(\mathbb{S}/\lambda^2)$. From Theorem 3.2(1) we can read off that $\pi_{189,3}(\mathbb{S}/\lambda^2) \cong \mathbb{Z}/4\{[h_6^3]\}$. Meanwhile, the synthetic analog of Moss' theorem (see [Bur23]) allows us to conclude that the image of $\langle\theta_6, 2, \theta_5\rangle$ in $\pi_{189,3}(\mathbb{S}/\lambda)$ is zero. As a consequence we have

$$\langle\theta_6, 2, \theta_5\rangle = 2a[h_6^3]$$

for some $a \in \{0, 1\}$. Now, with some further shuffling we may conclude that

$$\theta_6^2 = \langle 2a[h_6^3],\ \lambda\theta_5,\ \widetilde{2}\rangle \subseteq \langle\widetilde{2},\ a[h_6^3]\lambda^2\theta_5,\ \widetilde{2}\rangle = \langle\widetilde{2}, 0, \widetilde{2}\rangle = (\pi_{252,3}\,\mathbb{S}/\lambda^2) \cdot \widetilde{2} = 0$$

where the final step uses Lemma 7.11 which tells us $\pi_{252,3}\,\mathbb{S}/\lambda^2 = 0$. □

Corollary 7.19. *For $j \geq 4$ we have $\delta_{4,1}(h_{j+1}) = \widetilde{2}\theta_j$ in $\pi_{**}(\mathbb{S}/\lambda^3)$.*

Proof. Examining the map induced by $\widehat{\theta}_j$ on λ-Bocksteins in view of $(*)$ provides the desired conclusion. □

[21]As this is the farthest we lift h_j^2 in this paper we will hereafter fix our notation so that θ_j refers to the classes constructed in this proposition.

In the case of h_6^2 where we have a spherical class Θ_5 to work with we can provide a more refined statement.

Proposition 7.20. *The class h_6^2 survives to the E_{r+3}-page of the Adams spectral sequence if and only if $\eta\theta_5^2 = 0$ in $\pi_{**}(\mathbb{S}/\lambda^r)$.*

Proof. We examine the bottom two cells in the final diagram in the proof of Theorem 7.16. The (total) λ-Bockstein in the source of this map records the attaching map so

$$\delta_1(Q_0(b)) = \lambda\eta Q_0(a).$$

By naturality we now learn that

$$\delta_1(h_6^2) = \lambda\eta\Theta_5^2.$$

This implies that h_6^2 is a permanent cycle if and only if $\lambda\eta\Theta_5^2 = 0$ and more specifically that the λ-power divisibility of this product records the length of the Adams differential on h_6^2.

□

Using this proposition and our knowledge of the synthetic stable stems through topological degree 95 can give an analysis of the class $\eta\Theta_5^2$ and a corresponding lower bound on the length of an Adams differential on h_6^2.

Theorem 7.21. *The class $\eta\theta_5^2$ is divisible by λ^6. In particular, through the previous proposition we learn that h_6^2 survives to the Adams E_9-page.*

Proof. Classically, by Corollary 1.10 in [WX17], we have a strictly defined 4-fold Toda bracket for Θ_5:

$$\Theta_5 \in \langle 2, \Theta_4, \Theta_4, 2\rangle.$$

by Corollary 1.3 in [Xu16], we have $2\Theta_5 = 0$, therefore

$$\Theta_5^2 \in \Theta_5 \cdot \langle 2, \Theta_4, \Theta_4, 2\rangle \subseteq \langle\langle\Theta_5, 2, \Theta_4\rangle, \Theta_4, 2\rangle.$$

Working synthetically, we also have

$$2\theta_4 = 0 \in \pi_{30,2}\,\mathbb{S},\ \theta_4^2 = 0 \in \pi_{60,4}\,\mathbb{S},\ 2\theta_5 = 0 \in \pi_{62,2}\,\mathbb{S},$$

$$\langle 2, \theta_4, \theta_4\rangle = 0 \in \pi_{61,3}\,\mathbb{S},\ \theta_5 \in \langle 2, \theta_4, \theta_4, 2\rangle \subseteq \pi_{62,2}\,\mathbb{S},$$

by comparing with classical statements and using that these bidegrees are λ-torsion free. Therefore,

$$\theta_5^2 \in \theta_5 \cdot \langle 2, \theta_4, \theta_4, 2\rangle \subseteq \langle\langle\theta_5, 2, \theta_4\rangle, \theta_4, 2\rangle \subseteq \pi_{124,4}\,\mathbb{S}.$$

In order to run the main argument that proves the theorem we now need six additional facts whose proofs we defer for the moment.

(1) $\pi_{93,5}\,\mathbb{S} \subseteq \lambda^2 \cdot \pi_{93,5}\,\mathbb{S}$. In particular, any element in $\langle\theta_5, 2, \theta_4\rangle \subseteq \pi_{93,3}\,\mathbb{S}$ can be written in the form $\lambda^2 x$, where $x \in \pi_{93,5}\,\mathbb{S}$.
(2) For any x as in (1), we have $x \cdot \theta_4 = 0$ in $\pi_{123,7}\,\mathbb{S}$.
(3) For any x as in (1), any element in $\langle x, \theta_4, 2\rangle \subseteq \pi_{124,6}\,\mathbb{S}$ can be written in the form $\lambda^2 z$, where $z \in \pi_{124,8}\,\mathbb{S}$.
(4) $\pi_{124,8}\,\mathbb{S} \subseteq \lambda \cdot \pi_{124,9}\,\mathbb{S}$.
(5) $\eta \cdot \pi_{124,9}\,\mathbb{S} \subseteq \lambda \cdot \pi_{125,11}\,\mathbb{S}$.
(6) $\eta \cdot \pi_{31,1}\,\mathbb{S} \subseteq \lambda^4 \cdot \pi_{32,6}\,\mathbb{S}$.

Fact (1) lets us rewrite $\theta_5^2 \in \langle\langle\theta_5, 2, \theta_4\rangle, \theta_4, 2\rangle$ as

$$\theta_5^2 \in \langle\lambda^2 x, \theta_4, 2\rangle \text{ for some } x \in \pi_{93,5}\,\mathbb{S}\,.$$

Now, by (2), the bracket $\langle x, \theta_4, 2\rangle$ is well-defined. Choose a y in this bracket, then we have

$$\lambda^2 \cdot y \in \lambda^2 \cdot \langle x, \theta_4, 2\rangle \subseteq \langle\lambda^2 x, \theta_4, 2\rangle.$$

Since both θ_5^2 and $\lambda^2 \cdot y$ are contained in $\langle\lambda^2 x, \theta_4, 2\rangle$, their difference belongs to its indeterminacy, which is $\lambda^2 x \cdot \pi_{31,1}\,\mathbb{S} + 2 \cdot \pi_{124,4}\,\mathbb{S}$. Therefore, we have

$$\theta_5^2 \in \lambda^2 \cdot y + \lambda^2 x \cdot \pi_{31,1}\,\mathbb{S} + 2 \cdot \pi_{124,4}\,\mathbb{S}\,.$$

Using (3) and (4), we may write $y = \lambda^2 \cdot z = \lambda^2 \cdot \lambda \cdot w$ for some $z \in \pi_{124,8}\,\mathbb{S}$ and $w \in \pi_{124,9}\,\mathbb{S}$. Multiplying our expression for θ_5^2 by η we obtain

$$\eta\theta_5^2 \in \lambda^2 \cdot \eta \cdot y + \lambda^2 x \cdot \eta \cdot \pi_{31,1} = \lambda^5 \cdot \eta \cdot w + \lambda^2 x \cdot \eta \cdot \pi_{31,1}\,\mathbb{S}\,.$$

By (5), $\eta \cdot w$ is λ-divisible; by (6), $\eta \cdot \pi_{31,1}$ is λ^4-divisible. Therefore, $\eta\theta_5^2$ is λ^6-divisible.

We now return to proving the six facts from above.

(1) In the 93-stem of the classical Adams sseq both h_5^3 and $h_0 h_5^3$ support non-trivial differentials (see [IWX20b]). Synthetically, this means any element in $\pi_{93,3}\,\mathbb{S}$ is divisible by λ^2.

(2) Classically, $\langle\Theta_5, 2, \Theta_4\rangle \cdot \Theta_4 = \Theta_5 \cdot \langle 2, \Theta_4, \Theta_4\rangle \subseteq \Theta_5 \cdot \pi_{61} = 0$, where the last equality is due to $\pi_{61} S^0 = 0$ (see Theorem 1.9 in [WX17]). This means that synthetically $\lambda^2 x \cdot \theta_4$ is λ-torsion.

From Figure 12, we know that on the classical Adams E_2-page

$$\mathrm{Ext}_{\mathcal{A}}^{s,123+s} = 0 \text{ for } s \leq 7,$$

$$\mathrm{Ext}_{\mathcal{A}}^{s,124+s} = 0 \text{ for } s \leq 5.$$

These facts means that synthetically $\pi_{123,7}\,\mathbb{S}$ contains no non-trivial λ-torsion elements. Therefore, we must have $x \cdot \theta_4 = 0$ in $\pi_{123,7}\,\mathbb{S}$.

(3) It is clear from the classical Adams E_2 and E_3-pages in Figure 12 that the synthetic homotopy group

$$\pi_{124,6}(\mathbb{S}/\lambda^2) \cong \mathbb{F}_2\{a_{124,6},\ \underline{\lambda} a_{124,7},\ \underline{\lambda} b_{124,7}\},$$

where $a_{124,6}$, $a_{124,7}$, $b_{124,7}$ are the generators of the classical E_3-page in the corresponding degrees. We need to rule out these three elements in $\pi_{124,6}(\mathbb{S}/\lambda^2)$.

For $a_{124,6}$, note that x is detected by $h_1 g_3$ or it is further λ-divisible. Then it can be ruled out by the corresponding Massey product

$$\langle h_1 g_3, h_4^2, h_0\rangle = h_1 h_5 g_3 = 0 \neq a_{124,6}$$

in the Adams E_3-page and Moss's theorem (there are no crossing differentials).

For the other two elements, note that both $a_{124,7}$, $b_{124,7}$ on the E_3-page have nonzero h_2-multiples. Therefore, we only need to show that

$$\nu \cdot \langle x, \theta_4, 2\rangle \subseteq \pi_{127,7}(\mathbb{S}/\lambda^2)$$

does not contain (a linear combination of) $\nu \cdot \underline{\lambda} a_{124,7}$ and $\nu \cdot \underline{\lambda} b_{124,7}$.

In fact, since $h_1 g_3 \cdot h_2 = 0$ in $\mathrm{Ext}_{\mathcal{A}}^{6,96+6}$, and $\mathrm{Ext}_{\mathcal{A}}^{7,96+7} = 0$, we know that $\nu \cdot x$ is λ^2-divisible. Therefore, in $\pi_{127,7}(\mathbb{S}/\lambda^2)$,

$$\nu \cdot \langle x, \theta_4, 2\rangle \subseteq \langle\nu \cdot x, \theta_4, 2\rangle = \langle 0, \theta_4, 2\rangle = 2 \cdot \pi_{127,7}(\mathbb{S}/\lambda^2).$$

Factoring 2 as $\lambda\widetilde{2}$ we may use the isomorphism between the Adams E_3-page and the image of multiplication by λ on $\pi_{**}\,\mathbb{S}/\lambda^2$ to analyze 2-divisibility. In particular, we can read off that $\nu \cdot \underline{\lambda} a_{124,7}$ and $\nu \cdot \underline{\lambda} b_{124,7}$ are not 2-divisible in

$\pi_{127,7}(\mathbb{S}/\lambda^2)$ from the fact that $h_2a_{124,7}$ and $h_2b_{124,7}$ are not h_0-divisible on the E_3-page.

(4) This is clear from Figure 12 since the only nonzero element in $\mathrm{Ext}_{\mathcal{A}}^{8,124+8}$ supports a nonzero d_2-differential. Therefore, we have $\pi_{124,8} \subseteq \lambda \cdot \pi_{124,9}\,\mathbb{S}$.

(5) This is clear from Figure 12 since $h_1 \cdot \mathrm{Ext}_{\mathcal{A}}^{9,124+9} = 0$. Therefore, we have $\eta \cdot \pi_{124,9}\,\mathbb{S} \subseteq \lambda \cdot \pi_{125,11}\,\mathbb{S}$.

(6) It is clear from the classical Adams spectral sequence that $\pi_{31,1}\,\mathbb{S}$ is generated by $\lambda^2\eta\theta_4$ and λ^4-multiples. So we only need to show that $\eta \cdot \lambda^2\eta\theta_4$ is λ^4-divisible. This is true since it is zero. In fact, classically $\eta^2\Theta_4 = 0$, so synthetically $\eta^2\theta_4$ must be a λ-torsion or zero. For degree reasons, it must be zero. Therefore, we have $\eta \cdot \pi_{31,1}\,\mathbb{S} \subseteq \lambda^4 \cdot \pi_{32,6}\,\mathbb{S}$.

This completes the proof. □

7.3. A (synthetic) product of θ_j's.

Having constructed synthetic lifts θ_j in $\mathbb{S}/\lambda^4$ we make a short investigation of products among them. More specifically, we show that there is a (non-trivial) product

$$\theta_j\theta_{j-1} = \underline{\lambda}^2 h_0^2 g_{j-2} \neq 0$$

on the Adams E_4-page (i.e. in the homotopy of $\mathbb{S}/\lambda^3$)[22]. Though investigating this product may seem unmotivated, as it will turn out, knowledge of this product is essentially equivalent to Theorem 1.9 less the assertion that the differential is non-trivial.

Lemma 7.22. *For $j \geq 5$ and not 7 the $\mathbb{S}/\lambda^3$-linear 4-fold $\langle \widetilde{2}, \lambda\theta_j, \theta_j, 2\rangle$ is defined, contains θ_{j+1} and has no indeterminacy (except in the $j = 5$ case where $\underline{\lambda}^2 D_3(1)$ may be in the indeterminacy).*

Proof. We start with showing that the 4-fold is defined. In Proposition 7.18 we showed that $2\theta_j = 0$ and $\lambda^2\theta_j^2 = 0$ in $\mathbb{S}/\lambda^4$. This means that the 3-folds $\langle \widetilde{2}, \lambda\theta_j, \theta_j\rangle$ and $\langle \lambda\theta_j, \theta_j, \widetilde{2}\rangle$ are defined. After tensoring down to $\mathbb{S}/\lambda^3$ we observe that since $\pi_{2^{j+2}-3,3}(\mathbb{S}/\lambda^3) = \pi_{2^{j+2}-3,2}(\mathbb{S}/\lambda^3) = 0$ as proved in Lemma 7.10 and Lemma 7.11 both 3-folds are automatically zero.

In order to unambiguously identify the value of this 4-fold we begin by observing that the reduction map $\pi_{2^{j+2}-2,2}(\mathbb{S}/\lambda^3) \to \pi_{2^{j+2}-2,2}(\mathbb{S}/\lambda)$ is an isomorphism (surjective with kernel $\underline{\lambda}^2 D_3(1)$ in the $j = 5$ case). As a consequence it will suffice to understand the value of this bracket after reduction to $\mathbb{S}/\lambda$. The desired conclusion can now be obtained using the synthetic version of Moss' theorem for 4-folds [Bur23]. □

As it would be much more involved to argue that $\langle \widetilde{2}, \lambda\theta_7, \theta_7\rangle$ is zero in $\pi_{**}(\mathbb{S}/\lambda^3)$ we instead prove a slightly modified form of Lemma 7.22 in the $j = 7$ case.

Lemma 7.23. *In $\mathbb{S}/\lambda^3$ the matric 4-fold*

$$\left\langle \widetilde{2}, \begin{pmatrix} \lambda\theta_7 & 0 \end{pmatrix}, \begin{pmatrix} \theta_7 \\ \underline{\lambda}^2 \end{pmatrix}, 2 \right\rangle$$

is defined, contains θ_8 and has zero indeterminacy.

Proof. The proof of this lemma is quite similar to previous one, so we will mostly indicate the necessary modification. Each of the 2-fold products are zero as before. The right 3-fold is then defined and lands in a zero group, therefore it suffices to check that the left 3-fold contains zero. From Lemma 7.11 we know that $\pi_{509,3}(\mathbb{S}/\lambda^3) \cong \mathbb{Z}/2\{\underline{\lambda}^2 V_1'\}$ which

[22] We have passed from $\mathbb{S}/\lambda^4$ to $\mathbb{S}/\lambda^3$ here in order to avoid potential extra terms in the product.

implies that the left 3-fold is equal to its own indeterminacy and therefore it contains zero. In order to evaluate the value of this 4-fold we begin with some shuffling,

$$\left\langle \widetilde{2}, \begin{pmatrix}\lambda\theta_7 & 0\end{pmatrix}, \begin{pmatrix}\theta_7 \\ \underline{\lambda}^2\end{pmatrix}, 2 \right\rangle \subseteq \left\langle \widetilde{2}, \begin{pmatrix}\lambda\theta_7 & 0\end{pmatrix}, \begin{pmatrix}\theta_7\lambda \\ 0\end{pmatrix}, \widetilde{2} \right\rangle.$$

Next we remove the pair of zero maps through $\Sigma^{1,-3}\,\mathbb{S}/\lambda$ reducing to an ordinary 4-fold. Note that in doing this we must take into account the indeterminacy coming from composites which factor through $\Sigma^{1,-3}\,\mathbb{S}/\lambda$.

$$\left\langle \widetilde{2}, \begin{pmatrix}\lambda\theta_7 & 0\end{pmatrix}, \begin{pmatrix}\theta_7\lambda \\ 0\end{pmatrix}, \widetilde{2} \right\rangle \subseteq \left\langle \widetilde{2}, \lambda\theta_7, \theta_7\lambda, \widetilde{2} \right\rangle + [\Sigma^{510,2}\,\mathbb{S}/\lambda^3, \Sigma^{1,-3}\,\mathbb{S}/\lambda]_{\mathbb{S}/\lambda^3} \cdot [\Sigma^{1,-3}\,\mathbb{S}/\lambda, \mathbb{S}/\lambda^3]_{\mathbb{S}/\lambda^3}$$

Using the fact $\mathbb{S}/\lambda^3$ is a free $\mathbb{S}/\lambda^3$-module the adjunction between $\mathbb{S}$-modules and $\mathbb{S}/\lambda^3$-modules allows us to read off that for $f \in [\Sigma^{510,2}\,\mathbb{S}/\lambda^3, \Sigma^{1,-3}\,\mathbb{S}/\lambda]_{\mathbb{S}/\lambda^3}$ and $g \in [\Sigma^{1,-3}\,\mathbb{S}/\lambda, \mathbb{S}/\lambda^3]_{\mathbb{S}/\lambda^3}$ the composite $g \circ f$ depends only on the underlying $\mathbb{S}$-linear composite of f and g. From the cofiber sequence $\mathbb{S}^{1,-3} \to \Sigma^{1,-3}\,\mathbb{S}/\lambda \to \mathbb{S}^{2,-5}$ and the fact that $\pi_{1,-3}\,\mathbb{S}/\lambda^3 = 0$ and $\pi_{2,-5}\,\mathbb{S}/\lambda^3 = 0$ we can read off that $[\Sigma^{1,-3}\,\mathbb{S}/\lambda, \mathbb{S}/\lambda^3] = 0$. Thus, we may conclude that the composite $g \circ f$ is zero and our previous expression simplifies to

$$\left\langle \widetilde{2}, \begin{pmatrix}\lambda\theta_7 & 0\end{pmatrix}, \begin{pmatrix}\theta_7 \\ \underline{\lambda}^2\end{pmatrix}, 2 \right\rangle \subseteq \left\langle \widetilde{2}, \lambda\theta_7, \theta_7\lambda, \widetilde{2} \right\rangle.$$

At point this we can conclude as in the previous lemma by observing that the value of this bracket is uniquely identified after reduction to $\mathbb{S}/\lambda$ and then applying the synthetic version of Moss' theorem for 4-folds [Bur23]. □

Lemma 7.24. *In $\mathbb{S}/\lambda^3$ the product $\theta_j\theta_{j+1}$ is divisible by $\widetilde{2}$ for $j \geq 5$.*

Proof. We begin by handling the case $j \neq 5, 7$. From Lemma 7.11 and Lemma 7.10 we know that $\pi_{2^{j+2}-3,3}\,\mathbb{S}/\lambda^3 = 0$. This implies that $\langle \theta_j, 2, \theta_j \rangle$ is strictly zero. As a consequence of this relation we are able to shuffle the expression for θ_j given in Lemma 7.22 to produce the desired divisibility[23]

$$\langle \widetilde{2}, \lambda\theta_j, \theta_j, 2 \rangle \theta_j = \widetilde{2} \langle \lambda\theta_j, \theta_j, 2, \theta_j \rangle.$$

In the cases $j = 5, 7$ we need a different argument to conclude that $\langle \theta_j, 2, \theta_j \rangle$ contains zero. Since $2\theta_j = 0$ and $\mathbb{S}/\lambda^3$ is an $\mathbb{E}_\infty$-algebra (see Construction 7.8) this bracket contains the product $2 \cdot Q_1(\theta_j)$ where here we are referring to the spherical power operation Q_1 on even degree classes first identified by Toda [Tod62, p.27-28]. Since all elements of $\pi_{125,3}(\mathbb{S}/\lambda^3)$ and $\pi_{509,3}(\mathbb{S}/\lambda^3)$ are simple 2-torsion we know $2Q_1(\theta_j) = 0$ for $j = 5, 7$. This finishes the $j = 5$ case.

In the case $j = 7$ we follow the same basic strategy replacing Lemma 7.22 with Lemma 7.23. In order to shuffle the 4-fold in this case we need the pair of 3-folds

$$\langle \underline{\lambda}^2, 2, \theta_7 \rangle \qquad \text{and} \qquad \langle \theta_7, 2, \theta_7 \rangle$$

to be zero.[24] The first of these brackets is zero since it lands in a zero group and we have already checked the second contains zero. □

The restriction placed on the product of θ_j and θ_{j-1} by Lemma 7.24 allows us to obtain a corresponding restriction on the Adams differentials on h_j^3.

Proposition 7.25. *$\delta_{4,1}(h_j^3)$ is divisible by $\widetilde{2}^2$ for $j \geq 6$. Consequently, for $j \geq 6$,*

(1) $d_2(h_j^3) = 0$,
(2) $d_3(h_j^3)$ *is either* 0 *or* $h_0^2 g_{j-2}$ *and*

[23] Note that shuffling gives a one-to-one correspondence of values of brackets so indeterminacy issues do not arise in this argument.

[24] More precisely, we need that we can always pick the nullhomotopy of the product on the right to make the bracket zero.

(3) $d_4(h_j^3)$ *is either* 0 *or* $h_0^3 g_{j-2}$ *(if defined).*

Proof. From Construction 7.8 we know that $\delta_{4,1}$ arises as the cofiber of the map of rings $r_{4,1} : \mathbb{S}/\lambda^4 \to \mathbb{S}/\lambda$ and therefore that this map is $\mathbb{S}/\lambda^4$-linear. Using that θ_j lifts to $\mathbb{S}/\lambda^4$ (by Theorem 7.16) we may now perform the following manipulations

$$\delta_{4,1}(h_j^3) = \delta_{4,1}(\theta_j h_j) = \theta_j \delta_{4,1}(h_j) = \theta_j \cdot \widetilde{2}\theta_{j-1} \tag{2}$$

where we have used Corollary 7.19 to identify $\delta_{4,1}(h_j)$. Plugging the $\widetilde{2}$ divisibility of $\theta_j\theta_{j-1}$ from Lemma 7.24 into this equation completes the proof of the first claim.

Let $\delta_{4,1}(h_j^3) = \widetilde{2}^2 \cdot y$ with $y \in \pi_{3\cdot 2^j-4,3}\,\mathbb{S}/\lambda^3$. Using our knowledge of the E_2-page from [Che11] (cf. Theorem 3.2(1)) together with Example 1.20 which lets us lift g_{j-2} to $\mathbb{S}/\lambda^2$ (we denote the lift by $[g_{j-2}]$) we can read off that $\pi_{3\cdot 2^j-4,3}\,\mathbb{S}/\lambda^3 \cong \mathbb{Z}/4\{\underline{\lambda}[g_{j-2}]\}$.[25] It follows that $d_2(h_j^3) = 0$ and depending on the value of $y \in \mathbb{Z}/4\{\underline{\lambda}[g_{j-2}]\}$ we fall into one of the following cases

(a) y is a generator and $d_3(h_j^3) = h_0^2 g_{j-2}$,
(b) $y = 2\underline{\lambda}[g_{j-2}]$, $d_3(h_j^3) = 0$ and $d_4(h_j^3) = h_0^3 g_{j-2}$.
(c) $y = 0$, $d_2(h_j^3) = 0$, $d_3(h_j^3) = 0$ and $d_4(h_j^3) = 0$.

□

Now we can reverse the flow of information. From Theorem 1.9 we know that

$$d_4(h_j^3) = h_0^3 g_{j-2} \neq 0,$$

which, when combined with Equation (2), implies:

Corollary 7.26. *In* $\mathbb{S}/\lambda^3$*, there is a product relation,* $\theta_j\theta_{j-1} = \underline{\lambda}^2 h_0^2 g_{j-2} \neq 0$ *for* $j \geq 6$.

The family of products in this corollary are one of the first infinite families of hidden products in the Adams spectral sequence. It seems likely that they bear a close relations to the hidden products θ_j^2 which identify the HHR differentials.

Appendix A. Code

In this appendix we provide a short script which was used in Section 6. This script runs within the computer algebra system Sage and evaluates the cobar differential in the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ modulo v_1 and a power of 2 on terms which include only t_1 and t_2. Ultimately, we know this script gives the correct answer in our case of interest since we have checked the outputs by hand.

- The ring K is the base ring for our computation. We use $\mathbb{Z}/8$ since that is what is relevant for us, but other choices work as well.
- The rings A and B represent the third and fourth layers of the cobar complex respectively.
- The modifiers a, b, c or w, x, y, z indicate which tensor factor a given variable comes from.
- The cobar complex is a cosimplicial ring and the various di's are relevant face maps of this cosimplicial ring.
- The term E corresponds to the cocycle $T_j|T_{j+1}$ from Section 6.
- The final line (which is expanded and printed on evaluation) is the cocycle $T_j|T_{j+1} + d(c_j)$.
- We have annotated the various terms which make up c_j with the length of the May differential they are involved in (see Section 6.3).

[25] In the case $j = 6$ there is an additional summand $\mathbb{Z}/2\{\underline{\lambda}^2(h_7 D_3(0))\}$, but this class is $\widetilde{2}$-torsion and so does not affect the value of $\widetilde{2}^2 \cdot y$.

```
K = Zmod(8);
A.<t1a, t1b, t1c, t2a, t2b, t2c> = K[];
B.<t1w, t1x, t1y, t1z, t2w, t2x, t2y, t2z> = K[];

d0 = A.hom([t1x, t1y, t1z, t2x, t2y, t2z], B);
d1 = A.hom([t1w + t1x, t1y, t1z, t2w - t1w * t1x^2 + t2x, t2y, t2z], B);
d2 = A.hom([t1w, t1x + t1y, t1z, t2w, t2x - t1x * t1y^2 + t2y, t2z], B);
d3 = A.hom([t1w, t1x, t1y + t1z, t2w, t2x, t2y - t1y * t1z^2 + t2z], B);
d4 = A.hom([t1w, t1x, t1y, t2w, t2x, t2y], B);

def cobar_diff(x):
    return d0(x) - d1(x) + d2(x) - d3(x) + d4(x);

def make_theta_left(k):
    return 4 * t1w^(1 * 2^(k)) * t1x^(7 * 2^(k))
         + 6 * t1w^(2 * 2^(k)) * t1x^(6 * 2^(k))
         + 4 * t1w^(3 * 2^(k)) * t1x^(5 * 2^(k))
         + 3 * t1w^(4 * 2^(k)) * t1x^(4 * 2^(k))
         + 4 * t1w^(5 * 2^(k)) * t1x^(3 * 2^(k))
         + 6 * t1w^(6 * 2^(k)) * t1x^(2 * 2^(k))
         + 4 * t1w^(7 * 2^(k)) * t1x^(1 * 2^(k));

def make_theta_right(k):
    return 4 * t1y^(1 * 2^(k)) * t1z^(7 * 2^(k))
         + 6 * t1y^(2 * 2^(k)) * t1z^(6 * 2^(k))
         + 4 * t1y^(3 * 2^(k)) * t1z^(5 * 2^(k))
         + 3 * t1y^(4 * 2^(k)) * t1z^(4 * 2^(k))
         + 4 * t1y^(5 * 2^(k)) * t1z^(3 * 2^(k))
         + 6 * t1y^(6 * 2^(k)) * t1z^(2 * 2^(k))
         + 4 * t1y^(7 * 2^(k)) * t1z^(1 * 2^(k));

# The n which appears here is related to j by n = j - 3.
n = 2;

theta_left = make_theta_left(n);
theta_right = make_theta_right(n+1);

E = theta_left * theta_right;

term1 = t1a^(4 * 2^n) * t1b^(12 * 2^n) * t1c^(8 * 2^n);
term2 = 7 * t2a^(4 * 2^n) * t1b^(4 * 2^n) * t1c^(8 * 2^n);
term3 = 2 * t2a^(2 * 2^n) * t2b^(2 * 2^n) * t2c^(4 * 2^n);
term4 = 2 * t1a^(2 * 2^n) * t1b^(4 * 2^n) * t2c^(4 * 2^n)
          * (t2a^(2 * 2^n) + t2b^(2 * 2^n));
term5a = 2 * t1a^(4 * 2^n) * t1b^(4 * 2^n) * t1c^(4 * 2^n);
term5b = 2 * t1a^(2 * 2^n) * t1b^(2 * 2^n) * t1c^(8 * 2^n);
term5 = (term5a + term5b) * (t2a^(4 * 2^n) + t2b^(4 * 2^n) + t2c^(4 * 2^n));
term6 = 2 * t1a^(6 * 2^n) * t1b^(10 * 2^n) * t1c^(8 * 2^n);
term7 = 2 * t1a^(8 * 2^n) * t1b^(12 * 2^n) * t1c^(4 * 2^n)
      + 2 * t1a^(12 * 2^n) * t1b^(8 * 2^n) * t1c^(4 * 2^n);
term8 = 2 * t1a^(2 * 2^n) * t2b^(2 * 2^n) * t1c^(16 * 2^n)
      + 2 * t1a^(4 * 2^n) * t2b^(4 * 2^n) * t1c^(8 * 2^n);
```

```
correction = term1 + term2; # first row
correction += term3; # may d1
correction += term4; # may d0
correction += term5; # may d0
correction += term6; # may d0
correction += term7; # may d0
correction += term8; # may d1
E + cobar_diff(correction);
```

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