THE EQUIVARIANT COHOMOLOGY FOR SEMIDIRECT PRODUCT ACTIONS

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ABSTRACT. The rational Borel equivariant cohomology for actions of a compact connected Lie group is determined by restriction of the action to a maximal torus. We show that a similar reduction holds for any compact Lie group G when there is a closed subgroup K such that the cohomology of the classifying space BK is free over the cohomology of BG for field coefficients. We study the particular case when G is a semi-direct product and K is its maximal elementary abelian 2-subgroup for cohomology with coefficients in a field of characteristic two. This provides a different approach to investigate the syzygy order of the equivariant cohomology of a space with a torus action and a compatible involution, and we relate this description with results for 2-torus actions.

1. INTRODUCTION

Let G be a compact group and X be a finite G-CW complex. The G-equivariant cohomology of X with coefficients over a field $\Bbbk$ is defined as the singular cohomology of the homotopy quotient $H^*_G(X;\Bbbk) := H^*(EG \times_G X;\Bbbk)$. It becomes canonically a module over the cohomology of the classifying space $H^*(BG;\Bbbk)$ (cohomology coefficients will be omitted as long as there is no ambiguity). We say that X is G-equivariantly formal over $\Bbbk$ if the restriction map $H^*_G(X) \to H^*(X)$ is surjective. In this case, the Leray-Hirsch theorem implies that $H^*_G(X)$ is a free module over $H^*(BG)$.

Freeness of the equivariant cohomology has been generalized to the study of syzygy modules. A detailed discussion of this topic was started by Allday-Franz-Puppe [3] for torus actions with cohomology over a field of characteristic zero. Recall that a finitely generated module M over a commutative ring R is a j-th syzygy if there is an exact sequence

$$0 \to M \to F_1 \to \cdots \to F_j \tag{1.1}$$

of free modules F_k for $1 \leqslant k \leqslant j$. If R is a polynomial algebra in n variables over a field $\Bbbk$, then the n-th syzygy modules correspond to the free ones as a consequence of the Hilbert Syzygy theorem. In [3, Thm.5.7], the authors showed that the syzygy order of the equivariant cohomology of a space with a torus action is equivalent to the partial exactness of the Atiyah-Bredon sequence of such a space [5] [11]. This sequence is defined in the following way: for a T-space X where $T = (S^1)^n$, the filtration of X by the dimension of its orbits $X_0 = X^T \subseteq X_1 \subseteq \cdots \subseteq X_n = X$

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induces a sequence

$$0 \to H^*_T(X) \to H^*_T(X^T) \to H^{*+1}(X_1, X_0) \to \cdots \to H^{*+n}(X_n, X_{n-1}).$$

These results in equivariant cohomology for torus actions are key to extend the study of syzygies in equivariant cohomology for any compact connected Lie group actions by considering the restriction of the action to a maximal tori [22], and elementary p-abelian groups actions by restriction and transfer of the action to a torus one [4].

In this paper, we study a more general problem of characterizing syzygies in equivariant cohomology for compact Lie group actions in terms of the action to a suitable closed subgroup $K \subseteq G$, and considering coefficients over an arbitrary field $\Bbbk$. This generalizes the relation between compact connected Lie groups (resp. p-abelian groups) and their maximal tori in equivariant cohomology as discussed before. The methods developed in this paper also provide a new tool to recover these results.

Let us consider a compact Lie group G and let $K \subseteq G$ be a closed subgroup. We denote by $W = N_G(K)/K$ the Weyl group of K in G. Let us consider cohomology with coefficients over a field $\Bbbk$ (that we omit it in our notation). Suppose that the canonical map $H^*(BK) \to H^*(G/K)$ arising from the fibration $G/K \to BK \to BG$ is surjective and that there is an isomorphism of algebras $H^*(BK)^W \cong H^*(BG)$. The following result provides a general reduction in equivariant cohomology.

Theorem 1.1. *Let X be a G-space such that $H^*(X)^G = H^*(X)$, then W acts on the K-equivariant cohomology of X and there is a natural isomorphism of $H^*(BG)$-algebras $H^*_G(X) \cong H^*_K(X)^W$ and a natural isomorphism of $H^*(BK)$-algebras $H^*_K(X) \cong H_G(X) \otimes_{H^*(BG)} H^*(BK)$.*

Observe that the condition of the map $H^*(BK) \to H^*(G/K)$ being surjective allows us to describe the cohomology of the homogeneous spaces G/K in terms of the cohomology of the classifying space of G and K. This applies to the cohomology of homogeneous spaces of Lie groups [15], [29], [8] and for the equivariant cohomology of Hamiltonian actions of non-abelian compact connected Lie groups in symplectic geometry [7]. A pair of groups (G, K) satisfying this condition will be called a *free extension pair* following the notation introduced in [7].

As a consequence of Theorem 1.1, we can characterize syzygies in G-equivariant cohomology in terms of the K-equivariant cohomology.

Corollary 1.2. *The module $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$ if and only if $H^*_K(X)$ is a j-th syzygy over $H^*(BK)$.*

With our methods, we recover in equivariant cohomology classical results for compact connected Lie groups with cohomology over rational coefficients [30, Ch.III.Lem.4.12], or for finite groups with abelian p-Sylow subgroup with cohomology over a field of characteristic p [12, Ch.III.Thm.10.3]. Moreover, they also allow us to study the equivariant cohomology for actions of semi-direct product of groups or groups fitting in a group extension. For example, actions of matrix groups, dihedral and symmetric groups and torus action with compatible involutions. In particular, the latter case has been of interest in symplectic geometry. Namely, let M be a symplectic manifold with a symplectic action of a torus T and an antisymplectic compatible involution τ. The maximal elementary 2-abelian subgroup T_2 of T acts on the fixed point M^τ and if M is T-equivariantly formal over $\mathbb{Q}$, then M^τ is T_2-equivariantly formal over $\mathbb{F}_2$ [20],[6], [18].

This situation motivates our study of the equivariant cohomology for semi-direct product actions; in particular, we use it to approach the symplectic setting described above by considering the equivariant cohomology for actions of the group $T \rtimes \mathbb{Z}/2\mathbb{Z}$. We first prove a canonical description of the $\mathbb{F}_2$-cohomology of the classifying space $B(T \rtimes \mathbb{Z}/2\mathbb{Z})$ as a tensor product of the cohomologies of BT and $B\mathbb{Z}/2\mathbb{Z}$. Moreover, we describe the equivariant formality and syzygies in equivariant cohomology for actions of this group in terms of the maximal 2-elementary subgroup K and the invariants under the action of the Weyl group $W = N_G(K)/K$ as stated in the following result.

Theorem 1.3. *Let $G = T \rtimes \mathbb{Z}/2\mathbb{Z}$, let H be the maximal 2-elementary subgroup of G and W be the Weyl group of H in G. Then (G, H) is a free extension pair over $\mathbb{F}_2$, and there is an isomorphism of algebras $H^*(BG) \cong H^*(BH)^W$ that extends to equivariant cohomology for any G-space X. In particular, $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$ if and only if $H^*_H(X)$ is a j-th syzygy over $H^*(BH)$.*

As consequences of this result, we recover the equivariant formality of the real locus of conjugation spaces [26] and Hamiltonian torus actions on symplectic manifold when the cohomology of the space contains no 2-torsion.

This document is organized as follows: In Section 2, we discuss free extension pairs and the reduction of syzygies in equivariant cohomology for a pair of groups satisfying this property. In Section 3, we approach torus actions and compatible involutions by looking at the induced action of the semi-direct product of a torus and a 2-tori. Syzygies in equivariant cohomology for actions of such groups can be reduced to 2-torus actions as discussed in Section 4. In Section 5, we discuss a topological generalization of Hamiltonian actions on a symplectic manifold with an anti-symplectic compatible involution using the results discussed in the previous sections. Finally, in the last section of this document, we study a canonical semi-direct product action on the big polygon spaces to realize all possible syzygy orders in equivariant cohomology analogous to the torus actions case [21] and we will relate these results to the real big polygon spaces recovering some of the work discussed in [31].

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2. Free extension pairs

Let G be a compact connected Lie group and T be a maximal torus in T. The rational equivariant cohomology of a G-space X is completely determined by the induced action of T on X; namely, there is an isomorphism of $H^*(BT;\mathbb{Q})$-algebras

$$H^*_T(X;\mathbb{Q}) \cong H^*_G(X;\mathbb{Q}) \otimes_{H^*(BG;\mathbb{Q})} H^*(BT;\mathbb{Q}) \tag{2.1}$$

and an isomorphism of $H^*(BG;\mathbb{Q})$-algebras

$$H^*_G(X;\mathbb{Q}) \cong H^*_T(X;\mathbb{Q})^W \tag{2.2}$$

where $W = N_G(T)/T$ is the Weyl group of T in G, and the action of W on T by conjugation extends to one on the Borel construction X_T [28, Thm.2.2]. Such an isomorphism also holds over $\mathbb{Z}$ if $H^*(G;\mathbb{Z})$ contains no torsion and over $\mathbb{F}_p$ if $H^*(G;\mathbb{F}_p)$ contains no p-torsion as generalized in [10, Ch.VII]. These results from Leray and Borel are discussed for principal G-bundles and can be adapted to the equivariant cohomology setting,

The isomorphism (2.1) can be used to characterize free and torsion-free modules in G-equivariant cohomology, and more generally, the syzygy modules for any compact connected Lie group in terms of the restricted action to the maximal torus T in G [22, Prop.4.2]. The key ingredient is the fact that $H^*(BT;\mathbb{Q})$ becomes a free module over $H^*(BG;\mathbb{Q})$ via the map induced by the inclusion $T \to G$. We generalize this situation by introducing the following definition.

Definition 2.1. *Let G be a compact Lie group and $K \subseteq G$ be a closed subgroup. The pair (G,K) has the free extension property over a field $\Bbbk$ if the map $H^*(BK;\Bbbk) \to H^*(G/K;\Bbbk)$ is surjective.*

Observe that this is equivalent to the degeneracy at the E_2-term of the cohomological Serre spectral sequence associated to the fibration $G/K \to BK \to BG$ and the trivial action of G on G/K in cohomology. In particular, $H^*(BK;\Bbbk)$ becomes a finitely generated free $H^*(BG;\Bbbk)$-module by the Leray-Hirsch theorem. Recall that we will often omit the coefficient field $\Bbbk$ in our notation for cohomology.

Proposition 2.2. *Let $K \subseteq H \subseteq G$ be a sequence of groups such that (G,H) and (H,K) are free extension pairs. The following statements are equivalent.*

1. *(G,K) is a free extension pair.*
2. *The action of G on $H^*(G/K)$ is trivial.*
3. *The cohomological Serre spectral sequence arising from the fibration $H/K \to G/K \to H/K$ degenerates at E_2.*

Proof. For any space X, let $P_X(t)$ denote the Poincaré series of X with coefficients in the field $\Bbbk$. As (G,H) and (G,K) are free extension pairs, we get that $P_{BK}(t) = P_{BG}(t)P_{G/H}(t)P_{H/K}(t)$ and so $H^*(BK)$ is a free module of rank $b(G/H)b(H/K)$. This implies that the cohomological Serre spectral sequence arisen from the fibration $G/K \to BK \to BG$ degenerates at E_2 and $H^*(BK) \cong H^*(BG;H^*(G/K)) \cong H^*(BG) \otimes H^*(G/K)^G$ as $H^*(BG)$-modules. Then $P_{G/H}(t)P_{H/K}(t) = P_{G/K}(t)$ if and only if G acts trivially on the cohomology of G/K. □

Proposition 2.3. *Let $F \to E \xrightarrow{p} B$ be a fibration such that the map $H^*(E) \to H^*(F)$ is surjective. Let X be a connected space and $f\colon X \to B$ be a continuous map. Then in the pullback fibration $F \to X_f := f^*E \xrightarrow{q} X$, the map $H^*(X_f) \to H^*(F)$ is also surjective and there is an isomorphism of $H^*(E)$-modules $H^*(X_f) \cong H^*(X) \otimes_{H^*(B)} H^*(E)$.*

Proof. The surjectivity of the map $H^*(X_f) \to H^*(F)$ follows from the commutativity of the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & X_f \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ F \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

We can choose an additive section $\alpha\colon H^*(F) \to H^*(E)$ of the surjective map $i^*\colon H^*(E) \to H^*(F)$ that induces an isomorphism of $H^*(B)$-modules $\theta\colon H^*(B) \otimes H^*(F) \to H^*(E)$ given by $\theta(a \otimes t) = p^*(a)\alpha(t)$ using the Leray-Hirsch theorem. Similarly, the composite $\beta = g^* \circ \alpha$ is an additive section of $j^*\colon H^*(X_f) \to H^*(F)$ and there is an induced isomorphism of $H^*(X)$-modules $\phi\colon H^*(X) \otimes H^*(F) \to H^*(X_f)$ given by $\phi(b \otimes t) = q^*(b)\beta(t)$. Under these choices, there is a commutative diagram

$$\begin{tikzcd} H^*(B) \otimes H^*(F) \arrow[r, "\theta"] \arrow[d, "f^* \otimes id"] & H^*(E) \arrow[d, "g^*"] \\ H^*(X) \otimes H^*(F) \arrow[r, "\phi"] & H^*(X_f) \end{tikzcd}$$

Now we will show that the canonical map $K\colon H^*(X) \otimes_{H^*(B)} H^*(E) \to H^*(X_f)$ given by $b \otimes x \mapsto q^*(a)g^*(x)$ is an isomorphism. It follows from the commutative diagram

$$\begin{tikzcd} H^*(X) \otimes H^*(F) \arrow[rr, "\phi"] \arrow[d, "\cong"] & & H^*(X_f) \arrow[dd, equal] \\ H^*(X) \otimes_{H^*(B)} H^*(B) \otimes H^*(F) \arrow[d, "id \otimes \theta"] & & \\ H^*(X) \otimes_{H^*(B)} H^*(E) \arrow[rr, "K"] & & H^*(X_f) \end{tikzcd}$$

where all maps are isomorphisms. That the map K is one of $H^*(E)$-modules follows from naturality of the construction with respect to the map $f\colon X \to B$ and considering the particular case $id\colon B \to B$. □

The following result is a particular case of the previous proposition.

Proposition 2.4. *Let (G, K) be a free extension pair over $\Bbbk$ and X be a G-space. There is a natural isomorphism of $H^*(BK)$-modules.*

$$H^*_K(X) \cong H^*_G(X) \otimes_{H^*(BG)} H^*(BK) \tag{2.3}$$

where X is a K-space by restriction of the G-action.

Proof. The Borel constructions X_K and X_G sit in a pullback diagram

$$\begin{tikzcd} X_K \arrow[r] \arrow[d] & X_G \arrow[d] \\ BK \arrow[r] & BG \end{tikzcd}$$

where the horizontal maps are fibrations with fiber G/K. Since (G, K) is a free extension pair, the map $H^*(BK) \to H^*(G/K)$ is surjective and thus the isomorphism (2.3) follows by applying Proposition 2.3.

□

This result allows us to describe the syzygies in G-equivariant cohomology in terms of the K-equivariant cohomology analogously to the reduction from non-abelian compact connected Lie group actions to torus actions [22, Prop.4.2] as we state in the following result.

Proposition 2.5. *Let (G, K) be a free extension pair and X be a G-space. For any $j \geqslant 1$, $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$ if and only if $H^*_K(X)$ is a j-th syzygy over $H^*(BK)$.*

Proof. We use the characterization of syzygies via regular sequences [13, §16.E] and the following algebraic fact: Let R, S be rings such that S is a free finitely generated R-module. Let A be an S-algebra and B and R-algebra such that $A \cong B \otimes_R S$ as S-modules. Then A is a j-th syzygy over S if and only if B is a j-th syzygy over R. The result then follows by combining these facts, the remark after Definition 2.1 and Proposition 2.4. □

Besides the example where G is a compact connected Lie group and $K = T$ is the maximal torus, we can also find free extension pairs in the following situations.

Example 2.6. Let $\Bbbk$ denote field of characteristic p and $n > 1$ any integer. In the following cases (G, K) is a free extension pair.

- When G is a torus and K is the maximal elementary abelian p-subgroup of K [16, §5].
- When G is a finite abelian group and K is the subgroup isomorphic to the product of cyclic groups of order divisible by p in the elementary decomposition of G [12, Ch.III.§10].
- . Let $p = 2$. When $G = O(n), SO(n), U(n)$ or $SU(n)$ and K is the maximal elementary abelian p-subgroup of G [30, Ch.III.§4].
- Let $p = 2$. When $G = SU(2)$ and $H = Q_8$ is the quaternionic group [1, Ex.2.10].

Proposition 2.7. *Let $n \geqslant 1$ and G_n, K_n be one of the following pair of groups*
(a) Let $G_n = SO(n)$ and $K_n = O(n)$.
(b) Let $G_n = SU(n)$ and $K_n = U(n)$.
(c) Let $G_n = Sp(n)$ and $K_n = Sp(n) \rtimes Sp(1)$.
There is an embedding of K_n on G_{n+1} such that (G_{n+1}, K_n) is a free extension pair over $\mathbb{F}_2$ in case (a) and over an arbitrary field in cases (b) and (c).

Proof. Let $\Sigma_n = \mathbb{R}P^n, \mathbb{C}P^n$ or $\mathbb{H}P^n$ and $L = O(1), U(1)$ or $Sp(1)$ in cases (a), (b) or (c) respectively. There is a transitive action of G_{n+1} on Σ_n and thus the equivariant cohomology $H^*_{G_{n+1}}(\Sigma_n)$ is isomorphic to $H^*(B(G_{n+1})_x)$. Using homogeneous coordinates on Σ_n, we see that the isotropy group of $x = [0 : \cdots : 0 : 1]$ is given by $(G_{n+1})_x \cong K_n$ and thus the inclusion map $(G_{n+1})_x \to G_{n+1}$ induces an embedding of K_n into G_{n+1}. On the other hand, the restriction map $H^*_{G_{n+1}}(\Sigma_n) \to H^*(\Sigma_n)$ is the restriction of the first characteristic class to a finite approximation which shows that Σ_n is G_{n+1}-equivariantly formal. Combining both facts we have that (G_{n+1}, K_n) is a free extension pair and the cohomological Serre spectral sequence associated to the fibration $\Sigma_n \to BK_n \to BG_{n+1}$ degenerates at the E_2-page.

This argument can be generalized for any Grassmanian as follows. Let $\Sigma_{n.k} = Gr(n,k)$ be the Grassmannian of k-dimensional planes in $\mathbb{K}^n$. There is a canonical transitive action of K_n on $\Sigma_{n,k}$ and thus $H^*_{K_n}(\Sigma_{n,k}) \cong H^*(B(\Sigma_n)_X)$ for a chosen $X \in Gr(n,k)$. If $X = \langle e_1, \ldots, e_k \rangle$, we see that $(\Sigma_n)_X \cong S(G_k \times G_{n-k})$.

There is a short split exact sequence

$$0 \to S(G_k \times G_{n-k}) \to G_k \times G_{n-k} \to L \to 0$$

that induces an isomorphism in cohomology $H^*(BG_k) \otimes H^*(BG_{n-k}) \cong H^*(BL) \otimes H^*(BS(G_k \times G_{n-k})$ and $H^*(BS(G_k \times G_{n-k})) \cong (H^*(BG_k) \otimes H^*(BG_{n-k})) \otimes_{H^*(BL)} \mathbb{k}$. Recall that $H^*(\Sigma_{n,k}) \cong H^*(G_k)/I_n$ and so $\Sigma_{n,k}$ is K_n-equivariantly formal; in particular, this implies that $(K_n, S(G_k \times G_{n-k}))$ is a free extension pair. $\square$

Recall that for $K \subseteq G$ a closed subgroup of a Lie group G, the Weyl group of K in G is defined as $W = N_G(K)/K$ where $N_G(K)$ denotes the normalizer of K in G.

Theorem 2.8. *Let (G, K) be a free extension pair. Let W be the Weyl group of K in G and suppose that there is an isomorphism of algebras $H^*(BK)^W \cong H^*(BG)$. Then for any G-space X such that $H^*(X)^G = H^*(X)$, W acts on the K-equivariant cohomology of X and there is a natural isomorphism of $H^*(BG)$-modules $H^*_K(X)^W \cong H^*_G(X)$.*

Proof. The map c_g given by the conjugation of a chosen element $g \in G$ induces the identity map in the cohomology of $H^*(BG)$. This can be shown using Milnor's join construction of BG. Moreover, this map induces a map $X_G \to X_{\tilde{G}}$ where X is a $\tilde{G}$-space with the action of G induced by the map c_g. Notice that the $\tilde{G}$-equivariant cohomology of X is isomorphic to its G-equivariant cohomology. Since G acts trivially on the cohomology of X, a (Serre) spectral sequence argument shows that the map c_g induces the identity on $H^*_G(X)$. Therefore, there is a well defined action of W on both $H^*(BK)$ and $H^*_K(X)$ and the previous argument shows that $H^*_G(X) \subseteq H^*_K(X)^W$. Moreover, the canonical map $H^*_G(X) \otimes_{H^*(BG)} H^*(BK) \to H^*_K(X)$ of algebras is W-equivariant and an isomorphism by Proposition 2.4 and thus $H^*_K(X) \cong H^*_G(X)^W$ as $H^*(BG)$-modules since W acts trivially on $H^*_G(X)$ and $H^*(BK)^W \cong H^*(BG)$ by assumption. $\square$

Observe that Propositions 2.4 and Theorem 2.8 can be summarized in Theorem 1.1 as discussed at the beginning of this document. In the rest of this section, we discuss the case when G is a semi-direct product; we first start with the following result.

Proposition 2.9. *Let G and K be groups. Suppose that there is a subgroup $N \subseteq G$ and a group homomorphism $\phi \colon K \to Aut(G)$ such that $\phi(k)|_N$ is the identity for all $k \in K$. Then (G, N) is a free extension pair if and only if $(G \rtimes_\phi K, N \times K)$ also is.*

Proof. Under these assumptions, there are canonical isomorphisms $H^*(B(N \times K)) \cong H^*(BN) \otimes H^*(BK)$ and $H^*((G \rtimes_\phi K)/(N \times K)) \cong H^*(G/N)$. The commutative diagram

$$\begin{array}{ccc} H^*(BN) \otimes H^*(BK) & \longrightarrow & H^*((G \rtimes_\phi K)/(N \times K)) \\ \downarrow & & \downarrow \\ H^*(BN) & \longrightarrow & H^*(G/N) \end{array}$$

and the surjectivity of the vertical arrows implies that the top horizontal arrow is surjective if and only if the bottom also is. □

It is not difficult to check from the definition of free extension pairs and the fact that the classifying space functor preserves finite products, that the product of two free extension pairs is again a free extension pair.

In the following remark we discuss a more general criterion for the product property of free extension pairs.

Remark 2.10. *Let (G, N) and (K, L) be two free extension pairs. Suppose that there is a group homomorphism $\phi\colon K \to Aut(G)$ such that $\phi_k(N) \subseteq N$ then the map $H^*(B(N \rtimes L)) \to H^*(BN)$ is surjective if and only if $(G \rtimes_\varphi K, N \rtimes_\varphi L)$ is a free extension pair.*

Let $G = N \rtimes T$ be a semi-direct product group. The G-equivariant cohomology can be computed stepwise as in the direct product case; namely, for a a G-space X, there is an isomorphism of $\Bbbk$-algebras $H^*_G(X) \cong H^*_K(X_N)$ As a consequence of Proposition 2.9 and Remark 2.10, we can recover the free extension property for the matrix groups G_n and K_n of Proposition 2.7 since $K_n \cong G_n \rtimes K_n/G_n$. Notice that the group $L = K_n/G_n$ is unique up to isomorphism for all $n > 1$. We summarize it in the following corollary.

Corollary 2.11. *Let $L = K_n/G_n$. The pairs (G_{n+1}, L^n) and (K_n, L^n) are free extension pairs. In particular, for $\Bbbk = \mathbb{F}_2$, L^n can be replaced by the its maximal elementary abelian 2-subgroup.*

3. Torus actions and compatible involutions

In this section, we will consider cohomology with coefficients over $\Bbbk = \mathbb{F}_2$ (and we will omit it in our notation). Let X be a space with an action of a torus T and let $\tau\colon X \to X$ be an involution. We say that τ is *compatible* if $\tau(g \cdot x) = g^{-1} \cdot \tau(x)$ for any $g \in T$ and $x \in X$. Examples of such spaces appear naturally as toric varieties in algebraic geometry, Hamiltonian torus actions on symplectic manifolds and topological generalizations of these spaces as quasitoric manifolds, torus manifolds and moment angle complexes [17], [24], [14].

The aim of this section is to describe the equivariant cohomology for any T-space X with a compatible involution τ. Firstly, notice that in this case there is a well defined action of the group $G = T \rtimes_\tau \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts on T by inversion. Conversely, an action of G on X induces an action of T with a compatible involution τ on X. Therefore, the equivariant cohomology of a T-space with a compatible involution can be approached by studying the G-equivariant cohomology of X.

More generally, let m and n be positive integers. We will consider actions of the semidirect product group $G = T \rtimes K$ where $T = (S^1)^n$, $K = (\mathbb{Z}/2\mathbb{Z})^m$ and $\sigma \cdot g = g^{-1}$ for $g \in T$ and a generator $\sigma \in K$. By our assumption on the base field $\Bbbk$, the cohomology of the classifying spaces BT and BK are polynomial rings. In fact, we will show that under our assumptions there is a canonical isomorphism of algebras $H^*(BG) \cong H^*(BT) \otimes H^*(BK)$ and thus $H^*(BG)$ is a polynomial algebra in $(n+m)$-variables as we state in the following result.

Theorem 3.1. *There is a unique graded algebra isomorphism $H^*(BG) \cong H^*(BT) \otimes H^*(BK)$ such that*

- *The canonical map* $i^*\colon H^*(BG) \to H^*(BT)$ *induced by the inclusion is surjective and* $\ker(i^*) \cong (H^*(BK)^+)$ *is the ideal generated by the positive degree cohomology of* BK*.*
- *The canonical map* $p^*\colon H^*(BK) \to H^*(BG)$ *induced by the projection is injective and* $\operatorname{Coker}(p^*) \cong H^*(BT)$*.*
- *There is an algebra homomorphism* $\varphi : H^*(BT) \to H^*(BG)$*, such that the composite* $i^* \circ \varphi$ *is the identity over* $H^*(BT)$ *and* $\operatorname{Coker}(\varphi) \cong H^*(BK)$*.*
- *The map* $j^*\colon H^*(BG) \to H^*(BK)$ *induced by the inclusion* $j\colon \{e\} \times K \to G$ *has kernel* $(H^*(BT)^+)$ *and the composite* $j^* \circ p^*$ *is the identity over* $H^*(BK)$*.*

Proof. To compute the cohomology of BG, notice that the short exact sequence

$$1 \to T \to G \to K \to 1$$

yields a fibration of classifying spaces

$$BT \to BG \to BK. \tag{3.1}$$

Observe that the action of $\pi_1(BK)$ on $H^*(BT)$ is induced by the action of K on T and hence on BT. In fact, each generator $\sigma \in K$, induces an action on the integral cohomology $H^*(BT;\mathbb{Z})$ given by multiplication by -1 and thus trivial in cohomology over $\mathbb{F}_2$. Therefore, the E_2-term of the Serre spectral sequence associated to the fibration 3.1 is given by

$$E_2^{p,q} \cong H^p(BK; H^q(BT)) \Rightarrow H^*(BG)$$

and then we have an isomorphism of algebras

$$E_2 \cong H^*(BK) \otimes H^*(BT)$$

We will show that this spectral sequence degenerates at this term by induction on $m = \operatorname{rank} K$. Let us assume then that $m = 1$. By degree reasons, the only possible non-zero differential d_3 is determined by $d_3 : E_3^{0,2} \to E_3^{3,0}$. Choose generators $x_i \in H^2(BT)$ for $1 \leqslant i \leqslant n$ and $t \in H^1(BK)$. Under these identifications, we have that $d_3(x_i) = \alpha_i t^3$ with either $\alpha_i = 0$ or $\alpha_i = 1$.

The sub-extension

$$\begin{tikzcd}
T \arrow[r] & T \rtimes \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
1 \arrow[r] \arrow[u] & 1 \rtimes \mathbb{Z}/2 \arrow[r] \arrow[u] & \mathbb{Z}/2 \arrow[u]
\end{tikzcd}$$

induces a map of spectral sequences $E_s^{p,q} \to \widetilde{E}_s^{p,q}$, where $\widetilde{E}_2 \cong H^*(B\mathbb{Z}/2)$ is the $\widetilde{E}_2$ page of the spectral sequence associated to the bottom exact sequence. This implies that $d_3 = 0$ since the right vertical arrow is the identity map and $\widetilde{d}_3 = 0$ as the spectral sequence $\widetilde{E}_*$ degenerates at the E_2-term. We have then an isomorphism of $H^*(BK)$-modules

$$H^*(BK) \otimes H^*(BT) \cong H^*(BG).$$

On the other hand, since $H^*(BT)$ is a finitely generated polynomial algebra, we can choose a multiplicative section $\tilde{\varphi} : H^*(BT) \to H^*(BG)$ of the surjective map of $H^*(BG) \to H^*(BT)$ induced by the inclusion map. Therefore, such a map together with the canonical map $p^*\colon H^*(BK) \to H^*(BG)$ gives rise to an isomorphism of graded $H^*(BK)$-algebras

$$\tilde{\theta} : H^*(BK) \otimes H^*(BT) \to H^*(BG)$$

given by $\tilde{\theta}(\alpha \otimes \beta) = p^*(\alpha)\tilde{\varphi}(\beta)$.

Under this isomorphism, the canonical map induced by the inclusion $j : B(1 \times K) \to BG$ might satisfy $j(x_i) = w^2$ for some $1 \leqslant i \leqslant n$. If that is the case, then we consider the section $\varphi(x_i) = \tilde{\varphi}(x_i) + w^2$ if $\tilde{\varphi}(x_i) = w^2$ and $\varphi(x_i) = \tilde{\varphi}(x_i)$ if $\tilde{\varphi}(x_i) = 0$. As discussed before, it induces an isomorphism of algebras

$$\theta : H^*(BK) \otimes H^*(BT) \to H^*(BG);$$

furthermore, such a section is unique since it is determined by the condition $j^*\varphi = 0$, and thus it makes the isomorphism θ unique as well. Therefore, the composite

$$j^*\theta\colon H^*(BK) \otimes H^*(BT) \to H^*(BG) \to H^*(BK)$$

has kernel $H^*(BT)^+$. Now notice that the composite

$$H^*(BT) \xrightarrow{\varphi} H^*(BG) \xrightarrow{i^*} H^*(BT)$$

where i^* is induced by the inclusion $T \to G$ is the identity on $H^*(BT)$ since $i^*(w) = 0$ and φ was constructed as a section of this map. This implies that the maps

$$H^*(BT) \xrightarrow{\varphi} H^*(BG) \text{ and } H^*(BG) \xrightarrow{i^*} H^*(BT)$$

coincide with the canonical inclusion and restriction respectively. Using a similar argument over the composite $H^*(BK) \xrightarrow{p^*} H^*(BG) \xrightarrow{j^*} H^*(BK)$, which is the identity over $H^*(BK)$, we conclude that the map $H^*(B\mathbb{Z}/2) \xrightarrow{p^*} H^*(BG)$ has kernel $H^*(BT)^+$.

The inductive argument follows in a similar fashion, noticing that $G \cong (T \rtimes \hat{K}) \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\operatorname{rank}(K) = m-1$ and looking at the fibration $B(T \rtimes \hat{K}) \to BG \to B\mathbb{Z}/2\mathbb{Z}$. $\square$

Now we will study the algebraic properties of the G-equivariant cohomology as a module over $H^*(BG)$. Notice that for any G-space X, there is an induced involution τ on the space X_T; moreover, the Borel constructions X_G and $(X_T)_\tau$ are homotopic. Using this remark we prove the following result.

Proposition 3.2. *Let X be a G-space and assume that X is T-equivariantly formal. Then X is G-equivariantly formal if and only if the Borel construction X_T is G/T-equivariantly formal.*

Proof. Write $G = T \rtimes K$ where $K = \langle\tau\rangle \cong G/T$. Firstly, let us suppose that X is G-equivariantly formal, by Theorem 3.1 and the above remark we get isomorphisms

$$\begin{aligned} H^*_\tau(X_T) \cong H^*_G(X) &\cong H^*(BG) \otimes H^*(X) \\ &\cong H^*(BK) \otimes H^*(BT) \otimes H^*(X) \\ &\cong H^*(BK) \otimes H^*(X_T) \end{aligned}$$

and so X_T is K-equivariantly formal. Reversing the above sequence of isomorphisms, the converse of the statement holds. However, we need to be careful with the $H^*(BG)$-module structure of $H^*_G(X)$ and the $H^*(BK)$-module structure of $H^*_K(X_T)$. From the diagram

$$\begin{tikzcd}
H^*_G(X) \arrow[r, "\cong"] & H^*_K(X_T) \\
H^*(BG) \arrow[u] \arrow[r, "\cong"] & H^*_\tau(BT) \arrow[u] \\
H^*(B\tau) \arrow[u] \arrow[r, equal] & H^*(B\tau) \arrow[u]
\end{tikzcd}$$

and the canonical isomorphism $H^*(BG) \cong H^*(B\tau) \otimes H^*(BT)$ constructed in Theorem 3.1, the $H^*(B\tau)$-module structure on $H^*_\tau(X_T)$ coincides with the restriction of the $H^*(BG)$-module structure on $H^*_G(X)$ to the action of those elements of the form $\alpha \otimes 1 \in H^*(B\tau) \otimes H^*(BT) \cong H^*(BG)$. □

Now we will apply this theorem to the *conjugation spaces* introduced by Haussmann-Holm-Puppe [26]; among these spaces, complex Grassmannian, toric manifolds, polygon spaces and some symplectic manifolds fit. A conjugation space X satisfies that $H^{odd}(X) = 0$ by definition and thus it becomes T-equivariantly formal for any action a torus T. They also satisfy the following property [26, Thm.7.5].

Theorem 3.3. *Let X be a conjugation space with conjugation τ. Suppose that a torus T acts on X and that the action is compatible with τ. Then X_T is a conjugation space where the conjugation on X_T is the one induced by τ.*

This theorem shows that X_T is τ-equivariantly formal. Then immediately from Theorem 3.2 we obtain the following result.

Corollary 3.4. *Let X be a T-space which is also a conjugation space with a compatible involution τ. Then X is G-equivariantly formal.*

4. Reduction to 2-torus actions

In this section, we will use the results from section 2 on free extension pairs to study the equivariant cohomology for torus actions and compatible involutions by reducing to the maximal elementary 2-abelian subgroup (or 2-torus). Let $H = T_2 \times K \cong (\mathbb{Z}/2\mathbb{Z})^{n+m}$ denote the maximal 2-torus subgroup in $G = T \rtimes K$ where $T_2 \leqslant T$ is the maximal 2-torus subgroup in T. Since $(S^1, \mathbb{Z}/2\mathbb{Z})$ is a free extension pair over $\mathbb{F}_2$, by Proposition 2.9, Remark 2.10 and Theorem 3.1 it follows that (G, H) is a free extension pair as well. Let us choose generators $H^*(BT) \cong \mathbb{F}_2[x_1, \cdots .x_n]$ and $H^*(BH) \cong \mathbb{F}_2[t_1, \ldots, t_n, y_1, \ldots, y_n]$ so that $H^*(BT_2) \cong \mathbb{F}_2[t_1, \ldots, t_n]$ and the map induced by the inclusion $T_2 \to T$ maps x_i to t_i^2 for all $1 \leqslant i \leqslant n$. We now compute explicitly the module structure of $H^*(BH)$ over $H^*(BG)$ as stated in the following lemma.

Lemma 4.1. *The map $i^* \colon H^*(BG) \to H^*(BH)$ induced by the inclusion $i \colon H \to G$ is given by $i^*(c_i) = t_i^2 + t_i(w_1 + \cdots w_m)$ for all $1 \leqslant i \leqslant n$ and $i^*(w_j) = w_j$ for all $1 \leqslant j \leqslant m$.*

Proof. By theorem 3.1 we can assume that $m = n = 1$ and so $H^*(BG) \cong \mathbb{F}_2[x, w]$ and $H^*(BH) \cong \mathbb{F}_2[t, w]$. Notice that the statement $i^*(w) = w$ is clear as it follows from the map induced by the inclusion of $\mathbb{Z}/2\mathbb{Z}$ into the second factor of $S^1 \rtimes \mathbb{Z}/2\mathbb{Z}$ which factors through $H^*(BH)$, Now write $i^*(x) = \alpha t^2 + \beta tw + \gamma w^2$ for $\alpha, \beta, \gamma \in \mathbb{F}_2$. As before, the inclusion of $\mathbb{Z}/2\mathbb{Z}$ into the first and second factor of G show that $\alpha = 1$ and $\gamma = 0$ respectively. To compute β, we consider the inclusion of G into $SO(3)$ by identifying G with $O(2)$ as in Proposition 2.7 . Recall that $H^*(BSO(3)) \cong \mathbb{F}_2[\omega_2, \omega_3]$ where $|\omega_i| = i$ for $i = 2, 3$ and the inclusion $H \to SO(3)$ induces the map $\phi\colon H^*(BSO(3)) \to H^*(BH)$ given by $\phi(\omega_2) = t^2 + tw + w^2$ and $\phi(\omega_3) = t^2w + wt^2$. Since ϕ factors through i^*, this implies that $\beta = 1$ and so $i^*(x) = t^2 + tw$. □

In Theorem 3.1 we showed that the cohomology of $H^*(BG)$ behaves as the cohomology of the classifying space of the direct product $T \times K$. However, Lemma 4.1 implies that their cohomology as modules over the Steenrod algebra are not.

Remark 4.2. *The mod2-cohomology of the classifying spaces BG and $B(T \times K)$ is isomorphic as $\mathbb{F}_2$-algebras but not as modules over the Steenrod algebra.*

Proof. As before, we may assume $n = m = 1$. For $x \in H^2(BG)$ generator, write $Sq^1(x) = \alpha xw + \beta w^3$ for $\alpha, \beta \in \mathbb{F}_2$. By naturality of the Steenrod operations, we have that $i^*(Sq^1(x)) = Sq^1(i^*(x))$ where i^* is the map induced by the inclusion $H \to G$. Therefore, $\alpha(t^2w + tw^2) + \beta w^3 = Sq^1(t^2 + tw) = t^2w + wt^2$ by Lemma 4.1 and so $\alpha = 1, \beta = 0$. On the other hand, a similar argument applied to the inclusion $j\colon H \to T \times K$ shows that $Sq^1(x) = 0$ as $j^*(x) = t^2$. □

Notice that for $m \geqslant 2$, there is a group isomorphism $G \cong ((S^1)^n \rtimes (\mathbb{Z}/2\mathbb{Z})^{m-1}) \rtimes \mathbb{Z}/2\mathbb{Z}$ where the action of the last factor of $\mathbb{Z}/2\mathbb{Z}$ is trivial on $(\mathbb{Z}/2\mathbb{Z})^{m-1}$. Therefore, as in the proof of Theorem 3.1, for the results of this section we may assume that $m = 1$ and the general statement will follow by induction.

Let S be any commutative ring and consider the polynomial ring $S[a, b]$. Let $z = a^2 + ab \in S[a, b]$. Since $a^2 = z - ab$, one can check that

$$S[a, b]/S[z, b] = aS[z, b] = \{p(z, b) \in S[z, b] : a | p(z, b)\}. \tag{4.1}$$

We will use this fact to prove the following proposition.

Proposition 4.3. *$H^*(BH)$ is a free module of rank 2^n over $H^*(BG)$; moreover, it is freely generated by the elements of the form $t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_n^{\epsilon_n}$ where $\epsilon_i \in \{0, 1\}$ for all i, and the canonical multiplicative structure of $H^*(BH)$ as an $\mathbb{F}_2$-algebra induces an $H^*(BG)$-algebra structure completely determined by multiplication of the elements of this basis.*

Proof. Let $R = \Bbbk[c_1, \cdots, c_n, w] \cong H^*(BH)$, $M = \Bbbk[t_1, \ldots, t_n, w] \cong H^*(BG)$ and set $y_i = t_i^2 + t_i w \in M$ for $i = 1, \ldots, n$. Recall that M is an R-module by extending the action $c_i \cdot 1 = y_i$ and $w \cdot 1 = w$. Consider the filtration of M by the R-submodules

$$F_0 = \Bbbk[y_1, y_2, \ldots, y_n, w] \subseteq F_1 \subseteq \cdots \subseteq F_n = M \tag{4.2}$$

where $F_i = \Bbbk[t_1, \ldots, t_i, y_{i+1}, \ldots y_n, w]$ for $i = 1, \ldots n-1$. We will prove by induction that F_i is a free R-module of rank 2^i for $i = 0, \ldots, n$. The statement for $i = 0$ is immediate. For $i > 0$, set $S_i = \Bbbk[t_1, \ldots, t_{i-1}, y_{i+1}, \ldots, y_n]$ and notice that

$$F_i/F_{i-1} \cong S_i[t_i, w]/S_i[y_i, w] \cong t_i S[y_i, w] = t_i F_{i-1} \tag{4.3}$$

as in (4.1) and thus it is a free R-module by induction. This implies that the short exact sequence of R-modules

$$0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0 \tag{4.4}$$

splits and hence $F_i \cong F_{i-1} \oplus F_i/F_{i-1}$. Finally, we have then that F_i is a free R-module of rank $2^{i-1} + 2^{i-1} = 2^i$ by induction again. The claim about the basis elements follows also by iterating (4.1) and (4.3) in the decomposition

$$F_k = F_0 \oplus F_1/F_0 \oplus \cdots \oplus F_k/F_{k-1} \tag{4.5}$$

for all $k = 1, \ldots, n$.

□

Combining Propositions 2.5 and 4.3 we can state the following result.

Corollary 4.4. *Let X be a G-space. $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$ if and only if $H^*_H(X)$ is a j-th syzygy over $H^*(BH)$.* □

Proposition 2.4 shows that the H-equivariant cohomology of X is determined by the G-equivariant cohomology of X. As in the case for compact connected Lie groups and their maximal torus for rational coefficients, we can also describe the G-equivariant cohomology of X in terms of the Weyl invariants of the H-equivariant cohomology of X. Recall that the Weyl group of H in G is defined as the quotient $W = N_G(H)/H$ where $N_G(H)$ denotes the normalizer of H in G. We first proof the following proposition.

Proposition 4.5. *Let $W = N_G(H)/H$ be the Weyl group of H in G. Then $W \cong (\mathbb{Z}/2)^n$ and there is an isomorphism of algebras $H^*(BG) \cong H^*(BH)^W$ where the action on the cohomology of $H^*(BH)$ is induced by the conjugation action of W on H.*

Proof. Write $H = \langle (g_1, e), \ldots, (g_n, e), (1, \tau)\rangle$ where $g_i^2 = 1$ in the i-th factor S^1 of T. We claim that $N_G(H) \cong (\mathbb{Z}/4)^n \rtimes \mathbb{Z}/2\mathbb{Z}$ where $(\mathbb{Z}/4)^n = \langle \theta_1, \ldots, \theta_n\rangle$ is generated by elements $\theta_i^2 = g_i$ and $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/4$ by inversion. Notice that for any $(g, \sigma) \in G$ where $g \in T$ and $\sigma \in \langle \tau \rangle$, (g, σ) commutes with every element in H of the form (g_i, e) and so we only need to look at the conjugation of the element $(1, e) \in H$ by (g, σ). Namely, if $(g, \sigma) \in N_G(H)$ we have that $(g, \sigma)(1, \tau)(g^\sigma, \sigma) = (g^2, \tau) \in H$ and thus we get $g \in \langle \theta_1, \ldots, \theta_n\rangle$. This implies that $W \cong (\mathbb{Z}/2\mathbb{Z})^n$ is generated by the cosets $(\theta_i, e)H$ for $i = 1, \ldots, n$.

Recall that for any topological group the map induced in the cohomology of the classifying space by the conjugation of a fixed element is the identity map [2, Ch.II Thm.1.9] and so $i^*(H^*(BG)) \subseteq H^*(BH)^W$. It only remains to check the reverse inclusion to finish the proof. We now compute the induced action on the cohomology of $H^*(BH)$. Choose a decomposition $H^*(BH) \cong \Bbbk[t_1, \dots, t_n, w]$ where the variables t_i are dual to the generators g_i and w is to τ in $\mathbb{F}_2[H]$. For a fixed $i \in \{1, \dots, n\}$, notice that any $(\theta_i, e)H \in W$ acts trivially on the generators $(g_j, e) \in H$; on the other hand, we have that $(\theta_i, e)H \cdot (1, \tau) = (\theta_i, e)(1, \tau)(\theta_i g_i, \tau) = (g_i, \tau)$ This implies that the induced map φ_i by the action of $(\theta_i, e)H$ on the cohomology ring $\mathbb{F}_2[t_1, \dots, t_n, w]$ is given by $\varphi_i(t_j) = t_j$ for $j \neq i$, $\varphi_i(t_i) = t_i + w$ and $\varphi_i(w) = w$. This follows, for instance, from a group cohomology argument.

By Proposition 4.3, consider an element $P = \sum_{I \in \Lambda} P_I t_I \in H^*(BH)^W$ where $P_I \in H^*(BG)$ are uniquely determined. We will show that $P_I = 0$ if $I(k) \neq 0$ for some $1 \leqslant k \leqslant n$. Let $I \in \Lambda$ be such that $I(k) \neq 0$, then $\varphi_k(P_I t_I) = P_I t_I + w P_I t_{I_k}$ where $I_k(j) = I(j)$ if $j \neq k$ and $I_k(k) = 0$. Under this notation, we have that $\varphi_k(t_{I_k}) = t_{I_k}$ and then the equation $P = \varphi_k(P)$ implies that $P_{I_k} + w P_I = P_{I_k}$ and so $P_I = 0$ as desired. □

Actually, the isomorphism of Proposition 4.5 can be extended to a natural isomorphism in equivariant cohomology as we state in the following consequence of theorem 2.8

Corollary 4.6. *Let X be a G-space, H the maximal 2-torus in G and W the Weyl group of H in G. Suppose that G acts trivially on the cohomology of X. Then there is a natural isomorphism of $H^*(BG)$-algebras*

$$H_G^*(X) \cong H_H^*(X)^W$$

induced by the inclusion $H \to G$.

5. Equivariant Cohomology for the real locus

Let M be a symplectic manifold with an action of a torus T. A consequence of the work of Frankel [20], Atiyah [6] and Kirwan [27] in equivariant cohomology for Hamiltonian torus actions is that the action on M is Hamiltonian if and only if M is T-equivariantly formal. Moreover, if M admits a compatible anti-symplectic compatible involution τ, the *real locus* M^τ inherits a canonical action of T_2 and M^τ is T_2-equivariantly formal as shown in [18], and extended later in [9]. This can be summarized in the following result.

Theorem 5.1. *Let M be a symplectic manifold with a symplectic action of a torus T and a compatible anti-symplectic involution τ. If M is T-equivariantly formal, the real locus M^τ is T_2-equivariantly formal.*

In this section we generalize the notion of spaces with a torus action and a compatible involution to a large class of groups, and we study the equivariant cohomology for the fixed point subspace under the compatible involution to generalize Theorem 5.1 into a topological setting. We first introduce the following definition motivated by the case when X is a complex variety and the involution is induced by the complex conjugation.

Definition 5.2. *Let X be a space with involution τ. The real locus of X is defined as the fixed point subspace X^τ.*

Let G be a compact group, X be a G-space and τ_X be an involution on X. We say that τ_X is a *compatible involution* of X if there is a group homomorphism $\tau_G \colon G \to G$ such that $\tau_G^2 = id$ and $\tau_X(g \cdot x) = \tau_G(g) \cdot \tau_X(x)$ for any $g \in G$ and

$x \in X$. The condition of compatibility is equivalent to an action of the group $G_\tau = G \rtimes_\tau \mathbb{Z}/2\mathbb{Z}$ on X. To simplify our notation, the involutions τ_X and τ_G will be both referred as τ, and their domain can be inferred from the context. Notice that the subgroup G^τ of τ-fixed points of G acts on the real locus X^τ.

Definition 5.3. *Let H be a τ-invariant subgroup of G. We say that (G, H) is a τ-free extension if both (G, H) and (G^τ, H^τ) are free extensions.*

Notice that if τ acts trivially on H, then X^L is the real locus of X. We proceed to prove the following result.

Theorem 5.4. *Let G be a compact group and let X be a G-space with a compatible involution involution τ. Suppose that there is a τ-invariant 2-torus H in G such that (G, H) is a τ-free extension. For any splitting $H_\tau \cong H^\tau \times L$ and for any integer $j \geqslant 1$, if $H^*_{G_\tau}(X)$ is a j-th syzygy over $H^*(BG_\tau)$, then so is $H^*_{G^\tau}(X^L)$ as a module over $H^*(BG^\tau)$.*

Proof. As H is a 2-torus, (G, H) is a free extension if and only if (G_τ, H_τ) is. In fact, it follows from the commutativity of the diagram

$$\begin{array}{ccc} H^*(BH_\tau) & \longrightarrow & H^*(G_\tau/H_\tau) \\ \downarrow & & \downarrow \\ H^*(BH) & \longrightarrow & H^*(G/H) \end{array}$$

where the map $H^*(G_\tau/H_\tau) \to H^*(G/H)$ is an isomorphism and $H^*(BH_\tau) \to H^*(BH)$ is surjective. If X is a j-th syzygy over $H^*(BG_\tau)$, it follows from Proposition 2.4 that it also is a j-th syzygy as a module over $H^*(BH_\tau) \cong H^*(B(H \times \tau))$. We can use now the tools for syzygies for 2-torus actions discussed in [16] In fact, from Theorem [16, Thm.2.1] applied to the subgroup $L \subseteq H_\tau$, we obtain that $H^*_{H_\tau/L}(X^L) \cong H^*_H(X^L)$ is a j-th syzygy over $H^*(B(H_\tau/L)) \cong H^*(BH^\tau)$. Finally, as (G, H) is a τ-free extension pair, from Proposition 2.5 we get that X is also a j-th syzygy over $H^*(BG^\tau)$. $\square$

This theorem applies, for instance, to the groups $G = T \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ for any $n \geqslant 0$ which generalize torus actions and torus actions with compatible involutions where H is the maximal 2-torus in G. It also applies to $SO(n)$ with the canonical τ-action that makes the isomorphism $SO(n) \rtimes_\tau \mathbb{Z}/2\mathbb{Z} \cong O(n)$. In this case, H is the maximal 2-torus in $SO(n)$. In particular, we have a generalization of Theorem 5.1 given by the following result.

Theorem 5.5. *Let $G = T \rtimes \mathbb{Z}/2\mathbb{Z}$ and X be a G-space. If $H^*_G(X)$ is a j-th syzygy over $H^*(BG)$, then so is $H^*_{T_2}(X^\tau)$ as a module over $H^*(BT_2)$. In particular, if X is G-equivariantly formal, then the real locus X^τ is T_2-equivariantly formal.*

Example 5.6. Let X be a T-space. Suppose X is also a conjugation space with a compatible conjugation τ. Then from Theorem 5.5 and corollary 3.4 we have that the real locus X^τ is T_2-equivariantly formal.

The assumptions of Theorem 5.5 cannot be weakened. For example, If X is a G-space such that it is simultaneously T-equivariantly formal and τ-equivariantly formal, it is not necessarily true that X is G-equivariantly formal or that its real locus X^τ is T_2-equivariantly formal as the next example shows.

Example 5.7. Let $X = \{(u,z) \in \mathbb{C} \times \mathbb{R} : |u|^2 + |z|^2 = 1\} = S^2$, let $T = S^1$ act on X by $g \cdot (u,z) = (gu, z)$; more precisely, by scalar multiplication in the first factor. Let τ be the involution $\tau(u,z) = (\bar{u}, -z)$ which is compatible with the torus action. Notice that $X^T = \{(0,1),(0,-1)\} \cong S^0$ and $X^\tau = \{(-1,0),(1,0)\} \cong S^0$. Therefore, the action of T_2 on X^τ is the multiplication by ± 1 and thus it is a free T_2-space. This implies that its T_2-equivariant cohomology is not free over $H^*(BT_2)$. On the other hand, $H^*_T(X)$ is a free $H^*(BT)$-module since X and X^T have the same Betti sum.

One of the main issues of this example is that $X^G = \varnothing$. Even requiring $X^G \neq \varnothing$, a counterexample can be found and its construction will be motivated by [32, Sec. 5]. First we recall the following construction of topological spaces.

Definition 5.8. *Let $f : X \to Y$ be a G-map between G-spaces X and Y. The mapping cylinder is defined as the G-space $M_f = (X \times [0,1]) \sqcup Y/\sim$ where $(x,1) \sim f(x)$, with the action given by $g \cdot (x,t) = (gx,t)$ for $(x,t) \in X \times [0,1]$ and the action on Y. Notice that it is well defined at the points of the form $(x,1)$ since f is a G-map.*

The space M_f is G-homotopic to Y and therefore $H^*(M_f) \cong H^*(Y)$. Also, the fixed point subspace $(M_f)^G \cong M_{f^G}$ where $f^G \colon X^G \to Y^G$. Now let $g : X \to Z$ be a G-map and M_g the corresponding mapping cylinder. Then the space $M_{f,g} = M_f \cup_{X \times \{0\}} M_g$ has cohomology groups fitting in the long exact sequence

$$0 \to H^0(M_{f,g}) \to H^0(Y) \oplus H^0(Z) \to H^0(X) \to H^1(M_{f,g}) \to \cdots$$

following from the Mayer-Vietoris long exact sequence. Moreover, $M_{f,g}$ becomes a G-space and $(M_{f,g})^G \cong M_{f^G,g^G}$. In particular, we have

Proposition 5.9. *Let m,n,r be distinct positive integers, $h : S^m \to S^n$ a map between spheres and consider $f = h \times id \colon S^m \times S^r \to S^n \times S^r$ and $g \colon S^m \times S^r \to S^m$ the projection on the first factor. Then $H^*(M_{f,g})$ is free over $\mathbb{Z}/2\mathbb{Z}$ where a copy of $\mathbb{Z}/2\mathbb{Z}$ happens in degrees $0, n, m+r+1, n+r$ and it is zero otherwise. In particular, $b(M_{f,g}) = 4$.*

Using this construction, we have the following proposition.

Proposition 5.10. *There is a topological space M with an action of a torus T and a compatible involution τ such that $M^G \neq \varnothing$, M is T-equivariantly formal and $\mathbb{Z}/2\mathbb{Z}$-equivariantly formal, but the real locus M^τ is not T_2-equivariantly formal with respect to the induced action of the 2-torus $T_2 \subseteq T$ on M^τ.* □

Proof. Let $X = S^3$, $Y = S^2$ and $h : X \to Y$ be the Hopf map, which can be written as $h(u,z) = (2u\bar{z}, |u|^2 - |z|^2)$. Here S^3 is seen as the unit sphere in $\mathbb{C}^2$ and S^2 as the unit sphere in $\mathbb{C} \times \mathbb{R}$. Let $T = S^1$ act on S^3 and S^2 as the complex multiplication in the first component respectively, and τ be the involution on S^3 and S^2 given by the complex conjugation in the first component respectively. Then τ is compatible with the torus action and $X^T \cong S^1$, $X^\tau \cong S^2$, $Y^T \cong S^0$ and $Y^\tau \cong S^1$. Now let $Z = S^5$ be the unit sphere in $\mathbb{C}^3$, let T act on Z by multiplication in the first component and τ be the involution on Z given by the complex conjugation in the first component, and multiplication by -1 in the second and third component. Then $Z^T \cong S^3$ and $Z^\tau \cong S^0$; moreover, the action of the 2-torus $T_2 \subseteq T$ on Z^τ is free.

Let $M = M_{f,g}$ be the construction of Proposition 5.9. We have that the Betti sums $b(M) = b(M^T) = b(M^\tau) = 4$ and thus M is T-equivariantly formal but M^τ is not T_2-equivariantly formal since $b((M^\tau)^{T_2}) = 2 < b(M^\tau)$. □

6. Actions on big polygon spaces

Big polygon spaces provide remarkable examples for the study of torus equivariant cohomology since their equivariant cohomology is not free over the cohomology of the classifying space of the torus but realize all other possible syzygy order. They will also allow us to realize all possible syzygies in G-equivariant cohomology. These spaces where introduced in [21] where their non-equivariant and T-equivariant cohomology was determined and an upper bound for their syzygy order was conjectured. This was proved later in [23]. They generalize chain spaces and polygon spaces studied in different contexts in [19] and [25] for instance. The real analogous of these spaces is also studied in [31] for the case of 2-torus actions and cohomology with $\mathbb{F}_2$-coefficients. Before discussing these spaces, we will review the construction of a cohomology class that will be useful for the results of this section. We use the *equivariant homology* for 2-torus actions; for its construction and properties we follow the work in [4].

Let G be a 2-torus, M be a closed G-manifold of dimension m and $N \subseteq M$ be a closed G-invariant submanifold of M of dimension n. Let $j^G_*\colon H^G_*(N) \to H^G_*(M)$ denote the map induced in equivariant homology by the inclusion $j\colon N \to M$; similarly, $j^*_G\colon H^*_G(M) \to H^*_G(N)$ denotes the map induced in equivariant cohomology. As M and N satisfy Poincaré duality (cohomology with coefficients over a field of characteristic two), there are isomorphism $PD_M\colon H^*_G(M) \to H^G_{m-*}(M)$ and $PD_N\colon H^*_G(N) \to H^G_{n-*}(N)$. Consider the composite map

$$\nu_{N,M}\colon H^*_G(N) \xrightarrow{PD_N} H^G_{n-*}(N) \xrightarrow{j^G_*} H^G_{n-*}(M) \xrightarrow{PD_M^{-1}} H_G^{m-n+*}(M) \xrightarrow{j^*_G} H_G^{m-n+*}(N).$$

Under this construction, we introduce the following definition.

Definition 6.1. *The G-**equivariant Euler class** of N with respect to M denoted by $e_G(N \subseteq M)$ is defined as the cohomology class $\nu_{N,M}(1) \in H^{m-n}_G(N)$.*

Consider the following example (compare with [21, Lem.4.2]).

Example 6.2. Let $G = G_1 \times G_2$ be a 2-torus of rank 2 where $G_1 = \{1, g\}, G_2 = \{1, \tau\}$. Let G act on $\mathbb{C}$ where g acts as the multiplication by -1 and τ as the complex conjugation. Let x, w denote the generators of $H^*(BG_1)$ and $H^*(BG_2)$ dual to g and τ respectively. Then the equivariant Euler class $e_G(0 \subseteq \mathbb{C}) = \alpha x^2 + \beta xw + \gamma w^2 \in H^2(BG)$. Let $K = \{1, s\}$ and t be the generator of $H^*(BK)$. Consider the following cases

- Let s act on $\mathbb{C}$ in the same fashion as g and let $j_1 : K \to G$ be the map sending s to g, then the induced map in cohomology is given by $j_1^*(x) = t$ and $j_2^*(w) = 0$. Notice that $e_K(0 \subseteq \mathbb{C}) = t^2$ since g acts non-trivially in both components of $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$ and the Euler class is multiplicative. From the naturality of the Euler class we get $\alpha t^2 = j_1^*(e_G(0 \subseteq \mathbb{C})) = e_{G_1}(0 \subseteq \mathbb{C}) = t^2$; therefore, $\alpha = 1$.
- Let s act on $\mathbb{C}$ in the same fashion as τ. As before, the map $j_2 : K \to G$ sending s to τ induces the map in cohomology mapping x to 0 and w to t.

In this case, $e_K(0 \subseteq \mathbb{C}) = 0$ since τ acts trivially on one real factor of $\mathbb{C}$. Therefore, by naturality, we obtain $\gamma = 0$

- Finally, let s act on $\mathbb{C}$ as $g\tau$, and $j_3 : K \to G$ sends s to (g, τ) and, in cohomology, both x, w are sent to t. Since s acts trivially on one real factor of $\mathbb{C}$, $e_K(0 \subseteq \mathbb{C}) = 0$ and by naturality, $t^2 + \beta t^2 = j_3^*(e_G(0 \subseteq \mathbb{C})) = e_K(0 \subseteq \mathbb{C}) = 0$. Therefore, $\beta = 1$.

So we have proved that $e_G(0 \subseteq \mathbb{C}) = x(x + w)$.

Definition 6.3. *Let a, b, n be positive integers and $M = (S^{2a+2b-1})^n \subseteq (\mathbb{C}^a \times \mathbb{C}^b)^n$. Let $\ell = (l_1, \ldots, l_n) \in \mathbb{R}^n$ be such that $l_i > 0$ for all i and such that it cannot be split as the sum of two vectors ℓ_1 and ℓ^2 of equal sum. The big polygon space is defined as*

$$X_{a,b}(\ell) = \left\{ (u, z) \in M : \sum_{i=1}^{n} l_i u_i = 0 \right\}$$

The space $X_{a,b}(\ell)$ inherits an action of an n-dimensional torus T by componentwise complex multiplication on the variables z_i. In this case, $X_{a,b}(\ell)$ becomes a compact orientable T-manifold of dimension $(2a+2b-1)n - 2a$ and its equivariant diffeomorphic type depends on ℓ [21, Lem.2.1]. Moreover, the complex conjugation on M induces a compatible involution τ on $X_{a,b}(\ell)$ and its real locus is the real big polygon space.

The first property of the G-equivariant cohomology of $X_{a,b}(\ell)$ is that it is not free over the cohomology of $H^*(BG)$ as we show in the following result.

Proposition 6.4. *$H^*_G(X)$ is not free over $H^*(BG)$. In fact, $H^*_G(X)$ is not a j-th syzygy for $j \geqslant (n+1)/2$.*

As a consequence of Theorem 5.5, we can recover a theorem of Puppe [31, Thm.1.2] that bounds the syzygy order of the real big polygon spaces.

Corollary 6.5. *The equivariant cohomology of the real big polygon space X^τ under the action of the 2-torus T_2 is a j-th syzygy $j \leqslant (n+1)/2$.*

Proof. By Corollary 4.4, it is enough to restrict to the action of the maximal 2-torus H of G. Let us denote by X the big polygon space to simplify notation. On the one hand, the integer cohomology of X is free and its Betti sum is $b(X) = 2^n$ [21, Prop.3.3]. On the other hand, when $a > 2$, the $\mathbb{F}_2$-cohomology of the fixed point subspace X^H is isomorphic to a quotient of an exterior algebra on n-generators by a non-trivial ideal [19, Prop. 4.2] and so $b(X^G) < 2^n$. The same bound holds when $a = 2$ by using that $X^H \cong S^1 \times X^T/SO(2)$ and the computation of the Betti sum. This shows that $b(X^H) < b(X)$ and thus X is not G-equivariantly formal by the Betti sum criterion for 2-torus actions [33, Prop.III.4.16]. The last assertion of the Corollary follows from the fact that X is a compact manifold and it satisfies Poincaré duality over $\mathbb{F}_2$-cohomology. $\square$

To bound the syzygy order of the G-equivariant cohomology of the big polygon spaces we use a similar approach as in the torus case. Firstly, we need to explicitly describe the generators of the non-equivariant and equivariant homology of the spaces M and $M \backslash X$. Namely, for any subset $J \subseteq \{1, 2, \ldots, n\}$, write $J^c = \{1, \ldots, n\} \backslash J$ and $J \cup j = J \cup \{j\}$, and define $\ell(J) = \sum_{j \in J} l_J$. We say that J is short if $\ell(J) < \ell(J^c)$. We also define the manifolds

$$V_J = \{(u, z) \in M : \forall j \notin J \ (u_j, z_j) = *\}$$

$$W_J = \{(u,z) \in M : \forall j,k \notin J,\ u_j = u_k, z_j = z_k = 0\}$$

where $* \in S^{2a+2b-1} \cap (\mathbb{C}^a \times \{0\})$ is a chosen base point. Notice that V_J is homeomorphic to a product of $|J|$ spheres of dimension d and $W_J \cong V_J \times S^{2a-1}$. These homeomorphisms imply that $V_J \subseteq W_J$, $\dim V_J = |J|d$ and $\dim W_J = |J|d + (2a-1)$.

Let $[V_J], [W_J]$ be the respective homological orientation classes of V_J and W_J and $[V_J]_H$, $[W_J]_H$ their equivariant lifting. Then $H_*(M)$ is free with basis $\{[V_J] : J \subseteq \{1,\dots,n\}\}$ and $H_*(M\backslash X)$ is free with basis $\{[V_J],[W_J] : J \text{ short}\}$ [21, Lem. 3.2].

Analogously to [21, Lem. 4.5, Prop.4.6] we have the following description in H-equivariant cohomology.

Proposition 6.6. *Let $\iota\colon M\backslash X \to M$ be the inclusion map.*

(i) $H_*^H(M)$ is a free $H^*(BH)$-module with basis $\{[V_J]_H,\ J \subseteq \{1,\dots,n\}\}$.
(ii) $H_*^H(M\backslash X)$ is a free $H^*(BH)$-module with basis $\{[V_J]_H,\ [W_J]_H,\ J$ short $\}$.
(iii) $\iota_*^H([V_J]_H) = [V_J]_H$ and $\iota_*^H([W_J]_H) = \sum_{j\notin J} w_j^b (w_j + w)^b [V_{J\cup j}]_H$.

Proof. Notice that $b(M) = b(M^H)$ as $M^H \cong (S^{2a-1})^n$ and so M is H-equivariantly formal. Moreover, we obtain that the restriction map

$$H_*^H(M) \to H_*(M)$$

which is the edge homomorphism of the homological spectral sequence with E_2-term given by $E_2 = H_*(M) \otimes H^*(BH)$ and converging to $H_*^H(M)$ is surjective since the basic elements $[V_J]$ have a lifting in $H_H^*(M)$. Therefore, as in the Leray-Hirsch Theorem, the spectral sequence collapses and so $\{[V_J]_H,\ J \subseteq \{1,\dots,n\}\}$ is a basis of $H_*^H(M)$ over $H^*(BH)$, thus proving (i). The proof of (ii) follows in a similar fashion.

To prove (iii) we will use the H-equivariant Euler class to compute explicitly the map ι_*^H on the generators of $H_*^H(M\backslash X)$.

Let $K = K_1 \times K_2$ where $K_1 = \{1, g\}, K_2 = \{1, \tau\}$, g denotes the action induced by multiplication by -1 and τ the complex conjugation in $\mathbb{C}$, and let x, w denote the canonical generators of $H^*(BK_1)$ and $H^*(BK_2)$ dual to the generators of K_1 and K_2 respectively. Similarly to 6.2, we get $e_K(S^{K_1} \subseteq S) = x^b(x+w)^b$, or equivalently, $[S^{K_1}]_K = x^b(x+w)^b[S]_K$.

Now the proof for the case of the torus action on the big polygon space found in [21, Lem.4.5] can be imitated in our situation to show that (iii) holds. Firstly, the identity $i_*^H([V_J]_H) = [V_J]_H$ follows from the naturality of the equivariant homology, that is, from the commutative diagram

$$\begin{tikzcd} H_*^H(V_J) \arrow[d] \arrow[dr] & \\ H_*^H(M\backslash X) \arrow[r, "\iota_*^H"] & H_*^H(M) \end{tikzcd} .$$

To compute $i_*^H([W_J])$, we need to "enlarge" the acting group. For $J \subseteq [n]$, define τ_J to be the involution on M given by the complex conjugation on the variables $u_j : j \in J$ and $z_j : j \in J$, and write $\sigma_J = \tau_{J^c}$. Set $H_J = T_2 \times \tau_J \times \sigma_J$ and $H \to H_J$ the map induced by the identity on T_2 and the map which sends τ to (τ_J, σ_J). Thus we get a map $H^*(BH_J) = \Bbbk[t_1,\dots,t_n,w_\tau,w_\sigma] \to H^*(BH) =$

$\Bbbk[t_1, \ldots, t_n, w]$ sending w_τ and w_σ to w which is the identity in the other variables. Moreover, we have maps in equivariant homology

$$H_*^{H_J}(M) \to H_*^H(M)$$

Notice that the H_J-action on M induces an action of H on M; such an action coincides with the initial action of H on M described at the beginning of the section. Also, we have similar restriction maps for the H_J-invariant submanifolds $X, M\backslash X \subseteq M$.

Set $\tilde{M} = M \cap (\mathbb{C}^a \times \{0\})^n \cong (S^{2a-1})^n$. For $J \subseteq [n]$, let Δ_J be the inclusion of S^{2a-1} into the factors $j \in J$ of $\tilde{M}$. Notice that there is a homeomorphism $W_J \cong V_J \times \Delta_{J^c}$; moreover, such homeomorphism yields to an equivariant decomposition $H_J = (K_J \times \tau_J) \times (K_{J^c} \times \sigma_J)$ where $K_J \subseteq T_2$ is the 2-subtorus of non-trivial factors in the position $j \in J$. Therefore, by the Künneth theorem in equivariant homology for 2-torus actions (compare with [21, Prop.4.1]) we have that

$$[W_J]_{H_J} = [V_J]_{K_J \times \tau_J} \times [\Delta_{J^c}]_{K_{J^c} \times \sigma_J}$$

By naturality of the Euler class, we have

$$i_*^{H_J}([W_J]_{H_J}) = i_*^{K_J \times \tau_J}([V_J]_{K_J \times \tau_J}) \times i_*^{K_{J^c} \times \sigma_J}([\Delta_{J^c}]_{K_{J^c} \times \sigma_J}) \tag{6.1}$$

As above, it is straightforward to check that $i_*^{K_J \times \tau_J}([V_J]_{K_J \times \tau_J}) = [V_J]_{K_J \times \tau_J}$, so it only remains to compute the last term of (6.1). Without loss of generality we can assume that $J = \varnothing$, so $\Delta_{J^c} = \Delta$ is the diagonal of $\tilde{M}$, $\sigma_J = \tau$, τ_J is trivial and $H_J = H$. So we need to compute $i_*^H([\Delta]_H)$. Since in $H_*(\tilde{M})$ we have that $[\Delta] = \sum_{j=1}^n [\Delta_j]$ and $\tilde{M}$ is H-equivariantly formal, we have then in equivariant homology that $[\Delta]_H = \sum_{j=1}^n [\Delta_j]_H$. Consider the inclusion $K_1 \to H$ into the j-th factor of T_2 and denote the image by K_j. This map induces in cohomology an identification of x with t_j. Observe that $\Delta_j = V_j^{T_2} = V_j^{K_j}$ and thus $[\Delta_j]_H = [V_j^{K_j}]_{K_j \times \tau}$. We obtain by naturality of the Euler class and the above computation that $[\Delta_j]_H = t_j^b (t_j + w)^b [V_j]_H$. Finally this implies that

$$i_*^H([\Delta]_H) = \sum_{j=1}^n t_j^b (t_j + w)^b [V_j]_H \tag{6.2}$$

For the general case, using this computation, for any J we have again by (6.1) that

$$\begin{aligned} i_*^{H_J}([W_J]_{H_J}) &= i_*^{K_J \times \tau_J}([V_J]_{K_J \times \tau_J}) \times i_*^{K_{J^c} \times \sigma_J}([\Delta_{J^c}]_{K_{J^c} \times \sigma_J}) \\ &= [V_J]_{K_J \times \tau_J} \times \sum_{j \notin J} t_j^b (t_j + w_\sigma)^b [V_j]_{K_{J^c} \times \sigma_J} \\ &= \sum_{j \notin J} t_j^b (t_j + w_\sigma)^b [V_{J \cup \{j\}}]_{H_J} \end{aligned}$$

The computation for the H-equivariant cohomology follows by naturality and using the restriction map $H^*(BH_J) \to H^*(BH)$ which sends w_σ to w.

□

Let $R = H^*(BH) = \mathbb{F}_2[t_1, \ldots, t_n, w]$ and write $y_j = t_j(t_j + w)$. Let $n = 2m + 1$ and $\ell = (1, \ldots, 1)$. We will use the Koszul resolution of $L = R/(y_1^b, \ldots, y_n^b)$ to identify the H-equivariant cohomology of the big polygon space $X = X_{a,b}(\ell)$ with the Koszul syzygies appearing in such resolution. The assumption on ℓ is made so for $J \subseteq [n]$ is short if and only if $\ell(J) < m$.

The following result follows from the analogous case of equivariant cohomology for torus actions on the big polygon spaces [21, §5] and only an outline of the proof will be presented.

Theorem 6.7. *Let $n = 2m+1$, $m \geqslant 1$. The G-equivariant cohomology of the equilateral big polygon space*

$$X = \left\{ (u,z) \in (S^{2a+2b-1})^n : \sum_{i=1}^{n} u_i = 0 \right\}$$

is an m-th syzygy but not an $(m+1)$-st syzygy.

Proof. Let $\iota : M\backslash X \to M$ be the inclusion and let ι_*^H be the induced map in equivariant homology. For simplicity set $d = 2a+2b-1$; The equivariant Poincaré-Alexander-Lefschetz duality [4, Thm.7.6] implies that there is a short exact sequence

$$0 \to \operatorname{Coker} i_*^H[nd] \to H_H^*(X) \to \ker i_*^H[nd-1] \to 0. \tag{6.3}$$

From Proposition 6.6 we have that $H_H^*(M\backslash X) \cong \bigoplus_{|J|\leqslant m} (R\cdot[V_J]_H \oplus R\cdot[W_J]_H)$ and $H_H^*(M) \cong \oplus_{J\subseteq[n]} R\cdot[V_J]_H$ as R-modules. By Proposition 6.6 (iii), the Kernel of the map

$$\iota_*^H\colon \bigoplus_{|J|\leqslant m} R\cdot[V_J]_H \oplus \bigoplus_{|J|<m} R\cdot[W_J]_H \to H_*^H(M)$$

is the free R-submodule of $H_*^H(M\backslash X)$ generated by the elements $[W_J]_H - \sum_{j\in J} y_j^b[V_{J\cup j}]_H$ where $|J| < m$ since $\iota_*^H([V_J]_H) = [V_J]_H$ and $\iota_*^H([W_J]_H) = \sum_{j\notin J} t_j^b(t_j+w)^b[V_{J\cup j}]_H$. On the other hand, the map

$$\iota_*^H\colon \bigoplus_{|J|=m} R\cdot[W_J]_H \to \bigoplus_{|J|=m+1} R\cdot[V_J] \subseteq H_*^H(M)$$

can be identified with the map d_{m+1} in the Koszul resolution of $L = R/(y_1^b,\dots,y_n^b)$ described above whose kernel is the Koszul syzygy K_{m+2}. So we obtain that

$$\ker(\iota_*^H) \cong \bigoplus_{|J|<m} R[-|J|d - \bar{d}] \oplus K_{m+2}[-md - \bar{d} + 2]$$

The degree shifts follows from the fact that $\dim W_J = |J|d + \bar{d}$ and $\dim V_J = |J|d$ and the convention that the Koszul syzygies are generated in degree 0.

Similarly, we can see that $\operatorname{im}(\iota_*^H) \cong \bigoplus_{|J|\leqslant m} R\cdot[V_J]_H \oplus \operatorname{im}(d_{m+1})$ and thus

$$\operatorname{Coker}(\iota_*^H) = H_*^H(M)/\operatorname{im}(\iota_*^H) \cong \bigoplus_{|J|>m+1} R\cdot[V_J]_H \oplus \operatorname{Coker}(d_{m+1}).$$

Notice that from the Koszul resolution it follows that $\operatorname{Coker}(d_{m+1}) \cong \operatorname{im}(d_{m+2}) = K_m$ the m-th Koszul syzygy of L. Summarizing, we obtained that

$$\operatorname{Coker}(\iota_*^H) \cong \bigoplus_{|J|>m+1} R[-|J|d] \oplus K_m[-(m+1)d].$$

and thus both $\ker(\iota_*^H)$ and $\operatorname{Coker}(\iota_*^H)$ are m-th syzygies. To finish the proof, it is enough to show that the sequence (6.3) splits. This will follow from [31, Lem.3.12] and using that the singular Cartan model as a free R-model for the G-equivariant cohomology. □

Finally, from 5.5 we can obtain the syzygy order of the equilateral real big polygon spaces recovering also one of the main results in [31, Thm.1.2].

Corollary 6.8. *The equivariant cohomology of the equilateral real big polygon space X^τ under the action of the 2-torus T_2 is an m-th syzygy but not an $(m+1)$-st syzygy.*

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