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Homology of the double and triple loop space of SO(n)

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1 Introduction

Let G be a compact, connected, simple Lie group and let $\pi:P \longrightarrow S^4$ be a principal G-bundle over S^4. Since $\pi_4(BG) = \pi_3(G) = Z$, we can classify the principal bundle P_k over S^4 by the map $S^4 \longrightarrow BG$ of degree k. As Atiyah and Jones [1] pointed out, $\mathscr{C}_k(G) = A_k/\mathscr{G}^b(P_k)$ is homotopy equivalent to $\Omega^3_k G \simeq \Omega^4_k BG$, that is, $\Omega^3 G \simeq \mathscr{C}(G)$, where A_k is the space of the all connections on P_k and $\mathscr{G}^b(P_k)$ is the group of all base-point preserving automorphisms on P_k. In this paper, we study the homology with coefficient $\mathbf{Z}/(p)$ of the double loop space and the triple loop space of $SO(n)$. Especially the homology of the triple loop space of $SO(n)$ was one of the questions in [3] because it contains the homological informations of $\mathscr{M}_k(SO(n))$, the moduli space of instantons for $SO(n)$ with instanton number k, by the natural inclusion $\iota_k : \mathscr{M}_k(SO(n)) \rightarrow \mathscr{C}_k(SO(n))$. For more informations we refer to [4].

Harris [6] proved that for p odd

$$\begin{aligned} SU(2n) &\simeq_p SU(2n)/Sp(n) \times Sp(n) \\ SU(2n+1) &\simeq_p SU(2n+1)/SO(2n+1) \times SO(2n+1) \end{aligned}$$

where $\simeq_p$ means the homotopy equivalence localized at p. But we already know $H_*(\Omega^k SU(n); Z/(p))$ when $k = 2, 3$ [8],[9]. From above facts we can get $H_*(\Omega^k SO(n); Z/(p))$ easily for odd p. Therefore we concentrate on the case at $p = 2$. Since $Spin(n)$ is the double covering space of $SO(n)$, $\Omega^2 Spin(n) \simeq \Omega^2 SO(n)$. Here we will study $Spin(n)$ instead of $SO(n)$.

First we compute the cohomology of $\Omega Spin(n)$, and then using the the Serre spectral sequence for the following fibraton

$$\Omega^2 Spin(n-1) \longrightarrow \Omega^2 Spin(n) \longrightarrow \Omega^2 S^{n-1}$$

we compute $H_*(\Omega^2 Spin(n); \mathbf{Z}/(2))$, and determine some of the Steenrod actions on $H_*(\Omega^2 Spin(n); \mathbf{Z}/(2))$. By the Bockstein spectral sequence, we get also the

2–torsion information for $H_*(\Omega^2 Spin(n); Z)$. The interesting fact of these computations is that the structures of $H_*(\Omega^2 Spin(n); \mathbf{Z}/(2))$ depend on the congruence of n mod 8. Similarly we compute the homology of $\Omega_0^3 Spin(n) \simeq \Omega_0^3 SO(n)$.

2 The basic facts and $H^*(\Omega Spin(n); \mathbf{Z}/(2))$

Let $E(x)$ be the exterior algebra on x and $P(x)$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power algebra on x which is free over $\gamma_i(x)$ with coproduct

$$\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$$

and the product

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x).$$

For $(n+1)$–fold loop spaces, there are homology operations

$$Q_i : H_q(\Omega^{n+1}X; \mathbf{Z}/(2)) \longrightarrow H_{2q+i}(\Omega^{n+1}X; \mathbf{Z}/(2))$$

defined for $0 \le i \le n$ which is natural for $(n+1)$–fold loop spaces. Let Q_i^a be the iterated operation $Q_i \dots Q_i (a$ times). If G is a Lie group, G is homotopy equivalent to ΩBG. Hence Q_2 is defined in $H_*(\Omega^2 G; \mathbf{Z}/(2))$ and Q_3 is defined in $H_*(\Omega^3 G; \mathbf{Z}/(2))$. Throughout this paper, the subscript of an element always denotes the degree of an element, i.e. ,i is the degree of x_i. We also recall the following. Let $V(x_{i_1}, \dots, x_{i_t})$ be the commutative associative algebra over $\mathbf{Z}/(2)$ such that

1. $\{(x_{i_1})^{\epsilon_i}, \dots, (x_{i_t})^{\epsilon_i} : \epsilon_i = 0, 1\}$ is a basis.
2. $(x_{i_q})^2 = x_{i_s}$ if $2i_q = i_s$ for some $1 \le s \le t$
   $(x_{i_q})^2 = 0$ otherwise.

Choose s such that $2^s < n \le 2^{s+1}$. Then

$$\begin{aligned} H^*(Spin(n); \mathbf{Z}/(2)) &= V(x_i | 3 \le i \le n-1 \text{ and } i \ne 2^j) \otimes E(z), \\ Sq^r(x_i) &= \tbinom{i}{r} x_{i+r}. \end{aligned} \tag{2.1}$$

where $|z| = 2^{s+1} - 1$. In fact we have the Steenrod actions on z [7]. But we do not need it here. For small values of n, it is well known that

$$\begin{aligned} Spin(3) &\simeq S^3 \\ Spin(4) &\simeq S^3 \times S^3 \\ Spin(5) &\simeq Sp(2) \\ Spin(6) &\simeq SU(4) \\ Spin(7)_{(2)} &\simeq (G_2 \times S^7)_{(2)} \\ Spin(8)_{(2)} &\simeq (Spin(7) \times S^7)_{(2)} \end{aligned}$$

Now we will compute $H^*(\Omega Spin(n); \mathbf{Z}/(2))$.

Lemma 2.2 $H^*(\Omega Spin(8n);\mathbf{Z}/(2))$, $n>0$, *is*

$$P(a_{4i-2}:1\le i\le n)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+2+4k}:0\le k\le(n-1))$$
$$\otimes\Gamma(c_{8n-2+2k}:0\le k\le(4n-2),\ k\not\equiv 3\ mod\ 4)$$
where ν_i *is the power of* 2 *such that* $8n\le\nu_i(4i-2)\le 16n-8$.

$H^*(\Omega Spin(8n+1);\mathbf{Z}/(2))$, $n>0$, *is*

$$P(a_{4i-2}:1\le i\le n)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+2+4k}:0\le k\le(n-1))$$
$$\otimes\Gamma(c_{8n+2k}:0\le k\le(4n-1),\ k\not\equiv 2\ mod\ 4)$$
where ν_i *is the power of* 2 *such that* $8n\le\nu_i(4i-2)\le 16n-8$.

$H^*(\Omega Spin(8n+2);\mathbf{Z}/(2))$, $n>0$, *is*

$$P(a_{4i-2}:1\le i\le n)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+2+4k}:0\le k\le(n-1))$$
$$\otimes\Gamma(c_{8n+2+2k}:0\le k\le(4n-2),\ k\not\equiv 1\ mod\ 4)$$
$$\bigotimes_{i\ge 0}P(\gamma_{2^i}(d_{8n}))/((\gamma_{2^i}(d_{8n}))^4)$$
where ν_i *is the power of* 2 *such that* $8n+8\le\nu_i(4i-2)\le 16n$

$H^*(\Omega Spin(8n+3);\mathbf{Z}/(2))$ *is*

$$P(a_{4i-2}:1\le i\le n)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+2+4k}:0\le k\le n-1)$$
$$\otimes\Gamma(c_{8n+2+2k}:0\le k\le 4n,\ k\not\equiv 1\ mod\ 4)$$
where ν_i *is the power of* 2 *such that* $8n+8\le\nu_i(4i-2)\le 16n$.

$H^*(\Omega Spin(8n+4);\mathbf{Z}/(2))$ *is*

$$P(a_{4i-2}:1\le i\le n)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+2+4k}:0\le k\le n)$$
$$\otimes\Gamma(c_{8n+2+2k}:0\le k\le 4n,\ k\not\equiv 1\ mod\ 4)$$
where ν_i *is the power of* 2 *such that* $8n+8\le\nu_i(4i-2)\le 16n$.

$H^*(\Omega Spin(8n+5);\mathbf{Z}/(2))$ *is*

$$P(a_{4i-2}:1\le i\le n)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+2+4k}:0\le k\le n)$$
$$\otimes\Gamma(c_{8n+6+2k}:0\le k\le 4n,\ k\not\equiv 3\ mod\ 4)$$
where ν_i *is the power of* 2 *such that* $8n+8\le\nu_i(4i-2)\le 16n$.

$H^*(\Omega Spin(8n+6);\mathbf{Z}/(2))$ *is*

$$P(a_{4i-2}:1\le i\le n+1)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+6+4k}:0\le k\le n-1)$$
$$\otimes\Gamma(c_{8n+6+2k}:0\le k\le 4n,\ k\not\equiv 3\ mod\ 4)$$
$$\bigotimes_{i\ge 0}P(\gamma_{2^i}(b_{8n+4}))/((\gamma_{2^i}(b_{8n+4}))^4)$$
where ν_i *is the power of* 2 *such that* $8n+8\le\nu_i(4i-2)\le 16n+8$.

$H^*(\Omega Spin(8n+7);\mathbf{Z}/(2))$ *is*

$$P(a_{4i-2}:1\le i\le n+1)/(a_{4i-2}^{\nu_i})\otimes\Gamma(a_{4n+6+4k}:0\le k\le n-1)$$
$$\otimes\Gamma(c_{8n+6+2k}:0\le k\le 4n+2,\ k\not\equiv 3\ mod\ 4)$$
where ν_i *is the power of* 2 *such that* $8n+8\le\nu_i(4i-2)\le 16n+8$.

Proof. Let $H^*(\Omega S^n;\mathbf{Z}/(2)) = \Gamma(a_{n-1})$. We will prove this lemma by induction on k for $H^*(\Omega Spin(k);\mathbf{Z}/(2))$. Assume that it hold for $k \leq 8n+3$. Remind that $\Omega Spin(3) \simeq \Omega S^3$. For $H^*(\Omega Spin(8n+4);\mathbf{Z}/(2))$, we have the following fibration

$$\Omega Spin(8n+3) \longrightarrow \Omega Spin(8n+4) \longrightarrow \Omega S^{8n+3}.$$

Since both $H^*(\Omega Spin(8n+3);\mathbf{Z}/(2))$ and $H^*(\Omega S^{8n+3};\mathbf{Z}/(2))$ are even dimensional, the Serre spectral sequence collapses. There is no extension problem by the dimension reason.

For next step consider the following fibration

$$\Omega Spin(8n+4) \longrightarrow \Omega Spin(8n+5) \longrightarrow \Omega S^{8n+4}.$$

It is well known that $H_*(\Omega Spin(8n+5);\mathbf{Z}/(2))$ concentrates in the even dimensions [2]. Therefore so does $H^*(\Omega Spin(8n+5);\mathbf{Z}/(2))$. Since $H^*(\Omega S^{8n+4};\mathbf{Z}/(2))$ contains an $(8n+3)$ dimensional element, we have the first non–zero differential which comes from an $(8n+2)$–dimensional generator in $H^*(\Omega Spin(8n+4);\mathbf{Z}/(2))$ and goes to a_{8n+3}. But in $H^*(\Omega Spin(8n+4);\mathbf{Z}/(2))$ we have two generators a_{8n+2}, c_{8n+2} of that dimension. So consider the morphism of fibrations

$$\begin{array}{ccccc}
\Omega Spin(8n+3) & \longrightarrow & \Omega Spin(8n+5) & \longrightarrow & \Omega Spin(8n+5)/Spin(8n+3) \\
f\downarrow & & \downarrow & & \downarrow \\
\Omega Spin(8n+4) & \longrightarrow & \Omega Spin(8n+5) & \longrightarrow & \Omega S^{8n+4} \\
g\downarrow & & \downarrow & & h\downarrow \\
\Omega S^{8n+3} & \longrightarrow & * & \longrightarrow & S^{8n+3}.
\end{array}$$

From the naturality of the differential we have

$$\begin{aligned}
\tau(g^*(a_{8n+2})) &= h^*(\tau(a_{8n+2})) \\
&= h^*(x_{8n+3}) \\
&= 0
\end{aligned}$$

,where $H^*(S^{8n+3};\mathbf{Z}/(2)) = E(x_{8n+3})$ and τ is the transgression. Hence we have the differential with the source c_{8n+2} to a_{8n+3} and from $\gamma_2(c_{8n+2})$ to $c_{8n+2}a_{8n+3}$ and so on. $\gamma_{2^{i+1}}(a_{8n+3})$ survives permanently for $i \geq 0$. Put $\gamma_2(a_{8n+3}) = c_{16n+6}$.

For $H^*(\Omega Spin(8n+6))$ consider the following fibration

$$\Omega Spin(8n+5) \longrightarrow \Omega Spin(8n+6) \longrightarrow \Omega S^{8n+5}.$$

By the same reason as the case $H^*(\Omega Spin(8n+4);\mathbf{Z}/(2))$, the spectral sequence collapses. So we get that the E_∞–term for $H^*(\Omega Spin(8n+6);\mathbf{Z}/(2))$ is

$$P(a_{4i-2} : 1 \leq i \leq n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2}, a_{4n+2}, \ldots, a_{8n+2}) \otimes \Gamma(a_{8n+4})$$
$$\otimes \Gamma(c_{8n+6+2k} : 0 \leq k \leq 4n,\ k \not\equiv 3 mod 4)$$
$$\textit{where } \nu_i \textit{ is the power of 2 such that } 8n+8 \leq \nu_i(4i-2) \leq 16n.$$

But in this case there are extension problems. We claim that $(a_{4n+2})^2 = a_{8n+4}$. From $H^*(Spin(8n+6);\mathbf{Z}/(2))$ we can compute $\mathrm{Tor}_{H^*(Spin(8n+6))}(\mathbf{Z}/(2),\mathbf{Z}/(2))$. Since $Sq^{4n+2}x_{4n+3} = \binom{4n+3}{4n+2}x_{8n+5} = x_{8n+5}$ in $H^*(Spin(8n+6);\mathbf{Z}/(2))$ by (2.1), $(a_{4n+2})^2 = Sq^{4n+2}a_{4n+2} = Sq^{4n+2}\sigma(x_{4n+3}) = \sigma(Sq^{4n+2}x_{4n+3}) = \sigma(x_{8n+5}) = a_{8n+4}$ where σ is the cohomology suspension. So $(\gamma_{2^i}(a_{4n+2}))^2 = \gamma_{2^i}(a_{8n+4})$ for each $i \geq 0$ and $\Gamma(a_{4n+2}) \otimes \Gamma(a_{8n+4})$ produces $\otimes_{i\geq 0} P(\gamma_{2^i}(a_{4n+2}))/((\gamma_{2^i}(a_{4n+2}))^4)$ as an algebra. Let $\otimes_{i\geq 0} P(\gamma_{2^i}(a_{4n+2}))/((\gamma_{2^i}(a_{4n+2}))^4) = P(a_{4n+2})/(a_{4n+2}^4) \otimes_{i\geq 0} P(\gamma_{2^{i+1}}(a_{4n+2}))/((\gamma_{2^{i+1}}(a_{4n+2}))^4)$ and let $\gamma_2(a_{4n+2}) = b_{8n+4}$. Hence we extend the conditions: $1 \leq i \leq n+1$, $\nu_i(4i-2) \leq 16n+8$.

Consider the next fibration

$$\Omega Spin(8n+6) \longrightarrow \Omega Spin(8n+7) \longrightarrow \Omega S^{8n+6}.$$

Since $H^*(\Omega S^{8n+6})$ contains a_{8n+5}, we have the first nonzero differential from b_{8n+4} to a_{8n+5} and the next differentials from $\gamma_2(b_{8n+4})$ to $a_{8n+5}\cdot b_{8n+4}$ and so on. Then $(\gamma_{2^i}(b_{8n+4}))^2$ survives permanently for each $i \geq 0$ but in fact, by the previous step $(\gamma_{2^i}(b_{8n+4}))^2 = (\gamma_{2^{i+1}}(a_{4n+2}))^2 = \gamma_{2^{i+1}}(a_{8n+4})$ for $i \geq 0$. $\gamma_{2^{i+1}}(a_{8n+5})$ is also permanent for each $i \geq 0$. Let $(\gamma_1(b_{8n+4}))^2 = c_{16n+8}$ and $\gamma_2(a_{8n+5}) = c_{16n+10}$.

We can prove the other cases in similar way. The induction from $H^*(\Omega Spin(8n+i);\mathbf{Z}/(2))$ to $H^*(\Omega Spin(8n+1+i);\mathbf{Z}/(2))$ is almost same as that from $H^*(\Omega Spin(8n+4+i);\mathbf{Z}/(2))$ to $H^*(\Omega Spin(8n+5+i);\mathbf{Z}/(2))$. However, compared with $H^*(\Omega Spin(8n+6);\mathbf{Z}/(2))$, we have little different extension problems for $H^*(\Omega Spin(8n+2);\mathbf{Z}/(2))$. Note that in $H^*(Spin(8n+2);\mathbf{Z}/(2))$ $Sq^{4n}x_{4n+1} = x_{8n+1}$, $Sq^{2n}x_{2n+1} = x_{4n+1}$. So $a_{8n} = \sigma(x_{8n+1}) = \sigma(Sq^{4n}x_{4n+1}) = Sq^{4n}\sigma(x_{4n+1}) = Sq^{4n}(a_{4n}) = (a_{4n})^2 = (\sigma(x_{4n+1}))^2 = (\sigma(Sq^{2n}x_{2n+1}))^2 = (Sq^{2n}a_{2n})^2 = a_{2n}^4$. In fact, the difference come from the property of the number: $8n = 2^2 2n$, $8n+4 = 2(4n+2)$.

□

Remark 2.3 If we use the Eilenberg–Moore spectral sequence of Steenrod modules converging to $H^*(\Omega Spin(n);\mathbf{Z}/(2))$ with $E_2 = \mathrm{Tor}_{H^*(Spin(n);\mathbf{Z}/(2))}(\mathbf{Z}/(2),\mathbf{Z}/(2))$, then $E_2 = E_\infty$ and after solving algebra extension problems by the Steenrod actions we get the same result. So we can choose the primitive generators a_i, b_i, c_i such that $\sigma(x_i) = a_j^{2^k}$ where $2^k j = i-1$ or $\sigma(x_i) = b_{i-1}$ according to the dimension and $\sigma(z_i) = c_{i-1}$ and $\rho(x_i^{2^k}) = c_{2^k i-2}$ where σ is the cohomology suspension and $\rho(x_i^{2^k})$ is the transpotence of $x_i^{2^k}$. Note that a_i becomes the stable element.

3 The homology of $\Omega^2 Spin(n)$

Theorem 3.1 *There are choices of the primitive generators u_i, v_i, w_i such that as a Hopf algebra*
$H_*(\Omega^2 Spin 8n;\mathbf{Z}/(2))$, $n > 0$, is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n-1) \otimes P(v_{8n+8k-2} : 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a u_{4n+4k+1} : a \ge 0, 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a w_{8n-3+2k} : a \ge 0, 0 \le k \le 4n-2 \text{ and } k \not\equiv 3 \bmod 4)$$

$H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$, $n > 0$, is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n-1) \otimes P(v_{8n+8k-2} : 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a u_{4n+4k+1} : a \ge 0, 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a w_{8n-1+2k} : a \ge 0, 0 \le k \le 4n-1 \text{ and } k \not\equiv 2 \bmod 4)$$

$H_*(\Omega^2 Spin(8n+2); \mathbf{Z}/(2))$, $n > 0$, is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n-1) \otimes P(v_{8n+8k+6} : 0 \le k \le n-2) \bigotimes$$
$$P(Q_1^a u_{4n+4k+1} : a, b \ge 0, 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a w_{8n+2k+1} : a \ge 0, 0 \le k \le 4n-2 \text{ and } k \not\equiv 1 \bmod 4)$$
$$\otimes E(Q_1^a w_{8n-1} : a \ge 0) \otimes P(Q_2^a v_{16n-2} : a \ge 0)$$

$H_*(\Omega^2 Spin(8n+3); \mathbf{Z}/(2))$ is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n-1) \otimes P(v_{8n+8k+6} : 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a u_{4n+4k+1} : a \ge 0, 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a w_{8n+2k+1} : a \ge 0, 0 \le k \le 4n \text{ and } k \not\equiv 1 \bmod 4)$$

$H_*(\Omega^2 Spin(8n+4); \mathbf{Z}/(2))$ is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n-1) \otimes P(v_{8n+8k+6} : 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a u_{4n+4k+1} : a \ge 0, 0 \le k \le n) \bigotimes$$
$$P(Q_1^a w_{8n+2k+1} : a \ge 0, 0 \le k \le 4n \text{ and } k \not\equiv 1 \bmod 4)$$

$H_*(\Omega^2 Spin(8n+5); \mathbf{Z}/(2))$ is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n-1) \otimes P(v_{8n+8k+6} : 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a u_{4n+4k+1} : a \ge 0, 0 \le k \le n) \bigotimes$$
$$P(Q_1^a w_{8n+5+2k} : a \ge 0, 0 \le k \le 4n \text{ and } k \not\equiv 3 \bmod 4)$$

$H_*(\Omega^2 Spin(8n+6); \mathbf{Z}/(2))$ is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n) \otimes P(v_{8n+8k+6} : 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a u_{4n+4k+5} : a \ge 0, 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a w_{8n+5+2k} : a \ge 0, 0 \le k \le 4n \text{ and } k \not\equiv 3 \bmod 4)$$
$$\otimes E(Q_1^{a+1} u_{4n+1} : a \ge 0) \otimes P(Q_2^a v_{16n+6} ; a \ge 0)$$

$H_*(\Omega^2 Spin(8n+7); \mathbf{Z}/(2))$ is isomorphic to

$$E(u_{4k+1} : 0 \le k \le n) \otimes P(v_{8n+8k+6} : 0 \le k \le n) \bigotimes$$
$$P(Q_1^a u_{4n+4k+5} : a \ge 0, 0 \le k \le n-1) \bigotimes$$
$$P(Q_1^a w_{8n+5+2k} : a \ge 0, 0 \le k \le 4n+2 \text{ and } k \not\equiv 3 \bmod 4)$$

Proof. Recall that there is a choice of a generator ι_{n-2} such that $H_*(\Omega^2 S^n; \mathbf{Z}/(2))$ is isomorphic to $P(Q_1^a \iota_{n-2} | a \geq 0), n > 2$ as a Hopf algebra. We will compute $H_*(\Omega^2 Spin(m))$ by induction on m by studying the Serre spectral sequence for the fibration

$$\Omega^2 Spin(m) \longrightarrow \Omega^2 Spin(m+1) \longrightarrow \Omega^2 S^m.$$

Note that $\Omega^2 Spin(3) \simeq \Omega^2 S^3$. Hence we can start the induction.

(Case 1). From $H_*(\Omega^2 Spin(8n+3); \mathbf{Z}/(2))$ to $H_*(\Omega^2 Spin(8n+4); \mathbf{Z}/(2))$. Consider the map of fibrations

$$\begin{array}{ccccc} \Omega^3 S^{8n+3} & \longrightarrow & * & \longrightarrow & \Omega^2 S^{8n+3} \\ \downarrow & & \downarrow & & \| \\ \Omega^2 Spin(8n+3) & \longrightarrow & \Omega^2 Spin(8n+4) & \longrightarrow & \Omega^2 S^{8n+3}. \end{array}$$

We know that the source of the first non–trivial differential is an indecompasable element and the target is a primitive element in the spectral sequence of a Hopf algebra. But in $H_*(\Omega^2 Spin(8n+3); \mathbf{Z}/(2))$ there is no $8n$–dimensional primitive element. So in the Serre spectral sequence for the second row, $\tau(\iota_{8n+1}) = 0$. From the commutativity of the diagram and the naturality of the Dyer– Lashof operation, the spectral sequence of the second row fibration collapses and we let $\iota_{8n+1} = u_{8n+1}$. Note that $Spin4 \simeq S^3 \times S^3$.

(Case 2). From $H_*(\Omega^2 Spin(8n+4); \mathbf{Z}/(2))$ to $H_*(\Omega^2 Spin(8n+5); \mathbf{Z}/(2))$. Consider the map of fibrations

$$\begin{array}{ccccc} \Omega^3 S^{8n+4} & \longrightarrow & * & \longrightarrow & \Omega^2 S^{8n+4} \\ {\scriptstyle f}\downarrow & & \downarrow & & \| \\ \Omega^2 Spin(8n+4) & \xrightarrow{\Omega^2 i} & \Omega^2 Spin(8n+5) & \xrightarrow{\Omega^2 \pi} & \Omega^2 S^{8n+4} \end{array} \qquad (3.2)$$

We will show that the first differential of the spectral sequence of the second row fibration is not zero. Assume that it is zero. Then we have a surjection $\Omega^2\pi_*$ from $H_*(\Omega^2 Spin(8n+5); \mathbf{Z}/(2))$ onto $H_*(\Omega^2 S^{8n+4}; \mathbf{Z}/(2))$ sending $(8n+2)$ dimensional element, we call it x_{8n+2}, to ι_{8n+2}. But we have the map of fibrations

$$\begin{array}{ccc} \Omega^2 Spin(8n+5) & \longrightarrow & \Omega^2 S^{8n+4} \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \\ \downarrow & & \downarrow \\ \Omega Spin(8n+5) & \xrightarrow{\Omega\pi} & \Omega S^{8n+4}. \end{array}$$

By naturality,

$$(\Omega\pi)_*(\sigma(x_{8n+2})) = \sigma(\iota_{8n+2}) \neq 0$$

Therefore $\sigma(x_{8n+2})$ should be non-zero odd dimensional primitive element in $H_*(\Omega Spin(8n+5); \mathbf{Z}/(2))$. But $H_*(\Omega Spin(8n+5); \mathbf{Z}/(2))$ concentrates in even dimensions, so this is a contradiction. Thus we have nonzero first differential from ι_{8n+2} to a $(8n+1)$ dimensional primitive element, however, we have two primitive elements u_{8n+1}, w_{8n+1} of $8n+1$ dimension in $H_*(\Omega^2 Spin(8n+4); \mathbf{Z}/(2))$. Consider the morphism of fibrations

$$\begin{array}{ccccc}
\Omega^2 Spin(8n+3) & \longrightarrow & \Omega^2 Spin(8n+5) & \longrightarrow & \Omega^2 Spin(8n+5)/Spin(8n+3) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 Spin(8n+4) & \longrightarrow & \Omega^2 Spin(8n+5) & \longrightarrow & \Omega^2 S^{8n+4} \\
g\downarrow & & \downarrow & & h\downarrow \\
\Omega^2 S^{8n+3} & \longrightarrow & * & \longrightarrow & \Omega S^{8n+3}.
\end{array}$$

$g_*(\tau(\iota_{8n+2})) = \tau(h_*(\iota_{8n+2}))$. We can check easily from the Serre spectral sequence of the third column fibration that $h_*(\iota_{8n+2}) = 0$. So $g_*(\tau(\iota_{8n+2})) = 0$. From the Case 1 we know that $g_*(u_{8n+1}) = \iota_{8n+1}$. Hence we should choose w_{8n+1} for the target of the first differential in the second row spectral sequence. Since $\tau(Q_0^a(\iota_{8n+2}) = f_*(Q_1^a \iota_{8n+1}) = Q_1^a(f_*(\iota_{8n+1})) = Q_1^a w_{8n+1}$ in (3.2) , $P(Q_1^a w_{8n+1} : a \geq 0)$ is contained in $ker(\Omega^2 i)_*$. Next we claim that $Q_2(w_{8n+1}) = 0$. If so, in 3.2 $\tau(Q_1(\iota_{8n+2}) = f_*(Q_2\iota_{8n+1}) = Q_2(f_*(\iota_{8n+1})) = Q_2 w_{8n+1} = 0$. Then we get the conclusion as we expect. From now on we will show that $Q_2(w_{8n+1}) = 0$. Consider the following fibration

$$\Omega^2 Spin(8n+5) \longrightarrow \Omega^2 Spin \xrightarrow{f} \Omega^2 Spin/Spin(8n+5).$$

By the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 Spin(8n+5); \mathbf{Z}/(2))$

$$\begin{aligned}
E_2 &= Cotor^{H_*(\Omega^2 Spin/Spin(8n+5);\mathbf{Z}/(2))}(H_*(\Omega^2 Spin; \mathbf{Z}/(2)), \mathbf{Z}/(2)) \\
&= Cotor^{H_*(\Omega^2 Spin/Spin(8n+5);\mathbf{Z}/(2))//f_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \\
&\quad \otimes H_*(\Omega^2 Spin; \mathbf{Z}/(2))\backslash\backslash f_* .
\end{aligned} \tag{3.3}$$

This is a spectral sequence of Hopf algebras but it depends on the coalgebra structure.

Now we will compute $H_*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))$. First consider the following fibration

$$Spin(8n+5) \longrightarrow Spin \longrightarrow Spin/Spin(8n+5).$$

Since $H^*(Spin(8n+5); \mathbf{Z}/(2)) = V(x_i | 3 \leq i \leq 8n+4 \text{ and } i \neq 2^j) \otimes E(z)$ and $H^*(Spin; \mathbf{Z}/(2)) = V(x_i | i \geq 3 \text{ and } i \neq 2^j)$, $H^*(Spin/Spin(8n+5); \mathbf{Z}/(2)) = V(x_i | i \geq 8n+5 \text{ and } i \neq 2^j) \otimes P(z')$, where $|z| = 2^{s+1} - 1$, $2^s < 8n+5 \leq 2^{s+1}$ and $\tau(z) = z'$. So $8n+5 \leq |z'| < 16n+10$. From the Steenrod actions on x_i (2.1) we get

$$H^*(Spin/Spin(8n+5); \mathbf{Z}/(2)) = P(x_{8n+5+2k} | k \geq 0) \otimes P(y_{8n+6+2k} | 0 \leq k \leq 4n+1)$$

where we put $x_{8n+6+2k} = y_{8n+6+2k}$ and $z' = y_{2^s+1}$. Using the Eilenberg–Moore spectral sequence with the path loop fibration converging to $H^*(\Omega Spin/Spin(8n+5);\mathbf{Z}/(2))$,

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(Spin/Spin(8n+5);\mathbf{Z}/(2))}(\mathbf{Z}/(2),\mathbf{Z}/(2)) \\ &= E(a_{8n+4+2k}\,|k \geq 0)\otimes \\ &\quad\ E(w_{8n+5+2k}\,|0 \leq k \leq 4n+1)\,. \end{aligned}$$

By the bidegree reason the spectral sequence collapses from E_2–term. But since the Eilenberg–Moore spectral sequence preserves the Steenrod actions, we have the following extensions. $Sq^{8n+4+2k}a_{8n+4+2k} = a_{16n+8+4k}$, that is, $a^2_{8n+4+2k} = a_{16n+8+4k}$ for $k \geq 0$. Hence we get

$$\begin{aligned} &H^*(\Omega Spin/Spin(8n+5);\mathbf{Z}/(2)) = P(a_{8n+6+4k} : k \geq 0)\otimes \\ &P(z_{8n+4+4k} : 0 \leq k \leq 2n) \otimes E(w_{8n+5+2k}|0 \leq k \leq 4n+1) \end{aligned}$$

where we put $a_{8n+4+4k} = z_{8n+4+4k}$. For the next step consider the morphism of fibrations

$$\begin{array}{ccccc} \Omega^2 Spin & \longrightarrow & \Omega^2 Spin/Spin(8n+5) & \longrightarrow & \Omega Spin(8n+5) \\ \| & & \downarrow & & \downarrow \\ \Omega^2 Spin & \longrightarrow & * & \longrightarrow & \Omega Spin \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \Omega Spin/Spin(8n+5) & \longrightarrow & \Omega Spin/Spin(8n+5). \end{array}$$

From Lemma 2.2 $H^*(\Omega Spin(8n+5);\mathbf{Z}/(2))$ is

$$\begin{gathered} P(a_{4i-2} : 1 \leq i \leq n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \leq k \leq n) \\ \otimes\Gamma(c_{8n+6+2k} : 0 \leq k \leq 4n,\ k\not\equiv 3\,mod\,4) \\ \textit{where } \nu_i \textit{ is the power of } 2 \textit{ such that } 8n+8 \leq \nu_i(4i-2) \leq 16n \end{gathered}$$

and we know that $H^*(\Omega Spin;\mathbf{Z}/(2)) = P(a_{4i-2} : i \geq 1)$ and
$H^*(\Omega^2 Spin;\mathbf{Z}/(2)) = E(e_{4i-3} : i \geq 1)$ where $\sigma(a_{4i-2}) = e_{4i-3}$.

Studying the behaviors of the the Serre spectral sequence of the second row fibration and the third column fibration and the naturality of the differentials, we have

$$\tau(e_{4j-3}) = \begin{cases} a_{4j-2} & ,1 \leq j \leq (2n+1) \\ 0 & ,j > (2n+1) \end{cases}$$

in the Serre spectral sequence converging to $H^*(\Omega^2 Spin/Spin(8n+5);\mathbf{Z}/(2))$ of the top row fibration and $a_{4i-2}^{\nu_i-1}e_{4i-3}$ survives permanently for $1 \leq i \leq n$. We put $a_{4i-2}^{\nu_i-1}e_{4i-3} = q_{(4i-2)\nu_i-1}$, $1 \leq i \leq n$. $a_{4i-2}e_{4i-3}$ is also permanent for $n+1 \leq i \leq 2n+1$ and let $a_{4i-2}e_{4i-3} = q_{8i-5}$. We also have a permanent element $\gamma_2(a_{4i-2})$ for $n+1 \leq i \leq 2n+1$ and let $\gamma_2(a_{4i-2}) = c_{8i-4}$. Then $\Gamma(c_{8i-4})$ is also permanent, $n+1 \leq i \leq 2n+1$. From above, we get the following E_∞–term for $H^*(\Omega^2 Spin/Spin(8n+5);\mathbf{Z}/(2))$ in the Serre spectral sequence for the top row fibration

$$E_\infty = E(e_{8n+5+4k} : k \geq 0) \otimes E(q_{8i-5} : n+1 \leq i \leq 2n+1)$$
$$E(q_{(4i-2)\nu_i - 1} : 1 \leq i \leq n) \otimes \Gamma(c_{8n+4+2k} : 0 \leq k \leq 4n+1).$$

Here we can check that $\{q_{(4i-2)\nu_i-1} : 1 \leq i \leq n\}$ is $\{q_{8n+7}, q_{8n+15}, \dots, q_{16n-1}\}$. In fact, in the Serre spectral sequence of the second column path loop fibration

$$\begin{aligned} \sigma(a_{8n+6+4k}) &= e_{8n+5+4k} \\ \sigma(z_{8n+4+4k}) &= q_{8n+3+4k} \\ \sigma(w_{8n+5+2k}) &= c_{8n+4+2k}. \end{aligned}$$

Now we will solve the extension problem. By the dimension reason only possibility is whether $q_{8n+3}^2 = 0$ or not. Assume that $q_{8n+3}^2 \neq 0$. Then q_{8n+3}^2 should be c_{16n+6}, that is, $Sq^{8n+3}q_{8n+3} = c_{16n+6}$. Since $Sq^{8n+3} = Sq^1 Sq^{8n+2}$, $Sq^{8n+2}q_{8n+3} \neq 0$. But e_{16n+5} is the only primitive element of that dimension. The fact that $Sq^{8n+2}q_{8n+3} = e_{16n+5}$ imply that $Sq^{8n+2}z_{8n+4} = a_{16n+6}$ in $H^*(\Omega Spin/Spin(8n+5); \mathbf{Z}/(2))$. This implies that $Sq^{8n+2}x_{8n+5} = x_{16n+7}$ in $H^*(Spin/Spin(8n+5); \mathbf{Z}/(2))$. However from the Steenrod action (2.1) in $H^*(Spin; \mathbf{Z}/(2))$ we have $Sq^{8n+2}x_{8n+5} = \binom{5}{2}x_{16n+7} = 0$. This is a contradiction. Hence there is no extension and we get $H^*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))$. Since every generator in $H^*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))$ is the image of the cohomology suspension, it is primitive. Passing to homology, we get

$$\begin{aligned} H_*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2)) &= E(u_{8n+5+4k} : k \geq 0) \otimes \\ & \quad E(s_{8n+3+4k} : 0 \leq k \leq 2n) \otimes \\ & \quad P(d_{8n+4+2k} : 0 \leq k \leq 4n+1) \end{aligned}$$

,where $< u_{8n+5+4k}, e_{8n+5+4k} >= 1$, $< s_{8n+3+4k}, q_{8n+3+4k} >= 1$, $< d_{8n+4+2k}, c_{8n+4+2k} >= 1$. Here $<,>$ is the natural pairing of H_* and H^*. Hence every generator in $H_*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))$ is primitive. So back to (3.3) we have

$$H_*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))//f_* =$$
$$E(s_{8n+3+4k} : 0 \leq k \leq 2n) \otimes P(d_{8n+4+2k} : 0 \leq k \leq 4n+1),$$
$$H_*(\Omega^2 Spin; \mathbf{Z}/(2)) \backslash\backslash f_* = E(u_{4k+1} : 0 \leq k \leq 2n).$$

Hence

$$\begin{aligned} E_2 &= Cotor^{H_*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))}(H_*(\Omega^2 Spin; \mathbf{Z}/(2)), \mathbf{Z}/(2)) \\ &= Cotor^{H_*(\Omega^2 Spin/Spin(8n+5); \mathbf{Z}/(2))//f_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \\ & \quad \otimes H_*(\Omega^2 Spin; \mathbf{Z}/(2)) \backslash\backslash f_* \\ &= P(v_{8n+2+4k} : 0 \leq k \leq 2n) \otimes \\ & \quad P(Q_1^a w_{8n+3+2k} : a \geq 0,\ 0 \leq k \leq 4n+1) \otimes E(u_{4k+1} : 0 \leq k \leq 2n). \end{aligned}$$

For some technical reason, we express E_2 like

$$\begin{aligned} & E(u_{4k+1} : 0 \leq k \leq n-1) \otimes E(u_{4n+1+4k} : 0 \leq k \leq n) \otimes P(v_{8n+2+8k} : 0 \leq k \leq n) \otimes \\ & P(v_{8n+6+8k} : 0 \leq k \leq n-1) \otimes P(Q_1^a w_{8n+3+8k} : a \geq 0,\ 0 \leq k \leq n) \\ & \otimes P(Q_1^a w_{8n+5+2k} : a \geq 0, 0 \leq k \leq 4n \text{ and } k \not\equiv 3 \text{ mod } 4). \end{aligned} \tag{3.4}$$

This is the same size as the E_∞-term of the previous Serre spectral sequence converging to $H_*(\Omega^2 Spin(8n+5)$ in (3.2) under the assumption that $Q_2(w_{8n+1}) = 0$. Now we go back to the original question of deciding whether $Q_2(w_{8n+1})$ is 0 or not for w_{8n+1} in $H_*(\Omega^2 Spin(8n+4); \mathbf{Z}/(2))$. Assume that it is not zero. Then $Q_2(w_{8n+1}) = (u_{4n+1})^4$ because $(u_{4n+1})^4$ is only the primitive element at that dimension. So in the bottom row fibration of (3.2), we have

$$\tau(Q_1(\iota_{8n+2}) = Q_2(w_{8n+1}) = (u_{4n+1})^4.$$

That means that the Eilenberg–Moore spectral sequence of (3.4) have a differential from w_{16n+5} to $(v_{8n+2})^2$. But the bidegrees of w_{16n+5} and $(v_{8n+2})^2$ are $(-1, 16n+6)$ and $(-2, 16n+6)$. So there can not exist a differential from w_{16n+5} to $(v_{8n+2})^2$. Therefore $Q_2(w_{8n+1}) = 0$. Hence we finish the proof of Case 2. In fact the result says that the above the Eilenberg–Moore spectral sequence collapses from E_2 but has extensions, $(u_{4n+4k+1})^2 = v_{8n+8k+2}$ for $0 \le k \le n$ and we have the choices of the primitive generators $u_{4n+4k+1}$ so that $E(u_{4n+4k+1}) \otimes P(v_{8n+8k+2}) \otimes P(Q_1^a w_{8n+8k+3})$ produces $P(Q_1^a u_{4n+4k+1})$ for $0 \le k \le n$ in $H_*(\Omega^2 Spin(8n+5); \mathbf{Z}/(2))$.

(Case 3). From $H_*(\Omega^2 Spin(8n+5); \mathbf{Z}/(2))$ to $H_*(\Omega^2 Spin(8n+6); \mathbf{Z}/(2))$. Consider the morphism of fibrations

$$\begin{array}{ccccc}
\Omega^3 Spin/Spin(8n+5) & \xrightarrow{f} & \Omega^2 Spin(8n+5) & \longrightarrow & \Omega^2 Spin \\
\downarrow & & \downarrow & & \| \\
\Omega^3 Spin/Spin(8n+6) & \longrightarrow & \Omega^2 Spin(8n+6) & \longrightarrow & \Omega^2 Spin \\
\downarrow & & \downarrow & & \| \\
\Omega^2 S^{8n+5} & = & \Omega^2 S^{8n+5} & \longrightarrow & *
\end{array}$$

Look at the spectral sequence of the first column fibration. By the connectivity of $\Omega^3 Spin/Spin(8n+5)$ and $\Omega^3 Spin/Spin(8n+6)$ we have non-zero differential from ι_{8n+3} in $H_*(\Omega^2 S^{8n+5}; \mathbf{Z}/(2))$ to the $(8n+2)$ dimensional element, we call it t_{8n+2}, in $H_*(\Omega^3 Spin/Spin(8n+5); \mathbf{Z}/(2))$. Consider the spectral sequence of the first row fibration. Since there does not exist $8n+3$ dimensional indecompasable element in $H_*(\Omega^2 Spin; \mathbf{Z}/(2))$, t_{8n+2} survives ,i.e. ,$f_*(t_{8n+2}) \neq 0$. So in the spectral sequence for the second column fibration

$$\Omega^2 Spin(8n+5) \longrightarrow \Omega^2 Spin(8n+6) \longrightarrow \Omega^2 S^{8n+5}, \qquad (3.5)$$

by the naturality of the differential, we have nonzero first differential from ι_{8n+3} to $f_*(t_{8n+2})$. Since the target of the first differential is the primitive element, the only possible element is $(u_{4n+1})^2$ by the dimension reason. From the Cartan formula for the Dyer–Lashof operations (See p 217 [5]),

$$\begin{aligned}
Q_1((u_{4n+1})^2) &= 2Q_1(u_{4n+1})Q_0(u_{4n+1}) = 0 \\
Q_2(u_{4n+1}^2) &= 2Q_2(u_{4n+1})Q_0(u_{4n+1}) + Q_1(u_{4n+1})^2 \\
&\quad + u_{4n+1}\lambda_2(u_{4n+1}, u_{4n+1})u_{4n+1} \\
&= Q_1(u_{4n+1})^2.
\end{aligned}$$

Similarly

$$\begin{aligned} Q_2((Q_1^a u_{4n+1})^2) &= (Q_1^{a+1} u_{4n+1})^2 \ , a \geq 0 \\ Q_1((Q_1^a u_{4n+1})^2) &= 0 \ , a \geq 0. \end{aligned}$$

Note that Q_2 is the top operation. Thus we should consider the Browder operation λ_2. But if $p = 2$, $\lambda_2(x, x) = 0$. So we get the following differentials in the Serre spectral sequence for the fibration (3.5).

$$\begin{aligned} \tau(Q_1^a \iota_{8n+3}) &= Q_2^a(u_{4n+1}^2) = (Q_1^a u_{4n+1})^2 \ , a \geq 0 \\ \tau((Q_1^a i_{8n+3})^2) &= 0 \, , a \geq 0. \end{aligned}$$

This imply that $P((Q_1^a \iota_{8n+3})^2 \, : \, a \geq 0)$ and $E(Q_1^a u_{n+1} \, : \, a \geq 0)$ are the permenant cycle in the spectral sequence. Let $(i_{8n+3})^2 = v_{16n+6}$. Hence we get the $H_*(\Omega^2 Spin(8n+6); \mathbf{Z}/(2))$.

(Case 4). From $H_*(\Omega^2 Spin(8n+6); \mathbf{Z}/(2))$ to $H_*(\Omega^2 Spin(8n+7); \mathbf{Z}/(2))$. Consider the following fibration

$$\Omega^2 Spin(8n+6) \longrightarrow \Omega^2 Spin(8n+7) \longrightarrow \Omega^2 S^{8n+6}. \tag{3.6}$$

Using the same method as case 2 or case 3, we can show that we have the first nonzero differential from i_{8n+4} in $H_*(\Omega^2 S^{8n+6}; \mathbf{Z}/(2))$ to $Q_1 u_{4n+1}$, since $Q_1 u_{4n+1}$ is the only $(8n+3)$ dimensional primitive element in $H_*(\Omega^2 Spin(8n+6); \mathbf{Z}/(2))$. From the commutativity of the Dyer-Lashof operation with the homology suspension and the naturality of the Dyer-Lashof operation,

$$\tau(Q_0^a \iota_{8n+4}) = Q_1^{a+1} u_{4n+1} \ , a \geq 0.$$

Since there is no $(16n+8)$ dimensional primitive element, $Q_2(Q_1 u_{4n+1}) = 0$. So $Q_1(\iota_{8n+4})$ is the permanent cycle and let $Q_1(\iota_{8n+4}) = w_{16n+9}$. Since $(Q_1^a u_{4n+1})^2 = 0$ for $a \geq 0$ in $H_*(\Omega^2 Spin(8n+6); \mathbf{Z}/(2))$, $Q_1^{a+1} u_{4n+1} Q_0^a \iota_{8n+4}$, $a \geq 0$, are also permanent cycles and

$$(Q_1^{a+1} u_{4n+1} Q_0^a \iota_{8n+4})^2 = 0.$$

Let $Q_1 u_{4n+1} \iota_{8n+4} = w_{16n+7}$, so $Q_1^{a+1} u_{4n+1} Q_0^a \iota_{8n+4} = Q_1^a w_{16n+7}$. Hence we get that E_∞ is

$$\begin{gathered} E(u_{4k+1} : 0 \leq k \leq n) \otimes P(v_{8n+8k+6} : 0 \leq k \leq n) \bigotimes \\ P(Q_1^a u_{4n+4k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes P(Q_1^a w_{16n+9} : a \geq 0) \otimes \\ P(Q_1^a w_{8n+5+2k} : a \geq 0, 0 \leq k \leq 4n \text{ and k} \not\equiv 3 \bmod 4) \\ E(Q_1^a w_{16n+7} : a \geq 0) \otimes P(Q_2^{a+1} v_{16n+6}; a \geq 0) \, . \end{gathered} \tag{3.7}$$

We claim that there are the following extensions:

$$(Q_1^a w_{16n+7})^2 = (Q_2^{a+1} v_{16n+6}) \ , a \geq 0.$$

From Lemma 2.2, $H^*(\Omega Spin(8n+7); \mathbf{Z}/(2))$ is

$$P(a_{4i-2} : 1 \leq i \leq n+1)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+6+4k} : 0 \leq k \leq n-1)$$
$$\otimes \Gamma(c_{8n+6+2k} : 0 \leq k \leq 4n+2, k \not\equiv 3 mod 4)$$
whereν_i *is the power of* 2 *such that* $8n+8 \leq \nu_i(4i-2) \leq 16n+8$.

Using the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 Spin(8n+7); \mathbf{Z}/(2))$,

$$\begin{aligned} E_2 &= Ext_{H^*(\Omega Spin(8n+7);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \\ &= E(u_{4k+1} : 0 \leq k \leq n) \otimes \\ & \quad P(v_{8n+8k+6} : a \geq 0, 0 \leq k \leq n) \otimes \\ & \quad P(Q_1^a u_{4n+4k+5} : a \geq 0, 0 \leq k \leq n-1) \otimes \\ & \quad P(Q_1^a w_{8n+5+2k} : a \geq 0, 0 \leq k \leq 4n+2 \text{ and } k \not\equiv 3 \text{ mod } 4). \end{aligned}$$

However the size of this E_2–term is the same as the E_∞–term of the Serre spectral sequence (3.7). This means that above the Eilenberg–Moore spectral sequence collapses from the E_2–term and in other side, the E_∞–term of the Serre spectral sequence have the extensions as we claimed. So we get the conclusion. Note that $Q_2 v_{16n+6} = (w_{16n+7})^2$.

The other four cases is almost same as the previous four cases. In case 7 if we keep the track of the computation we can observe that $Q_2(v_{8n-2}) = w_{8n-1}^2$ in $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$. □

Remark 3.8 In fact, if we use the Eilenberg–Moore spectral sequence with $E_2 = Ext_{H^*(\Omega Spin(n);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ for $H_*(\Omega^2 Spin(n); \mathbf{Z}/(2))$, the above theorem says that the Eilenberg–Moore spectral sequence collapses from E_2–term. So we can choose u_i, v_i, w_i such that $< u_i, \sigma(a_{i+1}) >= 1$, $< w_i, \sigma(c_{i+1}) >= 1$, $< v_{2^k i-2}, \rho(a_i^{2^k}) >= 1$ where a_i and c_i are the elements of Lemma 2.2 and σ is a cohomology suspension and ρ is a transpotence.

Next we will determine some of the Steenrod actions for $H_*(\Omega^2 Spin(n); \mathbf{Z}/(2))$ as follows.

Lemma 3.9

$$\begin{aligned} Sq_*^{4i} u_m &= \binom{m-4i+2}{4i} u_{m-4i} \\ Sq_*^{2(4i+1)} w_{2m+1} &= \binom{m-4i+1}{4i+1} Q_1 u_{m-4i-1} \\ Sq_*^{2i} w_{2m+1} &= \binom{m-i+2}{i} w_{2m+1-2i} \qquad , i \equiv 0,2,3 \pmod 4 \\ Sq_*^1 w_{8m+7} &= v_{8m+6} \,. \end{aligned}$$

Proof. First, Steenrod actions for the stable element u_m is come from Steenrod actions for $H_*(\Omega^2 Spin; \mathbf{Z}/(2)) = H_*(U/Sp; \mathbf{Z}/(2))$. The relation between v and w come from the following argument. As we mentioned in last part of the proof for Theorem 3.1, we can observe that $Q_2(v_{8i+6}) = (w_{8i+7})^2$. By the Nishida relation,

$$\begin{aligned} Sq_*^2 Q_2 v_{8i+6} &= \textstyle\sum_j \binom{8i+6}{2-2j} Q_{2j} Sq_*^j v_{8i+6} \\ &= (v_{8i+6})^2 + Q_2 Sq_*^1 v_{8i+6}. \end{aligned}$$

But $Q_2 Sq_*^1 v_{8i+6} = 0$. For if it were not zero, by the dimension reason the only possible case would be that $Sq_*^1 v_{8i+6} = w_{8i+5}$ and $Q_2 Sq_*^1 v_{8i+6} = (v_{8i+6})^2$. By the Nishida relation $Sq_*^2 Q_2 w_{8i+5} = Q_2 Sq_*^1 w_{8i+5} = 0$, since there is no $(8i+4)$ dimensional primitive in each case. On the other hands $Sq_*^2 v_{8i+6}^2 = (Sq_*^1 v_{8i+6})^2 = (w_{8i+5})^2$. This is a contradiction. Now $Sq_*^2 (w_{8i+7})^2 = Sq_*^2 Q_2 (v_{8i+6}) = (v_{8i+6})^2$. Since $(Sq_*^1 w_{8i+7})^2 = Sq_*^2 (w_{8i+7})^2$, $Sq_*^1 w_{8i+7} = v_{8i+6}$.

Now turn to other relations. The Lemma 2.2 and Theorem 3.1 say that if we use the Eilenberg–Moore spectral sequence twice with $E_2 = \mathrm{Tor}_{H^*(Spin(n);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ for $H^*(\Omega Spin(n); \mathbf{Z}/(2))$ and with $E_2 = \mathrm{Tor}_{H^*(\Omega Spin(n);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ for $H^*(\Omega^2 Spin(n); \mathbf{Z}/(2))$, both the Eilenberg–Moore spectral sequences collapse from E_2–terms. Moreover the Eilenberg–Moore spectral sequence is the spectral sequence of Steenrod modules. So we can prove the other relations from the Steenrod actions for $H^*(Spin(n); \mathbf{Z}/(2))$ and the Nishida relations. Here we assume that above relations of the Steenrod actions hold for $H_*(\Omega^2 Spin(k); \mathbf{Z}/(2))$ for $k \leq 8n$ and will prove the Steenrod actions for $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$. The other inductive steps are almost same. We will determine the Steenrod actions for w_{16n-3} using the naturality of the Steenrod operations for the following fibration

$$\Omega^2 Spin(4n+1) \longrightarrow \Omega^2 Spin(8n+1) \xrightarrow{f} \Omega^2 Spin(8n+1)/Spin(4n+1).$$

By the same computation as Theorem 3.1 we have choices of the generators

$$\begin{aligned} &H_*(\Omega^2 Spin(8n+1)/Spin(4n+1); \mathbf{Z}/(2)) \\ &= P(Q_1^a z_{4n-1+i} : a \geq 0, 0 \leq i \leq 4n-1). \end{aligned}$$

From the Steenrod actions for $H^*(SO(n); \mathbf{Z}/(2))$ we can get Steenrod actions for $H_*(\Omega^2 Spin(8n+1)/Spin(4n+1); \mathbf{Z}/(2)) = H_*(\Omega^2 SO(8n+1)/SO(4n+1); \mathbf{Z}/(2))$:

$$\begin{array}{rcl} Sq_*^j z_{4n-1+i} &=& \binom{4n+1+i-j}{j} z_{4n-1+i-j} \ \ 0 \leq i \leq 4n-1 \ \text{,especially} \\ Sq_*^1 z_{4n+2k} &=& z_{4n+2k-1}, 0 \leq k \leq 2n-1. \end{array} \tag{3.10}$$

From above fact and the knowledge of $H_*(\Omega^2 Spin(4n+1); \mathbf{Z}/(2))$ and $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$ we have the following differentials

$$\begin{array}{rcl} \tau(z_{4n-1}) &=& \begin{cases} v_{4n-2} & , n\text{:even} \\ u_{2n-1}^2 & , n\text{:odd} \end{cases} \\ \tau(z_{4n}) &=& \begin{cases} w_{4n-1} & , n\text{:even} \\ Q_1 u_{2n-1} & , n\text{:odd} \end{cases} \\ \tau(z_{4n+1}) &=& 0 \\ \tau(z_{4n+2}) &=& w_{4n+1} \\ & \vdots & \end{array}$$

Then $z_{4n-1+4i}^2$, $Q_1^{a+1} z_{4n-1+4i}$, $Q_1^{a+1} z_{4n+4i}$, $Q_1^a z_{4n+1+4i}$ and $Q_1^{a+1} z_{4n+2+4i}$ survive and become $v_{8n+8i-2}$, $Q_1^a w_{8n+8i-1}$, $Q_1^a w_{8n+8i+1}$, $Q_1^a u_{4n+4i+1}$ and $Q_1^a w_{8n+8i+5}$, for $a \geq 0$, $0 \leq i \leq n-1$ in $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$. First we claim that $Sq_*^1 w_{16n-3} = 0$. If it is not zero, the only possibility is $Sq_*^1 w_{16n-3} = v_{8n-2}^2$. Then $Sq_*^1 f_*(w_{16n-3}) = f_*(v_{8n-2}^2)$, so $Sq_*^1 Q_1 z_{8n-2} = (z_{4n-1})^4$. But by the Nishida relation, in $H_*(\Omega^2 Spin(8n+1)/Spin(4n+1); \mathbf{Z}/(2))$

$$\begin{aligned} Sq^1_* Q_1 z_{8n-2} &= \tbinom{8n-2}{1-2j} Q_{2j} Sq^j_* z_{8n-2} \\ &= (8n-2) z^2_{8n-2} \\ &= 0. \end{aligned}$$

Hence $Sq^1_* w_{16n-3} = 0$. So $Sq^{2i+1}_* w_{16n-3} = Sq^{2i}_* Sq^1_* w_{16n-3} = 0$. For $Sq^{2i}_* w_{16n-3}$ we consider

$$\begin{aligned} Sq^{2i}_* Q_1 z_{8n-2} &= \textstyle\sum_j \tbinom{8n-1-2i}{2i-2j} Q_{1+2j-2i} Sq^j_* z_{8n-2} \\ &= Q_1 Sq^i_* z_{8n-2} \\ &= \tbinom{8n-i}{i} Q_1 z_{8n-i-2} \text{ by (3.10).} \end{aligned}$$

Hence by the naturality of the Steenrod operation

$$Sq^{2i}_* w_{16n-3} = \begin{cases} \tbinom{8n-i}{i} w_{16n-2i-3} & i \equiv 0,2,3 \bmod 4 \\ \tbinom{8n-i}{i} Q_i u_{8n-i-2} & i \equiv 1 \bmod 4. \end{cases}$$

□

Corollary 3.11 *The 2–torsions of $H_*(\Omega^2 Spin(8n+i);Z)$ are of order 2 if $i \neq 2,6$ and $H_*(\Omega^2 Spin(8n+i);Z)$ has the 2–torsions of order 2 and order 2^2 if $i = 2,6$.*

Proof. We will prove this by the Bockstein spectral sequence converging to $H_*(\Omega^2 Spin(8n);Z)$ with $E_1 = H_*(\Omega^2 Spin(8n);\mathbf{Z}/(2))$. By the Nishida relation

$$\begin{aligned} Sq^1_* Q_1^{a+1} u_{4n+4k+1} &= Q_0 Q_1^a u_{4n+4k+1} \quad, a \geq 0, 0 \leq k \leq n-1 \\ Sq^1_* Q_1^{a+1} w_{8n-3+2k} &= Q_0 Q_1^a w_{8n-3+2k} \;, a \geq 0, 0 \leq k \leq 4n-2 \text{ and } k \not\equiv 3 \bmod 4. \end{aligned}$$

And by Lemma 3.9

$$Sq^1_* w_{8n+8k-1} = v_{8n+8k-2} \quad, 0 \leq k \leq n-1.$$

Hence

$$\begin{aligned} E_2 &= E(u_{4k+1} : 0 \leq k \leq n-1) \otimes E(u_{4n+4k+1} : 0 \leq k \leq n-1) \\ &\quad \otimes E(w_{8n-3+4k} : 0 \leq k \leq 2n-1). \end{aligned}$$

Therefore $E_2 = E_\infty$. So the 2–torsions of $H_*(\Omega^2 Spin(8n);Z)$ are of order 2. We can prove the other $H_*(\Omega^2 Spin(8n+i);Z)$ for $i = 1,3,4,5,7$ in the same ways.

For $H_*(\Omega^2 Spin(8n+2);Z)$, $E_1 = H_*(\Omega^2 Spin(8n+2);\mathbf{Z}/(2))$.
Like above case we get

$$\begin{aligned} E_2 &= E(u_{4k+1} : 0 \leq k \leq n-1) \otimes E(u_{4n+4k+1} : 0 \leq k \leq n-1) \otimes \\ &\quad E(w_{8n+1+4k} : 0 \leq k \leq 2n-1) \otimes \\ &\quad E(Q_1^a w_{8n-1} \otimes P(Q_2^a v_{16n-2} : a \geq 0). \end{aligned}$$

Consider the following fibration

$$\Omega^2 Spin(8n+1) \longrightarrow \Omega^2 Spin(8n+2) \longrightarrow \Omega^2 S^{8n+1}$$

The behaviors of the Serre spectral sequence for the above fibration are exactly same as the Case 3 of the proof for Theorem 3.1, i.e. , we have

$$\begin{aligned}\tau(\iota_{8n-1}) &= v_{8n-2}\\ \tau(Q_1^{a+1}\iota_{8n-1}) &= (Q_1^a w_{8n-1})^2\ , a \geq 0.\end{aligned}$$

Note that $Q_2(v_{8n-2}) = (w_{8n-1})^2$ in $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$. Here $(Q_1^a \iota_{8n-1})^2$, $a \geq 0$, survives and become $Q_2^a v_{16n-2}$, $a \geq 0$, in $H_*(\Omega^2 Spin(8n+2); \mathbf{Z}/(2))$. Since $Sq_*^1 Q_1^{a+1} w_{8n-1} = (Q_1^a w_{8n-1})^2$ in $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$ and $Sq_*^1 Q_1^{a+1}\iota_{8n-1} = (Q_1^a \iota_{8n-1})^2$ in $H_*(\Omega^2 S^{8n+1}; \mathbf{Z}/(2))$, by the Bockstein Lemma we get

$$\beta_*^2((Q_1^{a+1} w_{8n-1})) = (Q_2^a v_{16n-2}) \quad a \geq 0. \tag{3.12}$$

Therefore

$$\begin{aligned}E_3 = {} & E(u_{4k+1} : 0 \leq k \leq n-1) \otimes E(u_{4n+4k+1} : 0 \leq k \leq n-1) \otimes \\ & E(w_{8n+1+4k} : 0 \leq k \leq 2n-1) \otimes E(w_{8n-1})\,.\end{aligned}$$

So $E_3 = E_\infty$. That means that $H_*(\Omega^2 Spin(8n+2); Z)$ has the 2–torsions of order 2 and order 2^2. We can also prove this for $H_*(\Omega^2 Spin(8n+6); Z)$ by the same method. □

The proof of the above Corollary implies the following well–known fact.

Corollary 3.13

$$\begin{aligned}SO(2n+1) &\simeq_Q S^3 \times S^7 \times \cdots \times S^{4n-1}\\ SO(2n+2) &\simeq_Q S^3 \times S^7 \times \cdots \times S^{4n-1} \times S^{2n+1}\,.\end{aligned}$$

4 The homology of $\Omega_0^3 Spin(n)$

In this section we will compute $H_*(\Omega_0^3 Spin(n); \mathbf{Z}/(2))$ by studying the Serre spectral sequence for the fibration

$$\Omega^3 Spin(m) \longrightarrow \Omega^3 Spin(m+1) \longrightarrow \Omega^3 S^m.$$

Recall that $H_*(\Omega_0^3 S^3; \mathbf{Z}/(2)) = P(Q_1^a Q_2^b[1] * [-2^{a+b}] : a, b \geq 0)$, where $\Omega_0^3 S^3$ is the zero component in $\Omega^3 S^3$ and [1] is the image of the generator in $\tilde{H}_0(S^0; \mathbf{Z}/(2))$ for the map: $S^0 \longrightarrow \Omega^3 S^3$ and $*$ is the loop sum pontryagin product. Let $H_*(\Omega^3 S^n; \mathbf{Z}/(2)) = P(Q_1^a Q_2^b \iota_{n-3} : a, b \geq 0)$, $n > 3$.

Theorem 4.1 *There are choices of the generators x_i, y_i, z_i such that as an algebra $H_*(\Omega_0^3 Spin 8n; \mathbf{Z}/(2))$, $n > 0$, is isomorphic to*

$$\begin{aligned}P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k-3} : a \geq 0, 0 \leq k \leq n-1) \otimes \\ P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n-1) \otimes \\ P(Q_1^a Q_2^b z_{8n-4+2k} : a, b \geq 0, 0 \leq k \leq 4n-2 \text{ and } k \not\equiv 3 \text{ mod } 4)\end{aligned}$$

$H_*(\Omega_0^3 Spin(8n+1); \mathbf{Z}/(2))$, $n > 0$, *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k-3} : a \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n-2+2k} : a, b \geq 0, 0 \leq k \leq 4n-1 \text{ and } k \not\equiv 2 \text{ mod } 4)$$

$H_*(\Omega_0^3 Spin(8n+2); \mathbf{Z}/(2))$, $n > 0$, *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-2) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n+2k} : a, b \geq 0, 0 \leq k \leq 4n-2 \text{ and } k \not\equiv 1 \text{ mod } 4)$$
$$\otimes P(Q_2^a z_{8n-2} : a \geq 0) \otimes P(Q_1^a Q_3^b y_{16n-3} : a, b \geq 0)$$

$H_*(\Omega_0^3 Spin(8n+3); \mathbf{Z}/(2))$ *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n+2k} : a, b \geq 0, 0 \leq k \leq 4n \text{ and } k \not\equiv 1 \text{ mod } 4)$$

$H_*(\Omega_0^3 Spin(8n+4); \mathbf{Z}/(2))$ *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n+2k} : a, b \geq 0, 0 \leq k \leq 4n \text{ and } k \not\equiv 1 \text{ mod } 4)$$

$H_*(\Omega_0^3 Spin(8n+5); \mathbf{Z}/(2))$ *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n+4+2k} : a, b \geq 0, 0 \leq k \leq 4n \text{ and } k \not\equiv 3 \text{ mod } 4)$$

$H_*(\Omega_0^3 Spin(8n+6); \mathbf{Z}/(2))$ *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k+4} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n+4+2k} : a, b \geq 0, 0 \leq k \leq 4n \text{ and } k \not\equiv 3 \text{ mod } 4)$$
$$\otimes P(Q_2^{a+1} x_{4n} : a \geq 0) \otimes P(Q_1^a Q_3^b y_{16n+5} : a, b \geq 0)$$

$H_*(\Omega_0^3 Spin(8n+7); \mathbf{Z}/(2))$ *is isomorphic to*

$$P(x_{4k} : 1 \leq k \leq n) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n) \bigotimes$$
$$P(Q_1^a Q_2^b x_{4n+4k+4} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes$$
$$P(Q_1^a Q_2^b z_{8n+4+2k} : a, b \geq 0, 0 \leq k \leq 4n+2 \text{ and } k \not\equiv 3 \text{ mod } 4)$$

When $n = 0$,

$$
\begin{aligned}
P(Q_1^a Q_2^b x_0 : a,b \geq 0) &= P(Q_1^a Q_2^b[1] * [-2^{a+b}] : a,b \geq 0)\\
P(Q_1^a Q_2^b z_0 : a,b \geq 0) &= P(Q_1^a Q_2^b[1] * [-2^{a+b}] : a,b \geq 0)\\
P(Q_2^{a+1} x_0 : a \geq 0) &= P(Q_2^a(Q_2[1] * [-2]) : a \geq 0)
\end{aligned}
$$

In fact, if we use the Eilenberg–Moore spectral sequence with $E_2 = \mathrm{Cotor}_{H_*(\Omega^2 Spin(n);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$, *the above results say that spectral sequence collapses from the* E_2*–term. So we can choose the generator* x_i, y_i, z_i *such that*
$\sigma(x_i) = u_{i+1}$, $\sigma(y_i) = v_{i+1}$, $\sigma(z_i) = w_{i+1}$.

Proof. We will prove this theorem by the induction on k, i.e., from $H_*(\Omega_0^3 Spin(8n+k);\mathbf{Z}/(2))$ to $H_*(\Omega_0^3 Spin(8n+k+1);\mathbf{Z}/(2))$. Like the double loop case we will prove four cases when $k = 0,1,2$ and 3. The proofs of the remain 4 cases, when $k = 4,5,6$ and 8, are almost same as above $k = 0,1,2$ and 3 cases. Consider the morphism of fibrations

$$
\begin{array}{ccccc}
\Omega^4 Spin/Spin(8n+k) & \xrightarrow{f} & \Omega_0^3 Spin(8n+k) & \longrightarrow & \Omega_0^3 Spin\\
\downarrow & & \downarrow & & \|\\
\Omega^4 Spin/Spin(8n+k+1) & \longrightarrow & \Omega_0^3 Spin(8n+k+1) & \longrightarrow & \Omega_0^3 Spin\\
\downarrow & & \downarrow & & \downarrow\\
\Omega^3 S^{8n+k} & = & \Omega^3 S^{8n+k} & \longrightarrow & *
\end{array}
$$

By the connectivity of $H_*(\Omega^4 Spin/Spin(8n+k+1))$ we have the non–trivial differential from ι_{8n-3+k} to a $(8n-4+k)$–dimensional element, we call it c_{8n-4+k}, in $H_*(\Omega^4 Spin/Spin(8n+k);\mathbf{Z}/(2))$ for the Serre spectral sequence of the first column fibration. Here we exclude the case from $Spin3$ to $Spin4$. In that case the result comes from the fact $Spin4 \simeq Spin3 \times Spin3$. Since there is no $(8n-3+k)$ dimensional generator in $H_*(\Omega^3 Spin)$ for $k = 0,1,2$ $f_*(c_{8n-4+k}) \neq 0$, $k = 0,1,2$. So by the naturality of the differential there is nonzero differential from ι_{8n+k-3} to a $(8n+k-4)$ dimensional primitive element in $H_*(\Omega_0^3 Spin(8n+k);\mathbf{Z}/(2))$ for $k = 0,1,2$ for the following fibration

$$\Omega_0^3 Spin(8n+k) \xrightarrow{\Omega^3 i} \Omega_0^3 Spin(8n+1+k) \xrightarrow{\Omega^3 \pi} \Omega^3 S^{8n+k}.$$

(Case 1) $k = 0$. We have the nonzero differential from ι_{8n-3} to a $(8n-4)$ dimensional primitive element in $H_*(\Omega_0^3 Spin(8n);\mathbf{Z}/(2))$. But we have two possible elements x_{8n-4}, z_{8n-4} in $H_*(\Omega_0^3 Spin(8n);\mathbf{Z}/(2))$. By the same method as Case 2 in the proof of Theorem 3.1, we should choose z_{8n-4}. Since $H_*(\Omega^3 S^{8n}) = P(Q_1^a \iota_{8n-3} : a \geq 0) \otimes P(Q_1^a Q_2^{b+1} \iota_{8n-3} : a,b \geq 0)$,

$$
\begin{aligned}
\tau(Q_0^a(\iota_{8n-3})) &= Q_1^a(z_{8n-4}), a \geq 0\\
\tau(Q_1^a(\iota_{8n-3})) &= Q_2^a(z_{8n-4}), a \geq 0.
\end{aligned}
\tag{4.13}
$$

For next we will prove that $Q_3(z_{8n-4}) = 0$. Assume that it is not zero. Since $Q_3 z_{8n-4}$ is primitive, by the dimension reason the only possible case is that $Q_3(z_{8n-4}) = Q_1 y_{8n-3}$. By the Nishida relation,

$$\begin{aligned} Sq^1_* Q_3 z_{8n-4} &= \sum_j \binom{8n-2}{1-2j} Q_{2+2j} Sq^j_* z_{8n-4} + \lambda_3(Sq^1_* z_{8n-4}, z_{8n-4}) \\ &= (8n-2) Q_2 z_{8n-4} = 0 . \end{aligned}$$

Note that $Sq^1_* z_{8n-4} = 0$ because there is no $(8n-5)$ dimensional primitive element in $H_*(\Omega^3_0 Spin(8n); \mathbf{Z}/(2))$. But

$$\begin{aligned} Sq^1_* Q_1 y_{8n-3} &= \sum_j \binom{8n-3}{1-2j} Q_{2j} Sq^j_* y_{8n-3} \\ &= (8n-3) Q_0 y_{8n-3} = (y_{8n-3})^2 \neq 0 . \end{aligned}$$

This is a contradiction. So we get $Q_3(z_{8n-4}) = 0$. Hence $Ker(\Omega^3 i)_* = Q_1^a Q_2^b z_{8n-4}$, $a, b \geq 0$, and $Q_1^a Q_2^b (Q_2 \iota_{8n-3})$ are permanent cycles for $a, b \geq 0$. Let $Q_2(\iota_{8n-3}) = z_{16n-4}$.

(Case 2) $k = 1$. Since y_{8n-3} is the only $8n-3$ dimensional primitive element in $H_*(\Omega^3_0 Spin(8n+1); \mathbf{Z}/(2))$, there is the nonzero differential from ι_{8n-2} to y_{8n-3}.

$$\tau(Q_0^a(\iota_{8n-2}) \quad = \quad Q_1^a(y_{8n-3})\, a \geq 0.$$

We claim that $Q_3 y_{8n-3} \neq 0$.

$$\begin{aligned} Sq^2_* Q_3 y_{8n-3} &= \sum_j \binom{8n-2}{2-2j} Q_{1+2j} Sq^j_* y_{8n-3} \\ &\quad + \lambda_3(Sq^1_* y_{8n-3}, Sq^1_* y_{8n-3}) \\ &= \binom{8n-2}{2} Q_1 y_{8n-3} + \binom{8n-2}{0} Q_3 Sq^1_* y_{8n-3} \\ &= Q_1 y_{8n-3} \neq 0. \end{aligned}$$

Hence $Q_3(y_{8n-3}) \neq 0$. Note that $Sq^1_* y_{8n-3} = 0$. If it is not zero, $Sq^1_* y_{8n-3} = x_{8n-4}$ by the dimension reasion. Then in $H_*(\Omega^2 Spin(8n+1); \mathbf{Z}/(2))$ $Sq^1_* v_{8n-2} = Sq^1_* \sigma(y_{8n-3}) = \sigma(Sq^1_* y_{8n-3}) = \sigma(x_{8n-4}) = u_{8n-3}$, where σ is the homology suspension. However from Lemma 3.9 $Sq^1_* w_{8n-1} = v_{8n-2}$. Since $Sq^1_* Sq^1_* = 0$, $0 = Sq^1_* Sq^1_* w_{8n-1} = Sq^1_* v_{8n-2} = u_{8n-3}$. This is a contradiction. So $Sq^1_* y_{8n-3} = 0$. By the dimension reason $Q_3(y_{8n-3}) = Q_1(z_{8n-2})$.

Next we claim that $Q_2(y_{8n-3}) = 0$. By the Nishida relation, we have

$$\begin{aligned} Sq^1_* Q_3 y_{8n-3} &= \sum_j \binom{8n-1}{1-2j} Q_{2+2j} Sq^j_* y_{8n-3} \\ &\quad + \lambda_3(Sq^1_* y_{8n-3}, y_{8n-3}) \\ &= (8n-1) Q_2 y_{8n-3} \\ &= Q_2 y_{8n-3}. \end{aligned}$$

On the other hand,

$$\begin{aligned} Sq^1_* Q_3 y_{8n-3} &= Sq^1_* Q_1 z_{8n-2} \\ &= \sum_j \binom{8n-2}{1-2j} Q_{2j} Sq^j_* z_{8n-2} \\ &= (8n-2) Q_0 z_{8n-2} \\ &= 0. \end{aligned}$$

For next we will prove that $Q_3(Q_3 y_{8n-3}) \neq 0$.

$$\begin{aligned} Sq^2_* Q_3(Q_3 y_{8n-3}) &= \sum_j \binom{16n-2}{2-2j} Q_{1+2j} Sq^j_*(Q_3 y_{8n-3}) \\ &\quad +\lambda_3(Sq^1_*(Q_3 y_{8n-3}), Sq^1_*(Q_3 y_{8n-3})) \\ &= \binom{16n-2}{2} Q_1 Q_3 y_{8n-3} + \binom{16n-2}{0} Q_3 Sq^1_* Q_3 y_{8n-3} \\ &= Q_1 Q_3 y_{8n-3} \\ &= Q_1^2(z_{8n-2}) \neq 0 . \end{aligned}$$

Hence $Q_3(Q_3 y_{8n-3}) \neq 0$. Note that $Sq^1_*(Q_3 y_{8n-3}) = 0$. Then by the dimension reason $Q_3(Q_3 y_{8n-3}) = Q_1(Q_2 z_{8n-2})$.

Next we claim that $Q_2(Q_3 y_{8n-3}) = 0$. By the Nishida relation, we have

$$\begin{aligned} Sq^1_* Q_3(Q_3 y_{8n-3}) &= \sum_j \binom{16n-1}{1-2j} Q_{2+2j} Sq^j_*(Q_3 y_{8n-3}) \\ &\quad +\lambda_3(Sq^1_*(Q_3 y_{8n-3}), Q_3 y_{8n-3}) \\ &= (16n-1) Q_2(Q_3 y_{8n-3}) = Q_2(Q_3 y_{8n-3}) , \end{aligned}$$

and

$$\begin{aligned} Sq^1_* Q_3(Q_3 y_{8n-3}) &= Sq^1_* Q_1(Q_2 z_{8n-2}) \\ &= \sum_j \binom{16n-2}{1-2j} Q_{2j} Sq^j_*(Q_2 z_{8n-2}) \\ &= (16n-2) Q_0(Q_2 z_{8n-2}) \\ &= 0 . \end{aligned}$$

In the same method we can prove that

$$\begin{aligned} Q_3^{a+1}(y_{8n-3}) &= Q_1 Q_2^a(z_{8n-2}) \ , a \geq 0 \\ Q_2(Q_3^a y_{8n-3}) &= 0 \ , a \geq 0. \end{aligned}$$

So we have for $a, b \geq 0$

$$\begin{aligned} \tau(Q_0^a Q_2^b(\iota_{8n-2})) &= Q_1^a Q_3^b(y_{8n-3}) \\ \tau(Q_1^{a+1} Q_2^b(\iota_{8n-2})) &= 0 . \end{aligned}$$

Hence $Ker\,\Omega^3 i_*$ contains $P(Q_1^a Q_3^b y_{8n-3} : a, b \geq 0)$, i.e. , $P(Q_1^a y_{8n-3} : a \geq 0)$ and $P(Q_1^{a+1} Q_2^b z_{8n-2} : a, b \geq 0)$. $Q_2^a z_{8n-2}$ are permanent cycles for $a \geq 0$. $Q_1^{a+1} Q_2^b \iota_{8n-2}$ are also permanent cycles for $a, b \geq 0$. By the same method as above we can show that $Q_1^{a+1} Q_2^b \iota_{8n-2} = Q_1^a Q_3^b Q_1 \iota_{8n-2}$. Let $Q_1 \iota_{8n-2} = y_{16n-3}$. In fact, by the Adem relation $Q_3 Q_1 \iota_{8n-2} = Q_1 Q_2 \iota_{8n-2}$ and $Q_3^2 Q_1 \iota_{8n-2} = Q_3(Q_3 Q_1 \iota_{8n-2}) = Q_3(Q_1 Q_2 \iota_{8n-2}) = Q_3 Q_1(Q_2 \iota_{8n-2}) = Q_1 Q_2(Q_2 \iota_{8n-2})$. Inductively we also get $Q_1^{a+1} Q_2^b \iota_{8n-2} = Q_1^a Q_3^b Q_1 \iota_{8n-2}$. So we get the conclusion.

(Case 3) $k = 2$. We have the differential from ι_{8n-1} to z_{8n-2}. Then

$$\tau(Q_1^a(\iota_{8n-1}) = Q_2^a(z_{8n-2}) .$$

We will show that $Q_1 z_{8n-2} = 0$. Assume that $Q_1 z_{8n-2} \neq 0$. By the dimension argument $Q_1 z_{8n-2} = y_{16n-3}$. By the Nishida relation

$$\begin{aligned} Sq^1_* Q_2 z_{8n-2} &= \sum_j \binom{8n-1}{1-2j} Q_{1+2j} Sq^j_* z_{8n-2} \\ &= Q_1 z_{8n-2} = y_{16n-3} . \end{aligned}$$

This would imply that in $H_*(\Omega^2 Spin(8n+2); \mathbf{Z}/(2))$
$Sq_*^1 Q_1 w_{8n-1} = Sq_*^1 \sigma(Q_2 z_{8n-2}) = \sigma(Sq_*^1 Q_2 z_{8n-2}) = \sigma(y_{8n-3}) = v_{16n-2}$.
But from (3.12), we know that $\beta_*^2 Q_1 w_{8n-1} = v_{16n-2}$. Hence $Q_1 z_{8n-2} = 0$. Since $Q_3 z_{8n-2} = 0$ by the dimension reason, $\tau(Q_2 \iota_{8n-1}) = 0$. Let $Q_2 \iota_{8n-1} = z_{16n}$ and $Q_3 y_{16n-3} = y_{32n-3}$. Thus we get that the E_∞–term for $H_*(\Omega_0^3 Spin(8n+3); \mathbf{Z}/(2))$ is

$$\begin{gathered} P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes \\ P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes \\ P(Q_1^a Q_2^b z_{8n+2k} : a, b \geq 0, 0 \leq k \leq 4n-2 \text{ and } k \not\equiv 1 \mod 4) \otimes \\ P(Q_2^a((\iota_{8n-1})^2 : a \geq 0) \otimes P(Q_1^a Q_3^b y_{32n-3} : a, b \geq 0) \otimes P(Q_1^a Q_2^b z_{16n} : a, b \geq 0). \end{gathered} \tag{4.2}$$

In other sides using the Eilenberg–Moore spectral sequence converging to $H_*(\Omega_0^3 Spin(8n+3); \mathbf{Z}/(2))$

$$\begin{aligned} E_2 &= Cotor^{H_*(\Omega^2(Spin(8n+3)<3>);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \\ &= Cotor^{E(u_{4k+1}:1\leq k\leq n-1)\otimes P(v_{8n+8k+6}:0\leq k\leq n-1)\otimes P(Q_1^a u_{4n+4k+1}:a\geq 0,0\leq k\leq n-1)\otimes P(Q_1^a w_{8n+2k+1}:a\geq 0,0\leq k\leq 4n \text{ and } k\not\equiv 1 \mod 4)}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \end{aligned}$$

where $Spin(8n+3) < 3 >$ is the 3–connected cover of $Spin(8n+3)$. Hence we get E_2–term is

$$\begin{gathered} P(x_{4k} : 1 \leq k \leq n-1) \otimes P(Q_1^a y_{8n+8k+5} : a \geq 0, 0 \leq k \leq n-1) \bigotimes \\ P(Q_1^a Q_2^b x_{4n+4k} : a, b \geq 0, 0 \leq k \leq n-1) \bigotimes \\ P(Q_1^a Q_2^b z_{8n+2k} : a, b \geq 0, 0 \leq k \leq 4n \text{ and } k \not\equiv 1 \mod 4). \end{gathered} \tag{4.3}$$

This E_2–term is the same size as the E_∞–term of the previous spectral sequence (4.2). This implies that the Eilenberg–Moore spectral sequence(4.3) collapses from the E_2–term and we get the result as we want. In fact, there is a choice of generator z_{16n-2} such that $P(Q_2^a(\iota_{8n-1})^2 : a \geq 0) \otimes P(Q_1^a Q_3^b y_{32n-3} : a, b \geq 0)$ becomes $P(Q_1^a Q_2^b z_{16n-2} : a, b \geq 0)$ in $H_*(\Omega_0^3 Spin(8n+3); \mathbf{Z}/(2))$.

(Case 4) $k = 3$. There is no $8n-1$ primitive element in $H_*(\Omega_0^3 Spin(8n+3); \mathbf{Z}/(2))$. Therefore the Serre spectral sequence collapses from E_2–term. □

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