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Group-Like Structures in General Categories I
Multiplications and Comultiplications

By

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1. Introduction

The starting point of the investigation whose results are to be presented in a series of three papers, of which this is the first, was an attempt to give precise meaning to and to answer the question how group structures arise in homotopy theory and why these group structures satisfy the familiar conditions of naturality. This problem, set in the restricted context of homotopy theory, was discussed in [2], but it was already clear that, by restricting the category of study in this way, we were disguising the generality of the approach and of the results. A more general treatment was indicated in Chapter 14 of [6] but not taken very far.

A group is a set A with a multiplication $m: A \times A \to A$ satisfying certain axioms. The basic idea of this paper is to consider categories $\mathfrak{C}$ rich enough in objects and maps to enable us to formulate a set of axioms which, in the case where $\mathfrak{C}$ is the category $\mathfrak{S}$ of (based) sets, are equivalent to the group axioms. Of course these axioms are formulated entirely in terms of the maps of the category $\mathfrak{C}$. It is then a basic observation that if (A, m) is a "group" in $\mathfrak{C}$ or, as we shall prefer to say, a $\underline{G}$-object, and if $H(X, A)$ is the set of maps from X to A, then $H(X, A)$ acquires a group-structure, in the familiar sense, from the structure map m; and the group-structure is commutative if m is "commutative" in a sense applicable to the category $\mathfrak{C}$. Moreover the group-structure is natural with respect to maps $X \to Y$ in $\mathfrak{C}$ in the sense that, for such a map f, the induced set-transformation $f^*: H(Y, A) \to H(X, A)$ is a homomorphism; and it may be shown (see [2] or Theorem 4.3 of this paper) that all such natural group-structures are induced from $\underline{G}$-structures m. The cohomology groups $H^n(X; G)$ of a polyhedron X give us an example of such a natural group structure. We may identify $H^n(X; G)$ with $\Pi(X, K(G, n))$, the set of homotopy classes of continuous maps of X into the Eilenberg-MacLane space $K(G, n)$ and the natural group structure of $H^n(X; G)$ is then acquired from a $\underline{G}$-structure or group-like multiplication on $K(G, n)$ in the category $\mathfrak{T}_h$ of (based) spaces and homotopy classes[1]).

By working in an arbitrary category $\mathfrak{C}$, we achieve, of course, a gain in generality. However, other advantages also accrue which are worth mentioning. The first advantage, which we will discuss in some detail, is the availability of

[1]) *Added in proof:* Thus $H^n(\;\; ; G)$ is a "representable" functor in the sense of GROTHENDIECK; many of the notions of these three papers are related to those in the categorical foundations of GROTHENDIECK's work.

the formal duality principle. With every category $\mathfrak{C}$ is associated its *dual category* $\mathfrak{C}^*$. The objects of $\mathfrak{C}^*$ are the objects of $\mathfrak{C}$ and the set of maps $H^*(B,A)$ in $\mathfrak{C}^*$ is precisely $H(A, B)$; moreover if we, temporarily, indicate composition of maps in $\mathfrak{C}$ by $g \circ f$, $f \in H(A, B)$, $g \in H(B, C)$, then composition in $\mathfrak{C}^*$ is given by

$$f \circ^* g = g \circ f. \tag{1.1}$$

It is easy (but vital) to observe that $\mathfrak{C}^*$, so defined, is a category whose identity maps coincide with those of $\mathfrak{C}$; if $\mathfrak{C}$ has zero-maps (see section 2) so has $\mathfrak{C}^*$ and the zero-map in $H^*(B, A)$ is just the zero-map of $H(A, B)$. Further $\mathfrak{C}^{**} = \mathfrak{C}$.

The construction of $\mathfrak{C}^*$ is a formal device to enable us to dualize axioms, definitions and theorems in the theory of categories. Two statements in $\mathfrak{C}$, in terms of objects and maps, are called *dual* if they differ only in the direction of the maps involved[1a]). More precisely, if S is a statement which is meaningful in any category, let $S(\mathfrak{C})$ be the statement S applied to the category $\mathfrak{C}$. If we *interpret* $S(\mathfrak{C}^*)$ as a statement about the objects and maps of $\mathfrak{C}$ we get a statement S^*, meaningful in any category, given by $S^*(\mathfrak{C}) = S(\mathfrak{C}^*)$. Then S^* is the *dual* of S. It is in this precise sense that we will speak of dual axioms, dual definitions and dual theorems. If the proof of a theorem belongs to the theory of categories the dual theorem is automatically true, being, in fact, logically equivalent to it.

This duality principle is exploited repeatedly in this series of papers. If a theorem T is proved for all categories satisfying some axioms A, then theorem T^* automatically holds for all categories satisfying axiom A^*. If one works in a single category $\mathfrak{C}_0$ (say, the category of groups $\mathfrak{G}$ or the category of based sets $\mathfrak{S}$) then $\mathfrak{C}_0$ may satisfy axiom A but not axiom A^* so the duality principle does not allow us to deduce the truth of T^* in $\mathfrak{C}_0$ even if the proof of theorem $T(\mathfrak{C}_0)$ has been made in category-theoretic terms. It may also happen that while theorem T^* does hold in $\mathfrak{C}_0$ it is trivial or even vacuous there; and that the interesting categories for the applications of theorems T and T^* are certainly not identical. Thus, for example, comultiplications with two-sided units are definable only on the one-element sets of $\mathfrak{S}$ and so are totally uninteresting. However they are of great interest in $\mathfrak{G}$ and in the category $\mathfrak{T}_h$ of (based) spaces and homotopy classes, and thus the theorems about multiplicative structures in general categories yield on dualization theorems of interest in group theory and homotopy theory.

We should at this stage make a remark about notation in connection with the application of the duality principle. In previous publications (e.g. [2], [3], [4]) we have indicated the dual of a statement or concept by attaching a prime $'$. Thus we have talked of H'-spaces and referred to Theorem $X \cdot YZ'$. The choice of which of the two notions to regard as "basic" and which as "the dual" was determined in each case on various grounds-traditional, psychological, and pedagogic- and could not, in the nature of things, be systematic.

[1a]) Indeed we speak of obtaining the dual of statement S by "reversing the arrows" occurring in S.

Here we have preferred to start with a completely unprejudiced viewpoint and have sought a notation appropriate to that viewpoint. Thus two notions which are dual to each other are indicated by the same letter, underlined in the one case and overlined in the other; for example a multiplication on an object A is called an $\underline{\mathrm{M}}$-structure and a comultiplication is called an $\overline{\mathrm{M}}$-structure. Where a notation is represented by a symbol rather than a letter we have chosen a comparable symbol for the dual notion; for example the direct product of A_1 and A_2 is written $A_1 \times A_2$ and the component form of a map $X \to A_1 \times A_2$ is written $\{f_1, f_2\}$, while the inverse product of A_1 and A_2 is written $A_1 * A_2$ and the component form of a map $A_1 * A_2 \to X$ is written $\langle f_1, f_2 \rangle$. The break with tradition is not complete, of course, as our choice of the phrase "direct product" indicates; and since we refer to direct products and inverse products we have felt compelled to refer to a category with direct products as a D-category and a category with inverse products as an I-category although these concepts are dual to each other. We do not index dual theorems in the way referred to above. Usually only one of a pair of dual theorems is enunciated and in the exceptional cases where we think it preferable to give both statements explicitely each statement receives the enumeration appropriate to its position in the text.

A second advantage of working in a general category which we mention briefly is that of the functorial approach. Since our theorems consist of assertions about categories in which certain constructions may be carried out, attention is naturally directed to those functors which respect the constructions. Such functors effect the transport of the structures we are studying from one category to another.

The contents of the present paper are as follows. Following a brief section describing categories with zero-maps, section 3 contains the theory of *direct products* in general categories and the dual theory of *inverse products* [2]). This provides a preparation for section 4 wherein multiplicative structures in categories are discussed. Simultaneously with the introduction of such a concept we naturally define the notion of *homomorphism* or, as we prefer to say to avoid confusion, the notion of a *primitive* map which is a map of the category from one object with multiplication to another which is compatible with the multiplications. Various axioms are considered to which the multiplicative structures may be subjected, in particular, axioms producing group-like structures. Section 5 consists of examples of the fundamental notions of the paper; one of the examples treated briefly here, namely that of the category of groups $\mathfrak{G}$, is dealt with more extensively in a separate publication [5]. Section 6 is concerned with the relation of Kan's theory of adjoint functors (see [7]) to the theory presented here. It turns out that the condition of possessing an adjoint has important implications for the structure preserving properties of the functor. Section 7 contains an example of a category with direct products in which there exist epimorphisms whose direct product is not an epimorphism. We would mention that the theory of operators and cooperators in general

[2]) For further details of this theory, see [9], [10].

categories, appropriate though it is to this paper, has not been discussed as it appeared in sufficient generality in [3].

It may help the reader if we summarize the contents of the second and third papers of this series. The second paper is concerned with generalized unions and intersections in arbitrary categories and with a consequent notion of the *length* of an object which generalizes the multiplicative and comultiplicative structures of this paper; this generalization includes and dualizes, e.g., Lusternik-Schnirelmann "category" of spaces. The third paper is concerned with what we call *primitive* categories, namely, categories in which the objects are multiplicative objects of a given category $\mathfrak{C}$ and in which the maps are primitive maps. We show how various phenomena familiar in group theory and abelian group theory (as well as certain less familiar ones) are explicable as properties of primitive categories, and we derive a curious relation of symmetry between the categories $\mathfrak{S}$ and $\mathfrak{G}$.

We wish to acknowledge the benefit of very fruitful correspondence with T. GANEA on the topics covered by these three papers.

2. Categories with zero-maps

Let $\mathfrak{C}$ be a category; following [9] we will denote the set of maps associated with the ordered pair of objects (A, B) of $\mathfrak{C}$ by $H(A, B)$, and we will permit ourselves to write $f: A \to B$ for a map in $H(A, B)$ and call f a map from A to B. We will denote the identity map of $H(A, A)$ by 1_A, frequently abbreviated to 1, and the dual category of $\mathfrak{C}$ by $\mathfrak{C}^*$ (cf. 1).

A map $f: A \to B$ is called an *equivalence* (or *invertible*) if there is a map f': $B \to A$ such that $f'f = 1_A$ and $ff' = 1_B$; in that case A and B are called *equivalent*. Of course the equivalence f determines its inverse f' and we may write f^{-1} for f'. The map f is called an *epimorphism* if for any C and any v_i: $B \to C$, $i = 1, 2$, the relation $v_1 f = v_2 f$ implies $v_1 = v_2$. The map f is called a *monomorphism* if for any D and any $w_i: D \to A$, $i = 1, 2$, the relation $f w_1 = f w_2$ implies $w_1 = w_2$. Notice that if f is an equivalence then f is both an epimorphism and a monomorphism, but the converse is in general false.

We shall suppose — though this is by no means necessary for all our subsequent definitions and results — that $\mathfrak{C}$ possesses *zero-maps*. That is, we suppose that for any two objects A, B of $\mathfrak{C}$ the set $H(A, B)$ is non-empty and contains a distinguished element, $0 = 0_{AB}$, such that

$$(2.1) \qquad f 0_{AB} = 0_{AC} \quad \text{for all } C \text{ and all } \quad f \in H(B, C),$$
$$0_{AB} g = 0_{DB} \quad \text{for all } D \text{ and all } \quad g \in H(D, A).$$

Note that the zero-maps are unique; for if also

$$f \tilde{0}_{AB} = \tilde{0}_{AC}, \tilde{0}_{AB} g = \tilde{0}_{DB}, \quad \text{all} \quad f, g,$$

then

$$0_{AB} = \tilde{0}_{BB} 0_{AB} = \tilde{0}_{AB}.$$

We list some elementary but important observations in the following propositions.

Proposition 2.2. (i) *The map f is an equivalence in $\mathfrak{C}$ if and only if it is an equivalence in $\mathfrak{C}^*$;*

(ii) *The map f is a monomorphism in $\mathfrak{C}$ if and only if it is an epimorphism in $\mathfrak{C}^*$;*

(iii) *$\mathfrak{C}$ possesses zero-maps if and only if $\mathfrak{C}^*$ possesses zero-maps, and the zero-map in $H(A, B)$ coincides with the zero-map in $H^*(B, A)$.*

Proposition 2.3. (i) *If f is an epimorphism and $vf = 0$ then $v = 0$.*

(ii) *If f is a monomorphism and $fw = 0$ then $w = 0$.*

In the light of Prop. 2.2 (ii) the two parts of this proposition are dual to each other. Note that the converse of Prop. 2.3 is, in general, false.

Since in our discussion all categories will be assumed to possess zero-maps we will henceforth simply use the word "category" to denote a category with zero-maps and the word "functor" to denote a (unary) functor which preserves zero-maps. Examples of such categories and functors are given below (§ 5).

3. Direct and inverse products

Let $A_1, A_2, \ldots, A_n$ be a finite collection of objects of $\mathfrak{C}$. A *direct product* $(P; p_1, p_2, \ldots, p_n)$, abbreviated to $(P; p_j)$ or even P, of $A_1, A_2, \ldots, A_n$ is an object P of $\mathfrak{C}$ and a system of maps $p_j\colon P \to A_j$, $j = 1, 2, \ldots, n$, with the property[3]): (D) For any object X of $\mathfrak{C}$ and any system of maps $f_j\colon X \to A_j$, $j = 1, 2, \ldots, n$, there exists a unique map $f\colon X \to P$ with $p_j f = f_j$.

The maps p_j are called the *projections* of P, and the maps f_j are called the *components* of f; we write $f = \{f_1, f_2, \ldots, f_n\}$, so that

$$p_j\{f_1, f_2, \ldots, f_n\} = f_j\,. \tag{3.1}$$

Notice that

$$\{0_1, 0_2, \ldots, 0_n\} = 0\,, \tag{3.2}$$

in view of (3.1) and the uniqueness of f.

Suppose that $(P'; p'_j)$ is a direct product of the objects $A'_1, A'_2, \ldots, A'_n$ and let maps $g_j\colon A_j \to A'_j$ be given, $j = 1, 2, \ldots, n$. The map

$$g = \{g_1 p_1, g_2 p_2, \ldots, g_n p_n\}\colon P \to P' \tag{3.3}$$

will frequently be written $g_1 \times g_2 \times \cdots \times g_n$. In particular setting $A'_j = A_j$, $g_j = 1$, $j = 1, 2, \ldots, n$, we have

$$1 = 1 \times 1 \times \cdots \times 1 = \{p_1, p_2, \ldots, p_n\}\colon P \to P\,. \tag{3.4}$$

We now list some rules which may be deduced immediately from the definitions.

Proposition 3.5. *Given $h\colon X' \to X$ and $\{f_1, f_2, \ldots, f_n\}\colon X \to P$, then*

$$\{f_1, f_2, \ldots, f_n\}h = \{f_1 h, f_2 h, \ldots, f_n h\}\colon X' \to P\,.$$

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[3]) There is, of course, no difficulty in generalizing this definition and the subsequent discussion to arbitrary collections of objects of $\mathfrak{C}$, but we are content to leave this generalization to the reader.

Proposition 3.6. *Given* $\{f_1, f_2, \ldots, f_n\} : X \to P$ *and* $g_1 \times g_2 \times \cdots \times g_n : P \to P'$, *then*

$$(g_1 \times g_2 \times \cdots \times g_n)\{f_1, f_2, \ldots, f_n\} = \{g_1 f_1, g_2 f_2, \ldots, g_n f_n\} : X \to P'.$$

Proposition 3.7. *Given*

$$g_1 \times g_2 \times \cdots \times g_n : P \to P' \quad \text{and} \quad h_1 \times h_2 \times \cdots \times h_n : P' \to P'',$$

where $(P''; p_j'')$ *is a direct product of* $A_1'', A_2'', \ldots, A_n''$, *then*

$$(h_1 \times h_2 \times \cdots \times h_n)(g_1 \times g_2 \times \cdots \times g_n) = h_1 g_1 \times h_2 g_2 \times \cdots \times h_n g_n : P \to P''.$$

From Prop. 3.6 we immediately infer

Theorem 3.8. *If* $g_j : A_j \to A_j'$ *is a monomorphism for each* j, *then* $g = g_1 \times g_2 \times \cdots \times g_n : P \to P'$ *is a monomorphism.*

(The corresponding statement for epimorphisms is false; see the appendix.)

From Prop. 3.7 and (3.4.) we immediately infer

Theorem 3.9. *If* $g_j : A_j \to A_j'$ *is an equivalence for each* j, *then* $g = g_1 \times g_2 \times \cdots \times g_n : P \to P'$ *is an equivalence such that*

$$p_j' g = g_j p_j, \quad j = 1, 2, \ldots, n; \tag{3.10}$$

and g *is uniquely determined by* (3.10).

By taking $A_j' = A_j$ in this theorem and all $g_j = 1$, we see that if $(P; p_j)$, $(P'; p_j')$ are both direct products of $A_1, A_2, \ldots, A_n$ then there is a unique equivalence $g : P \to P'$ such that

$$p_j' g = p_j; \tag{3.11}$$

we say that g is the *canonical* equivalence between $(P; p_j)$ and $(P'; p_j')$. Conversely if $(P; p_j)$ is a direct product of $A_1, A_2, \ldots, A_n$ and $g : P \to Q$ is an equivalence then $(Q; p_j g^{-1})$ is a direct product of $A_1, A_2, \ldots, A_n$ and g is the canonical equivalence between $(P; p_j)$ and $(Q; p_j g^{-1})$.

The definition of a direct product refers to an unordered[4]) set of objects so that it is, *per definitionem*, commutative as a function of two objects. On the other hand if we wish to form the direct product of objects presented as $A, B, C, \ldots$, it is necessary to assign to them a definite but arbitrary order so that we may be able to refer to a map $f : X \to P$ by means of its components $f_1, f_2, \ldots, f_n$ without ambiguity. Once this order is assigned the notations $f = \{f_1, f_2, \ldots, f_n\}$ and p_j for the projections become unambiguous.

Let us take the case of two objects which is quite typical. If then we say that $(P; k, l)$ *is a direct product of* A *and* B it is to be understood that k, l are maps in $H(P, A)$, $H(P, B)$ respectively; and if we refer to a map $f : X \to P$ by means of its components, $f = \{g, h\}$, it is to be understood that $g \in H(X, A)$, $h \in H(X, B)$ and $kf = g$, $lf = h$. Now let $(P'; k', l')$ be a direct product of B and A. Then $(P'; l', k')$ is a direct product of A and B so that there is a canonical equivalence

$$\tau : (P; k, l) \to (P'; l', k')$$

[4]) An alternative procedure is to define the direct product for *ordered* sets of objects. This procedure leads, of course, to a different development of the basic concepts.

such that

$$l'\tau = k, \quad k'\tau = l. \tag{3.12}$$

If we regard τ as a map from P to $(P'; k', l')$ it follows from (3.12) that we may write it

$$\tau = \{l, k\} : P \to (P'; k', l'). \tag{3.13}$$

Of course we may have $P' = P$, $\tau = 1$, $l' = k$, $k' = l$. On the other hand *in the particular case* $A = B$ we may take $P' = P$, $k' = k$, $l' = l$, giving rise to an equivalence,

$$\tau = \{l, k\} : P \to (P; k, l) \tag{3.14}$$

which is, in general, different from the identity map.

We shall henceforth adopt the notation $(A \times B \times C \times \ldots; p_1, p_2, p_3, \ldots)$, usually contracted to $A \times B \times C \times \ldots$, for an arbitrary representative of the class of canonically equivalent direct products of the objects $A, B, C, \ldots$, Indeed by writing the object $A \times B \times C \times \ldots$ in this way we are exhibiting the range of the jth projection p_j and rendering unambiguous the expression by components of a map into $A \times B \times C \times \ldots$. In this notation the symbol p_j becomes a generic symbol for the projection of a direct product onto its jth factor and assumes the nature of an operator which, when applied to a map $f = \{f_1, f_2, f_3, \ldots\}$ yields its jth component (see (3.1)). With these notational conventions (3.13) assumes the form

$$\tau = \{p_2, p_1\} : A_1 \times A_2 \to A_2 \times A_1 \tag{3.15}$$

and (3.14) becomes

$$\tau = \{p_2, p_1\} : A \times A \to A \times A. \tag{3.16}$$

We call the map τ in (3.15) or (3.16) the *switching map* or switch; it switches components in the precise sense that

$$\tau\{f_1, f_2\} = \{f_2, f_1\}, \tag{3.17}$$

$f_i \in H(X, A_i)$, $i = 1, 2$. We can, of course, have $A_1 = A_2 = A$ in this formula, as in the following theorem.

Theorem 3.18. *Let* $\alpha_i : A_i \to B_i$, $i = 1, 2$. *Then, assuming the direct products to exist, the diagram*

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, "\alpha_1 \times \alpha_2"] \arrow[d, "\tau"'] & B_1 \times B_2 \arrow[d, "\tau"] \\ A_2 \times A_1 \arrow[r, "\alpha_2 \times \alpha_1"] & B_2 \times B_1 \end{tikzcd}$$

is commutative.

For $\tau(\alpha_1 \times \alpha_2) = \{p_2, p_1\}(\alpha_1 \times \alpha_2)$

$= \{p_2(\alpha_1 \times \alpha_2), p_1(\alpha_1 \times \alpha_2)\}$ by Prop. 3.5,

$= \{\alpha_2 p_2, \alpha_1 p_1\}$ by definition of $\alpha_1 \times \alpha_2$,

$= (\alpha_2 \times \alpha_1)\{p_2, p_1\}$ by Prop. 3.6,

$= (\alpha_2 \times \alpha_1)\tau.$

We will call $\mathfrak{C}$ a *category with direct products* or *D-category* if any *finite*[5]) collection of objects of $\mathfrak{C}$ has a direct product. We prove

Theorem 3.19. *$\mathfrak{C}$ is a D-category if and only if any two objects of $\mathfrak{C}$ have a direct product.*

This follows immediately from

Lemma 3.20. $((A_1 \times \cdots \times A_r) \times (A_{r+1} \times \cdots \times A_n);\ p_1p_1, \ldots, p_rp_1, p_1p_2, \ldots, p_{n-r}p_2)$ *is a direct product of* $A_1, A_2, \ldots, A_n$.

Proof. Suppose given $f_i : X \to A_i$, $i = 1, \ldots, n$. Then there exist $f' : X \to A_1 \times \cdots \times A_r$ such that $p_j f' = f_j$, $j = 1, \ldots, r$, and $f'' : X \to \to A_{r+1} \times \cdots \times A_n$ such that $p_k f'' = f_{k+r}$, $k = 1, \ldots, n - r$. Consequently there exists $f : X \to (A_1 \times \cdots \times A_r) \times (A_{r+1} \times \cdots \times A_n)$ such that $p_1 f = f'$, $p_2 f = f''$, whence

$$p_j p_1 f = f_j,\ j = 1, \ldots, r;\quad p_k p_2 f = f_{k+r},\ k = 1, \ldots, n - r\,.$$

Now suppose that $p_j p_1 f = p_j p_1 g$, $j = 1, \ldots, r$, and $p_k p_2 f = p_k p_2 g$, $k = 1, \ldots, n - r$. Then $p_1 f = p_1 g$, $p_2 f = p_2 g$, so that $f = g$ and the lemma is proved.

The lemma implies also the *associativity* of direct products:

Theorem 3.21. *There is a canonical equivalence* $a : (A_1 \times A_2) \times A_3 \to A_1 \times \times (A_2 \times A_3)$ *such that*

$$a\{\{f_1, f_2\}, f_3\} = \{\{f_1, \{f_2, f_3\}\}$$

where $f_i \in H(X, A_i)$, $i = 1, 2, 3$.

Note that the map a may be written $\{p_1p_1, p_2 \times 1\}$ and a^{-1} may be written $\{1 \times p_1, p_2p_2\}$. There is also a canonical equivalence

$$b : (A_1 \times A_2) \times A_3 \to A_1 \times A_2 \times A_3\,,$$

given by

$$b = \{p_1p_1, p_2p_1, p_2\}\,,\quad b^{-1} = \{\{p_1, p_2\}, p_3\}\,. \tag{3.22}$$

We may use such equivalences as b implicitly to introduce brackets into a direct product object. Thus, for example, we may refer to the map

$$1 \times \tau : A_1 \times A_2 \times A_3 \to A_1 \times A_3 \times A_2$$

as a shorthand notation for $c(1 \times \tau)c^{-1}$, where c is the canonical equivalence

$$c = ba^{-1} = \{p_1, p_1p_2, p_2p_2\} : A_1 \times (A_2 \times A_3) \to A_1 \times A_2 \times A_3\,.$$

In fact in calculations involving implicit bracketing it is usually preferable to present the maps by means of their components *in the unbracketed form.* Thus, for example the map $1 \times \tau$ above is given in components by

$$1 \times \tau = \{p_1, p_3, p_2\} \tag{3.23}$$

To prove (3.23) we must show that

$$\{p_1, p_1p_2, p_2p_2\}\,(1 \times \{p_2, p_1\}) = \{p_1, p_3, p_2\}\,\{p_1, p_1p_2, p_2p_2\}\,. \tag{3.24}$$

[5]) See footnote 3.

Now $1 \times \{p_2, p_1\} = \{p_1, \{p_2, p_1\}p_2\} = \{p_1, \{p_2p_2, p_1p_2\}\}$. Thus

$$\begin{aligned}\{p_1, p_1p_2, p_2p_2\}\,(1 \times \{p_2, p_1\}) &= \{p_1, p_1\{p_2p_2, p_1p_2\}, p_2\{p_2p_2, p_1p_2\}\}\\ &= \{p_1, p_2p_2, p_1p_2\}\\ &= \{p_1, p_3, p_2\}\,\{p_1, p_1p_2, p_2p_2\}\,.\end{aligned}$$

In this calculation we have made repeated use of (3.1) and Prop. 3.5; in future we will state the component forms of such maps as $1 \times \tau$ without proof, leaving the verification to the reader. We illustrate the notation in the following proposition which will be used in a later paper of the series.

Proposition 3.25. The diagram

$$\begin{tikzcd} A \times B \times C \arrow[d, "\tau \times 1"'] \arrow[dr, "\tau"] & \\ B \times A \times C \arrow[r, "1 \times \tau"] & B \times C \times A \end{tikzcd}$$

commutes.

Notice that the description of the maps in the diagram involves implicit bracketing. However in component form the maps are given by

$$\begin{aligned}\tau \times 1 &= \{p_2, p_1, p_3\}\,,\\ 1 \times \tau &= \{p_1, p_3, p_2\}\,,\\ \tau &= \{p_2, p_3, p_1\}\,,\end{aligned}$$

and in this form the commutativity is trivial.

Frequent use will be made of the following special maps related to direct products. If $A_1, A_2, \ldots, A_n$ are objects of a D-category $\mathfrak{C}$ then there exists for each j, $1 \leqq j \leqq n$, a map

(3.26) $$\iota_j = \{0, \ldots, 0, 1, 0, \ldots, 0\} : A_j \to A_1 \times A_2 \times \cdots \times A_n\,,$$

where the jth component is 1 and the rest are zero. Since $p_j\iota_j = 1$ it follows that each p_j is an epimorphism and each ι_j is a monomorphism. If $A_1 = A_2 = \cdots = A_n$ there is a map

(3.27) $$\underline{d} = \{1, 1, \ldots, 1\} : A \to A \times A \times \cdots \times A\,,$$

called the *diagonal map*; it is evidently a monomorphism, and, moreover

(3.28) $$\{f_1, f_2, \ldots, f_n\} = (f_1 \times f_2 \times \cdots \times f_n)\underline{d}\,,$$

for maps $f_j \in H(A, B_j)$, $j = 1, 2, \ldots, n$.

Proposition 3.29. The diagram

$$\begin{tikzcd} A \times B \arrow[d, "\underline{d} \times \underline{d}"'] \arrow[dr, "\underline{d}"] & \\ A \times A \times B \times B \arrow[r, "1 \times \tau \times 1"] & A \times B \times A \times B \end{tikzcd}$$

is commutative.

For, in terms of components, the assertion is that

$$\{p_1, p_3, p_2, p_4\}\{p_1, p_1, p_2, p_2\} = \{p_1, p_2, p_1, p_2\},$$

which is obviously true.

We turn now to the concept dual to that of direct product; this will be called the *inverse product* and it is sufficient of course to establish terminology and notation. Thus an inverse product of the objects $A_1, A_2, \ldots, A_n$ is an object Q and a system of maps $q_j : A_j \to Q$, $j = 1, 2, \ldots, n$, with the property: (I) For any object X of $\mathfrak{C}$ and any system of maps $f_j : A_j \to X$, $j = 1, 2, \ldots, n$, there exists a unique map $f : Q \to X$ with $f q_j = f_j$.

The maps q_j are called the *injections* into Q, and the maps f_j are called the *components* of f; we write $f = \langle f_1, f_2, \ldots, f_n \rangle$, so that

$$\langle f_1, f_2, \ldots, f_n \rangle q_j = f_j \tag{3.30}$$

If $(Q'; q'_j)$ is an inverse product of $A'_1, A'_2, \ldots, A'_n$ and maps $g_j : A'_j \to A_j$, $j = 1, 2, \ldots, n$, are given, the map $g = \langle q_1 g_1, q_2 g_2, \ldots, q_n g_n \rangle : Q' \to Q$ will be written $g_1 * g_2 * \cdots * g_n$. Further we shall write $A_1 * A_2 * \cdots * A_n$ for an arbitrary representative of the class of canonically equivalent inverse products of the objects $A_1, A_2, \ldots, A_n$. If in $\mathfrak{C}$ any two objects and hence any finite number of objects have an inverse product we say that $\mathfrak{C}$ is a *category with inverse products* or an *I-category*.

Dual to the maps ι_j we have the maps

$$\pi_j = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle : A_1 * A_2 * \cdots * A_n \to A_j \tag{3.31}$$

in an I-category; and if $A_1 = A_2 = \cdots = A_n$ we have the *folding map*

$$\bar{d} = \langle 1, 1, \ldots, 1 \rangle : A * A * \cdots * A \to A,$$

such that

$$\langle f_1, f_2, \ldots, f_n \rangle = \bar{d}(f_1 * f_2 * \cdots * f_n), \tag{3.32}$$

for maps $f_j \in H(B_j, A)$, $j = 1, 2, \ldots, n$.

If no ambiguity is to be feared we will write d for $\underline{d}$ or $\bar{d}$.

A category with direct and inverse products is called a DI-category. We prove the following theorem for such categories.

Theorem 3.33. *Let $A_1, \ldots, A_m$; $B_1, \ldots, B_n$ be objects and $f_{ij} : A_i \to B_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, maps in the DI-category $\mathfrak{C}$. Then*

$$\bar{f} = \underline{f} : A_1 * \cdots * A_m \to B_1 \times \cdots \times B_n,$$

where $\bar{f} = \langle \{f_{11}, \ldots, f_{1n}\}, \ldots, \{f_{m1}, \ldots, f_{mn}\} \rangle$, *and* $\underline{f} = \{\langle f_{11}, \ldots, f_{m1} \rangle, \ldots, \langle f_{1n}, \ldots, f_{mn} \rangle\}$.

For $p_j \bar{f} = \langle p_j \{f_{11}, \ldots, f_{1n}\}, \ldots, p_j \{f_{m1}, \ldots, f_{mn}\} \rangle$, by the dual of Prop. 3.5,
$= \langle f_{1j}, \ldots, f_{mj} \rangle$
$= p_j \underline{f}$, $1 \leqq j \leqq n$.

In particular we may take $m = n$, $A_j = B_j$, $j = 1, \ldots, n$, and $f_{ii} = 1$, $f_{ij} = 0$, $i \neq j$. We thus obtain a map

$$\varkappa = \langle \iota_1, \ldots, \iota_n \rangle = \{\pi_1, \ldots, \pi_n\} : A_1 * \cdots * A_n \to A_1 \times \cdots \times A_n, \tag{3.34}$$

called the *canonical* map, from inverse to direct product. It is natural in an obvious sense.

We close this section with a brief discussion of the functors which are of special interest to us in considering D-categories and I-categories. Since our categories will always be categories with zero-maps, *we shall always insist that a functor preserve zero maps.* Now let $F : \mathfrak{C} \to \mathfrak{C}'$ be a covariant functor from the D-category $\mathfrak{C}$ to the D-category $\mathfrak{C}'$. We will say that F is a *D-functor* if it preserves direct products in the following sense: If $(P; p_j)$ is a direct product of $A_1, A_2, \ldots, A_n$, then $(F(P); F(p_j))$ is a direct product of $F(A_1)$, $F(A_2), \ldots, F(A_n)$. We shall write this property briefly as

$$F(A_1 \times A_2 \times \cdots \times A_n) = F(A_1) \times F(A_2) \times \cdots \times F(A_n)\,,$$
$$F(p_j) = p'_j\,.$$

Proposition 3.35. *If $F : \mathfrak{C} \to \mathfrak{C}'$ is a D-functor and $f_j \in H(X, A_j)$, $j = 1, 2, \ldots, n$, then*

$$F\{f_1, f_2, \ldots, f_n\} = \{Ff_1, Ff_2, \ldots, Ff_n\}\,.$$

For
$$\begin{aligned} p'_j F\{f_1, f_2, \ldots, f_n\} &= F(p_j) F\{f_1, f_2, \ldots, f_n\} \\ &= F(p_j\{f_1, f_2, \ldots, f_n\}) \\ &= F(f_j) \\ &= p'_j\{Ff_1, Ff_2, \ldots, Ff_n\}\,. \end{aligned}$$

We omit the proofs of the following propositions, in which F is understood to be a D-functor.

Proposition 3.36. $F(\tau) = \tau' : F(A_1) \times F(A_2) \to F(A_2) \times F(A_1)$.

Proposition 3.37. (i) $F(\iota_j) = \iota'_j : F(A_j) \to F(A_1) \times F(A_2) \times \cdots \times F(A_n)$

(ii) $F(d) = d' : F(A) \to F(A) \times F(A) \times \cdots \times F(A)$.

Of course an I-functor from the I-category $\mathfrak{C}$ to the I-category $\mathfrak{C}'$ is defined similarly; in particular a D-functor from $\mathfrak{C}$ to $\mathfrak{C}'$ can be interpreted as an I-functor from $\mathfrak{C}^*$ to $\mathfrak{C}'^*$. A contravariant functor from the I-category $\mathfrak{C}$ to the D-category $\mathfrak{C}'$ will be called an *I-functor* (or, more explicitly, a contravariant I-functor) if it is an I-functor from $\mathfrak{C}$ to the I-category $\mathfrak{C}'^*$; a contravariant D-functor is defined analogously. If $\mathfrak{C}$ and $\mathfrak{C}'$ are both DI-categories then we may define covariant and contravariant DI-functors from $\mathfrak{C}$ to $\mathfrak{C}'$ in an evident way; and we have

Proposition 3.38. If $F : \mathfrak{C} \to \mathfrak{C}'$ is a DI-functor then

$$F(\varkappa) = \varkappa' : F(A_1) * F(A_2) * \cdots * F(A_n) \to F(A_1) \times F(A_2) \times \cdots \times F(A_n)\,.$$

We are deferring our principal examples to section 5, but we mention here two very important examples of the special functors we have been discussing. Let $\mathfrak{S}$ be the category of based sets and let $\mathfrak{C}$ be any category (with zero maps). Then for each fixed object A of $\mathfrak{C}$ the transformation

$$X \to H(A, X)$$

induces a (covariant) functor $\overline{F}$ from $\mathfrak{C}$ to $\mathfrak{S}$, if, for $f : X \to Y$,

$$\overline{F}(f) : H(A, X) \to H(A, Y)$$

is defined by the rule

$$\overline{F}(f)\,(g) = fg\,,\ g \in H(A, X)\,.$$

The zero maps of $\mathfrak{S}$ are just the functions sending sets to base elements and the base element of $H(A, X)$ is the zero map from A to X. Thus $\overline{F}$ certainly preserves zero maps. We prove

Theorem 3.39. *If $\mathfrak{C}$ is a D-category then $\overline{F} = H(A,\ \ )$ is a D-functor from $\mathfrak{C}$ to $\mathfrak{S}$.*

Proof. We must show that if $X_1, X_2, \ldots, X_n \in \mathfrak{C}$ then $H(A, X_1 \times X_2 \times \cdots \cdots \times X_n; \overline{F}(p_j))$ is the direct product of $H(A, X_n)$, $j = 1, 2, \ldots, n$, in $\mathfrak{S}$.

Let $Y \in \mathfrak{S}$ and let $f_j : Y \to H(A, X_j)$ be maps in $\mathfrak{S}$. Writing X for $X_1 \times X_2 \times \cdots \times X_n$, we define $f : Y \to H(A, X)$ by the rule

$$f(y) = \{f_1(y), f_2(y), \ldots, f_n(y)\},\ y \in Y\,.$$

Then f is a map of based sets by (3.2). Moreover

$$(\overline{F}(p_j)f)\,(y) = p_j \circ f(y) = f_j(y)\,,\quad \text{all}^{6)}\quad y \in Y\,,$$

so that

$$\overline{F}(p_j)f = f_j\,.$$

Now suppose $\overline{F}(p_j)f = \overline{F}(p_j)g$. Then $p_j \circ f(y) = p_j \circ g(y)$, all $y \in Y$, so that $f(y) = g(y)$, $f = g$. This proves the theorem.

In an obviously analogous way we obtain, by fixing $A \in \mathfrak{C}$, a contravariant functor $H(\ \ , A)$ from $\mathfrak{C}$ to $\mathfrak{S}$. The duality principle ensures

Theorem 3.40. *If $\mathfrak{C}$ is an I-category then $\underline{F} = H(\ \ , A)$ is a (contravariant) I-functor from $\mathfrak{C}$ to $\mathfrak{S}$.*

4. Multiplicative structures

Let $\mathfrak{C}$ be a D-category and let A be an object of $\mathfrak{C}$. Then an $\underline{\mathrm{M}}$-*structure (or multiplication) on* A is simply a map $m : A \times A \to A$ in $\mathfrak{C}$ and the pair (A, m) is called an $\underline{\mathrm{M}}$-*object*. If (A, m) and (A', m') are two $\underline{\mathrm{M}}$-objects a map $g : A \to A'$ is called *primitive* or *homomorphic* with respect to the given $\underline{\mathrm{M}}$-structures if $m'(g \times g) = gm : A \times A \to A'$; that is, if the diagram

$$\begin{tikzcd} A \times A \arrow[r, "m"] \arrow[d, "g \times g"] & A \arrow[d, "g"] \\ A' \times A' \arrow[r, "m'"] & A' \end{tikzcd}$$

commutes. We observe that identity and zero maps are primitive. It is easy to prove:

Proposition 4.1. *If (A, m), (A', m'), (A'', m'') are $\underline{\mathrm{M}}$-objects and $g : A \to A'$, $h : A' \to A''$ are maps then* (i) *if g and h are primitive so is hg*; (ii) *if hg and h are primitive and h is a monomorphism then g is primitive.*

---

6) We use the notation $\circ$ to indicate composition of maps where its omission might lead to confusion.

Given an $\underline{\mathrm{M}}$-object (A, m) and an arbitrary $X \in \mathfrak{C}$, the map m induces a composition in the set $H(X, A)$ by the rule

$$(4.2) \qquad f + g = m\{f, g\} : X \to A \times A \to A, f, g \in H(X, A) .$$

Thus $(H(X, A), +)$ is an $\underline{\mathrm{M}}$-object in the category $\mathfrak{S}$, or, as we may say, an $\underline{\mathrm{M}}$-*set*. If $h : X \to Y$ in $\mathfrak{C}$, we write h^* for $\underline{F}(h)$ where $\underline{F} : \mathfrak{C} \to \mathfrak{S}$ is the contravariant functor of Theorem 3.40 and prove

Theorem 4.3. *$h^* : H(Y, A) \to H(X, A)$ is primitive with respect to the $\underline{\mathrm{M}}$-structures in $H(Y, A)$, $H(X, A)$ induced by the $\underline{\mathrm{M}}$-structure in A. Conversely, if for each $X \in \mathfrak{C}$ an $\underline{\mathrm{M}}$-structure $+$ is defined on $H(X, A)$ in such a way that h^* is primitive for every map h of $\mathfrak{C}$, then A admits a unique $\underline{\mathrm{M}}$-structure m such that* (4.2) *holds.*

Proof. Notice that to assert primitivity for h^* is simply to assert that

$$(4.4) \qquad h^*(f_1 + f_2) = h^*(f_1) + h^*(f_2), f_1, f_2 \in H(Y, A) .$$

Now $h^*(f_1 + f_2) = (f_1 + f_2)h = m\{f_1, f_2\}h = m\{f_1 h, f_2 h\} = f_1 h + f_2 h = h^*(f_1) + h^*(f_2)$, proving the first part of the theorem.

Conversely let each set $H(X, A)$ have an $\underline{\mathrm{M}}$-structure $+$ and let h^* be primitive with respect to $+$ for all h. Take in particular $X = A \times A$ and consider the map $p_1 + p_2 : A \times A \to A$. If now X is an arbitrary object of $\mathfrak{C}$ and $f_1, f_2 \in H(X, A)$, then $\{f_1, f_2\} : X \to A \times A$, so, by (4.4),

$$\begin{aligned}(p_1 + p_2)\{f_1, f_2\} = \{f_1, f_2\}^*(p_1 + p_2) &= \{f_1, f_2\}^*(p_1) + \{f_1, f_2\}^*(p_2) \\ &= p_1\{f_1, f_2\} + p_2\{f_1, f_2\} \\ &= f_1 + f_2 .\end{aligned}$$

This means that the $\underline{\mathrm{M}}$-structure in $H(X, A)$ is induced by the map $m = p_1 + p_2 : A \times A \to A$. If the map $n : A \times A \to A$ induces the same $\underline{\mathrm{M}}$-structure $+$ in $H(X, A)$ for each $X \in \mathfrak{C}$, then

$$p_1 + p_2 = n\{p_1, p_2\} = n\, 1_{A \times A} = n ,$$

so that the uniqueness is proved.

We may express the conclusion of Theorem 4.3 in the following way. Let $\mathfrak{M}$ be the category of $\underline{\mathrm{M}}$-sets and primitive maps. There is then an evident functor U from $\mathfrak{M}$ to $\mathfrak{S}$ which simply divests an $\underline{\mathrm{M}}$-set of its $\underline{\mathrm{M}}$-structure, and the rule (4.2) sets up a one-to-one correspondence between $\underline{\mathrm{M}}$-structures m on A and contravariant functors $\underline{F}_m : \mathfrak{C} \to \mathfrak{M}$ such that $U\underline{F}_m = \underline{F}$.

We now dualize. Let $\mathfrak{C}$ be an I-category and let A be an object of $\mathfrak{C}$. Then an $\overline{\mathrm{M}}$-*structure* (or *comultiplication*) on A is a map $m : A \to A * A$ and the pair (A, m) is called an $\overline{\mathrm{M}}$-*object*. If (A, m) and (A', m') are two $\overline{\mathrm{M}}$-objects a map $g : A' \to A$ is called *primitive* or homomorphic if the diagram

$$\begin{tikzcd} A' \arrow[r, "m'"] \arrow[d, "g"'] & A' * A' \arrow[d, "g * g"] \\ A \arrow[r, "m"] & A * A \end{tikzcd}$$

commutes. We omit the dual of Prop. 4.1 and pass to the observation that, given an $\overline{\mathrm{M}}$-object (A, m) and an arbitrary $X \in \mathfrak{C}$, the map m induces an $\underline{\mathrm{M}}$-structure in the set $H(A, X)$ by the rule

$$(4.5) \qquad f + g = \langle f, g \rangle m : A \to A * A \to X, f, g \in H(A, X).$$

We write h_* for $\overline{F}(h)$ where $\overline{F} : \mathfrak{C} \to \mathfrak{S}$ is the covariant functor of Theorem 3.39 and $h : X \to Y$; the duality principle yields

Theorem 4.6. *$h_* : H(A, X) \to H(A, Y)$ is primitive with respect to the $\underline{\mathrm{M}}$-structures in $H(A, X)$, $H(A, Y)$ induced by the $\overline{\mathrm{M}}$-structure on A. Conversely if for each X an $\underline{\mathrm{M}}$-structure $+$ is defined on $H(A, X)$ in such a way that h_* is primitive for every map h of $\mathfrak{C}$, then A admits a unique $\overline{\mathrm{M}}$-structure m such that* (4.5) *holds.*

We may rephrase this by saying that (4.5) sets up a one-to-one correspondence between $\overline{\mathrm{M}}$-structures m on A and covariant functors $\overline{F}_m : \mathfrak{C} \to \mathfrak{M}$ such that $U \overline{F}_m = \overline{F}$.

We next enunciate two theorems whose duals will remain implicit.

Theorem 4.7. *Let (A, m), (A', m') be two $\underline{\mathrm{M}}$-objects and let $g : A \to A'$ be primitive; then $g_* : H(X, A) \to H(X, A')$ is primitive with respect to the $\underline{\mathrm{M}}$-structures induced by m, m'.*

Proof. Let $f_1, f_2 \in H(X, A)$. Then

$$\begin{aligned} g_*(f_1 + f_2) = g(f_1 + f_2) &= g m\{f_1, f_2\} \\ &= m'(g \times g)\{f_1, f_2\}, \quad \text{since } g \text{ is primitive}, \\ &= m'\{g f_1, g f_2\}, \quad \text{by Prop. 3.6}, \\ &= g f_1 + g f_2 \\ &= g_*(f_1) + g_*(f_2). \end{aligned}$$

It is obvious that the converse of Theorem 4.7 holds: if $g_* : H(X, A) \to H(X, A')$ is primitive for all $X \in \mathfrak{C}$ then g is primitive.

Theorem 4.8. *Let (A_1, m_1), (A_2, m_2) be two $\underline{\mathrm{M}}$-objects. Then there exists a unique $\underline{\mathrm{M}}$-structure m on $A_1 \times A_2$ such that p_1 and p_2 are primitive. Moreover if (X, m') is an $\underline{\mathrm{M}}$-object then $\{f_1, f_2\} : X \to A_1 \times A_2$ is primitive (with respect to m' and m) if f_1 and f_2 are primitive.*

Proof. Define $m : A_1 \times A_2 \times A_1 \times A_2 \to A_1 \times A_2$ by $m = \{m_1\{p_1, p_3\}, m_2\{p_2, p_4\}\}$; using implicit bracketing m may be described as $(m_1 \times m_2)(1 \times \tau \times 1)$. Now the map $p_1 \times p_1 : A_1 \times A_2 \times A_1 \times A_2 \to A_1 \times A_1$ is given in components by $\{p_1, p_3\}$. Thus

$$p_1 m = m_1(p_1 \times p_1)$$

and, similarly,

$$p_2 m = m_2(p_2 \times p_2),$$

so that p_1 and p_2 are primitive. The uniqueness of m follows from the fact that the primitivity of $p_1(p_2)$ determines the first (second) component of m.

By Prop. 4.1 f_1 and f_2 are primitive if $\{f_1, f_2\}$ is primitive. Conversely let f_1, f_2 be primitive. Then

$$\begin{aligned} p_j m(\{f_1, f_2\} \times \{f_1, f_2\}) &= m_j(p_j \times p_j)(\{f_1, f_2\} \times \{f_1, f_2\}) \\ &= m_j(f_j \times f_j) \\ &= f_j m', \quad \text{since } f_j \text{ is primitive} \\ &= p_j\{f_1, f_2\}m', j = 1, 2, \end{aligned}$$

and $\{f_1, f_2\}$ is primitive.

Corollary 4.9. *If $g_j : A_j \to B_j, j = 1, 2$, are primitive then $g_1 \times g_2 : A_1 \times A_2 \to$ $\to B_1 \times B_2$ is primitive.*

The converse of this corollary also holds since $g_j = p_j(g_1 \times g_2)\iota_j$.

We call the $\underline{\mathrm{M}}$-object $(A_1 \times A_2, m)$ the *direct product* of the $\underline{\mathrm{M}}$-objects (A_j, m_j), $j = 1, 2$. This terminology is amply justified since it is indeed their direct product in the category of $\underline{\mathrm{M}}$-objects and primitive maps.

We now consider axioms which we might wish to impose on an $\underline{\mathrm{M}}$-structure. These axioms which are natural generalizations of those employed in group theory are as follows; they relate to the $\underline{\mathrm{M}}$-object (A, m).

I (zero as unit) $m\{1, 0\} = m\{0, 1\} = 1 : A \to A \times A \to A$;

II (associativity)

$$m(m \times 1) = m(1 \times m)a : (A \times A) \times A \to A \times (A \times A) \to A \times A \to A;$$

III (existence of an inverse) there exists $s : A \to A$ such that

$$m\{1, s\} = m\{s, 1\} = 0 : A \to A \times A \to A;$$

IV (commutativity) $m = m\tau : A \times A \to A \times A \to A$.

In II the map a is the canonical equivalence of Theorem 3.21. We will generally omit it, writing simply $m(m \times 1) = m(1 \times m)$. In terms of components $m \times 1 = \{m\{p_1, p_2\}, p_3\}$ and $1 \times m = \{p_1, m\{p_2, p_3\}\}$.

An $\underline{\mathrm{M}}$-structure satisfying I will be called an $\underline{\mathrm{H}}$-structure

An $\underline{\mathrm{M}}$-structure satisfying I, II will be called an $\underline{\mathrm{AH}}$-structure.

An $\underline{\mathrm{M}}$-structure satisfying I, II, III will be called a $\underline{\mathrm{G}}$-structure.

An $\underline{\mathrm{M}}$-structure satisfying I, II, III, IV will be called a $\underline{\mathrm{CG}}$-structure.

An $\underline{\mathrm{M}}$-structure satisfying I, II, IV will be salled an $\underline{\mathrm{ACH}}$-structure.

Similar conventions apply to $\underline{\mathrm{M}}$-objects and $\underline{\mathrm{M}}$-sets, except that a $\underline{\mathrm{G}}$-set will usually be given its familiar name "group" and a $\underline{\mathrm{CG}}$-set its familiar name "commutative group" or "abelian group". We now prove a basic theorem.

Theorem 4.10. *Let (A, m) be an $\underline{\mathrm{M}}$-object and let $H(X, A)$ have the induced $\underline{\mathrm{M}}$-structure for each X. Then m satisfies axiom $K(K = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ if and only if the induced $\underline{\mathrm{M}}$-structures satisfy axiom K. Indeed zero is a right (left) unit in A if and only if it is a right (left) unit in $H(X, A)$ and a right (left) inverse exists in A if and only if it exists in $H(X, A)$.*

Proof. I. Let $m\{1, 0\} = 1$. Then $f + 0 = m\{f, 0\} = m\{1, 0\}f = f$. Conversely if 0 is a right unit in $H(A, A)$ then $1 = 1 + 0 = m\{1, 0\}$. Similarly for left units.

II. If $f_i \in H(X, A)$, $i = 1, 2, 3$, then $(f_1 + f_2) + f_3 = m\{m\{f_1, f_2\}, f_3\}$, $f_1 + (f_2 + f_3) = m\{f_1, m\{f_2, f_3\}\}$. Now

$$\{m\{f_1, f_2\}, f_3\} = \{m\{p_1, p_2\}, p_3\}\{f_1, f_2, f_3\}\,.$$

For
$$\begin{aligned}\{m\{p_1, p_2\}, p_3\}\{f_1, f_2, f_3\} &= \{m\{p_1, p_2\}\{f_1, f_2, f_3\}, p_3\{f_1, f_2, f_3\}\}\\ &= \{m\{p_1\{f_1, f_2, f_3\}, p_2\{f_1, f_2, f_3\}\}, p_3\{f_1, f_2, f_3\}\}\\ &= \{m\{f_1, f_2\}, f_3\}\,.\end{aligned}$$

Similarly $$\{f_1, m\{f_2, f_3\}\} = \{p_1, m\{p_2, p_3\}\}\{f_1, f_2, f_3\}\,.$$

Thus if m satisfies axiom II $H(X, A)$ is associative and if, in particular, $H(A \times A \times A, A)$ is associative then m satisfies axiom II.

III. Let $m\{1, s\} = 0$. Then $f + sf = m\{f, sf\} = m\{1, s\}f = 0$. Conversely if s is a right inverse of 1 in $H(A, A)$ then $0 = 1 + s = m\{1, s\}$. Similarly for left inverses.

IV. Let $m\tau = m$. Then if $f_1, f_2 \in H(X, A)$, $f_1 + f_2 = m\{f_1, f_2\} = m\tau\{f_1, f_2\} = m\{f_2, f_1\} = f_2 + f_1$. Conversely if $H(A \times A, A)$ is commutative then $m = p_1 + p_2 = p_2 + p_1 = m\{p_2, p_1\} = m\tau$.

Theorem 4.10 enables us to transfer certain theorems from group theory to general $\underline{\mathrm{G}}$-objects. Thus let $I_r (I_l)$ be the axiom that zero is a right (left) unit and let $\mathrm{III}_r (\mathrm{III}_l)$ be the axiom asserting the existence of right (left) inverses. Then we have

Corollary 4.11. *If the $\underline{\mathrm{M}}$-structure $m : A \times A \to A$ satisfies axioms I_r, II, III_r, (or I_l, II, III_l) then it is a $\underline{\mathrm{G}}$-structure. Moreover the right inverse s is uniquely determined by m.*

For if m satisfies I_r, II, III_r so does the $\underline{\mathrm{M}}$-structure $+$ in $H(A, A)$ with s as the right inverse of 1. Thus, by classical group theory, $H(A, A)$ is a group, where $0 + 1 = 1$, $s + 1 = 0$, and s is unique. The first two conclusions assert that m satisfies axioms I and II and the third asserts that m determines s.

Again we may prove by using Theorem 4.10

Corollary 4.12. *Let $(A_1 \times A_2, m)$ be the direct product of the $\underline{\mathrm{M}}$-objects (A_1, m_1), (A_2, m_2). Then if the structures m_1, m_2 satisfy axiom K, so does the structure m.*

Proof. Instead of a direct proof we use Theorem 4.10 and refer to known (or obvious) facts in the category $\mathfrak{S}$. We show that if $H(X, A_1)$, $H(X, A_2)$ have the $\underline{\mathrm{M}}$-structures induced by m_1, m_2, then, in the direct product $H(X, A_1) \times H(X, A_2) = H(X, A_1 \times A_2)$, the direct product $\underline{\mathrm{M}}$-structure coincides with the structure induced by m. For if $f_i, f_i' \in H(X, A_i)$, $i = 1, 2$, then

$$\begin{aligned}m\{f_1, f_2, f_1', f_2'\} &= \{m_1\{p_1, p_3\}, m_2\{p_2, p_4\}\}\{f_1, f_2, f_1', f_2'\}\\ &= \{m_1\{p_1, p_3\}\{f_1, f_2, f_1', f_2'\}, m_2\{p_2, p_4\}\{f_1, f_2, f_1', f_2'\}\}\\ &= \{m_1\{f_1, f_1'\}, m_2\{f_2, f_2'\}\}\\ &= \{f_1 + f_1', f_2 + f_2'\}\,,\end{aligned}$$

or

$$\{f_1, f_2\} + \{f_1', f_2'\} = \{f_1 + f_1', f_2 + f_2'\}\,,$$

where $+$ on the left is induced by m. We now conclude the proof of the Corollary by claiming that the assertion is trivial in $\mathfrak{S}$ and then inferring the assertion in an arbitrary category by means of Theorem 4.10.

We next prove the general associativity law for $\underline{\mathrm{M}}$-structures satisfying axiom II. This could also be achieved by referring to properties of the category $\mathfrak{S}$ but we prefer a direct proof. We define an *n-product* in the $\underline{\mathrm{M}}$-object (A, m) as follows.

(a) An n-product in (A, m) is a map in $H(A^n, A)$, where A^n is the direct product of n copies of A;

(b) The unique 1-product in (A, m) is the identity map in $H(A, A)$;

(c) Assume n-products defined for $n < k$. Then a k-product is any map of the form

$$A^k \xrightarrow{w} A^q \times A^r \xrightarrow{f \times g} A \times A \xrightarrow{m} A \,,$$

where $q + r = k$, w is the canonical equivalence, f is a q-product and g is a r-product. Notice that (by a simple inductive argument) there are n-products for every $n \geqq 1$.

Theorem 4.13. *If (A, m) satisfies axiom* II *then the n-product in (A, m) is unique.*

Proof. We argue by induction on n, the assertion being true by definition if $n = 1$.

Suppose the assertion true for $n < k$ and consider two k-products

$$A^k \xrightarrow{w} A^q \times A^r \xrightarrow{f \times g} A \times A \xrightarrow{m} A \,,$$

$$A^k \xrightarrow{w} A^{q'} \times A^{r'} \xrightarrow{f' \times g'} A \times A \xrightarrow{m} A \,;$$

notice that we use the generic symbol w for the canonical equivalence. If $q = q'$ it is clear by the inductive hypothesis that these two k-products coincide. Thus we may suppose without real loss of generality that $q > q'$, say $q = q' + s$. Let $h : A^s \to A$ be an s-product. Then consider

$$\begin{tikzcd}
A^k \arrow[r, "w"] \arrow[dr, "w"'] & A^q \times A^r \arrow[r, "w \times 1"] & (A^{q'} \times A^s) \times A^r \arrow[r, "(f' \times h) \times g"] \arrow[d, "a"] & (A \times A) \times A \arrow[r, "m \times 1"] \arrow[d, "a"] & A \times A \arrow[r, "m"] & A \\
& A^{q'} \times A^{r'} \arrow[r, "1 \times w"] & A^{q'} \times (A^s \times A^r) \arrow[r, "f' \times (h \times g)"] & A \times (A \times A) \arrow[r, "1 \times m"] & A \times A \arrow[ur, "m"'] &
\end{tikzcd}$$

where a is the canonical equivalence of Theorem 3.21. This diagram is commutative because (i) the canonical equivalence $A^k \to A^{q'} \times (A^s \times A^r)$ is unique, (ii) a is natural, (iii) m satisfies axiom II. But

$$m(m \times 1)\,((f' \times h) \times g)\,(w \times 1)w = m\big((m(f' \times h)w) \times g\big)w = m(f \times g)w \,,$$

by the inductive hypothesis, and similarly

$$m(1 \times m)\,(f' \times (h \times g))\,(1 \times w)w = m\big(f' \times (m(h \times g)w)\big)w = m(f' \times g')w \,.$$

This completes the induction. We will later adopt the notation $m^n : A^n \to A$ $(m^2 = m)$, for the unique n-product in an associative $\underline{\mathrm{M}}$-object (A, m).

That part of Theorem 4.10 which asserts that the $\underline{\mathrm{M}}$-structure in $H(X, A)$ inherits properties from the $\underline{\mathrm{M}}$-structure m on A admits an evident generalization. For let F be a covariant D-functor from the category $\mathfrak{C}$ to the category $\mathfrak{C}'$. Then if m is an $\underline{\mathrm{M}}$-structure on A, $F(m)$ is evidently an $\underline{\mathrm{M}}$-structure on $F(A)$ and it is easy to show

Theorem 4.14. *The* $\underline{\mathrm{M}}$*-structure* $F(m)$ *satisfies axiom* K *if the* $\underline{\mathrm{M}}$*-structure* m *satisfies axiom* K, ($K = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$).

The following elementary result has important consequences when the question of uniqueness of structure is under discussion.

Proposition 4.15. *Let* (A, m) *be an* $\underline{\mathrm{M}}$*-object and let* $\tau : A \times A \to A \times A$ *be the switch. Then* $m\tau$ *is an* $\underline{\mathrm{M}}$*-structure on* A *and* m *satisfies axiom* K *if and only if* $m\tau$ *satisfies axiom* K ($K = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$).

We next prove

Proposition 4.16. *Let* (A, m), (A', m') *be two* $\underline{\mathrm{G}}$*-objects and let* $g : A \to A'$ *be primitive. Then* g *commutes with inverses; i.e., if* $s : A \to A$, $s' : A' \to A'$ *are inverses with respect to the structures* m, m', *then* $gs = s'g$.

Proof. Consider $g_* : H(X, A) \to H(X, A')$. By Theorems 4.7 and 4.10 g_* is a homomorphism of groups and so maps inverses to inverses. Moreover if $f \in H(X, A)$ its inverse if sf; similarly the inverse of f' in $H(X, A')$ is $s'f'$.

Take $X = A$. Then in $H(A, A)$, s is the inverse of 1 so that $g_*(s)$ is the inverse of $g_*(1)$ in $H(A, A')$, or gs is the inverse of g in $H(A, A')$. Thus $gs = s'g$.

Assertions 4.7—4.16 may all be dualized, but we will not make the duals explicit. However the following important theorem (which is self-dual) relates to DI-categories.

Theorem 4.17. *Let* (A, m_1) *be an* $\overline{\mathrm{H}}$*-object and* (B, m_2) *an* $\underline{\mathrm{H}}$*-object in the* DI*-category* $\mathfrak{C}$; *and let* m_1 *induce the* $\underline{\mathrm{H}}$*-structure* $+_1$ *in* $H(A, X)$, m_2 *induce the* $\underline{\mathrm{H}}$*-structure* $+_2$ *in* $H(Y, B)$. *Then* $+_1 = +_2$ *in* $H(A, B)$, *and is a commutative* $\underline{\mathrm{H}}$*-structure.*

Proof. Let $f, g \in H(A, B)$. Then by Theorem 3.33

$$\langle\{f, 0\}, \{0, g\}\rangle = \{\langle f, 0\rangle, \langle 0, g\rangle\} : A * A \to B \times B$$

But

$$\begin{aligned} m_2\langle\{f, 0\}, \{0, g\}\rangle m_1 &= \langle m_2\{f, 0\}, m_2\{0, g\}\rangle m_1, \quad \text{by the dual of Prop. 3.5} \\ &= \langle f, g\rangle m_1, \quad \text{since } m_2 \text{ is an } \underline{\mathrm{H}}\text{-structure} \\ &= f +_1 g; \end{aligned}$$

while

$$\begin{aligned} m_2\{\langle f, 0\rangle, \langle 0, g\rangle\} m_1 &= m_2\{\langle f, 0\rangle m_1, \langle 0, g\rangle m_1\}, \quad \text{by Prop. 3.5} \\ &= m_2\{f, g\}, \quad \text{since } m_1 \text{ is an } \overline{\mathrm{H}}\text{-structure} \\ &= f +_2 g. \end{aligned}$$

To prove commutativity we simply consider the $\underline{\mathrm{M}}$-structure $\tilde{m}_2 = m_2\tau$ on B. By Prop. 4.15 $\tilde{m}_2$ is an $\underline{\mathrm{H}}$-structure since m_2 is, and, if $\tilde{m}_2$ induces $\tilde{+}_2$ in $H(Y, B)$ then

$$f \,\tilde{+}_2\, g = g +_2 f, \quad f, g \in H(Y, B).$$

But, by what we have already proved, if $Y = A$ then $f \,\tilde{+}_2\, g = f +_1 g$. Thus

$$f +_1 g = g +_1 f, \quad f, g \in H(A, B).$$

We close this section with some brief remarks about $\underline{\mathrm{H}}$-structures in DI-categories.

Proposition 4.18. *Let $\mathfrak{C}$ be a DI-category and let $\varkappa : A * A \to A \times A$ be the canonical map* (3.34). *Then an $\underline{\mathrm{M}}$-structure m on A is an $\underline{\mathrm{H}}$-structure if and only if*

$$m\varkappa = \langle 1, 1\rangle : A * A \to A .$$

For $m\varkappa = m\langle \iota_1, \iota_2\rangle = \langle m\iota_1, m\iota_2\rangle$, and m is an $\underline{\mathrm{H}}$-structure if and only if $m\iota_1 = 1$, $m\iota_2 = 1$. We apply this proposition in proving

Theorem 4.19. *Let $\mathfrak{C}$ be a DI-category and let $\varkappa : A * A \to A \times A$ be an epimorphism. Then if A admits an $\underline{\mathrm{H}}$-structure that structure is unique and commutative. If also B admits an $\underline{\mathrm{H}}$-structure then every map in $H(A, B)$ is primitive.*

Proof. If m, m' are two $\underline{\mathrm{H}}$-structures on A, then by Prop. 4.18 $m\varkappa = m'\varkappa$. Thus, $\varkappa$ being an epimorphism, $m = m'$. The commutativity of m follows from Prop. 4.15, since $m\tau$ is also an $\underline{\mathrm{H}}$-structure on A.

Now let (B, m) be an $\underline{\mathrm{H}}$-object and $\varphi : A \to B$. Consider the diagram

$$\begin{tikzcd} A * A \arrow[r, "\varkappa"] \arrow[d, "\varphi * \varphi"] & A \times A \arrow[r, "m"] \arrow[d, "\varphi \times \varphi"] & A \arrow[d, "\varphi"] \\ B * B \arrow[r, "\varkappa"] & B \times B \arrow[r, "m"] & B \end{tikzcd}$$

The canonical map $\varkappa$ is certainly natural so $\varkappa(\varphi * \varphi) = (\varphi \times \varphi)\varkappa$. On the other hand

$$\varphi m\varkappa = \varphi\langle 1, 1\rangle = \langle \varphi, \varphi\rangle = \langle 1, 1\rangle (\varphi * \varphi) = m\varkappa(\varphi * \varphi)$$

Thus

$$\varphi m\varkappa = m\varkappa(\varphi * \varphi) = m(\varphi \times \varphi)\varkappa .$$

But $\varkappa : A * A \to A \times A$ is an epimorphism so $\varphi m = m(\varphi \times \varphi)$ and φ is primitive.

There is, in fact, a stronger result underlyng Theorem 4.19. For the conclusion of the theorem holds in any D-category $\mathfrak{C}$in which maps $f\colon A \times A \to X$ are determined by $f\iota_1$ and $f\iota_2$; and if $A * A$ exists this condition is equivalent to the conditions that $\varkappa\colon A * A \to A \times A$ be an epimorphism.

5. Examples

5.1. *The category $\mathfrak{S}$ of based sets.* We have already discussed this category extensively in connection with the sets $H(A, B)$ and the functors $\underline{F}, \overline{F}$. We have remarked that it is a D-category, the direct product of the based sets S_1, S_2 being the Cartesian product $S_1 \times S_2$ based at the direct product of the base-points of S_1 and S_2. It is indeed a DI-category, $S_1 * S_2$ being the union of S_1 and S_2 with base-points identified, and $\varkappa : S_1 * S_2 \to S_1 \times S_2$ is given by $\varkappa(x_1) = (x_1, o)$, $\varkappa(x_2) = (o, x_2)$, $x_i \in S_i$, where o is the base-point[7]) of S_1 or S_2. Notice that $\varkappa$ is a monomorphism in $\mathfrak{S}$.

As remarked $\underline{\mathrm{G}}$-objects of $\mathfrak{S}$ are just *groups* and $\underline{\mathrm{C}}\,\underline{\mathrm{G}}$-objects are *commutative groups*. On the other hand the notion of $\overline{\mathrm{H}}$-object is uninteresting in $\mathfrak{S}$; for

[7]) It is convenient to adopt a common notation for all base-elements in $\mathfrak{S}$ and all base-points in $\mathfrak{T}$.

Proposition 5.1.1. *The only* $\overline{\mathrm{H}}$*-objects in* $\mathfrak{S}$ *are the one-point sets.*

Proof. Let (S, m) be an $\overline{\mathrm{H}}$-object in $\mathfrak{S}$, so that m is a map from S to $S * S$. By the dual of Prop. 4.16 $\varkappa m = \{1, 1\} : S \to S \times S$. Now $\varkappa(S * S)$ intersects the diagonal in $S \times S$ in the single point (o, o) where o is the base-point of S. Thus for any $x \in S$, $(x, x) = \varkappa m(x) = (o, o)$ or $x = o$.

5.2. *The category* $\mathfrak{T}$ *of based topological spaces.* The objects of $\mathfrak{T}$ are topological spaces with a base-point[7]) o, the maps continuous maps preserving the base-point. An equivalence is a homeomorphism. As in $\mathfrak{S}$ there exist direct and inverse products: the direct product of X_1 and X_2 is the Cartesian product $X_1 \times X_2$ with the usual product topology and base-point, and the inverse product is the union $X_1 \vee X_2$ with base-points identified, topologized so that a subset is closed if and only if it intersects X_i in a closed subset of X_i, $i = 1, 2$. The map $\varkappa : X_1 * X_2 \to X_1 \times X_2$ is defined as in $\mathfrak{S}$ and is a monomorphism. The functor from $\mathfrak{T}$ to $\mathfrak{S}$ which associates with every space the underlying set is a DI-functor and the only $\overline{\mathrm{H}}$-objects in $\mathfrak{T}$ are the one-point spaces.

5.3. *The category* $\mathfrak{T}_h$ *of based topological spaces and based homotopy classes.* The objects of $\mathfrak{T}_h$ are the same as those of $\mathfrak{T}$; the elements of $H(A, B)$ in $\mathfrak{T}_h$ are, however, the homotopy classes of continuous maps $A \to B$, the base-points o being preserved by maps and homotopies. The element $0 \in H(A, B)$ is the class of nullhomotopic maps; an equivalence $A \to B$ in $\mathfrak{T}_h$ is a (based) homotopy equivalence. Direct (and inverse) products in $\mathfrak{T}$ and $\mathfrak{T}_h$ coincide in the sense that the objects are the same in the two categories and the projections (injections) in $\mathfrak{T}_h$ are the homotopy classes of the projections (injections) in $\mathfrak{T}$; but the canonical map $\varkappa : X_1 \vee X_2 \to X_1 \times X_2$ in $\mathfrak{T}_h$ (which is the homotopy class of the canonical map in $\mathfrak{T}$) is, in general, neither a monomorphism nor an epimorphism. The functor $h : \mathfrak{T} \to \mathfrak{T}_h$ which puts each map in its homotopy class is a DI-functor, and we sometimes permit ourselves to use this functor implicitly[8]) in referring to notions in $\mathfrak{T}_h$.

An $\underline{\mathrm{H}}$-object (A, m) in $\mathfrak{T}_h$ is usually called an H-space (a space with a continuous multiplication having o as two-sided homotopy-unit). A $\underline{\mathrm{G}}$-object in $\mathfrak{T}_h$ is a homotopy-associative H-space with homotopy-inverse. All topological groups are $\underline{\mathrm{G}}$-objects in $\mathfrak{T}_h$; and loop-spaces furnished with the multiplication given by composition of loops are also $\underline{\mathrm{G}}$-objects in $\mathfrak{T}_h$. The loop-space functor Ω, regarded as a functor from $\mathfrak{T}$ to $\mathfrak{T}$ or from $\mathfrak{T}_h$ to $\mathfrak{T}_h$, is a D-functor but not an I-functor.

In $\mathfrak{T}_h$ there exist non-trivial $\overline{\mathrm{H}}$-objects, indeed $\overline{\mathrm{G}}$-objects, namely the suspensions $\Sigma X (= I \times X/(0) \times X \cup (1) \times X \cup I \times (0))$. The comultiplication in ΣX is the homotopy class of the map $m : \Sigma X \to \Sigma X * \Sigma X$, given by

$$m(t, x) = q_1(2t, x),\ 0 \leqq t \leqq \frac{1}{2},\ x \in X\,,$$

$$= q_2(2t - 1, x),\ \frac{1}{2} \leqq t \leqq 1,\ x \in X\,;$$

[8]) As in the next paragraph where, for example, we refer to $\underline{\mathrm{H}}$-spaces; strictly, an $\underline{\mathrm{H}}$-space is an $\underline{\mathrm{M}}$-object (A, m) of $\mathfrak{T}$ such that $(A, h(m))$ is an $\underline{\mathrm{H}}$-object of $\mathfrak{T}_h$.

here q_1, q_2 are the injections of ΣX into the inverse product $\Sigma X * \Sigma X$. With respect to this comultiplication the inverse is the homotopy class of the map $s: \Sigma X \to \Sigma X$ given by

$$s(t, x) = (1 - t, x) .$$

There exist in $\mathfrak{T}_h$ $\overline{\text{H}}$-objects which are not $\overline{\text{G}}$-objects; if we confine attention to polyhedra then a space A admits an $\overline{\text{H}}$-structure in $\mathfrak{T}_h$ if and only if it is of Lusternik-Schnirelmann category $\leqq 2$ and there exist (see [1]) non-suspensions (even non-$\overline{\text{G}}$-objects) with this property. The suspension functor Σ, regarded as a functor from $\mathfrak{T}$ to $\mathfrak{T}$ or from $\mathfrak{T}_h$ to $\mathfrak{T}_h$, is an I-functor but not a D-functor.

Double loop-spaces $\Omega^2 X$ are $\underline{\text{CG}}$-objects; indeed ΩA is a $\underline{\text{CG}}$-object if A is an $\underline{\text{H}}$-object. Similarly double suspensions $\Sigma^2 X$, and suspensions ΣA where A is an $\overline{\text{H}}$-object, are $\overline{\text{CG}}$-objects. We refer again to the functors Ω and Σ in the appendix on adjoint functors.

We will adopt the notation of [2] and write $\Pi(A, B)$ for $H(A, B)$ when we are working in the category $\mathfrak{T}_h$. If (A, m) is a $\overline{\text{G}}$-object in $\mathfrak{T}_h$ then by the dual of Theorem 4.10 the induced $\underline{\text{M}}$-structure in $\Pi(A, B)$ is a group structure. In particular if A is the n-sphere S_n with its suspension structure then $\Pi(A, B) = \pi_n(B)$, the nth homotopy group of B; and if $A = \Sigma P$ then $\Pi(A, B) = \Pi_1(P, B)$; see [2]. From Theorem 4.17 we deduce that if B admits an $\underline{\text{H}}$-structure then the group structure in $\Pi_1(P, B)$ is commutative and may be obtained from the $\overline{\text{H}}$-structure in B. This is a classical theorem if $P = S_0$ and B is a topological group. We also infer from Theorems 4.6 and 4.10 that the only natural group structure which could be introduced into the sets $\pi_n(X)$, $n > 1$, is the *homotopy group* structure, and the only two natural group structures which could be introduced into the sets $\pi_1(X)$ are the *fundamental group* structure and its anti-isomorph. For S_n, $n > 1$, admits a unique $\overline{\text{G}}$-structure in $\mathfrak{T}_h$, and S_1 admits only the usual $\overline{\text{G}}$-structure[9]) $m: S_1 \to S_1 * S_1$ and the $\overline{\text{G}}$-structure $\tau m: S_1 \to S_1 * S_1$.

5.4. *The category $\mathfrak{G}$ of groups.* The objects of $\mathfrak{G}$ are groups and $H(A, B)$ consists of homomorphisms from A to B; we write $\operatorname{Hom}(A, B)$ for $H(A, B)$ in $\mathfrak{G}$. The zero-map in $H(A, B)$ is the trivial homomorphism mapping A to the unit element e of B. An equivalence in $\mathfrak{G}$ is an isomorphism of groups.

Direct products in $\mathfrak{G}$ are those considered as usual in group theory; inverse products in $\mathfrak{G}$ are just free products of groups. The projections $p_1: A \times B \to A$, $p_2: A \times B \to B$ are given by $p_1(a, b) = a$, $p_2(a, b) = b$; the injections $q_1: A \to A * B$, $q_2: B \to A * B$ by $q_1(a) = a$, $q_2(b) = b$ (and $A * B$ is the group generated without further relations by $q_1(A)$ and $q_2(B)$). The theory of $\underline{\text{G}}$- and $\underline{\text{H}}$-objects in $\mathfrak{G}$ and their duals has been discussed in a separate paper [5]; here we merely list a few elementary facts, together with some results from [5].

First we note that for all $A, B \in \mathfrak{G}$ the canonical map $\varkappa: A * B \to A \times B$, given by $\varkappa(a) = (a, e)$, $\varkappa(b) = (e, b)$, $a \in A$, $b \in B$, is an epimorphism. Thus by Theorem 4.19 an $\underline{\text{H}}$-structure on $A \in \mathfrak{G}$ is unique and commutative. Now if

[9]) Actually S_n, $n > 1$, admits only one $\overline{\text{H}}$-structure, but S_1 admits infinitely many H-structures.

(A, m) is an $\underline{\mathrm{H}}$-object then

(5.4.1) $\quad m(a_1, a_2) = m(a_1, e) \cdot m(e, a_2) = a_1 a_2$ by axiom I, $a_1, a_2 \in A$,

whence, m being commutative, $a_1 a_2 = a_2 a_1$; conversely if A is an abelian group the rule (5.4.1) does give an $\underline{\mathrm{H}}$-structure on A. Thus

Theorem 5.4.2. *(A, m) is an $\underline{\mathrm{H}}$-object in $\mathfrak{G}$ if and only if A is an abelian group, the $\underline{\mathrm{H}}$-structure m being the group operation in A. If A, B are abelian groups every element of* $\mathrm{Hom}(A, B)$ *is primitive.*

Of course the $\underline{\mathrm{H}}$-structures in $\mathfrak{G}$ are, in fact, $\underline{\mathrm{G}}$-structures.

Turning to $\overline{\mathrm{H}}$-structures in $\mathfrak{G}$ we quote from [5] (see also [8]).

Theorem 5.4.3. *A group A admits an $\overline{\mathrm{H}}$-structure if and only if it is free. There are, in a free group, several $\overline{\mathrm{H}}$-structures. The $\overline{\mathrm{AH}}$-structures in A are in one-to-one correspondence with the sets of free generators of A. There are no non-trivial commutative $\overline{\mathrm{H}}$-structures in $\mathfrak{G}$. If (A, m), (B, m') are $\overline{\mathrm{AH}}$-objects in $\mathfrak{G}$ an element of* $\mathrm{Hom}(A, B)$ *is primitive if and only if it maps each generator of the generating set of A corresponding to the $\overline{\mathrm{AH}}$-structure m to $e \in B$ or to a generator of the generating set of B corresponding to m'.*

The functor $\mathfrak{G} \to \mathfrak{S}$ which associates with each group its underlying set is a D-functor but not an I-functor. A more interesting functor is the fundamental group functor π_1 which may be regarded either as a functor $\mathfrak{T} \to \mathfrak{G}$ or as a functor $\mathfrak{T}_h \to \mathfrak{G}$; we will take the latter view. Then π_1 is a D-functor; if we impose some restriction on the spaces we study, e.g., if we consider spaces of the based homotopy type of CW-complexes, then π_1 not only remains a D-functor but indeed is a DI-functor. We infer immediately from this and the results above

Theorem 5.4.4. *If X is an $\underline{\mathrm{H}}$-space then $\pi_1(X)$ is commutative; if X is a space of the based homotopy type of a* CW*-complex and if X admits an $\overline{\mathrm{H}}$-structure then $\pi_1(X)$ is free.*

The first assertion of this theorem has already been deduced from Theorem 4.15; the second assertion, restricted to suspension spaces, is fairly well-known and may be proved, for example, by purely combinatorial methods.

5.5. *The category $\mathfrak{A}$ of abelian groups.* The objects of $\mathfrak{A}$ are (additive) abelian groups, and $H(A, B)$, which we again write as $\mathrm{Hom}(A, B)$, consists of homomorphisms from A to B. The direct sum $A + B$ of the two abelian groups A and B has the properties both of the direct product and of the inverse product; the injections of A and B into $A + B$ are given by $q_1 = \{1, 0\}$, $q_2 = \{0, 1\}$. Thus the canonical map $\varkappa$ is the identity. All objects B admit a unique $\underline{\mathrm{H}}$-structure, given by the group operation, and this structure is a $\underline{\mathrm{CG}}$-structure; similarly all objects A admit a unique $\overline{\mathrm{H}}$-structure, given by $a \to (a, a)$, $a \in A$, and this structure is a $\overline{\mathrm{CG}}$-structure. The abelian group structure induced by either of these structures in $\mathrm{Hom}(A, B)$ is the usual abelian group structure in $\mathrm{Hom}(A, B)$. These remarks apply, of course, equally to the category $\mathfrak{M}$ of (right) Λ-modules where Λ is any ring. In particular, it follows from Prop. 4.18 that in any DI-category in which $\varkappa$ is the identity each object admits a unique $\underline{\mathrm{H}}$-structure $m = \langle 1, 1 \rangle$ and a unique

$\overline{\mathrm{H}}$-structure $m = \{1, 1\}$. On the other hand we show in the third paper of the series how tha features of the category $\mathfrak{A}$ may be derived by considering its special relation to the category $\mathfrak{G}$, and the latter category's special relation to the category $\mathfrak{S}$.

The functors H_n (nth reduced homology group, $n = 0, 1, 2, \ldots$) are I-functors from $\mathfrak{T}$ or $\mathfrak{T}_h$ to $\mathfrak{A}$; the functors H^n (nth reduced cohomology group, $n = 0, 1, 2, \ldots$) are contravariant I-functors from $\mathfrak{T}$ or $\mathfrak{T}_h$ to $\mathfrak{A}$; the functors π_n (nth homotopy group, $n = 2, 3, \ldots$) are D-functors from $\mathfrak{T}$ or $\mathfrak{T}_h$ to $\mathfrak{A}$. The failure of H_n to be a D-functor is measured by the Künneth formula, but no such universal formula has yet been found for measuring the failure of π_n to be an I-functor.

5.6. *The category of pairs of a given category.* Given any category $\mathfrak{C}$ we may form a new category $\mathfrak{P} = \mathfrak{P}(\mathfrak{C})$ in which the objects are the maps of $\mathfrak{C}$ and in which $H(f, g)$, where f, g are objects of $\mathfrak{P}$, consists of maps (a, b) such that $ga = bf$, i.e., such that the diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "b"] & D \end{tikzcd}$$

is commutative. If also $(a', b'): g \to h$ then, by definition,

$$(a', b')\,(a, b) = (a'a, b'b)\ .$$

It is plain that, $\mathfrak{C}$ being a category with zero maps, $\mathfrak{P}$ is also a category with zero maps, with $0 = (0, 0)$.

Now let $g_j : A_j \to B_j$, $j = 1, 2, \ldots, n$ in the D-category $\mathfrak{C}$. Then

Theorem 5.6.1. *$\mathfrak{P}(\mathfrak{C})$ is a D-category in which the direct product of $g_1, g_2, \ldots, g_n$ is $g_1 \times g_2 \times \cdots \times g_n$, the projections being the pairs of projections $(p_j(A), p_j(B))$.*

We leave the proof to the reader.

Let (A, m_A) and (B, m_B) be $\underline{\mathrm{M}}$-objects in $\mathfrak{C}$ and let $g : A \to B$ be a map. Then $(m_A, m_B) \in H(g \times g, g)$ if and only if g is primitive; and every $\underline{\mathrm{M}}$-structure in $\mathfrak{P}$ is such a pair (m_A, m_B) of $\underline{\mathrm{M}}$-structures, belonging to a set $H(g \times g, g)$ where g is primitive with respect to m_A and m_B. We again leave to the reader the proof of

Theorem 5.6.2. *The $\underline{\mathrm{M}}$-structure (m_A, m_B) satisfies axiom K ($K =$ I, II, III, IV) if and only if m_A and m_B each satisfy axiom K.*

Notice, too, that the categories $\mathfrak{P}^*(\mathfrak{C})$ and $\mathfrak{P}(\mathfrak{C}^*)$ are isomorphic; for if f, g are maps of $\mathfrak{C}$ and if (a, b) is a map from f to g in $\mathfrak{P}^*$ then (b, a) is a map from f to g in $\mathfrak{P}(\mathfrak{C}^*)$.

Categories of pairs play an important role in topology and algebra and have been used in homotopy theory (see [2], [4]).

5.7. *The category of functors $\mathfrak{C} \to \mathfrak{D}$.* Let $\mathfrak{C}, \mathfrak{D}$ be two categories and let $\mathfrak{F} = \mathfrak{F}(\mathfrak{C}, \mathfrak{D})$ be the category whose objects are the covariant functors $F, G, \ldots : \mathfrak{C} \to \mathfrak{D}$ and whose maps are the natural transformations[10]): $F \to G$.

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[10]) See, for example, [9].

It is plain that if $\mathfrak{D}$ is a category with zero-maps, so is $\mathfrak{F}$; precisely $0: F \to G$ is given by $0(X) = 0: F(X) \to G(X)$, $X \in \mathfrak{C}$.

Theorem 5.7.1. *If $\mathfrak{D}$ is a* D*-category so is $\mathfrak{F}(\mathfrak{C}, \mathfrak{D})$.*

Proof. Given functors $F_1, F_2: \mathfrak{C} \to \mathfrak{D}$, define $F_1 \times F_2: \mathfrak{C} \to \mathfrak{D}$ by $(F_1 \times F_2)(X) = F_1(X) \times F_2(X)$, $(F_1 \times F_2)(f) = F_1(f) \times F_2(f)$. Clearly $F_1 \times F_2$ is a functor. Define natural transformations $p_j: F_1 \times F_2 \to F_j$ by $p_j(X) = p_j: F_1(X) \times F_2(X) \to F_j(X)$, $j = 1, 2$. These transformations are indeed natural since if $f: X \to Y$ in $\mathfrak{C}$ then $F_j(f) \circ p_j(X) = p_j(Y) \circ (F_1(f) \times F_2(f))$.

Now let J be an arbitrary functor and let $\theta_j: J \to F_j$ be natural transformations, $j = 1, 2$. Define $\theta: J \to F_1 \times F_2$ by $\theta(X) = \{\theta_1(X), \theta_2(X)\}$, $X \in \mathfrak{C}$. Then θ is natural; for if $f: X \to Y$ then

$$\begin{aligned}
\theta(Y) \circ J(f) &= \{\theta_1(Y), \theta_2(Y)\} \circ J(f) \\
&= \{\theta_1(Y) \circ J(f), \theta_2(Y) \circ J(f)\} \\
&= \{F_1(f) \circ \theta_1(X), F_2(f) \circ \theta_2(X)\}, \text{ by the naturality of } \theta_1, \theta_2 \\
&= (F_1(f) \times F_2(f)) \circ \{\theta_1(X), \theta_2(X)\} \\
&= (F_1 \times F_2)(f) \circ \theta(X).
\end{aligned}$$

Moreover $p_j \theta = \theta_j$, evidently.

Finally let $p_j \varphi = \theta_j$ for $\varphi: J \to F_1 \times F_2$. Then $p_j \varphi(X) = \theta_j(X)$ so that $\varphi(X)$ is uniquely determined and hence so too is φ.

Now an $\underline{\mathrm{M}}$-structure on the functor F is a natural transformation $\mu: F \times F \to F$. This is a collection of $\underline{\mathrm{M}}$-structures $\mu(X)$ on $F(X)$ for each $X \in \mathfrak{C}$ such that $F(f)$ is primitive for each $f: X \to Y$ in $\mathfrak{C}$. We discuss such $\underline{\mathrm{M}}$-structures in the next section where we refer to them as *natural families* of $\underline{\mathrm{M}}$-structures.

We remark that 5.6 is a special case of the notion of category of functors: we take for $\mathfrak{C}$ a category with two objects (which we may call "domain" and "range") and, apart from identities, one map (which we may call "arrow", going from "domain" to "range"). Then an object of $\mathfrak{F}(\mathfrak{C}, \mathfrak{D})$ is a map of $\mathfrak{D}$ and a map of $\mathfrak{F}(\mathfrak{C}, \mathfrak{D})$ is a pair of maps yielding a commutative diagram.

Notice that, in general,

$$\mathfrak{F}^*(\mathfrak{C}, \mathfrak{D}) = \mathfrak{F}(\mathfrak{C}^*, \mathfrak{D}^*). \tag{5.7.2}$$

Thus we may deduce from Theorem 5.7.1.

Theorem 5.7.3. *If $\mathfrak{D}$ is an* I*-category so is $\mathfrak{F}(\mathfrak{C}, \mathfrak{D})$.*

6. Adjoint functors

Kan [7] has given the following definition of adjoint functors. Let $\mathfrak{C}, \mathfrak{D}$ be two categories and let $S: \mathfrak{C} \to \mathfrak{D}$, $T: \mathfrak{D} \to \mathfrak{C}$ be covariant functors. If A, B are objects of $\mathfrak{C}$ then $H(A, B)$ is an element of the category $\mathfrak{S}$ and we write[11])

[11]) The notion of *direct product* of two categories presents no difficulty. Objects of $\mathfrak{C} \times \mathfrak{D}$ are pairs (A, B), $A \in \mathfrak{C}$, $B \in \mathfrak{D}$, and similarly maps of $\mathfrak{C} \times \mathfrak{D}$ are pairs of maps (f, g), f in $\mathfrak{C}$, g in $\mathfrak{D}$; moreover composition of maps in $\mathfrak{C} \times \mathfrak{D}$ is componentwise. Direct products of (covariant) functors are defined in the obvious way.

M_1 for the associated functor $\mathfrak{C} \times \mathfrak{C} \to \mathfrak{S}$; similarly M_2 is a functor $\mathfrak{D} \times \mathfrak{D} \to \mathfrak{S}$. We also have functors $S \times 1 : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{D} \times \mathfrak{D}$ and $1 \times T : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{C} \times \mathfrak{C}$ and we declare S to be *left-adjoint* to T (T to be *right-adjoint* to S) if there is a natural equivalence η between the functors $M_2(S \times 1)$ and $M_1(1 \times T)$ from $\mathfrak{C} \times \mathfrak{D}$ to $\mathfrak{S}$. In other words, given $A \in \mathfrak{C}$, $B \in \mathfrak{D}$, there exists a one-to-one correspondence $\eta = \eta_{AB}$ between the sets $H(SA, B)$ and $H(A, TB)$ such that, for all $f : A' \to A$ in $\mathfrak{C}$ and $g : B \to B'$ in $\mathfrak{D}$,

$$(6.1) \qquad \eta(g \circ \alpha \circ Sf) = Tg \circ \eta(\alpha) \circ f, \quad \alpha \in H(SA, B)\,.$$

We call η the *adjugant* of S and T. Of particular interest in homotopy theory are the adjoint functors Σ and Ω, regarded either as functors from $\mathfrak{T}$ to $\mathfrak{T}$ or from $\mathfrak{T}_h$ to $\mathfrak{T}_h$. If $A, B \in \mathfrak{T}$ the adjugant $\eta : H(\Sigma A, B) \to H(A, \Omega B)$ is given by $\eta(f)\,(a)\,(t) = f(a, t)$; the adjugant of the functors Σ and Ω in $\mathfrak{T}_h$ is the map of homotopy classes induced by the adjugant in $\mathfrak{T}$. We remarked in 5.3 that Σ is an I-functor and Ω a D-functor. That this is merely a consequence of their adjointness is attested by

Theorem 6.2. *Let $\mathfrak{C}, \mathfrak{D}$ be D-categories and let $T : \mathfrak{D} \to \mathfrak{C}$ be a covariant functor. Then if T admits a left-adjoint, T is a D-functor.*

Proof. We have to show that $(T(A_1 \times A_2)\,;\, Tp_1, Tp_2)$ is a direct product of TA_1 and TA_2 in $\mathfrak{C}$ for all A_1, A_2 in $\mathfrak{D}$. Let $S : \mathfrak{C} \to \mathfrak{D}$ be left-adjoint to T and let η be the adjugant of S and T. Given $f_j : X \to TA_j$, $j = 1, 2$, in $\mathfrak{C}$ let $g_j = \eta^{-1}(f_j) : SX \to A_j$, and consider the map $\eta\{g_1, g_2\} : X \to T(A_1 \times A_2)$. Then

$$\begin{aligned} Tp_j \circ \eta\{g_1, g_2\} &= \eta(p_j\{g_1, g_2\}) \quad \text{by} \quad (6.1) \\ &= \eta(g_j) \\ &= f_j, \quad j = 1, 2\,. \end{aligned}$$

It remains to show that if $f, f' : X \to T(A_1 \times A_2)$ are such that $Tp_j \circ f = Tp_j \circ f'$, $j = 1, 2$, then $f = f'$. Let $f = \eta(g)$, $f' = \eta(g')$. Then $Tp_j \circ f = \eta(p_j g)$, $Tp_j \circ f' = \eta(p_j g')$ so $p_j g = p_j g'$, $j = 1, 2$, whence $g = g'$, $f = f'$, and the theorem is proved.

Next we relate the two facts that every loop-space carries a "natural" multiplication and every suspension a "natural" comultiplication. Let $\mathfrak{C}$ be a D-category and $T : \mathfrak{D} \to \mathfrak{C}$ a covariant functor. We denote by m_T a family of $\underline{\mathrm{M}}$-structures, one for each TB as B ranges over the objects of $\mathfrak{D}$ and we say that m_T is *natural*[12]) if Tf is primitive for each map f in $\mathfrak{D}$. We say that m_T satisfies axiom K if m_{TB} satisfies axiom K for each $B \in \mathfrak{D}$. Similar definitions apply to a functor $S : \mathfrak{C} \to \mathfrak{D}$ where $\mathfrak{D}$ is an I-category, and m_S denotes a family of $\overline{\mathrm{M}}$-structures.

In fact we apply these definitions when $\mathfrak{C}$ is a D-category, $\mathfrak{D}$ is an I-category and $S : \mathfrak{C} \to \mathfrak{D}$ is left-adjoint to $T : \mathfrak{D} \to \mathfrak{C}$ with adjugant η. Let m_T be a family of $\underline{\mathrm{M}}$-structures as above and let $A \in \mathfrak{C}$. Then we may define an $\overline{\mathrm{M}}$-structure on SA,

$$n = n_{SA} : SA \to SA * SA\,,$$

[12]) See 5.7.

by the rule

$$\eta(n) = m_{TB_0} \circ \{\eta q_1, \eta q_2\} : A \to TB_0 \times TB_0 \to TB_0 , \tag{6.3}$$

where $B_0 = SA * SA$.

We suppose in enunciating the next theorem that each TB, $B \in \mathfrak{D}$, is endowed with its $\underline{\mathrm{M}}$-structure m_{TB} and each SA, $A \in \mathfrak{C}$, is endowed with its $\overline{\mathrm{M}}$-structure n_{SA}. We prove

Theorem 6.4. *Let the family m_T be natural. Then*

(i) *$\eta : H(SA, B) \to H(A, TB)$ is an isomorphism of $\underline{\mathrm{M}}$-sets for each $A \in \mathfrak{C}$, $B \in \mathfrak{D}$;*

(ii) *the family n_S is natural;*

(iii) *n_S satisfies axiom K if m_T satisfies axiom K ($K =$ I, II, III, IV).*

Proof. Fix A and give $H(A, TB)$ the $\underline{\mathrm{M}}$-structure induced by m_{TB} for each $B \in \mathfrak{D}$. If we define an $\underline{\mathrm{M}}$-structure in each $H(SA, B)$ by the rule

$$\eta(g_1 + g_2) = \eta(g_1) + \eta(g_2), \; g_1, g_2 \in H(SA, B) , \tag{6.5}$$

then it follows from the naturality of m_T and Theorem 4.7 that f_* is primitive with respect to the $\underline{\mathrm{M}}$-structures (6.5) for each $f : B \to B'$ in $\mathfrak{D}$. Hence by Theorem 4.6 there exists an $\overline{\mathrm{M}}$-structure n'_{SA} on SA which induces the given $\underline{\mathrm{M}}$-structures in $H(SA, B)$, $B \in \mathfrak{D}$. Moreover since $h^* : H(A, TB) \to H(A', TB)$ is primitive for all $h : A' \to A$ in $\mathfrak{C}$ and since, by (6.1),

$$\eta((Sh)^* g) = h^* \eta(g) ,$$

it follows from the converse of Theorem 4.7 that Sh is primitive with respect to the $\overline{\mathrm{M}}$-structures $n'_{SA'}$, n'_{SA} so that the family n'_S is natural. Thus, in the light of Theorem 4.10, Theorem 6.4 is proved when we have shown that $n'_S = n_S$. But $n' = q_1 + q_2 \in H(SA, SA * SA)$ so

$$\eta(n') = \eta(q_1) + \eta(q_2) = m_{TB_0} \circ \{\eta q_1, \eta q_2\} , \quad B_0 = SA * SA$$

and the theorem is proved.

Notice that, of course, the family n_S determines the family m_T by the rule dual to (6.3) provided the family m_T is natural. It is easy to verify that the natural structures in ΩB and ΣA, $A, B \in \mathfrak{T}$ or $\mathfrak{T}_h$, are related exactly as in (6.3).

We now put the extra condition on the category $\mathfrak{D}$ that it be also a D-category (and so, in fact, a DI-category). Let (B, μ) be an $\underline{\mathrm{M}}$-object in $\mathfrak{D}$. Then by theorem 4.14 $(TB, T\mu)$ is an $\underline{\mathrm{H}}$-object in $\mathfrak{C}$ if μ is an $\underline{\mathrm{H}}$-structure; here again T is a covariant functor $\mathfrak{D} \to \mathfrak{C}$ with left-adjoint S and adjugant η, but for the moment we do not postulate a natural family m_T. We remark that for any $D \in \mathfrak{D}$, $A \in \mathfrak{C}$, $f_j : SA \to D$ in $\mathfrak{D}$, $j = 1, 2$,

$$\eta\{f_1, f_2\} = \{\eta f_1, \eta f_2\} . \tag{6.6}$$

For T is a D-functor, $Tp_j = p_j$, whence

$$p_j \circ \eta\{f_1, f_2\} = Tp_j \circ \eta\{f_1, f_2\} = \eta(p_j\{f_1, f_2\}) = \eta f_j, j = 1, 2 .$$

Proposition 6.7. *Let $H(SA, B)$ be given the $\underline{\mathrm{M}}$-structure induced by μ and $H(A, TB)$ the $\underline{\mathrm{M}}$-structure induced by $T\mu$. Then $\eta : H(SA, B) \to H(A, TB)$ is an isomorphism of $\underline{\mathrm{M}}$-sets.*

$$\begin{aligned} \text{For } \eta(\mu \circ \{f_1, f_2\}) &= T\mu \circ \eta\{f_1, f_2\} \quad \text{by (6.1)} \\ &= T\mu \circ \{\eta f_1, \eta f_2\} \quad \text{by (6.6)}\,. \end{aligned}$$

Now let m_T be a natural family of $\underline{\mathrm{H}}$-structures in $\mathfrak{C}$ and let μ be an $\underline{\mathrm{H}}$-structure. We prove

Theorem 6.8. *Under these hypotheses $m_{TB} = T\mu$ and is commutative.*

Proof. Let A be a fixed but arbitrary object of $\mathfrak{C}$. Then $H(SA, B)$ receives an $\underline{\mathrm{H}}$-structure either from the $\overline{\mathrm{H}}$-structure n_{SA} in SA given by (6.3) or from the $\underline{\mathrm{H}}$-structure μ in B. By Theorem 4.17 these two $\underline{\mathrm{H}}$-structures coincide in $H(SA, B)$ and are commutative. On the other hand $\eta : H(SA, B) \to H(A, TB)$ is an isomorphism of $\underline{\mathrm{H}}$-sets either if we use the structures n_{SA} and m_{TB} (Theorem 6.4) or if we use the structures μ and $T\mu$ (Prop. 6.7). It follows that the structures m_{TB} and $T\mu$ induce the *same commutative* structure in $H(A, TB)$. Since this is true for every A in $\mathfrak{C}$, we must have $m_{TB} = T\mu$ and each is commutative.

The assertion of Theorem 6.8 is familiar for the functor Ω. Our proof yields simultaneously, by the duality principle, the corresponding conclusion for Σ; these conclusions were referred to near the end of 5.3.

7. Appendix: a counterexample

In this appendix we show that there exist D-categories in which the direct product of epimorphisms is not always an epimorphism (compare Theorem 3.8); it will be shown in the third paper of this series that, in an important class of D-categories (namely, the *primitive* categories), the direct product of epimorphisms is an epimorphism.

The category of our example is a subcategory of the category $\mathfrak{S}$. Let $A = (a, o)$, $B = (b, b', o)$. Then the objects of our category $\mathfrak{C}$ are precisely all the finite direct products $A^m \times B^n$, $m \geqq 0$, $n \geqq 0$, of copies of A and B. A map $A^m \times B^n \to A^r \times B^s$ of $\mathfrak{S}$ is in $\mathfrak{C}$ if and only if its components are in $\mathfrak{C}$ (this will ensure that $\mathfrak{C}$ is a D-category) so it remains to describe those maps $A^m \times B^n \to A$, $A^m \times B^n \to B$ of $\mathfrak{S}$ which belong to $\mathfrak{C}$. For this description we will write an element of A^m as a "vector" $\mathbf{a}$, and an element of B^n as a "vector" $\mathbf{b}$; and the fact that a is among the components of $\mathbf{a}$ will be denoted by $a \in \mathbf{a}$; similarly $b \in \mathbf{b}$, $b' \in \mathbf{b}$.

The maps of $\mathfrak{C}$ are determined as follows. A map $f : A^m \times B^n \to A$ is in $\mathfrak{C}$ if and only if the equation $f(\mathbf{a}, \mathbf{b}) = a$ implies that f is a projection *or* $b, b' \in \mathbf{b}$ *or* $a \in \mathbf{a}$, $b' \in \mathbf{b}$. A map $g : A^m \times B^n \to B$ is in $\mathfrak{C}$ if and only if the equation $g(\mathbf{a}, \mathbf{b}) = b$ implies that g is a projection *or* $b, b' \in \mathbf{b}$ *or* $a \in \mathbf{a}$ and the equation $g(\mathbf{a}, \mathbf{b}) = b'$ implies that g is a projection.

To illustrate this definition we describe the maps for small values of m, n. Thus we exclude from $\mathfrak{C}$ the map $A \to B$ sending a to b'; we include only the zero map $B \to A$; we include only the zero and identity maps $B \to B$; we include in $\mathfrak{C}$ only the maps $0, p_1, k : A \times B \to A$, where $k(a, b') = a$ and maps the rest of $A \times B$ to o; the maps $A \times B \to B$ in $\mathfrak{C}$ are p_2 and any map sending (o, b) and (o, b') to o; the maps $A \times A \to A$ in $\mathfrak{C}$ are $0, p_1, p_2$; the maps $A \times A \to B$ in $\mathfrak{C}$ are all those whose image does not contain b'; the maps $B \times B \to A$ in $\mathfrak{C}$ are those which map all elements except (b, b') and (b', b) — and possibly these elements, too — to o; and the maps $B \times B \to B$ in $\mathfrak{C}$ are p_1 and p_2 and those which map all elements except (b, b') and (b', b) to o and map (b, b') and (b', b) to o or b. Notice that $\mathfrak{C}$ contains all projections $A^m \times B^n \to A$, $A^m \times B^n \to B$.

Proposition 7.1. *$\mathfrak{C}$ is a category.*

Proof. We call the maps $f : A^m \times B^n \to A$, $g : A^m \times B^n \to B$ *admissible* if they are in $\mathfrak{C}$. Thus we must show that if $u = \{f_1, \ldots, f_m, g_1, \ldots, g_n\} : A^r \times B^s \to A^m \times B^n$ has admissible components and if $f : A^m \times B^n \to A$, $g : A^m \times B^n \to B$ are admissible then fu and gu are admissible.

Let $u(\mathbf{a}, \mathbf{b}) = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, and suppose $fu(\mathbf{a}, \mathbf{b}) = a$. Then $f(\boldsymbol{\alpha}, \boldsymbol{\beta}) = a$ so f is a projection or $b, b' \in \boldsymbol{\beta}$ or $a \in \boldsymbol{\alpha}$, $b' \in \boldsymbol{\beta}$. If $f = p_j$ then $fu = f_j$ and is admissible; If $b = \beta_i$, $b' = \beta_j$, then g_j is a projection, so $b' \in \mathbf{b}$ and g_i is a projection or $b, b' \in \mathbf{b}$ or $a \in \mathbf{a}$. Thus $b, b' \in \mathbf{b}$ or $a \in \mathbf{a}$, $b' \in \mathbf{b}$ if $b, b' \in \boldsymbol{\beta}$. If $a = \alpha_i$, $b' = \beta_j$ then $b' \in \mathbf{b}$ and f_i is a projection or $b, b' \in \mathbf{b}$ or $a \in \mathbf{a}$, $b' \in \mathbf{b}$. Thus $a \in \mathbf{a}$, $b' \in \mathbf{b}$ or $b, b' \in \mathbf{b}$ if $a \in \boldsymbol{\alpha}$, $b' \in \boldsymbol{\beta}$. This shows that fu is admissible.

Now suppose that $gu(\mathbf{a}, \mathbf{b}) = b$. Then $g(\boldsymbol{\alpha}, \boldsymbol{\beta}) = b$ so g is a projection or $b, b' \in \boldsymbol{\beta}$ or $a \in \boldsymbol{\alpha}$. If $g = p_{m+j}$ then $gu = g_j$ and is admissible. If $b, b' \in \boldsymbol{\beta}$ then, as above $b, b' \in \mathbf{b}$ or $a \in \mathbf{a}$, $b' \in \mathbf{b}$, so certainly $b, b' \in \mathbf{b}$ or $a \in \mathbf{a}$. If $a = \alpha_i$ then, as above, f_i is a projection or $b, b' \in \mathbf{b}$ or $a \in \mathbf{a}$, $b' \in \mathbf{b}$. Thus, if $a \in \alpha$, $a \in \mathbf{a}$ or $b, b' \in \mathbf{b}$.

Finally suppose that $gu(\mathbf{a}, \mathbf{b}) = b'$. Then $g(\boldsymbol{\alpha}, \boldsymbol{\beta}) = b'$ so g is a projection, say $g = p_{m+j}$. Then $gu = g_j$ and $g_j(\mathbf{a}, \mathbf{b}) = b'$ so g_j is a projection. Thus gu is a projection and the proposition is proved. Plainly $\mathfrak{C}$ is a D-category.

Proposition 7.2. *Let $h : A \to B$ be the map given by $h(a) = b$. Then h is an epimorphism in $\mathfrak{C}$ but $h \times 1 : A \times B \to B \times B$ is not an epimorphism in $\mathfrak{C}$.*

Proof. To test whether h is an epimorphism it is sufficient to compose it with maps $B \to A$, $B \to B$. Since only $0 : B \to A$ is in $\mathfrak{C}$, it is trivial that $v_1 h = v_2 h \Rightarrow v_1 = v_2$ for $v_1, v_2 : B \to A$ in $\mathfrak{C}$. Since only $0, 1 : B \to B$ are in $\mathfrak{C}$ and $0h \neq 1h : A \to B$, it is plain that $v_1 h = v_2 h \Rightarrow v_1 = v_2$ for $v_1, v_2 : B \to B$. Thus h is an epimorphism in $\mathfrak{C}$. On the other hand let $v : B \times B \to A$ be the map sending all of $B \times B$ to o except that (b', b) is mapped to a. Then v is in $\mathfrak{C}$ and $v \neq 0$. On the other hand $v(h \times 1) = 0(h \times 1) = 0$ so that $h \times 1$ is not an epimorphism.

We should remark that a far simpler example is available of this phenomenon if we do not insist on D-categories; it is comparatively trivial to find a category with two epimorphisms whose direct product is not an epimorphism.

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