On the Asphericity of One-Point Unions of Cones

by

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ONE-POINT UNIONS OF CONES

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Abstract. We prove that the one-point union of two copies of the cone over the Hawaiian earring is aspherical.

1. Introduction and definitions

The one-point union $C\mathbb{H} \lor C\mathbb{H}$ of two copies of the cone over the Hawaiian earring $\mathbb{H}$ is not simply connected [9]. This is a well-known example of a non-contractible one-point union of two contractible spaces [15, p. 59]. The non-triviality of its fundamental group follows from the presentation of the group given by H. B. Griffiths in [10], a flaw in which was remedied in [13]. Another proof was suggested by R. H. Fox in his review of [9] and is proved in detail in [3, Theorem 2].

On the other hand, the Hawaiian earring and, more generally, every planar or one-dimensional space are aspherical in the sense that all homotopy groups of dimension at least 2 is trivial [17], [2],

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and [1]. In [4], the authors constructed a 2-dimensional, simply-connected, cell-like Peano continuum \(SC(S^1)\) such that the second homotopy group \(\pi_2(SC(S^1))\) is non-trivial. In [5], the authors demonstrated variants of \(SC(S^1)\)-construction which, on one hand, produces a space homotopy equivalent to \(SC(S^1)\) [5, Theorem 4.3(2)] and, on the other hand, produces a space homotopy equivalent to \(CH\lor CH\) [5, Theorem 4.3(3)]. This leads to a question of whether the space \(CH\lor CH\) is aspherical. The present paper answers this question in the affirmative.

For a Hausdorff space \(X\), \(CX\) denotes the cone over \(X\)

\[ CX = X \times [0, 1]/X \times \{1\}, \]

with the quotient topology. The peak point of \(CX\) is the point represented by \(X \times \{1\}\), and is denoted by \(p\). The space \(X\) is identified with the subspace \(X \times \{0\}\). Let \(X_0\) and \(X_1\) be two Hausdorff spaces with two points \(o_0 \in X_0\) and \(o_1 \in X_1\). For \(i = 0, 1\), the peak point of \(CX_i\) is denoted by \(p_i\). The one-point union \(CX_0 \lor CX_1\) is the space obtained from the topological sum \(CX_0 \oplus CX_1\) with the points \(o_0\) and \(o_1\) being identified with a point \(o\).

**Theorem 1.1.** Let \(X_0\) and \(X_1\) be one-dimensional compact metric spaces. Then \(\pi_n(CX_0 \lor CX_1)\) is trivial for each \(n \geq 2\).

Consequently, we have an answer to the question above.

**Corollary 1.2.** Let \(H_0\) and \(H_1\) be copies of the Hawaiian earring \(H\). Then \(\pi_n(CH_0 \lor CH_1)\) is trivial for each \(n \geq 2\).

Since the cone construction makes a space contractible, it does not seem that “the coning” adds any complexity to one-point unions.

**Question 1.3.** Let \(X_0\) and \(X_1\) be path-connected (Hausdorff) spaces such that the \(n\)-th homotopy group \(\pi_n(X_0 \lor X_1)\) is trivial. Then is the group \(\pi_n(CX_0 \lor CX_1)\) also trivial?

At the time of this writing, we can answer the above question only for \(n = 2\).

**Theorem 1.4.** Let \(X_0\) and \(X_1\) be path-connected Hausdorff spaces such that the second homotopy group \(\pi_2(X_0 \lor X_1)\) is trivial. Then the group \(\pi_2(CX_0 \lor CX_1)\) is also trivial.

All spaces are assumed to be Hausdorff and all maps are assumed to be continuous unless otherwise stated. The word “components”
means “path-connected components.” The reader is referred to [15] for undefined notions.

2. Proofs of Theorems 1.1 and 1.4

Let \( K \) be a polyhedron with a triangulation \( \mathcal{T} \). By abuse of notation, the subcomplex of \( \mathcal{T} \) that defines a subpolyhedron \( L \) of \( K \) is denoted by the same symbol \( L \). For an \( n \)-dimensional PL submanifold \( Q \) of \( S^n \) (with the standard triangulation), the manifold boundary of \( Q \) coincides with the topological boundary of \( Q \) in \( S^n \) and is denoted by \( \partial Q \). Also, \( \text{Int} Q = Q \setminus \partial Q \).

The following result seems to be well known and a proof is provided for completeness of the argument. Let \( n \) be an integer such that \( n \geq 2 \). Note that, for \( n = 2 \), we make no assumption on the space \( X \) other than its path-connectivity.

**Lemma 2.1.** Let \( X \) be a path-connected space with base point \( o \) such that \( \pi_i(X, o) = 0 \) for each \( i = 2, \cdots, n-1 \). Let \( P \) be a compact \( n \)-dimensional PL submanifold of \( S^n \) and let \( f : P \to X \) be a map such that

1. for each map \( g : S^1 \to \partial P \), the composition \( f \circ g : S^1 \to X \) is null homotopic.

Then the map \( f \) admits an extension to a map \( \tilde{f} : S^n \to X \).

**Proof:** Let \( P_0, \cdots, P_k \) be the components of \( P \) and let \( \{C_{ij} | j = 0, \cdots, l_i\} \) be the components of \( \partial P_i \). We take a sufficiently fine triangulation \( \mathcal{T} \) of \( S^n \) such that

1. each \( P_i \) is a subpolyhedron with respect to \( \mathcal{T} \), and
2. no 1-simplex of \( \mathcal{T} \) connects distinct components \( C_{ij} \) and \( C_{i'j'} \).

We define an extension \( \tilde{f} \) of \( f \) by an induction on the skeleton \( \mathcal{T}^{(m)} \). At the outset, we fix a maximal tree \( T_{ij} \subseteq C_{ij} \subseteq (\partial P_i)^{(1)} \) and a vertex \( v_{ij} \in T_{ij} \) for each \( C_{ij} \). Additionally, we choose and fix a path \( p_{ij} \) from \( f(v_{ij}) \) to \( o \). For a 1-simplex with vertices \( u \) and \( v \), \( (u, v) \) denotes the 1-simplex endowed with the orientation from \( u \) toward \( v \).

Define \( \tilde{f}(v) = f(v) \) for each vertex \( v \in P \) and \( f(v) = o \) for \( v \notin P \). For a 1-simplex \( \sigma \notin P \) with vertices \( v_0 \) and \( v_1 \), we define \( \tilde{f} \) on \( \sigma \) as follows:
(1.1) if $\sigma \cap P = \emptyset$, then let $\bar{f}$ on $\sigma$ be the constant map $c_o$ to the point $o$, and

(1.2) if $v_0 \in C_{ij}$ and $v_1 \notin P$, take the unique path $q_{v_0}$ in $T_{ij}$ from $v_0$ to $v_{ij}$ and let $\bar{f}(v_0,v_1)$ be a map defined by the concatenation $(f \circ q_{v_0}) \ast p_{ij}$ of the paths $f \circ q_{v_0}$ (from $f(v_0)$ to $f(v_{ij})$) and $p_{ij}$ (from $f(v_{ij})$ to $o$). Notice that $\bar{f}(v_0) = f(v_0)$ and $\bar{f}(v_1) = o$.

Next, we take a 2-simplex $\sigma$ with vertices $v_0, v_1, v_2$. If $\sigma \cap P = \emptyset$, then let $\bar{f}|\sigma$ be the constant map $c_o$. Assume that $\sigma$ intersects with $P$.

(2.1) If $v_0, v_1 \notin P$ and $v_2 \in C_{ij}$, then the restriction $\bar{f}|\partial \sigma = f|(v_0,v_1,v_2)$ is null homotopic because it is represented by the concatenation $c_o \ast (f \circ q_{v_2} \ast p_{ij})^{-1} \ast (f \circ q_{v_2} \ast p_{ij})$. Thus, $\bar{f}|\partial \sigma$ admits an extension on $\sigma$.

(2.2) If $v_0 \notin P$ and $v_1, v_2 \in C_{ij}$, then let $g : \partial \sigma \to C_{ij} \subset P_i \subset P$ be a map defined by the loop $q_{v_1}^{-1} \ast (v_1,v_2) \ast q_{v_2}$ at $v_{ij}$. Then $\bar{f}|\partial \sigma$ is a map defined by the path $p_{ij}^{-1} \ast (f \circ q_{v_1})^{-1} \ast f|(v_1,v_2) \ast (f \circ q_{v_2}) \ast p_{ij}$ which is freely homotopic to the map $f \circ (q_{v_1}^{-1} \ast (v_1,v_2) \ast q_{v_2}) \simeq f \circ g \simeq 0$ by the hypothesis (1). Hence, $\bar{f}|\partial \sigma$ is null homotopic and it extends to a map on $\sigma$.

The above completes an extension procedure of $f$ to the 2-skeleton $T^{(2)}$ and thus completes the proof for $n = 2$. For $n > 2$, we can make use of the triviality of $\pi_1(X,o)$ to continue the extension process and, at the $n$-th step, obtain the desired extension $\bar{f}$ on $S^n$. \qed

The proof of Theorem 1.1 relies on the following lemma. The idea of using the monotone-light factorization theorem is due to M. L. Curtis and M. K. Fort, Jr. [2] and was applied in [6]. A local dendrite (a dendrite, respectively) is a one-dimensional locally connected compact connected metric space containing at most finitely many (no, respectively) simple closed curves. A map $h : S \to T$ between compact metric spaces is said to be monotone (light, respectively) if every point inverse of $h$ is connected (zero-dimensional, respectively).

**Lemma 2.2.** Let $f : N \to X$ be a map of a compact polyhedron $N$ to a compact metric space $X$ such that $\dim X \leq 1$. Then there exist
a compact metric space $G$ and maps $m : N \to G$ and $l : G \to X$ such that

1. $f = l \circ m$,
2. the map $m$ is monotone and the map $l$ is light, and
3. the space $G$ has finitely many components, each of which is a local dendrite or a singleton.

Proof: Applying the monotone-light factorization [16, Chap. VIII, section 4] to the map $f$, we find a monotone map $m : N \to G$ and a light map $l : G \to X$ satisfying conditions (1) and (2). We show that the space $G$ satisfies condition (3). Since $l$ is a light map, by [8, Theorem 3.3.10] and the hypothesis, we see $\dim G \leq \dim X + 0 = 1$.

By the monotonicity of $m$, every component of $N$ is of the form $m^{-1}(S)$ where $S$ is a component of $G$. The space $N$ has finitely many components and so does $G$. Enumerate the components of $G$ as $\{G_j\}$ and let $N_j = m^{-1}(G_j)$. Each $N_j$ is a component of $N$ and the restriction $m|N_j : N_j \to G_j$ is monotone. By the Hahn-Mazurkiewicz Theorem, $G_j$, as a continuous image of a locally connected compact connected metric space $N_j$, is locally connected. Furthermore, by the monotonicity of $m|N_j$, the induced homomorphisms $(m|N_j)^* : \check{H}^1(G_j; \mathbb{Z}) \to \check{H}^1(N_j; \mathbb{Z})$ is a monomorphism [12] to a finitely generated abelian group. Hence, $\check{H}^1(G_j; \mathbb{Z})$ is finitely generated. By [11, section 52], every one-dimensional locally connected compact connected metric space with finitely generated first Čech cohomology is a local dendrite. Hence, we obtain the desired conclusion (3). 

Proof of Theorem 1.1: Fix an integer $n \geq 2$ and take a map $f : \mathbb{S}^n \to CX_0 \vee CX_1$. Notice that the set $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ consists of at most countably many connected components, each of which is open in $\mathbb{S}^n$. Among these components, at most finitely many of them meet $f^{-1}(\{p_0, p_1\})$ and neither of them intersects both of $f^{-1}(\{p_0\})$ and $f^{-1}(\{p_1\})$.

We construct a map $g : \mathbb{S}^n \to CX_0 \vee CX_1$ such that

1. $g$ is homotopic to $f$, and
2. $g(\mathbb{S}^n) \subset CX_0 \vee CX_1 \setminus \{p_0, p_1\}$.

Let us assume, for a moment, that we have the above map $g$. Since $X_0 \vee X_1$ is a strong deformation retract of $CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, $g$ is homotopic to a map from $\mathbb{S}^n$ to $X_0 \vee X_1$. Since $\dim(X_0 \vee X_1) = 1$,
\(\pi_n(X_0 \vee X_1)\) is trivial by [2] and [1] and hence, \(g\) is null homotopic. Consequently, we conclude that \(f\) is null homotopic, as desired.

The map \(g\) and the homotopy between \(f\) and \(g\) are defined on each component of \(f^{-1}(CX_0 \vee CX_1 \setminus \{o\})\). If a component \(O\) does not meet \(f^{-1}(\{p_0, p_1\})\), then \(g|O = f|O\) and the homotopy \(H_O : O \times [0, 1] \rightarrow CX_0 \vee CX_1\) is given by \(H(x, t) = f(x)\) for each point \((x, t) \in O \times [0, 1]\).

Next, take a component \(O\) of \(f^{-1}(CX_0 \vee CX_1 \setminus \{o\})\) such that \(O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset\). Without loss of generality, we may assume that \(O \cap f^{-1}(p_0) \neq \emptyset = O \cap f^{-1}(p_1)\). Take a compact PL submanifold \(N\) of \(S^n\) such that \(S^n \setminus O \subset \text{Int}N\) and \(N \cap (f^{-1}(\{p_0\}) \cap O) = \emptyset\). Define a map \(f_O : S^n \rightarrow CX_0\) by \(f_O(x) = f(x)\) for \(x \in O\) and \(f_O(x) = o\) otherwise. Let \(r : CX_0 \vee CX_1 \setminus \{p_0, p_1\} \rightarrow X_0 \vee X_1\) be the standard homotopy equivalence.

Applying Lemma 2.2 to the composition \(r \circ f_O|N : N \rightarrow X_0 \vee X_1\), we obtain a compact metric space \(G\), a monotone map \(m : N \rightarrow G\), and a light map \(l : G \rightarrow X_0 \vee X_1\) such that \(r \circ f_O|N = l \circ m\) and

(3) the space \(G\) has finitely many components \(G_j\), each of which is a local dendrite or a singleton.

Let \(C_j = l^{-1}(\{o\}) \cap G_j\) and note \(l^{-1}(o) = \bigcup_j C_j\). Since \(\dim C_j = 0\), the above condition (3) implies that there exists a closed neighborhood \(D_j\) of \(C_j\) such that \(D_j\) is the disjoint union of finitely many dendrites, each of which intersects with \(C_j\). In particular, \(D_j\) contains no simple closed curve and hence,

(4) the inclusion \(i_j : D_j \rightarrow G_j\) is null homotopic.

Observe that \(S^n \setminus O \subseteq (f_O|N)^{-1}(\{o\}) \subseteq (r \circ f_O|N)^{-1}(\{o\}) = (l \circ m)^{-1}(\{o\})\). There exists a compact PL submanifold \(P\) of \(N\) with the components \(P_0, \ldots, P_k\) such that \(\overline{S^n \setminus P}\) is a PL submanifold and also

(5) \(S^n \setminus O \subseteq (l \circ m)^{-1}(\{o\}) \subseteq \text{Int}P\), \((l \circ m)^{-1}(\{o\}) \cap P_i \neq \emptyset\) for each \(i = 0, \ldots, k\), and each \(m(P_i)\) is a subset of some \(D_j\).

Then, for each \(h : S^1 \rightarrow P_i\), the composition \(r \circ (f_O|P) \circ h = l \circ m \circ h\) is null homotopic by (4). Since \(r\) is a homotopy equivalence, the map \((f_O|P) \circ h\) is null homotopic as well. By Lemma 2.1, \(f_O|P\) extends to a map \(g_0 : S^n \rightarrow CX_0 \setminus \{p_0\}\). Define \(g_1 : S^n \rightarrow CX_0 \vee CX_1\) by \(g_1(x) = g_0(x)\) for \(x \in O\) and \(g_1(x) = f(x)\) otherwise. Then \(g_1|P = f|P\). Since \(P\) and \(\overline{S^n \setminus P}\) are compact PL-submanifolds
and \( CX_0 \) is contractible, we see that \( f \) and \( g_1 \) are homotopic relative to \( P \) and \( g_1(\bar{O}) \cap \{p_0, p_1\} = \emptyset \).

We iterate this procedure for every component \( O \) of \( f^{-1}(CX_0 \lor CX_1 \setminus \{o\}) \). The continuity of \( f \) implies that there are at most finitely many such components. Carrying out all these procedures, we obtain the desired map \( g : S^n \to CX_0 \lor CX_1 \setminus \{p_0, p_1\} \), satisfying conditions (1) and (2).

The next lemma is for the proof of Theorem 1.4.

**Lemma 2.3.** Let \( K_0, K_1 \) be disjoint closed subsets of \( S^2 \) and \( X \) be a path-connected space with a point \( o \in X \) specified. There exists a compact surface \( P \subset S^2 \) with boundary such that

1. \( K_0 \subset \text{Int} P \) and \( K_1 \cap P = \emptyset \),
2. each component of the boundary \( \partial P \) is a polygonal simple closed curve, and
3. for each map \( f : P \to X \) with \( f(K_0) = \{o\} \), the restriction \( f|\partial P : \partial P \to X \) is null homotopic.

**Proof:** Take a compact surface \( P \) satisfying (1) and (2) above and let \( P_0, \ldots, P_k \) be the components of \( P \). We may assume that

4. each component of \( S^2 \setminus (K_0 \cap P_i) \) contains at most one component of \( S^2 \setminus P_i \) for every \( i \).

Indeed, if a component of \( S^2 \setminus (K_0 \cap P_i) \) contains two components of \( S^2 \setminus P_i \), then by cutting open \( P_i \) along an arc connecting these components, we have a smaller neighborhood \( P'_i \subset P_i \) of \( P_i \cap K_0 \) so that these components are contained in a single component of \( S^2 \setminus P'_i \). Iterating this procedure, we can make \( P \) satisfy condition (4).

For a map \( f : P \to X \) satisfying the hypothesis of (3), we show that \( f|\partial P_i \) is null homotopic for each component \( P_i \), which follows from

\[ f|C \text{ is null homotopic for each component } C \text{ of } \partial P_i. \]

Let \( O \) be a component of \( S^2 \setminus K_0 \) which intersects with the component of \( S^2 \setminus P_i \) whose boundary is equal to \( C \). The curve \( C \) divides \( S^2 \) into two components. Let \( U \) be the component of \( S^2 \setminus C \) containing \( \text{Int}(P_i) \).
The closure $\overline{U} = U \cup C$ is the closed disk such that $\overline{U} \supset P \cap O$.

Define $g : \overline{U} \to X$ by

$$g(u) = \begin{cases} f(u) & \text{for } u \in P \cap O, \\ o & \text{for } u \in U \setminus O. \end{cases}$$

By (4), $g$ is actually defined on $\overline{U}$ and is a continuous extension of $f|C$ and hence, $f|C$ is null homotopic.

**Proof of Theorem 1.4:** Let $f : S^2 \to C_{X_0} \cup C_{X_1}$ be a map. As in the proof of Theorem 1.1, we construct a map $g : S^2 \to C_{X_0} \cup C_{X_1}$ such that

(i) $g(S^2) \subset C_{X_0} \cup C_{X_1} \setminus \{p_0, p_1\}$, and

(ii) $g \simeq f : S^2 \to C_{X_0} \cup C_{X_1}$.

Having constructed such a map $g$, the proof is completed as follows: Let $r : C_{X_0} \cup C_{X_1} \setminus \{p_0, p_1\} \to X_0 \cup X_1$ be the standard retraction. Then the hypothesis $\pi_2(X_0 \cup X_1) = 0$, together with (ii), implies $f \simeq g \simeq r \circ g \simeq 0$.

Choose a component $O$ of $f^{-1}(C_{X_0} \cup C_{X_1} \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$ and assume, without loss of generality, $O \cap f^{-1}(\{p_0\}) \neq \emptyset$. We construct a map $g_1 : S^2 \to C_{X_0} \cup C_{X_1}$ such that

(1) $g_1$ is homotopic to $f$, and

(2) $g_1|S^2 \setminus O = f|S^2 \setminus O$ and $g_1(\overline{O}) \cap \{p_0, p_1\} = \emptyset$.

First we apply Lemma 2.3 to $K_0 = S^2 \setminus O, K_1 = f^{-1}(\{p_0\}) \cap O$ and obtain a compact surface $P$ with polygonal boundary such that $S^2 \setminus O \subset \text{Int}P$ and

(3) for each map $\varphi : P \to C_{X_0} \setminus \{p_0\}$ with $\varphi(S^2 \setminus O) = \{o\}$, the restriction $\varphi|\partial P : \partial P \to X$ is null homotopic.

Define $f_O : P \to C_{X_0}$ by

$$f_O(x) = \begin{cases} f(x) & \text{for } x \in P \cap O, \\ o & \text{for } x \in S^2 \setminus O. \end{cases}$$

Condition (3) above guarantees that $f_O : P \to C_{X_0} \setminus \{p_0\}$ satisfies hypotheses (1) and (2) of Lemma 2.1 and hence admits an extension $g_0 : S^2 \to C_{X_0} \setminus \{p_0\}$. Define $g_1 : S^2 \to C_{X_0} \cup C_{X_1}$ by $g_1(x) = g_0(x)$ for $x \in O$ and $g_1(x) = f(x)$ otherwise. Then $g_1|P = f|P$. Since $P$ and $S^2 \setminus P$ are compact surfaces and $C_{X_0}$ is contractible, we see that $f$ and $g_1$ are homotopic relative to $P$. By the definition, we also have $g_1(\overline{O}) \cap \{p_0, p_1\} = \emptyset.$
As in the proof of Theorem 1.1, we obtain the desired map \( g : \mathbb{S}^2 \rightarrow C X_0 \vee C X_1 \setminus \{p_0, p_1\} \) by iterating the above procedure at most finitely many times on each component of \( f^{-1}(C X_0 \vee C X_1 \setminus \{o\}) \) such that \( O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset \). □

For more information on homology and homotopy groups on one-point unions of cones, see [3] and [7].

3. Remark on Lemma 2.3

The proof of Lemma 2.3 shows the following: for each compact subset \( K_0 \) of \( \mathbb{R}^2 \) and for each neighborhood \( U \) of \( K_0 \), there exists a compact surface \( P \) such that

1. \( K_0 \subset \text{Int} P \subseteq U \), and
2. for each map \( f : P \rightarrow X \) with \( f(K_0) = \{o\} \), the restriction \( f|\partial P : \partial P \rightarrow X \) is null homotopic.

The following result illustrates that the 3-dimensional analogue of the above result does not hold. This is the main technical obstacle to answering Question 1.3 in its full generality.

**Proposition 3.1.** Let \( ST \) be a solid torus in \( \mathbb{R}^3 \) which contains Antoine’s necklace \( K_0 \) in its interior in the standard way [14, pp. 71-72]. Let \( P \) be a compact 3-manifold-neighborhood of \( K_0 \) in \( ST \) and \( Y_0 \) be the quotient space \( P/K_0 \) with the quotient map \( q : P \rightarrow Y_0 \). Then, the restriction \( q|\partial P : \partial P \rightarrow Y_0 \) is not null-homotopic.

To prove the above, it is convenient to make the following lemma.

**Lemma 3.2.** Let \( X \) be a simply-connected PL manifold and \( Y \) be a connected PL submanifold of \( X \). Then for each component \( Z \) of \( X \setminus Y \), the topological boundary of \( Z \) is path-connected.

**Proof:** It suffices to verify the conclusion when \( \dim Y = \dim X \). Suppose the topological boundary \( \partial Z \) of \( Z \) is not path-connected. Then we have two points \( p \) and \( q \) in \( \partial Z \) which are not joined by arcs in \( \partial Z \). We have, on one hand, an arc \( A \) in \( Y \) connecting \( p \) and \( q \) and, on the other hand, an arc \( B \) in \( Z \) connecting \( p \) and \( q \) such that \( A \cap B = \{p, q\} \). The union \( A \cup B \) is a simple closed curve in \( X \) which is not null homotopic. This contradicts the assumption. □

**Proof of Proposition 3.1:** For simplicity, \( ST \setminus K_0 \) is regarded as a subspace of \( Y_0 \) via the homeomorphism \( q|ST \setminus K_0 \). Let \( y_0 \) be the point with \( \{y_0\} = q(K_0) \).
Suppose that there exists a homotopy \( H : \partial P \times I \to Y_0 \) such that
\[
H(x, 0) = x \quad \text{and} \quad H(x, 1) = y_0 \quad \text{for } x \in \partial P.
\]
Let \( S \) be a component of \( \partial P \). By making use of the homotopy \( H|S \times I \) between the inclusion \( S \to Y_0 \) and the constant map to \( y_0 \), we show that
\[
(\ast) \text{ there exists a homotopy } \overline{H} : S \times I \to P \text{ between the inclusion } S \to P \text{ to a constant map.}
\]
Take the component \( O \) of \( (H|S \times I)^{-1}(Y_0 \setminus \{y_0\}) \) which contains \( S \times \{0\} \). Define
\[
H_0(x, t) = H(x, t) \text{ if } (x, t) \in O, \quad H_0(x, t) = y_0, \text{ otherwise.}
\]
Then \( H_0 \) is also a homotopy from the inclusion \( S \to Y_0 \) to the constant map. Hence, we assume that \( O := (H|S \times I)^{-1}(Y_0 \setminus \{y_0\}) \) is connected and let \( C = S \times I \setminus O = (H|S \times I)^{-1}(\{y_0\}) \). Then we have \( S \times \{0\} \subseteq O \) and \( S \times \{1\} \subseteq C \). In the next lemma, \( H|S \times I \) is abbreviated to \( H \).

**Lemma 3.3.** Let \( C_0 \) be a component of \( C \). Then there exists a unique \( u \in K_0 \) such that \( H_1 : O \cup C_0 \to P \) defined by \( H_1|O = H|O \) and \( H_1(x, t) = u \) for \( (x, t) \in C_0 \) is continuous.

**Proof:** We show that there exists a unique point \( u \in K_0 \) such that for each sequence \( \{p_n\} \subset O \) with \( \lim_{n \to \infty} p_n \in \partial C_0 \), the sequence \( \{H(p_n)\} \) accumulates to \( u \). It is easily seen that \( u \) is the desired point.

To show this by contradiction, suppose there exist two points \( a, b \in \partial C_0 \) and sequences \( \{a_n\}, \{b_n\} \subset O \) such that \( \lim_{n \to \infty} a_n = a \), \( \lim_{n \to \infty} b_n = b \), and further, \( \lim_{n \to \infty} H(a_n) \) and \( \lim_{n \to \infty} H(b_n) \) are distinct points of \( K_0 \). Since \( K_0 \) is 0-dimensional, we have open sets \( U \) and \( V \) in \( P \) such that \( \lim_{n \to \infty} H(a_n) \in U \), \( \lim_{n \to \infty} H(b_n) \in V \), \( U \cap V = \emptyset \), and \( K_0 \subseteq U \cup V \). There exists a PL-manifold-neighborhood \( P_1 \) of \( C \) such that \( H(P_1) \subseteq \{y_0\} \cup (U \setminus K_0) \cup (V \setminus K_0) \).

Let \( P_2 \) be the component of \( P_1 \) containing \( C_0 \). Choose \( a_n \) and \( b_n \) so that \( a_n, b_n \in P_2 \). Notice that
\[
(\ddagger) \text{ there is no arc connecting } a_n \text{ and } b_n \text{ in } P_2 \cap O,
\]
because \( H(P_2 \cap O) \subseteq (U \cup V) \setminus K_0 \). In other words, \( C \) separates the connected manifold \( P_2 \).

As \( S \) is a surface in \( ST \), \( S \times I \) is naturally embedded in \( \mathbb{R}^3 \) by “thickening \( S \)” Under this embedding, the topological boundary
of $S \times I$ in $\mathbb{R}^3$ is $S \times \{0, 1\}$. We apply Lemma 3.2 to $X = \mathbb{R}^3$ and $Y = P_2$. If $P_2$ does not intersect with $S \times \{1\}$, then the topological boundary of $P_2$ in $\mathbb{R}^3$ is contained in $O$. If $P_2$ meets $\partial P \times \{1\}$, then it contains $S \times \{1\}$ and the topological boundary of $P_2$ in $\mathbb{R}^3$ is contained in the disjoint union of $S \times \{1\}$ and $O$. Hence, we have the following remark:

$$S \times I \setminus P_2$$

consists of finitely many components and the topological boundary of each component in $S \times I$ is path-connected and is contained in $O$.

We have a polygonal arc $A$ in $O$ which connects $a_n$ and $b_n$. There exist finitely many pairwise disjoint subarcs $B_1, \ldots, B_r$ of $A$ such that the endpoints of each $B_j$ belong to $\partial P_2$, each $B_j$ is contained in the union of $P_2$ and a unique component of the complement of $P_2$, and $A \setminus \cup B_j \subset P_2$. By the preceding remark, for each $B_j$, we have an arc on the boundary of $P_2$ which connects the endpoints of $B_j$. Hence, we obtain an arc in $P_2 \cap O$ connecting $a_n$ and $b_n$, which contradicts $(\sharp)$.

Proof of Proposition 3.1 (continued): Applying the above lemma to each component of $C$, we have a map $\overline{H}_S : S \times I \to P$ such that $\overline{H}_S|O \cup C_0$ is continuous for each component $C_0$ of $C$. To see the continuity of $\overline{H}_S$ on $S \times I$, it suffices to show the following.

(**) Let $\{p_n\}$ be a sequence of $C$ such that $\lim_{n \to \infty} p_n = p \in C$ and let $C_n$ ($C_0$, respectively) be the component of $C$ containing $p_n$ ($p$, respectively). Take the unique points $u_n$ for $C_n$ and $u$ for $C_0$ as in the previous lemma. Then

$$\lim_{n \to \infty} u_n = u.$$

To show the above, we may assume that $p_n \in \partial C_n$ and $p \in \partial C$. By the definition of $\overline{H}_S$, $\overline{H}_S(p_n) = u_n$ and $\overline{H}_S(p) = u$. By the continuity of $\overline{H}_S|O \cup C_n$, we may take $a_n \in O$ so close to $p_n$ that $\lim_{n \to \infty} a_n = p$ and $\lim_{n \to \infty} \overline{H}_S(a_n) = \lim_{n \to \infty} \overline{H}_S(p_n)$. Then, by the uniqueness of $u$, we obtain $\lim_{n \to \infty} \overline{H}_S(a_n) = u$. Thus, $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \overline{H}_S(p_n) = \lim_{n \to \infty} \overline{H}_S(a_n) = u$. This proves (**)) and hence completes the proof of (*).

Taking the union $\overline{H} := \cup \overline{H}_S$ over all components $S$ of $\partial P$, we have a homotopy $\overline{H} : \partial P \times I \to P$ such that $\overline{H}(x, 0) = x, x \in \partial P$, $\overline{H}(\partial P \times \{1\}) \subseteq K_0$. It is easy to see that for each component $P_0$ of $P$, we have $\overline{H}(\partial P_0 \times I) \subseteq P_0$. 

By the construction of Antoine’s necklace, there exists a component $P_0$ of $P$ such that the inclusion $\partial P_0 \to P_0$ is not null-homotopic. Then there exists a component $S$ of $\partial P_0$ such that the inclusion $S \to P_0$ is not null-homotopic. However, the restriction $\overline{H}|S \times I$ provides a homotopy between the inclusion $S \to P_0$ and a constant map because $\overline{H}(S \times \{1\})$, as a connected set of the zero-dimensional $K_0$, is a singleton. This contradiction completes the proof of the proposition.

For the tame Cantor set $K$ in $\mathbb{R}^3$, there exists an arbitrarily small neighborhood $P$ of $K$ which is the disjoint union of 3-balls. For the quotient map $q: P \to P/K$, the restriction $q|\partial P : \partial P \to P/K$ is null-homotopic, since the restriction $q|\partial P_0$ is easily seen to be null-homotopic for each component $P_0$ of $P$.

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