# PERFECT EVEN MODULES AND THE EVEN FILTRATION 

PIOTR PSTRĄGOWSKI


#### Abstract

Inspired by the work of Hahn-Raksit-Wilson, we introduce a variant of the even filtration which is naturally defined on $\mathbf{E}_{1}$-rings and their modules. We show that our variant satisfies flat descent and so agrees with the Hahn-Raksit-Wilson filtration on ring spectra of arithmetic interest, showing that various "motivic" filtrations are in fact invariants of the $\mathbf{E}_{1-}$ structure alone. We prove that our filtration can be calculated via appropriate resolutions in modules and apply it to the study of even cohomology of connective $\mathbf{E}_{1}$-rings, proving vanishing above the Milnor line, base-change formulas, and explicitly calculating cohomology in low weights.


## Contents

1. Introduction 2
1.1. Acknowledgments 7
2. The even filtration $\quad 7$
2.1. Perfect even modules $\quad 7$
2.2. Even sheaves and homology of perfect evens 8
2.3. The even filtration and even cohomology 10
2.4. Example: Modules with even homotopy groups 11
2.5. Example: Rings with even homotopy groups 12
3. Multiplicative properties of the even filtration 13
3.1. Monoidal structure on additive sheaves 13
3.2. Synthetic spectra and even filtration of the sphere 14
$\begin{array}{ll}3.3 \text {. Monoidality of the even filtration } & 15\end{array}$
4. Calculus of evenness 16
4.1. Even flat modules 17
4.2. Homotopy even envelopes 18
4.3. Homologically even modules 21
4.4. Evenness over connective and homotopy even rings 22
5. Homological resolutions 23
6. Base-change and descent 25
6.1. Evenness and extension/restriction of scalars 26
6.2. Faithfully flat descent 29
7. Comparison with the $\mathbf{E}_{\infty}$-even filtration 33
8. Even cohomology of connective rings 36
8.1. Vanishing above the Milnor line 36
8.2. Cohomology in low weights 38
8.3. Base-change around the Milnor line 39

References 40

[^0]
## 1. Introduction

In [HRW22], Hahn-Raksit-Wilson introduced the even filtration attached to a commutative algebra in spectra. They show that when applied to ring spectra of arithmetic interest, such as the sphere or THH of commutative rings, their construction recovers various important filtrations, such as the Bhatt-Morrow-Scholze filtration of [BMS19], implying that these filtrations are invariants of the $\mathbf{E}_{\infty}$-ring structure alone. The Bhatt-Morrow-Scholze filtration can be used to define prismatic and syntomic cohomology of commutative rings, and the even filtration allows one to extend this construction to the context of $\mathbf{E}_{\infty}$-ring spectra. This provides strong evidence towards the longstanding conjecture of Rognes on the existence of a motivic filtration on algebraic $K$-theory of commutative ring spectra [Rog14].

In this paper, we introduce a variant of the even filtration which informally measures the complexity of the $\infty$-category of perfect complexes with even cells. The filtration we construct, which one could call the perfect even filtration, has the following properties:
(1) it is naturally defined on $\mathbf{E}_{1}$-ring spectra and their modules, rather than only in $\mathbf{E}_{\infty}$-case,
(2) it satisfies flat descent, and as a consequence agrees with the even filtration in the examples considered in [HRW22], showing in particular that various motivic filtrations are invariants of the $\mathbf{E}_{1}$-ring structure alone,
(3) in the examples where they disagree, for example on free $\mathbf{E}_{\infty}$-algebras, the perfect even filtration gives more reasonable answers than the even filtration,
(4) the perfect even filtration has an essentially linear definition and so can be efficiently computed by resolutions of modules,
(5) the perfect even cohomology groups have excellent formal properties, especially in the case of connective $\mathbf{E}_{1}$-rings: they vanish above the Milnor line, satisfy a base-change formula in a neighbourhood of it, and in low weights can be calculated explicitly.
Our construction of the perfect even filtration is of categorical nature: we work with the $\infty$ category of modules, together with its notion of evenness, rather than the ring itself. This has the advantage that it naturally lends itself to generalization into other contexts, such as equivariant or motivic homotopy theory. We briefly discuss the possible generalizations at the end of the introduction.

Note that for $\mathrm{THH}(R)$ to have an $\mathbf{E}_{1}$-ring structure and hence have an induced perfect even filtration, $R$ needs to be only an $\mathbf{E}_{2}$-ring. Thus, the main construction of this paper suggests the existence of a good theory of prismatic cohomology in the context of $\mathbf{E}_{2}$-ring spectra, a topic we will pursue in the sequel to the current work written jointly with Raksit [PR23].

Terminology. The above few paragraphs are the only part of the paper where we use the term perfect even filtration. This is an apt name, but for brevity in the main body of the text we refer to the filtration introduced in the current work simply as the even filtration. To distinguish it from the one introduced by Hahn-Raksit-Wilson, we will refer to the latter as the $\mathbf{E}_{\infty}$-even filtration.

We now discuss our results in more detail. Let $R$ be an $\mathbf{E}_{1}$-algebra in spectra. We say that a left $R$-module $A$ is perfect even if it belongs to the smallest subcategory

$$
\operatorname{Perf}(R)_{e v} \subseteq \operatorname{Perf}(R)
$$

containing $R$ and closed under even (de)suspensions, retracts and extensions. We say that a map of perfect even $R$-modules is an even epimorphism if its fibre is again perfect even. Declaring even epimorphisms as coverings endows $\operatorname{Perf}(R)_{e v}$ with a Grothendieck topology. An $R$-module $M$ determines through the Yoneda embedding an additive spectral sheaf

$$
Y(M) \in \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)
$$

given by the formula $Y(M)(A):=\operatorname{Map}_{\mathcal{M o d}_{R}}(A, M)$, the mapping spectrum.

Definition 1.1. The even filtration on $M$ is the filtered spectrum given by

$$
\operatorname{fil}_{e v}^{n}(M):=\Gamma_{\operatorname{Perf}(R)_{e v}}\left(R, \tau_{\geq 2 n} Y(M)\right),
$$

where $\tau_{\geq 2 n}$ denotes the connective cover in sheaves of spectra.
Our first result shows that this filtration is compatible with the ring structure.
Theorem 1.2 (3.15). Let $R$ be an $\mathbf{E}_{1}$-ring. Then, the even filtration fil ${ }_{e v}^{*}(R)$ has a canonical lift to an $\mathbf{E}_{1}$-algebra in filtered spectra and the $R$-module even filtration canonically lifts to a functor

$$
\begin{equation*}
\mathrm{fil}_{e v / R}^{*}: \operatorname{M}_{o d}(\mathcal{S} p) \rightarrow{\mathcal{M} o d_{\mathrm{fil}_{e v}^{*}(R)}(\mathrm{FilS} p) .} \tag{1.1}
\end{equation*}
$$

If $R$ is an $\mathbf{E}_{n}$-ring for $n \geq 2$, then fil ${ }_{e v}^{*}(R)$ can be canonically refined to a filtered $\mathbf{E}_{n}$-algebra and (1.1) can be canonically refined to a lax $\mathbf{E}_{n-1}$-monoidal functor.

We in fact show something slightly stronger, namely that filev ${ }_{e v}^{*}(R)$ is an $\mathbf{E}_{1}$-algebra over fil ${ }_{e v}^{*}\left(S^{0}\right)$, the even filtered sphere. As we discuss below, the latter can be identified with the Adams-Novikov filtration of the sphere, so that $\mathrm{fil}_{e v}^{*}(R)$ has a canonical lift to an algebra in synthetic spectra or, after $p$-completion, in the category of $\mathbb{C}$-motivic spectra [GIKR21, Pst22].

We show that the even filtration is always given by the Whitehead tower $M \mapsto \tau_{\geq 2 *} M$ if either the ring $R$ or the module $M$ has homotopy concentrated in even degree. In particular, it follows that when restricted to $\mathbf{E}_{\infty}$-rings, there is a canonical comparison natural transformation

$$
\mathrm{fil}_{e v}^{*}(-) \rightarrow \mathrm{fil}_{\mathbf{E}_{\infty}-e v}^{*}(-)
$$

into the Hahn-Raksit-Wilson $\mathbf{E}_{\infty}$-even filtration. As the main method of calculating the $\mathbf{E}_{\infty}$-even filtration is via flat descent, to establish that the comparison map is an equivalence for a large class of rings we first prove descent for the even filtration of Definition 1.1.

We say that an $R$-module $M$ is even flat if it can be written as a filtered colimit of perfect evens; these modules can be characterized in several different ways, see Proposition 4.12, and so can be effectively detected. If $R$ is an $\mathbf{E}_{2}$-ring ${ }^{1}$, then we say that an $\mathbf{E}_{1}$ - $R$-algebra is faithfully even flat if both $S$ and $\operatorname{cofib}(R \rightarrow S)$ are even flat as $R$-modules. This definition is motivated by a classical observation that a monomorphism of classical commutative rings is faithfully flat if both the target and the cokernel are flat. In Proposition 6.19, we show that in the $\mathbf{E}_{\infty}$-context, our notion of faithfully even flat is equivalent to that of Hahn-Raksit-Wilson for connective rings, and strictly stronger in general.
Theorem 1.3 (6.24). Let $R$ be an $\mathbf{E}_{2}$-ring and let $S$ be faithfully even flat an $\mathbf{E}_{1}-R$-algebra. Then for any $R$-module $M$ the canonical map

$$
\mathrm{fil}_{e v / R}^{*}(M) \rightarrow \operatorname{lim~fil}_{e v / S^{\otimes_{R}} \bullet}^{*}\left(S^{\otimes_{R} \bullet} \otimes_{R} M\right)
$$

is an equivalence of filtered spectra after completion.
Since the $\mathbf{E}_{\infty}$-even filtration also satisfies flat descent, we deduce the following.
Theorem 1.4 (7.3). Let $R$ be an $\mathbf{E}_{\infty}$-ring which admits a faithfully even flat map $R \rightarrow S$ into an $\mathbf{E}_{\infty}$-ring with $\pi_{*} S$ even. Then the comparison functor

$$
\mathrm{fil}_{e v}^{*}(R) \rightarrow \mathrm{fil}_{\mathbf{E}_{\infty}-e v}^{*}(R)
$$

is an equivalence of filtered spectra after completion.
Note that the even filtration considered in this paper is always exhaustive in the sense that $\underset{\longrightarrow}{\lim \operatorname{fil}_{e v}^{*}}(R) \simeq R$, but it is not always complete. Thus, even in cases where the two filtrations agree up to completion, fil ev $(R)$ can be considered as a naturally occuring "decompletion" of the Hahn-Raksit-Wilson filtration.

[^1]The comparison of Theorem 1.3 covers most of examples considered in [HRW22]. In particular, it applies to $S^{0}, \operatorname{HH}(R / k), \operatorname{THH}(R)$ and $\operatorname{THH}(R)_{p}^{\wedge}$, where by results of Hahn-Raksit-Wilson one recovers, respectively, the Adams-Novikov filtration, the Hochschild-Kostant-Rosenberg filtration, the Bhatt-Lurie filtration and the filtration of Bhatt-Morrow-Scholze. However, we warn the reader that it does not apply to the the motivic filtrations on $\mathrm{TC}, \mathrm{TC}^{-}$and TP , as here an appropriate definition of the $\mathbf{E}_{\infty}$-even filtration requires one to take the $S^{1}$-action into account. We believe that a suitable $S^{1}$-equivariant variant of Definition 1.1 would recover these remaining motivic filtrations, but we do not consider this problem here.

The even filtration and the $\mathbf{E}_{\infty}$-even filtration do not agree in general. In Example 7.6, due to Robert Burklund, we give an instructive instance of this phenomena for a connective $\mathbf{E}_{\infty}$-ring. However, one could argue that when they disagree, it is the even filtration of Definition 1.1 which gives more reasonable answers: in Burklund's example, we are able to determine the structure of the even filtration completely, but the nature of the $\mathbf{E}_{\infty}$-even filtration seems somewhat difficult.

Outside of the connective context, the situation is even more striking: in Warning 7.7, we describe a periodic $\mathbf{E}_{\infty}$-ring whose $\mathbf{E}_{\infty}$-even filtration is identically zero, but whose even filtration is exhaustive, complete and easy to calculate. This simplicity comes down to the fact that even if one starts with an $\mathbf{E}_{\infty}$-ring $R$, it is much easier to produce $\mathbf{E}_{1}-R$-algebras rather than $\mathbf{E}_{\infty}-R$ algebras, which allows one to apply Theorem 1.3.

We now describe how one can calculate the even filtration in practice. Since the even filtration is defined in terms of Postnikov towers in sheaves, it is controlled by sheaf cohomology. For any half-integer $q$, we write

$$
\mathcal{F}_{M}(q):=\pi_{2 q} Y(M)
$$

for the sheaf of homotopy groups and call it the even sheaf of weight $q$. We say that $M$ is homologically even if these sheaves vanish for $q \in \mathbb{Z}+1 / 2$. For example, all perfect even modules are homologically even, in particular $R$ itself.

Definition 1.5. The even cohomology of $R$ with coefficients in $M$ is given by sheaf cohomology

$$
\mathrm{H}_{e v}^{p, q}:=\mathrm{H}_{\operatorname{Perf}(R)_{e v}}^{p}\left(R, \mathcal{F}_{M}(q)\right) .
$$

Note that if $M$ is homologically even ${ }^{2}$, then the associated graded object of the even filtration can be described in terms of even cohomology in the sense that

$$
\pi_{2 q-p} \operatorname{gr}_{e v}^{q}(M) \simeq \mathrm{H}_{e v}^{p, q}(R, M)
$$

The even filtration thus induces a spectral sequence of signature

$$
\mathrm{H}_{e v}^{p, q}(R, M) \Rightarrow \pi_{2 q-p}(M),
$$

which we call the even spectral sequence.
As any form of sheaf cohomology, even cohomology can be efficiently computed using injective resolutions inside the category of sheaves of abelian groups. To do so, one needs access to injective sheaves, a bountiful source of which are $R$-modules themselves. This means that even cohomology (and hence the even filtration) can be efficiently computed through resolutions in the $\infty$-category of $R$-modules, as we now explain.

Let $M$ be homologically even. By iteratively attaching even cells in $R$-modules along each odd degree homotopy class we construct a map $M \rightarrow E_{0}$ into a module with even homotopy groups with the property that the cofibre $C_{0}:=\operatorname{cofib}\left(M \rightarrow E_{0}\right)$ is homologically even. We can then

[^2]attach even cells to $C_{0}$ to obtain a map $C_{0} \rightarrow E_{1}$ with $\pi_{*} E_{1}$ even and $C_{2}:=\operatorname{cofib}\left(C_{0} \rightarrow E_{1}\right)$ homologically even. Proceeding inductively in this form we produce a diagram of $R$-modules

whose top row is a chain complex in the homotopy category. The following shows that this diagram encodes the even cohomology of $M$.

Theorem 1.6 (5.3). If $M$ is homologically even and (1.2) a diagram as described above, then there is a canonical isomorphism

$$
\mathrm{H}_{e v}^{p, q}(R, M) \simeq \mathrm{H}^{p}\left(\pi_{2 q} E_{\bullet}\right),
$$

between the $(p, q)$-th even cohomology of $M$ and the cohomology of the cochain complex

$$
\pi_{2 q} E_{0} \rightarrow \pi_{2 q} E_{1} \rightarrow \pi_{2 q} E_{2} \rightarrow \ldots
$$

computed in the p-th spot.
We also prove a more refined version of Theorem 1.6 which gives a description of the even filtration itself as a décalage of an appropriate cosimplicial resolution through modules with even homotopy, see Proposition 5.6 and Remark 5.8.

We show that Theorem 1.6 gives an effective way of calculating even cohomology, particularly in the connective case. To see this, note that if both $R$ and $M$ are connective, then in the construction of $E_{0}$ of (1.2) one needs to use only positive-dimensional cells. Iterating this construction thus leads to a raising connectivity of the diagram, giving a vanishing line.

Theorem 1.7 (8.1, 8.2, 8.3). Let $R$ be a connective $\mathbf{E}_{1}$-ring and $M$ a connective, homologically even $R$-module. Then the even cohomology groups of $M$ vanish above the Milnor line; that is, we have

$$
\mathrm{H}_{e v}^{p, q}(R, M)=0
$$

for $p>q$. In particular, the even filtration $\mathrm{fil}_{\text {ev/R }}^{*}(M)$ is complete and the even spectral sequence

$$
\mathrm{H}_{e v}^{p, q}(R, M) \Rightarrow \pi_{2 q-p} M
$$

is strongly convergent.
Strikingly, in low weights we are able to calculate the even cohomology groups explicitly with virtually no assumptions on the ring $R$.

Theorem 1.8 (8.5, 8.6). Let $R$ be a connective $\mathbf{E}_{1}$-ring and $M$ a connective, homologically even $R$-module. Then there are canonical isomorphisms
(1) $\mathrm{H}_{e v}^{0,0}(R, M) \simeq \pi_{0} M$,
(2) $\mathrm{H}_{e v}^{0,1}(R, M) \simeq \operatorname{coker}\left(\pi_{1} R \otimes_{\mathbb{Z}} \pi_{1} M \rightarrow \pi_{2} M\right)$,
(3) $\mathrm{H}_{e v}^{1,1}(R, M) \simeq \pi_{1} M$,
(4) $\mathrm{H}_{e v}^{2,2}(R, M) \simeq \operatorname{im}\left(\pi_{1} R \otimes_{\mathbb{Z}} \pi_{1} M \rightarrow \pi_{2} M\right)$.

Before stating our last main result, we mention that the various notions of evenness one can attach to an $R$-module introduced in this article (perfect even, even flat, homologically even, as well as the classical notion of having homotopy groups concentrated in even degrees) interact with each other in interesting ways, and $\S 4$ is devoted to the study of their relationships. For an example of the kind of result we prove, we show that if $E$ is a right $R$-module with $\pi_{*} E$ even and $M$ is homologically even, then

$$
\pi_{*}\left(E \otimes_{R} M\right)
$$

is concentrated in even degree if at least one of $E$ or $M$ is even flat, see Proposition 4.12 and Theorem 4.14. Moreover, these kind of tensor properties characterize these classes with respect to each other, giving more evidence that our notion of even flatness is the right one.

These results are important in showing how even cohomology behaves under either extension or restriction of scalars attached to a map of $\mathbf{E}_{1}$-rings, which we do in detail in $\S 6.1$. As an application of these methods, we prove the following base-change result which describes behaviour of even cohomology in a neighbourhood of the Milnor line $p=q$.

Theorem 1.9 (8.7). Let $f: R \rightarrow S$ be a map of connective $\mathbf{E}_{1}$-rings such that $S$ is homologically even as a right $R$-module and let $M$ be an even flat $R$-module. Then the base-change of the canonical comparison map

$$
\pi_{0} S \otimes_{\pi_{0} R} \mathrm{H}_{e v}^{p, q}(R, M) \rightarrow \mathrm{H}_{e v}^{p, q}\left(S, S \otimes_{R} M\right)
$$

is a surjection for $p \geq q-1$.

As mentioned at the beginning of the introduction, a tantalizing prospect of having an $\mathbf{E}_{1}$-even filtration is that it allows one to define a "motivic" filtration on $\operatorname{THH}(R)$ and its variants as soon as $R$ is an $\mathbf{E}_{2}$-ring. This would be important in applications, since many chromatically important spectra (such as the Brown-Peterson spectrum or its truncated variants) cannot be made $\mathbf{E}_{\infty}$, but can often be made $\mathbf{E}_{2}$ [Sen17, Law18, HW22].

As one piece of evidence that the even filtration introduced in the present work is the right way to define prismatic cohomology of $\mathbf{E}_{2}$-ring spectra, we observe in Example 6.25 that the descent filtration associated to $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \rightarrow \mathrm{THH}(\mathrm{BP}\langle n\rangle / \mathrm{MU})$, used by Hahn-Wilson in their proof of Lichtenbaum-Quillen conjectures for $\mathrm{BP}\langle n\rangle$ [HW22], coincides with the even filtration $\mathrm{fil}_{e v}^{*}(\mathrm{THH}(\mathrm{BP}\langle n\rangle))$. In particular, it is canonically attached to $\mathrm{BP}\langle n\rangle$ as a $\mathbf{E}_{2}$-ring spectrum and does not depend on the structure of an MU-algebra. To keep this article focused and at manageable length, we do not pursue the idea of prismatic cohomology of $\mathbf{E}_{2}$-ring spectra in the current work and instead will pick it up in upcoming joint work with Raksit [PR23].

With a nod towards future applications, observe that the only input needed to define the even filtration of Definition 2.19 is the $\infty$-category of modules, together with the notion of a perfect even. This suggests a natural generalization of our construction to filtrations defined in other contexts, using an appropriate notion of "evenness". For example:
(1) In equivariant homotopy theory, the role of the Postnikov filtration of a spectrum is played by the equivariant slice filtration [Dug05, Hil12]. As the generating objects for the slice filtration are essentially representation spheres, this suggests that in $C_{2}$-equivariant homotopy theory, the role of perfect even $R$-modules should be played by modules built out of $\Sigma^{n \sigma} R$, where $\sigma$ is the regular representation of $C_{2}$.
(2) As shown by Hahn-Raksit-Wilson, the even filtration relative to the $\infty$-category of spectra is essentially the Adams-Novikov filtration relating stable homotopy theory to formal groups. The work of Bachmann-Kong-Wang-Xu on the Chow-Novikov $t$-structure shows that similar phenomena are visible in the stable motivic category $\mathrm{SH}(k)$ if as generating objects one takes the suspension spectra of smooth projective varieties [BKWX22]. This suggests that the right notion of a "perfect even" in the motivic world should perhaps be that of a motive of a smooth projective variety.

Ideas related to the second observation above will be used in upcoming joint work with Haine to construct spectral refinements of the weight filtration on cohomology of algebraic varieties [HP23a].

### 1.1. Acknowledgments

I would like to thank Ben Antieau, Robert Burklund, Jeremy Hahn, Lars, Hesselholt, Jacob Lurie, Arpon Raksit, Noah Riggenbach and Dylan Wilson for insightful conversations related to this work. I would also like to thank Princebucks ${ }^{3}$, where most of this paper was written.

## 2. The even filtration

This section is devoted to the construction of the even filtration and its most basic properties.
Notation 2.1. Throughout, $R$ denotes an $\mathbf{E}_{1}$-algebra in spectra. By an $R$-module we mean a left module in spectra.

### 2.1. Perfect even modules

Definition 2.2. We say that an $R$-module $A$ is perfect even if it belongs to the smallest subcategory

$$
\operatorname{Perf}(R)_{e v} \subseteq \mathcal{M}_{o d_{R}}(\mathcal{S} p)
$$

which contains $\Sigma^{2 k} R$ for $k \in \mathbb{Z}$ and is closed under extensions and retracts.
Warning 2.3. Beware that $\operatorname{Perf}(R)_{e v}$ is additive and admits even (de)suspensions, but it is usually not a stable $\infty$-category.

Notation 2.4. If the ring $R$ is understood, we write $\operatorname{Perf}_{e v}:=\operatorname{Perf}(R)_{e v}$.
Definition 2.5. We say that a map $f: A \rightarrow B$ of perfect even $R$-modules is an even epimorphism if its fibre in $R$-modules is also perfect even. A family of maps $\left\{f_{i}: A_{i} \rightarrow B\right\}$ of perfect even modules is covering if it consists of a single even epimorphism.

Note that even epimorphisms are closed under pullback, so that coverings define a Grothendieck topology. As a consequence of the fact that covering families are singleton, we have the following elegant characterization of additive sheaves.
Theorem 2.6. If $\mathcal{A}$ is an additive, presentable $\infty$-category, then the following holds:
(1) an $\mathcal{A}$-valued presheaf $X: \operatorname{Perf}_{e v}^{o p} \rightarrow \mathcal{A}$ is additive and a sheaf with respect to the even epimorphism topology if and only if for every even epimorphism $f: A \rightarrow B$, the sequence

$$
X(B) \rightarrow X(A) \rightarrow X(\operatorname{fib}(f))
$$

is fibre,
(2) the sheafication functor on $\mathcal{A}$-valued presheaves preserves additive presheaves so that its restriction

$$
L: \mathcal{P}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A}\right) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A}\right)
$$

to additive presheaves is an exact localization.
Proof. This is [Pst22, Theorem 2.8] and [Pst22, Proposition 2.5].
We will be mainly interested in the situation where the additive $\infty$-category is given either by abelian groups or spectra. The two are closely related:

Corollary 2.7. The $\infty$-category of additive sheaves of spectra admits a uniquet-structure in which $X: \operatorname{Perf}(R)_{e v}^{o p} \rightarrow \mathcal{S} p$ is coconnective if and only if $X(A) \in \mathcal{S} p_{\leq 0}$ for all perfect even $A$. Moreover:
(1) the sheafication functor

$$
L: \mathcal{P}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)
$$

is both left and right t-exact,

[^3](2) taking homotopy groups induces a canonical equivalence
$$
\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)^{\complement} \simeq \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{A} b\right)
$$
between the heart and the category of additive sheaves of abelian groups and
(3) the $t$-structure on $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{R}, e v ; S p\right)$ is compatible with filtered colimits; that is, coconnective sheaves are closed under filtered colimits.

Proof. The first property follows from (2) of Theorem 2.6. The second is a consequence of the first and the description $\mathcal{P}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{S} p\right)^{\mathscr{C}} \simeq \mathcal{P}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A} b\right)$. The third property follows from part (1) of Theorem 2.6, since fibre sequences are stable under filtered colimits.

### 2.2. Even sheaves and homology of perfect evens

The spectral Yoneda embedding associated to an $R$-module $M$ is a presheaf

$$
Y(M): \operatorname{Perf}(R)_{e v}^{o p} \rightarrow \delta p
$$

of spectra given by the formula

$$
Y(M)(A):=\operatorname{Map}(A, M)
$$

where the mapping spectrum is calculated in $\mathcal{M o d}_{R}$. As a consequence of Theorem 2.6, this is in fact a sheaf with respect to the even epimorphism topology. Of particular importance are its sheaves of homotopy groups, so that we give them a dedicated name:

Definition 2.8. The even sheaf associated to $M$ is given by

$$
\mathcal{F}_{M}:=\pi_{0} Y(M)
$$

It is an additive sheaf of abelian groups on perfect even $R$-modules. More generally, for any $q \in 1 / 2 \mathbb{Z}$, the even sheaf of weight $q$ is given by

$$
\mathcal{F}_{M}(q):=\pi_{2 q} Y(M)
$$

Remark 2.9. Our grading convention in terms of half-integer Serre twists is inspired by algebraic geometry, and it is compatible with the one employed in joint work with Hesselholt on geometry of graded-commutative rings [HP22a, HP23b].

In more detail, in various cohomology theories of algebraic geometry, a single twist $\mathbb{Z}(1)$ usually denotes the reduced cohomology of $\mathbb{P}^{1}$. Since the latter is topologically a 2 -sphere, it follows that weight should be in correspondence with twice the topological dimension.
Remark 2.10. Concretely, $\mathcal{F}_{M}:=L([-, M])$ is given by as the sheafication of the presheaf of abelian groups given by

$$
A \mapsto \pi_{0} \operatorname{Map}_{\mathcal{M o d}_{R}}(A, M)
$$

We have a canonical isomorphism

$$
\mathcal{F}_{M}(-1 / 2) \simeq \mathcal{F}_{\Sigma M}
$$

so that the even sheaves of non-zero weight can be identified with sheaves associated to (de)suspensions of $M$.

Proposition 2.11. The even sheaf functor

$$
\mathcal{F}_{-}:{\operatorname{M} o d_{R}} \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v} ; \mathcal{A} b\right)
$$

is homological; that is, if $M_{1} \rightarrow M_{2} \rightarrow M_{3}$ is a cofibre sequence of $R$-modules, then

$$
\mathcal{F}_{M_{1}} \rightarrow \mathcal{F}_{M_{2}} \rightarrow \mathcal{F}_{M_{3}}
$$

is exact in the middle as a sequence of sheaves of abelian groups.
Proof. This is clear from Remark 2.10, since sheafication is exact.

Remark 2.12. Note that by extending $M_{1} \rightarrow M_{2} \rightarrow M_{3}$ using (de)suspensions, we see that associated to a cofibre sequence we in fact have a long exact sequence of the form

$$
\ldots \rightarrow \mathcal{F}_{M_{2}}(1 / 2) \rightarrow \mathcal{F}_{M_{3}}(1 / 2) \rightarrow \mathcal{F}_{M_{1}} \rightarrow \mathcal{F}_{M_{2}} \rightarrow \mathcal{F}_{M_{3}} \rightarrow \mathcal{F}_{M_{1}}(-1 / 2) \rightarrow \mathcal{F}_{M_{2}}(-1 / 2) \rightarrow \ldots,
$$

where we use the isomorphism of Remark 2.10.
Construction 2.13 (Local grading of the sheaf $\infty$-category). The double suspension functor on $\operatorname{Perf}(R)_{e v}$ induces via precomposition a $t$-exact autoequivalence of the $\infty$-category of sheaves which we denote by

$$
(-)(1): \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}(R), \mathcal{S} p\right) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}(R), \mathcal{S} p\right)
$$

For $n \in \mathbb{Z}$, we denote the $n$-fold composite of this functor with itself by $(-)(n)$. Explicitly, for any sheaf $X$ and any $A \in \operatorname{Perf}_{e v}(R)$ we have

$$
(X(n))(A):=X\left(\Sigma^{2 n} A\right)
$$

Remark 2.14. Our notation concerning the local grading is compatible with that of Definition 2.8 in the sense that if $M$ is an $R$-module then

$$
\mathcal{F}_{M}(k / 2)(n) \simeq \mathcal{F}_{M}(k / 2+n)
$$

We will be mainly interested in the even filtration in case of the following class of modules.
Definition 2.15. We say that an $R$-module $M$ is homologically even if $\mathcal{F}_{M}(q)=0$ for any half-weight $q \in 1 / 2+\mathbb{Z}$.

Remark 2.16. By Remark 2.14, $M$ is homologically even if and only if $\mathcal{F}_{M}(-1 / 2)=0$.
Lemma 2.17. Let $A$ be a perfect even $R$-module. Then $A$ is homologically even. In particular, $R$ is homologically even as a module over itself.
Proof. By Remark 2.16, it is enough to show the vanishing of $\mathcal{F}_{A}(-1 / 2)$, which we can identify with the sheafication of the presheaf

$$
\left(B \in \operatorname{Perf}_{e v}\right) \mapsto\left(\pi_{0} \operatorname{Map}_{\operatorname{Mod}_{R}}(B, \Sigma A) \in A b\right)
$$

Thus, we have to show that every homotopy class of maps $B \rightarrow \Sigma A$ is locally zero in the even epimorphism topology. However, we have a cofibre sequence

$$
A \rightarrow F \rightarrow B \rightarrow \Sigma A
$$

and $F \rightarrow B$ is the required even epimorphism.
We also verify that the notion of an even epimorphism introduced in Definition 2.5 is the one detected by even sheaves.

Lemma 2.18. A map $B \rightarrow A$ of perfect even $R$-modules is an even epimorphism (that is, has a perfect even fibre) if and only if $\mathcal{F}_{B} \rightarrow \mathcal{F}_{A}$ is an epimorphism in $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A} b\right)$.

Proof. If $B \rightarrow A$ is an even epimorphism, then by definition it is a singleton covering family in the even epimorphism topology. It follows that the induced map of sheaves is an epimorphism.

Conversely, suppose that $\mathcal{F}_{B} \rightarrow \mathcal{F}_{A}$ is an epimorphism of sheaves. It follows that there exists a commutative diagram of perfect even spectra

with $p$ an even epimorphism. We can extend $p$ and $q$ to a larger diagram

where both lower rows and two right columns are cofibre sequences of $R$-modules. Since $p$ is is assumed to be an even epimorphism, $D$ is perfect even and hence so is $A \times_{B} C$ as an extension. The map $s$ in the original triangle provides a splitting of $q^{\prime}$ and we deduce that $F$ is a retract of $A \times_{B} C$ and hence it is also perfect even, which is what we wanted to show.

### 2.3. The even filtration and even cohomology

If $M$ is an $R$-module, we have a canonical identification of spectra

$$
\Gamma_{\text {Perf }_{e v}}(R, Y(M)):=Y(M)(R) \simeq \operatorname{Map}_{\mathcal{M o d}_{R}}(R, M) \simeq M
$$

As the left hand side is given by sections of a sheaf of spectra, it has a canonical filtration induced by the $t$-structure.

Definition 2.19. Let $R$ be an $\mathbf{E}_{1}$-ring and $M$ be an $R$-module. The even filtration of $M$ is given by the filtered spectrum

$$
\operatorname{fil}_{e v}^{n}(M):=\Gamma_{\operatorname{Perf}_{e v}(R)}\left(R, \tau_{\geq 2 n} Y(M)\right)
$$

where the connective cover is calculated in the sheaf $\infty$-category.
Remark 2.20. The use of connective covers in Definition 2.19 is similar to the construction of the Bhatt-Morrow-Scholze filtration on THH and its variants through the use of the quasisyntomic site [BMS19]. The difference is that in the present case the site $\operatorname{Perf}(R)_{e v}$ used to define the filtration depends on the ring itself; on the other hand, it is somewhat linear in nature.

We record that the even filtration is exhaustive and commutes with filtered colimits.
Proposition 2.21. The canonical maps

$$
\pi_{k} \operatorname{fil}_{e v}^{n}(M) \rightarrow \pi_{k} M
$$

induced by $\tau_{\geq 2 n} Y(M) \rightarrow Y(M)$ are an isomorphism for $k \geq 2 n$ and injective for $k=2 n-1$. In particular

$$
\underset{\longrightarrow}{\lim } \mathrm{fi}_{e v}^{n} M \simeq M .
$$

Proof. Since sheafication is right $t$-exact, the cofibre of $\tau_{\geq 2 n} Y(M) \rightarrow Y(M)$ in sheaves is (2n-1)coconnective. As coconnectivity in sheaves is detected levelwise, the claim follows.

Lemma 2.22. Let $M \simeq \underset{\longrightarrow}{\lim } M_{\alpha}$ be a filtered colimit of $R$-modules. Then $\mathrm{fil}_{e v}^{*}(M) \simeq \underset{\longrightarrow}{\lim } \mathrm{fil}_{e v}^{*}\left(M_{\alpha}\right)$. Proof. This is immediate from property (3) in Corollary 2.7.

As the even filtration is induced by the Postnikov filtration in sheaves, its associated graded object can be described in terms of sheaf cohomology, for which we introduce dedicated notation.
Definition 2.23. If $M$ is an $R$-module, the even cohomology of $R$ with coefficients in $M$ is given by sheaf cohomology

$$
\mathrm{H}_{e v}^{p, q}(R, M):=\operatorname{H}_{\mathrm{Perf}_{e v}}^{s}\left(R, \mathcal{F}_{M}(q)\right) \simeq \operatorname{Ext}_{\mathrm{Perf}_{e v}}^{p}\left(\mathcal{F}_{R}, \mathcal{F}_{M}(q)\right)
$$

If the ring $R$ is understood, we write

$$
\mathrm{H}_{e v}^{p, q}(M):=\mathrm{H}_{e v}^{p, q}(R, M) .
$$

Remark 2.24 (Even cohomology and the even filtration). Let $M$ be a homologically even $R$ module, so that $\mathcal{F}_{M}(k / 2)=0$ for all odd $k$. In this case, the Whitehead tower of $Y(M)$ of the form

has associated graded given by (de)suspensions of the even sheaves of Definition 2.8, considered as objects of the heart $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{S} p\right)^{\complement} \simeq \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A} b\right)$.

Passing to sections over $R$, we deduce that for any $n, t \in \mathbb{Z}$ we have a canonical isomorphism

$$
\pi_{t} \operatorname{gr}_{e v}^{n} M \simeq \mathrm{H}_{e v}^{2 n-t}\left(R, \mathcal{F}_{M}(n)\right) \simeq \mathrm{H}_{e v}^{2 n-t, n}(R, M)
$$

between the homotopy groups of the associated graded and even cohomology. This can be rewritten in a more pleasant form as

$$
\mathrm{H}_{e v}^{p, q}(R, M) \simeq \pi_{2 q-p} \operatorname{gr}_{e v}^{q}(M) .
$$

Definition 2.25. If $M$ is homologically even, we call the spectral sequence associated to the even filtration of $M$ the even spectral sequence. It is of signature

$$
E_{2}^{p, q}:=\mathrm{H}_{e v}^{p, q}(R, M) \Rightarrow \pi_{2 q-p}(M)
$$

with differentials of bidegree $\left|d_{r}\right|=(2 r-1, r-1)$.
Remark 2.26 (Integer and half-integer grading). In principle, it is possible to consider the even filtration as half-integer graded; that is, it makes sense to talk of

$$
\mathrm{fil}_{e v}^{n}:=\Gamma_{\operatorname{Perf}_{e v}(R)}\left(R, \tau_{\geq 2 n} Y(M)\right)
$$

for $n \in 1 / 2 \mathbb{Z}$. In practice, we are mainly interested in the even filtration for homologically even $M$. In this case, which includes $R$ itself by Lemma 2.17, we have

$$
\mathrm{fi}_{e v}^{n}(M) \simeq \mathrm{fil}_{e v}^{n-1 / 2}(M)
$$

for all $n \in \mathbb{Z}$, so that it is more convenient to consider the even filtration as only integer-graded. Our choice of notation is dictated by compatibility with the even filtration of Hahn-Raksit-Wilson [HRW22] and subsequently, the various motivic filtrations.

That being said, the convention of Definition 2.19 is really most appropriate only when $M$ is homologically even. In the general case, it is preferable to work with the half-integer graded even filtration, whose associated graded is always given by the even cohomology groups (which now might be non-zero in half-integer weight).

Similar tension exists in the classical case of MU-homology, since the category of even graded $\mathrm{MU}_{*} \mathrm{MU}$-comodules has a beautiful geometric interpretation as the category of quasi-coherent sheaves on the moduli stack of formal groups. However, the language of quasi-coherent sheaves is less convenient when talking about spectra whose MU-homology is not concentrated in even degrees, as one then needs to keep track of a pair of sheaves as in [Lur10, Lecture 11].

### 2.4. Example: Modules with even homotopy groups

In case of modules with even homotopy, even cohomology takes a particularly simple form.
Lemma 2.27. Let $E$ be an $R$-module and suppose that $\pi_{*} E$ is concentrated in even degree. Then
(1) $E$ is homologically even,
(2) we have fil ${ }_{e v}^{n} E \simeq \tau_{\geq 2 n} E$ for all $n \in \mathbb{Z}$.

Proof. Since $\pi_{*} E$ is concentrated in even degree, so is the $R$-linear cohomology group

$$
E_{R}^{*}(A) \simeq \pi_{-*} \operatorname{Map}_{\mathcal{M o d}_{R}}(A, E)
$$

for any perfect even $A$. Since the half-weight even sheaves are defined as the sheafication of the presheaf of odd homotopy groups, which vanish, we deduce that they are all zero, proving (1).

For (2), observe that since all three groups are concentrated in even degree, long exact sequence of cohomology shows that if

$$
A \rightarrow B \rightarrow C
$$

is a cofibre sequence of perfect evens, then

$$
0 \rightarrow E_{R}^{*}(C) \rightarrow E_{R}^{*}(B) \rightarrow E_{R}^{*}(A) \rightarrow 0
$$

is short exact. It follows from the criterion of Theorem 2.6 that the presheaf of spectra on $\operatorname{Perf}_{e v}$ defined by

$$
A \mapsto \tau_{\geq 2 n}(\Gamma(A, E)) \simeq \tau_{\geq 2 n}(\operatorname{Map}(A, E))
$$

is a sheaf and thus

$$
\operatorname{fil}_{e v}^{n} E:=\Gamma\left(R, \tau_{\geq 2 n} Y(E)\right) \simeq \tau_{\geq 2 n}(\Gamma(R, Y(E))) \simeq \tau_{\geq 2 n}(\operatorname{Map}(R, E)) \simeq \tau_{\geq 2 n} E,
$$

as claimed.
Corollary 2.28. If $E$ is an $R$-module with $\pi_{*} E$ even, then its even cohomology groups are given by

$$
\mathrm{H}_{e v}^{0, q}(R, E) \simeq \pi_{2 q} E
$$

in cohomological degree zero and vanish otherwise.
Proof. This follows from Lemma 2.27 and Remark 2.24.

### 2.5. Example: Rings with even homotopy groups

In the previous section, we have seen that the even filtration of an $R$-module $E$ such that $\pi_{*} E$ is even coincides with the Postnikov filtration. We now verify that this conclusion can be extended to all $R$-modules assuming that the ring itself satisfies this condition.
Lemma 2.29. Let $R$ be a $\mathbf{E}_{1}$-ring such that $\pi_{*} R$ is even. Then every cofibre sequence

$$
A \rightarrow B \rightarrow C
$$

of perfect even $R$-modules splits.
Proof. Since the condition of having even homotopy groups is stable under retracts, extensions and even (de)suspensions, we deduce that $\pi_{*} M$ is even for any perfect even $M$. Applying the same reasoning to $\pi_{*} \operatorname{Map}_{\mathcal{M o d}_{R}}(N, M)$, we deduce that all odd degree maps between perfect even $R$-modules are null-homotopic. In particular, this applies to the boundary map $C \rightarrow \Sigma A$, so that the cofibre sequence splits.
Corollary 2.30. Let $R$ be a $\mathbf{E}_{1}$-ring such that $\pi_{*} R$ is even. Then, every additive presheaf $X: \operatorname{Perf}(R)_{e v}^{o p} \rightarrow X$ is a sheaf with respect to the even epimorphism topology. In particular, connective covers of additive sheaves are calculated levelwise.
Proof. The first part is immediate from Lemma 2.29 and the characterization of additive sheaves given in Theorem 2.6. The second follows from the fact that additive presheaves are closed under taking levelwise connective covers.

Proposition 2.31. Let $R$ be a $\mathbf{E}_{1}$-ring such that $\pi_{*} R$ is even. Then for every $R$-module $M$ we have

$$
\operatorname{fil}_{e v}^{n}(M) \simeq \tau_{\geq 2 n} M
$$

Proof. This follows from Corollary 2.30, since

$$
\Gamma\left(R, \tau_{\geq 2 n} Y(M)\right) \simeq \tau_{\geq 2 n}\left(\operatorname{Map}_{\mathcal{M}^{\prime} o d_{R}}(R, M)\right) \simeq \tau_{\geq 2 n} M
$$

## 3. Multiplicative properties of the even filtration

In this section we establish that the even filtration of an $\mathbf{E}_{n}$-ring can be canonically promoted to an $\mathbf{E}_{n}$-algebra in filtered spectra. Our analysis requires some study of the monoidal properties of the $\infty$-categories of additive sheaves introduced in $\S 2$.

The $\infty$-category $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \delta p\right)$ has a canonical set of compact generators as a stable, presentable $\infty$-category. These are not given by the image of perfect evens under the spectral Yoneda embedding, but rather their connective covers, for which we introduce dedicated notation:

Notation 3.1. If $M \in \mathcal{M} o d_{R}(\delta p)$, we write

$$
\nu_{R}(M):=\tau_{\geq 0} Y(R) \in \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)
$$

Remark 3.2. Note that if $A \in \operatorname{Perf}(R)_{e v}$, then the Yoneda lemma yields an equivalence

$$
\operatorname{Map}(\nu(A), X) \simeq \Gamma_{\operatorname{Perf}_{e v}}(A, X)
$$

for any $X \in \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)$, where the left hand side is the internal mapping spectrum. In particular, $\nu(A)$ generate the $\infty$-category of additive sheaves under colimits and desuspensions.

### 3.1. Monoidal structure on additive sheaves

Construction 3.3. If $R$ is an $\mathbf{E}_{n}$-ring, them the $\infty$-category $\operatorname{Perf}(R)$ has a canonical $\mathbf{E}_{n-1^{-}}$ monoidal structure given by the $R$-linear tensor product. Since the tensor products preserves colimits in each variable separately and $R \otimes_{R} R \simeq R$, the subcategory

$$
\operatorname{Perf}(R)_{e v} \subseteq \operatorname{Perf}(R)
$$

is closed under the tensor product and hence inherits a unique $\mathbf{E}_{n-1}$-monoidal structure. This monoidal structure is compatible with the Grothendieck topology of even epimorphisms, so that we have an induced monoidal structure on the $\infty$-category $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)$ of additive sheaves given by Day convolution. It is uniquely determined by the properties that:
(1) the tensor product of sheaves preserves colimits separately in each variable,
(2) the connective Yoneda embedding $\nu_{R}(-)=\tau_{\geq 0} Y(-): \operatorname{Perf}(R)_{e v} \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)$ is strongly $\mathbf{E}_{n-1}$-monoidal.

Definition 3.4. If $R$ is an $\mathbf{E}_{n}$-ring, we refer to the $\mathbf{E}_{n-1}$-monoidal structure of Construction 3.3 as the canonical monoidal structure.

We record that the canonical monoidal structure is compatible with that of $R$-modules in the following sense:
Proposition 3.5. If $R$ is an $\mathbf{E}_{n}$-ring, then the functor

$$
\nu_{R}(-): \mathcal{M}_{o d}(S p) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, S p\right)
$$

admits a canonical lax $\mathbf{E}_{n-1}$-monoidal structure extending the strongly $\mathbf{E}_{n-1}$-monoidal structure of its restriction to $\operatorname{Perf}(R)_{e v}$.
Proof. Since the image of $\nu_{R}$ is contained in connective sheaves, which are closed under the tensor product, it is enough to work with the connective part $\operatorname{Shv}{ }_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{S} p\right)_{\geq 0}$. The universal property of the canonical monoidal structure yields a unique strongly $\mathbf{E}_{n-1}$-monoidal functor

$$
\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{S} p\right)_{\geq 0} \rightarrow \operatorname{Mod}_{R}(\mathcal{S} p)
$$

with the property that $\nu_{R}(A) \mapsto A$ for any $A \in \operatorname{Perf}(R)_{e v}$. By construction, the functor $\nu_{R}$ can be identified with its right adjoint, and hence it inherits a canonical lax $\mathbf{E}_{n-1}$-monoidal structure as an adjoint of an $\mathbf{E}_{n-1}$-monoidal functor.

### 3.2. Synthetic spectra and even filtration of the sphere

Introduced in [Pst22], the $\infty$-category of synthetic spectra is an $\infty$-categorical deformation which categorifies the Adams spectral sequence. We have the following relationship between sheaves on finite even spectra and synthetic spectra based on complex bordism MU.
Proposition 3.6. There exists a canonical symmetric monoidal equivalence

$$
\operatorname{Syn}_{\mathrm{MU}}^{e v} \simeq \operatorname{Sh} v_{\Sigma}\left(\operatorname{Perf}\left(S^{0}\right)_{e v}, \mathcal{S} p\right)
$$

between additive sheaves on finite even spectra and even MU-based synthetic spectra, uniquely determined by

$$
\left(\nu(P) \in \operatorname{Syn}_{\mathrm{MU}}^{e v}\right) \mapsto\left(\nu_{S^{0}}(P) \in \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}\left(S^{0}\right)_{e v}, \mathcal{S} p\right)\right)
$$

when $P \in \operatorname{Perf}\left(S^{0}\right)_{e v}$, the left hand side is the synthetic analogue of [Pst22, Definition 4.3] and the right hand side is as in Notation 3.1.
Proof. By definition, even synthetic spectra are additive sheaves on the site $\mathcal{S} p_{\mathrm{MU}}^{f p e}$ of finite spectra with even, projective $M U_{*}$-homology, where the topology is given by $M U_{*}$-epimorphisms [Pst22, Definition 5.10]. We claim that we in fact have an equivalence of $\infty$-sites

$$
\operatorname{Perf}\left(S^{0}\right)_{e v} \simeq \mathcal{S} p_{\mathrm{MU}}^{f p e}
$$

This comes down to the following two easy observations:
(1) a finite spectrum is perfect even if and only if it has even, projective $M U_{*}$-homology,
(2) a map between finite spectra with even, projective $\mathrm{MU}_{*}$-homology is a $\mathrm{MU}_{*}$-epimorphism if and only if its fibre also has even, projective $\mathrm{MU}_{*}$-homology.
One can verify that the equivalence of sheaf $\infty$-categories induced by this equivalence of sites has the above property; it is the unique one since synthetic spectra of the form $\nu(P)$ generate $\operatorname{Syn}_{\mathrm{MU}}^{e v}$ under colimits and desuspensions.

By a result of Gheorge-Isaksen-Krause-Ricka [GIKR22], MU-synthetic spectra can be identified with modules in filtered spectra. Keeping in mind Proposition 3.6, we present their construction in terms of additive sheaves on perfect evens.
Definition 3.7. We define a functor

$$
\Gamma^{*}: \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}\left(S^{0}\right)_{e v}, \mathcal{S} p\right) \rightarrow \operatorname{Fil}(\mathcal{S} p)
$$

by the formula

$$
\Gamma^{n}(X):=\Sigma^{2 n}\left(\Gamma\left(S^{2 n}, X\right)\right)
$$

where the connecting maps

$$
\Gamma^{n+1}(X)=\Sigma^{2 n+2}\left(\Gamma\left(S^{2 n+2}, X\right)\right) \rightarrow \Sigma^{2 n}\left(\Gamma\left(S^{2 n}, X\right)\right)=\Gamma^{n}(X)
$$

are adjoint to the the canonical colimit-to-limit comparison map induced by the identification

$$
\Sigma^{2}\left(S^{2 n}\right) \simeq S^{2 n+2}
$$

Remark 3.8. Let $S$ be a spectrum, $Y(S):=\operatorname{Map}_{\mathcal{S}_{p}}(-, S)$ the associated spectral presheaf, and $\nu_{S^{0}}(S):=\tau_{\geq 0} Y(S)$ its connective cover, where we're using Notation 3.1. Then

$$
\operatorname{fil}_{e v}^{n}(S):=\Gamma\left(S^{0}, \tau_{\geq 2 n} Y(S)\right) \simeq \Sigma^{2 n} \Gamma\left(S^{2 n}, \nu_{S^{0}}(S)\right) \simeq \Gamma^{n}\left(\nu_{S^{0}}(S)\right)
$$

so that

$$
\Gamma^{*} \circ \nu_{S^{0}}(-) \simeq \mathrm{fil}_{e v}^{*}(-)
$$

as functors $\mathcal{S} p \rightarrow$ FilS $p$. In particular, the image under $\Gamma^{*}(-)$ of monoidal unit $\nu_{S^{0}}\left(S^{0}\right)$ of additive sheaves is given by the even filtered sphere.

The functor $\Gamma^{*}(-)$ can be given a canonical lax symmetric monoidal structure, so that it lifts to a functor valued in modules over the image of the unit, which by Remark 3.8 is the even filtered sphere. The work of Gheorghe-Isaksen-Krause-Ricka then yields the following:

Proposition 3.9 ([GIKR22, 6.13]). The functor $\Gamma^{*}$ of Definition 3.7 lifts to a a symmetric monoidal equivalence

$$
\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}\left(S^{0}\right)_{e v}, \mathcal{S} p\right) \simeq{\mathcal{M} o d_{\mathrm{fil}_{e v}}^{*}\left(S^{0}\right)}(\mathrm{FilS} p)
$$

between additive sheaves on perfect even spectra and modules over the filtered even sphere.
Remark 3.10. The identification of even MU-synthetic spectra with modules in filtered even spectra is somewhat formal. Instead, the difficult part of [GIKR22] corresponds to the identification of the even filtration on the sphere with the Adams-Novikov filtration. This was independently proven for the $\mathbf{E}_{\infty}$-even filtration by Hahn-Raksit-Wilson [HRW22, Theorem 1.1.5] and we prove it in the same way for the variant of the even filtration studied here in Corollary 7.4.

### 3.3. Monoidality of the even filtration

Construction 3.11. We will construct an action of the $\infty$-category of modules over the even filtered sphere on the $\infty$-category of additive sheaves on perfect even $R$-modules.

Since the $\infty$-category of $R$-modules is stable and presentable, it carries a canonical action of the $\infty$-category of spectra, given on objects by

$$
\left((M, S) \in \mathcal{M}_{\operatorname{Mod}}^{R}(\mathbb{S} p) \times \mathcal{S} p\right) \mapsto\left(M \otimes_{S^{0}} S \in \mathcal{M} o d_{R}(\mathcal{S} p)\right)
$$

This action restricts to an action

$$
\operatorname{Perf}(R)_{e v} \times \operatorname{Perf}(S)_{e v} \rightarrow \operatorname{Perf}(R)_{e v}
$$

of perfect even spectra on perfect even $R$-modules. The restricted action preserves cofibre sequences in each variable and so induces an action

$$
\otimes: \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right) \times \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}\left(S^{0}\right)_{e v}, \mathcal{S} p\right) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)
$$

uniquely determined by the properties that
(1) it preserves colimits separately in each variable and
(2) $\nu_{R}(M) \otimes \nu_{S^{0}}(A) \simeq \nu_{R}\left(P \otimes_{S^{0}} A\right)$ whenever $M \in \operatorname{Perf}(R)_{e v}$ and $A \in \operatorname{Perf}\left(S^{0}\right)_{e v}$, where we're using Notation 3.1.
The first property implies that this action presents its target as a module over

$$
\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}\left(S^{0}\right)_{e v}, S p\right) \simeq{\mathcal{M} o d_{\mathrm{fil}_{e v}}\left(S^{0}\right)}(\operatorname{FilS} p)
$$

in the $\infty$-category $\operatorname{Pr}^{L}$ of presentable $\infty$-categories and cocontinuous functors of [Lur, $\left.\S 4.8\right]$.
Remark 3.12. As any tensoring induced by a module structure in $\operatorname{Pr}^{L}$, the tensor structure of Construction 3.11 is closed. That is, for any $A, B \in \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)$, there exists an internal mapping object

$$
\left.\underline{\operatorname{Map}}^{*}(A, B) \in \mathcal{M}^{( }\right) d_{\mathrm{fil}_{e v}^{*}\left(S^{0}\right)}(\operatorname{FilS} p)
$$

which is uniquely specified by the property of representing the functor

$$
\left(M \in \mathcal{M}_{\operatorname{lod}}^{\mathrm{fi}_{e v}^{*}\left(S^{0}\right)}\right) \mapsto\left(\operatorname{map}_{\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, s p\right)}(A \otimes M, B) \in \mathcal{S}\right)
$$

where $-\otimes$ - denotes the action. In the case of $A=B$, the internal mapping object Map ${ }^{*}(A, A)$ inherits a canonical structure of an $\mathbf{E}_{1}$-monoid in modules over the filtered sphere.

The closed tensor structure of Construction 3.11 can be used to define an $R$-module analogue of the $\Gamma^{*}$ functor studied in $\S 3.2$.

Definition 3.13. The functor $\left.\Gamma_{R}^{*}: \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, S p\right)\right) \rightarrow{\mathcal{M} o d_{\mathrm{fil}_{e v}( }\left(S^{0}\right)}(\operatorname{FilS} p)$ is given by

$$
\Gamma_{R}^{*}(X):=\underline{\operatorname{Map}}^{*}\left(\nu_{R}(R), X\right)
$$

where Map* is the internal mapping object of Remark 3.12 and $\nu_{R}(R)$ is the connective Yoneda embed $\overline{d i n g}$ of the unit.
Proposition 3.14. For any $M \in \mathcal{M}_{\operatorname{Cod}}^{R}(\mathcal{S} p)$, there's a canonical equivalence of filtered spectra

$$
\Gamma_{R}^{*}\left(\nu_{R}(M)\right) \simeq \mathrm{fil}_{e v / R}^{*}(M)
$$

Proof. For brevity, we denote the free filtered fil ${ }_{e v}^{*}\left(S^{0}\right)$-module generated in filtration $n \in \mathbb{Z}$ by $F_{n}$. By direct inspection, under the equivalence of Proposition 3.9, $F_{n}$ corresponds to the additive sheaf $\Sigma^{-2 n} \nu_{S^{0}}\left(S^{2 n}\right)$. It follows that

$$
\Gamma_{R}^{n}(M) \simeq \operatorname{Map}_{\mathcal{M o d}_{\mathrm{fil}_{e v}^{*}\left(S^{0}\right)}}\left(F_{n}, \underline{\operatorname{Map}}^{*}\left(\nu_{R}(R), \nu_{R}(M)\right)\right) \simeq \operatorname{Map}\left(\nu_{R}(R) \otimes \Sigma^{-2 n} \nu_{S^{0}}\left(S^{2 n}\right), \nu_{R}(M)\right)
$$

where the right hand side is the internal mapping spectrum in additive sheaves. Using the defining property of the tensor product we can rewrite this further as
$\operatorname{Map}\left(\nu_{R}(R) \otimes \Sigma^{-2 n} \nu_{S^{0}}\left(S^{2 n}\right), \nu_{R}(M)\right) \simeq \operatorname{Map}\left(\Sigma^{-2 n} \nu_{R}\left(\Sigma^{2 n} R\right), \nu_{R}(M)\right) \simeq \Sigma^{2 n}\left(\Gamma\left(\Sigma^{2 n} R, \nu_{R}(M)\right)\right)$,
where the last equivalence is the Yoneda lemma, and

$$
\Sigma^{2 n}\left(\Gamma\left(\Sigma^{2 n} R, \nu_{R}(M)\right)\right) \simeq \Sigma^{2 n}\left(\Gamma\left(\Sigma^{2 n} R, \tau_{\geq 0} Y_{R}(M)\right)\right) \simeq \Gamma\left(R, \tau_{\geq 2 n} Y_{R}(M)\right) \simeq \operatorname{fil}_{e v / R}^{n}(M)
$$

as needed.
Theorem 3.15. For any $\mathbf{E}_{1}$-ring $R$, the even filtration fil ${ }_{e v}^{*}(R)$ has a canonical lift to a filtered $\mathrm{fil}_{\text {ev }}^{*}\left(S^{0}\right)-\mathbf{E}_{1}$-algebra. Moreover, the $R$-module even filtration functor canonically lifts to

If $R$ is an $\mathbf{E}_{n}$-ring for $n \geq 2$, then fil $_{e v}^{*}(R)$ can be canonically refined to a filtered fil ${ }_{e v}^{*}\left(S^{0}\right)-\mathbf{E}_{n}$ algebra and (3.1) can be canonically refined to a lax $\mathbf{E}_{n-1}$-monoidal functor.

Proof. By Proposition 3.14, the claim is equivalent to saying that the internal mapping object Map ${ }^{*}\left(\nu_{R}(R), \nu_{R}(R)\right)$ of Remark 3.12 can be promoted to an $\mathbf{E}_{1}$-monoid and that Map ${ }^{*}\left(\nu_{R}(R),-\right)$ takes values in left modules. Both statements are clear.

Now suppose that $R$ is an $\mathbf{E}_{n}$-ring for $n \geq 2$. As $R$ is an $\mathbf{E}_{n}$ - $S^{0}$-algebra in spectra, $\operatorname{Perf}(R)_{e v}$ is a $\operatorname{Perf}\left(S^{0}\right)_{e v}$ - $\mathbf{E}_{n-1}$-algebra in $\infty$-categories. It follows that Construction 3.11 induces on $\operatorname{Shv} \Sigma\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)$
 $\mathbf{E}_{n-1}$-algebra, the internal mapping spectrum $\underline{\operatorname{Map}}^{*}\left(\nu_{R}(R), \nu_{R}(R)\right)$ can be canonically promoted to an $\mathbf{E}_{n}$-algebra and the functor

$$
\underline{\operatorname{Map}}^{*}\left(\nu_{R}(R),-\right): \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right) \rightarrow{\mathcal{M} o d_{\mathrm{fil}_{e v}^{*}}^{*}(R)}^{(\operatorname{FilS} p)}
$$

to a lax $\mathbf{E}_{n-1}$-monoidal functor. Since $\nu_{R}(-)$ can also be made lax symmetric monoidal by Proposition 3.5, the second claim follows.

Remark 3.16. By an argument using compact generators, it is not difficult to see that the functor

$$
\underline{\operatorname{Map}}^{*}\left(\nu_{R}(R),-\right): \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right) \rightarrow{\mathcal{M} o d_{\mathrm{fil}_{e v}^{*}(R)}^{*}(\operatorname{FilS} p)}
$$

is an equivalence of $\infty$-categories. In other words, the $\infty$-category of additives sheaves is essentially encoded by the even filtration of $R$. We will not need this result, so we leave the proof to an interested reader.

## 4. Calculus of evenness

This section is devoted to the study of the various notions of evenness one can attach to an $R$-module, as well as their relationships.

### 4.1. Even flat modules

In this subsection, we introduce the notion of an even flat $R$-module, which informally is a module "flat from the point of view of modules with even homotopy groups". We will make the latter characterization precise in Proposition 4.12.
Proposition 4.1. The following two conditions are equivalent for an $R$-module $M$ :
(1) it can be written $M \simeq \underset{\longrightarrow}{\lim } M_{\alpha}$ as a filtered colimit of perfect evens,
(2) any map $P \rightarrow M$ from $\vec{a}$ perfect $R$-module into $M$ factors through a perfect even.

Proof. Since perfect $R$-modules are compact as objects of $\mathcal{M} \operatorname{Cod}$, certainly $(1 \rightarrow 2)$, and we only have to prove the converse. We can write $M$

$$
M \simeq \underset{P \in \operatorname{Perf}(R)_{-/ M}}{\lim _{\longrightarrow}} P
$$

as a colimit of perfect $R$-modules indexed by the overcategory $\operatorname{Perf}(R)_{/ M}$. We claim that if $M$ satisfies the condition (2), then the inclusion

$$
\operatorname{Perf}(R)_{-/ M}^{e v} \rightarrow \operatorname{Perf}(R)_{-/ M}
$$

is cofinal, which will finish the argument. By Quillen's Theorem A [Lur09, 4.1.3.1], we have to check that for any map $f: P \rightarrow M$ from a perfect $R$-module, the under-over-category

$$
\operatorname{Perf}(R)_{P /-/ M}^{e v}
$$

whose objects are commutative triangles

with $A$ perfect even, is weakly contractible. We will show that it is filtered, which implies weak contractibility. Suppose that

$$
p: K \rightarrow \operatorname{Perf}(R)_{P /-/ M}^{e v}
$$

is a finite diagram. As $\operatorname{Perf}(R)$ admits finite colimits, $p$ admits a colimit $C$ in the larger $\infty$-category $\operatorname{Perf}(R)_{P /-/ M}$, where the middle module as in (4.1) is perfect, but not necessarily even. Since the map $C \rightarrow M$ factors through a finite even $C^{\prime}$ by assumption, declaring $\widetilde{p}(\triangleright):=C^{\prime}$ provides the necessary extension of $p$ to a diagram $\widetilde{p}: K^{\triangleright} \rightarrow \operatorname{Perf}(R)_{A /-/ M}^{e v}$.
Definition 4.2. We say that an $R$-module $M$ is even flat if it satisfies the equivalent conditions of Proposition 4.1.

We record that the class of even flats has good closure properties.
Proposition 4.3. The full subcategory $\mathcal{M} o d_{R}^{e b} \subseteq \mathcal{M} \operatorname{Lod}{ }_{R}$ spanned by even flat modules is closed under extensions, retracts and filtered colimits.
Proof. The property (2) of Proposition 4.1 is clearly closed under retracts and filtered colimits. We will verify that it is also closed under extensions. Suppose we have a cofibre sequence

$$
M^{\prime} \rightarrow M \rightarrow M^{\prime \prime}
$$

of $R$-modules such that $M^{\prime}$ and $M^{\prime \prime}$ are even flat. Let $P \rightarrow M$ be a map from a perfect; we have to show that it factors through a perfect even. Using that $M^{\prime \prime}$ is even flat, we can factor the composite $P \rightarrow M^{\prime \prime}$ through a perfect even $A^{\prime \prime}$, obtaining a diagram

where both rows are cofibre and where $A^{\prime \prime}$ is perfect even. Using that $M^{\prime}$ is even flat, we can factor the map $\operatorname{fib}\left(P \rightarrow A^{\prime \prime}\right) \rightarrow M^{\prime}$ through a perfect even $A^{\prime}$. Finally, let $A$ be defined by a pushout diagram

so that the universal property of the pushout gives a factorization $P \rightarrow A \rightarrow M$. As by construction we have a cofibre sequence

$$
A^{\prime} \rightarrow A \rightarrow A^{\prime \prime}
$$

$A$ is perfect even, as needed.
Remark 4.4. As perfect even modules are generated under retracts and extensions by $\Sigma^{2 n} R$ for $n \in \mathbb{Z}$, as a consequence of Proposition 4.3 the subcategory $\mathcal{M}_{\text {od }}^{\text {eb }} \subseteq \mathcal{M} \operatorname{\mathcal {M}} d_{R}$ of even flats can be characterized as the smallest subcategory containing $\Sigma^{2 n} R$ and closed under filtered colimits, retracts and extensions.

### 4.2. Homotopy even envelopes

As modules with even homotopy groups have a particularly simple even filtration by Lemma 2.27, a useful way to prove results about the even filtration of an arbitrary $R$-module is to map it into a module with even homotopy groups. In this section, we study the following particularly wellbehaved way to do so:

Definition 4.5. Let $M$ be an $R$-module. We say that a map $M \rightarrow E$ is a $\pi_{*}$-even envelope if
(1) the cofibre $\operatorname{cofib}(M \rightarrow E)$ is even flat,
(2) $\pi_{*} E$ is a concentrated in even degrees,
(3) every map $M \rightarrow F$ into an $R$-module such that $\pi_{*} F$ is even can be completed to a commutative diagram


Remark 4.6. Note that the condition (3) of Definition 4.5 is equivalent to saying that for any $\pi_{*}$-even $F$, the relative $R$-linear cohomology groups

$$
F_{R}^{*}(E, M) \simeq F_{R}^{*}(\operatorname{cofib}(M \rightarrow E)) \simeq \pi_{-*} \operatorname{Map}_{\mathcal{M o d}_{R}}(\operatorname{cofib}(M \rightarrow E), F)
$$

are concentrated in even degrees.
Remark 4.7. If $M \rightarrow E$ is $\pi_{*}$-even envelope, then the map $\mathcal{F}_{M} \rightarrow \mathcal{F}_{E}$ is a monomorphism. Indeed, we have an exact sequence

$$
\mathcal{F}_{\operatorname{cofib}(M \rightarrow N)}(1 / 2) \rightarrow \mathcal{F}_{M} \rightarrow \mathcal{F}_{N} .
$$

where the left hand side vanishes, since the cofibre is even flat.
Remark 4.8. Since even flat (resp. homologically flat) $R$-modules are closed under extensions, if $M \rightarrow E$ is a $\pi_{*}$-even envelope and $M$ is even flat (resp. homologically flat), then so is $E$.

Proposition 4.9 (Existence of homotopy even envelopes). Let $M$ be an $R$-module. Then there exists a $\pi_{*}$-even envelope $M \rightarrow E$. Moreover, if $R$ and $M$ are connective, then there exists an envelope such that $E$ is also connective and $\pi_{0} M \simeq \pi_{0} E$.

Proof. We will use the small object argument to construct $E$. Consider the set of homotopy classes of maps

$$
\Sigma^{k} R \rightarrow M
$$

where $k$ is odd. Taking a direct sum of all such maps, we obtain a cofibre sequence

$$
\bigoplus \Sigma^{k_{\alpha}} R \rightarrow M \rightarrow M^{\prime}
$$

We now inductively declare $M_{0}:=M$ and $M_{n+1}:=\left(M_{n}\right)^{\prime}$ as above. We claim that $E:=\underset{\longrightarrow}{\lim } M_{n}$ has the required properties.

We first verify that $E$ is an $\pi_{*}$-even envelope. For property (1), since

$$
\operatorname{cofib}(M \rightarrow E) \simeq \underset{\longrightarrow}{\lim } \operatorname{cofib}\left(M \rightarrow M_{n}\right)
$$

and even flat modules are stable under filtered colimits, it is enough to show that each of $\operatorname{cofib}(M \rightarrow$ $M_{n}$ ) is even flat. We argue by induction. For $n=0$, this cofibre vanishes and so is even flat. For $n>0$ we have a cofibre sequence

$$
\operatorname{cofib}\left(M \rightarrow M_{n}\right) \rightarrow \operatorname{cofib}\left(M \rightarrow M_{n_{1}}\right) \rightarrow \bigoplus \Sigma^{k_{\alpha}+1} R
$$

As the right hand side term is a direct sum of perfect evens and even flat modules are closed under extensions by Proposition 4.3, we deduce that the middle term is also even flat.

For property (2), observe that since any map $\Sigma^{k} R \rightarrow E$ where $k$ is odd factors through some $M_{n}$, it vanishes in $M_{n+1}$ and hence in $E$. Thus, $\pi_{*} E$ is even as needed.

We move on to property (3). Using Remark 4.6, it is enough to show that if $\pi_{*} F$ is even, then the relative $R$-linear cohomology groups $F_{R}^{*}(E, M)$ are concentrated in even degrees. Since

$$
\operatorname{cofib}(M \rightarrow E) \simeq \xrightarrow{\lim } \operatorname{cofib}\left(M \rightarrow M_{n}\right),
$$

we have a Milnor exact sequence

$$
0 \rightarrow R^{1} \varliminf_{\longleftarrow} F_{R}^{*-1}\left(M_{n}, M\right) \rightarrow F_{R}^{*}(E, M) \rightarrow \varliminf_{\check{l}} F_{R}^{*}\left(M_{n}, M\right) \rightarrow 0 .
$$

Since the left hand side term vanishes on Mittag-Leffler sequences, vanishing in odd degrees will follow if we can show that
(1) $F_{R}^{k}\left(M_{n}, M\right)$ vanishes for $k$ odd and
(2) $F_{R}^{k}\left(M_{n+1}, M\right) \rightarrow F_{R}^{k}\left(M_{n}, M\right)$ is an epimorphism for $k$ even.

Both follow at once (in the first case, by induction) from the long exact sequence

$$
\ldots \rightarrow F_{R}^{*}\left(M_{n+1}, M\right) \rightarrow F_{R}^{*}\left(M_{n}, M\right) \rightarrow F_{R}^{*}\left(\bigoplus R^{k_{\alpha}}\right) \rightarrow F_{R}^{*+1}\left(M_{n+1}, M\right) \rightarrow F_{R}^{*+1}\left(M_{n}, M\right) \rightarrow \ldots
$$

since

$$
F_{R}^{*}\left(\bigoplus R^{k_{\alpha}}\right) \simeq \prod F_{R}^{*}\left(R^{k_{\alpha}}\right) \simeq \prod \pi_{*-k_{\alpha}}(F)
$$

vanishes in even degrees by assumption as all $k_{\alpha}$ are odd.
Finally, suppose that $R$ and $M$ are connective. Then in the inductive construction above it is enough to consider homotopy clases of maps $\Sigma^{k} R \rightarrow M$ such that $k$ is odd and positive. If the above inductive construction is performed with this restriction, the result will be connective with $\pi_{0} M \simeq \pi_{0} E$.

Of particular importance is the $\pi_{*}$-even envelope of $R$ itself, which generates envelopes of any perfect even in the following sense:

Proposition 4.10. Let $R \rightarrow E$ and $A \rightarrow E$ be $\pi_{*}$-even envelopes, where $A$ is perfect even. Then, $E^{\prime}$ belongs to the subcategory generated by $E$ under direct sums, even (de)suspensions and retracts.

Proof. Let $f: \bigoplus \Sigma^{k_{\alpha}} R \rightarrow E^{\prime}$ be a $\pi_{*}$-surjective map from a direct sum of even (de)suspensions of $R$ which exists by the assumption that $\pi_{*}\left(E^{\prime}\right)$ is even. By assumption, we can extend this map to a commutative diagram


We claim that the map $p$ admits a section, which will imply that $E^{\prime}$ is a retract of $\bigoplus \Sigma^{k_{\alpha}} E$, proving the result. This happens precisely when the map $E^{\prime} \rightarrow C$ vanishes, where

$$
C:=\operatorname{cofib}\left(\bigoplus \Sigma^{k_{\alpha}} E \rightarrow E^{\prime}\right)
$$

Since $f$ is $\pi_{*}$-surjective on homotopy groups, so is $p$, and thus $\pi_{*} C$ is concentrated in odd degree. Since $A$ is perfect even,

$$
\left(\bigoplus \Sigma^{k_{\alpha}} E\right)_{R}^{*}(A) \rightarrow E_{R}^{*}(A)
$$

is surjective and thus the structure map $A \rightarrow E^{\prime}$ lifts to a map into $\bigoplus \Sigma^{k_{\alpha}} E$, so that the composite $A \rightarrow E^{\prime} \rightarrow C$ vanishes. Thus, we have a factorization

$$
E^{\prime} \rightarrow \operatorname{cofib}\left(A \rightarrow E^{\prime}\right) \rightarrow C
$$

The second map vanishes by Remark 4.6 since $\pi_{*} C$ is odd, proving the result.
Corollary 4.11 (Weak uniqueness of the $\pi_{*}$-even envelope of the unit). Any two $\pi_{*}$-even envelopes $R \rightarrow E$ and $R \rightarrow E^{\prime}$ generate the same subcategory of $\mathcal{M o d}_{R}$ under direct sums, retracts and even (de)suspensions. This subcategory contains $\pi_{*}$-even envelopes of any perfect even.

As a corollary of the construction of $\pi_{*}$-even envelopes, we have the following characterization of even flat modules in terms of tensor products:

Proposition 4.12 (Lazard's Theorem). The following are equivalent for an $R$-module $M$ :
(1) $M$ is even flat in the sense of Definition 4.2 or
(2) for any right $R$-module $E$ such that $\pi_{*} E$ is even, $\pi_{*}\left(E \otimes_{R} M\right)$ is even.

Moreover, if $R$ is connective, then it is enough to verify the second condition when $E$ is also connective.

Proof. $(1 \Rightarrow 2)$ Since the subcategory of those $R$-modules $M$ such that $\pi_{*}\left(E \otimes_{R} M\right)$ is even is closed under retracts, filtered colimits and contains $\Sigma^{2 k} R$, it necessarily contains all even flat modules.
$(2 \Rightarrow 1)$ By Proposition 4.1, it is enough to show that every map $P \rightarrow M$ from a perfect $R$ module factors through a perfect even. By dualizing, we can identify the given map with a map of spectra $S^{0} \rightarrow P^{\vee} \otimes_{R} M$.

Let $P^{\vee} \rightarrow E$ be a map such that its suspension is a $\pi_{*}$-even envelope of right $R$-modules, so that the fibre $F:=\mathrm{fib}(P \rightarrow E)$ is even flat and $\pi_{*} E$ is concentrated in odd degrees. By assumption, $\pi_{*}\left(E \otimes_{R} M\right)$ is also concentrated in odd degrees, so that the composite $S^{0} \rightarrow E \otimes_{R} M$ vanishes. It follows that we have a lift $S^{0} \rightarrow F \otimes_{R} M$ which since $F$ is even flat factors through $S^{0} \rightarrow A \otimes_{R} M$, where $A$ is a perfect even right $R$-module. After dualizing, this provides the needed factorization

$$
P \rightarrow A \rightarrow M
$$

If $R$ is connective, then the homotopy groups of $P$ as above are bounded from below, so that by a sufficiently large even suspension we can assume that $P$ is connective. In this case, $E$ can also chosen to be connective by Proposition 4.9.

Remark 4.13. Observe that any cofibre sequence $E \rightarrow F \rightarrow G$ of $\pi_{*}$-even right $R$-modules is necessarily short exact on homotopy; that is,

$$
0 \rightarrow \pi_{*} E \rightarrow \pi_{*} F \rightarrow \pi_{*} G \rightarrow 0
$$

is exact. By Proposition 4.12, the same is true for

$$
0 \rightarrow \pi_{*}\left(E \otimes_{R} M\right) \rightarrow \pi_{*}\left(F \otimes_{R} M\right) \rightarrow \pi_{*}\left(G \otimes_{R} M\right) \rightarrow 0
$$

when $M$ is even flat since again everything is even. It is in this sense that an even flat module behaves in a "flat" manner, but only from the perspective of modules with even homotopy groups.

### 4.3. Homologically even modules

Recall that we say that an $R$-module $M$ is homologically even if $\mathcal{F}_{M}(1 / 2+k)$ for all $k \in \mathbb{Z}$. The goal of this section is to characterize these modules in a variety of different ways.

Theorem 4.14. For an $R$-module $M$, the following conditions are equivalent:
(1) $M$ is homologically even,
(2) every map $A \rightarrow \Sigma M$ from a perfect even spectrum into suspension of $M$ factors through $A \rightarrow \Sigma B$, where $B$ is perfect even,
(3) for any even flat, $\pi_{*}$-even right $R$-module $E, \pi_{*}\left(E \otimes_{R} M\right)$ is even,
(4) if $A \rightarrow E$ is $\pi_{*}$-even envelope of a perfect even right $R$-module, then $\pi_{*}\left(E \otimes_{R} M\right)$ is even.
(5) if $R \rightarrow E$ is $\pi_{*}$-even envelope of the unit in right $R$-modules, then $\pi_{*}\left(E \otimes_{R} M\right)$ is even.

Proof. ( $1 \Leftrightarrow 2$ ) The homotopy class of $A \rightarrow \Sigma M$ determines an element of $\pi_{0} \operatorname{Map}(A, \Sigma M)$. Since $\mathcal{F}_{M}(1 / 2)$ is defined as the sheafication of the presheaf

$$
\pi_{0} \operatorname{Map}(-, \Sigma M): \operatorname{Perf}(R)_{e v}^{o p} \rightarrow \mathcal{A} b
$$

it vanishes if for every such element there exists an even epimorphism $f: B \rightarrow A$ such that the composite

$$
B \rightarrow A \rightarrow \Sigma M
$$

is zero. This happens precisely when the second map factors through $\Sigma(\operatorname{fib}(f))$, which is a suspension of a perfect even as needed.
$(2 \Rightarrow 3)$ Suppose we have a homotopy class of maps $S^{k} \rightarrow E \otimes_{R} M$ with $k$ odd. Since $E$ is even flat, it is a filtered colimit of perfect evens, so that the map factors through $S^{k} \rightarrow A \otimes_{R} M$, where $A$ is a perfect even right $R$-module. This is determined by a map $\Sigma^{k} A^{\vee} \rightarrow M$ from the dual. As $A^{\vee}$ is perfect even and $k$ is odd, by assumption this map factors as

$$
\Sigma^{k} A^{\vee} \rightarrow B \rightarrow M
$$

where $B$ is perfect even. By dualizing again, we obtain a commutative diagram


Since $B$ is perfect even, the spectrum $\pi_{*}\left(E \otimes_{R} B\right)$ is concentrated in even degree by Proposition 4.12. We deduce that the given map vanishes in $\pi_{k}\left(E \otimes_{R} M\right)$, too.
$(3 \Rightarrow 4)$ and $(4 \Rightarrow 5)$ are clear. $(5 \Rightarrow 4)$ follows immediately from Proposition 4.10.
$(4 \Rightarrow 2)$. Let $\Sigma^{k} A \rightarrow M$ be a map from an odd suspension of a perfect even spectrum, which we can identify with a map of spectra $S^{k} \rightarrow A^{\vee} \otimes_{R} M$. Let $A^{\vee} \rightarrow E$ be a $\pi_{*}$-even envelope in right $R$-modules which exists by Proposition 4.9 and which we can complete to a cofibre sequence.

$$
A^{\vee} \rightarrow E \rightarrow C
$$

By assumption, $\pi_{*}\left(E \otimes_{R} M\right)$ is even, so that the map $S^{k} \rightarrow A^{\vee} \otimes_{R} M$ lifts to $S^{k} \rightarrow\left(\Sigma^{-1} C\right) \otimes_{R} M$. As $C$ is even flat, we obtain a factorization

$$
S^{k} \rightarrow \Sigma^{-1} B \otimes_{R} M
$$

where $B$ is perfect even. Dualizing, this gives a factorization of $\Sigma^{k} A \rightarrow M$ through the perfect even $\Sigma^{k+1} B$, which is what we wanted to show.

Remark 4.15. Note that it follows from characterizations (2) of Theorem 4.14 and Proposition 4.1 that any even flat module is homologically even. Alternatively, we can also observe that $\mathcal{F}_{-}(-1 / 2)$ commutes with filtered colimits and vanishes on perfect evens by Lemma 2.17.
Warning 4.16. Beware that the implication of Remark 4.15 cannot be reversed in general: that is, not every homologically even $R$-module is even flat. For a specific example, by Proposition 4.18 below, $\mathbb{Z} / p$ is homologically even as a $\mathbb{Z}$-module in spectra, but it is not even flat.

Remark 4.17. Note that parts (2) of Proposition 4.12 and part (3) of Theorem 4.14 can be combined in the following elegant manner: the homotopy groups $\pi_{*}\left(E \otimes_{R} M\right)$ of a tensor product of a $\pi_{*}$-even right $R$-module $E$ and a homologically even left $R$-module $M$ are concentrated in even degree if at least one of the two is even flat. This mimics the classical observation that for $\operatorname{Tor}_{\mathbb{Z}}^{1}(A, B)$ to vanish it is enough for one of the $A$ or $B$ to be a flat as an abelian group.

### 4.4. Evenness over connective and homotopy even rings

In this section, we describe the various notions of evenness over rings which are connective or have homotopy groups concentrated in even degrees.

Proposition 4.18. Let $R$ be an $\mathbf{E}_{1}$-ring such that $\pi_{*} R$ is concentrated in even degrees. Then an $R$-module $M$
(1) is even flat if and only if $\pi_{*} M$ is a flat $\pi_{*} R$-module concentrated in even degrees,
(2) is homologically even if and only if $\pi_{*} M$ is concentrated in even degrees.

Proof. In both cases, the forward implication is clear. We begin with backwards implication for (1), so suppose that $\pi_{*} M$ is flat and concentrated in even degrees. By Proposition 4.12, it is enough to show that for every right $R$-module $E$ with homotopy in even degrees, the same is true for $\pi_{*}\left(E \otimes_{R} M\right)$. However, by [Lur, 7.2.1.19] we have a conditionally convergent Künneth spectral sequence

$$
\operatorname{Tor}_{\pi_{*}}^{*, *}\left(\pi_{*} E, \pi_{*} M\right) \Rightarrow \pi_{*}\left(E \otimes_{R} M\right)
$$

Observe that the left hand side is concentrated in Tor-degree zero as $\pi_{*} M$ is flat. As it is concentrated in even internal degree by assumption, we deduce that the same is true for the right hand side, which is what we wanted to show.

The backwards implication for (2) is identical, using characterization (3) of Theorem 4.14.
Theorem 4.19. Let $R$ be a connective $\mathbf{E}_{1}$-ring and let us write $R_{\leq 0}:=\pi_{0} R$, considered as an $\mathbf{E}_{1}$-algebra in spectra. Then the following are equivalent for an $R$-module $M$ :
(1) $M$ is even flat,
(2) $R_{\leq 0} \otimes_{R} M$ is even flat as a $R_{\leq 0}$-module,
(3) $\pi_{*}\left(R_{\leq 0} \otimes_{R} M\right)$ is a flat $\pi_{0} R$-module, concentrated in even degrees.

Proof. Observe that $(1 \Rightarrow 2)$ is clear, since even flat modules are closed under base-change, and $(2 \Leftrightarrow 3)$ follows from Proposition 4.18. Thus, we only have to show $(2 \Rightarrow 1)$.

Suppose that $R_{\leq 0} \otimes_{R} M$ is even flat as a $R_{\leq 0}$-module. By Proposition 4.12, it is enough to verify that for every connective $R$-module $E$ with homotopy in even degrees, $\pi_{*}\left(E \otimes_{R} M\right)$ is also concentrated in even degrees. As the truncation map $E \rightarrow \tau_{\leq 2 n} E$ induces an isomorphism

$$
\pi_{*}\left(E \otimes_{R} M\right) \rightarrow \pi_{*}\left(E \otimes_{R} M\right)
$$

in degrees $* \leq 2 n$, we can assume that $E$ is also bounded from above. In this case, it admits a finite filtration whose subquotients are even suspensions objects of $N \in \mathcal{M} o d_{R}^{\varrho} \simeq \mathcal{M} o d_{R_{\leq 0}}^{\varrho}$. In this case, we can write

$$
N \otimes_{R} M \simeq N \otimes_{R_{\leq 0}} \otimes\left(R_{\leq 0} \otimes_{R} M\right)
$$

and as the right hand side has even homotopy groups by assumption, so does the left hand side. By a filtration argument we deduce that the same is true for $\pi_{*}\left(E \otimes_{R} M\right)$.

Warning 4.20. Beware that the criterion of Theorem 4.19 does not apply to homological even modules; in fact, the base-change along $R \rightarrow R_{\leq 0}$ need not even preserve homological evens. For a specific counterexample, let $k$ be a field and $k[x]$ a free $\mathbf{E}_{1}$-algebra on a variable in degree two, so that $\pi_{*}(k[x])$ is a polynomial ring. Then

$$
k \simeq \operatorname{cofib}\left(x: \Sigma^{2} k[x] \rightarrow k[x]\right)
$$

is homologically even as a $k[x]$-module as a consequence of Proposition 4.18, but $k \otimes_{k[x]} k \simeq k \oplus \Sigma^{3} k$ is not homologically even over $k$.

## 5. Homological Resolutions

In this section, we describe how even cohomology of an $R$-module $M$ can be calculated by resolving it through $R$-modules with even homotopy groups.

Remark 5.1 (Adams resolutions). The various constructions described in this section are standard in homotopy theory; essentially, we study a particular class of Adams resolutions with respect to the homological functor $\mathcal{F}_{-}: \mathcal{M o d}_{R}(\mathcal{S} p) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A} b\right)$. For more background on Adams resolutions, see [Ada95, §III.15] or [PP21, §2.2]. Since our aim is for the paper to be accessible to a possibly large audience, we do not assume any knowledge of Adams resolutions or the associated Adams spectral sequence.

Observe that if $M_{1} \rightarrow M_{2} \rightarrow M_{3}$ is a cofibre sequence of homologically even $R$-modules, then the long exact sequence of even sheaves shows that

$$
0 \rightarrow \mathcal{F}_{M_{1}}(q) \rightarrow \mathcal{F}_{M_{2}}(q) \rightarrow \mathcal{F}_{M_{2}}(q) \rightarrow 0
$$

is short exact for any $q \in \mathbb{Z}$. Applying $\mathrm{H}_{e v}^{*}(R,-)$ we obtain a long exact sequence of even cohomology

$$
\ldots \rightarrow \mathrm{H}_{e v}^{p-1, q}\left(R, M_{1}\right) \rightarrow \mathrm{H}_{e v}^{p, q}\left(R, M_{1}\right) \rightarrow \mathrm{H}_{e v}^{p, q}\left(R, M_{2}\right) \rightarrow \mathrm{H}_{e v}^{p, q}\left(R, M_{3}\right) \rightarrow \mathrm{H}_{e v}^{p+1, q}\left(R, M_{1}\right) \rightarrow \ldots
$$

Since even cohomology of $\pi_{*}$-even modules is particularly simple by Corollary 2.28 , it follows that to compute cohomology with coefficients in an $R$-module $M$, one can map it into a $\pi_{*}$-even module, which can be done in a particularly nice way as studied in $\S 4.2$. One elementary way to exploit this idea was described in the introduction, and we do so again here:

Construction 5.2. Let $M$ be homologically even. Choose a $\pi_{*}$-envelope $M \rightarrow E_{0}$, which can be done by Proposition 4.9, and set $C_{0}:=\operatorname{cofib}\left(M \rightarrow E_{0}\right)$. Inductively letting $C_{i} \rightarrow E_{i+1}$ be a $\pi_{*}$-even envelope and setting $C_{i+1}:=\operatorname{cofib}\left(C_{i} \rightarrow E_{i}\right)$ leads to a diagram of $R$-modules

where in the upper row the composite of any two maps is null.
Proposition 5.3. If $M$ is homologically even and (5.1) is a diagram as in Construction 5.2, then there is a canonical isomorphism

$$
\mathrm{H}_{e v}^{p, q}(R, M) \simeq \mathrm{H}^{p}\left(\pi_{2 q} E_{\bullet}\right),
$$

between the $(p, q)$-th even cohomology of $M$ and the cohomology of the cochain complex

$$
\pi_{2 q} E_{0} \rightarrow \pi_{2 q} E_{1} \rightarrow \pi_{2 q} E_{2} \rightarrow \ldots
$$

computed in the p-th spot.
Proof. By construction, all of these $R$-modules are homologically even, so that for each $q \in \mathbb{Z}$ and $i \geq-1$ we have a long exact sequence

$$
\ldots \rightarrow \mathrm{H}_{e v}^{p-1, q}\left(R, C_{i+1}\right) \rightarrow \mathrm{H}_{e v}^{p, q}\left(R, C_{i}\right) \rightarrow \mathrm{H}_{e v}^{p, q}\left(R, E_{i}\right) \rightarrow \mathrm{H}_{e v}^{p, q}\left(R, C_{i+1}\right) \rightarrow \ldots
$$

where $C_{-1}:=M$. Splicing these together leads to an exact couple and hence a spectral sequence

$$
\mathrm{H}_{e v}^{*, q}\left(E_{\bullet}\right) \Rightarrow \mathrm{H}_{e v}^{*, q}(M) .
$$

By Corollary 2.28, the first page is concentrated in cohomological degree zero, where we have $\mathrm{H}_{e v}^{0, q}\left(E_{\bullet}\right) \simeq \pi_{2 q} E_{\bullet}$. It follows that the spectral sequence collapses on the second page, inducing the needed isomorphism.

While Proposition 5.3 gives an effective way of calculating even cohomology, it is often convenient to have a more refined version of Construction 5.2, based on cosimplicial objects. This has the advantage that it gives a limit description of the even filtration itself, rather than just the calculation of the even cohomology.

Definition 5.4. We say that an augmented semicosimplicial object $X: \Delta_{s,+} \rightarrow \operatorname{M}_{\operatorname{Mod}}^{R}$ is a homological resolution if for every $q \in 1 / 2 \mathbb{Z}$, the induced Moore cochain complex

$$
0 \rightarrow \mathcal{F}_{X_{-1}}(q) \rightarrow \mathcal{F}_{X_{0}}(q) \rightarrow \mathcal{F}_{X_{1}}(q) \rightarrow \ldots
$$

is exact as a complex in the abelian category $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A} b\right)$.
Remark 5.5. Let $\mathcal{A}$ be a Grothendieck abelian category, and suppose that we have an augmented semicosimplicial diagram $C: \Delta_{s,+} \rightarrow \mathcal{A}$. Then the following two conditions are equivalent:
(1) the associated Moore cochain complex $0 \rightarrow C_{-1} \rightarrow C_{0} \rightarrow C_{1} \rightarrow \ldots$ is exact,
(2) the composite $i \circ C: \Delta_{s,+} \rightarrow \mathcal{D}(\mathcal{A})$, where $i$ is the inclusion of the heart of the derived $\infty$-category, is a limit diagram.
It follows that a diagram $X: \Delta_{s,+} \rightarrow \mathcal{M} o d_{R}$ is a homological resolution if and only if for each weight $q$

$$
\mathcal{F}_{X_{-1}}(q) \simeq \lim _{\rightleftarrows} \mathcal{F}_{X_{m}}(q)
$$

in the derived $\infty$-category of additive sheaves, where the limit is taken over $[m] \in \Delta_{s}$.
The usefulness of homological resolutions comes down to the following simple observation:
Proposition 5.6. Let $X: \Delta_{s,+} \rightarrow \mathcal{M o d}_{R}$ be a homological resolution. Then we have an equivalence of graded spectra

$$
\operatorname{gr}_{e v}^{*}\left(X_{-1}\right) \simeq \lim _{\leftrightarrows} \operatorname{gr}_{e v}^{*}\left(X_{m}\right),
$$

where the limit is taken over $[m] \in \Delta_{s}$. Thus, the canonical map of filtered spectra

$$
\operatorname{fil}_{e v}^{*}\left(X_{-1}\right) \rightarrow \lim _{\check{L}}\left(\mathrm{fil}_{e v}^{*}\left(X_{m}\right)\right)
$$

is an equivalence after completion.
Proof. Using Remark 2.24, for any homologically even $M$ we have an identification

$$
\operatorname{gr}_{e v}^{n}(M) \simeq \operatorname{Map}_{\mathcal{D}\left(\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v}, \mathcal{A} b\right)\right)}\left(\mathcal{F}_{R}, \Sigma^{2 n} \mathcal{F}_{M}(n)\right)
$$

Thus, the statement is immediate from Remark 5.5. For an arbitrary $M$, we instead have a fibre sequence of spectra

$$
\operatorname{Map}\left(\mathcal{F}_{R}, \Sigma^{2 n+1} \mathcal{F}_{M}(n+1 / 2)\right) \rightarrow \operatorname{gr}_{e v}^{n}(M) \rightarrow \operatorname{Map}\left(\mathcal{F}_{R}, \Sigma^{2 n} \mathcal{F}_{M}(n)\right)
$$

and the result follows in the same way.

As in the context of Proposition 5.3, homological resolutions are most useful when they consists of modules with only even homotopy groups.

Theorem 5.7. Any homologically even $R$-module $M$ can be completed to a homological resolution $X: \Delta_{s,+} \rightarrow \operatorname{Mod}_{R}$ with $X_{-1}=M$ and $\pi_{*} X_{m}$ even for $m \geq 0$.
Proof. We recall that augmented semicosimplicial objects can be constructed inductively, by choosing appropriate maps out of matching objects, see [Lur09, A.2.9.15, A2.9.16].

We define $X_{-1}:=M$, so that the 0 -th matching object is given by $M_{0} X \simeq X_{-1} \simeq M$. By Proposition 4.9, there exists a $\pi_{*}$-even envelope

$$
M_{0} X \rightarrow E_{0}
$$

and we set $X_{0}:=E_{0}$. This extends $X$ to a diagram indexed by $\Delta_{s,+, \leq 0}$, so that the matching object $M_{1} X$ is well-defined. We then let $X_{1}$ be a $\pi_{*}$-even envelope of $M_{1} X$, extending our diagram to $\Delta_{s,+, \leq 1}$. Proceeding inductively in this manner, we obtain an augmented semicosimplicial object which by construction has $\pi_{*} X_{m}$ even for $m \geq 0$. We will show that it is a homological resolution.

We first argue by induction that $M_{m} X$ is homologically even for all $m \geq 0$. The base-case is clear, since $M_{0} X \simeq M$, so suppose that we know that each of $M_{k} X$ is homologically even for $k<m$. As by construction, the map $M_{k} X \rightarrow X_{k}$ has homologically even cofibre, this will also show that for in this range $X_{k}$ is homologically even and $\mathcal{F}_{M_{k} X} \rightarrow \mathcal{F}_{X_{k}}$ is a monomorphism.

The long exact sequence of homology shows that homologically even modules are closed under pushouts along $\mathcal{F}_{-}$-monomorphisms, and that on this subcategory the association $M \mapsto \mathcal{F}_{M}$ commutes with such pushouts. It follows from [Pst22, A.2: Proposition 5 and Remark 4] and the inductive step that $M_{m} X$ is homologically even and $\mathcal{F}_{M_{m} X} \simeq M_{m}\left(\mathcal{F}_{X}\right)$ is a monomorphism. The first conclusions ends the induction.

Since the cofibre of $M_{m} X \rightarrow X_{m}$ is homologically even, the second conclusion from the previous paragraph shows that

$$
M_{m}\left(\mathcal{F}_{X}\right) \simeq \mathcal{F}_{M_{m} X} \rightarrow \mathcal{F}_{X_{m}}
$$

is a monomorphism for all $m \geq 0$. If follows that if $I$ is an injective cogenerator of $\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}_{e v} ; \mathcal{A} b\right)$, then $\operatorname{Hom}\left(\mathcal{F}_{X}, I\right)$ defines a hypercover in $\mathcal{D}_{\geq 0}(\mathbb{Z})$ and hence its Moore complex is exact. We deduce that the Moore complex of $\mathcal{F}_{X}$ itself is exact. Since the integral weight twists can be obtained from $\mathcal{F}_{-}$using Remark 2.14 and the half-weight twists vanish, we deduce the exactness of the Moore complex of $\mathcal{F}_{X}(q)$ for all $q$.

Remark 5.8. Suppose that $M$ is homologically even, so that by Theorem 5.7 we can complete it to a homological resolution $X: \Delta_{s,+} \rightarrow \mathcal{M} o d_{R}$ such that $\pi_{*} X_{m}$ is even for all $m \geq 0$. Then by a combination of Proposition 5.6 and Lemma 2.27 we see that the canonical map

$$
\mathrm{fil}_{e v}^{*} M \rightarrow \underset{\longleftarrow}{\lim } \tau_{\geq 2 n} X_{m}
$$

is an equivalence after completion. Here, the right hand side is Deligne's décalage of the cosimplicial spectrum $X$, see [Del71].

## 6. Base-change and descent

Suppose that $R$ is an $\mathbf{E}_{2}$-ring and $S$ is an $\mathbf{E}_{1}-R$-algebra, so that we have an associated cobar diagram of $\mathbf{E}_{1}$-rings

$$
R \longrightarrow S \Longrightarrow S \otimes_{R} S \Longrightarrow \ldots,
$$

where each coboundary map is induced by the unit $R \rightarrow S$, see [MNN17, Construction 2.7]. In this context, it is natural to ask if the even filtration associated to $R$ can be recovered from the even filtration of $\mathbf{E}_{1}$-rings $S^{\otimes_{R} n}$, perhaps up to some form of completion.

Warning 6.1. We will think of the cobar diagram as an augmented semicosimplicial diagram $\Delta_{s,+} \rightarrow \operatorname{Alg}_{\mathbf{E}_{1}}(\delta p)$. While it can be naturally extended to a cosimplicial diagram, the coboundary maps do not respect multiplication and so the resulting extension is only a diagram of spectra.

As the notion of homological resolution of Definition 5.4 gives us some natural conditions under which a semicosimplicial diagram of $R$-modules induces a limit on even filtrations, the main problem to tackle is how the even filtration of $S^{\otimes_{R} n}$ considered as a module over itself differs from the one where we consider it as a module over $R$.

### 6.1. Evenness and extension/restriction of scalars

Associated to a morphism $f: R \rightarrow S$ of $\mathbf{E}_{1}$-rings we have an adjunction

$$
S \otimes_{R}-\dashv R_{S . R}: \mathcal{M o d}_{R} \rightleftarrows{\mathcal{M} o d_{S}}
$$

where $R_{S . R}$ denotes the forgetful functor, and it is natural to ask how these two functors relate to the various notions introduced in the present work.
Lemma 6.2. The functor $S \otimes_{R}-: \mathcal{M}_{\operatorname{Mod}}^{R} \boldsymbol{\rightarrow} \rightarrow \mathcal{M}$ od ${ }_{S}$ preserves perfect even and even flat modules.
Proof. This is clear, since $S \otimes_{R} R \simeq S$ and both classes are defined as closure of the unit under certain kinds of colimits.

Remark 6.3. Note that we had seen in Warning 4.20 that the extension of scalars does not in general preserve homologically even modules.

To get further properties, we need to make some assumptions on the map. Compatibility with the even filtration can be thought of as a form of exactness, so as motivation, let us analyze the classical situation, when $f: S \rightarrow R$ is a map of classical rings. In this case we have an induced extension/restriction of scalars adjunction between the categories of modules in abelian groups

$$
\operatorname{Tor}_{R}^{0}(S,-) \dashv R_{S, R}: \mathcal{M} o d_{R}(\mathcal{A} b) \rightleftarrows \mathcal{M}^{( } o d_{S}(\mathcal{A} b)
$$

where we write $\operatorname{Tor}_{R}^{0}(S,-)$ to emphasize that here we mean the classical rather than derived tensor product. In this case
(1) the restriction of scalars $R_{S, R}$ is always exact, as (co)limits in either $R$ or $S$-modules can be both calculated in abelian groups,
(2) the extension of scalars $\operatorname{Tor}_{R}^{0}(S,-)$ is exact precisely when $S$ is flat as a right $R$-module.

In the context of $\mathbf{E}_{1}$-rings, the behaviour of extension of scalars is somewhat similar to the one above. However, the situation with the forgetful functor is more subtle, as the even filtration varies with the ring. In particular, whether something is "exact with respect to the even filtration" depends on more than the underlying spectra. Instead, the behaviour of the forgetful functor depends on the structure of $S$ as a left $R$-module. Thus, to get the best of both worlds we are forced to think of $S$ as both a left and right $R$-module.

Definition 6.4. We say that a map $f: R \rightarrow S$ of $\mathbf{E}_{1}$-rings is
(1) left even flat if $S$ is even flat as a left $R$-module,
(2) right even flat if $S$ is even flat as a right $R$-module,
(3) left homologically even if $S$ is homologically even as a left $R$-module.
(4) left homologically even if $S$ is homologically even as a right $R$-module.

Remark 6.5. If $R$ is $\mathbf{E}_{2}$ and $f: R \rightarrow S$ can be promoted to a unit map of an $\mathbf{E}_{1}$ - $R$-algebra structure, then $f$ is left even flat if and only if it is right even flat, and similarly for homological evenness. In particular, this happens whenever $f$ can be promoted to a map of $\mathbf{E}_{2}$-rings.

Lemma 6.6. Let $f: R \rightarrow S$ be a map of $\mathbf{E}_{1}$-rings. Then the forgetful functor $\mathcal{M}_{\operatorname{cod}}^{S} \rightarrow\left(\mathcal{M} o d_{R}\right.$
(1) preserves even flat modules if $f$ is left even flat and
(2) preserves homologically even modules if $f$ is left homologically even.

Proof. The first part is clear, since the forgetful functor sends the unit $S$ to an even flat module by assumption, and even flat modules are defined as a closure of the unit under various colimits which commute with the forgetful functor.

For the second, by Theorem 4.14 it is enough to show that if $N$ is a homologically even $S$ module and $E$ is a $\pi_{*}$-even, even flat right $R$-module, then $\pi_{*}\left(E \otimes_{R} N\right)$ is concentrated in even degrees. We can rewrite this tensor product as

$$
E \otimes_{R} N \simeq\left(E \otimes_{R} S\right) \otimes_{S} N
$$

Note that $E \otimes_{R} S$ is even flat as a right $S$-module by Lemma 6.2 and $\pi_{*}\left(E \otimes_{R} S\right)$ is concentrated in even degrees by Proposition 4.12 and the assumption that $S$ is homologically even as a left $R$-module. Thus, $\pi_{*}\left(\left(E \otimes_{R} S\right) \otimes_{S} N\right)$ is even, as needed.

Lemma 6.7. Let $f: R \rightarrow S$ be right even flat. Then $S \otimes_{R}-: \mathcal{M o d}_{R} \rightarrow \mathcal{M}_{\operatorname{Lod}}^{S}$ preserves homologically even modules.

Proof. Let $M$ be a homologically even $R$-module. By Theorem 4.14, to show that $S \otimes_{R} M$ is homologically even, it is enough to verify that if $F$ is a $\pi_{*}$-even, even flat right $S$-module, then $\pi_{*}\left(F \otimes_{S} S \otimes_{R} M\right)$ is concentrated in even degrees. We have

$$
F \otimes_{S} S \otimes_{R} M \simeq F \otimes_{R} M
$$

and the right hand side has homotopy groups concentrated in even degrees since $F$ is even flat as a right $R$-module by (the right module variant of) Lemma 6.6.

We will also need the following variant for bimodules:
Lemma 6.8. Let $B$ be an R-bimodule which is homologically even as a left $R$-module and even flat as a right $R$-module. Then the functor

$$
B \otimes_{R}-:{\mathcal{M} o d_{R}} \rightarrow \mathcal{M}_{\operatorname{Mod}}^{R}
$$

Proof. This is a combination of the arguments of Lemma 6.6 and Lemma 6.7.
Warning 6.9. Beware that if $f: R \rightarrow S$ is merely right homologically even rather than even flat, then $S \otimes_{R}-: \mathcal{M} o d_{R} \rightarrow \mathcal{M} o d_{S}$ need not preserve homologically even modules. For a specific counterexample, consider the unit map $\eta: S^{0} \rightarrow \mathbb{Z} / p$.

As MU admits an even cell structure and has even homotopy groups, the unit $S^{0} \rightarrow \mathrm{MU}$ is easily seen to be a $\pi_{*}$-even envelope in spectra and from Theorem 4.14 we deduce that a spectrum $M$ is homologically even if and only if $\mathrm{MU}_{*} M$ is concentrated in even degrees. Since

$$
\mathrm{MU}_{*}(\mathbb{Z} / p) \simeq \mathbb{Z} / p\left[b_{1}, b_{2}, \ldots\right]
$$

is a polynomial algebra concentrated in even degrees, we see that $\mathbb{Z} / p$ is homologically even as a spectrum. However, the base-change $\mathbb{Z} / p \otimes_{S^{0}} \mathbb{Z} / p$ is not homologically even as a $\mathbb{Z} / p$-module as a consequence of Proposition 4.18, since the dual Steenrod algebra

$$
\mathcal{A}_{*} \simeq \pi_{*}\left(\mathbb{Z} / p \otimes_{S^{0}} \mathbb{Z} / p\right)
$$

is not concentrated in even degrees.
Notation 6.10. Since we are interested in comparing even filtrations relative to different rings, we will sometimes write

$$
\mathrm{fil}_{e v / R}^{*}(M):=\operatorname{fil}_{e v}^{*}(M)
$$

to emphasize that to calculate this even filtration we consider $M$ as an $R$-module.
We record that there is a canonical map comparing even filtrations over different rings.

Construction 6.11. Let $f: R \rightarrow S$ be a map of $\mathbf{E}_{1}$-rings and let $N$ be an $S$-module. We will construct a canonical map of filtered spectra

$$
\mathrm{fil}_{e v, R}^{*}(N) \rightarrow \mathrm{fil}_{e v, S}^{*}(N),
$$

which after passing to homotopy groups of the associated graded object induces a map

$$
\mathrm{H}_{e v}^{p, q}(R, N) \rightarrow \mathrm{H}_{e v}^{p, q}(S, N)
$$

of even cohomology groups.
By Lemma 6.2, extensions of scalars restricts to a functor

$$
S \otimes_{R}-: \operatorname{Perf}(R)_{e v} \rightarrow \operatorname{Perf}(R)_{e v}
$$

This functor preserves cofibre sequences, so that it is a morphism of sites with respect to the even epimorphism Grothendieck topology. It follows that we have an induced adjunction

$$
f^{*} \dashv f_{*}: \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right) \rightleftarrows \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(S)_{e v}, \mathcal{S} p\right)
$$

where $f^{*}$ is the left Kan extension and $f_{*}$ is given by precomposition. Since the latter preserves levelwise coconnective sheaves, it is right $t$-exact.

Let us write $Y_{S}(N)$ for the representable presheaf of spectra on $\operatorname{Perf}(S)_{e v}$ and analogously $Y_{R}(N)$ for the presheaf on $\operatorname{Perf}(R)_{e v}$. Since
$\Gamma\left(A, f_{*} Y_{S}(N)\right) \simeq \Gamma\left(S \otimes_{R} A, Y_{S}(N)\right) \simeq \operatorname{Map}_{\mathcal{M o d}_{S}}\left(S \otimes_{R} A, N\right) \simeq \operatorname{Map}_{\mathcal{M o d}_{R}}(A, N) \simeq \Gamma\left(A, Y_{R}(N)\right)$ we have a functorial equivalence

$$
f_{*} Y_{S}(N) \simeq Y_{R}(N)
$$

Consider the cofibre sequence

$$
f_{*}\left(\tau_{\geq 2 n} Y_{S}(N)\right) \rightarrow f_{*}\left(Y_{S}(N)\right) \rightarrow f_{*}\left(\tau_{\leq 2 n-1} Y_{S}(N)\right)
$$

Since the last term is $2 n-1$-coconnective as $f_{*}$ is right $t$-exact, the canonical map

$$
\tau_{\geq 2 n}\left(Y_{R}(N)\right) \simeq \tau_{\geq 2 n}\left(f_{*} Y_{S}(N)\right) \rightarrow f_{*} Y_{S}(N)
$$

lifts uniquely to a morphism

$$
\tau_{\geq 2 n}\left(Y_{R}(N)\right) \rightarrow f_{*}\left(\tau_{\geq 2 n} Y_{S}(N)\right)
$$

Passing to sections over $R \in \operatorname{Perf}(R)_{e v}$ gives the required morphism of filtered spectra.
Construction 6.12. Suppose that $f: R \rightarrow S$ is a map of $\mathbf{E}_{1}$-rings and that $M$ is an $R$-module. Then we can consider the composite

$$
\mathrm{fil}_{e v / R}^{*}(M) \rightarrow \mathrm{fil}_{e v / R}^{*}\left(S \otimes_{R} M\right) \rightarrow \mathrm{fil}_{e v / S}^{*}\left(S \otimes_{R} M\right)
$$

where the first map is induced by the unit map $M \rightarrow S \otimes_{R} M$ of $R$-modules and the second is that of Construction 6.11. Passing to homotopy groups of the associated graded, this yields a canonical base-change map

$$
\mathrm{H}_{e v}^{p, q}(R, M) \rightarrow \mathrm{H}_{e v}^{p, q}\left(S, S \otimes_{R} M\right)
$$

in even cohomology.
Remark 6.13. Observe that after passing to colimits the comparison map

$$
\begin{equation*}
\xrightarrow[\longrightarrow]{\lim \mathrm{fil}_{e v, R}^{*}}(N) \rightarrow \underset{\longrightarrow}{\lim } \mathrm{fil}_{e v, S}^{*}(N) \tag{6.1}
\end{equation*}
$$

of Construction 6.11 can be identified with the identity $N \rightarrow N$. It follows that it is an equivalence if and only if it is an equivalence between associated graded; that is, when it induces an isomorphism

$$
\mathrm{H}_{e v}^{*, *}(R, N) \simeq \mathrm{H}_{e v}^{*, *}(S, N)
$$

between even cohomology groups. Indeed, if the latter holds, then the cofibre of the comparison map is a filtered spectrum whose associated graded object vanishes (hence it is a constant filtered spectrum) and whose colimit is zero.

Lemma 6.14. Let $f: R \rightarrow S$ be an left homologically even map of $\mathbf{E}_{1}$-rings. Then

$$
S \otimes_{R}-: \operatorname{Perf}(R)_{e v} \rightarrow \operatorname{Perf}(S)_{e v}
$$

has the covering lifting property with respect to the Grothendieck topologies of even epimorphisms.
Proof. We have to show that any pair of $A \in \operatorname{Perf}(R)_{e v}$ and an even epimorphism $q: M \rightarrow S \otimes_{R} A$ in $\operatorname{Perf}(S)_{e v}$ can be completed to a commutative diagram

where $p$ is an even epimorphism of perfect even $R$-modules.
Since the fibre fib $(q)$ is homologically even as an $R$-module by Lemma 6.6, long exact sequence of homology shows that the map $\mathcal{F}_{M} \rightarrow \mathcal{F}_{S \otimes_{R} A}$ is an epimorphism of sheaves on $\operatorname{Shv}{ }_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{A} b\right)$. It follows that that there exists an even epimorphism $p: B \rightarrow A$ of perfect even $R$-modules which can be completed to a commutative diagram


The induced map $S \otimes_{R} p: S \otimes_{R} B \rightarrow S \otimes_{R} A$ factors through $M$, as needed.
Theorem 6.15. Let $f: R \rightarrow S$ be a homologically even map of $\mathbf{E}_{1}$-rings. Then for any $S$-module $N$ the canonical comparison map

$$
\mathrm{fil}_{e v, R}^{*}(N) \rightarrow \mathrm{fil}_{e v, S}^{*}(N)
$$

of Construction 6.11 is an equivalence. In particular,

$$
\mathrm{H}_{e v}^{*, *}(R, N) \simeq \mathrm{H}_{e v}^{*, *}(S, N)
$$

Proof. Since $S \otimes_{R}: \operatorname{Perf}(R)_{e v} \rightarrow \operatorname{Perf}(S)_{e v}$ has the covering lifting property by Lemma 6.14 , the precomposition functor

$$
f_{*}: \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(S)_{e v}, \mathcal{S} p\right) \rightarrow \operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{S} p\right)
$$

is left $t$-exact by [Pst22, Remark 2.23]. It follows that the canonical map

$$
\tau_{\geq 2 n} Y_{R}(N) \rightarrow f^{*}\left(\tau_{\geq 2 n} Y_{S}(N)\right)
$$

used in Construction 6.11 is an equivalence.

### 6.2. Faithfully flat descent

In Theorem 6.15 , we had shown that if $f: R \rightarrow S$ is a homologically even map of $\mathbf{E}_{1}$-rings, then for any $S$-module its even filtration over $S$ agrees with the one relative to $R$. This shows that information can be moved "up" along a map of rings; that is, what happens over $S$ is already determined by $R$.

More commonly, we instead want to move information "down"; that is, to deduce results about $R$-modules from their base-change $S \otimes_{R}-$. As Grothendieck's theory of descent shows, this usually requires some variant of faithful flatness. As we discussed in $\S 6.1$, in the context of even filtration this requires some control over $S$ as both a left and right $R$-module.

Definition 6.16. We say that a map $f: R \rightarrow S$ of $\mathbf{E}_{1}$-rings is (left) faithfully even flat if both $S$ and cofib $(f)$ are even flat as right $R$-modules and homologically even as left $R$-modules.

Remark 6.17. The motivation for Definition 6.16 is given by the following classical observation: a monomorphism $R \hookrightarrow S$ of (discrete) commutative rings is faithfully flat if and only if both $S$ and $\operatorname{coker}(R \rightarrow S)$ are flat as $R$-modules, see [HP22b, Addendum 3.9].

Remark 6.18. Note that there is an obvious "opposite" notion of a right faithfully even flat morphism, which would be relevant in the context of working with right $R$-modules.

Note that our definition of faithfully even flat is distinct from the notion of "evenly faithfully flat" given by Hahn-Raksit-Wilson in the context of $\mathbf{E}_{\infty}$-rings [HRW22, 2.2.13]. In this paper, by faithfully even flat we will always mean the notion introduced in Definition 6.16. The following shows that they agree on connective $\mathbf{E}_{\infty}$-rings:

Proposition 6.19. A map $R \rightarrow S$ of $\mathbf{E}_{\infty}$-rings which is faithfully even flat in the sense of Definition 6.16 is also evenly faithfully flat in the sense of Hahn-Raksit-Wilson; that is, for every map $R \rightarrow E$ into a $\pi_{*}$-even $\mathbf{E}_{\infty}$-ring, the base-change $E \otimes_{R} S$ is also $\pi_{*}$-even and the map $\pi_{*}(E) \rightarrow \pi_{*}\left(E \otimes_{R} S\right)$ of classical commutative rings is faithfully flat. If $R$ is connective, the converse holds as well.

Proof. Since by Proposition 4.12 a tensor product of an even flat module and a $\pi_{*}$-even module is $\pi_{*}$-even, if $f: R \rightarrow S$ is faithfully even flat then the sequence

$$
0 \rightarrow \pi_{*}(E) \rightarrow \pi_{*}\left(E \otimes_{R} S\right) \rightarrow \pi_{*}\left(E \otimes_{R} \operatorname{cofib}(f)\right) \rightarrow 0
$$

is short exact and concentrated in even degrees. As the middle and right terms are flat over $\pi_{*} E$ by Proposition 4.18, we deduce that the first map is a faithfully flat map of classical rings. It follows that $f$ is evenly faithfully flat in the sense of Hahn-Raksit-Wilson.

Conversely, suppose that $R$ is connective and that $f$ is evenly faithfully flat in the sense of Hahn-Raksit-Wilson. Let us write $R_{\leq 0} \simeq \pi_{0} R$ for the 0 -truncation. By assumption, the first map in the the sequence

$$
0 \rightarrow \pi_{*}\left(R_{\leq 0}\right) \rightarrow \pi_{*}\left(R_{\leq 0} \otimes_{R} S\right) \rightarrow \pi_{*}\left(R_{\leq 0} \otimes_{R} \operatorname{cofib}(f)\right) \rightarrow 0
$$

is faithfully flat, hence injective. It follows from that the sequence is short exact, concentrated in even degrees, and that the third term is also flat as $\pi_{*}\left(R_{\leq 0}\right) \simeq \pi_{0} R$-module. It follows from a combination of Proposition 4.18 and Theorem 4.19 that $S$ and $\operatorname{cofib}(f)$ are even flat as $R$-modules, which is what we wanted to show.

Warning 6.20. For non-connective $\mathbf{E}_{\infty}$-rings, a map which is evenly faithfully flat in the sense of [HRW22] need not be faithfully even flat in the sense of Definition 6.16, see Remark 7.8.

Lemma 6.21. Let $f: R \rightarrow S$ be a left faithfully even flat map of $\mathbf{E}_{1}$-rings. If $M$ is an $R$-module, then for any weight $q$ the sequence

$$
0 \rightarrow \mathcal{F}_{M}(q) \rightarrow \mathcal{F}_{S \otimes_{R} M}(q) \rightarrow \mathcal{F}_{\operatorname{cofib}(f) \otimes_{R} M}(q) \rightarrow 0
$$

of abelian sheaves on $\operatorname{Perf}(R)_{\text {ev }}$ is short exact.
Proof. Assume first that $M$ is homologically even or a suspension of one. In this case, so are $S \otimes_{R} M$ and $\operatorname{cofib}(f) \otimes_{R} M$ by Lemma 6.8. It follows that the above sequence is short exact.

In the case of general $M$, since $\mathcal{F}_{M}(q) \simeq \mathcal{F}_{\Sigma^{-2 q} M}$, it is enough to do the case when $q=0$. We want to prove that the boundary map

$$
\mathcal{F}_{\mathrm{cofib}(f) \otimes_{R} M} \rightarrow \mathcal{F}_{M}(-1 / 2)
$$

is zero. By taking an appropriately large direct sum of perfect evens, we can find a map $N \rightarrow M$ of $R$-modules where $N$ is perfect even and $\mathcal{F}_{N} \rightarrow \mathcal{F}_{M}$ is a surjection. It follows that the cofibre
$C$ is a suspension of a homological even and that $\mathcal{F}_{M}(-1 / 2) \rightarrow \mathcal{F}_{C}(-1 / 2)$ is a monomorphism. We thus have a commutative diagram


The bottom horizontal arrow is zero by the first paragraph. Since the right vertical arrow is a monomorphism by construction, we deduce that the top horizontal arrow is also zero, as needed.

Corollary 6.22. Let $f: R \rightarrow S$ be a left faithfully even flat. Then a map $M \rightarrow M^{\prime}$ of $R$-modules is a $\mathcal{F}_{-}$-monomorphism if and only if $S \otimes_{R} M \rightarrow S \otimes_{R} M^{\prime}$ is.

Proof. A map is an $\mathcal{F}_{-}$-monomorpism if and only if its cofibre is homologically even, so the forward direction follows from Lemma 6.8. For the converse, consider the commutative diagram

where both vertical arrows are monomorphisms by Lemma 6.21.
If $f: R \rightarrow S$ is a map of $\mathbf{E}_{1}$-rings, then the extension-restriction of scalars adjunction induces a monad on $R$-modules which we also denote by $S \otimes_{R}$-. It follows that every $R$-module $M$ determines an augmented cosimplicial diagram

$$
S^{\otimes_{R} \bullet} \otimes_{R} M: \Delta_{s,+} \rightarrow{\mathcal{M} o d_{R}}
$$

of the form

$$
M \longrightarrow S \otimes_{R} M \Longrightarrow S \otimes_{R} S \otimes_{R} M \Longrightarrow \cdots
$$

which we call the cobar resolution. In good cases, this is a limit diagram, giving a way to understand $M$ through its base-change. The associated spectral sequence

$$
\pi_{*}\left(S^{\otimes_{R} \bullet} \otimes_{R} M\right) \Rightarrow \pi_{*}(M)
$$

is called the Adams spectral sequence associated to $f$, or the descent spectral sequence.
Theorem 6.23 (Faithfully flat descent). Let $f: R \rightarrow S$ be an left faithfully even flat map of $\mathbf{E}_{1}$-rings and let $M$ be an $R$-module. Then the canonical map

$$
\operatorname{gr}_{e v / R}^{*}(M) \rightarrow \operatorname{gr}_{e v / R}^{*}\left(S^{\otimes_{R} \bullet} \otimes_{R} M\right)
$$

induced by the cobar resolution is an equivalence of graded spectra. Thus,

$$
\operatorname{fil}_{e v / R}^{*}(M) \rightarrow \lim _{\check{c} \mathrm{fil}_{e v / R}^{*}}\left(S^{\otimes_{R} \bullet} \otimes_{R} M\right)
$$

induced by the cobar resolution is an equivalence of filtered spectra after completion.
Proof. By Proposition 5.6, it is enough to verify that $S^{\otimes_{R} \bullet} \otimes_{R} M: \Delta_{s,+} \rightarrow \operatorname{Mod}_{R}$ is a homological resolution; that is, that the Moore cochain complex

$$
\begin{equation*}
0 \rightarrow \mathcal{F}_{M}(q) \rightarrow \mathcal{F}_{S \otimes_{R} M}(q) \rightarrow \mathcal{F}_{S \otimes_{R} S \otimes_{R} M}(q) \rightarrow \ldots \tag{6.2}
\end{equation*}
$$

is exact for any $q$. By replacing $M$ by a suitable (de)suspension we can assume that $q=0$. To avoid multiple tensor products in notation, it will be convenient to write $C:=S^{\otimes_{R} \bullet} \otimes_{R} M$ for the cobar resolution.

Let us instead consider the Moore cochain complex associated to the tensored up cobar resolution $S \otimes_{R} C$, of the form

$$
\begin{equation*}
0 \rightarrow \mathcal{F}_{S \otimes_{R} C_{-1}} \rightarrow \mathcal{F}_{S \otimes_{R} C_{0}} \rightarrow \mathcal{F}_{S \otimes_{R} C_{1}} \rightarrow \ldots \tag{6.3}
\end{equation*}
$$

By [Lur, 4.7.2.7], the tensored cobar resolution is split as an augmented cosimplicial object. It follows from [Lur, 4.7.2.4] it is an absolute limit; in particular, that it forms a limit in $\mathcal{D}\left(\operatorname{Shv}_{\Sigma}\left(\operatorname{Perf}(R)_{e v}, \mathcal{A} b\right)\right)$ after applying $\mathcal{F}_{-}$. Thus, by Remark 5.5, (6.3) is exact.

Let us go back to (6.2), the Moore complex of the cobar resolution itself, which we have to show is exact. Exactness in degree zero is to verify that the map $\mathcal{F}_{C_{-1}} \rightarrow \mathcal{F}_{C_{0}}$ is injective. This follows from Corollary 6.22 , as $\mathcal{F}_{S \otimes_{R} C_{-1}} \rightarrow \mathcal{F}_{S \otimes_{R} C_{0}}$ is a monomorphism since (6.3) is exact. We deduce that

$$
\mathcal{F}_{\mathrm{cofib}\left(C_{-1} \rightarrow C_{0}\right)} \simeq \operatorname{coker}\left(\mathcal{F}_{C_{-1}} \rightarrow \mathcal{F}_{C_{0}}\right),
$$

so to check exactness of (6.2) in degree one, it is enough to verify that the induced map

$$
\mathcal{F}_{\mathrm{cofib}\left(C_{-1} \rightarrow C_{0}\right)} \rightarrow \mathcal{F}_{C_{1}}
$$

is a monomorphism. This again follows from Corollary 6.22 and the exactness of (6.3). Proceeding inductively, we see that (6.2) is exact, as needed.

Note that in Theorem 6.23, all of the even filtrations considered are relative to $R$. This is necessary, since for a general map of $\mathbf{E}_{1}$-rings, the tensor products $S \otimes_{R} \ldots \otimes_{R} S$ do not have a natural ring structure if they involve more than one factor. Thus, if we want to consider a variant of faithfully flat descent where the ring varies, we need to assume more structure on our map.

Theorem 6.24. Let $R$ be an $\mathbf{E}_{2}$-ring and let $S$ be an $\mathbf{E}_{1}$ - $R$-algebra whose unit map is faithfully even flat as map of $\mathbf{E}_{1}$-rings. Then for any $R$-module $M$ the canonical map

$$
\mathrm{fil}_{e v / R}^{*}(M) \rightarrow \lim _{幺} \mathrm{fil}_{e v / S^{*} \otimes_{R}}^{*} \cdot\left(S^{\otimes_{R} \bullet} \otimes_{R} M\right)
$$

is an equivalence of filtered spectra after completion. In particular, this is true for

$$
\mathrm{fil}_{e v / R}^{*}(R) \rightarrow \lim _{\leftarrow} \mathrm{fil}_{e v / S^{\otimes_{R}} \bullet}^{*}\left(S^{\otimes_{R} \bullet}\right)
$$

Proof. Note that since $S$ is an $\mathbf{E}_{1}$ - $R$-algebra, the left and right $R$-module structures can be identified. It follows that it is even flat as both a left and right $R$-module.

Keeping in mind Theorem 6.23, we just have to show that for any $m \geq 0$ the canonical comparison map

$$
\mathrm{fil}_{e v / R}^{*}\left(S^{\otimes_{R} m} \otimes_{R} M\right) \rightarrow \mathrm{fil}_{e v / S^{*} \otimes_{R^{m}}}\left(S^{\otimes_{R} m} \otimes_{R} M\right)
$$

is an equivalence. By Theorem 6.15, we just have to show that $S^{\otimes_{R} m}$ is homologically even which is clear since it is a tensor product of even flat $R$-modules.

Example 6.25. Let $\mathrm{BP}\langle n\rangle$ be the truncated Brown-Peterson spectrum with

$$
\pi_{*}(\mathrm{BP}\langle n\rangle) \simeq \mathbb{Z}_{(p)}\left[v_{1}, \ldots, v_{n}\right]
$$

which can be made into an $\mathbf{E}_{3}$-MU-algebra by the work of Hahn-Wilson [HW22, Theorem A]. By the main results of [HW22], the algebraic $K$-theory spectrum $\mathrm{K}(\mathrm{BP}\langle n\rangle)$ is of height $n+1$ and satisfies an analogue of Lichtenbaum-Quillen conjectures. A key step in the proof of this remarkable theorem is the analysis of $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ via descent along the map

$$
\operatorname{THH}(\mathrm{BP}\langle n\rangle) \rightarrow \operatorname{THH}(\mathrm{BP}\langle n\rangle / \mathrm{MU})
$$

into the relative topological Hochschild homology. As observed in [HRW22, Example 4.2.3, 4.2.4], the resulting descent filtration can be identified with the $\mathbf{E}_{\infty}$-even filtration

$$
\mathrm{fil}_{\mathbf{E}_{\infty}-e v / \operatorname{THH}(\mathrm{MU})}^{*}(\mathrm{THH}(\mathrm{BP}\langle n\rangle))
$$

relative to $\mathrm{THH}(\mathrm{MU})$. We claim that this filtration is in fact the even filtration

$$
\operatorname{fil}_{e v}^{*}(\operatorname{THH}(\mathrm{BP}\langle n\rangle))
$$

and so is an invariant of the ring spectrum $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ itself, providing evidence that our filtration is the right way to obtain a good theory of motivic cohomology of $\mathbf{E}_{2}$-ring spectra. To see this, we need the following two facts observed in [HRW22, Example 4.2.4]:
(1) $\mathrm{THH}(M U) \rightarrow \mathrm{MU}$ is faithfully even flat (by Proposition 6.19) and hence so is its basechange $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \rightarrow \mathrm{THH}(\mathrm{BP}\langle n\rangle / \mathrm{MU})$
(2) $\pi_{*} \mathrm{THH}(\mathrm{BP}\langle n\rangle / \mathrm{MU})$ is even.

The result then follows from Theorem 6.24.

## 7. Comparison with the $\mathbf{E}_{\infty}$-Even filtration

In this section, we compare the even filtration studied in the current work with the even filtration of $\mathbf{E}_{\infty}$-rings introduced by Hahn-Raksit-Wilson [HRW22]. To avoid confusion, we refer to the latter as the $\mathbf{E}_{\infty}$-even filtration.

We first recall the definition of the $\mathbf{E}_{\infty}$-even filtration; for details, see [HRW22, §2]. Let $\mathcal{M} o d$ denote the $\infty$-category of pairs $(A, M)$, where $A \in \operatorname{CAlg}(\mathcal{S} p)$ and $M \in \mathcal{M} o d_{A}(\mathcal{S} p)$; we write $\mathcal{M} o d^{e v} \subseteq \mathcal{M} o d$ for the full subcategory spanned by those pairs such that $\pi_{*} A$ is even.

Definition 7.1. The $\mathbf{E}_{\infty}$-even filtration

$$
\mathrm{fil}_{\mathbf{E}_{\infty}-e v /-}^{*}(-): \mathcal{M} o d \rightarrow \operatorname{FilS} p
$$

is the right Kan extension of the functor $U_{*}: \mathcal{M} o d^{e v} \rightarrow$ FilS $p$ given by

$$
U_{n}(A, M):=\tau_{\geq 2 n} M .
$$

along the inclusion $\mathcal{M o d}^{e v} \hookrightarrow \mathcal{M}$ od.
Concretely, if $(A, M) \in \mathcal{M} o d$, then the $\mathbf{E}_{\infty}$-even filtration is given by the limit

$$
\operatorname{fil}_{\mathbf{E}_{\infty}-e v / A}^{n}(M) \simeq \lim _{\check{c}} \tau_{\geq 2 n}\left(B \otimes_{A} M\right)
$$

taken over all $\mathbf{E}_{\infty}$-ring maps $A \rightarrow B$ with $\pi_{*} B$ even.
Construction 7.2. The even filtration of Definition 2.19 is more generally defined on pairs of an $\mathbf{E}_{1}$-algebra and a module, but by restriction defines a functor on the $\infty$-category $\mathcal{M}$ od. We will describe a canonical natural transformation

$$
\begin{equation*}
\mathrm{fil}_{e v /-}^{*}(-) \rightarrow \mathrm{fil}_{\mathbf{E}_{\infty}-e v /-}^{*}(-) \tag{7.1}
\end{equation*}
$$

of functors $\mathcal{M} o d \rightarrow$ FilS $p$.
Since the right hand side of (7.1) is defined as a right Kan extension, to construct the needed natural transformation it is enough to define it on the subcategory $\mathcal{M} o d^{e v}$. We claim that on this subcategory, the two filtrations are canonically equivalent, providing the needed natural transformation. Indeed, if $\pi_{*} A$ is even, then both filtrations are given by by

$$
(A, M) \mapsto \tau_{\geq 2 *} M
$$

by definition in the case of the $\mathbf{E}_{\infty}$-even filtration and by Proposition 2.31 in the case of the even filtration of Definition 2.19.

Theorem 7.3. Let $R$ be an $\mathbf{E}_{\infty}$-ring which admits a faithfully even flat map $R \rightarrow S$ in the sense of Definition 6.16 into a $\pi_{*}$-even $\mathbf{E}_{\infty}$-ring $S$. Then for any $R$-module $M$ the canonical comparison map

$$
\mathrm{fil}_{e v / R}^{*}(M) \rightarrow \mathrm{fil}_{\mathbf{E}_{\infty}-e v / R}^{*}(M)
$$

exhibits the target as a completion of the source. In particular, it is an equivalence after completion.

Proof. The $\mathbf{E}_{\infty}$-even filtration is always complete [HRW22, Remark 2.1.6], so it is enough to show that the canonical map is an equivalence after passing to the associated graded objects. As both filtrations satisfy faithfully even flat descent, the even filtration by Theorem 6.24 and the $\mathbf{E}_{\infty}$-even filtration by a combination of Proposition 6.19 and [HRW22, Corollary 2.2.14], the comparison map can be identified with

$$
\lim _{\leftarrow}^{\operatorname{gr}} \mathrm{r}_{e v / S^{\otimes_{R}[m]}}^{*}\left(S^{\otimes_{R}[m]} \otimes_{R} M\right) \rightarrow \varliminf_{幺}^{\lim } \operatorname{gr}_{\mathbf{E}_{\infty}-e v / S^{*} \otimes_{R}[m]}^{*}\left(S^{\otimes_{R}[m]} \otimes_{R} M\right)
$$

where the limit is taken over $[m] \in \Delta$. Thus, we can reduce to the case when $\pi_{*} R$ is even, in which case the two filtration agree as observed in Construction 7.2.

In [HRW22, §5], Hahn-Raksit-Wilson show that the $\mathbf{E}_{\infty}$-even filtration of various rings recovers various classically studied and important filtrations, implying that they are in fact a functorial invariant of the $\mathbf{E}_{\infty}$-ring itself. In particular, they show that this is true for:
(1) the sphere, for which the $\mathbf{E}_{\infty}$-even filtration is the Adams-Novikov filtration,
(2) $\mathrm{HH}(R / k)$, where $k \rightarrow R$ is a quasi-lci map of commutative rings, where one recovers the Hochschild-Kostant-Roserberg filtration,
(3) $\operatorname{THH}(R)_{p}^{\wedge}$, where $R$ is a $p$-quasisyntomic, $p$-complete commutative ring, where one recovers the Bhatt-Morrow-Scholze filtration of [BMS19],
(4) $\mathrm{THH}(R)$ for $R$ a quasisyntomic ring, where one recovers the Bhatt-Lurie filtration of [BL22]
Together with their work, our comparison result of Theorem 7.3 shows that these filtrations are in fact invariants of the $\mathbf{E}_{1}$-ring structure of each of their rings.

Corollary 7.4. In each of the above four examples, the respective filtration coincides with the even filtration of Definition 2.19.

Proof. The relevant comparison results with the $\mathbf{E}_{\infty}$-filtration are all proven in [HRW22] using eff descent. By Theorem 7.3, it is enough to verify that each of the maps appearing in the proof is in fact faithfully even flat in the sense of Definition 6.16. This follows from Proposition 6.19 since all of the relevant $\mathbf{E}_{\infty}$-rings are connective.

Remark 7.5. In [HRW22], Hahn-Raksit-Wilson also prove comparison results with motivic filtrations on $\mathrm{TC}^{-}(-), \mathrm{TC}(-)$ and $\mathrm{TP}(-)$. However, in these cases their definition of an appropriate $\mathbf{E}_{\infty}$-even filtration is different from the one we recalled in Definition 7.1, as one has to take the $S^{1}$ action into account. We expect that after appropriate modifications are made to Definition 2.19, the comparison can be extended to cover $\mathrm{TC}^{-}(-), \mathrm{TC}(-)$ and $\mathrm{TP}(-)$.

Beware that for a general $\mathbf{E}_{\infty}$-ring, the $\mathbf{E}_{\infty}$-even and even filtrations can disagree. We learned the following instructive example from Robert Burklund:

Example 7.6. Let $R:=\mathbb{F}_{2} \otimes \Sigma_{+}^{\infty} \Omega^{\infty} S^{1}$ be the free $\mathbb{F}_{2}$ - $\mathbf{E}_{\infty}$-algebra on a class $x \in \pi_{1} R$. The homotopy groups of $R$ can be described in terms of Dyer-Lashof operations, namely

$$
\pi_{*} R \simeq \mathbb{F}_{2}\left[Q^{J} x\right]
$$

forms a polynomial ring in generators

$$
Q^{I} x:=Q^{j_{1}} Q^{j_{2}} \ldots Q^{j_{p}} x
$$

satisfying $j_{i} \leq 2 j_{i+1}$ and $j_{1}-j_{2}-\ldots-j_{p}>|x|=1$, see [Law20, Example 1.5.10].
Observe that any $\mathbf{E}_{\infty}$-ring map $R \rightarrow S$ into a ring with $\pi_{*} S$ even necessarily sends $x$ to zero, from which it follows that it factors as

$$
R \rightarrow \mathbb{F}_{2} \rightarrow S
$$

so that all elements of positive degree are in the kernel of the induced map on homotopy. It follows that if $y \in \pi_{2 n} R$ is an element of positive even degree, then it maps to zero in

$$
\operatorname{gr}_{\mathbf{E}_{\infty}-e v}^{n}(R):=\underset{R \rightarrow S}{\lim _{\overparen{R}}} \pi_{2 n} S
$$

On the other hand, the structure of the even filtration of Definition 2.19 is more straightforward; in particular, all polynomial generators in $\pi_{2 n} R$ are detected in $\operatorname{gr}_{e v}^{n}(R)$.

To see this, we will show the following more general statement: Let $A$ be an $\mathbf{E}_{2}-\mathbb{F}_{2}$-algebra such that $\pi_{*} A \simeq \mathbb{F}_{2}\left[x_{i}, y_{j}\right]$ is a polynomial algebra in odd degree variables $x_{i}$ and even degree variables $y_{j}$. Then

$$
\begin{equation*}
\mathrm{H}_{e v}^{*, *}(A) \simeq \mathbb{F}_{2}\left[\widetilde{x}_{i}, \widetilde{y}_{j}\right] \tag{7.2}
\end{equation*}
$$

with $\widetilde{x}_{i} \in \mathrm{H}_{e v}^{1,\left|x_{i}\right|+1 / 2}(A)$ and $\widetilde{y}_{j} \in \mathrm{H}_{e v}^{0,\left|y_{j}\right| / 2}(A)$ are detecting the corresponding elements of $\pi_{*} A$. In particular, the even spectral sequence of $A$ collapses.

From now on, let us not distinguish in notation between even and odd degree generators and denote the totality of both by $\left(z_{i}\right)_{i \in I}$, with the index set $I$ implicitly well-ordered. Observe that any finite subset $A \subseteq I$ determines a map of $\mathbf{E}_{1}-\mathbb{F}_{2}$-algebras

$$
\bigotimes_{\mathbb{F}_{2}, i \in A} \mathbb{F}_{2}\left[z_{i}\right] \rightarrow \bigotimes_{\mathbb{F}_{2}} A \rightarrow A
$$

from the tensor product of free $\mathbf{E}_{1}-\mathbb{F}_{2}$-algebras, where the second map is the multiplication of $A$, which is $\mathbf{E}_{1}$ as $A$ is $\mathbf{E}_{2}$. We deduce that as an $\mathbf{E}_{1}$-ring, $A$ can be written as

$$
\begin{equation*}
A \simeq \underset{\overrightarrow{A \subseteq I}}{\lim _{i \in A}}\left(\bigotimes_{i \in I} \mathbb{F}_{2}\left[z_{i}\right]\right) \tag{7.3}
\end{equation*}
$$

a filtered colimit of finite tensor products. By Proposition 5.3, the even filtration of an $\mathbf{E}_{1}$-ring can be calculated by constructing an appropriate chain complex of modules

$$
E_{0} \rightarrow E_{1} \rightarrow \ldots
$$

where each $E_{i}$ has even homotopy groups (among other properties, see Construction 5.2), and taking cohomology of $\pi_{*} E_{\mathbf{\bullet}}$. By (7.3), the resolution of $A$ can be obtained as a filtered colimit of tensor products of resolutions of $\mathbb{F}_{2}\left[z_{i}\right]$. Note that here we use that we're working over $\mathbb{F}_{2}$, as this guarantees that a tensor product of modules with even homotopy groups also has even homotopy groups. We deduce that we have a Künneth-style isomorphism

$$
\mathrm{H}_{e v}^{*, *}(A) \simeq \underset{A \subseteq I}{\lim }\left(\bigotimes_{i \in A} \mathrm{H}_{e v}^{*, *}\left(\mathbb{F}_{2}\left[z_{i}\right]\right)\right)
$$

which reduces us to the case of a free $\mathbf{E}_{1}$-algebra on a single generator. If the generator $z$ is of even degree, then the even cohomology is as claimed in (7.2) by Corollary 2.28. If $z$ is of odd degree, then we have a resolution of the form

where $a=|z|+1$. Each of the diagonal arrows is surjective on homotopy groups, from which we deduce that the horizontal arrows are zero after passing to homotopy. We deduce that $\mathrm{H}_{e v}^{*, *}\left(\mathbb{F}_{2}[z]\right) \simeq \mathbb{F}_{2}[\widetilde{z}]$ with $|\widetilde{z}|=(1, a / 2)$, as claimed in (7.2).

Warning 7.7. The following variation on the Example 7.6 shows that the even and $\mathbf{E}_{\infty}$-even filtrations can diverge even more drastically once we leave the world of connective $\mathbf{E}_{\infty}$-rings.

Let $R$ be as in Example 7.6, so that $\pi_{*} R \simeq \mathbb{F}_{2}\left[Q^{j} x\right]$ is a polynomial algebra and let $e \in \pi_{2 n} R$ be an even degree polynomial generator; for example, we can take $Q^{3} x \in \pi_{4} R$. As a localization,
$R\left[e^{-1}\right]$ acquires a canonical $\mathbf{E}_{\infty^{-}}$-ring structure with $\pi_{*}\left(R\left[e^{-1}\right]\right) \simeq\left(\pi_{*} R\right)\left[e^{-1}\right]$. Observe that any $\operatorname{map} f: R \rightarrow S$ into an even $\mathbf{E}_{\infty}$-ring sends $x$ to zero, so that also $f(e)=0$. We deduce that the only map $R\left[e^{-1}\right] \rightarrow S$ from the localization to an even $\mathbf{E}_{\infty}$-ring is the zero map and consequently

$$
\mathrm{fil}_{\mathbf{E}_{\infty^{-}-e v}^{*}}^{*}\left(R\left[e^{-1}\right]\right)=0
$$

On the other hand, an analysis analogous to the one given in Example 7.6 shows that the the even filtration is complete and

$$
\mathrm{H}_{e v}^{*, *}\left(R\left[e^{-1}\right]\right) \simeq \mathbb{F}_{2}\left[\widetilde{e}_{j}, \widetilde{v}_{j}\right]\left[\widetilde{e}^{-1}\right]
$$

where $\widetilde{e}$ is the Hurewicz image of $e$.
Remark 7.8. Observe that the zero map $R\left[e^{-1}\right] \rightarrow 0$ from the $\mathbf{E}_{\infty}$-ring appearing in Warning 7.7 is evenly faithfully flat in the sense of Hahn-Raksit-Wilson [HRW22, 2.2.13], but it is not faithfully even flat in the sense of Definition 6.16.

## 8. Even cohomology of connective Rings

In this section we study even cohomology of connective rings.

### 8.1. Vanishing above the Milnor line

We first show that even cohomology of connective rings vanishes above the "Milnor line" $p=q$.
Theorem 8.1. Let $R$ be connective and let $M$ be connective, homologically even. Then
(1) $\mathrm{H}_{e v}^{0,0}(R, M) \simeq \pi_{0} M$,
(2) $\mathrm{H}_{e v}^{p, q}(R, M)=0$ for $p>q$.

Proof. We recall from Proposition 5.3 a recipe for calculating the even cohomology of $M$. Using Construction 5.2, we can find a diagram

with the properties that $C_{-1}=M$ and that each $C_{i} \rightarrow E_{i} \rightarrow C_{i+1}$ is a cofibre sequence with the first map a $\pi_{*}$-even envelope. We then have a canonical isomorphism

$$
\begin{equation*}
\mathrm{H}_{e v}^{p, q}(M) \simeq \mathrm{H}^{p}\left(\pi_{2 q} E_{\bullet}\right) \tag{8.2}
\end{equation*}
$$

By Proposition 4.9, a $\pi_{*}$-even envelope of a connective $R$-module can be chosen so that the cofibre is 2-connective. Since $M$ is connective, it follows that we can choose a diagram (8.1) with the property that $E_{i}$ is (2i)-connective for each $i \geq 0$. For such a diagram, both parts follow from (8.2), as the groups $\pi_{2 q} E_{p}$ vanish for $p>q$ and we have an isomorphism $\pi_{0} M \simeq \pi_{0} E_{0}$.

Theorem 8.2 (Completness of the even filtration). Let $R$ be connective and let $M$ be connective, homologically even. Then
(1) fil $_{\text {ev }}^{n}(M) \simeq M$ for $n<0$ and
(2) the even filtration filev $(M)$ is complete; that is, $\lim _{\leftrightarrows} \mathrm{fil}_{e v}^{*}(M) \simeq 0$.

Proof. We start with the first part. Recall from Remark 2.24 that the associated graded object of the even filtration satisfies

$$
\pi_{t} \operatorname{gr}_{e v}^{n} M \simeq \mathrm{H}_{e v}^{-2 n-t, n}(R, M)
$$

By Theorem 8.1, the even cohomology of $M$ vanishes in negative weight, so that the maps

$$
\operatorname{fil}_{e v}^{0}(M) \rightarrow \operatorname{fil}_{e v}^{-1}(M) \rightarrow \operatorname{fil}_{e v}^{-2}(M) \rightarrow \ldots
$$

are all equivalences. As their colimit is equivalent to $M$ by Proposition 2.21, we deduce that $\mathrm{fil}_{e v}^{n}(M) \simeq M$ for all $n<0$.

We now show that $\lim _{\leftrightarrows} \operatorname{fil}_{e v}^{n}(M)$ is $(-2)$-connective. Using the Milnor exact sequence associated to an inverse limit, it is enough to show that fil ${ }_{e v}^{n}(M)$ is $(-1)$-connective for each $n>0$. We've already seen that this is true when $n=0$, and we'll prove it for positive $n$ via induction.

The identification between even cohomology and homotopy groups of the associated graded object together with Theorem 8.1 show that for each $n \geq 0, \operatorname{gr}_{e v}^{n}(M)$ is 0 -connective (in fact, $n$-connective, but we will not need it). Since we have a cofibre sequence

$$
\Sigma^{-1} \operatorname{gr}_{e v}^{n}(M) \rightarrow \operatorname{fil}_{e v}^{n+1}(M) \rightarrow \operatorname{fil}_{e v}^{n}(M)
$$

the ( -1 )-connectivity of $\mathrm{fil}_{e v}^{n+1}$ follows from that of fil ${ }^{n}$, ending the inductive argument. This shows that $\lim _{\mathrm{fil}}^{e v} n(M)$ is $(-2)$-connective, as claimed.

We will now inductively show that for any $k \in \mathbb{Z}$, $\lim _{\mathrm{il}}^{\mathrm{ev}}{ }^{n}(M)$ is $k$-connective. The case $k=-2$ was proved above, so we assume that $k \geq-1$. Using Proposition 4.9, we can find a cofibre sequence

$$
M \rightarrow E \rightarrow C
$$

where both $E$ and $C$ are also homologically flat, $\pi_{*} E$ is even and $C$ is 2 -connective. This yields a cofibre sequence

$$
\lim _{\rightleftarrows} \mathrm{fil}_{e v}^{*}(M) \rightarrow \lim _{\leftrightarrows} \mathrm{fi}_{e v}^{*}(E) \rightarrow \lim _{\leftrightarrows} \mathrm{fil}_{e v}^{*}(C)
$$

The middle term vanishes by Lemma 2.27, giving an equivalence

$$
\begin{equation*}
\lim _{\leftrightarrows} \mathrm{fi}_{e v}^{*}(M) \simeq \Sigma^{-1}\left(\lim _{\leftrightarrows} \mathrm{fil}_{e v}^{*}(C)\right) \simeq \Sigma\left(\lim _{\leftrightarrows} \mathrm{fi}_{e v}^{*}\left(\Sigma^{-2} C\right)\right) \tag{8.3}
\end{equation*}
$$

where we use that

$$
\mathrm{fil}_{e v}^{n}\left(\Sigma^{-2} C\right) \simeq \Sigma^{-2} \operatorname{fil}^{n+1}(C)
$$

which follows immediately from the definition. $\mathrm{As} \Sigma^{-2} C$ is connective, the right hand side of (8.3) is $k$-connective by the inductive assumption. We deduce that the same is true for the left hand side. It follows that $\lim _{\leftrightarrows} \mathrm{fi}_{e v}^{n}(M)$ is $k$-connective for all $k \in \mathbb{Z}$ and thus must vanish, proving the second part.

Corollary 8.3. If $R$ is connective and $M$ is homologically even, connective, then the even spectral sequence

$$
\mathrm{H}_{e v}^{p, q}(R, M) \Rightarrow \pi_{2 q-p}(M)
$$

converges completely.
Proof. Since the even spectral sequence of Definition 2.25 is the spectral sequence associated to the filtered spectrum filev ${ }_{e}^{*}(M)$, conditional convergence in the sense of Boardman follows from completeness of the even filtration, which we've shown in Theorem 8.2.

As the differentials are of bidegree $\left|d_{r}\right|=(2 r-1, r-1)$, the vanishing line of Theorem 8.1 implies that the group of elements in any given bidegree can receive and support only finitely many differentials, complete convergence follows.

Remark 8.4 (The case of the $\mathbf{E}_{\infty}$-even filtration). The Hahn-Raksit-Wilson filtration attached to an $\mathbf{E}_{\infty}$-ring $R$ is always complete by construction, and the question of whether the associated spectral sequence converges to $\pi_{*} R$ instead depends on whether the filtration is exhaustive; that is, whether $\xrightarrow{\lim } \operatorname{fil}_{\mathbf{E}_{\infty_{-}}}^{*}(R) \simeq R$. This was shown to be the case if $R$ is connective by Achim Krause and Robert Burklund, giving an analogue of Theorem 8.2 also in this context ${ }^{4}$.

[^4]
### 8.2. Cohomology in low weights

In Theorem 8.1, we have shown that the weight zero even cohomology of a connective $R$-module is concentrated in cohomological degree zero, where $\mathrm{H}_{e v}^{0,0}(M) \simeq \pi_{0} M$. In this section, we give a calculation of groups in weight one and deduce a calculation of $\mathrm{H}_{e v}^{2,2}(M)$.

Proposition 8.5. Let $R$ be connective and $M$ homologically even and connective. Then
(1) $\mathrm{H}_{e v}^{0,1}(M) \simeq \operatorname{coker}\left(\pi_{1} R \otimes_{\mathbb{Z}} \pi_{1} M \rightarrow \pi_{2} M\right)$,
(2) $\mathrm{H}_{e v}^{1,1}(M) \simeq \pi_{1} M$.

Proof. Each homotopy class $m \in \pi_{1} M$ determines a map $\Sigma R \rightarrow M$. We will consider the direct sum

$$
\bigoplus_{m \in \pi_{1} M} \Sigma R \rightarrow M
$$

of all of these maps and the corresponding cofibre sequence

$$
M \rightarrow E \rightarrow C
$$

where $C \simeq \Sigma^{2} R$ is the suspension of the direct sum above. This is a homologically even $R$-modules, so that we have a short exact sequence

$$
0 \rightarrow \mathcal{F}_{M} \rightarrow \mathcal{F}_{E} \rightarrow \mathcal{F}_{C} \rightarrow 0
$$

This induces a long exact sequence of even cohomology

$$
\begin{equation*}
0 \rightarrow \mathrm{H}_{e v}^{0,1}(M) \rightarrow \mathrm{H}_{e v}^{0,1}(E) \rightarrow \mathrm{H}_{e v}^{0,1}(C) \rightarrow \mathrm{H}_{e v}^{1,1}(M) \rightarrow \mathrm{H}_{e v}^{1,1}(E) \rightarrow 0 \tag{8.4}
\end{equation*}
$$

Since $E$ and $C$ are 2-connective, from Theorem 8.1 we know that $H_{e v}^{1,1}(E)=0, H_{e v}^{0,1}(E) \simeq \pi_{2} E$ and $\mathrm{H}_{e v}^{0,1}(C) \simeq \pi_{2} C$. Taking this into account, (8.4) becomes

$$
0 \rightarrow \mathrm{H}_{e v}^{0,1}(M) \rightarrow \pi_{2} E \rightarrow \pi_{2} C \rightarrow \mathrm{H}_{e v}^{1,1}(M) \rightarrow 0
$$

Since $C \simeq \bigoplus \Sigma^{2} R$ and

$$
\ldots \rightarrow \pi_{3} C \rightarrow \pi_{2} M \rightarrow \pi_{2} E \rightarrow \pi_{2} C \rightarrow \pi_{1} M \rightarrow 0
$$

is exact we deduce that

$$
\mathrm{H}_{e v}^{0,1}(M) \simeq \operatorname{ker}\left(\pi_{2} E \rightarrow \pi_{2} C\right) \simeq \operatorname{coker}\left(\bigoplus \pi_{1} R \rightarrow \pi_{2} M\right) \simeq \operatorname{coker}\left(\pi_{1} R \otimes_{\mathbb{Z}} \pi_{1} M \rightarrow \pi_{2} M\right)
$$

This shows the first needed isomorphism. Similarly, we have

$$
\mathrm{H}_{e v}^{1,1}(M) \simeq \operatorname{coker}\left(\pi_{2} E \rightarrow \pi_{2} C\right) \simeq \pi_{1} M
$$

Corollary 8.6. Let $R$ be connective and $M$ homologically even, connective. Then

$$
\mathrm{H}_{e v}^{2,2}(M) \simeq \operatorname{im}\left(\pi_{1} R \otimes_{\mathbb{Z}} \pi_{1} M \rightarrow \pi_{2} M\right) .
$$

Proof. By Corollary 8.3, in the case at hand the even spectral sequence

$$
\mathrm{H}_{e v}^{p, q}(M) \Rightarrow \pi_{2 q-p}(M)
$$

converges completely. As the differentials are of bidegree $(2 r-1, r-1)$, the vanishing line of Theorem 8.1 implies that elements in weights $q \leq 2$ can neither receive or support differentials. It follows that we have a short exact sequence

$$
0 \rightarrow \mathrm{H}_{e v}^{2,2}(M) \rightarrow \pi_{2}(M) \rightarrow \mathrm{H}_{e v}^{0,1}(M) \rightarrow 0
$$

and the identification of the last group in Proposition 8.5 yields the needed result.

### 8.3. Base-change around the Milnor line

As we have seen in Theorem 8.1, if $M$ is a homologically even, connective module over a connective $\mathbf{E}_{1}$-ring $R$, then $\operatorname{gr}_{e v}^{n}(M)$ is always $n$-connective, with lowest homotopy group given

$$
\pi_{n} \operatorname{gr}_{e v}^{n}(M) \simeq \mathrm{H}_{e v}^{n, n}(R, M)
$$

Since the $n$-th homotopy group detects effective epimorphisms in the $\infty$-category of $n$-connective spectra, the Milnor line group $\mathrm{H}_{e v}^{n, n}(R, M)$ plays a large role in the study of connective modules. In this section, we prove the following result about how the even cohomology groups vary in a neighbourhood of the Milnor line:

Theorem 8.7. Let $f: R \rightarrow S$ be a right homologically even map of connective $\mathbf{E}_{1}$-rings and let $M$ be an even flat $R$-module. Then the base-change of the canonical comparison map

$$
\pi_{0} S \otimes_{\pi_{0} R} \mathrm{H}_{e v}^{p, q}(R, M) \rightarrow \mathrm{H}_{e v}^{p, q}\left(S, S \otimes_{R} M\right)
$$

is a surjection for $p \geq q-1$.
Proof. We will show this by induction on weight. The case of $q=0$ follows from Theorem 8.1.
Let us assume that $q>0$ and let $M \rightarrow E$ be a $\pi_{*}$-even envelope, which by Proposition 4.9 we can choose so that the cofibre $C$ is 2 -connective. Since both $M$ and $C$ are even flat, so is $E$. Since $S$ is homologically even as a right $R$-module, it follows from Theorem 4.14 that $\pi_{*}\left(S \otimes_{R} E\right)$ is even. In particular, $S \otimes_{R} E$ is homologically even as an $S$-module. Consider the map of cofibre sequences


This induces a map of long exact sequences

where the vertical maps are the base-change maps of Construction 6.12. As $p \geq q-1 \geq 0$, the two groups in the right-most column vanish by Corollary 2.28 . Thus, in the square

the two horizontal maps are surjective. As a base-change of an epimorphism is an epimorphism, the same is true in


Since $\mathrm{H}_{e v}^{p, q}(R, C) \simeq \mathrm{H}_{e v}^{p, q}\left(R, \Sigma^{-2} C\right)$ and $\Sigma^{-2} C$ is connective, the left vertical map is surjective by the inductive assumption. It follows that so is the right vertical map, ending the argument.

## References

[Ada95] John Frank Adams, Stable homotopy and generalised homology, University of Chicago press, 1995.
[BKWX22] Tom Bachmann, Hana Jia Kong, Guozhen Wang, and Zhouli Xu, The chow t-structure on the $\infty$ category of motivic spectra, Ann. Math 195 (2022), 707-773.
[BL22] Bhargav Bhatt and Jacob Lurie, Absolute prismatic cohomology, arXiv preprint arXiv:2201.06120 (2022).
[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, Topological hochschild homology and integral p-adic hodge theory, Publications mathématiques de l'IHÉS 129 (2019), no. 1, 199-310.
[Del71] Pierre Deligne, Théorie de Hodge, II, Publications Mathématiques de l'Institut des Hautes Études Scientifiques 40 (1971), no. 1, 5-57.
[Dug05] Daniel Dugger, An atiyah-hirzebruch spectral sequence for $k$ r-theory, K-Theory 3 (2005), no. 35, 213-256.
[GIKR21] Bogdan Gheorghe, Daniel C Isaksen, Achim Krause, and Nicolas Ricka, $\mathbb{C}$-motivic modular forms, Journal of the European Mathematical Society (2021).
[GIKR22] , C-motivic modular forms, Journal of the European Mathematical Society 24 (2022), no. 10, 3597-3628.
[Hil12] Michael A Hill, The equivariant slice filtration: a primer, Homology, Homotopy and Applications 14 (2012), no. 2, 143-166.
[HP22a] Lars Hesselholt and Piotr Pstragowski, Dirac geometry i: Commutative algebra, arXiv preprint arXiv:2207.09256 (2022).
[HP22b] , Dirac geometry i: Commutative algebra, arXiv preprint arXiv:2207.09256 (2022).
[HP23a] Peter J. Haine and Piotr Pstrągowski, Spectral weight filtrations, To appear (2023).
[HP23b] Lars Hesselholt and Piotr Pstragowski, Dirac geometry ii: Coherent cohomology, arXiv preprint arXiv:2303.13444 (2023).
[HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson, A motivic filtration on the topological cyclic homology of commutative ring spectra, arXiv preprint arXiv:2206.11208 (2022).
[HW22] Jeremy Hahn and Dylan Wilson, Redshift and multiplication for truncated brown-peterson spectra, Annals of Mathematics 196 (2022), no. 3, 1277-1351.
[Law18] Tyler Lawson, Secondary power operations and the brown-peterson spectrum at the prime 2, Annals of Mathematics 188 (2018), no. 2, 513-576.
[Law20] , E n-spectra and dyer-lashof operations, Handbook of homotopy theory, Chapman and Hall/CRC, 2020, pp. 793-849.
[Lur] Jacob Lurie, Higher algebra, http://www.math.harvard.edu/~lurie/papers/HA.pdf.
[Lur09] , Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
[Lur10] Jacob Lurie, Chromatic homotopy theory, Lecture series (2010).
[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel, Nilpotence and descent in equivariant stable homotopy theory, Advances in Mathematics 305 (2017), 994-1084.
[PP21] Irakli Patchkoria and Piotr Pstragowski, Adams spectral sequences and franke's algebraicity conjecture, arXiv preprint arXiv:2110.03669 (2021).
[PR23] Piotr Pstragowski and Arpon Raksit, Motivic cohomology of $\mathbf{E}_{2}$-ring spectra, In preparation (2023).
[Pst22] Piotr Pstrągowski, Synthetic spectra and the cellular motivic category, Inventiones mathematicae (2022), 1-129.
[Rog14] John Rognes, Algebraic k-theory of strict ring spectra, arXiv preprint arXiv:1403.5998 (2014).
[Sen17] Andrew Senger, The brown-peterson spectrum is not $E_{2\left(p^{2}+2\right)}$ at odd primes, arXiv preprint arXiv:1710.09822 (2017).

Harvard University, USA, and Institute for Advanced Study, USA
Email address: pstragowski.piotr@gmail.com


[^0]:    The author received support from NSF Grant number DMS-1926686 and Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy - EXC-2047/1 - 390685813.

[^1]:    ${ }^{1}$ In the main body of the text we prove descent for maps of $\mathbf{E}_{1}$-rings, see Theorem 6.23 , but we stick to algebras in the introduction for simplicity.

[^2]:    ${ }^{2}$ The assumption that $M$ is homologically even can be removed, but to get the most elegant results in this case it is preferable to introduce a refinement of the even filtration into a filtration indexed by half-integers, see Remark 2.26. We mainly work in the homologically even case as it covers all of the examples we are after, since $R$ itself is homologically even.

[^3]:    ${ }^{3} 100$ Nassau St, Princeton, NJ 08542

[^4]:    ${ }^{4}$ Proviate communication with Achim Krause and Robert Burklund.

