Geometry & Topology 12 (2008) 103–176 **103**

Equivariant homotopy theory for pro–spectra

HALVARD FAUSK

We extend the theory of equivariant orthogonal spectra from finite groups to profinite groups, and more generally from compact Lie groups to compact Hausdorff groups. The G–homotopy theory is "pieced together" from the G/U–homotopy theories for suitable quotient groups G/U of G; a motivation is the way continuous group cohomology of a profinite group is built out of the cohomology of its finite quotient groups. In the model category of equivariant spectra Postnikov towers are studied from a general perspective. We introduce pro–G–spectra and construct various model structures on them. A key property of the model structures is that pro–spectra are weakly equivalent to their Postnikov towers. We discuss two versions of a model structure with "underlying weak equivalences". One of the versions only makes sense for pro–spectra. In the end we use the theory to study homotopy fixed points of pro–G–spectra.

55P91; 18G55

1 Introduction

This paper is devoted to the exploration of some aspects of equivariant homotopy theory of G–equivariant orthogonal spectra when G is a profinite group. We develop the theory sufficiently to be able to construct homotopy fixed points of G–spectra in a natural way. A satisfactory theory of G–spectra, when G is a profinite group, requires the generality of pro–G–spectra. The results needed about model structures on pro–categories are presented in two papers joint with Daniel Isaksen [20; 21]. Most of the theory also works for compact Hausdorff groups and discrete groups.

We start out by considering model structures on G–spaces. This is needed as a starting point for the model structure on G–spectra. A set of closed subgroups of G is said to be a collection if it is closed under conjugation. To any collection $\mathcal{C}$ of subgroups of G, we construct a model structure on the category of G–spaces such that a G–map f is a weak equivalence if and only if f^H is a underlying weak equivalence for $H \in \mathcal{C}$.

The collections of subgroups of G that play the most important role in this paper are the cofamilies, ie collections of subgroups that are closed under passing to larger subgroups. The example to keep in mind is the cofamily of open subgroups in a profinite group.

Published: 8 February 2008 DOI: 10.2140/gt.2008.12.103

We present the foundation for the theory of orthogonal G–spectra, indexed on finite orthogonal G–representations, with minimal assumptions on the group G and the collection $\mathcal{C}$. Most of the results extend easily from the theory developed for compact Lie groups by Michael Mandell and Peter May [35]. We include enough details to make our presentation readable, and provide new proofs when the generalizations to our context are not immediate. Equivariant K–theory and stable equivariant cobordism theory both extend from compact Lie groups to general compact Hausdorff groups. A generalization of the Atiyah–Segal completion theorem is studied in [18].

Let R be a symmetric monoid in the category of orthogonal G–spectra indexed on a universe of G–representations. In Theorem 4.4 the category of R–modules, denoted $\mathcal{M}_R$, is given a stable model structure; the weak equivalences are maps whose H–fixed points are stable equivalences for all H in a suitable collection $\mathcal{C}$. For example $\mathcal{C}$ might be the smallest cofamily containing all normal subgroups H of G such that G/H is a compact Lie group. A stable G–equivariant theory of spectra, for a profinite group G, is also given by Gunnar Carlsson in [6].

We would like to have a notion of "underlying weak equivalences" even when the trivial subgroup is not included in the collection $\mathcal{C}$. We consider a more general framework. In Theorem 5.4 we show that for two reasonable collections, $\mathcal{W}$ and $\mathcal{C}$, of subgroups of G such that WU is in $\mathcal{C}$, whenever $W \in \mathcal{W}$ and $U \in \mathcal{C}$, there is a model structures on $\mathcal{M}_R$ such that the cofibrations are retracts of relative $\mathcal{C}$–cell complexes and the weak equivalences are maps f such that $\Pi_*^W(f) = \operatorname{colim}_{U\in\mathcal{C}} \pi_*^{WU}(f)$ is an isomorphism for every $W \in \mathcal{W}$. For example, $\mathcal{C}$ can be the collection of open subgroups of a profinite group G and $\mathcal{W}$ the collection, $\{1\}$, consisting of the trivial subgroup in G.

In the rest of this introduction we assume that $\Pi_n^U(R) = 0$ whenever $n < 0$ and $U \in \mathcal{C}$. We can then set up a good theory of Postnikov sections in $\mathcal{M}_R$. The Postnikov sections are used in our construction of the model structures on pro–$\mathcal{M}_R$. Although we are mostly interested in the usual Postnikov sections that cut off the homotopy groups at the same degree for all subgroups $W \in \mathcal{W}$, we give a general construction that allows the cutoff to take place at different degrees for different subgroups.

In Theorem 9.4 we construct a stable model structure, called the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure, on pro–$\mathcal{M}_R$. It can be thought of as the localization of the strict model structure on pro–$\mathcal{M}_R$, where we invert all maps from a pro–spectrum to its levelwise Postnikov tower, regarded as a pro–spectrum. Here is one characterization of the weak equivalences: the class of weak equivalences in the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure is the class of pro–maps that are isomorphic to a levelwise map $\{f_s\}_{s\in S}$ such that f_s becomes arbitrarily highly connected (uniformly with respect to the collection $\mathcal{W}$) as s increases [20, 3.2].

In Theorem 9.23 we give an Atiyah–Hirzebruch spectral sequence. It is constructed using the Postnikov filtration of the target pro–spectrum. The spectral sequence has good convergence properties because any pro–spectrum can be recovered from its Postnikov tower in our model structure.

The category pro–$\mathcal{M}_R$ inherits a tensor product from $\mathcal{M}_R$. This tensor structure is not closed, and it does not give a well-defined tensor product on the whole homotopy category of pro–$\mathcal{M}_R$ with the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure.

The Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$ is a stable model structure. But the associated homotopy category is not an axiomatic stable homotopy category in the sense of Hovey–Palmieri–Strickland [27].

We discuss two model structures on pro–$\mathcal{M}_R$ with two different notions of "underlying weak equivalences". Let G be a finite group and let $\mathcal{C}$ be the collection of all subgroups of G. There are many different, but Quillen equivalent, $\mathcal{W}$–$\mathcal{A}$–model structures on $\mathcal{M}_R$ with $\mathcal{W}=\{1\}$ and $\{1\}\subset\mathcal{A}\subset\mathcal{C}$. Two extreme model structures are the cofree model structure, with $\mathcal{A}=\mathcal{C}$, and the free model structure, with $\mathcal{A}=\mathcal{W}=\{1\}$. The cofibrant objects in the free model structure are retracts of relative G–free cell spectra.

Now let G be a profinite group and let $\mathcal{C}$ be the collection of all open subgroups of G. In this case the situation is more complicated. The $\{1\}$–weak equivalences are maps f such that $\Pi^1_*(f)=\operatorname{colim}_{U\in\mathcal{C}}\pi^U_*(f)$ is an isomorphism. We call these maps the $\mathcal{C}$–underlying weak equivalences. The Postnikov $\{1\}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$ is the closest we get a cofree model structure. It is given in Theorem 9.5. Assume G is a nonfinite profinite group. Certainly, it is not sensible to have a model structure with cofibrant objects relative free G–cell complexes, because $S^n\wedge G_+$ is equivalent to a point. In pro–$\mathcal{M}_R$, unlike $\mathcal{M}_R$, we can form an arbitrarily good approximation to the free model structure by letting the cofibrations be retracts of levelwise relative G–cell complexes that become "eventually free". That is, as we move up the inverse system of spectra, the stabilizer subgroups of the relative cells become smaller and smaller subgroups in the collection $\mathcal{C}$. The key idea is that the cofibrant replacement of the constant pro–spectrum $\Sigma^\infty S^0$ should be the pro–spectrum

$$\{\Sigma^\infty EG/N_+\},$$

indexed by the normal subgroups N of G in $\mathcal{C}$, ordered by inclusions. We use the theory of filtered model categories, developed in [20], to construct the free model structure on pro–$\mathcal{M}_R$. This $\mathcal{C}$–free model structure is given in Theorem 10.2.

The $\mathcal{C}$–free and $\mathcal{C}$–cofree model structures on pro–$\mathcal{M}_R$ are Quillen adjoint, via the identity functors, but there are fewer weak equivalences in the free than in the cofree model structure. Thus, we actually get two different homotopy categories. We relate this

to the failure of having an internal hom functor in the pro–category. Let Ho(pro–$\mathcal{M}_R$) denote the homotopy category of pro–$\mathcal{M}_R$ with the Postnikov $\mathcal{C}$–model structure. Assume that X is cofibrant and that Y is fibrant in the Postnikov $\mathcal{C}$–model structure on pro–$\mathcal{M}_R$. Then Theorem 10.10 says that the homset of maps from X to Y in the homotopy category of the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$ is:

$$\mathrm{Ho}(\text{pro–}\mathcal{M}_R)\ (X \wedge \{EG/N_+\}, Y)$$

while the homset in the homotopy category of the $\mathcal{C}$–cofree model structure on pro–$\mathcal{M}_R$ is:

$$\mathrm{Ho}(\text{pro–}\mathcal{M}_R)\ (X, \mathrm{hocolim}_N\ F(EG/N_+, Y),$$

where the colimit is taken levelwise.

The Postnikov model structures are well-suited for studying homotopy fixed points. For definiteness, let G be a profinite group, let $\mathcal{C}$ be the collection of open subgroups of G, and let R be a non-equivariant $\mathcal{S}$–cell spectrum with trivial homotopy groups in negative degrees. The homotopy fixed points of a pro–G–spectrum $\{Y_t\}$ is defined to be the G–fixed points of a fibrant replacement in the Postnikov $\mathcal{C}$–cofree model structure. It is equivalent, in the Postnikov model structure on R–spectra, to the pro–spectrum

$$\mathrm{hocolim}_N\ F(EG/N_+, P_n Y_t)^G$$

indexed on n and t. The spectrum associated to the homotopy fixed point pro–spectrum (take homotopy limits) is equivalent to

$$\mathrm{holim}_{t,m}\,\mathrm{hocolim}_N\ F((EG/N)^{(m)}_+, Y_t)^G.$$

These expressions resemble the usual formula for homotopy fixed points.

The appropriate notion of a ring spectrum in pro–$\mathcal{M}_R$ is a monoid in pro–$\mathcal{M}_R$. This is more flexible than a pro–monoid. The second formula for homotopy fixed point spectra shows that if Y is a (commutative) fibrant monoid in pro–$\mathcal{M}_R$ with the strict $\mathcal{C}$–model structure, then the associated homotopy fixed point spectrum is a (commutative) monoid in $\mathcal{M}_R$.

Under reasonable assumptions there is an iterated homotopy fixed point formula. This appears to be false if one defines homotopy fixed points in the strict $\mathcal{C}$–model structure on pro–$\mathcal{M}_R$. We obtain a homotopy fixed point spectral sequence as a special case of the Atiyah–Hirzebruch spectral sequence.

The explicit formulas for the homotopy fixed points, the good convergence properties of the homotopy fixed point spectral sequence and the iterated homotopy fixed point formula are all reasons for why it is convenient to work in the Postnikov $\mathcal{C}$–model structure.

A general theory of homotopy fixed point spectra for actions by profinite groups was first studied by Daniel Davis in his Ph.D. thesis [9]. His theory was inspired by a homotopy fixed point spectral sequence for E_n, with an action by the extended Morava stabilizer group, constructed by Ethan Devinatz and Michael Hopkins [14]. We show that our definition of homotopy fixed point spectra agrees with Davis' when G has finite virtual cohomological dimension. Our theory applies to the example of E_n above, provided we follow Davis and use the "pro–spectrum $K(n)$–localization" of E_n rather than (the $K(n)$–local spectrum) E_n itself.

1.1 Acknowledgements

The model theoretical foundation for this paper is joint work with Daniel Isaksen. I am grateful to him for many discussions on the foundation of this paper. I am also grateful to Andrew Blumberg and especially Daniel Davis for helpful comments on the paper. Finally, I would like to thank the referee for comments.

2 Unstable equivariant theory

We associate to a collection, $\mathcal{W}$, of closed subgroups of G a model structure on the category of based G–spaces. The weak equivalences in this model structure are maps f such that the H–fixed points map f^H is a non-equivariant weak equivalence for each $H \in \mathcal{W}$.

2.1 G–Spaces

We work in the category of compactly generated weak Hausdorff spaces. Let G be a topological group. A G–space X is a topological space together with a continuous left action by G. The stabilizer of $x \in X$ is $\{g \in G \mid gx = x\}$. This is a closed subgroup of G since it is the preimage of the diagonal in $X \times X$ under the map $g \mapsto x \times gx$. Let Z be any subset of X. The stabilizer of Z is the intersection of the stabilizers of the points in Z, hence a closed subgroup of G. Similarly, for any subgroup H of G the H–fixed points, $X^H = \{x \in X \mid hx = x \text{ for each } h \in H\}$, of a G–space X is a closed subset of X. The stabilizer of X^H contains H and is a closed subgroup of G. So $X^H = X^{\overline{H}}$, for any subgroup H of G, where $\overline{H}$ denotes the closure of H in G. Hence, we consider closed subgroups of G only. The H–fixed point functor commutes with pushout along a closed inclusion.

A based G–space is a G–space together with a G–fixed basepoint. We denote the category of based G–spaces and basepoint preserving continuous G–maps by $G\mathcal{T}$. The category of based G–spaces $G\mathcal{T}$ is complete and cocomplete.

We denote the category of based G–spaces and continuous basepoint preserving maps by $\mathcal{T}_G$. The space of continuous maps is given a G–action by $(g\cdot f)(x)=gf(g^{-1}x)$ (and topologized as the Kellyfication of the compact open topology). The action of G on $\mathcal{T}_G(X,Y)$ is continuous, since the adjoint of the action map, $G\times\mathcal{T}_G(X,Y)\times X\to Y$, is continuous. The corresponding categories of unbased G–spaces are denoted $G\mathcal{U}$ and $\mathcal{U}_G$.

The category $G\mathcal{T}$ is a closed symmetric tensor category, where S^0 is the unit object, the smash product $X\wedge Y$ is the tensor product, and the G–space $\mathcal{T}_G(X,Y)$ is the internal hom functor.

Define a functor $G\mathcal{U}\to G\mathcal{T}$ by attaching a disjoint basepoint, $X\mapsto X_+$. This functor is left adjoint to the forgetful functor $G\mathcal{T}\to G\mathcal{U}$. The morphism set $G\mathcal{U}(X,Y)$ is naturally a retract of $G\mathcal{T}(X_+,Y_+)$. More precisely, we have that

$$G\mathcal{T}(X_+,Y_+)=\coprod_Z G\mathcal{U}(Z,Y)$$

where the sum is over all open and closed G–subsets Z of X. Let $f\colon X_+\to Y_+$ be a map in $G\mathcal{T}$. Then the corresponding unbased map is $f|Z\colon Z\to Y$ where $Z=X_+-f^{-1}(+)$.

2.2 Collections of subgroups of G

This paper is mostly concerned with cofamilies of subgroups.

Definition 2.1 A collection $\mathcal{W}$ of subgroups of G is a nonempty set of closed subgroups of G such that if $H\in\mathcal{W}$, then $gHg^{-1}\in\mathcal{W}$ for any $g\in G$. A collection $\mathcal{W}$ is a normal collection if for all $H\in\mathcal{W}$ there exists a $K\in\mathcal{W}$ such that $K\leq H$ and K is a normal subgroup of G.

Definition 2.2 A collection $\mathcal{W}$ of subgroups of G is a cofamily if $K\in\mathcal{W}$ implies that $L\in\mathcal{W}$ for all subgroups $L\geq K$. A collection $\mathcal{C}$ of subgroups of G contained in a cofamily $\mathcal{W}$ is a family in $\mathcal{W}$, if, for all $K\in\mathcal{C}$ and $H\in\mathcal{W}$ such that $H\leq K$, we have that $H\in\mathcal{C}$.

Let $\mathcal{W}$ be a collection of subgroups of G. The smallest cofamily of closed subgroups of G containing $\mathcal{W}$ is called the cofamily closure of $\mathcal{W}$ and is denoted $\overline{\mathcal{W}}$. A cofamily is called a normal cofamily if it is the cofamily closure of a collection of normal subgroups of G.

We now give some important cofamilies.

Example 2.3 The collection of all subgroups U of G such that G/U is finite and discrete is a cofamily. This collection of subgroups is closed under finite intersection since $G/U \cap V \subseteq G/U \times G/V$. A finite index subgroup of G has only finitely many G–conjugate subgroups of G. Hence, if U is a finite index subgroup of G, then $\cap_{g\in G} gUg^{-1}$ is a normal subgroup of G such that $G/\cap_{g\in G} gUg^{-1}$ is a finite discrete group. Let $\mathrm{fnt}(G)$ be the collection of all normal subgroups U of G such that G/U is a finite discrete group.

Example 2.4 Define $\mathrm{dsc}(G)$ to be the collection of all normal subgroups U of G such that G/U is a discrete group. This collection is closed under intersection. We call a collection that is contained in the cofamily closure of $\mathrm{dsc}(G)$ a discrete collection of subgroups of G.

Example 2.5 Let $\mathrm{Lie}(G)$ be the collection of all normal subgroups U of G such that G/U is a compact Lie group. This collection is closed under intersection since a closed subgroup of a compact Lie group is a compact Lie group. We call a collection that is contained in the cofamily closure of $\mathrm{Lie}(G)$ a Lie collection of subgroups of G.

Lemma 2.6 *Let G be a compact Hausdorff group, and let K be a closed subgroup of G. Then $\{U \cap K \mid U \in \mathrm{Lie}(G)\}$ is a subset of $\mathrm{Lie}(K)$, and for every $H \in \mathrm{Lie}(K)$ there exists a $U \in \mathrm{Lie}(G)$ such that $U \cap K \subset H$.*

Proof Let $U \in \mathrm{Lie}(G)$. The subgroup $U \cap K$ is in $\mathrm{Lie}(K)$ since $K/K \cap U$ is a closed subgroup of the compact Lie group G/U.

Let H be a subgroup in $\mathrm{Lie}(K)$. We have that $\cap_{U\in \mathrm{Lie}(G)} U = 1$ by Corollary A.3. Hence $U \cap K/H$ for $U \in \mathrm{Lie}(G)$ is a collection of closed subgroups of the compact Lie group K/H whose intersection is the unit element. Since $\mathrm{Lie}(G)$ is closed under finite intersections, the descending chain property for closed subgroups of a compact Lie group (tom Dieck [40, 1.25, Exercise 15]) gives that there exists a $U \in \mathrm{Lie}(G)$ such that $U \cap K$ is contained in H. □

We order $\mathrm{fnt}(G)$ and $\mathrm{Lie}(G)$ by inclusions. We recall the following facts.

Proposition 2.7 *A topological group G is a profinite group precisely when*

$$G \to \lim_{U\in \mathrm{fnt}(G)} G/U$$

is a homeomorphism. A topological group G is a compact Hausdorff group precisely when

$$G \to \lim_{U\in \mathrm{Lie}(G)} G/U$$

is a homeomorphism.

Proof These facts are well-known. The second claim is proved in Proposition A.2. □

Even though we are mostly interested in actions by profinite groups, we find it natural to study actions by compact Hausdorff groups whenever possible.

2.3 Model structures on the category of G–spaces

We associate to a collection $\mathcal{W}$ of closed subgroups of G a model structure on the category of based G–spaces.

Definition 2.8 Let $f\colon X \to Y$ be a map in $G\mathcal{T}$. The map f is said to be a $\mathcal{W}$–equivalence if the underlying unbased maps $f^U\colon X^U \to Y^U$ are weak equivalences for all $U \in \mathcal{W}$.

Definition 2.9 Let $p\colon E \to B$ be a map in $G\mathcal{T}$. We say that p is a $\mathcal{W}$–fibration if the underlying unbased maps $p^U\colon E^U \to B^U$ are Serre fibrations for all $U \in \mathcal{W}$.

We next define the generating cofibrations and generating acyclic cofibrations. We use the conventions that S^{-1} is the empty set and D^0 is a point.

Definition 2.10 Let $\mathcal{W}I$ be the set of maps

$$\{(G/U \times S^{n-1})_+ \to (G/U \times D^n)_+\},$$

for $n \geq 0$ and $U \in \mathcal{W}$. Let $\mathcal{W}J$ be the set of maps

$$\{(G/U \times D^n)_+ \to (G/U \times D^n \times [0,1])_+\},$$

for $n \geq 0$ and $U \in \mathcal{W}$.

The following model structure is called the $\mathcal{W}$–model structure on $G\mathcal{T}$. For the definition of relative cell complexes see Hirschhorn [24, 10.5].

Proposition 2.11 *There is a proper model structure on $G\mathcal{T}$ with weak equivalences $\mathcal{W}$–weak equivalences, fibrations $\mathcal{W}$–fibrations, and cofibrations retracts of relative $\mathcal{W}I$–cell complexes. The set $\mathcal{W}I$ is a set of generating cofibrations and $\mathcal{W}J$ is a set of generating acyclic cofibrations.*

Proof A map $p\colon E \to B$ in $G\mathcal{T}$ is a $\mathcal{W}$–fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{W}J$. A map f is a $\mathcal{W}$–acyclic fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{W}I$. This follows from the corresponding non-equivariant result and by the fixed point adjunction (Hovey

[26, 2.4]). The verifications of the model structure axioms follow as in [26, Section 2.4]. The model structure is both left and right proper. This follows from the corresponding non-equivariant results since pullbacks commute with fixed points and since pushouts along closed inclusions also commute with fixed points. □

An alternative way to set up the model structure on $G\mathcal{T}$ is given in Mandell–May [35, Section III.1]. Let $\mathcal{W}G\mathcal{T}$, or simply $\mathcal{W}\mathcal{T}$, denote $G\mathcal{T}$ with the $\mathcal{W}$–model structure, and let $Ho(\mathcal{W}G\mathcal{T})$ denote its homotopy category.

Proposition 2.12 *Let X be a retract of a $\mathcal{W}I$–cell complex, and let Y be a G–space. Then the set $Ho(\mathcal{W}\mathcal{T})(X,Y)$ is isomorphic to the set of based G–homotopy classes of maps from X to Y.*

Proof All objects are fibrant and a retract of a $\mathcal{W}I$–cell complex is cofibrant in the $\mathcal{W}$–model structure. The cylinder object of (a cofibrant object) X in the $\mathcal{W}$–model structures is $X \wedge [0,1]_+$. □

The next result has also been proved by Bill Dwyer [15, 4.1]. Note that a G–cell complex X is a $\mathcal{W}I$–cell complex if and only if all its isotropy groups are in $\mathcal{W}$.

Corollary 2.13 *Let X and Y be $\mathcal{W}I$–cell complexes. If a map $f\colon X \to Y$ is a $\mathcal{W}$–weak equivalence, then f is a based G–homotopy equivalence.*

To get a topological model structure on our model category we need some assumptions on the collection $\mathcal{W}$.

Definition 2.14 Let $\mathcal{E}$ and $\mathcal{W}$ be two collections of subgroups of G. Then $\mathcal{E}$ is called a $\mathcal{W}$–Illman collection if $(G/U \times G/H)_+$ is a $\mathcal{W}I$–cell complex for any $U \in \mathcal{W}$ and $H \in \mathcal{E}$. A collection $\mathcal{W}$ of subgroups of G is called an Illman collection if $\mathcal{W}$ is a $\mathcal{W}$–Illman collection.

In particular, if $\mathcal{E}$ is a $\mathcal{W}$–Illman collection, then $U \cap H \in \mathcal{W}$, for all $U \in \mathcal{W}$ and $H \in \mathcal{E}$, because $U \cap H$ is an isotropy group of $G/U \times G/H$. The collection $\{G\}$ is a $\mathcal{W}$–Illman collection of subgroups of G for any collection $\mathcal{W}$ of subgroups of G.

Lemma 2.15 *If $\mathcal{W}$ is a discrete or a Lie collection of subgroups of G and $\mathcal{W}$ is closed under intersection, then $\mathcal{W}$ is an Illman collection of subgroups of G.*

Proof The statement is clear when $\mathcal{W}$ is contained in $\overline{\mathrm{dsc}(G)}$. When $\mathcal{W}$ is contained in $\overline{\mathrm{Lie}(G)}$, then the claim follows from a result of Illman [28]. □

The next lemma shows, in particular, that if $\mathcal{W}$ is an Illman collection of subgroups of G, then the smash product of two $\mathcal{W}I$–cell complexes is again a $\mathcal{W}I$–cell complex.

Lemma 2.16 *Let $\mathcal{E}$ and $\mathcal{W}$ be collections of subgroups of G such that $\mathcal{E}$ is a $\mathcal{W}$–Illman collection. If X is a $\mathcal{W}I$–cell complex and Y is a $\mathcal{E}I$–cell complex, then $X \wedge Y$ is (homeomorphic to) a $\mathcal{W}I$–cell complex.*

Proof It suffices to show that

$$(S^{n-1} \times S^{m-1} \times G/U \times G/U' \to D^n \times D^m \times G/U \times G/U')_+$$

is a relative $\mathcal{W}I$–cell complex, for $U \in \mathcal{W}$ and $U' \in \mathcal{E}$. This is so since $(G/U \times G/U')_+$ is a relative $\mathcal{W}I$–cell complex. □

We follow the treatment of a topological model structure given in [35, Section III.1]. Note that the G–fixed points of the mapping spaces in $\mathcal{T}_G$ are the mapping spaces in $G\mathcal{T}$. Let $\mathcal{M}_G$ be a category enriched in $G\mathcal{T}$. Let $G\mathcal{M}$ be the G–fixed category of $\mathcal{M}_G$. Simplicial structures are defined by Hirschhorn [24, 9.1.1,9.1.5]. We modify the definition of a simplicial structure by model theoretically enriching $\mathcal{M}_G$ in the model category $\mathcal{W}\mathcal{T}$ instead of the model category of simplicial sets.

Let $i\colon A \to X$ and $p\colon E \to B$ be two maps in $\mathcal{M}_G$. Let

$$\mathcal{M}_G(i^*, p_*)\colon \mathcal{M}_G(X, E) \to \mathcal{M}_G(A, E) \times_{\mathcal{M}_G(A,B)} \mathcal{M}_G(X, B)$$

be the G–map induced by precomposing with i and composing with p.

Definition 2.17 Let $\mathcal{M}_G$ be enriched over $G\mathcal{T}$. A model structure on $G\mathcal{M}$ is said to be $\mathcal{E}$–topological if it is G–topological (see [24, 9.1.2]) and the following holds:

(1) There is a tensor functor $X\square T$ and a cotensor functor $F_\square(T, X)$ in $\mathcal{M}_G$, for $X \in \mathcal{M}_G$ and $T \in \mathcal{T}_G$, such that there are natural isomorphisms of based G–spaces

$$\mathcal{M}_G(X\square T, Y) \cong \mathcal{T}_G(T, \mathcal{M}_G(X, Y)) \cong \mathcal{M}_G(X, F_\square(T, Y)),$$

for $X, Y \in \mathcal{M}_G$ and $T \in \mathcal{T}_G$.

(2) The map $\mathcal{M}_G(i^*, p_*)$ is a $\mathcal{E}$–fibration in $G\mathcal{T}$ whenever i is a cofibration and p is a fibration in $G\mathcal{M}$, and if i or p in addition is a weak equivalence in $G\mathcal{M}$, then $\mathcal{M}_G(i^*, p_*)$ is a $\mathcal{E}$–equivalence in $G\mathcal{T}$.

Remark 2.18 The G–fixed points of $\mathcal{M}_G(i^*, p_*)$ is $G\mathcal{M}(i^*, p_*)$. So if $\{G\} \in \mathcal{E}$, then a $\mathcal{E}$–topological model structure on $G\mathcal{M}$ gives a topological model structure.

If $\mathcal{W}$ is an Illman collection, then the following Lemma implies the pushout–product axiom [39, 2.1,2.3].

Lemma 2.19 *Let $\mathcal{E}$ be a $\mathcal{W}$–Illman collection. Assume that $f\colon A \to B$ is in $\mathcal{E}I$ and $g\colon X \to Y$ is in $\mathcal{W}I$, then $f\square g\colon (A\wedge Y)\cup_{A\wedge X}(B\wedge X)\to B\wedge Y$ is a $\mathcal{W}$–cofibration. Moreover, if f is in $\mathcal{E}J$ instead of $\mathcal{E}I$ or g is in $\mathcal{W}J$ instead of $\mathcal{W}I$, then $f\square g$ is a $\mathcal{W}-$ acyclic cofibration.*

Proof This reduces to our assumption on $\mathcal{E}$ and $\mathcal{W}$. See also [35, II.1.22]. □

Proposition 2.20 *Let $\mathcal{E}$ be a $\mathcal{W}$–Illman collection of subgroups of G. Then the $\mathcal{W}$–model structure on $G\mathcal{T}$, from Proposition 2.11, is a $\mathcal{E}$–topological model structure.*

Proof This follows from [35, III.1.15–1.21] and Lemma 2.19. □

The $\mathcal{W}$–model structure on $G\mathcal{M}$ is a topological model structure for any collection $\mathcal{W}$ by Remark 2.18.

Remark 2.21 A based topological model category $\mathcal{M}$ has a canonical based simplicial model structure. In the topological model structure denote the mapping space by $\mathrm{Map}\,(M,N)$, the tensor by $M\square X$, and the cotensor by $F_{\square}(X,M)$. Here X is a based space, and M and N are objects in $\mathcal{M}$. The singular simplicial set functor, sing, is right adjoint to the geometric realization functor $|-|$. The corresponding based simplicial mapping space is given by $\mathrm{sing}\,(\mathrm{Map}\,(M,N))$. The simplicial tensor and cotensor are $M\square|K|$ and $F_{\square}(|K|,M)$, respectively, where K is a based simplicial set and M and N are objects in $\mathcal{M}$. We use that $|K\wedge L|\cong|K|\wedge|L|$.

A based simplicial structure gives rise to an unbased simplicial structure. We get a unbased simplicial structure by forgetting the basepoint in the based simplicial mapping space, and by adding a disjoint basepoint to unbased simplicial sets in the definition of the tensor and the cotensor. Hence we can apply results about (unbased) simplicial model structures to a topological model category.

2.4 Some change of groups results for spaces

Let $\phi\colon G_1\to G_2$ be a continuous group homomorphism between compact Hausdorff groups. Let $j\colon G_2\mathcal{T}\to G_1\mathcal{T}$ be defined by restricting the G_2–action to a G_1–action along ϕ. This functor has a left adjoint given by sending X to $(G_2)_+\wedge_{G_1}X$ and a right adjoint given by sending X to $\mathcal{T}_{G_1}((G_2)_+,X)$.

In general these three functors do not behave well with respect to the model structures on the categories of G_1–spaces and G_2–spaces. We give some conditions that guarantee that they are Quillen adjoint functors. Let $\mathcal{W}_1$ be a collection of subgroups of G_1 and let $\mathcal{W}_2$ be a collection of subgroups of G_2. Let $\phi\mathcal{W}_1$ be the smallest collection of subgroups of G_2 containing (the closures of) $\phi(H)$, for all $H \in \mathcal{W}_1$. Let $\phi^{-1}\mathcal{W}_2$ be the smallest collection of subgroups of G_1 containing $\phi^{-1}(K)$, for all $K \in \mathcal{W}_2$.

Lemma 2.22 *The functor $j\colon \mathcal{W}_2G_2\mathcal{T} \to \mathcal{W}_1G_1\mathcal{T}$ is a right Quillen adjoint functor if $\phi\mathcal{W}_1 \subset \mathcal{W}_2$ and a left Quillen adjoin functor if, in addition, $\phi^{-1}\mathcal{W}_2 \subset \mathcal{W}_1$ and $\mathcal{W}_2 \subset \overline{\mathrm{Lie}(G_2)}$.*

Proof The left adjoint functor $(G_2)_+ \wedge_{G_1} X$ is a Quillen left adjoint functor if it respects the generators in Definition 2.10. Hence the first claim follows since $(G_2)_+ \wedge_{G_1} G_1/H_+ \cong (G_2/\overline{\phi(H)})_+$. Let K be in $\overline{\mathrm{Lie}(G_2)}$. Then the restriction of G_2/K along ϕ is a G_1–cell complex with stabilizers in $\phi^{-1}(gKg^{-1})$, for $g \in G_2$ [28]. Hence j respects cofibrations if $\phi^{-1}\mathcal{W}_2 \subset \mathcal{W}_1$. The H–fixed points of $j(X)$ is $X^{\phi(H)}$. Hence j respects acyclic cofibrations if in addition $\phi\mathcal{W}_1 \subset \mathcal{W}_2$. □

Any group homomorphism between compact Hausdorff groups is a composite of a surjective identification homomorphism followed by a closed inclusion of a subgroup. So for compact Hausdorff groups it suffices to consider these two types of group homomorphisms. Let K be a subgroup of G. Then the forgetful functor $G\mathcal{T} \to K\mathcal{T}$ has a left adjoint given by sending X to $G_+ \wedge_K X$ and a right adjoint given by sending X to $\mathcal{T}_K(G_+, X)$. Let N be a normal subgroup of G. Then the functor $G/N\mathcal{T} \to G\mathcal{T}$ has a left adjoint given by the N–orbit functor and a right adjoint given by the N–fixed point functor.

Example 2.23 Let K be a subgroup of G. The forgetful functor from $\overline{\mathrm{Lie}(G)}\, G\mathcal{T}$ to $\overline{\mathrm{Lie}(K)}\, K\mathcal{T}$ is both a left and a right Quillen adjoint functor if K is in $\overline{\mathrm{Lie}(G)}$. It is neither a left nor a right Quillen adjoint functor if K is not in $\overline{\mathrm{Lie}(G)}$.

Let N be a normal subgroup of G. Then the functor from $\overline{\mathrm{Lie}(G/N)}\, G/N\mathcal{T}$ to $\overline{\mathrm{Lie}(G)}\, G\mathcal{T}$ is both a left and a right Quillen adjoint functor.

3 Orthogonal G–Spectra

Equivariant orthogonal spectra for compact Lie groups were introduced by Michael Mandell and Peter May in [35]. We generalize their theory to allow more general groups. We develop the theory with minimal assumptions on the collection of subgroups used to define cofibrations and weak equivalences. We follow Chapters 2 and 3 of their work closely.

3.1 $\mathcal{J}_G^{\mathcal{V}}$–spaces

We define universes of G–representations.

Definition 3.1 A G–universe $\mathcal{U}$ is a countable infinite direct sum $\oplus_{i=1}^{\infty}\mathcal{U}'$ of a real G–inner product space $\mathcal{U}'$ satisfying the following: (1) the one-dimensional trivial G–representation is contained in $\mathcal{U}'$; (2) $\mathcal{U}$ is topologized as the union of all finite dimensional G–subspaces of $\mathcal{U}$ (each with the norm topology); and (3) the G–action on all finite dimensional G–subspaces V of $\mathcal{U}$ factors through a compact Lie group quotient of G.

If G is a compact Hausdorff group, then the G–action on a finite dimensional G–representation factors through a compact Lie group quotient of G by Lemma A.1. This is not true in general (consider the representation $\mathbb{Q}/\mathbb{Z} < S^1$). We only use the finite dimensional G–subspaces of $\mathcal{U}$, so one might as well assume that $\mathcal{U}'$ is a union of such.

Definition 3.2 Let S^V denote the one-point compactification of a finite dimensional G–representation V.

The last assumption in Definition 3.1 is added to guarantee that spaces like S^V have the homotopy type of a finite $\overline{\mathrm{Lie}(G)}I$–cell complex.

Definition 3.3 If the G–action on $\mathcal{U}$ is trivial, then $\mathcal{U}$ is called a trivial universe. If each finite dimensional orthogonal G–representations is isomorphic to a G–subspace of $\mathcal{U}$, then $\mathcal{U}$ is called a complete G–universe.

All compact Hausdorff groups have a complete universe. However, it might not be possible to find a complete universes with a countable dimension. Traditionally, the universes have often been assumed to have countable dimension (May [37, Definition IX.2.1]).

Remark 3.4 A complete G–universe suffices to construct a sensible equivariant homotopy theory for compact Hausdorff groups with the weak equivalences determined by the cofamily closure of $\mathrm{Lie}(G)$. For example, there are Spanier–Whitehead duals of suspension spectra of finite cell–complexes with stabilizers in $\overline{\mathrm{Lie}(G)}$ (see Proposition 4.7). Transfer maps can then be constructed as in [37, XVII. Section 1].

We recall some definitions from [35, Chapter II].

Definition 3.5 Let $\mathcal{U}$ be a universe. An indexing representation is a finite dimensional G–inner product subspace of $\mathcal{U}$. If V and W are two indexing representations and $V \subset W$, then the orthogonal complement of V in W is denoted by $W - V$. The collection of all real G–inner product spaces that are isomorphic to an indexing representation in $\mathcal{U}$ is denoted $\mathcal{V}(\mathcal{U})$.

When $\mathcal{U}$ is understood, we write $\mathcal{V}$ instead of $\mathcal{V}(\mathcal{U})$ to make the notation simpler.

Definition 3.6 Let $\mathcal{J}_G^{\mathcal{V}}$ be the unbased topological category with objects $V \in \mathcal{V}$ and morphisms linear isometric isomorphisms. Let $G\mathcal{J}^{\mathcal{V}}$ denote the G–fixed category $(\mathcal{J}_G^{\mathcal{V}})^G$.

Definition 3.7 A continuous G–functor $X\colon \mathcal{J}_G^{\mathcal{V}} \to \mathcal{T}_G$ is called a $\mathcal{J}_G^{\mathcal{V}}$–space. (The induced map on hom spaces is a continuous unbased G–map.) Denote the category of $\mathcal{J}_G^{\mathcal{V}}$–spaces and (enriched) natural transformations by $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$. Let $G\mathcal{J}^{\mathcal{V}}\mathcal{T}$ denote the G–fixed category $(\mathcal{J}_G^{\mathcal{V}}\mathcal{T})^G$.

Definition 3.8 Let $\mathcal{S}_G^{\mathcal{V}}\colon \mathcal{J}_G^{\mathcal{V}} \to \mathcal{T}_G$ be the $\mathcal{J}_G^{\mathcal{V}}$–space defined by sending V to the one point compactification S^V of V. For simplicity we sometimes denote $\mathcal{S}_G^{\mathcal{V}}$ by $\mathcal{S}$.

The external smash product

$$\overline{\wedge}\colon \mathcal{J}_G^{\mathcal{V}}\mathcal{T} \times \mathcal{J}_G^{\mathcal{V}}\mathcal{T} \to (\mathcal{J}_G^{\mathcal{V}} \times \mathcal{J}_G^{\mathcal{V}})\mathcal{T}$$

is defined to be $X\overline{\wedge}Y(V,W) = X(V) \wedge Y(W)$ for $X, Y \in \mathcal{J}_G^{\mathcal{V}}$ and $V, W \in \mathcal{V}$. The direct sum of finite dimensional real G–inner product spaces gives $\mathcal{J}_G^{\mathcal{V}}$ the structure of a symmetric tensor category. A topological left Kan extension gives an internal smash product on $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$ (Mandell–May–Schwede–Shipley [36, 21.4, 21.6]). We give an explicit description of the smash product. Let W be a real N–dimensional G–representation in $\mathcal{V}(\mathcal{U})$. Choose G–representations V_n and V_n' of dimension n in $\mathcal{V}(\mathcal{U})$ for $n = 0, 1, \dots, N$. For example let $V_n = V_n'$ be the trivial n–dimensional G–representation, $\mathbb{R}^n$. Then we have a canonical equivalence

$$X \wedge Y(W) \cong \vee_{n=0}^{N} \mathcal{J}_G^{\mathcal{V}}(W, V_n \oplus V_{N-n}') \wedge_{\mathcal{O}(V_n)\times\mathcal{O}(V_{N-n}')} X(V_n) \wedge Y(V_{N-n}').$$

The internal hom from X to Y is the $\mathcal{J}_G^{\mathcal{V}}$–space

$$V \mapsto \mathcal{J}_G^{\mathcal{V}}\mathcal{T}(X(-), Y(V \oplus -))$$

given by the space of continuous natural transformation of $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$–functors. The internal smash product and the internal hom functor give $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$ the structure of a closed symmetric tensor category [35, II.3.1,3.2]. The unit object is the functor that sends the indexing representation V to S^0 when $V = 0$, and to a point when $V \neq 0$. By passing to fixed points we also get a closed symmetric tensor structure on $G\mathcal{J}^{\mathcal{V}}\mathcal{T}$.

3.2 Orthogonal R–modules

For the definition of monoids and modules over a monoid in tensor categories see Mac Lane [34, VII.3 and 4]. The functor $\mathcal{S}_G^{\mathcal{V}}$ is a strong symmetric functor. Hence the $\mathcal{J}_G^{\mathcal{V}}$–space $\mathcal{S}_G^{\mathcal{V}}$ is naturally a symmetric monoid in $G\mathcal{J}^{\mathcal{V}}\mathcal{T}$. The following definition is from [35, II.2.6].

Definition 3.9 An orthogonal G–spectrum X is a $\mathcal{J}_G^{\mathcal{V}}$–space $X\colon \mathcal{J}_G^{\mathcal{V}} \to \mathcal{T}_G$ together with a left module structure over the symmetric monoid $\mathcal{S}_G^{\mathcal{V}}$ in $G\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$. Denote the category of G–spectra by $\mathcal{J}_G^{\mathcal{V}}\mathcal{S}$. Let $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$ be the G–fixed category $(\mathcal{J}_G^{\mathcal{V}}\mathcal{S})^G$.

The smash product and internal hom functors of orthogonal spectra are the smash product and internal hom functors of $\mathcal{S}_G^{\mathcal{V}}$–modules. So the category of orthogonal G–spectra, $\mathcal{J}_G^{\mathcal{V}}\mathcal{S}$, is itself a closed symmetric tensor category with $\mathcal{S}_G^{\mathcal{V}}$ as the unit object [35, II.3.9]. The fixed point category $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$ inherits a closed tensor structure from $\mathcal{J}_G^{\mathcal{V}}\mathcal{S}$. Explicit formulas for the tensor and internal hom functors are obtained from the formulas after Definition 3.8 and [35, II.3.9]. The monoid $\mathcal{S}_G^{\mathcal{V}}$ is symmetric so a left $\mathcal{S}_G^{\mathcal{V}}$–module has a natural right module structure.

Definition 3.10 We call a monoid R in $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$ an algebra. We say that R is a commutative algebra, or simply a ring, if it is a symmetric monoid in $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$. We sometimes add: orthogonal, G, and spectrum, to avoid confusion.

Let R be an orthogonal algebra spectrum. We assume that the basepoint of each based space $R(V)$ is nondegenerated. This assumption can be circumvented for the stable model structure when $\mathcal{W}$ is an Illman collection [35, Theorem III.3.5, Section III.7].

Definition 3.11 An R–module is a left R–module in the category of orthogonal spectra. Let $\mathcal{M}_R$ denote the category of R–modules.

The category of R–modules is complete and cocomplete. If R is a commutative monoid, then the category $\mathcal{M}_R$ is a closed symmetric tensor category. A monoid T in the category of R–modules is called an R–algebra. Any R–algebra is an $\mathcal{S}$–algebra. Let T be an R–algebra. Then the category of T–modules, in the category of R–modules, is equivalent to the category of T–modules, in the category of $\mathcal{S}$–modules, when T is regarded as an $\mathcal{S}$–algebra.

We now give pairs of adjoint functors between orthogonal G–spectra and G–spaces. The V–evaluation functor

$$\Omega^{\infty}_V\colon \mathcal{J}_G^{\mathcal{V}}\mathcal{S} \to \mathcal{T}_G$$

is given by $X \mapsto X(V)$. We abuse language and let Ω^∞_V also denote the functor Ω^∞_V precomposed with the forgetful functor from R–modules to orthogonal spectra. There is a left adjoint, denoted Σ^R_V, of the V–evaluation functor $\Omega^\infty_V\colon \mathcal{M}_R \to \mathcal{T}_G$. The R–module $\Sigma^R_V Z$, for a G–space Z, sends $W \in \mathcal{V}(\mathcal{U})$ to

$$\Sigma^R_V Z(W) = Z \wedge \mathcal{O}(W)_+ \wedge_{\mathcal{O}(W-V)} R(W-V) \tag{3–12}$$

when $W \supset V$, and to a point otherwise [36, 4.4]. This functor is called the V–shift desuspension spectrum functor and is also denoted F_V and Σ^∞_V (when $R = \mathcal{S}$) in [35]. When $V = 0$ we denote this functor by Σ^∞_R. We have that $\Sigma^R_V Z \cong \Sigma^{\mathcal{S}}_V Z \wedge R$.

3.3 Fixed point and orbit spectra

We define fixed point and orbit spectra. The results on adjoint functors and change of universes from [35, Section V.1] extend to our setting. The change of groups results in SubSection 2.4 extends to the model structures on the category of orthogonal spectra (constructed in later sections). We do not make those results explicit.

Let X be an orthogonal spectrum and let H be a subgroup of G. Then the quotient X/H is defined to be $X/H(V) = X(V)/H$ with structure maps

$$X/H(V) \wedge S^W \to X(V)/H \wedge S^W/H \cong (X(V) \wedge S^W)/H \to X(V \oplus W)/H.$$

The H–orbit spectrum is an $N_G H$–spectrum with trivial H–action. The orbit spectrum X/H restricted to the the universe $\mathcal{U}^H$ is an $N_G H/H$–spectrum.

Let H be a subgroup of G. Let X be a $\mathcal{U}$–spectrum. Then the H–fixed point spectrum X^H indexed on $\mathcal{U}^H$ is defined by $X^H(V) = (X(V))^H$ for $V \in \mathcal{V}(\mathcal{U}^H)$, and the structure map is

$$X^H(V) \wedge S^W \cong (X(V) \wedge S^W)^H \to X^H(V \oplus W).$$

This is a $N_G H/H$–spectrum. One can also define geometric fixed point spectra as in [35, Section V.4].

Let N be a closed normal subgroup of G, and let $G/N\mathcal{M}$ be the category of G/N–spectra indexed on $\mathcal{U}^N$. The N–orbit and N–fixed point functors are left and right adjoint functor to the restriction functor $G/N\mathcal{M} \to G\mathcal{M}$, respectively [35, Section V.3].

Remark 3.13 Let H be a subgroup of G. There is a restriction functor $G\mathcal{M}_R \to H\mathcal{M}_R$. A G–spectrum is regarded as an H–spectrum indexed on $\mathcal{V}(\mathcal{U})|H$. This is not a spectrum indexed on $\mathcal{V}(\mathcal{U}|H)$ because not all indexing H–representations in $\mathcal{U}$

are obtained as the restriction of an indexing G–representation in $\mathcal{U}$. There is a change of indexing functor associated to the full inclusion $\mathcal{V}(\mathcal{U})|H \to \mathcal{V}(\mathcal{U}|H)$. They define the same homotopy theory so for simplicity we use $\mathcal{V}(\mathcal{U})|H$ in this paper. See [35, Section V.2].

3.4 Examples of orthogonal G–Spectra

Let $T\colon \mathcal{T}_G \to \mathcal{T}_G$ be a continuous G–functor. Then we define the corresponding $\mathcal{J}_G^{\mathcal{V}}$–space by $T \circ \mathcal{S}_G^{\mathcal{V}}\colon \mathcal{V}(\mathcal{U}) \to \mathcal{T}_G$. This $\mathcal{J}_G^{\mathcal{V}}$–space is given an orthogonal G–spectrum structure by letting $S^W \wedge T(S^V) \to T(S^{V\oplus W}) \cong T(S^V \wedge S^W)$ be the adjoint of the map

$$S^W \to \mathcal{T}_G(S^V, S^W \wedge S^V) \xrightarrow{T} \mathcal{T}_G(T(S^V), T(S^W \wedge S^V))$$

where the first map is a G–map adjoint to the identity on $S^W \wedge S^V$.

We can define a G–equivariant orthogonal K–theory spectrum for a compact Hausdorff group G. If X is a compact G–space, then $K_G(X)$ is the Grothendieck construction on the semiring of isomorphism classes of finitely generated real bundles on X. The Atiyah–Segal completion theorem generalizes to compact Hausdorff groups if we make use of a suitable completion functor (see [18]).

Let G be a compact Hausdorff group. We define a Thom spectrum as $TO_G(V) = \operatorname{colim}_U TO_{G/U}(V)$ where the colimit is over $U \in \mathrm{Lie}(G)$ such that V has a trivial U–action. For more details see [18].

3.5 The levelwise $\mathcal{W}$–model structure on orthogonal G–Spectra

We make some minor modifications to the discussion of model structures in [35, III].

Let R be an algebra. The category of R–modules can be described as the category of continuous $\mathcal{D}$–spaces for an appropriate diagram category $\mathcal{D}$. The objects are the same as those of $\mathcal{J}_G^{\mathcal{V}}$, but the morphisms are more elaborate. See [35, Section II.4] and [36, Section 23] for details. Interpreted as a continuous diagram category in $G\mathcal{T}$, we give $\mathcal{M}_R$ the model structure with weak equivalences and fibrations inherited from the $\mathcal{W}$–model structure on $G\mathcal{T}$ [24, 11.3.2]. Recall Definition 2.10.

Definition 3.14 Let $\Sigma_R^\infty \mathcal{W}I$ denote the collection of $\Sigma_V^R i$, for all $i \in \mathcal{W}I$ and all indexing representations V in $\mathcal{U}$. Let $\Sigma_R^\infty \mathcal{W}J$ denote the collection of $\Sigma_V^R j$, for all $j \in \mathcal{W}J$ and all indexing representation V in $\mathcal{U}$.

Lemma 3.15 *If $f\colon X \to Y$ is a relative $\Sigma_R^\infty \mathcal{W}I$–cell complex, then $X(V) \to Y(V)$ is a G–equivariant Hurewicz cofibration (satisfies the homotopy extension property), for every $V \in \mathcal{V}$. All Hurewicz cofibrations are closed inclusions.*

Proof The adjunction between Σ^R_V and Ω^∞_V gives that the maps in the classes $\Sigma^\infty_R\mathcal{W}I$ and $\Sigma^\infty_R\mathcal{W}J$ are Hurewitz cofibrations. Hence relative $\mathcal{W}$–cell complexes, and their retracts, are Hurewitz cofibrations. The last claim follows since our spaces are weak Hausdorff. □

Lemma 3.16 *Let $\mathcal{E}$ be a $\mathcal{W}$–Illman collection. Assume that $f\colon A\to B$ is in $\Sigma^\infty_R\mathcal{E}I$ and $g\colon X\to Y$ is in $\Sigma^\infty_R\mathcal{W}I$, then $f\square g\colon (A\wedge Y)\cup_{A\wedge X}(B\wedge X)\to B\wedge Y$ is a relative $\Sigma^\infty_R\mathcal{W}I$–cell complex. Moreover, if f is in $\Sigma^\infty_R\mathcal{E}J$ instead of $\Sigma^\infty_R\mathcal{E}I$ or g is in $\Sigma^\infty_R\mathcal{W}J$ instead of $\Sigma^\infty_R\mathcal{W}I$, then $f\square g$ is a relative $\Sigma^\infty_R\mathcal{W}J$–cell complex.*

Proof This reduces to the analogue for spaces, Lemma 2.19, since Σ^R_V respects colimits and smash products from spaces to R–modules. □

The following model structure on the category of orthogonal G–spectra is called the levelwise $\mathcal{W}$–model structure.

Proposition 3.17 *Let $\mathcal{W}$ be a collection of subgroups of G. Then the category of R–modules has a compactly generated proper model structure with levelwise $\mathcal{W}$–weak equivalences and levelwise $\mathcal{W}$–fibrations (as $\mathcal{J}^{\mathcal{V}}_G$–diagrams). The cofibrations are generated by $\Sigma^\infty_R\mathcal{W}I$, and the acyclic cofibrations are generated by $\Sigma^\infty_R\mathcal{W}J$. If $\mathcal{E}$ is a $\mathcal{W}$–Illman collection, then the model structure is $\mathcal{E}$–topological.*

Proof The source of the maps in $\Sigma^\infty_R\mathcal{W}I$ and $\Sigma^\infty_R\mathcal{W}J$ are small since Ω^∞_V respects sum. Let f be a relative $\Sigma^\infty_R\mathcal{W}J$–cell complexes. Then $\Omega^\infty_V f$, for any indexing representation V in $\mathcal{U}$, is the colimit of a sequence of equivariant homotopy equivalences that are (closed) Hurewicz cofibrations, hence a levelwise $\mathcal{W}$–equivalences of G–spaces. It follows that $\mathcal{M}_R$ inherits a model structure from $G\mathcal{T}$ via the set of right adjoint functors Ω^∞_V, for indexing representations V in $\mathcal{U}$. The model structure is right proper since $\mathcal{W}\mathcal{T}$ is right proper. It is left proper by Lemma 3.15 because fixed points respects pushout along a closed inclusion and since pushout of a weak equivalence along a closed Hurewicz cofibration of spaces is again a weak equivalence. The last claim follows from Lemma 3.16. See also [36, 6.5]. □

Definition 3.18 The V–loop space $\Omega^V Z$ of a G–space Z is $\mathcal{T}_G(S^V, Z)$. A spectrum X is called a $\mathcal{W}$–Ω–spectrum if the adjoint of the structure maps, $X(V)\to\Omega^{W-V}X(W)$ are $\mathcal{W}$–equivalences of spaces for all pairs $V\subset W$ in $\mathcal{V}(\mathcal{U})$.

4 The stable $\mathcal{W}$–model structure on orthogonal G–spectra

We define stable equivalences between orthogonal G–spectra and localize the levelwise $\mathcal{W}$–model structure with respect to these equivalences [35, III.3.2]. Recall that if Z is a based G–space, then $\pi_n^H(Z)$ denotes the nth homotopy group of Z^H. We say that a map of based spaces, $f\colon X \to Y$, is a based $\mathcal{W}$–equivalence if $\pi_n^U(f)$ is an isomorphism for all $U \in \mathcal{W}$ and all $n \geq 0$. Let $f\colon X \to Y$ be a based G–map. If f is a unbased $\mathcal{W}$–equivalence as defined in Definition 2.8, then f is a based $\mathcal{W}$–equivalence. If f is a based $\mathcal{W}$–equivalence, then $\Omega f\colon \Omega X \to \Omega Y$ is a unbased $\mathcal{W}$–equivalence.

Definition 4.1 The nth homotopy group of an orthogonal G–spectrum X at a subgroup H of G is

$$\pi_n^H(X) = \operatorname{colim}_V \pi_n^H(\Omega^V X(V))$$

for $n \geq 0$, and

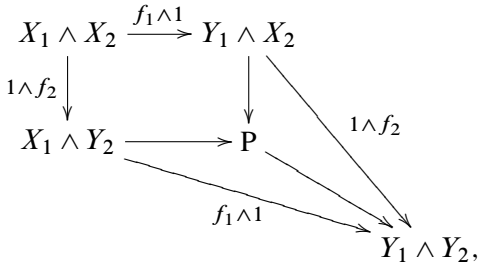
$$\pi_{-n}^H(X) = \operatorname{colim}_{V \supset \mathbb{R}^n} \pi_0^H(\Omega^{V-\mathbb{R}^n} X(V))$$

for $n \geq 0$, where the colimit is over indexing representations in $\mathcal{U}$. A map $f\colon X \to Y$ of orthogonal G–spectra is a stable $\mathcal{W}$–equivalence if $\pi_n^H(f)$ is an isomorphism for all $H \in \mathcal{W}$ and all $n \in \mathbb{Z}$.

We follow our program of giving model structures to the category of orthogonal G–spectra with minimal assumptions on the collection of subgroups used. We need to impose conditions on the stabilizers of the indexing representations. Let $\mathrm{st}(\mathcal{U})$ be the collection of stabilizers of elements in the G–universe $\mathcal{U}$. We need $\mathrm{st}(\mathcal{U})$ to be a $\mathcal{W}$–Illman collection. If $\mathcal{U}$ is a trivial G–universe, then $\mathrm{st}(\mathcal{U})$ is a $\mathcal{W}$–Illman collection for any collection $\mathcal{W}$ (see Definition 2.14). If $\mathcal{U}$ is a complete universe, then $\mathrm{st}(\mathcal{U})$ is a $\mathcal{W}$–Illman collection if $\mathcal{W}$ is a family in the cofamily closure of $\mathrm{Lie}(G)$ (see Definition 2.2). If $\mathcal{W}$ is an Illman collection of subgroups of G containing G itself, then $\mathrm{st}(\mathcal{U})$ is $\mathcal{W}$–Illman if and only if $\mathrm{st}(\mathcal{U})$ is contained in $\mathcal{W}$.

The category $\mathcal{M}_R$ is a closed symmetric tensor category. We follow Schwede–Shipley [39, 2] when considering the interaction of model structures and tensor structures. A model structure is said to be tensorial if the following pushout–product axiom is valid.

Definition 4.2 Pushout–product axiom [39, 2.1]: Let $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be cofibrations. Then the map from the pushout, P, to $Y_1 \wedge Y_2$ in the diagram



is again a cofibration. If, in addition, one of the maps f_1 or f_2 is a weak equivalence, then $\mathrm{P} \to Y_1 \wedge Y_2$ is also a weak equivalence.

Definition 4.3 Monoid axiom [39, 2.2]: Any acyclic cofibration tensored with an arbitrary object in $\mathcal{M}$ is a weak equivalence. Moreover, arbitrary pushouts and transfinite compositions of such maps are weak equivalences.

The following model structure on $\mathcal{M}_R$ is called the (stable) $\mathcal{W}$–model structure. We sometimes denote $\mathcal{M}_R$ together with the $\mathcal{W}$–model structure by $\mathcal{W}\mathcal{M}_R$.

Theorem 4.4 *Let R be an algebra. Assume that $\mathrm{st}(\mathcal{U})$ is a $\mathcal{W}$–Illman collection of subgroups of G. The category of R–modules is a compactly generated proper model category such that the weak equivalences are the stable $\mathcal{W}$–equivalences, the cofibrations are retracts of relative $\Sigma^\infty_R \mathcal{W} I$–cell complexes, and a map $f\colon X \to Y$ is a fibration if and only if the map $f(V)\colon X(V) \to Y(V)$ is a $\mathcal{W}$–fibration and the obvious map from $X(V)$ to the pullback of the diagram*

$$\begin{array}{ccc} & & \Omega^W X(V \oplus W) \\ & & \downarrow \\ Y(V) & \longrightarrow & \Omega^W Y(V \oplus W) \end{array}$$

is a unbased $\mathcal{W}$–equivalence of spaces for all $V, W \in \mathcal{V}$. A map $f\colon X \to Y$ is an acyclic fibration if and only if f is a levelwise acyclic fibration.

Assume R is symmetric. If $\mathcal{E}$ is a $\mathcal{W}$–Illman collection, then the model structure is $\mathcal{E}$–topological. If $\mathcal{W}$ is an Illman collection, then the model structure satisfies the pushout–product axiom and the monoid axiom.

Remark 2.18 and Remark 2.21 imply that the $\mathcal{W}$–model structure is simplicial. The fibrant spectra are exactly the $\mathcal{W}$–Ω–spectra. If X is a cofibrant $\mathcal{S}$–module, then a fibrant replacement is given by

$$QX(V) = \operatorname{hocolim}_W \Omega^W X(V \oplus W) \tag{4–5}$$

together with the natural transformation $1 \to Q$.

The stable homotopy group $\pi_n^H(X)$ is isomorphic to $\pi_n^H(QX')$, where X' is a cofibrant replacement of X in the category of $\mathcal{S}$–modules with the levelwise $\mathcal{W}$–model structure. Hence we get the following Lemma.

Lemma 4.6 *Let H be in $\mathcal{W}$. The stable homotopy group π_n^H is corepresented by $\Sigma^R_{\mathbb{R}^{-n}} G/H_+$, for $n \leq 0$, and by $\Sigma^\infty_R G/H_+ \wedge S^n$, for $n \geq 0$, in the homotopy category of $\mathcal{M}_R$ with the stable $\mathcal{W}$–model structure. Hence π_n^H is a homology theory which satisfies the colimit axiom.*

The colimit axiom says that $\operatorname{colim}_a \pi_*^H(X_a) \to \pi_*^H(X)$ is an isomorphism, where the colimit is over all finite subcomplexes X_a of the cell complex X.

Proposition 4.7 *Let $\mathcal{U}$ be a complete G–universe. Let $\mathcal{W}$ be a family in $\overline{\mathit{Lie}(G)}$. Then the dualizable objects in the homotopy category of $\mathcal{WM}_R$ (with the $\mathcal{W}$–model structure) are precisely retracts of $\mathcal{V}(\mathcal{U})$–desuspensions of finite $\mathcal{W}$–cell complexes.*

Proof The proof in May [37, XVI 7.4] goes through with modifications to allow for general R–modules instead of $\mathcal{S}$–modules. □

4.1 Verifying the model structure axioms

Lemma 4.8 *Let X, Y and Z be based G–spaces. Let $\mathcal{E}$ be a $\mathcal{W}$–Illman collection. If Z is a $\mathcal{E}I$–cell complex and $f\colon X \to Y$ is a $\mathcal{W}$–equivalence, then $\mathcal{T}_G(Z,X) \to \mathcal{T}_G(Z,Y)$ is a $\mathcal{W}$–equivalence.*

Proof Since $S^k \wedge S^n_+$ is a based CW–complex and $(G/H \times G/L)_+$ is a $\mathcal{W}I$–cell complex, for all $H \in \mathcal{W}$ and $L \in \mathcal{E}$, it follows that $S^k \wedge G/H_+ \wedge Z$ is a $\mathcal{W}I$–cell based complex. The smash product and internal hom adjunction gives that

$$\pi_k^H(\mathcal{T}_G(Z,X)) \cong [S^k \wedge G/H_+ \wedge Z, X]_G$$

where the square brackets are G–homotopy classes of maps (see Proposition 2.12). Since $\pi_k^K(f) \cong [S^k \wedge G/K_+, f]_G$ is an isomorphism for all $K \in \mathcal{W}$ and $k \geq 0$, the result follows by using the higher lim spectral sequence. □

Corollary 4.9 *Assume st($\mathcal{U}$) is a $\mathcal{W}$–Illman collection of subgroups of G. A levelwise $\mathcal{W}$–equivalence of G–spectra is a stable $\mathcal{W}$–equivalence.*

Proof By Definition 3.1, any finite dimensional G–representation V in the universe $\mathcal{U}$ is a G/U–representation for a compact Lie group quotient G/U of G. Since the collection st($\mathcal{U}$) is $\mathcal{W}$–Illman, it follows from Lemma 4.8 that

$$\Omega^V f(V') \cong \mathcal{T}_G(S^V,\ f(V'))$$

is a $\mathcal{W}$–equivalence for all $V, V' \in \mathcal{V}$. □

Note that if st($\mathcal{U}$) is a $\mathcal{W}$–Illman collection of subgroups of G, A is a based $\mathcal{W}I$–cell complex, and V is an indexing representation in $\mathcal{U}$, then $A \wedge S^V$ is again a based $\mathcal{W}I$–cell complex by Lemma 2.16. The next result, together with Corollary 4.9, show that a map between Ω–$\mathcal{W}$–spectra is a levelwise $\mathcal{W}$–equivalence if and only if it is a $\mathcal{W}$–equivalence. This fundamental result is an extension of [35, III.9].

Proposition 4.10 *Assume that st($\mathcal{U}$) is a $\mathcal{W}$–Illman collection of subgroups of G. Let $f\colon X \to Y$ be a map of $\mathcal{W}$–Ω–spectra. If*

$$f_*\colon \pi_*^H(X) \to \pi_*^H(Y)$$

is an isomorphism for each $H \in \mathcal{W}$, then $f(V)\colon X(V) \to Y(V)$ is a $\mathcal{W}$–equivalence for all indexing representations $V \subset \mathcal{U}$.

Proof We first prove that

$$f(V)_*\colon \pi_*^H(X(V)) \to \pi_*^H(Y(V))$$

is an isomorphism for all $H \in \mathcal{W}$. Let Z be the homotopy fiber of f. It is again an Ω–G–spectrum. We want to show that $\pi_*^H(Z) = 0$ for all $H \in \mathcal{W}$, implies that $\pi_*^H(Z(V)) = 0$ for any indexing representations V and any $H \in \mathcal{W}$. Fix an indexing representation V and a normal subgroup $N \in \mathrm{Lie}(G)$ such that N is a finite intersection of elements in st($\mathcal{U}$) and N acts trivially on V. With these choices $(\Omega^V Z(V))^H = \Omega^V(Z(V)^H)$ for all $H \leq N$. Hence $\pi_{*+|V|}^H(Z(V))$ is isomorphic to $\pi_*^H(\Omega^V Z(V))$ for all $H \leq N$ in $\mathcal{W}$. Since Z is an Ω–G–spectrum, an easy argument gives that $\pi_*^H(Z(V)) = 0$ for all $H \in \mathcal{W}$ such that $H \leq N$ [35, III.9.2].

We now prove the result for subgroups H in $\mathcal{W}$ that are not necessarily contained in N. Fix a subgroup $H \in \mathcal{W}$. Assume by induction that $\pi_*^K(Z(V)) = 0$ for all subgroups $K \in \mathcal{W}$ such that K is properly contained in H. If L is a stabilizer of V, then $H \cap gLg^{-1}$ is in $\mathcal{W}$ for all $g \in G$ since st($\mathcal{U}$) is a $\mathcal{W}$–Illman collection. The

argument given in [35, Section III.9] implies that $\pi_*^H(Z(V)) = 0$. We now justify that we can make the inductive argument. The quotient group $H/H \cap N$ is isomorphic to $H \cdot N/N$, which is a subgroup of the compact Lie group G/N. Hence the partially ordered set of closed subgroups of H containing $H \cap N$ satisfies the descending chain property. We have that $\pi_*^K(Z(V)) = 0$ for all $K \leq H \cap N$ in $\mathcal{W}$. We start the induction with the subgroup $H \cap N$ which by assumption is in $\mathcal{W}$. For more details see [35, Section III.9].

Since $X(V)^H$ and $Y(V)^H$ are weakly equivalent to $\Omega^{\mathbb{R}} X(V \oplus \mathbb{R})^H$ and $\Omega^{\mathbb{R}} Y(V \oplus \mathbb{R})^H$, respectively, and $\pi_{k+1}\colon \pi_{k+1}(X(V \oplus \mathbb{R})^H) \to \pi_{k+1}(X(V \oplus \mathbb{R})^H)$ is an isomorphism for each $H \in \mathcal{W}$, $k \geq 0$ and indexing representation V, it follows that $f(V)$ is a $\mathcal{W}$–equivalence. So f is a levelwise $\mathcal{W}$–equivalence. □

A set of generating cofibrations is $\Sigma_R^\infty \mathcal{W}I$. We give a set of generating acyclic cofibrations. Let

$$\lambda_{V,W}\colon \Sigma^R_{V\oplus W} S^W \to \Sigma^R_V S^0$$

be the adjoint of the map

$$S^W \to (\Sigma^R_V S^0)(V \oplus W) \cong \mathcal{O}(V \oplus W)_+ \wedge_{\mathcal{O}(W)} R(W)$$

given by sending an element w in S^W to $e \wedge \iota(W)(w)$ where e is the identity map in $\mathcal{O}(V \oplus W)$, and $\iota\colon \mathcal{S} \to R$ is the unit map. Let $k_{V,W}$ be the map from $\Sigma^R_{V\oplus W} S^W$ to the mapping cylinder, $M\lambda_{V,W}$, of $\lambda_{V,W}$. Let $\mathcal{W}K$ be the union of $\Sigma_R^\infty \mathcal{W}J$ and the set of maps of the form $i \square k_{V,W}$ for $i \in \Sigma_R^\infty \mathcal{W}I$ and indexing representations V, W in $\mathcal{U}$. The box denotes the pushout–product map. The map $\lambda_{V,W}$ is a $\mathcal{W}$–equivalence and $k_{V,W}$ is a levelwise $\mathrm{st}(\mathcal{U})$–cofibration for all indexing representations V, W in $\mathcal{U}$ [35, Lemma III.4.5]. Hence all maps in $\mathcal{W}K$ are both cofibrations and $\mathcal{W}$–equivalences. The set $\mathcal{W}K$ of maps in $\mathcal{M}_R$ is a set of generating acyclic cofibrations.

As in [35, III.4.8] and [36, 9.5], the following characterization of the maps that satisfy the right lifting property with respect to $\mathcal{W}K$ follows since the $\mathcal{W}$–model structure on $G\mathcal{T}$ is $\mathrm{st}(\mathcal{U})$–topological.

Proposition 4.11 *A map $p\colon E \to B$ satisfies the right lifting property with respect to $\mathcal{W}K$ if and only if p is a levelwise $\mathcal{W}$–fibration and the obvious map from $E(V)$ to the pullback of the diagram*

$$\begin{array}{ccc} & & \Omega^W E(V \oplus W) \\ & & \downarrow \\ B(V) & \longrightarrow & \Omega^W B(V \oplus W) \end{array}$$

is an unbased $\mathcal{W}$–equivalence of spaces for all $V, W \in \mathcal{V}(\mathcal{U})$.

Proof If k is a $\mathrm{st}(\mathcal{U})$–cofibration, and p is a levelwise $\mathcal{W}$–fibration, then by Proposition 2.20 $(\mathcal{M}_R)_G(k^*, p_*)$ is a $\mathcal{W}$–fibration of spaces. Hence the proof of [35, Proposition III.4.8] gives the result. □

The proof of Theorem 4.4 is similar to the proofs in [35, Section III.4],[35, III.7.4] and [36, Section 9]. Note that proofs of these results use a few lemmas, given in [35], that are not made explicit in this paper. Properness follows as in [36, 9.10]. The model structure is $\mathcal{E}$–topological as in [36, 9.9].

Remark 4.12 An alternative construction of this model structure is provided by the general framework of Olivier Renaudin [38]. The functorial fibrant replacement functor for the stable $\mathcal{W}$–model structure gives a homotopy idempotent functor in the levelwise model structure on $\mathcal{M}_R$. The description of the fibrations follows from the description of the fibrant replacement (of a cofibrant object) (4–5) and a result of Bousfield [5, Theorem 9.3].

If $\mathcal{W}$ is an Illman collection of subgroups of G, then the stable $\mathcal{W}$–model structure on $\mathcal{M}_R$ is a tensor model structure that satisfies the monoid axiom. This follows as in [35, Section III.7]

4.2 Positive model structures

We give some brief remarks about other model categories of spectra. Prespectra are defined by replacing the category $\mathcal{V}(\mathcal{U})$, in Definition 3.6 and Definition 3.9, by the smaller category consisting of the indexing representations and their inclusions. There is a stable $\mathcal{W}$–model structure on the category of prespectra. This model category is Quillen equivalent to the stable $\mathcal{W}$–model structure on G–orthogonal spectra [35, III.4.16].

We can also consider model structures on the category of algebras. We need to remove some of the cofibrant and acyclic cofibrant generators to make sure the free symmetric algebra construction takes acyclic cofibrant generators to stable $\mathcal{W}$–equivalences. Let $\Sigma^R_+\mathcal{W}I$ and $\Sigma^R_+\mathcal{W}J$ consist of all V–desuspensions of elements in $\mathcal{W}I$ and $\mathcal{W}J$ by indexing representations V in $\mathcal{U}$ such that $V^G \neq 0$. The positive levelwise $\mathcal{W}$–model structure on the category of orthogonal spectra is the model structure obtained by replacing $\Sigma^\infty_R\mathcal{W}I$ and $\Sigma^\infty_R\mathcal{W}J$ by $\Sigma^R_+\mathcal{W}I$ and $\Sigma^R_+\mathcal{W}J$, respectively. The positive stable $\mathcal{W}$–model structure on orthogonal spectra is obtained by replacing $\mathcal{W}K$ by the set $\mathcal{W}K_+$ consisting of the union of $\Sigma^R_+\mathcal{W}J$ and the maps $i\square k_{V,W}$ with $i \in \mathcal{W}I$ and $V^G \neq 0$. The discussion of the positive model structure goes through as in [35, Sections III.5, III.8].

Proposition 4.13 *Let R be a commutative monoid in the category of G–orthogonal spectra. Let $\mathcal{W}$ be an Illman collection containing $\mathrm{st}(\mathcal{U})$. Then there is a compactly generated $\mathcal{W}$–topological model structure on the category of R–algebras such that the fibrations and weak equivalences are created in the underlying positive $\mathcal{W}$–model category of orthogonal G–spectra. The same applies to the category of commutative R–algebras.*

4.3 Fibrations

We consider the behavior of fibrations in the stable model structure under restriction to subgroups. See Remark 3.13.

Definition 4.14 Let $\mathcal{W}$ be a collection of subgroups of G, and let K be a subgroup of G. The intersection $K \cap \mathcal{W}$ is defined to be the collection of all subgroups $H \in \mathcal{W}$ such that $H \leq K$.

If $\mathcal{W}$ is a Lie collection of subgroups of G that is closed under intersections and $\mathrm{st}_G(\mathcal{U})$ is a $\mathcal{W}$–Illman collection of subgroups of G, then, for every $K \in \mathcal{W}$, the collection of K–stabilizers $\mathrm{st}_K(\mathcal{U}|K)$ is a $K \cap \mathcal{W}$–Illman collection of subgroups of K.

Lemma 4.15 *Let $\mathrm{st}(\mathcal{U})$ be a $\mathcal{W}$–Illman collection of subgroups of G and let $K \in \mathcal{W}$. Let $f\colon X \to Y$ be a fibration in $\mathcal{W}$–$G\mathcal{M}_R$. Then f regarded as a map of K–spectra is a fibration in $(K \cap \mathcal{W})$–$K\mathcal{M}_R$.*

Proof This follows from the explicit description of fibrations in Theorem 4.4. (Alternatively, check that $G \wedge_K -$ is left Quillen adjoint to the forgetful functor from G–spectra to K–spectra.) □

Lemma 4.15 need not remain true when the subgroup K is not in $\mathcal{W}$. For applications in Section 11 we give some conditions that guarantee that the result remains true even when $K \notin \mathcal{W}$.

Lemma 4.16 *Let $\mathrm{st}(\mathcal{U})$ be a $\mathcal{W}$–Illman collection of subgroups of a compact Hausdorff group G. Let $f\colon X \to Y$ be a fibration in $\mathcal{W}\mathcal{M}_R$. Assume that both X and Y are $\mathcal{W}$–$\mathcal{S}$–cell complexes. Let K be any closed subgroup of G, and let $\mathcal{W}'$ be a collection of subgroups of K such that $\mathcal{W}'\mathcal{W} \subset \mathcal{W}$ and $\mathrm{st}_K(\mathcal{U}|K)$ is a $\mathcal{W}'$–Illman collection of subgroups of K. Then f, regarded as a map of K–spectra, is a fibration in the $\mathcal{W}'$–model structure on $K\mathcal{M}_R$ (indexed on $\mathcal{V}(\mathcal{U})|K$, and R regarded as a K–spectrum).*

Note that X and Y are required to be $\mathcal{W}$–S–cell complexes not just $\mathcal{W}$–R–cell complexes. This holds if they are $\mathcal{W}$–R–cell complexes and R is a $\mathcal{W}$–S–cell complex.

Proof Let $f\colon X \to Y$ be a $\mathcal{W}$–fibration between $\mathcal{W}$–cofibrant objects in $\mathcal{W}G\mathcal{M}_R$. Since $\mathrm{st}_K(\mathcal{U}|K)$ is a $\mathcal{W}'$–Illman collection, it suffices, by Theorem 4.4 and Proposition 4.11, to show that for any $L \in \mathcal{W}'$: (1) the map $f(V)^L\colon X(V)^L \to Y(V)^L$ is a fibration, for $V \in \mathcal{V}(\mathcal{U})|K$, and (2) the map from $X(V)^L$ to the pullback of the diagram

(4–17)
$$\begin{array}{ccc} & & Y(V)^L \\ & & \downarrow \\ (\Omega^W X(V\oplus W))^L & \longrightarrow & (\Omega^W Y(V\oplus W))^L \end{array}$$

is a weak equivalence of spaces, for $V, W \in \mathcal{V}(\mathcal{U})|K$.

We now prove that maps from a compact space C to the L–fixed points of $X(V)$ and $Y(V)$ factor through the UL–fixed points of $X(V)$ and $Y(V)$ for some $U \in \mathcal{W}$. Since X and Y are $\mathcal{W}$–$\mathcal{S}$–cell complexes, and a map from a compact space C into a $\mathcal{W}$–$\mathcal{S}$–cell complex factors through a finite sub cell complex, it suffices to verify the claim for individual cells. Recall, from (3–12), that $\Sigma^\infty_{V'} G/H_+ \wedge D^n_+(V)$ is the space $G/H_+ \wedge D^n_+ \wedge \mathcal{O}(V)_+ \wedge_{\mathcal{O}(V-V')} S^{V-V'}$, for $V \supset V'$, and a point otherwise. This is a finite $\mathcal{W}$–cell complex by Lemma 2.16 and Illman [28]. We conclude that a map from a compact space C into the L–fixed points of $X(V)$ and $Y(V)$ factor through $X(V)^{UL}$ and $Y(V)^{UL}$, respectively, for some $U \in \mathcal{W}$.

We are now ready to prove claim (1). Let

(4–18)
$$\begin{array}{ccc} D^n_+ & \longrightarrow & X(V)^L \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f(V)^L} \\ (D^n\times I)_+ & \longrightarrow & Y(V)^L \end{array}$$

be a diagram of based spaces. There exists a $U \in \mathcal{W}$ such that the map from j to $f(V)^L$ factors through $f(V)^{UL}$. Since $f(V)^{UL}$ is a fibration we get a lift in the diagram (4–18). Hence $f(V)^L$ is a fibration.

The proof of claim (2) is similar. We note that a map from a compact space C to $(\Omega^W X(V\oplus W))^L$, composed with the inclusion into $\Omega^W X(V\oplus W)$, is adjoint to a based map from $C_+ \wedge S^W$ to $X(V\oplus W)$. Hence it factors through $X(V\oplus W)^{U'}$, for some $U' \in \mathcal{W}$. By choosing a smaller $U \leq U'$ such that U acts trivially on W, the map from C factors through $(\Omega^W X(V\oplus W))^{UL}$. Hence to check that the map

from $X(V)^L$ to the pullback of (4–17) is a weak equivalence, it suffices to check this on all UL–fixed points for $U \in \mathcal{W}$. This follows by our assumptions. □

Lemma 4.19 *Let st($\mathcal{U}$) be a $\mathcal{W}$–Illman collection of subgroups of a compact Hausdorff group G. Let $f\colon X \to Y$ be a (co–)n–equivalence in $\mathcal{M}_R$ between $\mathcal{W}$–$\mathcal{S}$–cell complexes X and Y which are also fibrant objects in the $\mathcal{W}$–model structure on $G\mathcal{M}_R$. Let K be any closed subgroup of G, and let $\mathcal{W}'$ be a collection of subgroups of K such that $\mathcal{W}'\mathcal{W} \subset \mathcal{W}$ and $st_K(\mathcal{U}|K)$ is $\mathcal{W}'$–Illman. Then f regarded as a map of K–spectra is a (co–)n–equivalence in the $\mathcal{W}'$–model structure on $K\mathcal{M}_R$.*

Proof The spectra X and Y are also fibrant in the $\mathcal{W}'$–model structure on $K\mathcal{M}_R$ by Lemma 4.16. Hence it suffices to prove that $X(\mathbb{R}^m)^L \to Y(\mathbb{R}^m)^L$ is a (co–)$(n-m)$–equivalence for any $L \in \mathcal{W}'$ and $n \geq m \geq 0$. Since both X and Y are fibrant in the $\mathcal{W}$–model structure on $G\mathcal{M}_R$, it follows that

$$X(\mathbb{R}^m) \to Y(\mathbb{R}^m)$$

is a $\mathcal{W}$–(co–)$(n-m)$–equivalence. The proof of Lemma 4.16 gives the result. □

5 The $\mathcal{W}$–$\mathcal{C}$–model structure on orthogonal G–spectra

Let R be a ring and assume that st($\mathcal{U}$) is a $\mathcal{C}$–Illman collection of subgroups of G. We define K–equivalences in the $\mathcal{C}$–model structure on the category of R–modules, $\mathcal{M}_R$, for K not necessarily in $\mathcal{C}$. This is used to construct a model structure with weak equivalences detected by a collection $\mathcal{W}$ of subgroups of G which is not necessarily contained in $\mathcal{C}$. We start by briefly describing the resulting $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$ in the case when $\mathcal{W}$ is contained in $\mathcal{C}$.

Let H be in $\mathcal{C}$. Then π_*^H is a homology theory which satisfies the colimit axiom by Lemma 4.6. The direct sum

$$h = \bigoplus_{K \in \mathcal{W}, n \in \mathbb{Z}} \pi_n^K$$

is also a homology theory which satisfies the colimit axiom. We can now (left) Bousfield localize $\mathcal{C}\mathcal{M}_R$ with respect to the homology theory h [4] [24, 13.2.1]. Hence for any subcollection $\mathcal{W}$ in $\mathcal{C}$ there is a model structure on G–spectra such that the cofibrations are retracts of relative $\mathcal{C}$–cell complexes and the weak equivalences are maps f such that $\pi_n^H(f)$ is an isomorphism for all $H \in \mathcal{W}$ and $n \in \mathbb{Z}$.

5.1 The construction of $\mathcal{WCM}_R$

Assume that $\mathcal{C}$ is an Illman collection of subgroups of G.

Definition 5.1 Let K be a subgroup of G such that the closure $\overline{UK} \in \mathcal{C}$, for all $U \in \mathcal{C}$. The nth stable homotopy group at K is defined to be $\Pi_*^K(X) = \operatorname{colim}_{U\in\mathcal{C}} \pi_*^{\overline{UK}}(X)$.

The colimit is over the category with objects U in $\mathcal{C}$ and with morphisms containment of subgroups. The colimit is directed since $\mathcal{C}$ is an Illman collection. If $K \in \mathcal{C}$, then Π_*^K and π_*^K are canonically isomorphic functors.

Definition 5.2 Let $\mathcal{C}$ and $\mathcal{W}$ be two collections of subgroups of G. Then the product collection $\mathcal{CW}$ has elements the closure $\overline{UH}$ of the product subgroup UH in G, for all $U \in \mathcal{C}$ and all $H \in \mathcal{W}$.

The collection $\mathcal{W} = \{1\}$ satisfies $\mathcal{CW} \subset \mathcal{C}$ for any collection $\mathcal{C}$. If $\mathcal{C}$ is a cofamily, then $\mathcal{CW} \subset \mathcal{C}$ for any collection $\mathcal{W}$.

Definition 5.3 Let $\mathcal{W}$ be a collection of subgroups of G such that $\mathcal{CW} \subset \mathcal{C}$. Then a map f between orthogonal spectra is a $\mathcal{W}$–equivalence if $\Pi_n^K(f)$ is an isomorphism for all $K \in \mathcal{W}$ and all integers n.

Directed colimits of abelian groups respect direct sums and exact sequences. So Π_*^K is a homology theory which satisfies the colimit axiom by Lemma 4.6. The direct sum

$$h = \bigoplus_{K\in\mathcal{W}, n\in\mathbb{Z}} \Pi_n^K$$

is again a homology theory which satisfies the colimit axiom. Hence we can Bousfield localize with respect to h.

Theorem 5.4 *Let $\mathcal{C}$ be an Illman collection of subgroups of G such that $\mathrm{st}(\mathcal{U}) \subset \mathcal{C}$. Let $\mathcal{W}$ be any collection of subgroups of G such that $\mathcal{CW} \subset \mathcal{C}$. Then there is a cofibrantly generated proper simplicial model structure on $\mathcal{M}_R$ such that the weak equivalences are $\mathcal{W}$–equivalences and the cofibrations are retracts of relative $\mathcal{C}$–cell complexes.*

Proof There exists a set $\mathcal{K}$ of relative $\mathcal{C}$–G–cell complexes with sources $\mathcal{C}$–G–cell complexes such that a map p has the right lifting property with respect to all h–acyclic cofibrations with cofibrant source, if and only if p has the right lifting property with respect to $\mathcal{K}$. To find such a set of maps $\mathcal{K}$ we use the cardinality argument of Bousfield, taking into account both the cardinality of G and the cardinality of $\prod_V R(V)$, where

the product is over indexing representations in the universe $\mathcal{U}$ [4]. The class of h–equivalences is closed under pushout along $\mathcal{C}$–cofibrations. Hence we can apply [24, 13.2.1] to conclude that if p has the right lifting property with respect to the maps in the set $\mathcal{K}$, then it has the right lifting property with respect to all h–acyclic cofibrations. Hence there is a cofibrantly generated left proper model structure on $\mathcal{M}_R$ with the specified class of cofibrations and weak equivalence [24, 4.1.1]. It remains to show that the model structure is right proper and simplicial.

The model structure is right proper by comparing homotopy fibers in pullback diagrams [36, 9.10]. We show that the model structure is simplicial. See Remark 2.21. The tensor and cotensor functors are given by $\Sigma^\infty_R|K_+|\wedge X$ and $F(\Sigma^\infty_R|K_+|, X)$, respectively, for a simplicial set K and an R–module X. The simplicial hom functor is given by sing $G\mathcal{M}_R(X,Y)$. The pushout–product map applied to a simplicial cofibration and a $\mathcal{C}$–cofibration in $\mathcal{M}_R$ is again a $\mathcal{C}$–cofibration. If the simplicial cofibration is acyclic, then the pushout–product map is a $\mathcal{C}$–acyclic cofibration. It suffices to show that if $X_2 \to Y_2$ is a $\mathcal{W}$–$\mathcal{C}$–acyclic cofibration with $\mathcal{C}$–cofibrant source, then the map from the pushout of

$$\begin{array}{ccc} \Sigma^\infty_R S^{n-1}_+ \wedge X_2 & \longrightarrow & \Sigma^\infty_R D^n_+ \wedge X_2 \\ \downarrow & & \\ \Sigma^\infty_R S^{n-1}_+ \wedge Y_2 & & \end{array}$$

to $\Sigma^\infty_R D^n_+ \wedge Y_2$ is again a $\mathcal{W}$–$\mathcal{C}$–acyclic cofibration [39, 2.3]. This is the case since our weak equivalences are given by a homology theory on the homotopy category of the tensor $\mathcal{C}$–model structure on $\mathcal{M}_R$ (see Theorem 4.4). □

This model structure is called the $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$. The $\mathcal{W}$–model structure is the $\mathcal{W}$–$\mathcal{W}$–model structure. We sometimes denote $\mathcal{M}_R$ together with the $\mathcal{W}$–$\mathcal{C}$–model structure by $\mathcal{WCM}_R$.

Proposition 5.5 *Let $\mathcal{C}_1 \subset \mathcal{C}_2$ be two $\mathcal{U}$–Illman collections of subgroups of G containing the trivial subgroup, 1, and let $\mathcal{W}$ be a collection of subgroups of G such that $\mathcal{C}_1\mathcal{W} \subset \mathcal{C}_1$ and $\mathcal{C}_2\mathcal{W} \subset \mathcal{C}_2$. Then the identity functors $\mathcal{WC}_1\mathcal{M}_R \to \mathcal{WC}_2\mathcal{M}_R$ and $\mathcal{WC}_2\mathcal{M}_R \to \mathcal{WC}_1\mathcal{M}_R$ are left and right Quillen adjoint functors, respectively. Hence a Quillen equivalence.*

5.2 The $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$

The $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$ is of particular interest when $\mathcal{W} = \{1\}$.

Definition 5.6 We say that f is a $\mathcal{C}$–underlying equivalence if

$$\Pi^1_*(f) = \operatorname{colim}_{U\in\mathcal{C}} \pi^U_*(f)$$

is an equivalence.

The name, $\mathcal{C}$–underlying equivalence, is justified by the next Lemma.

Lemma 5.7 *Assume that G is a compact Hausdorff group and let $\mathcal{C}$ be the collection $\overline{\mathrm{Lie}(G)}$. Let R be the sphere spectrum $\mathcal{S}$. Then a map $f\colon X \to Y$ between cofibrant objects (retracts of $\mathcal{C}$–cell complexes) is a $\mathcal{C}$–underlying equivalence if and only if f is a non-equivariant weak equivalence.*

Proof This follows as in the proof of Lemma 4.16. □

Theorem 5.8 *Assume that $\mathcal{U}$ is a trivial G–universe. Let $\mathcal{C}$ be an Illman collection of subgroups of G. Then there is a cofibrantly generated proper simplicial model structure on $\mathcal{M}_R$ such that the weak equivalences are $\mathcal{C}$–underlying equivalences and the cofibrations are retracts of relative $\mathcal{C}$–cell complexes.*

Proof This is a special case of Theorem 5.4. □

We refer to this model structure as the $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$.

Remark 5.9 The universe is required to be trivial as part of the definition of the $\mathcal{C}$–cofree model structure. When $\{1\}$ is in $\mathcal{C}$, or $\mathcal{C}$ is a family in $\overline{\mathrm{Lie}(G)}$, then there is no loss of generality in making this assumption. In this case the $\mathcal{C}$–cofree model categories of G–spectra for a universe $\mathcal{U}$ and its fixed point universe $\mathcal{U}^G$ are Quillen equivalent [35, V.1.7]. This is so because the homotopy groups π^1_* are isomorphic for the universes $\mathcal{U}^G$ and $\mathcal{U}$.

6 A digression: G–spectra for noncompact groups

In this section we consider an example of a model structure on orthogonal G–spectra where the homotopy theory is "pieced together" from the genuine homotopy theory of the compact Lie subgroups of G. This example is inspired by conversations with Wolfgang Lück. This section plays no role later in the paper.

The model structure we construct below in Proposition 6.5 is in many ways opposite to the model structure (to be discussed) in Theorem 9.4: Compact Lie subgroups versus

compact Lie quotient groups, ind-spectra versus pro–spectra, pro-universes versus ind-universes. The difficulties here lie in dealing with inverse systems of universes for the compact Lie subgroups of G.

Let G be a topological group, and let $\mathcal{X}$ be a trivial G–universe. Let R be a symmetric monoid in the category of orthogonal G–spectra indexed on $\mathcal{X}$. Let $\mathcal{M}$ denote the category of R–modules indexed on $\mathcal{X}$.

Definition 6.1 Let $\mathcal{F}_G$ denote the family of compact Lie subgroups of G.

If G is a discrete group or a profinite group then $\mathcal{F}_G$ is the family of finite subgroups of G. The results in this section remain true if each $\mathcal{F}_G$ is replaced by a subfamily $\mathcal{F}'_G$ of $\mathcal{F}_G$, such that $\mathcal{F}'_J = J \cap \mathcal{F}'_G$ whenever $J < G$.

By Proposition 3.17 there is a cofibrantly generated model structure on $\mathcal{M}$ such that the cofibrations are retracts of relative $\Sigma^\infty_R \mathcal{F}_G I$–cell complexes and the weak equivalences are levelwise $\mathcal{F}_G$–equivalences. We would like to stabilize $\mathcal{M}$ with respect to H–representations for all compact Lie subgroups H of G. An H–representation might not be a retract of a G–representation restricted to H (there might not be any nontrivial H–representations of this form).

Our approach is to localize $\mathcal{M}$ with respect to stable H–homotopy groups defined using a complete H–universe, one universe for each H in $\mathcal{F}_G$.

Definition 6.2 An $\mathcal{F}_G$–universe consists of an H–universe $\mathcal{U}_H$, for each $H \in \mathcal{F}_G$, such that whenever $H \leq K$, then $\mathcal{U}_K|H$ is a subuniverse of $\mathcal{U}_H$. For any subgroups $H \leq K \leq L$ the three resulting inclusions of universes are required to be compatible.

We say that the $\mathcal{F}_G$–universe, $\{\mathcal{U}_H\}$, is complete if $\mathcal{U}_H$ is a complete H–universe for each $H \in \mathcal{F}_G$.

Lemma 6.3 *There exists a complete $\mathcal{F}_G$–universe.*

Proof Choose a complete H–universe $\mathcal{U}'_H$ for each $H \in \mathcal{F}_G$. Let $\mathcal{U}_H$ be defined to be

$$\bigoplus_{K \geq H} (\mathcal{U}'_K|H)$$

where the sum is over all $K \in \mathcal{F}_G$ that contains H. □

Let H be a compact Lie group. Then there is a stable model structure on orthogonal H–spectra, indexed on a trivial H–universe, which is Quillen equivalent to the "genuine" model structure on orthogonal H–spectra indexed on a complete H–universe. This is

proved in [35, V.1.7] (note that the condition $\mathcal{V} \subset \mathcal{V}'$ is not necessary). Let H be a compact Lie group and let $\mathcal{V}$ and $\mathcal{V}'$ be collections of H–representations containing the trivial H–representations. Typically, $\mathcal{V}$ is the collection $\mathcal{V}(\mathcal{U})$ of all H–representations that are isomorphic to some indexing representation in an H–universe $\mathcal{U}$. There is a change of indexing functor

$$I^{\mathcal{V}}_{\mathcal{V}'}\colon \mathcal{J}^{\mathcal{V}'}_G\mathcal{S} \to \mathcal{J}^{\mathcal{V}}_G\mathcal{S}$$

defined in [35, V.1.2]. The functor $I^{\mathcal{V}}_{\mathcal{V}'}$ induces an equivalence of categories with $I^{\mathcal{V}'}_{\mathcal{V}}$ as the inverse functor. The functor $I^{\mathcal{V}}_{\mathcal{V}'}$ is a strong symmetric tensor functor. These, and other, claims are proved in [35, V.1.5].

Lemma 6.4 *For each compact Lie subgroup H of G the functor*

$$\pi^H_*(I^{\mathcal{V}(\mathcal{U}_H)}_{\mathcal{V}(\mathcal{X})}-)$$

is a homology theory on $\mathcal{M}$ with the levelwise $\mathcal{F}_G$–$\mathcal{X}$–model structure, that satisfies the colimit axiom.

Proof This follows since $I^{\mathcal{V}(\mathcal{U}_H)}_{\mathcal{V}(\mathcal{X})}$ respects homotopy colimits and weak equivalences since $\mathcal{V}(\mathcal{X}) \subset \mathcal{V}(\mathcal{U}_H)$ [35, V.1.6]. The claim follows from Lemma 4.6. □

We localize the stable $\mathcal{F}_G$–$\mathcal{X}$–model category with respect to the homology theory given by

$$h = \bigoplus_H \pi^H_*(I^{\mathcal{V}(\mathcal{U}_H)}_{\mathcal{V}(\mathcal{X})}-),$$

where the sum is over all $H \in \mathcal{F}_G$.

Proposition 6.5 *Given an $\mathcal{F}_G$–universe $\{\mathcal{U}_H\}$. There is a cofibrantly generated stable model structure on $\mathcal{M}$ such that the cofibrations are retracts of relative $\Sigma^\infty_R\mathcal{F}_G I$–cell complexes (for the trivial universe $\mathcal{X}$) and the weak equivalences are the h–equivalences. If $\mathcal{F}_G$ is an Illman collection of subgroups of G (see Definition 2.14), then the model structure satisfies the pushout–product axiom.*

This model structure is called the stable $\{\mathcal{U}_H\}_{H\in\mathcal{F}_G}$–model structure on $\mathcal{M}$.

Proof See the proof of Theorem 5.4. □

The cofibrant replacement of $\Sigma^\infty_R S^0$ in this model structure (regardless of $\{\mathcal{U}_H\}$) is given by $\Sigma^\infty_R(E\mathcal{F}_G)_+$, where $E\mathcal{F}_G$ is an $\mathcal{F}_G$–cell complex such that $(E\mathcal{F}_G)^H$ is contractible whenever $H \in \mathcal{F}_G$, and empty otherwise [35, IV.6].

Lemma 6.6 *Assume G is a discrete group. If X is an arbitrary G–cell complex spectrum, then $X \wedge \Sigma^\infty_R(E\mathcal{F}_G)_+$ is a cofibrant replacement of X.*

Proof Note that $G/J_+ \wedge G/H_+$ is an $\mathcal{F}_G$–cell complex, whenever $H \in \mathcal{F}_G$ and J is an arbitrary subgroup of G. The collapse map $(E\mathcal{F}_G)_+ \to S^0$ induces an $\mathcal{F}_G$–$\{\mathcal{U}_H\}$–equivalence $X \wedge (E\mathcal{F}_G)_+ \to X$ □

Lemma 6.7 *If G has no compact Lie subgroups besides 1 (eg torsion-free discrete groups), then the stable $\{\mathcal{U}_1\}$–model structure $\mathcal{M}$ is the stable $\{1\}$–model structure.*

Lemma 6.8 *If G is a compact Lie group, then the stable $\{\mathcal{U}_G|H\}$–model structure on $\mathcal{M}$ is Quillen equivalent to the $\{all\}$–$\mathcal{U}_G$–model structure on $\mathcal{M}$.*

Let J be a subgroup of G. Let R be a monoid in the category of orthogonal G–spectra. Let $\mathcal{M}_G$ denote the category of R–modules in the category of orthogonal G–spectra indexed on $\mathcal{X}$, and let $\mathcal{M}_J$ denote the category of $R|J$–modules in the category of orthogonal J–spectra indexed on $\mathcal{X}$. Let $\{\mathcal{U}_H\}$ be an $\mathcal{F}_G$–universe, and let $\{\mathcal{U}_H\}_{H\in\mathcal{F}_J}$ be the $\mathcal{F}_J$–universe.

Note that the condition in the next Lemma is trivially satisfied if G is a discrete group.

Lemma 6.9 *Assume that $(G/K_+)|J$ has the structure of a J–$\mathcal{F}_J$–cell complex for any $K \in \mathcal{F}_G$. Give $\mathcal{M}_G$ the stable $\mathcal{F}_G$–$\{\mathcal{U}_H\}$–model structure, and give $\mathcal{M}_J$ the stable $\mathcal{F}_J$–$\{\mathcal{U}_H\}_{H\in\mathcal{F}_J}$–model structure. Then the functor*

$$F_J(G_+, -)\colon \mathcal{M}_J \to \mathcal{M}_G$$

is right Quillen adjoint to the restriction functor

$$\mathcal{M}_G \to \mathcal{M}_J.$$

Proof The restriction functor from G–spectra to J–spectra respects weak equivalences by the definition of weak equivalences. Since G/K_+ is a J–$\mathcal{F}_J$–cell complex for all $K \in \mathcal{F}_G$, by our assumption, the relative G–$\mathcal{F}_G$–cell complexes are also relative J–$\mathcal{F}_J$–cell complexes. □

Lemma 6.10 *Assume G is a discrete group. Give $\mathcal{M}_G$ the stable $\mathcal{F}_G$–$\{\mathcal{U}_H\}$–model structure, and give $\mathcal{M}_J$ the stable $\mathcal{F}_J$–$\{\mathcal{U}_H\}_{H\in\mathcal{F}_J}$–model structure. Then the functor*

$$G_+ \wedge_J -\colon \mathcal{M}_J \to \mathcal{M}_G$$

is left Quillen adjoint to the restriction functor

$$\mathcal{M}_G \to \mathcal{M}_J.$$

Proof Since $G_+ \wedge_J J/K_+ \cong G/K_+$ and the functor $G_+ \wedge_J -$ respects colimits, it follows that $G_+ \wedge_J -$ respects cofibrations.

Let $f\colon X \to Y$ be a J–$\mathcal{F}_J$–equivalence. We observe that $G_+ \wedge_J X$ is isomorphic to

$$\bigvee_{HgJ \in H\backslash G/J} H_+ \wedge_{gJg^{-1}\cap H} gX$$

as an orthogonal H–spectrum. Hence the map $G_+ \wedge_J f$ is an $\mathcal{F}_G$–weak equivalence if each $H_+ \wedge_{gJg^{-1}\cap H} g(f)$ is an H–equivalence for $H \in \mathcal{F}_G$. This follows from [35, V.1.7,V.2.3] since $g(f)$ is a K–equivalence for every $K \leq gJg^{-1} \cap H$, because $K \in \mathcal{F}_G$ and $K \leq gJg^{-1}$ implies that $K \in \mathcal{F}_{gJg^{-1}}$. □

Lemma 6.11 *Assume G is a discrete group. Let X be a cofibrant object in $\mathcal{M}_J$ and let Y be a fibrant object in $\mathcal{M}_G$. Then*

$$[G_+ \wedge_J X, Y]_G \cong [X, (Y|J)]_J,$$

where the first hom-group is in the homotopy category of the $\{\mathcal{U}_H\}$–model structure on $\mathcal{M}_R$, and the second hom-group is in the homotopy category of the $\{\mathcal{U}_H\}_{H\in\mathcal{F}_J}$–model structure on $\mathcal{M}_J$.

In particular, if X and Y are G–spectra and $H \in \mathcal{F}_G$, then

$$[G/H_+ \wedge X, Y]_G \cong [X, Y]_H.$$

Remark 6.12 A better understanding of the fibrations would be useful. They are completely understood when G is a compact Lie group [35, III.4.7,4.12]. Calculations in the stable $\{\mathcal{U}_H\}$–homotopy theory reduces to calculations in the stable homotopy categories for the compact Lie subgroups of G (via a spectral sequence). This follows from Lemma 6.11 using a cell filtration of a cofibrant replacement of the source by a $\mathcal{F}_G$–cell complex.

7 Homotopy classes of maps between suspension spectra

We first give a concrete description of the set of morphisms between suspension spectra in the $\mathcal{W}$–stable homotopy category on $\mathcal{M}_R$. Then we prove some results about vanishing of negative stems. These results are needed Section 8.

Recall that $\mathcal{WT}$ denotes $G\mathcal{T}$ with the $\mathcal{W}$–model structure.

Lemma 7.1 *Let X and Y be two based G–spaces. Then there is a natural isomorphism*

$$Ho(\mathcal{WM}_R)(\Sigma^\infty_R X, \Sigma^\infty_R Y) \cong Ho(\mathcal{WT})(X, \mathrm{hocolim}_{\mathcal{W}}\, \Omega^W (R(W) \wedge Y)_{cof})$$

where the subscript, cof, indicates a $\mathcal{W}$–cofibrant replacement.

Proof Recall the description of Σ^∞_R in (3–12). The functors Σ^∞_R and Ω^∞_0 are a Quillen adjoint pair. The result follows by replacing X by a cofibrant object, X_{cof}, in $\mathcal{WT}$, and replacing $\Sigma^\infty_R Y$ by a cofibrant and fibrant object as in (4–5). □

Corollary 7.2 *Let X and Y be two based G–spaces. Then there is a natural isomorphism*

$$Ho(\mathcal{WM}_{\mathcal{S}})(\Sigma^\infty X, \Sigma^\infty Y) \cong Ho(\mathcal{WT})(X, \mathrm{hocolim}_{\mathcal{W}}\, \Omega^W \Sigma^W Y_{cof}).$$

In particular, if X is a finite $\mathcal{W}$–cell complex, then

$$Ho(\mathcal{WM}_{\mathcal{S}})(\Sigma^\infty X, \Sigma^\infty Y) \cong \mathrm{colim}_{\mathcal{W}}\, Ho(\mathcal{WT})(X \wedge S^W, Y \wedge S^W).$$

We next show that the negative stable stems are zero. In what follows homotopy means usual homotopy (a path in the space of maps).

Lemma 7.3 *Let V be a finite dimensional real G–representation with G–action factoring through a Lie group quotient of G. Let X be a based G–space, and let $n > 0$ be an integer. Then any based G–map*

$$S^V \to S^V \wedge X \wedge S^n$$

is G–null-homotopic.

Proof Assume the action on S^V factors through a compact Lie group quotient G/K. The problem reduces to show that $S^V \to S^V \wedge X^K \wedge S^n$ is G/K–null homotopic for all $n > 0$. Hence we can assume that G is a compact Lie group. By Illman's triangulation theorem S^V is a finite G–cell complex [28]. We choose a G–CW structure on S^V. Let $(G/H_i \times D^{n_i})_+$ be a cell of S^V. We compare the real manifold dimensions, denoted dim, of the H_i–fixed points of S^V and the cells in S^V. This gives that $n_i = \dim(V^{H_i}) - \dim(N_G H_i/H_i)$. To prove the Lemma it suffices to show that any given map $f\colon S^V \to S^V \wedge X \wedge S^n$ extends over the cone $S^V \wedge I$ of S^V. There is a sequence

$$S^V = Y_{-1} \to Y_0 \to Y_1 \to \cdots \to Y_N = S^V \wedge I$$

where Y_{n+1} is obtained from Y_n by a pushout

$$\begin{array}{ccc} \bigvee G/H_{i+} \wedge S^{n_i} & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \bigvee G/H_{i+} \wedge D^{n_i+1} & \longrightarrow & Y_{n+1} \end{array}$$

where the wedge sum is over all i such that $n_i = n$, and N satisfies $n_i \leq N$ for all i. Hence it suffices to show that any map $\vee G/H_{i+} \wedge S^{n_i} \to S^V \wedge X \wedge S^n$ is G–null homotopic for all i. This is equivalent to showing that $S^{n_i} \to S^{V^{H_i}} \wedge X^{H_i} \wedge S^n$ is null homotopic, which is true because $n_i = \dim(V^{H_i}) - \dim(N_G H_i/H_i) < \dim(V^{H_i}) + n$. □

Lemma 7.4 *Let $\mathcal{U}$ be any G–universe, and let $st(\mathcal{U})$ be a $\mathcal{W}$–Illman collection of subgroups of G. Then we have that*

$$\mathrm{Ho}(\mathcal{W}\mathcal{M}_{\mathcal{S}})(\Sigma^\infty G/H_+,\ \Sigma^\infty G/K_+ \wedge S^n) = 0,$$

for all $H, K \in \mathcal{W}$ and $n > 0$.

Proof The space $S^V \wedge G/H_+$ is homeomorphic to a finite $\mathcal{W}$–cell complex by Lemma 2.16 and compactness of $S^V \wedge G/H_+$. Corollary 7.2 gives that the group $\mathrm{Ho}(\mathcal{W}\mathcal{M}_{\mathcal{S}})(\Sigma^\infty G/H_+,\ \Sigma^\infty G/K_+ \wedge S^n)$ is isomorphic to

$$\mathrm{colim}_{V \in \mathcal{V}(\mathcal{U})} \mathrm{Ho}(\mathcal{W}\mathcal{T})(S^V \wedge G/H_+,\ S^V \wedge G/K_+ \wedge S^n).$$

It suffices to show that any map $S^V \wedge G/H_+ \to S^V \wedge G/K_+ \wedge S^n$ is G–null homotopic. This is equivalent to show that $S^V \to S^V \wedge (G/K)_+ \wedge S^n$ is H–null homotopic. This follows from Lemma 7.3. □

Lemma 7.4 allows the formation of $\mathcal{W}$–CW complex approximations.

Lemma 7.5 *Let $st(\mathcal{U})$ be a $\mathcal{W}$–Illman collection of subgroups of G. Let T be an $\mathcal{S}$–module such that $\pi_j^H T = 0$, for $j < n$ and $H \in \mathcal{W}$. Then there is a cell complex, T', built out of cells of the form $\Sigma^\infty_{\mathbb{R}^{k'}} S^{k-1} \wedge G/H_+ \to \Sigma^\infty_{\mathbb{R}^{k'}} D^k \wedge G/H_+$, for $k - k' \geq n$ and $H \in \mathcal{W}$, and a $\mathcal{W}$–weak equivalence $T' \to T$.*

Proof The approximation can be constructed as a $\mathcal{W}$–cell complex using Lemma 4.6 and Lemma 7.4. □

Lemma 7.6 *Let $\mathcal{W}$ be an Illman collection of subgroups of G, and assume that $st(\mathcal{U})$ is $\mathcal{W}$–Illman. Let R and T be two $\mathcal{S}$–modules. If $\pi_i^H R = 0$, for $i < m$ and $H \in \mathcal{W}$, and $\pi_j^H T = 0$, for $j < n$ and $H \in \mathcal{W}$, then $\pi_k^H(R \wedge T) = 0$ for $k < m + n$ and $H \in \mathcal{W}$.*

Proof Replace R and T by $\mathcal{W}I$–cell complexes made of cells in dimension greater or equal to m and n. This is possible by Lemma 7.5. The spectrum analogue of Lemma 2.16 gives that $R \wedge T$ is again a $\mathcal{W}$–cell complex made out of cells in dimension greater or equal to $m + n$. The result now follows from Lemma 7.4. □

Proposition 7.7 *Let $\mathcal{W}$ be an Illman collection of subgroups of G and assume that $st(\mathcal{U})$ is $\mathcal{W}$–Illman. Let R be a ring spectrum such that $\pi_n^H R = 0$ for all $n < 0$ and $H \in \mathcal{W}$. Then we have that*

$$Ho(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty G/H_+,\ \Sigma_R^\infty G/K_+ \wedge S^n) = 0,$$

for all $H, K \in \mathcal{W}$ and $n > 0$.

Proof The group $\mathrm{Ho}(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty G/H_+,\ \Sigma_R^\infty G/K_+ \wedge S^n)$ is isomorphic to

$$\mathrm{Ho}(\mathcal{W}\mathcal{M}_\mathcal{S})(\Sigma^\infty G/H_+, \Sigma^\infty G/K_+ \wedge R \wedge S^n).$$

The result follows from Lemma 7.3 and Lemma 7.6. (Let T be $\Sigma^\infty G/K_+ \wedge S^n$.) □

Lemma 7.8 *Let $\mathcal{W}$ be an Illman collection of subgroups of G and assume that $st(\mathcal{U})$ is $\mathcal{W}$–Illman. Let R be a ring such that $\pi_n^H R = 0$, whenever $n < 0$ and $H \in \mathcal{W}$. Let T be an R–module such that $\pi_j^H T = 0$, for $j < n$ and $H \in \mathcal{W}$. Then there is a cell complex, T', built out of cells of the form $\Sigma^R_{\mathbb{R}^{k'}} S^{k-1} \wedge G/H_+ \to \Sigma^R_{\mathbb{R}^{k'}} D^k \wedge G/H_+$, for $k - k' \geq n$ and $H \in \mathcal{W}$, and a $\mathcal{W}$–weak equivalence $T' \to T$.*

Proof This follows from Proposition 7.7 and the proof of Lemma 7.5. □

If the universe $\mathcal{U}$ is trivial and H is a not subconjugated to K in G, then there are no nontrivial maps from $\Sigma^\infty G/H_+ \wedge S^n$ to $\Sigma^\infty G/K_+ \wedge S^m$. We take advantage of this to strengthen Proposition 7.7.

Proposition 7.9 *Let $\mathcal{U}$ be a trivial universe. Let $\mathcal{W}$ be an Illman collection of subgroups of G. Let R be ring spectrum such that $\pi_n^H R = 0$ for all $n < 0$ and $H \in \mathcal{W}$. Then, for each H, K in $\mathcal{W}$, we have that*

$$Ho(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty G/H_+,\ \Sigma_R^\infty G/K_+ \wedge S^n) = 0,$$

whenever $n > 0$ or H is not subconjugated to K.

Proof If $n>0$, then the result follows from Proposition 7.7. If H is not subconjugated to K, then:

$$\mathrm{Ho}(\mathcal{WM}_R)(\Sigma^\infty_R G/H_+, \Sigma^\infty_R G/K_+\wedge S^n)\cong \mathrm{colim}_m \pi_0\Omega^m(G/K_+\wedge(R\wedge S^n)(\mathbb{R}^m))^H$$

This is 0 since G/K^H_+ is the basepoint. □

7.1 The Segal–tom Dieck splitting theorem

We consider homotopy groups of suspension spectra. Let G be a compact Hausdorff group, let $\mathcal{U}$ be a complete G–universe, and let $\mathcal{M}_S$ have the $\overline{\mathrm{Lie}(G)}$–model structure.

Proposition 7.10 *If Y is a G–space, then there is an isomorphism of abelian groups*

$$\textstyle\bigoplus_H\ \pi_*(\Sigma^\infty(EW_GH_+\wedge_{W_GH}\Sigma^{Ad(W_GH)}Y^H))\to\pi^G_*(\Sigma^\infty_S Y)$$

where the sum is over all G–conjugacy classes of subgroups H in $\overline{\mathrm{Lie}(G)}$ and $Ad(W_GH)$ is the Adjoint representation of W_GH.

Proof We have an isomorphism

$$\mathrm{colim}_{N\in\mathrm{Lie}(G)}\ \mathrm{colim}_{V\subset\mathcal{U}^N}\ \pi_*((\Omega^V S^V Y^N)^G)\to\pi^G_*(\Sigma^\infty_S Y)$$

where the rightmost colimit is over indexing representations V in $\mathcal{U}^N$. The universe $\mathcal{U}^N$ is G/N–complete. The result follows from the splitting theorem for compact Lie groups (Lewis–May–Steinberger [33, V.9.1]). □

If $\mathcal{U}$ is a complete G–universe, then $\mathcal{U}$ restricted to K is again a complete K–universe (see [17, Section 3]). So for any $K\in\overline{\mathrm{Lie}(G)}$, the stable homotopy groups at K of a G–space Y calculated in the G–homotopy category are isomorphic to those calculated in the K–homotopy category. Hence the calculation of the nth stable homotopy group at $K\in\overline{\mathrm{Lie}(G)}$ of a G–space Y reduces to Proposition 7.10 (with G replaced by K).

7.2 Self-maps of the unit object

The additive tensor category, $\mathrm{Ho}(\mathcal{WM}_R)$, is naturally enriched in the category of modules over the ring

$$\mathrm{Ho}(\mathcal{WM}_R)(\Sigma^\infty_R S^0,\Sigma^\infty_R S^0)$$

of self map of the unit object in the homotopy category. Let us denote this ring by $B^R_{\mathcal{W}}$, and denote $B^{S^0}_{\mathcal{W}}$ simply by $B_{\mathcal{W}}$. The ring $B^R_{\mathcal{W}}$ depends on G, $\mathcal{W}$, R, and the G–universe $\mathcal{U}$. If $G\in\mathcal{W}$, then we can identify $B^R_{\mathcal{W}}$ with $\pi^G_0(R)$.

If A is an R–algebra, then $B^A_{\mathcal{W}}$ is an $B^R_{\mathcal{W}}$–algebra. Since all algebras of orthogonal spectra are $\mathcal{S}$–algebras, it is important to understand the ring $B_{\mathcal{W}}$.

If G is a compact Lie group, the universe is complete, and $\mathcal{W}$ is the collection of all closed subgroups of G, then $B_{\mathcal{W}}$ is naturally isomorphic to the Burnside ring, $A(G)$, of G [37, XVII.2.1].

Lemma 7.11 *Let $\mathcal{U}$ be a complete G–universe and let $\mathcal{W}$ be the collection $\overline{\mathrm{Lie}(G)}$. Then the self-maps of $\Sigma^\infty S^0$, in the homotopy category of $\mathcal{WM}$, is naturally isomorphic to*

$$\mathrm{colim}_{U\in \mathrm{Lie}(G)}\ A(G/U),$$

where $A(G/U)\cong Ho(\mathcal{WM})(\Sigma^\infty G/U_+, \Sigma^\infty S^0)$ is the Burnside ring of the Lie group G/U and the maps in the colimit are induced by the quotient maps $G/U_+\to G/V_+$, for $U<V$ in $\mathrm{Lie}(G)$.

In general, it is difficult to determine $B_{\mathcal{W}}$. For example, when G is a finite group and $\mathcal{W}$ is a family, then the proof of the Segal conjecture gives that the ring $\mathcal{B}_{\mathcal{W}}$ is isomorphic to the Burnside ring $A(G)$ of G completed at the augmentation ideal

$$\cap_{H\in\mathcal{W}}\ker(A(G)\to A(H)),$$

where the maps $A(G)\to A(H)$ are the restriction maps [37, XX.2.5].

We give an elementary observation which shows that different collections $\mathcal{W}$ might give rise to isomorphic rings $\mathcal{B}_{\mathcal{W}}$.

Lemma 7.12 *Let N be a normal subgroup of a finite group G, and let $\mathcal{W}_N$ be the family of all subgroups contained in N. If $X\in G\mathcal{T}$ has a trivial G–action and $Y\in G\mathcal{T}$, then*

$$Ho(\{N\}\mathcal{M}_R)(\Sigma^\infty X,\Sigma^\infty Y)\to Ho(\mathcal{W}_N\mathcal{M}_R)(\Sigma^\infty X,\Sigma^\infty Y)$$

is an isomorphism.

In particular, $B_{\{N\}}$ is isomorphic to $B_{\mathcal{W}_N}$.

Proof Let X' be a cell complex replacement of X built out of cells with trivial G–actions. The space $E\mathcal{W}_N$ is $\overline{\mathrm{Lie}(G)}$–equivalent to $E(G/N)$. This is an $\{N\}$–cell complex. Hence $X'\wedge EG/N_+\to X'$ is a cofibrant replacement of X both in the $\{N\}$ and and in the $\mathcal{W}_N$–model categories. A $\mathcal{W}_N$–fibrant replacement Y' of Y is also an $\{N\}$–fibrant replacement. □

Remark 7.13 If X does not have trivial G–action, then the $\mathrm{Ho}(\mathcal{WM})(\Sigma^\infty X,\Sigma^\infty Y)$ are typically different for the collections $\{N\}$ and $\mathcal{W}_N$.

8 Postnikov t–model structures

We modify the construction of the $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$ by considering the n–$\mathcal{W}$–equivalences, for all n, instead of $\mathcal{W}$–equivalences. This is used when we give model structures to the category of pro–spectra, pro–$\mathcal{M}_R$, in Section 9.2.

The homotopy category of a stable model category is a triangulated category [26, 7.1]. We consider t–structures on this triangulated category together with a lift of the t–structures to the model category itself. The relationship between n–equivalences and t–structures is given below in Definition 8.4 and Lemma 8.10.

8.1 Preliminaries on t–model categories

We recall the terminology of a t–structure (Beilinson–Bernstein–Deligne [3, 1.3.1]) and of a t–model structure (Fausk–Isaksen [21, 4.1]).

Definition 8.1 A homologically graded t–structure on a triangulated category $\mathcal{D}$, with shift functor Σ, consists of two full subcategories $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$ of $\mathcal{D}$, subjected to the following three axioms:

(1) $\mathcal{D}_{\geq 0}$ is closed under Σ, and $\mathcal{D}_{\leq 0}$ is closed under Σ^{-1};

(2) for every object X in $\mathcal{D}$, there is a distinguished triangle

$$X' \to X \to X'' \to \Sigma X'$$

such that $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \Sigma^{-1}\mathcal{D}_{\leq 0}$; and

(3) $\mathcal{D}(X, Y) = 0$, whenever $X \in \mathcal{D}_{\geq 0}$ and $Y \in \Sigma^{-1}\mathcal{D}_{\leq 0}$.

For convenience we also assume that $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$ are closed under isomorphisms in $\mathcal{D}$.

Definition 8.2 Let $\mathcal{D}_{\geq n} = \Sigma^n \mathcal{D}_{\geq 0}$, and let $\mathcal{D}_{\leq n} = \Sigma^n \mathcal{D}_{\leq 0}$.

Remark 8.3 A homologically graded t–structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ corresponds to a cohomologically graded t–structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ as follows: $\mathcal{D}^{\geq n} = \mathcal{D}_{\leq -n}$ and $\mathcal{D}^{\leq n} = \mathcal{D}_{\geq -n}$.

Definition 8.4 The class of n–equivalences in $\mathcal{D}$, denoted W_n, consists of all maps $f\colon X \to Y$ such that there is a triangle

$$F \to X \xrightarrow{f} Y \to \Sigma F$$

with $F \in \mathcal{D}_{\geq n}$. The class of co$-n$–equivalences in $\mathcal{D}$, denoted coW_n, consists of all maps f such that there is triangle

$$X \xrightarrow{f} Y \to C \to \Sigma X$$

with $C \in \mathcal{D}_{\leq n}$.

If $\mathcal{D}$ is the homotopy category of a stable model category $\mathcal{K}$, then a map f in $\mathcal{K}$ is called a (co–)n–equivalence if the corresponding map f in the homotopy category, $\mathcal{D}$, is a (co–)n–equivalence. We use the same symbols W_n and coW_n for the classes of n–equivalences and co–n–equivalence in $\mathcal{K}$ and $\mathcal{D}$, respectively.

Definition 8.5 A t–model category is a proper simplicial stable model category $\mathcal{K}$ equipped with a t–structure on its homotopy category together with a functorial factorization of maps in $\mathcal{K}$ as an n–equivalence followed by a co–n–equivalence in $\mathcal{K}$, for every integer n.

t–model categories are discussed in detail in [21]. They give rise to interesting model structures on pro–categories.

8.2 The d–Postnikov t–model structure on $\mathcal{M}_R$

We construct a t–model structure on $\mathcal{M}_R$ such that the t–structure on the homotopy category of $\mathcal{M}_R$ is given by Postnikov sections. In Section 9 we use this t–model structure to produce model structures on the category of pro–spectra. We allow Postnikov sections where the cut-off degree of π_*^H depends on H. See also Lewis [32].

Construction 8.6 Assume that $\mathcal{D}$ is the homotopy category of a proper simplicial stable model category $\mathcal{M}$. Let $\mathcal{D}_{\geq 0}$ be a strictly full subcategory of $\mathcal{D}$ that is closed under Σ. Define $\mathcal{D}_{\geq n}$ to be $\Sigma^n \mathcal{D}_{\geq 0}$. Let W_n be as in Definition 8.4, and lift W_n to $\mathcal{M}$. Let C denote the class of cofibrations in $\mathcal{M}$, and define $C_n = W_n \cap C$ and $F_n = \text{inj}\, C_n$, the class of maps with the right lifting property with respect to C_n. Let $\mathcal{D}_{\leq n-1}$ be the full subcategory of $\mathcal{D}$ with objects isomorphic to hofib(g) for all $g \in F_n$. If there is a functorial factorization of any map in $\mathcal{M}$ as a map in C_n followed by a map in F_n, then $\mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}$ is a t–structure on $\mathcal{D}$. Hence the model category $\mathcal{M}$, the factorization, and the t–structure on $\mathcal{D}$ is a t–model structure on $\mathcal{M}$ (Fausk–Isaksen [21, 4.12]).

Let $\mathcal{C}$ and $\mathcal{W}$ be collections of subgroups of G such that $\mathcal{C}\mathcal{W} \subset \mathcal{C}$. Let R be a ring, and let $\mathcal{D}$ be the homotopy category of $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Definition 8.7 A class function on $\mathcal{W}$ is a function $d\colon \mathcal{W} \to \mathbb{Z} \cup \{-\infty, \infty\}$ such that $d(H) = d(gHg^{-1})$, for all $H \in \mathcal{W}$ and $g \in G$.

Definition 8.8 Let d be a class function. Define a full subcategory of $\mathcal{D}$ by

$$\mathcal{D}^d_{\geq 0} = \{X \mid \Pi^U_i(X) = 0 \text{ for } i < d(U), U \in \mathcal{W}\}.$$

Let $\mathcal{D}^d_{\leq -1}$ be the full subcategory of $\mathcal{D}$ given by Construction 8.6.

The next result is needed to get a t–model structure on $\mathcal{M}_R$.

Lemma 8.9 *Any map in $\mathcal{M}_R$ factors functorially as a map in C_n followed by a map in F_n. Moreover, there is a canonical map from the nth factorization to the $(n-1)$th factorization.*

Proof The proof is similar to the proof of Theorem 5.4. See also [19, Appendix]. □

Lemma 8.10 *Let d be a class function on $\mathcal{W}$. The $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$, the two classes $\mathcal{D}^d_{\geq 0}$ and $\mathcal{D}^d_{\leq 0}$, together with the factorization in Lemma 8.9 is a t–model structure.*

Proof This follows from Theorem 5.4 and Construction 8.6 [21, 4.12]. □

This t–model structure is called the d–Postnikov t–model structure on $\mathcal{WCM}_R$. We call the 0–Postnikov t–model structure simply the Postnikov t–model structure.

A map f of spectra is an n–equivalence with respect to the d–Postnikov t–structure, as defined in Definition 8.4, if and only if $\Pi^U_m(f)$ is an isomorphism for $m < d(U)+n$ and $\Pi^U_{d(U)+n}(f)$ is surjective for all $U \in \mathcal{W}$.

8.3 An example: Greenlees connective K–theory

To show that there is some merit to the generality of d–Postnikov t–structures, we recover Greenlees equivariant connective K–theory as the d–connective cover of equivariant K–theory for a suitable class function d. Let G be a compact Lie group, and let $\mathcal{W} = \mathcal{C}$ be the collection of all closed subgroups of G. Let P_n denote the nth Postnikov section functor, and let C_n denote the nth connective cover functor.

Lemma 8.11 *Let G be a compact Lie group. Let d be the class function such that $d(1) = 0$ and $d(H) = -\infty$ for all $H \neq 1$. Then*

$$X_{\leq n} = F(EG_+, P_n X)$$

is a functorial truncation functor for the d–Postnikov t–model structure on $\mathcal{WM}_R$.

The nth d–connective cover, $X_{\geq n}$, of X is such that the left most square in the following diagram is a homotopy pullback square

$$\begin{array}{ccccc} X_{\geq n} & \longrightarrow & X & \longrightarrow & F(EG_+, P_{n-1}X) \\ \downarrow & & \downarrow & & \| \\ F(EG_+, C_nX) & \longrightarrow & F(EG_+, X) & \longrightarrow & F(EG_+, P_{n-1}X). \end{array} \tag{8–12}$$

In particular, $(K_G)_{\geq 0}$ is Greenlees' equivariant connective K–theory [22, 3.1].

Proof Axiom 1 of a t–structure is satisfied since

$$\Sigma^{-1}F(EG_+, P_nX) \cong F(EG_+, \Sigma^{-1}P_nX) \cong F(EG_+, P_n(\Sigma^{-1}P_nX)).$$

We combine the verification of axioms 2 and 3 of a t–structure. Let $X_{\geq n}$ denote the homotopy fiber of the natural transformation $X \to F(EG_+, P_{n-1}X)$. Since $X \to F(EG_+, X)$ is a non-equivariant equivalence we conclude, using Equation (8–12), that $X_{\geq n}$ and C_nX are non-equivariant equivalent. Hence $X_{\geq n} \in \mathcal{D}_{\geq n}$ for all $X \in \mathcal{D}$. If $Y \in \mathcal{D}_{\geq n}$ and $X \in \mathcal{D}$, then

$$\mathcal{D}(Y, F(EG_+, P_{n-1}X)) = 0$$

since $Y \wedge EG_+$ is in $C_n\mathcal{D}$. □

This example can also be extended to arbitrary compact Hausdorff groups [18].

8.4 Postnikov sections

Suppose d is a constant function and R has trivial $\mathcal{C}$–homotopy groups in negative degrees. Then there is a useful description of the full subcategory $\mathcal{D}_{\leq 0}$ of the homotopy category $\mathcal{D}$ of $\mathcal{WCM}_R$.

Definition 8.13 We say that a spectrum R is $\mathcal{C}$–connective if $\Pi_n^U(R) = 0$ for all $n < 0$ and all $U \in \mathcal{C}$.

Proposition 8.14 *Let R be a $\mathcal{C}$–connective ring. Then there is a t–structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ on the homotopy category $\mathcal{D}$ of $\mathcal{WCM}_R$ such that:*

$$\mathcal{D}_{\geq 0} = \{X \mid \Pi_i^U(X) = 0 \text{ whenever } i < 0,\ U \in \mathcal{W}\}$$

and

$$\mathcal{D}_{\leq 0} \subset \{X \mid \Pi_i^U(X) = 0 \text{ whenever } i > 0,\ U \in \mathcal{W}\}.$$

The inclusion is an equivalence whenever $\mathcal{W} \subset \mathcal{C}$.

Proof Recall that an object Y is in $\mathcal{D}_{\leq -1}$ if and only if $\mathcal{D}(X,Y)=0$ for all $X\in\mathcal{D}_{\geq 0}$ [3, 1.3.4]. Proposition 7.7 gives that $\Sigma^\infty_R G/H_+\wedge S^n\in\mathcal{D}_{\geq 0}$, for all $n\geq 0$ and all $H\in\mathcal{C}$. This gives that

$$\mathcal{D}_{\leq -1}\subset\{Y\mid \Pi_i^U(Y)=0 \text{ whenever } U\in\mathcal{W}, i\geq 0\}.$$

We prove the converse inclusion when $\mathcal{W}\subset\mathcal{C}$. Assume that $X\in\mathcal{D}_{\geq 0}$ and that $\Pi_i^U(Y)=0$, whenever $U\in\mathcal{W}$ and $i\geq 0$. By Lemma 7.5 there is a $\mathcal{W}$–cell complex approximation $X'\to X$ such that X' is a cell complex built from cells in non-negative dimensions, and $X'\to X$ is a $\mathcal{W}$–isomorphism in non-negative degrees, hence a $\mathcal{W}$–equivalence. This implies that $\mathcal{D}(X,Y)=0$. Since $\mathcal{D}(X,Y)=0$ for all $X\in\mathcal{D}_{\geq 0}$ we conclude that $Y\in\mathcal{D}_{\leq -1}$. □

If a map g is a co$-n$–equivalence in the Postnikov t–structure, then $\Pi_m^U(g)$ is an isomorphism for $m>n$ and $\Pi_n^U(g)$ is injective for $U\in\mathcal{W}$.

When the universe is trivial we can give a similar description of the t–model structure for more general functions d. We say that a class function $d\colon \mathcal{W}\to\mathbb{Z}\cup\{-\infty,\infty\}$ is increasing if $d(H)\leq d(K)$ whenever $H\leq K$.

Proposition 8.15 *Assume the G–universe $\mathcal{U}$ is trivial, and let R be a $\mathcal{C}$–connective ring. Let d be an increasing class function. Then there is a t–structure $(\mathcal{D}_{\geq 0},\mathcal{D}_{\leq 0})$ on the homotopy category $\mathcal{D}$ of $\mathcal{WCM}_R$ such that:*

$$\mathcal{D}^d_{\geq 0}=\{X\mid \Pi_i^U(X)=0 \text{ whenever } i<d(U),\ U\in\mathcal{W}\}$$

and

$$\mathcal{D}^d_{\leq 0}\subset\{X\mid \Pi_i^U(X)=0 \text{ whenever } i>d(U), U\in\mathcal{W}\}.$$

Proof This follows from Proposition 7.9, Construction 8.6 and the proof of Proposition 8.14. □

Definition 8.16 Let X be a spectrum in $\mathcal{M}_R$. The nth Postnikov section of X is a spectrum P_nX together with a map $p_nX\colon X\to P_nX$ such that $\Pi_m^U(P_nX)=0$, for $m>n$ and $U\in\mathcal{W}$, and $\Pi_m^U(p_nX)\colon \Pi_m^U(X)\to\Pi_m^U(P_nX)$ is an isomorphism, for $m\leq n$ and $U\in\mathcal{W}$. A Postnikov system of X consists of a Postnikov factorization $p_n\colon X\to P_nX$, for every $n\in\mathbb{Z}$, together with maps $r_nX\colon P_nX\to P_{n-1}X$, for all $n\in\mathbb{Z}$, such that $r_nX\circ p_nX=p_{n-1}X$.

Dually, one defines the nth connected cover $C_nX\to X$ of X. The nth connected cover satisfies $\Pi_k^U(C_nX)=0$, for $k<n$, and $\Pi_k^U(C_nX)\to\Pi_k^U(X)$ is an isomorphism, for $k\geq n$.

Definition 8.17 A functorial Postnikov system on $\mathcal{M}_R$ consists of functors P_n, for each $n \in \mathbb{Z}$, and natural transformation $p_n\colon 1 \to P_n$ and, $r_n\colon P_n \to P_{n-1}$ such that $p_n(X)$ and $r_n(X)$, for $n \in \mathbb{Z}$, is a Postnikov system for any spectrum X.

Proposition 8.18 *Let R be a $\mathcal{C}$–connective ring. Then the category $\mathcal{WCM}_R$ has a functorial Postnikov system.*

Proof This follows from Lemma 8.9. □

Remark 8.19 It is often required that the maps $r_n X$ are fibrations for every n and X. We can construct a functorial Postnikov tower with this property if we restricted ourself to the full subcategory $\mathcal{D}_{\geq n}$ for some n [21, Section 7].

8.5 Coefficient systems

In this subsection we describe the Eilenberg–Mac Lane objects in the Postnikov t–structure. Let $\mathcal{C}$ be an Illman collection of subgroups of G and assume that $\mathrm{st}(\mathcal{U})$ is a $\mathcal{W}$–Illman collection. Let $\mathcal{W}$ be a collection such that $\mathcal{CW} \subset \mathcal{C}$. Let R be a $\mathcal{C}$–connective ring spectrum.

Definition 8.20 The heart of a t–structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ on a triangulated category $\mathcal{D}$ is the full subcategory $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$ of $\mathcal{D}$ consisting of objects that are isomorphic to objects both in $\mathcal{D}_{\geq 0}$ and in $\mathcal{D}_{\leq 0}$.

The heart of a t–structure is an abelian category [3, 1.3.6].

Definition 8.21 An R–module X is said to be an Eilenberg–Mac Lane spectrum if $\Pi_n^U(X) = 0$, for all $n \neq 0$ and all $U \in \mathcal{W}$.

Lemma 8.22 *Let $d\colon \mathcal{W} \to \mathbb{Z}$ be the 0–function. If $\mathcal{C}$ and $\mathcal{W}$ are collections of subgroup of G such that $\mathcal{CW} \subset \mathcal{C}$, then the heart of the homotopy category of $\mathcal{WCM}_R$ is contained in the full subcategory consisting of the Eilenberg–Mac Lane spectra. If $\mathcal{W} \subset \mathcal{C}$, then the heart is exactly the full subcategory consisting of the Eilenberg–Mac Lane spectra.*

Proof This follows from Proposition 8.14. □

We give a more algebraic description of the heart in terms of coefficient systems when $\mathcal{W} \subset \mathcal{C}$. Let $\mathcal{D}$ denote the homotopy category of $\mathcal{WCM}_R$.

Definition 8.23 The orbit category, $\mathcal{O}$, is the full subcategory of $\mathcal{D}$ with objects $\Sigma^\infty_R G/H_+$, for $H \in \mathcal{W}$.

The orbit category depends on G, $\mathcal{W}, \mathcal{C}$, and the G–universe $\mathcal{U}$.

Definition 8.24 A $\mathcal{W}$–R–coefficient system is a contravariant additive functor from $\mathcal{O}^{\mathrm{op}}$ to the category of abelian groups.

Denote the category of $\mathcal{W}$–R–coefficient systems by $\mathcal{G}$. This is an abelian category. An object Y in $\mathcal{D}$ naturally represents a coefficient system given by

$$\Sigma^\infty_R G/H_+ \mapsto \mathcal{D}(\Sigma^\infty_R G/H_+, Y).$$

Definition 8.25 Let X be an R–module spectrum. The nth homotopy coefficient system of X, $\pi_n^{\mathcal{W}}(X)$, is the coefficient system naturally represented by $X \wedge \Sigma^R_{\mathbb{R}^n} S^0$, for $n \geq 0$, and by $X \wedge \Sigma^\infty_R S^{-n}$, for $n \leq 0$.

Lemma 8.26 *There is a natural isomorphism*

$$\mathcal{G}(\pi_0^{\mathcal{W}}(\Sigma^\infty_R G/H_+), M) \cong M(G/H_+)$$

for any $H \in \mathcal{W}$ and any coefficient system $M \in \mathcal{G}$.

Proof This is a consequence of the Yoneda Lemma. □

Proposition 8.27 *Let R be a $\mathcal{C}$–connective ring spectrum. The functor $\pi_0^{\mathcal{W}}$ induces a natural equivalence from the full subcategory of Eilenberg–Mac Lane spectra in the homotopy category of $\mathcal{WCM}_R$, to the category of $\mathcal{W}$–R–coefficient systems.*

Proof We show that for every coefficient system M, there is a spectrum HM such that $\pi_0^{\mathcal{W}}(HM)$ is isomorphic to M as a coefficient system, and furthermore, that $\pi_0^{\mathcal{W}}$ induces an isomorphism $\mathcal{D}(HM, HN) \to \mathcal{G}^{\mathcal{W}}(M, N)$ of abelian groups.

We construct a functor, H, from $\mathcal{G}$ to the homotopy category of spectra. The natural isomorphism in Lemma 8.26 gives a surjective map of coefficient systems

$$f(M)\colon \bigoplus_{H\in\mathcal{W}} \bigoplus_{M(G/H_+)} \pi_0^{\mathcal{W}}(G/H_+) \to M.$$

This construction is natural in M. Let C_M be the kernel of $f(M)$ and repeat the construction with C_M in place of M. We get an exact sequence

(8–28)
$$\bigoplus_{K\in\mathcal{W}} \bigoplus_{C_M(G/K_+)} \pi_0^{\mathcal{W}}(G/K_+) \to \bigoplus_{H\in\mathcal{W}} \bigoplus_{M(G/H_+)} \pi_0^{\mathcal{W}}(G/H_+) \to M \to 0.$$

This sequence is natural in M. We have that $\bigoplus_{H\in\mathcal{W}}\bigoplus_{M(G/H_+)}\pi_0^{\mathcal{W}}(G/H_+)$ is naturally isomorphic to

$$\pi_0^{\mathcal{W}}(\bigvee_{H\in\mathcal{W}}\bigvee_{M(G/H_+)}G/H_+)$$

and

$$\mathcal{G}^{\mathcal{W}}\left(\pi_0^{\mathcal{W}}(G/K_+),\pi_0^{\mathcal{W}}(\bigvee_{H\in\mathcal{W}}\bigvee_{M(G/H_+)}G/H_+)\right)$$

is naturally isomorphic to

$$\mathcal{D}(G/K_+,\bigvee_{H\in\mathcal{W}}\bigvee_{M(G/H_+)}G/H_+).$$

Hence there is a map

$$h(M)\colon \bigvee_{K\in\mathcal{W}}\bigvee_{C_M(G/K_+)}G/K_+\to\bigvee_{H\in\mathcal{W}}\bigvee_{M(G/H_+)}G/H_+,$$

unique up to homotopy, so $\pi_0^{\mathcal{W}}(h(M))$ is isomorphic to the leftmost map in the exact sequence (8–28). Let Z be the homotopy cofiber of $h(M)$. Proposition 7.7 says that $\pi_n^{\mathcal{W}}(G/K_+)=0$, for all $n<0$ and $K\in\mathcal{W}$. So $\pi_n(Z)=0$, for $n<0$, and there is a natural isomorphism $\pi_0^{\mathcal{W}}(Z)\cong M$. Let HM be $P_0(Z)$, the 0th Postnikov section of Z. Then HM is an Eilenberg–Mac Lane spectrum and there is a natural isomorphism

$$\pi_0^{\mathcal{W}}(HM)\cong M.$$

This proves the first claim.

The map $Z\to P_0(Z)=HM$ induces an isomorphism

$$[HM,HN]\to[Z,HN].$$

This gives an exact sequence

$$\begin{aligned}0\to[HM,HN]&\to[\bigvee_{H\in\mathcal{W}}\bigvee_{M(G/H_+)}G/H_+,HN]\\&\to[\bigvee_{K\in\mathcal{W}}\bigvee_{C(G/K_+)}G/K_+,HN],\end{aligned}$$

where the rightmost map is induced by $h(M)$ and the leftmost map is injective. Applying $\pi_0^{\mathcal{W}}$ gives an isomorphism between the rightmost map and the map

$$\mathcal{G}\left(\bigoplus_{H\in\mathcal{W},M(G/H_+)}\pi_0^{\mathcal{W}}(G/H_+),N\right)\to\mathcal{G}\left(\bigoplus_{K\in\mathcal{W},C(G/K_+)}\pi_0^{\mathcal{W}}(G/K_+),N\right).$$

The kernel of this map is $\mathcal{G}(M,N)$, so $\pi_0^{\mathcal{W}}$ induces an isomorphism

$$[HM,HN]\to\mathcal{G}(M,N)$$

of abelian groups. This proves the second claim. □

Remark 8.29 When d is not a constant class function, then the homotopy groups of the objects in the heart need not be concentrated in one degree. For example the heart of the t–structure in Lemma 8.11 consists of spectra of the form $F(EG_+, HM)$, where M is an Eilenberg–Mac Lane spectrum. The heart of the Postnikov t–structure on $\mathcal{D}$ is not well understood for general functions d.

8.6 Continuous G–modules

When $\mathcal{W} \not\subset \mathcal{C}$ it is harder to describe the full subcategory of the homotopy category of $\mathcal{WCM}$ consisting of the Eilenberg–Mac Lane spectra as a category of coefficient systems. We give a description of the heart of the Postnikov t–structure on the homotopy category of the $\overline{\mathrm{Lie}(G)}$–cofree model structure on $\mathcal{M}_R$ when G is a compact Hausdorff group. Let R^0 denote the (continuous) G–ring $\mathrm{colim}_U\, \pi_0^U(R)$. The ring R^0 has the discrete topology. By a continuous R^0–G–module we mean an R^0–G–module with the discrete topology such that the action by G is continuous. The module $\Pi_0^1(X) \cong \mathrm{colim}_U\, \pi_0^U(X)$ is a continuous R^0–G–module, for any R–module X. The stabilizer of any element m in a continuous R^0–G–module M is in $\overline{\mathrm{fnt}(G)}$.

Proposition 8.30 *The heart of $\mathcal{D}$ is equivalent to the category of continuous R^0–G–modules and continuous G–homomorphisms between them.*

Proof Let M be a continuous R^0–G–module. We get a canonical surjective map

$$f\colon \textstyle\bigoplus_m R^0[G/\mathrm{st}(m)] \to M$$

where the sum is over all elements m in M. The map $R^0[G/\mathrm{st}(m)] \to M$, corresponding to the summand m, is given by sending the element (r, g) to $r \cdot gm$. This is a G–map since $g'r \cdot g'gm = g'(rgm)$, for $g' \in G$. Repeating this construction with the kernel of f gives a canonical right exact sequence of continuous R^0–G–modules

$$\textstyle\bigoplus R^0[G/U'] \xrightarrow{h} \bigoplus R^0[G/U] \to M \to 0. \tag{8–31}$$

We want to realize this sequence at the level of spectra. We have that

$$\Pi_0^1(R \wedge G/U_+) \cong R^0[G/U]$$

as R^0–G–modules, for all $U \in \overline{\mathrm{fnt}(G)}$. The map h is realized as Π_0^1 applied to a map

$$h(M)\colon \textstyle\bigvee R \wedge G/U'_+ \to \bigvee R \wedge G/U_+.$$

Let Z be the homotopy cofiber of $h(M)$ and let HM be $P_0(Z)$. The right exact sequence in (8–31) is naturally isomorphic to Π_0^1 applied to the sequence

$$\textstyle\bigvee R \wedge G/U'_+ \to \bigvee R \wedge G/U_+ \to HM.$$

Proposition 7.7 gives that $\Pi_n^1(HM) = 0$ when $n \neq 0$, and there is a natural isomorphism $\Pi_0^1(HM) \cong M$ of continuous R^0–G–modules, for any continuous R^0–G–module M. It remains to show that Π_0^1 is a full and faithful functor. The same argument as in the proof of Proposition 8.27 applies. This works since $\mathcal{D}(\Sigma_R^\infty G/U+, HM)$ is naturally isomorphic to M^U. □

9 Pro–G–spectra

In this section we use the $\mathcal{W}$–$\mathcal{C}$–Postnikov t–model structure on $\mathcal{M}_R$, discussed in Section 8, to give a model structure on the pro–category, pro–$\mathcal{M}_R$. For terminology and general properties of pro–categories see for example (Fausk–Isaksen [21], Isaksen [30]). We recall the following Definition.

Definition 9.1 Let M be a collection of maps in $\mathcal{C}$. A levelwise map $g = \{g_s\}_{s\in S}$ in pro–$\mathcal{C}$ is a levelwise M–map if each g_s belongs to M. A pro–map f is an essentially levelwise M–map if f is isomorphic, in the arrow category of pro–$\mathcal{C}$, to a levelwise M–map. A map in pro–$\mathcal{C}$ is a special M–map if it is isomorphic to a cofinite cofiltered levelwise map $f = \{f_s\}_{s\in S}$ with the property that for each $s \in S$, the map

$$M_s f\colon X_s \to \lim_{t<s} X_t \times_{\lim_{t<s} Y_t} Y_s$$

belongs to M.

Definition 9.2 Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between two categories $\mathcal{A}$ and $\mathcal{B}$. We abuse notation and let $F\colon$ pro–$\mathcal{A} \to$ pro–$\mathcal{B}$ also denote the extension of F to the pro–categories given by composing a cofiltered diagram in $\mathcal{A}$ by F. We say that we apply F levelwise to pro–$\mathcal{A}$.

9.1 Examples of pro–G–Spectra

We list a few examples of pro–spectra.

(1) The finite p–local spectra M_I constructed by Devinatz assemble to give an interesting pro–spectrum $\{M_I\}$ [12]. The pro–spectrum is more well behaved than the individual spectra. This pro–spectrum is important in understanding the homotopy fixed points of the spectrum E_n (Davis [11], Devinatz–Hopkins [14]).

(2) There is an approach to Floer homology that is based on pro–spectra (Cohen–Jones–Segal [7]).

(3) The spectrum $\mathbb{RP}^{\infty}_{-\infty}$, and more generally, the pro–Thom spectrum associated to a (virtual) vector bundle over a space X, are non-constant pro–spectra.

Construction 9.3 Let N be a normal subgroup of G in $\mathcal{C}$. Let EG/N denote the free contractible G/N–space constructed as the (one sided) bar construction of G/N. Then EG/N_+ is a cell complex built out of cells $(G/N \times D^m)_+$ for integers $m \geq 0$. The bar construction gives a functor from the category with objects quotient groups G/N, of G, and morphisms the quotient maps, to the category of unbased G–spaces. In particular, we get a pro–G–spectrum $\{\Sigma^\infty EG/N_+\}$ indexed on the directed set of normal subgroups $N \in \mathcal{C}$ ordered by inclusion. This pro–spectrum plays an important role in our theory. The notation is slightly ambiguous; the N–orbits of EG are denoted by $(EG)/N$.

9.2 The Postnikov model structure on pro–$\mathcal{M}_R$

The most immediate candidate for a model structure on pro–$\mathcal{M}_R$ is the strict model structure obtained from the $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$ [30]. In this model structure the cofibrations are the essentially levelwise $\mathcal{C}$–cofibrations and the weak equivalences are the essentially levelwise $\mathcal{W}$–equivalences.

Let $\{P_n\}$ denote a natural Postnikov tower (for $n \geq 0$) [21, Section 7]. The Postnikov towers in $\mathcal{M}_R$ extend to pro–$\mathcal{M}_R$ (Definition 9.2). A serious drawback of the strict model structure on pro–$\mathcal{M}_R$ is that $Y \to \operatorname{holim}_n P_n Y$ is not always a weak equivalence. We explain why. The homotopy limit in pro–$\mathcal{M}_R$, $\operatorname{holim}_n P_n\{Y_s\}$, of the Postnikov tower $P_n\{Y_s\}$, is strict weakly equivalent to the pro–object $\{P_n Y_s\}$, for any $\{Y_s\} \in$ pro–$\mathcal{M}_R$. Let Y be the constant pro–object $\vee_{n\geq 0} H\mathbb{Z}[n]$, where $H\mathbb{Z}[n]$ is the Eilenberg–Mac Lane spectrum of $\mathbb{Z}$ concentrated in degree n. Then the map $Y \to \{P_n Y\}$ is not a strict pro–equivalence because $\oplus_{n\geq 0}\, \pi_n$ applied to this map is not an isomorphism of pro–groups.

In this section we construct an alternative to the strict model structure to rectify the flaw that $Y \to \operatorname{holim}_n P_n Y$ is not always a weak equivalence. This model structure has the same class of cofibrations but more weak equivalences than the strict model structure. The class of weak equivalences is the smallest class of maps closed under composition and retract containing both the strict weak equivalences and maps $Y \to \{P_n Y\}_{n\in\mathbb{Z}}$, for $Y \in$ pro–$\mathcal{M}_R$. This model structure on pro–$\mathcal{M}_R$ is called the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure. We construct the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$ from the Postnikov t–structure on $\mathcal{WCM}_R$ using a general technique developed in [20; 21].

The benefits of replacing pro–spaces (and pro–spectra) by their Postnikov towers was already made clear by Artin–Mazur [2, Section 4]. Dwyer–Friedlander also made use of this replacement [16].

In the next theorem we give a general model structure, called the d–Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$. The Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure referred to above is the model structure obtained by letting d be the constant class function that takes the value 0.

Theorem 9.4 *Let $\mathcal{C}$ be a $\mathcal{U}$–Illman collection of subgroups of G and let $\mathcal{W}$ be a collection of subgroups of G such that $\mathcal{C}\mathcal{W} \subset \mathcal{C}$. Let R be a $\mathcal{C}$–connective ring spectrum. Let $d\colon \mathcal{W} \to \mathbb{Z} \cup \{-\infty, \infty\}$ be a class function. Then there is a proper simplicial stable model structure on pro–$\mathcal{M}_R$ such that:*

(1) *the cofibrations are essentially levelwise retracts of relative $\mathcal{C}$–cell complexes;*

(2) *The weak equivalences are essentially levelwise n–$\mathcal{W}$–equivalences in the d–Postnikov t–model structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$ for all integers n; and*

(3) *the fibrations are retracts of special F_∞–maps.*

Here F_∞ is the class of all maps that are both $\mathcal{W}$–$\mathcal{C}$–fibrations and co–n–$\mathcal{W}$–equivalences, for some n, in the d–Postnikov t–model structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Proof This is a consequence of Fausk–Isaksen [21, 6.3, 6.13], Isaksen [29, 16.2], and Lemma 8.10. □

We consider a particular example. Since $\{1\}\mathcal{C} = \mathcal{C}$ there is a d–Postnikov $\{1\}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$. If d is $+\infty$, then the model structure is the strict model structure on pro–$\mathcal{M}_R$ obtained from the $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$ and if d is $-\infty$, then all maps are weak equivalences. If d is an integer, then the model structure is independent of the integer d, so we omit it from the notation. We call this model structure the (Postnikov) $\mathcal{C}$–cofree model structure on pro–$\mathcal{M}_R$ and denote it $\mathcal{C}$–cofree pro–$\mathcal{M}_R$.

Theorem 9.5 *Let $\mathcal{U}$ be a trivial G–universe. Then there is a model structure on the category pro–$\mathcal{M}_R$ such that:*

(1) *the cofibrations are essentially levelwise retracts of relative $\mathcal{C}$–cell complexes;*

(2) *the weak equivalences are essentially levelwise $\mathcal{C}$–underlying n–equivalences, for all integers n; and*

(3) *the fibrations are retracts of special F_∞–maps.*

Here F_∞ is the class of all maps that are both $\{1\}$–$\mathcal{C}$–fibrations and $\mathcal{C}$–underlying co–n–equivalences, for some n, in the Postnikov t–model structure on $\{1\}\mathcal{C}\mathcal{M}_R$.

We return to the general situation. Let $\mathcal{D}$ denote the homotopy category of the $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$, and let $\mathcal{P}$ denote the homotopy category of the d–Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$.

An alternative description of the weak equivalences in the Postnikov model structure is given in [21, 9.13]. There is a t–structure on $\mathcal{P}$ as described in [21, 9.4].

Let Map denote the simplicial mapping space in $\mathcal{M}_R$ with the $\mathcal{W}$–$\mathcal{C}$–model structure. We give a concrete description of the homsets in the homotopy category of the d–Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$.

Proposition 9.6 *Let X and Y be objects in pro–$\mathcal{M}_R$ such that each X_a is cofibrant and each d–Postnikov section P_nY_b is fibrant in $\mathcal{WCM}_R$. Then the group of maps from X to Y, in the homotopy category of the d–Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$, is equivalent to*

$$\pi_0(\mathit{holim}_{n,b}\ \mathit{hocolim}_a\ \mathit{Map}(X_a, P_nY_b)).$$

Proof This follows from [21, 8.4]. □

Recall that the constant pro–object functor $c\colon \mathcal{M}_R \to \text{pro–}\mathcal{M}_R$ is a left adjoint to the inverse limit functor $\lim\colon \text{pro–}\mathcal{M}_R \to \mathcal{M}_R$. The composite functor $\lim \circ c$ is canonically isomorphic to the identity functor on $\mathcal{M}_R$.

Proposition 9.7 *Let $\mathcal{M}_R$ have the $\mathcal{W}$–$\mathcal{C}$–model structure, and let pro–$\mathcal{M}_R$ have the d–Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure. Then c is Quillen left adjoint to lim. If d is a uniformly bounded below class function ($d \geq n$ for some integer n), then the constant pro–object functor c induces a full embedding $c\colon \mathit{Ho}(\mathcal{M}_R) \to \mathit{Ho}(\text{pro–}\mathcal{M}_R)$.*

Proof It is clear that c respects cofibrations and acyclic cofibrations. Let X and Y be in $\mathcal{M}_R$. The assumption on d gives that $Y \to \operatorname{holim}_n P_nY$ is a $\mathcal{W}$–weak equivalence. The result follows from Proposition 9.6 since

$$\operatorname{holim}_n \operatorname{Map}(X, P_nY)) \to \operatorname{Map}(X, \operatorname{holim}_n P_nY))$$

is a weak equivalence of simplicial sets. □

Remark 9.8 If d is a uniformly bounded (above and below) class function on $\mathcal{W}$, then a map is an essentially levelwise $(n+d)$–equivalence for every integer n, if and only if it is an essentially levelwise n–equivalence for every integer n (with the constant function with value 0). Hence under this assumption the d–Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$ is the same as the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$.

Remark 9.9 It is not clear if there is an Adams isomorphism when G is not a compact Lie group or $\{1\}$ is not contained in $\mathcal{W}$. There are no free G–cell complexes (that are cofibrant) so the usual statements does not make sense. One might try to replace G by $\{G/N\}$, indexed on normal subgroups, N, of G in $\mathcal{W}$. The most naive implementation of this approach does not work.

Assume that G is a compact Hausdorff group which is not a Lie group, and let $\mathcal{C} = \mathcal{W} = \overline{\mathrm{Lie}(G)}$. Assume in addition that $\mathcal{U}$ is a complete G–universe. Proposition 7.10 and Proposition 9.6 applied to the pro–suspension spectrum $\{\Sigma^\infty EG/N_+\}$ give that $\pi_*^G(\{\Sigma^\infty EG/N_+\})$ is 0.

Example 9.10 It is harder to be an essentially levelwise $\pi_n^{\mathcal{W}}$–isomorphism than it is to be an essentially levelwise π_n^H–isomorphism for each $H \in \mathcal{W}$ individually. The difference is fundamental as the following example shows (in the category of spaces, or the category of spectra indexed on a trivial universe). Let $\mathcal{W}$ be a normal collection closed under intersection and without a smallest element. Let N be a normal subgroup of G. The fixed point space $(EG/N)^H$ is empty, for $H \not\leq N$, and it is EG/N, for $H \leq N$. The pro–map $\{\Sigma^\infty EG/N_+\} \to \{*\}$, where the first object is indexed on the directed set of normal subgroups of G contained in $\mathcal{W}$ ordered by inclusion, is a π_n^H–isomorphism, for all $H \in \mathcal{W}$ and any integer n. But this map is typically not an essentially levelwise $\pi_n^{\mathcal{W}}$–isomorphism for any n. The same conclusion applies when the universe is not trivial by (an (in)complete universe version of) Proposition 7.10 (see Remark 9.9).

We include a result about fibrations for use in Section 11.

Lemma 9.11 *Let $\mathrm{st}(\mathcal{U})$ be a $\mathcal{W}$–Illman collection of subgroups of a compact Hausdorff group G. Let $f\colon X \to Y$ be a fibration in pro–$\mathcal{M}_R$ between fibrant objects X and Y in the Postnikov $\mathcal{W}$–model structure on pro–$G\mathcal{M}_R$. Assume in addition that X and Y are levelwise $\mathcal{W}$–$\mathcal{S}$–cell complexes. Let K be any closed subgroup of G, and let $\mathcal{W}'$ be a collection of subgroups of K such that $\mathcal{W}'\mathcal{W} \subset \mathcal{W}$ and $\mathrm{st}_K(\mathcal{U})$ is $\mathcal{W}'$–Illman. Then f regarded as a map of pro–K–spectra is a fibration in the Postnikov $\mathcal{W}'$–model structure on pro–$K\mathcal{M}_R$.*

Proof This reduces to Lemma 4.16 and Lemma 4.19, since fixed points respect the limits used in the definition of special F_∞–maps (see Definition 9.1). □

9.3 Tensor structures on pro–$\mathcal{M}_R$

Let R be a ring. Then the category $\mathcal{M}_R$ is a closed symmetric tensor category. The category pro–$\mathcal{M}_R$ inherits a symmetric tensor structure. Let $\{X_s\}_{s\in S}$ and $\{Y_t\}_{t\in T}$ be two objects in pro–$\mathcal{M}_R$.

Definition 9.12 The smash product $\{X_s\}_{s\in S} \wedge \{Y_t\}_{t\in T}$ is defined to be the pro–spectrum $\{X_s \wedge Y_t\}_{s\times t\in S\times T}$.

The tensor product in pro–$\mathcal{M}_R$ is not closed. Worse, the smash product does not commute with direct sums in general [21, 11.2]. The tensor product does not behave well homotopy theoretically for general R–modules.

Definition 9.13 A pro–object Y is bounded below if it is isomorphic to a pro–object $X = \{X_s\}$ and there exists an integer n such that each $* \to X_s$ is an n–equivalence. A pro–object Y is levelwise bounded below if it is isomorphic to a pro–object $X = \{X_s\}$ and for every s there exists an integer n_s such that $* \to X_s$ is an n_s–equivalence.

The simplicial structure on $\mathcal{M}_R$ is compatible with the tensor structure [21, 12.2]. If $\mathcal{C}$ is an Illman collection of subgroups of G, then pro–$\mathcal{M}_R$, with the strict model structure, is a tensor model category [21, 12.7].

If R is $\mathcal{C}$–connective, then Lemma 7.6 gives that the Postnikov t–structure on $\mathcal{D}$ is compatible with the tensor product [21, 12.5]. This implies that the full subcategory of pro–$\mathcal{M}_R$, with the Postnikov model structure, consisting of essentially bounded below objects is a tensor model category [21, 12.10].

We can define a pro–spectrum valued hom functor. Let F denote the internal hom functor in $\mathcal{M}_R$.

Definition 9.14 We extend the definition of F to pro–$\mathcal{M}_R$ by letting $F(X, Y)$ be the pro–object

$$\{\operatorname{colim}_{s\in S} F(X_s, Y_t)\}_{t\in T}.$$

The pro–spectrum valued hom functor is not an internal hom functor in general. The next result shows that under some finiteness assumption the (derived) pro–spectrum valued hom functor behaves as an internal hom functor in the homotopy category.

Lemma 9.15 *Let $\{X_s\}$ be a pro–spectrum such that each X_s is a retract of a finite $\mathcal{C}$–cell spectrum. Let Y be an essentially bounded below pro–spectrum and let Z be a pro–spectrum. Then there is an isomorphism*

$$\mathcal{P}(X \wedge Y, Z) \cong \mathcal{P}(X, F(Y, Z))$$

Proof This follows from Proposition 9.6. □

9.4 Bredon cohomology and group cohomology

Assume that $\mathcal{W} \subset \mathcal{C}$ or that $\mathcal{W} = \{1\}$ and $\mathcal{C} = \overline{\mathrm{Lie}(G)}$. Under these assumptions the heart of the Postnikov t–structures are equivalent to categories of coefficient systems as described in Proposition 8.27 and Proposition 8.30. In the latter case the coefficient systems are continuous (discrete) G–modules.

Definition 9.16 The nth Bredon cohomology of a pro–spectrum X with coefficients in a pro–coefficient system $\{M_a\}$, is defined to be

$$\mathcal{P}(X, \Sigma^\infty_R S^n \wedge \{HM_a\}).$$

The Bredon cohomology of X with coefficients in $\{M_a\}$ is denoted by $H^n(X; \{M_a\})$. This is the cohomology functor with coefficients in the heart, in the terminology of [21, Definition 2.13] (when $\{HM_a\}$ is in the heart of $\mathcal{P}$ [21, Proposition 9.11, Lemma 9.12]).

Lemma 9.17 *Let M be a constant pro–coefficient system. Then the nth Bredon cohomology of a pro–spectrum $X = \{X_s\}$ is*

$$\mathrm{colim}_s H^n(X_s; M).$$

Proof This follows from Proposition 9.6. □

Definition 9.18 The nth Bredon homology of an essentially bounded below pro–spectrum X with coefficients in a pro–coefficient system $\{M_a\}$, is defined to be

$$\mathcal{P}(\Sigma^\infty_R S^n, X \wedge \{HM_a\}).$$

The Bredon homology of X with coefficients in $\{M_a\}$ is denoted $H_n(X; \{M_a\})$.

Remark 9.19 We get isomorphic groups if we use the strict model structure instead of the Postnikov model structure on pro–$\mathcal{M}_R$ to define the Bredon cohomology and Bredon homology. In the case of cohomology since $\{HM_\alpha\}$ is bounded above, and in the case of homology since we are mapping from a constant pro–object.

Let R be the sphere spectrum $\mathcal{S}$, $\mathcal{W} = \{1\}$ and $\mathcal{C} = \overline{\mathrm{Lie}(G)}$. Let $\{M_a\}$ be a pro–object of discrete G–R^0–modules, and let $\{HM_a\}$ be the associated Eilenberg–Mac Lane pro–spectrum.

Definition 9.20 The group cohomology of G with coefficients in $\{M_a\}$ is the Bredon cohomology of $\{EG/N_+\}$ with coefficients in $\{HM_a\}$.

We denote the nth group cohomology by $H^n_{\mathrm{cont}}(G;\{M_a\})$. If M is a constant coefficient system, then we recover the usual definition of group cohomology as

$$H^n_{\mathrm{cont}}(G;M) \cong \operatorname{colim}_N H^n(G/N;M^N),$$

where the colimit is over all subgroups $N \in \mathrm{Lie}(G)$. In general, there is a higher lim spectral sequence relating the group cohomology of a pro–coefficient system $\{M_a\}$ to the continuous group cohomology of the individual modules M_a.

Lemma 9.21 *A short exact sequence of pro–G–modules gives a long exact sequence in group cohomology.*

Proof This follows from the fact that a short exact sequence of pro–G–modules gives rise to a cofiber sequence, of the corresponding Eilenberg–Mac Lane pro–spectra, in the Postnikov $\overline{\mathrm{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$. □

Lemma 9.22 *The group cohomology functor in Definition 9.20, composed with the functor from towers of discrete G–modules and levelwise maps between them to pro–G–coefficient systems, agrees with Jannsen's group cohomology [31].*

Proof A comparison to Jannsen's cohomology follows from the proof of [31, Lemma 3.30]. □

9.5 The Atiyah–Hirzebruch spectral sequence

A t–structure on a triangulated category gives rise to an Atiyah–Hirzebruch spectral sequence [21, 10.1]. Let $\mathcal{C}$ be an Illman collection of subgroups of G and assume that $\mathrm{st}(\mathcal{U})$ is $\mathcal{W}$–Illman. Assume that $\mathcal{W} \subset \mathcal{C}$ or $\mathcal{W} = \{1\}$ and $\mathcal{C} = \overline{\mathrm{Lie}(G)}$. We can relax this assumption to $\mathcal{C}\mathcal{W} \subset \mathcal{C}$ if we work with objects in the heart instead of coefficient systems. Let R be a $\mathcal{C}$–connective ring spectrum. Let $\mathcal{P}$ denote the homotopy category of pro–$\mathcal{M}_R$ with the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structure. Let square brackets denote homotopy classes in $\mathcal{P}$. Recall Definition 9.13.

Theorem 9.23 *Let X and Y be any pro–G–spectra. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p(X, \Pi^{\mathcal{W}}_{-q}(Y))$$

converging to $[X,Y]_G^{p+q}$. The differentials have degree $(r,-r+1)$. The spectral sequence is conditionally convergent if:

(1) *X is bounded below; or*

(2) *X is levelwise bounded below and Y is a constant pro–G–spectrum.*

The E_2–term is the Bredon cohomology of X with coefficients in the pro–coefficient system $\Pi^{\mathcal{W}}_{-q}(Y)$ (Definition 9.16).

Proof This follows from [21, 10.3] and our identification of the heart in Proposition 8.27 and Proposition 8.30. □

Lemma 9.24 *If Y is a monoid in the homotopy category of pro–$\mathcal{M}_R$ with the strict model structure obtained from the $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$, then the spectral sequence is multiplicative.*

Proof By Lemma 7.6 the Postnikov t–structure respects the smash product in the sense of [21, 12.5]. The result follows from [21, 12.11]. □

Remark 9.25 If X is a bounded below $CW-R$–module, then it is possible to filter $F(X,Y)$ by the skeletal filtration of X instead of the Postnikov filtration of Y. The two filtrations give rise to two spectral sequences. When Y is a constant bounded above pro–spectrum the two spectral sequences are isomorphic by Greenlees–May [23, Appendix B]. However, in general we get two different spectral sequences. For a discussion of this see Davis [10].

10 The $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$

Suppose $\mathcal{C}$ is an Illman collection that does not contain the trivial subgroup 1. Then it is not possible to have a $\mathcal{W}$–$\mathcal{C}$–model structure on $\mathcal{M}_R$ such that the cofibrant objects are free G–cell complexes; this is so because $*\to G_+$ is a $\mathcal{W}$–equivalence for any collection $\mathcal{W}$ such that $\mathcal{C}\mathcal{W}\subset\mathcal{C}$. When $\cap_{H\in\mathcal{C}}H=1$, it turns out that it is possible to construct a model structure on pro–$\mathcal{M}_R$ with cofibrations that are arbitrarily close approximations to G–free cell complexes. The class of weak equivalences in this model structure is contained in the class of weak equivalences in the Postnikov $\{1\}$–$\mathcal{C}$–model structure on pro–$\mathcal{M}_R$.

In this section we assume that $\mathcal{U}$ is a trivial G–universe, and that $\mathcal{C}$ is a normal Illman collection of subgroups of G (see Definition 2.1 and Definition 2.14). We also assume that R is a $\mathcal{C}$–connective ring spectrum.

10.1 Construction of the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$

We use the framework of filtered model structures defined in [20, 4.1]. Define an indexing set A to be the product of $\mathcal{C}$, ordered by containment, and the integers, with the usual totally ordering.

Lemma 10.1 *There is a proper simplicial filtered model structure on $\mathcal{M}_R$, indexed on the directed set A, such that:*

(1) *$C_{U,n} = C_U$ is the class of retracts of relative G–cell complexes with cells of the form $\Sigma^R_V G/H_+ \wedge D^m_+$, for some integer m, indexing representation V and $H \in \mathcal{C}$ such that $H \leq U$;*

(2) *$F_{U,n}$ is the class of maps f such that f^H is a fibration and a co–n–equivalence, for each $H \in \mathcal{C}$ such that $H \leq U$; and*

(3) *$W_{U,n} = W_n$ is the class of maps f for which there exists an $H \in \mathcal{C}$ such that f^K is an n–equivalence for every $K \in \mathcal{C}$ such that $K \leq H$.*

Proof The directed set of classes C_U and W_n are decreasing and the directed set of classes $F_{U,n}$ is increasing. The verification of the proper filtered model structure axioms is similar to the verification of the axioms in the case of G–spaces. We omit the details and refer the reader to the detailed discussion given in [20, Section 8]. The simplicial structure follows as in the proof of Theorem 5.4. □

Let F_∞ denote the union $\cup_{U,n} F_{U,n}$. The following model structure on pro–$\mathcal{M}_R$ is a consequence of [20, Theorems 5.15, 5.16].

Theorem 10.2 *There is a proper simplicial model structure on pro–$\mathcal{M}_R$ such that:*

(1) *the cofibrations are maps that are retracts of essentially levelwise C_U–maps for every $U \in \mathcal{C}$;*

(2) *the weak equivalences are maps that are essentially levelwise W_n–maps for every $n \in \mathbb{Z}$; and*

(3) *the fibrations are special F_∞–maps.*

We call this model structure the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$. We denote the model category by $\mathcal{C}$–free pro–$\mathcal{M}_R$.

Lemma 10.3 *Let $\{\Sigma^\infty_R EG/N_+\}$ be indexed on all $N \in \mathcal{C}$ which are normal in G. If $X = \{X_s\}$ is cofibrant in the $\mathcal{C}$–model structure on pro–$\mathcal{M}_R$, then $X \wedge \{\Sigma^\infty_R EG/N_+\}$, is a cofibrant replacement of X in the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$.*

Proof By our assumption each X_s is a retract of a $\mathcal{C}$–cell complex. Hence, since $\mathcal{C}$ is an Illman collection, $X_s \wedge \Sigma^\infty_R EG/N_+$ is also a retract of a $\mathcal{C}$–cell complex built out of G/H–cells for $H \leq N$. Since $\mathcal{C}$ is a normal Illman collection, $* \to X \wedge \{\Sigma^\infty_R EG/N_+\}_{N \leq U}$ is an essentially levelwise C_U–map for every $U \in \mathcal{W}$. We conclude that $X \wedge \{\Sigma^\infty_R EG/N_+\}$ is a cofibrant replacement of X. □

In particular, $\{\Sigma^\infty_R EG/N_+\} \to R$ is a cofibrant replacement of R in the $\mathcal{C}$–free model structure on $\mathcal{M}_R$.

10.2 Comparison of the free and the cofree model structures

We compare the $\mathcal{C}$–cofree model structure on pro–$\mathcal{M}_R$, given in Theorem 9.5, with the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$, given in Theorem 10.2.

Clearly a $\mathcal{C}$–free cofibration is a $\mathcal{C}$–cofree cofibration, and a $W_{U,n}$–equivalence is a $\mathcal{C}$–underlying n–equivalence. Hence the identity functors give a Quillen adjunction

$$\mathcal{C}\text{–free pro–}\mathcal{M}_R \leftrightarrows \mathcal{C}\text{–cofree pro–}\mathcal{M}_R.$$

If 1 is in $\mathcal{C}$, then the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$ is the model structure obtained from the Postnikov t–model structure on $\{1\}\{1\}\mathcal{M}_R$. Hence the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$ is Quillen equivalent to the $\mathcal{C}$–cofree model structure on pro–$\mathcal{M}_R$ by Proposition 5.5. If 1 is not an element in $\mathcal{C}$, then the $\mathcal{C}$–free and the $\mathcal{C}$–cofree model structures on $\mathcal{M}_R$ are typically not Quillen equivalent, as shown by the next example.

Example 10.4 Let $f\colon \vee_N \Sigma^\infty_S EG/N_+ \to \vee_N \Sigma^\infty_S S^0$ be the sum of the collapse maps for all normal subgroups $N \in \mathcal{C}$. The map f is a $\mathcal{C}$–underlying equivalence, but if $\mathcal{C}$ does not contain a smallest element (ordered by subconjugation), then f is not a $\mathcal{C}$–free weak equivalence by Remark 9.9.

Let X be a cofibrant object and let Y be a fibrant object in pro–$\mathcal{M}_R$ with the Postnikov $\mathcal{C}$–model structure. Then, by Lemma 10.3, $X \wedge \{\Sigma^\infty_R EG/N_+\}$ is a cofibrant object in the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$, and Y is a fibrant object in the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$. The cofibrations in the $\mathcal{C}$–cofree model structure and the Postnikov $\mathcal{C}$–model structure on pro–$\mathcal{M}_R$ are the same. For the purpose of calculating mapping spaces in the $\mathcal{C}$–cofree model structure on pro–$\mathcal{M}_R$ it suffices to replace Y by a levelwise fibrant replacement in the $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$. This follows from [21, 5.3,7.3] since Y is essentially levelwise $\mathcal{W}$–bounded above. We choose a natural fibrant replacement for bounded above objects in the $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$ and denote it by adding a subscript f. We describe the fibrant replacement functor in the $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$ after the next Lemma.

Lemma 10.5 *Let G be a compact Lie group and let $\mathcal{C} = \overline{\mathrm{Lie}(G)}$. Let $L < K$ be two normal subgroups of G. Let $\tilde{M}$ be a $\mathcal{C}$–coefficient system. The group of equivariant (weak) homotopy classes of maps*

$$[\Sigma^\infty_R EG/K_+, H\tilde{M}]_G$$

is isomorphic to the group cohomology of G/K with coefficients in the G/K–module $M(G/K)$.

Proof This follows from the equivariant chain homotopy description of Bredon cohomology. See for example [17, 8.1]. □

Recall that F denotes the internal hom functor in $\mathcal{M}_R$.

Proposition 10.6 *Assume that G is a compact Hausdorff group, $\mathcal{C} = \overline{\mathrm{Lie}(G)}$, and the universe $\mathcal{U}$ is trivial. Let R be a $\mathcal{C}$–connective ring, and let Y be a $\mathcal{C}$–bounded above fibrant object in the $\mathcal{C}$–model structure on $\mathcal{M}_R$. Then the two maps*

$$\mathrm{hocolim}_N\ F(\Sigma^\infty_R EG/N_+, Y) \longrightarrow \mathrm{hocolim}_N\ F(\Sigma^\infty_R EG/N_+, Y_f) \longleftarrow Y_f$$

are weak equivalences in $\mathcal{C}\mathcal{M}_R$.

Proof We need to show that both maps induce isomorphisms on $\pi^{\mathcal{C}}_*$. For $K \in \mathcal{C}$ we get that π^K_n applied to the sequence above is isomorphic to

$$\mathrm{colim}_N\,[\Sigma^\infty_R EG/N_+ \wedge \Sigma^\infty_R G/K_+, Y] \longrightarrow \tag{10–7}$$

$$\mathrm{colim}_N\,[\Sigma^\infty_R EG/N_+ \wedge \Sigma^\infty_R G/K_+, Y_f] \longleftarrow [S^0 \wedge \Sigma^\infty_R G/K_+, Y_f]$$

where the square brackets denote the homomorphism groups in the homotopy category of $\mathcal{C}G\mathcal{M}_R$. The second map is an equivalence since

$$\Sigma^\infty_R G/K_+ \wedge \Sigma^\infty_R EG/N_+ \to \Sigma^\infty_R G/K_+$$

is an underlying equivalence of cofibrant objects in the $\mathcal{C}$–cofree model structure on $G\mathcal{M}_R$, for any normal subgroup N of G in $\mathcal{C}$.

We now prove that the first map is an isomorphism. We first consider the case when $K = G$. There is a map between conditionally convergent spectral sequences converging to the first map in (10–7). The spectral sequences are the Atiyah–Hirzebruch spectral sequences in the Postnikov $\mathcal{C}$–model structure on pro–$G\mathcal{M}_R$ [21, 10.3]; see also Section 8.6 and Section 9.5. The map between the E_2–terms is

$$\mathrm{colim}_N\ H^p_{\mathrm{cont}}(G/N, \Pi^N_{-q}(Y)) \to \mathrm{colim}_N\ H^p_{\mathrm{cont}}(G/N, \Pi^N_{-q}(Y_f))$$

induced by $Y \to Y_f$. This map is isomorphic to

$$\operatorname{colim}_N H^p_{\mathrm{cont}}(G, \Pi^N_{-q}(Y)) \to \operatorname{colim}_N H^p_{\mathrm{cont}}(G, \Pi^N_{-q}(Y_f)).$$

This is an isomorphism since group cohomology commutes with directed colimits of continuous G–modules and $\operatorname{colim}_N \Pi^N_{-q}(Y) \to \operatorname{colim}_N \Pi^N_{-q}(Y_f)$ is an isomorphism of continuous G–modules. The spectral sequences converges conditionally since $\{\Sigma^\infty_R EG/N_+\}$ is bounded below.

For general $K \in \mathcal{C}$, we use Lemma 2.6 and Lemma 4.15 together with the adjunction between $G \wedge_K -$ and the restriction functor $G\mathcal{M}_R \to K\mathcal{M}_R$ to reduce the problem to the case $G = K$. □

Corollary 10.8 *Let Y be in $\mathcal{M}_R$. Then the fibrant replacement, Y_f, of Y in the $\overline{Lie(G)}$–cofree model structure on $\mathcal{M}_R$ is equivalent to*

$$\operatorname{holim}_m \operatorname{hocolim}_N F(\Sigma^\infty_R EG/N_+, P_m Y)$$

in the $\overline{Lie(G)}$–model structure on $\mathcal{M}_R$, where the homotopy colimit is over $N \in Lie(G)$, and the homotopy limit and colimit are formed in the $\overline{Lie(G)}$–model structure.

Remark 10.9 Note that the proof of Proposition 10.6 only uses the Postnikov $\mathcal{W}$–$\mathcal{C}$–model structures on pro–$\mathcal{M}_R$. It does not depend on the existence of the $\mathcal{C}$–free model structure, which is technically more sophisticated.

The next result clarifies the relationship between the $\mathcal{C}$–free and the $\mathcal{C}$–cofree model structures on pro–$\mathcal{M}_R$. Let Map denote the simplicial mapping space in the $\mathcal{C}$–model structure on $\mathcal{M}_R$. Recall that $\{P_n\}$ denotes a natural Postnikov tower.

Theorem 10.10 *We assume that G is a compact Hausdorff group, $\mathcal{C} = \overline{Lie(G)}$, and the universe is trivial. Let R be a $\mathcal{C}$–connective ring. Let $\{X_s\}$ and $\{Y_t\}$ be objects in pro–$\mathcal{M}_R$ such that each X_s is cofibrant, and each $P_n Y_t$ is fibrant in $\mathcal{C}\mathcal{M}_R$. Then the homset in the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$ is isomorphic to*

$$\pi_0 \operatorname{holim}_{t,n} \operatorname{hocolim}_{s,N} \operatorname{Map}(X_s \wedge EG/N_+, P_n Y_t)),$$

and the homset in the $\mathcal{C}$–cofree model structure on $\mathcal{M}_R$ is isomorphic to

$$\pi_0 \operatorname{holim}_{t,n} \operatorname{hocolim}_s \operatorname{Map}(X_s, \operatorname{hocolim}_N F(EG/N_+, P_n Y_t)).$$

Proof This follows from the description of mapping spaces in [21, 5.3,7.3] and from Lemma 10.3 and Proposition 10.6. □

Remark 10.11 One can also let pro–$\mathcal{M}_S$ inherit a model structure from the $\mathcal{W}$–$\mathcal{C}$–model structure on $G\mathcal{M}_S$ along the (right adjoint) inverse limit functor [24, 12.3.2]. The weak equivalences are pro–maps $f\colon X \to Y$ such that $f\colon \lim_s X_s \to \lim_t Y_t$ are weak equivalences in $\mathcal{M}_R$. The fibrations are the pro–maps such that the inverse limits are fibrations in $\mathcal{M}_R$. (This follows from the right lifting property.) We have that c and lim are a Quillen adjoint pair between $\mathcal{M}_R$ and pro–$\mathcal{M}_R$. This model structure does not play any role in this paper.

11 Homotopy fixed points

We define homotopy fixed points of pro–G–spectra for closed subgroups of G, and show that they behave well with respect to iteration.

11.1 The homotopy fixed points of a pro–spectrum

Let G be a compact Hausdorff group, $\mathcal{U}$ a trivial G–universe, and $\mathcal{C}$ the cofamily closure, $\overline{\mathrm{Lie}(G)}$, of $\mathrm{Lie}(G)$. Let R be a $\mathcal{C}$–connective $\mathcal{S}$–cell complex ring with a trivial G–action. The last assumption guarantees that we can apply Lemma 4.16.

Definition 11.1 Let Y be a pro–G–R–module. The homotopy fixed point pro–spectrum Y^{hG} is defined to be the G–fixed points of a fibrant replacement of Y in the Postnikov $\overline{\mathrm{Lie}(G)}$–cofree model structure on pro–$\mathcal{M}_R$.

By choosing a fibrant replacement functor, $Y \mapsto Y_f$, we get a homotopy fixed point functor.

Lemma 11.2 *Let $Y = \{Y_b\}$ be a pro–G–R–module which is levelwise fibrant and bounded above in the $\overline{\mathrm{Lie}(G)}$–model structure on $\mathcal{M}_R$. Then the homotopy fixed point pro–spectrum Y^{hG} is weakly equivalent to*

$$\{(\mathrm{hocolim}_{N\in \mathrm{Lie}(G)}\ F(\Sigma^\infty_R EG/N_+, Y_b))^G\}$$

in the (non-equivariant) Postnikov model structure on pro–$\mathcal{M}_R$.

Proof This follows from Proposition 10.6. □

In particular, if G is a compact Lie group and Y is a fibrant G–spectrum, then the G–homotopy fixed point pro–spectrum Y^{hG} is equivalent to $F(\Sigma^\infty_R EG_+, \{P_n Y\})^G$ in the Postnikov model structure on pro–$\mathcal{M}_R$. The associated spectrum is equivalent to $F(\Sigma^\infty_R EG_+, Y)^G$ in $\mathcal{M}_R$. See Lemma 11.14.

If Y is a pro–G–spectrum and K is a subgroup of G, then we expect the K–homotopy fixed point pro–spectrum to be equipped with an action by $N_G K/K$. We make the following definition.

Definition 11.3 Let Y be a pro–G–R–module, and let K be a closed subgroup of G. The K–G–homotopy fixed point pro–spectrum $Y^{h_G K}$ of Y is defined to be

$$\operatorname{hocolim}_N ((Y_f)^{KN})$$

where the colimit is over $N \in \operatorname{Lie}(G)$, and Y_f is a fibrant replacement of Y in the Postnikov $\overline{\operatorname{Lie}(G)}$–cofree model structure on pro–$\mathcal{M}_R$.

The pro-spectrum $Y^{h_G K}$ is an $N_G K/K$–pro–spectrum. If $K \in \overline{\operatorname{Lie}(G)}$, then the canonical map $Y^{h_G K} \to (Y_f)^K$ is an equivalence in the Postnikov $\overline{\operatorname{Lie}(N_G K/K)}$–model structure on pro–$\mathcal{M}_R$. The restriction of Y_f to a subgroup $K \in \overline{\operatorname{Lie}(G)}$ is a fibrant object in the $\overline{\operatorname{Lie}(K)}$–cofree model structure on pro–$K\mathcal{M}_R$ by Lemma 4.15. Hence $Y^{h_G K}$ is equivalent to $(Y|K)^{hK}$ in the Postnikov model structure on pro–R–modules, for all subgroups $K \in \overline{\operatorname{Lie}(G)}$. This need not be true when $K \notin \overline{\operatorname{Lie}(G)}$. For example, consider the suspension G–spectrum $\Sigma^\infty_R G_+$ and $K = \{1\} \notin \overline{\operatorname{Lie}(G)}$. The next lemma shows that for certain pro–spectra Y, we still have that $Y^{h_G K}$ is equivalent to $(Y|K)^{hK}$ even when $K \notin \overline{\operatorname{Lie}(G)}$.

Lemma 11.4 *Let $K \lhd L$ be two closed subgroups of G. Let Y be a pro–G–spectrum that is both fibrant and cofibrant in the Postnikov $\overline{\operatorname{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$. Let Y' be $\operatorname{hocolim}_{N \in Lie(G)} F(\Sigma^\infty_R EG/N_+, Y)$ (levelwise hocolimit). Then $Y^{h_G K}$ is equivalent to $(Y'|L)^K$ in the Postnikov $\overline{\operatorname{Lie}(L/K)}$–model structure on pro–$L/K\mathcal{M}_R$.*

Proposition 10.6 says that Y' is equivalent to Y_f in the Postnikov $\overline{\operatorname{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$. Note that we don't claim that Y' is fibrant in the Postnikov $\overline{\operatorname{Lie}(G)}$–cofree model structure on pro–$\mathcal{M}_R$.

Proof In the following we use that homotopy colimits commute with fixed points. The pro–spectrum $\operatorname{hocolim}_{N'}(Y')^{JN'}$ is equivalent to $Y^{h_G J}$ in the Postnikov model structure, for every subgroup J of G containing K, where the colim is over $N' \in \overline{\operatorname{Lie}(G)}$. So it suffices to show that $(Y')^{h_G J}$ is equivalent to $(Y'|L)^J$ in the Postnikov model structure for $K \le J \le L$. Consider the map

$$\operatorname{hocolim}_{N' \in \operatorname{Lie}(G)} (\operatorname{hocolim}_{N \in \operatorname{Lie}(G)} F(\Sigma^\infty_R EG/N_+, Y))^{JN'} \to$$

$$(\operatorname{hocolim}_{N \in \operatorname{Lie}(G)} F(\Sigma^\infty_R EG/N_+,\ Y))^J.$$

Levelwise, Y is a bounded above fibrant object in the $\overline{\mathrm{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$. The map above is an equivalence by cofinality (take the homotopy colimit over $N = N'$), using that the nth skeleta of EG/N_+ are all finite cell complexes (hence small objects), and finally, by Lemma 4.19 which says that $\operatorname{hocolim}_{N\in\mathrm{Lie}(G)} Y^{NH} \to Y^H$ is an equivalence of pro–spectra in the Postnikov model structure, for any closed subgroup H of G. Note that the internal hom functor from a small object respects the functor that sends a spectrum to an Ω–spectrum (see (4–5)). □

Proposition 11.5 *Let G be a compact Hausdorff group and let K be a closed normal subgroup of G. Then there is an equivalence of pro–spectra*

$$(Y^{h_G K})^{hG/K} \simeq Y^{hG}$$

in the (non-equivariant) Postnikov model structure on pro–$\mathcal{M}_R$.

Proof We assume that Y is fibrant in the Postnikov–$\overline{\mathrm{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$. The pro–spectrum $(Y^{h_G K})^{hG/K}$ is equivalent to

$$\operatorname{hocolim}_W F(\Sigma^\infty_R EG/WK_+, \operatorname{hocolim}_{N,L} F(\Sigma^\infty_R EG/N_+, Y)^{LK})^G. \tag{11–6}$$

We use that the internal hom functor (from a small object) respects the functor that sends a spectrum to an Ω–spectrum (4–5). Since

$$\operatorname{hocolim}_{N,L} F(\Sigma^\infty_R EG/N_+, Y)^{LK}$$

is levelwise bounded above (since Y is) in the $\overline{\mathrm{Lie}(G/K)}$–model structure on $\mathcal{M}_R$, and $\Sigma^\infty_R EG/WK_+$ has a dualizable nth skeleton for all n, (11–6) is equivalent to

$$\operatorname{hocolim}_{N,L,W} F(\Sigma^\infty_R EG/WK_+, F(\Sigma^\infty_R EG/N_+, Y)^{LK})^G.$$

(See Proposition 11.17.) By cofinality, (11–6) is equivalent to the colimit over $N = L = W$. The fixed point adjunction and internal hom adjunction now give

$$(\operatorname{hocolim}_N F(\Sigma^\infty_R(EG/KN_+ \wedge EG/N_+), Y))^G.$$

This is equivalent to Y^{hG} by Lemma 11.2. □

11.2 Homotopy orbit and homotopy fixed point spectral sequences

In this subsection we work in the Postnikov $\overline{\mathrm{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$ for a compact Hausdorff group G and a trivial universe. We denote the homsets in the associated homotopy category by $[X, Y]_G$.

Definition 11.7 Let $X = \{X_a\}$ be a G–pro–spectrum. Then the G–homotopy orbit pro–spectrum X_{hG} of X is

$$X_{\mathrm{cof}} \wedge_G \{\Sigma^\infty_R EG/N_+\} = \{((X_a)_{\mathrm{cof}} \wedge \Sigma^\infty_R EG/N_+)/G\}_{N,a},$$

where $\{(X_a)_{\mathrm{cof}}\}_a$ is a cofibrant replacement of X. The Borel homology of X is $\pi_*(X_{hG})$.

Lemma 11.8 *Let G be a profinite group and set $\mathcal{C} = \overline{\mathrm{Lie}(G)}$. Then the G–homotopy orbits of a G–pro–spectrum X are the G–orbits of a cofibrant replacement of X in the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$. Let $f\colon X \to Y$ be a weak equivalence in the $\mathcal{C}$–free model structure on pro–$\mathcal{M}_R$. Then $f_{hG}\colon X_{hG} \to Y_{hG}$ is a weak equivalence.*

Proof The first claim follows from Lemma 10.3 since $\overline{\mathrm{Lie}(G)}$ is a normal Illman collection by Example 2.5. The acyclic cofibrations are maps that are essentially levelwise $C_U \cap W_n$–maps, for all $U \in \mathcal{C}$ and $n \in \mathbb{Z}$ [20, 5.11]. The G–orbit functor is a Quillen left adjoint functor from pro–G–$\mathcal{M}_R$ with the $\mathcal{C}$–free model structure to pro–$\mathcal{M}_R$ with the Postnikov model structure. (This follows since $(G/H)/G$ is a point for all H.) Hence the G–orbit functor respects weak equivalences between cofibrant objects [24, 8.5.7]. □

The next result is an instance of Theorem 9.23.

Proposition 11.9 *Let X and Y be objects in pro–$\mathcal{M}_R$ with the Postnikov $\overline{\mathrm{Lie}(G)}$–model structure. Assume that X is cofibrant and essentially bounded below. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p(X \wedge \{\Sigma^\infty_R EG/N_+\}; \Pi_{-q}^{\overline{\mathrm{Lie}(G)}}(Y))$$

converging conditionally to $[X \wedge \{\Sigma^\infty_R EG/N_+\}, Y]_G^{p+q}$. If Y is a monoid in the homotopy category of pro–$\mathcal{M}_R$ with the strict $\overline{\mathrm{Lie}(G)}$–model structure, then the spectral sequence is multiplicative.

The homotopy orbit and fixed point spectral sequences are special cases of the spectral sequence in Proposition 11.9. We first consider the homotopy orbit spectral sequence.

Corollary 11.10 *Let X and Y be two objects in pro–$\mathcal{M}_R$, and assume that X is cofibrant and bounded below, and that Y is a non-equivariant pro–spectrum. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p((X \wedge \{\Sigma^\infty_R EG/N_+\})/G; \Pi^1_{-q}(Y))$$

converging conditionally to $Y^{p+q}(X_{hG})$. If, in addition, Y is a monoid in the homotopy category of pro–$\mathcal{M}_R$ with the strict $\overline{\mathrm{Lie}(G)}$–model structure, then the spectral sequence is multiplicative.

We now consider the homotopy fixed point spectral sequence. If each X_s in $X=\{X_s\}$ is a retract of a finite $\overline{\mathrm{Lie}(G)}$–cell complex, then the abutment of the spectral sequence in Proposition 11.9 is naturally isomorphic to

$$[X, \mathrm{hocolim}_N\ F(\Sigma^\infty_R EG/N_+, Y)]^{p+q}_G$$

by Lemma 9.15. If, in addition, X is a pro–G/K–spectrum for some $K\in\mathcal{C}$ (made into a pro–G–spectrum), then the abutment is isomorphic to

$$[X, \mathrm{hocolim}_N\ F(\Sigma^\infty_R EG/N_+, Y)^K]^{p+q}_{G/K}.$$

Proposition 11.11 *Let Y be in pro–$\mathcal{M}_R$. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p_{cont}(G; \Pi^1_{-q}(Y))$$

converging conditionally to $\pi_{-p-q}(Y^{hG})$. If Y is a monoid in the homotopy category of pro–$\mathcal{M}_R$ with the strict $\overline{\mathrm{Lie}(G)}$–model structure, then the spectral sequence is multiplicative.

Proof This follows from Lemma 9.15 and Proposition 11.9. □

A spectral sequence of this type was first studied by Devinatz and Hopkins for the spectrum E_n, with an action by the extended Morava stabilizer group [14]. It has also been studied by Daniel Davis [9; 11; 10; 8].

We combine the homotopy fixed point spectral sequence and Proposition 11.5 to obtain a generalization of the Lyndon–Hochschild–Serre spectral sequence. A spectral sequence like this was obtained by Ethan Devinatz for E_n [13]. See also [8, 7.4].

Proposition 11.12 *Let $K\trianglelefteq L$ be closed subgroups of G. Let Y be any pro–G–R–module, where R is a $\overline{\mathrm{Lie}(G)}$–connective $\mathcal{S}$–cell complex ring with trivial G–action. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p_{cont}(L/K; \Pi^1_{-q}(Y^{h_G K}))$$

converging conditionally to $\pi_{-p-q}(Y^{h_G L})$. If Y is a monoid in the homotopy category of pro–$\mathcal{M}_R$ with the strict $\overline{\mathrm{Lie}(G)}$–model structure, then the spectral sequence is multiplicative.

Proof Without loss of generality assume that Y is both fibrant and cofibrant in the Postnikov $\overline{\mathrm{Lie}(G)}$–model structure on pro–$\mathcal{M}_R$. Apply Proposition 11.5 and Proposition 11.11 to the pro–spectrum $(Y')^{h_G K}$, where $Y' = \mathrm{hocolim}_{N\in \mathrm{Lie}(G)} F(\Sigma^\infty_R EG/N_+, Y)$ (levelwise hocolimit). The Lth homotopy fixed points of $Y'|L$ is equivalent to $Y^{h_G L}$ by Lemma 11.4. The replacement Y' respects monoids as in the proof of Corollary 11.19. □

We now give a more concrete description of the E_2–term of the homotopy fixed point spectral sequence for certain pro–spectra.

Proposition 11.13 *Let K be a closed subgroup of G. Let $\{Y_m\}$ be a countable tower in pro–$\mathcal{M}_R$. Then there is a short exact sequence*

$$0 \to lim^1_{m,n}\ H^{p-1}_{cont}(G/K; \Pi^1_{-q}((P_nY_m)^{h_G K})) \to E_2^{p,q}$$
$$\to lim_{m,n}\ H^p_{cont}(G/K; \Pi^1_{-q}((P_nY_m)^{h_G K})) \to 0$$

where $E_2^{p,q}$ is the E_2–term of the spectral sequence in Proposition 11.12.

11.3 Comparison to Davis' homotopy fixed points

In this subsection we show that the homotopy fixed points defined in Definition 11.1 agree with the classical homotopy fixed points when G is a compact Lie group. We also compare our definition of homotopy fixed points to a construction by Daniel Davis [11]. We work in homotopy categories. The next lemma says that if G is a compact Lie group, then the orthogonal G–spectrum associated to the homotopy fixed points in pro–$\mathcal{M}_S$, with the strict or with the Postnikov cofree model structures are equivalent.

Lemma 11.14 *Let G be a compact Lie group (or a discrete group), and let $\mathcal{C}$ be the collection of all (closed) subgroups of G. Let X be any orthogonal G–spectrum. Then the map*

$$F(\Sigma^\infty EG_+, X)^G \to holim_n\ F(\Sigma^\infty EG_+, P_nX)^G$$

is an equivalence.

Proof Since $\{1\} \in \mathcal{C}$, the pro–spectrum $\{\Sigma^\infty EG/N_+\}$, indexed on normal subgroups N in G, is equivalent to $\Sigma^\infty EG_+$. The spectrum $\Sigma^\infty EG_+$ is the homotopy colimit of the (G–cell complex) skeleta $\Sigma^\infty EG^{(m)}_+$, for $m \geq 0$. Hence, $F(\Sigma^\infty EG_+, Z)$ is equivalent to

$$F(\mathrm{hocolim}_m\ \Sigma^\infty EG^{(m)}_+, Z) \simeq \mathrm{holim}_m\ F(\Sigma^\infty EG^{(m)}_+, Z)$$

for any G–spectrum Z. The canonical map

$$F(\Sigma^\infty EG_+^{(m)}, X)^G \to F(\Sigma^\infty EG_+^{(m)}, P_n X)^G$$

is $(n-m-\dim G)$–connected. So

$$\operatorname{holim}_m F(\Sigma^\infty EG_+^{(m)}, X)^G \to \operatorname{holim}_{m,n} F(\Sigma^\infty EG_+^{(m)}, P_n X)^G$$

is a weak equivalence. This proves the claim. □

Daniel Davis defines homotopy fixed point spectra for towers of discrete simplicial Bousfield–Friedlander G–spectra for any profinite group G [11]. The main difference from our construction, translated into our terminology, is that he uses the strict model structure on pro–$\mathcal{M}_S$ obtained from the $\overline{\operatorname{fnt}(G)}$–cofree model structure on $G\mathcal{M}_S$, rather than the Postnikov $\overline{\operatorname{fnt}(G)}$–cofree model structure on $G\mathcal{M}_S$. He shows that if G has finite virtual cohomological dimension, then his definition of the homotopy fixed points of a (discrete) pro–G–spectrum $\{Y_b\}$ is equivalent to

$$(\operatorname{holim}_b \operatorname{Tot} \operatorname{hocolim}_N \Gamma^\bullet_{G/N}(Y_b))^G, \tag{11–15}$$

where N runs over all open normal subgroups of G. Here $\Gamma^\bullet_{G/N}(Y_b)$ is defined to be the cosimplicial object given by $F_\square(G/N^{\bullet+1}, Y_b)$, where $F_\square$ is the cotensor functor, and $G/N^{\bullet+1}$ is a simplicial object obtained from a group [9; 10]. When G has finite virtual cohomological dimension, one can use (11–15) as the definition of homotopy fixed points for categories of G–spectra other than the category of discrete simplicial G–spectra.

Since Tot is the homotopy inverse limit of the Tot_n, and Tot_n is a finite (homotopy) limit, we get that (11–15) is equivalent to

$$(\operatorname{holim}_{b,n} \operatorname{hocolim}_N \operatorname{Tot}_n \Gamma^\bullet_{G/N}(Y_b))^G.$$

By definition, $\operatorname{Tot}_n \Gamma^\bullet_{G/N}(Y_b)$ is equivalent to the internal hom $F(\Sigma^\infty (EG/N_+)^{(n)}, Y_b)$ in $\mathcal{M}_S$, where $(EG/N_+)^{(n)}$ denotes the nth skeleton of EG/N_+. Hence Davis' homotopy fixed points are equivalent to

$$\operatorname{holim}_{b,n} (\operatorname{hocolim}_N F(\Sigma^\infty (EG/N_+)^{(n)}, Y_b))^G. \tag{11–16}$$

We compare (11–16) to the spectrum associated to our definition of homotopy fixed points.

Proposition 11.17 *Let G be a profinite group. The canonical maps from*

$$\{\operatorname{hocolim}_N F(\Sigma^\infty (EG/N_+)^{(n)}, Y_b)^G\}_{b,n}$$

and

$$\{\mathrm{hocolim}_N\ F(\Sigma^\infty EG/N_+, P_mY_b)^G\}_{b,m}$$

to

$$\{\mathrm{hocolim}_N\ F(\Sigma^\infty (EG/N_+)^{(n)}, P_mY_b)^G\}_{b,m,n} \tag{11–18}$$

are both equivalences in pro–$\mathcal{M}_\mathcal{S}$ with the strict model structure.

Proof It suffices to prove the result when Y is a constant pro–spectrum. Since P_mY is co–m–connected, the skeletal inclusion gives an equivalence

$$F(\Sigma^\infty(EG/N_+), P_mY) \to F(\Sigma^\infty(EG/N_+)^{(n)}, P_mY)$$

when $n > m$. Hence the map from the second expression to (11–18) is an equivalence.

Since $(EG/N_+)^{(n)}$ only has cells in dimension less than or equal to n we get that

$$F(\Sigma^\infty(EG/N_+)^{(n)}, Y) \to F(\Sigma^\infty(EG/N_+)^{(n)}, P_mY)$$

is an equivalence when $n < m$. Hence the map from the first expression to (11–18) is an equivalence. □

Hence, the spectrum associated to our definition of the homotopy fixed point pro-spectrum agrees with Davis' definition when G is a profinite group with finite virtual cohomological dimension.

Corollary 11.19 *If Y is a (commutative) algebra in pro–$\mathcal{M}_\mathcal{S}$, then Y^{h_GK} is a (commutative) algebra in pro–$\mathcal{M}_\mathcal{S}$, for all K.*

Proof By Proposition 11.17 we get that Y^{h_GK} is equivalent to

$$\{\mathrm{hocolim}_U\ (\mathrm{hocolim}_N\ F(\Sigma^\infty(EG/N_+)^{(n)}, Y_b))^{UK}\}_{b,n}.$$

The result follows since the pro–category is cocomplete by [21, 11.4], directed colimits of algebras are created in the underlying category of modules, and fixed points preserves algebras. □

Appendix A Compact Hausdorff Groups

In this appendix we recall some well known properties of compact Lie groups. The relationship between compact Lie groups and compact Hausdorff groups is analogues to the relationship between finite groups and profinite groups.

We first note that if G is a compact Hausdorff group, then the finite dimensional G–representations are all obtained from suitable G/N–representations via the quotient map $G \to G/N$ where G/N is a compact Lie group quotient of G.

Lemma A.1 *Let V be a finite dimensional G–representation. Then the G–action on V factors through some compact Lie group quotient G/N of G.*

Proof A G–representation V is a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V).$$

The action factor through the image $\rho(G)$. Since G is a compact group $\rho(G)$, with the subspace topology from $\mathrm{GL}(V)$, is a closed subgroup of the Lie group $\mathrm{GL}(V)$. Hence $\rho(G)$ is itself a Lie group. Again, since G is compact Hausdorff, the subspace topology on $\rho(G)$ agrees with the quotient topology from ρ. Hence ρ gives a homeomorphism $G/\ker\rho \cong \rho(G)$, and $G/\ker\rho$ is a compact Lie group. □

Recall from Example 2.5 that $\mathrm{Lie}(G)$ denotes the collection of closed normal subgroups N of G such that G/N is a compact Lie group. We consider the inverse system of quotients G/N such that G/N is a compact Lie group. If G/N and G/K are compact Lie groups, then $G/N \cap K$ is again a compact Lie group, since it is a closed subgroup of $G/N \times G/K$. Hence the inverse system is a filtered inverse system.

In the next theorem it is essential that we work in the category of compactly generated weak Hausdorff topological spaces.

Proposition A.2 *Let X be a topological space with a (not necessarily continuous) G–action. Then the G–action on X is continuous if and only if the G–action on X/N is continuous for all subgroups $N \in \overline{\mathrm{Lie}(G)}$ and the canonical map*

$$\rho\colon X \to \lim_N X/N,$$

where the limit is over all $N \in \mathrm{Lie}(G)$, is a homeomorphism.

Proof Assume that ρ is a homeomorphism. Then the G–action on X is continuous since the G–action on $\lim_N X/N$ is continuous.

We now assume that the G–action on X is continuous. We first show that

$$\rho\colon X \to \lim_N X/N$$

is a bijection. The Peter–Weyl theorem for compact Hausdorff groups implies that there are enough finite dimensional real G–representations to distinguish any two

given elements in G (Adams [1, 3.39]). Hence $\cap_N N$ is 1, and ρ is injective. Now let $\{Nx_N\}$ be an element in $\lim_N X/N$. Since G is a compact group and since the G–action on X is continuous we get that Nx_N is a compact subset of X for every $N \in \mathrm{Lie}(G)$. In particular, Nx_N is a closed subset of the compact space Gx_G for every N in $\mathrm{Lie}(G)$. Since $\cap_N N = 1$ and $Nx_N \cap Lx_L \supset N \cap Lx_{N\cap L}$, we conclude that the intersection of the closed sets Nx_N, for $N \in \mathrm{Lie}(G)$, is a point. Call this point x. We then have that $\rho(x) = \{Nx_N\}$. So ρ is surjective.

We need to show that ρ is a closed map. This amounts to showing that for any closed set A of X, and for any $N \in \mathrm{Lie}(G)$ we have that $N \cdot A$ is a closed subset of X. When A is a compact (hence closed) subset of X this follows since $N \cdot A$ is the image of $N \times A$ under the continuous group action on X. Since we use the compactly generated topology the subset $N \cdot A$ of X is closed if for all compact subsets K of X the subset $(N \cdot A) \cap K$ is closed in X. This is true since

$$(N \cdot A) \cap K = (N \cdot (A \cap (N \cdot K))) \cap K$$

and $N \cdot K$ is a compact subset of X. Hence ρ is a homeomorphism. □

Corollary A.3 *Any compact Hausdorff group G is an inverse limit of compact Lie groups.*

Proof This follows from Proposition A.2 by letting X be G. □

The pro–category of compact Lie groups is equivalent to the category of compact Hausdorff groups. This follows since a closed subgroup of a compact Lie group is again a compact Lie group. The categories are equivalent as topological categories since both homspaces are compact Hausdorff spaces.

Groups which are inverse limits of Lie groups have been studied recently. See for example Hofmann–Morris [25].

Corollary A.4 *The category $G\mathcal{T}$ is a full subcategory and a retract of the pro–category of the full subcategory of $G\mathcal{T}$ consisting of G–spaces with a G–action factoring through G/N for some $N \in$ Lie(G).*

Proof A G–space X is sent to the pro–G–space $\{X/N\}$. The retract map is given by taking the inverse limit. By Proposition A.2 the composite is isomorphic to the identity map on $G\mathcal{T}$. Let X and Y be two G–spaces. Then the canonical map

$$G\mathcal{T}(X, Y) \to \lim_L \mathrm{colim}_N\, G\mathcal{T}(X/N, Y/L)$$

is a bijection. □

Remark A.5 Let G be a profinite group. We observe that in the category of sets, X is a continuous G–set if and only if $\operatorname{colim}_N X^N \to X$ is a bijection. On the other hand, in the category of compactly generated weak Hausdorff spaces, X is a continuous G–space if and only if $X \to \lim_N X/N$ is a continuous G–space.

References

[1] **J F Adams**, *Lectures on Lie groups*, W. A. Benjamin, New York-Amsterdam (1969) MR0252560

[2] **M Artin**, **B Mazur**, *Etale homotopy*, Lecture Notes in Mathematics 100, Springer, Berlin (1986) MR883959 Reprint of the 1969 original

[3] **A A Beĭlinson**, **J Bernstein**, **P Deligne**, *Faisceaux pervers*, from: "Analysis and topology on singular spaces, I (Luminy, 1981)", Astérisque 100, Soc. Math. France, Paris (1982) 5–171 MR751966

[4] **A K Bousfield**, *The localization of spaces with respect to homology*, Topology 14 (1975) 133–150 MR0380779

[5] **A K Bousfield**, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. 353 (2001) 2391–2426 MR1814075

[6] **G Carlsson**, *Structured stable homotopy theory and the descent problem for the algebraic K–theory of fields* (2003) http://math.stanford.edu/~gunnar/PDFpage.html

[7] **R L Cohen**, **J D S Jones**, **G B Segal**, *Floer's infinite-dimensional Morse theory and homotopy theory*, from: "The Floer memorial volume", Progr. Math. 133, Birkhäuser, Basel (1995) 297–325 MR1362832

[8] **D G Davis**, *Iterated homotopy fixed points for the Lubin–Tate spectrum* arXiv:math.AT/0610907

[9] **D G Davis**, *The Lubin–Tate spectrum and its homotopy fixed point spectra*, PhD thesis, Northwestern University (2003)

[10] **D G Davis**, *The E_2–term of the descent spectral sequence for continuous G–spectra*, New York J. Math. 12 (2006) 183–191 MR2242532

[11] **D G Davis**, *Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action*, J. Pure Appl. Algebra 206 (2006) 322–354 MR2235364

[12] **E S Devinatz**, *Small ring spectra*, J. Pure Appl. Algebra 81 (1992) 11–16 MR1173820

[13] **E S Devinatz**, *A Lyndon–Hochschild–Serre spectral sequence for certain homotopy fixed point spectra*, Trans. Amer. Math. Soc. 357 (2005) 129–150 MR2098089

[14] **E S Devinatz**, **M J Hopkins**, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology 43 (2004) 1–47 MR2030586

[15] **W G Dwyer**, *Homology decompositions for classifying spaces of finite groups*, Topology 36 (1997) 783–804 MR1432421

[16] **W G Dwyer**, **E M Friedlander**, *Algebraic and etale K–theory*, Trans. Amer. Math. Soc. 292 (1985) 247–280 MR805962

[17] **H Fausk**, *Artin and Brauer induction for compact Lie groups* arXiv:math.AT/0609641

[18] **H Fausk**, *Atiyah–Segal completion for profinite groups* in preparation

[19] **H Fausk**, *T–model structures on chain complexes of presheaves* arXiv:math.AG/0612414

[20] **H Fausk**, **D C Isaksen**, *Model structures on pro-categories*, Homology, Homotopy Appl. 9 (2007) 367–398 MR2299804

[21] **H Fausk**, **D C Isaksen**, *t–model structures*, Homology, Homotopy Appl. 9 (2007) 399–438 MR2299805

[22] **J P C Greenlees**, *Equivariant connective K–theory for compact Lie groups*, J. Pure Appl. Algebra 187 (2004) 129–152 MR2027899

[23] **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 113 (1995) viii+178 MR1230773

[24] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, American Mathematical Society, Providence, RI (2003) MR1944041

[25] **K H Hofmann**, **S A Morris**, *Projective limits of finite-dimensional Lie groups*, Proc. London Math. Soc. (3) 87 (2003) 647–676 MR2005878

[26] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI (1999) MR1650134

[27] **M Hovey**, **J H Palmieri**, **N P Strickland**, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. 128 (1997) x+114 MR1388895

[28] **S Illman**, *The equivariant triangulation theorem for actions of compact Lie groups*, Math. Ann. 262 (1983) 487–501 MR696520

[29] **D C Isaksen**, *A model structure on the category of pro-simplicial sets*, Trans. Amer. Math. Soc. 353 (2001) 2805–2841 MR1828474

[30] **D C Isaksen**, *Strict model structures for pro-categories*, from: "Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)", Progr. Math. 215, Birkhäuser, Basel (2004) 179–198 MR2039766

[31] **U Jannsen**, *Continuous étale cohomology*, Math. Ann. 280 (1988) 207–245 MR929536

[32] **L G Lewis, Jr**, *Change of universe functors in equivariant stable homotopy theory*, Fund. Math. 148 (1995) 117–158 MR1360142

[33] **L G Lewis, Jr**, **J P May**, **M Steinberger**, **J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics 1213, Springer, Berlin (1986) MR866482

[34] **S Mac Lane**, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics 5, Springer, New York (1998) MR1712872

[35] **M A Mandell**, **J P May**, *Equivariant orthogonal spectra and S–modules*, Mem. Amer. Math. Soc. 159 (2002) x+108 MR1922205

[36] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) 82 (2001) 441–512 MR1806878

[37] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC (1996) MR1413302

[38] **O Renaudin**, *Spectres en diagramme dans les catégories modèles*, Bull. Belg. Math. Soc. Simon Stevin 13 (2006) 1–30 MR2245974

[39] **S Schwede**, **B E Shipley**, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) 80 (2000) 491–511 MR1734325

[40] **T tom Dieck**, *Transformation groups*, de Gruyter Studies in Mathematics 8, Walter de Gruyter & Co., Berlin (1987) MR889050

Department of Mathematics, University of Oslo
1053 Blindern, 0316 Oslo, Norway

fausk@math.uio.no

Proposed: Haynes Miller
Seconded: Tom Goodwillie, Paul Goerss

Received: 20 December 2006
Revised: 16 April 2007