PERGAMON

Topology 40 (2001) 667–680

TOPOLOGY

www.elsevier.com/locate/top

On the Brown representability theorem for triangulated categories

Jens Franke

Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany

Received 15 October 1997; accepted 23 February 1998

1. Introduction

In this paper, we consider the Brown representability theorem for triangulated categories which are closed under coproducts of arbitrary size. We prove that every triangulated category which is, in a certain strong sense, generated by one of its objects and which is exhausted by an ascending sequence, indexed by all ordinal numbers, of small subcategories satisfying reasonable conditions, satisfies the Brown representability theorem. In particular, the unbounded derived category of a Grothendieck category and some of its full subcategories satisfy the Brown representability theorem.

There appear to be applications of these results to the truncation functors used in the construction of the Harder–Goresky–MacPherson-weighted cohomology [7] and also to the construction of $f^!$ for various Grothendieck–Verdier-style duality results.

The most natural way to prove the Brown representability for these triangulated categories would seem to be to extend the iterated attaching of cells used in the topological case by a transfinite induction procedure similar to the one used in some proofs of the existence of sufficiently many injective objects of Grothendieck categories. A general representability theorem based on this procedure is [8, Theorem 1.3]. An application of this result would require the existence of G-privileged (in the sense of [8]) weak colimits over sufficiently large regular cardinal numbers. It seems that this assumption is rarely satisfied for homotopy categories because of the following obstacles:

- If β is a higher cardinal number, it can often be shown that homotopy colimits over β have the property of being G-privileged. This easily implies that G-privileged weak colimits of diagrams in

E-mail address: franke@rhein.iam.uni-bonn.de (J. Franke).

PII: S0040-9383(99)00034-8

the homotopy category which lift to the category of models must be isomorphic to the corresponding homotopy colimits. However, these homotopy colimits may fail to be weak colimits for the following reason. In general in the case of triangulated categories one expects a spectral sequence

$$E_2^{p,q} = \lim_{\xi\in\beta}{}^p \mathrm{Hom}^q_{\mathscr{D}}(X_\xi, Y) \Rightarrow \mathrm{Hom}^{p+q}_{\mathscr{D}}(\operatorname*{hocolim}_{\xi\in\beta} X_\xi, Y). \tag{1}$$

The homotopy colimit should be expected to be a weak colimit only if the $\lim^p$-terms in its E_2-term vanish if $p > 1$. This, however, fails to be the case unless the cofinality of β is ω (cf. [11, Corollary 36.9]).

- A difficulty in literally satisfying the assumptions of Heller's theorem (but not necessarily in applying the idea of its proof) could also be caused by the possible existence of diagrams in the homotopy category which do not lift to the model categories. This will never happen for ω-diagrams, but it could well be the case for certain ω_1-diagrams. For such diagrams, constructing a G-privileged weak colimit would be even more complicated.

For these reasons, there is no straightforward way of applying Heller's result to triangulated categories like the derived category of a Grothendieck category.

The proof of the Brown representability theorem given in this paper avoids the procedure of attaching cells in favour of a different method which first proves the existence of a solution set for the representation problem. In the case of the unbounded derived category of a Grothendieck category, Keller has pointed out that one can also use the Gabriel–Popescu theorem to reduce the assertion to the case of the unbounded derived category of the category of modules over a ring (cf. Remark 2.3). However, this method does not seem to apply to subcategories of unbounded derived categories (like the subcategory of $\mathscr{D}(\mathscr{O}_X)$ consisting of all complexes of $\mathscr{O}_X$-modules on a prescheme with quasi-coherent cohomology) or to homotopy categories of sheaves of spectra.

Neeman [13] has proved a Brown representability theorem which does apply in such a situation. The relation of his theorem to our theorem is not clear. The main advantage of his theorem is that it is based on a much more careful categorical analysis of the representability problem and is also able to prove a representability theorem for covariant functors in certain cases. Our approach, however, is short and direct.

The author is indebted to B. Keller, A. Neeman and R. Thomason for helpful discussions. In particular, this article was motivated by [12].

2. The main representability theorem

Let $\mathscr{D}$ be a triangulated category which has coproducts of arbitrary size.

Definition 2.1. A subset $G \subset \mathfrak{Ob}(\mathscr{D})$ strongly generates $\mathscr{D}$ if there is a cardinal number κ such that $\mathscr{D}$ is the smallest full triangulated subcategory of $\mathscr{D}$ which is closed under coproducts of size $\leqslant \kappa$ and contains arbitrary coproducts of the form $\coprod_{\xi\in\alpha} B_\xi$, with all B_ξ belonging to G.

Definition 2.2. Let G be a subset of the class of objects of a $\mathscr{D}$. Let $\mathfrak{M}(G)$ be the class of all regular infinite cardinal numbers κ such that there exists a full triangulated subcategory $C(\kappa)$ of $\mathscr{D}$ which is closed under coproducts of size $< \kappa$, contains a small subcategory to which it is equivalent and has the property that for all $A \in G$ and all $X \in C(\kappa)$, we have $\mathrm{card}(\mathrm{Hom}_{\mathscr{D}}(A, X)) < \kappa$.

Remark 2.3. It should be pointed out that this definition has an equivalent reformulation in which $\mathfrak{M}(G)$ is the class of all ordinal numbers κ such that there exists a subset S of $\mathfrak{Ob}(\mathscr{D})$ such that the class $C(\kappa)$ of all objects isomorphic to an element of S has the property mentioned above. Therefore, the von Neumann–Bernays–Gödel Axioms guarantee the existence of the class $\mathfrak{M}(G)$.

Theorem 2.4. *Let $\mathscr{D}$ be a triangulated category which has coproducts of arbitrary size. Assume that $\mathscr{D}$ is strongly generated by a subset G of its class of objects such that $\mathfrak{M}(G)$ is a proper class* (*i.e., unbounded by any cardinal number*). *Then $\mathscr{D}$ satisfies the Brown representability theorem.*

The proof of this theorem depends on the following construction: Let $\mathscr{D} \xrightarrow{F} \mathfrak{Ab}$ be a contravariant cohomological functor which maps coproducts to products. Let $C \subset \mathscr{D}$ be a full triangulated subcategory. We denote by C_F the following category:

- Objects of C_F are pairs (B, f), where $B \in \mathfrak{Ob}(C)$ and $f \in F(B)$.
- Morphisms from (B, f) to $(\tilde{B}, \tilde{f})$ are morphisms $B \xrightarrow{\beta} \tilde{B}$ in $\mathscr{D}$ such that $F(\beta)\tilde{f} = f$.

Let us assume that C possesses a set of representatives for the isomorphism classes of objects. We denote by $\tilde{F}_C$ the functor defined by

$$\tilde{F}_C(X) = \operatorname*{colim}_{(B,f)\in\tilde{C}_F} \mathrm{Hom}_{\mathscr{D}}(X, B), \tag{2}$$

where $\tilde{C}_F$ is a full small equivalent subcategory of C_F. Obviously, (1) depends on the choice of $\tilde{C}_F$ only up to canonical isomorphism, which justifies the notation $\tilde{F}_C$. There is a natural transformation

$$\tilde{F}_C \xrightarrow{\phi_C} F \tag{3}$$

which, for $(B,f) \in \mathfrak{Ob}(C_F)$, maps the image of $\alpha \in \mathrm{Hom}_{\mathscr{D}}(X, B)$ in the colimit (1) to $F(\alpha)f \in F(X)$.

We now outline the proof of Theorem 2.4. For a cardinal $\kappa \in \mathfrak{M}(G)$, let $C(\kappa)$ be a subcategory of $\mathscr{D}$ with the properties described in Definition 2.2. In the first subsection of this section we will prove that $F_{C(\kappa)}$ is a cohomological functor which maps coproducts of size $< \kappa$ to products. Therefore, the full subcategory D_κ of all X for which $\tilde{F}_{C(\kappa)}(X[i]) \xrightarrow{\phi_{C(\kappa)}} F(X[i])$ is an isomorphism for all integers i is a full triangulated subcategory of $\mathscr{D}$, and is stable under coproducts of size $< \kappa$. In the second subsection, we will prove that D_κ contains arbitrary coproducts of objects B_ξ with the property that $\mathrm{card}(F(B_\xi[i])) < \kappa$ and $\mathrm{card}(\mathrm{Hom}_{\mathscr{D}}(B_\xi, X)) < \kappa$ for all ξ, i, and objects X of $C(\kappa)$. From this and from Definition 2.1, it follows that $\phi_{C(\kappa)}$ is a natural isomorphism for sufficiently large $\kappa \in \mathfrak{M}(G)$. But then a set of representatives for the objects of $C(\kappa)$ is a solution set for the problem of representing F, and this will imply that F is representable.

2.1. Cohomological properties of $\tilde{F}_C$

Our proof of the fact that $\tilde{F}_C$ is cohomological and maps certain coproducts to products depends on some general facts about directed categories.

Definition 2.5. Let κ be a regular infinite cardinal number. A category E is κ-directed if it satisfies the following condition:

- For every family of less than κ objects of E, $(X_\xi)_{\xi\in\tilde{\kappa}}$, $\tilde{\kappa} < \kappa$, there exist an object Y of E and morphisms $X_\xi \xrightarrow{\phi_\xi} Y$ for $\xi \in \tilde{\kappa}$.
- For two objects X, Y of E and every family $(\alpha_\xi)_{\xi\in\tilde{\kappa}}$ of less than κ morphisms from X to Y, there exists a morphism $Y \xrightarrow{\beta} Z$ such that $\beta\alpha_\xi = \beta\alpha_\upsilon$ for $\xi,\upsilon\in\tilde{\kappa}$.

Remark 2.6. Obviously, ω-directedness is the same as directedness in the sense of [1, Definition I.(8.1)]. This condition has been called "κ-kofiltrierend" in [5, Section 5].

Proposition 2.7. *Assume that E is a small category which is κ-directed for some infinite regular cardinal κ. Then the functor* $\mathfrak{Ab}^E \xrightarrow{\mathrm{colim}_E} \mathfrak{Ab}$ *is exact and commutes with products of size* $< \kappa$.

The proof is not given because it is straightforward and essentially contained in [5, Satz 5.2] and [1, Proposition I.(8.2)]. Now we return to our consideration of the triangulated category $\mathscr{D}$, the cohomological functor F, and the full triangulated subcategory C. Let κ be infinite.

Lemma 2.8. *Assume that $\mathscr{D}$ has coproducts of size $< \kappa$, that F maps such coproducts to products, and that C is closed under coproducts of size $< \kappa$. Then C_F is κ-directed.*

Proof. Indeed, if $X_\xi = (B_\xi, f_\xi \in F(B_\xi))$, $\xi \in \tilde{\kappa} < \kappa$ are objects of C_F, then all X_ξ have morphisms to $X = (B,f)$, where $B = \coprod_{\xi<\tilde{\kappa}} B_\xi$ and $f = (f_\xi)_{\xi<\tilde{\kappa}} \in \prod_{\xi\in\kappa} F(B_\xi) \cong F(B)$.

If $X = (B,f) \xrightarrow{\alpha_\xi} Y = (\tilde{B},\tilde{f})$ are morphisms in C_F for $\xi \in \tilde{\kappa} < \kappa$, then let

$$\coprod_{\xi<\upsilon<\tilde{\kappa}} B \xrightarrow{\Delta_\alpha = (\alpha_\xi - \alpha_\upsilon)_{\xi<\upsilon<\kappa}} \tilde{B} \xrightarrow{\beta} \bar{B} \coprod_{\xi<\upsilon<\tilde{\kappa}} B[1]$$

be a distinguished triangle. Since all α_ξ are morphisms in C_F, and F maps the coproduct to the corresponding product, we have $F(\Delta_\alpha)(\tilde{f}) = 0$. Since F is cohomological, there is $\bar{f} \in F(\bar{B})$ with $F(\beta)(\bar{f}) = \tilde{f}$. Then $Y \xrightarrow{\beta} Z = (\bar{B},\bar{f})$ is a morphism in C_F such that $\beta\alpha_\xi$ is independent of ξ. $\square$

Remark 2.9. Obviously, the conditions that $\mathscr{D}$ and C are triangulated and that F is cohomological can be replaced by the condition that $\mathscr{D}$ has weak coequalizers, that every pair of morphisms in

C has a weak coequalizer in $\mathscr{D}$ which belongs to C, and that F maps weak coequalizers to weak equalizers.

Corollary 2.10. *Under the assumptions of this proposition, $\tilde{F}_C$ is a cohomological functor which maps coproducts of size $<\kappa$ to products.*

Proof. Indeed, we have seen that the category over which the colimit is taken in (1) is κ-directed. Since the functor whose direct limit is the right-hand side of (1) is cohomological and maps coproducts in its first variable X to products, the assertion follows from Proposition 2.7. $\square$

2.2. Some properties of ϕ_C

Now we assume that $\mathscr{D}$ has arbitrary coproducts, and that F is a cohomological functor which maps coproducts to products. Let κ be a regular infinite cardinal.

Lemma 2.11. *Let C be a full triangulated subcategory of $\mathscr{D}$ which is closed under coproducts of size $<\kappa$. Let G be a subset of the class of objects of C such that* $\mathrm{card}(G)<\kappa$, *let β be an arbitrary cardinal number, and let $(A_\xi)_{\xi\in\beta}$ be a family of elements of G, and let $B=\coprod_{\xi\in\beta}A_\xi$.*

- *Assume that* $\mathrm{card}(F(A))<\kappa$ *for all $A\in G$. Then $\tilde{F}_C(B)\overset{\phi_C}{\to}F(B)$ is surjective.*
- *Assume that for every $X\in\mathfrak{Ob}(C)$ and every $A\in G$, we have* $\mathrm{card}(\mathrm{Hom}_{\mathscr{D}}(A,X))<\kappa$. *Then*

$$\tilde{F}_C(B)\overset{\phi_C}{\to}F(B)$$

is injective.

Proof. For the first point, let $f\in F(B)$. Let $(f_\xi)_{\xi\in\beta}$ be the image of f by the canonical isomorphism $F(B)\cong\prod_{\xi\in\beta}F(A_\xi)$. Let $\tilde{B}=\coprod_{A\in G,\upsilon\in F(A)}A\in C$, and let $\tilde{f}\in F(\tilde{B})$ be the element whose image under the canonical isomorphism $F(\tilde{B})\cong\prod_{A\in G,\upsilon\in F(A)}F(A)$ is $(\upsilon)_{A\in G,\upsilon\in F(A)}$. Then $(\tilde{B},\tilde{f})$ is an object of C_F. Let $A_\xi\overset{i_\xi}{\to}B$ and let $A\overset{j_{(A,\upsilon)}}{\to}\tilde{B}$ be the inclusion of the ξth summand and of the summand belonging to A and υ. By the properties of the coproduct, there is a unique morphism $B\overset{\mu}{\to}\tilde{B}$ with $\mu i_\xi=j_{A_\xi,f_\xi}$. Then $F(\mu)\tilde{f}=f$. This proves that f is in the image of ϕ_C.

For the second point, we first claim if (X,f) is an object of C_F with $f=0$, then for every $Y\in\mathfrak{Ob}\mathscr{D}$ the canonical morphism

$$\mathrm{Hom}_{\mathscr{D}}(Y,X)\to\operatorname*{colim}_{(\tilde{X},\tilde{f})\in C_F}\mathrm{Hom}_{\mathscr{D}}(Y,\tilde{X})=F_C(Y)$$

vanishes. This follows readily from the fact that $(X,f)\overset{0}{\to}(0,0)$ is a morphism in C_F.

Now let $X\in C$, $f\in F(X)$ and $B\overset{\mu}{\to}X$ be such that $F(\mu)f=0$. We have to prove that the image of μ in $F_C(B)$ is zero. Let i_ξ be the same as before, and let $\mu_\xi=\mu i_\xi$. Then $F(\mu_\xi)f=0$. By our assumption about card($\mathrm{Hom}_{\mathscr{D}}(A, X)$) for $A \in G$, the set Z_A of morphisms μ from A to X such that $F(\mu)f = 0$ has cardinality $< \kappa$. We put $\tilde{X} = \coprod_{A \in G, \mu \in Z_A} A$ and denote the embedding of the summand belonging to A and μ by $j_{A,\mu}$. There is a unique morphism $\tilde{X} \xrightarrow{\Theta} X$ with $\Theta j_{A,\mu} = \mu$, and $(\tilde{X}, 0) \xrightarrow{\Theta} (X, f)$ is a morphism in C_F. There is a unique morphism $B \xrightarrow{\chi} \tilde{X}$ with $\chi i_\xi = j_{A_\xi, \mu_\xi}$. Then $\Theta\chi = \mu$. Therefore, the images of $B \xrightarrow{\mu} X$ at (X,f) and of $B \xrightarrow{\chi} \tilde{X}$ at $(\tilde{X},0)$ in $F_C(B)$ agree. But the latter of these two images is zero, as we have seen a few moments ago. This proves the second point. □

Corollary 2.12. *Let G be the same as in Theorem* 2.4. *We assume that* $\kappa \in \mathfrak{M}(G)$ *is so large that* card($F(A[i])$) $< \kappa$ *for all* $i \in \mathbb{Z}$ *and all* $A \in G$. *We also assume that* $\mathscr{D}$ *coincides with its smallest full triangulated subcategory which is closed under coproducts of size* $< \kappa$ *and contains arbitrary coproducts of elements of G. Then for any subcategory C of* $\mathscr{D}$ *with the properties described in Definition* 2, ϕ_C *is a natural isomorphism.*

Proof. Let $\tilde{\mathscr{D}} \subseteq \mathscr{D}$ be the full subcategory of all X such that $F_C(X[i]) \xrightarrow{\phi_C} F(X[i])$ is an isomorphism for all $i \in \mathbb{Z}$. It follows from Lemma 2.8 and our assumptions that $\tilde{\mathscr{D}}$ contains arbitrary coproducts of copies of G, and from Corollary 1.1.1 that it is a full triangulated subcategory which is closed under coproducts of size $< \kappa$. We conclude $\tilde{\mathscr{D}} = \mathscr{D}$. □

2.3. End of the proof of Theorem 2.4

Lemma 2.13. *Let* $\mathscr{D}$ *be a triangulated category which has arbitrary coproducts, and let* $\mathscr{D} \xrightarrow{F} \mathfrak{Ab}$ *be a contravariant cohomological functor which maps coproducts to products. Assume that there is a small subcategory C of* $\mathscr{D}$ *such that the canonical morphism* $F_C \xrightarrow{\phi_C} F$ *is a natural isomorphism. Then F is representable.*

Since by our assumptions there are always cardinal numbers satisfying the assumptions of Corollary 2.12, this lemma completes the proof of Theorem 2.4.

Proof. The assumptions of the lemma imply that the functor F has a solution set. By [8, Theorem 1.4] it is hyporepresentable. By an observation of Bökstedt and Neeman [2, Proposition 3.2], every idempotent in $\mathscr{D}$ splits. But this implies that F is representable. One can also use these facts to apply [4, 1.834]. □

The proof of Theorem 2.4 is now complete.

Remark 2.14. Despite Remark 2.9, there is no obvious way to state an analogue of our Theorem 2.4 in the situation which Heller is considering. It would be absolutely necessary to have some replacement for the notion of a full triangulated subcategory.

3. Application to Grothendieck categories

This section is devoted to the proof of the following theorem:

Theorem 3.1. *The unbounded derived category of a Grothendieck category exists and satisfies the Brown representability theorem.*

To prove the theorem, let $\mathscr{A}$ be a Grothendieck category and let G be a generator of $\mathscr{A}$. Let us define the size of an object X of $\mathscr{A}$ by

$$\operatorname{size}(X) = \operatorname{card}(\operatorname{Hom}_{\mathscr{A}}(G, X))$$

and the size of a cochain complex as the supremum of the sizes of its individual terms. Let the subscript $\leqslant \kappa$ on $\mathscr{A}$, or $\mathscr{K}(\mathscr{A})$ denote the full subcategories of objects of size $\leqslant \kappa$. Let $\mathscr{D}(\mathscr{A})_{\leqslant \kappa}$ be the full subcategory of all cochain complexes which are quasi-isomorphic to a cochain complex of size $\leqslant \kappa$.

The existence of the derived category follows from the existence of sufficiently many $\mathscr{K}$-injective complexes for Grothendieck categories. This has in principle been proved, but not explicitly stated, by Spaltenstein [15, 4.6]. It will also be necessary to control the size of $\mathscr{K}$-injective resolutions. The following proposition allows us to do this.

Proposition 3.2. *For every object X of $\mathscr{K}(\mathscr{A})$, there exist an injective and $\mathscr{K}$-injective Y of $\mathscr{K}(\mathscr{A})$ and a monomorphism $X \to Y$ which is a quasi-isomorphism. Moreover, there is an infinite cardinal number κ_1 such that for every cardinal number λ with $\lambda^{\kappa_1} = \lambda$ the size of Y can be chosen to be $\leqslant \lambda$ if the size of X is $\leqslant \lambda$.*

Since $\mathscr{A}$ is an AB5 category, coproducts are exact. It follows from this and the calculus of fractions that coproducts in $\mathscr{D}(\mathscr{A})$ exist and coincide with usual coproducts of cochain complexes.

The Brown representability theorem for $\mathscr{D}(\mathscr{A})$ follows from Theorem 2.4, Proposition 3.2 and the following assertions:

Proposition 3.3. *$\mathscr{D}(\mathscr{A})$ coincides with its smallest full triangulated subcategory which contains coproducts of arbitrarily many copies of G and is closed under countable coproducts. In particular, it is strongly generated by G.*

Proposition 3.4. *There exists a cardinal number κ_1 such that for every cardinal number λ with $\lambda^{\kappa_1} = \lambda$ the following assertions hold*:

- *$\mathscr{D}(\mathscr{A})_{\leqslant \lambda}$ is closed under coproducts of size λ.*
- card $\operatorname{Hom}^i_{\mathscr{D}(\mathscr{A})}(G, X) \leqslant \lambda$ *if $X \in \mathscr{D}(\mathscr{A})_{\leqslant \lambda}$.*

It is easy to see that the class of λ which satisfy the assumptions of Proposition 3.2 is unbounded. For instance, α^{κ_1} satisfies these assumptions for every cardinal number α. Therefore, Propositions 3.3 and 3.4 prove that $\mathscr{D}(\mathscr{A})$ satisfies the assumptions of Theorem 2.4. Moreover, the following is true:

Proposition 3.5. *Let $\mathscr{E} \subset \mathscr{D}(\mathscr{A})$ be a full triangulated subcategory which is closed under arbitrary coproducts in $\mathscr{D}(\mathscr{A})$ and is strongly generated by a subset of its class of objects. Then the Brown representability theorem holds for $\mathscr{E}$.*

Proof. Let us first mention that, since Hom maps coproducts in its first argument to products, the following fact holds:

> Let κ and λ be cardinals such that $\kappa^\lambda = \kappa$, and let $X \in \mathfrak{Ob}\mathscr{D}(\mathscr{A})$. Then the class of all $Y \in \mathfrak{Ob}\mathscr{D}(\mathscr{A})$ with $\operatorname{card}\operatorname{Hom}^i_{\mathscr{D}(\mathscr{A})}(Y, X) < \kappa$ for all i is a full triangulated subcategory which is closed under coproducts of size $\leqslant \lambda$.

Now, let $\mathscr{E}$ be strongly generated by a subset M of its class of objects. There exists a cardinal number $\kappa_2 \geqslant \kappa_1$ such that M is contained in the smallest full triangulated subcategory of $\mathscr{D}(\mathscr{A})$ which contains G and is closed under coproducts of size $\leqslant \kappa_2$. It follows from the above observation and from Proposition 3.4 that for every λ with $\lambda^{\kappa_2} = \lambda$ we have $\operatorname{card}\operatorname{Hom}^i(X, Y) < \lambda$ for all i and for $X \in M$ and $Y \in \mathscr{D}(\mathscr{A})_{<\lambda}$. It follows that $\mathscr{E}$ satisfies the assumptions of Theorem 2.4. □

We now give the remaining proofs of the above propositions. The proof of Proposition 3.3 is easy: Let $\widetilde{\mathscr{D}(\mathscr{A})}$ be the smallest full triangulated subcategory stable under countable coproducts and containing coproducts of arbitrarily many copies of G. Since G is a generator, every object of $\mathscr{A}$ has a left resolution by objects which are coproducts of copies of G. Therefore, $\widetilde{\mathscr{D}(\mathscr{A})}$ contains arbitrary complexes concentrated in dimension zero. Since it is a triangulated subcategory, it contains arbitrary bounded complexes. Since it is stable under countable coproducts, it is closed under forming the union of an ascending sequence of subcomplexes since such a union is easily identified with a homotopy colimit in the sense of [2]. But every complex is a union of an ascending sequence of bounded subcomplexes. The proposition follows. The proof of the crucial Proposition 1 is longer and is best given in a separate subsection. Finally, Proposition 3 follows from Proposition 1 and (using the above-mentioned fact that the functor from cochain complexes over $\mathscr{A}$ to $\mathscr{D}(\mathscr{A})$ commutes with coproducts) Lemma 3.7 below.

Remark 3.6. B. Keller has pointed out to me that the Brown representability theorem for $\mathscr{D}(\mathscr{A})$ can also be derived from the Gabriel–Popescu theorem and the Brown representability theorem for the unbounded derived category of modules over a ring, which follows from the main abstract result of [12] (cf. [9, Theorem 5.2]). Indeed, the Gabriel–Popescu theorem identifies $\mathscr{D}(\mathscr{A})$ with a quotient of the unbounded derived category of modules over a ring, and this quotient inherits the Brown representability theorem.

3.1. Existence of sufficiently many $\mathscr{K}$-injective complexes

We first need some properties of the function size. It is clear that the class of objects of size $\leqslant \kappa$ is closed under subobjects and extensions, but there is no reason why it should be closed under quotients. It will be necessary to show that there are sufficiently many κ for which it is also closed under quotients. Let

$$G_\kappa = \coprod_\kappa G.$$

For an arbitrary object X of $\mathscr{A}$, let ν_X be the smallest infinite cardinal number such that there is no strictly increasing sequence $(X_\xi)_{\xi \in \nu_X}$ of subobjects of X. Let N_X be the smallest regular cardinal number which is $\geqslant \nu_X$.[1] We have

$$\nu_X \leqslant \ \mathrm{size}(X),$$

since for every strictly increasing sequence $(X_\xi \subset X)_{\xi \in \mu}$ and every $\xi \in \mu$ there is a morphism $G \to X$ which factors over $X_{\xi+1}$, but not over X_ξ. Also, it is easy to see that for every ordinal number α with $\mathrm{cf}(\alpha) \geqslant \nu_X$ (which by the regularity of $\mathrm{cf}(\alpha)$ is equivalent to $\mathrm{cf}(\alpha) \geqslant N_X$) every ascending sequence $(X_\xi)_{\xi \in \alpha}$ terminates.

Lemma 3.7. *Assume that $\kappa^{N_G} = \kappa$ and that every quotient of G_{N_G} has size $\leqslant \kappa$. Then the class of objects of size $\leqslant \kappa$ is stable under quotients, under coproducts and under colimits over ordinal numbers $\leqslant \kappa$.*

Proof. The third of these stability assertions follows from the other two. The stability under coproducts of size κ follows from the following facts:

- Every morphism of G into a coproduct of size κ factors over a sub-coproduct of size $\leqslant N_G$. This easily follows from [14, Theorem 2.8.6], cf. Lemma 3.9 below.
- There are at most $\kappa^{N_G} = \kappa$ subsets of κ of cardinality $\leqslant N_G$.
- Since $\kappa^{N_G} = \kappa$, every product of N_G objects of size $\leqslant \kappa$ has size $\leqslant \kappa$. By AB5, the coproduct of these objects is a subobject of the product, hence it also has size $\leqslant \kappa$.

To prove the stability under quotients, it suffices to show that every quotient of G_κ has size $\leqslant \kappa$. For every object is the quotient of $\coprod_{\mathrm{Hom}(G,X)} G$, and the size of the coproduct is $\leqslant \kappa$ if $\mathrm{size}(X) \leqslant \kappa$.

If Q is a quotient of G_κ and if M is a subset of κ, let Q_M be the image of $\coprod_{\xi \in M} G \subseteq G_\kappa$ in Q. It is easy to see (Lemma 3.9) that every morphism of G into Q factors over Q_M for some M with $\mathrm{card}(M) \leqslant N_G$. By our assumptions on κ, there are only κ of such subsets, and for each of them the size of Q_M, and hence the cardinality of the set of morphism $G \to Q$ which factor over Q_M, is $\leqslant \kappa$. This implies that there are at most κ morphisms from G to Q. $\square$

Corollary 3.8. *If κ satisfies the assumptions of the lemma, if $X \overset{p}{\to} Y$ is an epimorphism, and if $\mathrm{size}(Y) \leqslant \kappa$ then there exists a subobject $\tilde{X} \subseteq X$ with $\mathrm{size}(\tilde{X}) \leqslant \kappa$ and $p(\tilde{X}) = Y$.*

Proof. Since G is a generator, X is generated by the morphisms from G to X (in the sense that it coincides with the smallest subobject containing all their images), hence Y is generated by the

[1] Let $I = \bigcup_{k=1}^{\infty} \{k\} \times \omega_k$, totally ordered by $(k,\alpha) < (l,\beta)$ if and only if $k > l$ or $k = l$ and $\beta > \alpha$. The constant $\mathbb{Z}/2\mathbb{Z}$-Diagram is an object X of the category of I-diagrams of Abelian groups for which $\nu_X = \omega_\omega$ is singular.

morphisms from G to Y which lift to X. Since there are at most κ such morphisms, there is a morphism $G_\kappa \xrightarrow{f} X$ such that pf is an epimorphism. The image $\tilde{X}$ of f has the required properties. Its size is $\leqslant \kappa$ because it is a quotient of G_κ. $\square$

The following fact was used in the proof of Lemma 3.7:

Lemma 3.9. *Let $\coprod_{m\in M} X_m \xrightarrow{p} Y$ be an epimorphism. Then for every morphism $\tilde{G} \xrightarrow{f} Y$ with $\tilde{G} \subseteq G$ there exists a subset $N \subseteq M$ of cardinality $< N_G$ such that f factors over the image of $\coprod_{m\in N} X_N$ in Y.*

Proof. We proceed by transfinite induction on the cardinality of M, starting with the trivial case where this cardinality is $< N_G$. Assume that the assertion has been proved for morphisms from $\tilde{G}$ to the quotient of a coproduct of less than card(M) objects. We may assume $M = \mathrm{card}(M) = \mu$. For $\alpha \in \mu$, let Y_α be the image of $\coprod_{\beta\in\alpha} X_\beta$ in Y. If $\chi = \mathrm{cf}(\mu) \geqslant N_G$, then the sequence $(f^{-1}(Y_\alpha))_{\alpha\in\mu}$ of subobjects of $\tilde{G}$ stabilizes, and by AB5 (cf. [14, Theorem 2.8.6]) we have

$$G = f^{-1}\left(\sum_{\alpha\in\mu} Y_\alpha\right) = \sum_{\alpha\in\mu} f^{-1}(Y_\alpha),$$

hence there exists $\alpha \in \mu$ such that f factors over Y_α. If however $\chi < N_G$, let $(\xi_\alpha)_{\alpha\in\chi}$ be a cofinal sequence in μ. Applying the induction assumption with $\tilde{G}$ replaced by its subobject $f^{-1}(Y_{\xi_\alpha})$, we see that there is a subset $N_\alpha \subseteq \mu$ of cardinality $< N_G$ such that the intersection of Y_{ξ_α} and the image of f is contained by the image in Y of $\coprod_{\beta\in N_\alpha} X_\beta$.

Let $N = \bigcup_{\alpha\in\chi} N_\alpha$, and let $\tilde{Y}$ be the image of $\coprod_{\beta\in N} X_\beta$ in Y. This subobject contains the intersection of the image of f with Y_{ξ_α} for every $\alpha \in \chi$, and by an application of AB5 this implies that f factors over $\tilde{Y}$. On the other hand, card(N) $< N_G$ by the regularity of N_G. $\square$

Proposition 3.10. *If κ_0 is the smallest cardinal number which satisfies the assumptions of Lemma* 3.7, *then every non-zero acyclic cochain complex C^* over $\mathscr{A}$ has a non-zero acyclic subcomplex $\tilde{C}^*$ consisting of objects of size $\leqslant \kappa_0$.*

If in addition C^* is a quotient of a (*not necessarily acyclic*) *cochain complex D^*, then D^* has a subcomplex $\tilde{D}^*$ of size $\leqslant \kappa_0$ such that $\tilde{C}^*$ is the image of $\tilde{D}^*$ in C^*.*

Proof. Let $C^i \neq 0$. Let $\tilde{C}^i$ be a non-vanishing subobject of C^i of size $\leqslant \kappa_0$. Let $\tilde{C}^{i+1} = d(\tilde{C}^i)$, and let $\tilde{C}^j = 0$ for $j > i+1$.

As usual, let $Z^j \subseteq C^j$ be the kernel of d. Assume that $j \leqslant i$ and that a subobject $\tilde{C}^j \subseteq C^j$ of size $\leqslant \kappa_0$ has already been defined. Let $\tilde{Z}^j = Z^j \cap \tilde{C}^j$. Then let $\tilde{C}^{j-1}$ be a subobject of size $\leqslant \kappa$ of C^{j-1} such that $d\tilde{C}^{j-1} = \tilde{Z}^j$. Such a subobject can be found by an application of Corollary 1 to the morphism $d^{-1}\tilde{C}^j \to \tilde{Z}^j$, which is an epimorphism since C^* is acyclic.

The subcomplex $\tilde{C}^* \subseteq C^*$ has the desired properties. The second part follows by first constructing subobjects Δ^i of size $\leqslant \kappa_0$ of D^i which epimorphically project onto $\tilde{C}^i$, and then put $\tilde{D}^i = \Delta^i + d(\Delta^{i-1})$. $\square$

Recall that an object Y has the right lifting property with respect to a monomorphism $A \xrightarrow{j} B$ if every morphism $A \to Y$ extends to B. It is easy to see and probably well-known (see for instance [3, Proposition 1.3.3]) that a complex is a $\mathscr{K}$-injective complex of injective objects of $\mathscr{A}$ if and only if it has the right lifting property with respect to every monomorphism with acyclic cokernel. Let us call such a complex strictly injective.

Corollary 3.11. *Let κ_0 have the property described in Corollary 3.8. Assume that a complex I^* has the right lifting property with respect to every monomorphism $X^* \xrightarrow{j} Y^*$ such that Y is of size $\leqslant \kappa_0$ and such that* coker(j) *is acyclic. Then I^* has the right lifting property with respect to every monomorphism with acyclic cokernel. In particular, it is strictly injective.*

Proof. Let $X^* \xrightarrow{j} Y^*$ be an arbitrary monomorphism with acyclic cokernel, and let $X^* \xrightarrow{f} I^*$ be a morphism. Since $\mathscr{A}$ is AB5, by Zorn's lemma there is a maximal extension $\tilde{f}$ of f to a subcomplex $\tilde{X}^*$ of Y^* such that $Y^*/\tilde{X}^*$ is acyclic. We claim that $\tilde{X}^* = Y^*$. Otherwise, there are a non-zero acyclic subcomplex $C^* \subseteq Y^*/\tilde{X}^*$ and a subcomplex $D^* \subseteq Y^*$ which projects onto C^* such that C^* and D^* have size $\leqslant \kappa_0$. Then we can apply our assumption about I^* to the monomorphism $D^* \cap \tilde{X}^* \to D^*$ (whose cokernel is $\cong C^*$) and the restriction of $\tilde{f}$ to its source. We obtain an extension $\tilde{X}^* + D^* \xrightarrow{\hat{f}} Y$ of $\tilde{f}$. Since

$$Y^*/(\tilde{X}^* + D^*) \cong (Y^*/\tilde{X}^*)/C^*$$

is acyclic, this contradicts the maximality of $\tilde{f}$. This contradiction proves $\tilde{X}^* = Y^*$. Therefore, f extends to Y^*.

Lemma 3.12. *We have*

$$\operatorname{card} \operatorname{Hom}(X,Y) \leqslant \operatorname{size}(Y)^{\operatorname{size}(X)}.$$

Proof. Since

$$\operatorname{card}(\operatorname{Hom}(G_\lambda, Y)) = \operatorname{size}(Y)^\lambda,$$

this follows from the fact that X is a quotient of $G_{\operatorname{size}(X)}$. $\square$

Proof of Proposition 3.2. Let κ_1 be the maximum of κ_0 and the cardinality of a set of representatives

$$X_\xi \xrightarrow{i_\xi} Y_\xi, \quad \xi \in \kappa_1$$

for the isomorphism classes of monomorphisms $X \xrightarrow{i} Y$ such that size(Y) $\leqslant \kappa_0$ and such that coker(i) is acyclic.

Let A have size $\leqslant \lambda$, where $\lambda^{\kappa_1} = \lambda$. By transfinite induction, we define an inductive system $(A_\xi)_{\xi \leqslant \lambda}$ of cochain complexes of size $\leqslant \lambda$ and with transition morphisms which are monomorphisms and quasi-isomorphisms as follows: $A_0 = A$. If X_ξ has been defined, let $A_{\xi+1}$ be defined by attaching a copy of Y_ν/X_ν along each morphism $X_\nu \to A_\xi$ which does not extend to Y_ν. There are only κ_1 values for ν, and for each given ν there are by Lemma 3.7 at most λ morphisms $X_\nu \to A_\xi$. It follows from these facts, Lemma 3.7 and the stability of size under extensions that $A_{\xi+1}$ also has size $\leqslant \lambda$.

If ξ is a limit ordinal and if A_ν has been defined for $\nu < \xi$, we put $A_\xi = \operatorname{colim}_{\nu<\xi} A_\nu$. By Lemma 3.7, this object also has size $\leqslant \lambda$. By the exactness of filtered colimits, the transition morphisms $A_\nu \to A_\xi$ are quasi-isomorphisms for $\nu < \xi$.

Let $X_\xi \stackrel{\varphi}{\to} A_\lambda$ be a morphism. Since $\lambda^{\kappa_1} = \kappa_1$, the cofinality of λ is $\geqslant \nu_{X_\xi}$ since $\nu_{X_\xi} \leqslant \operatorname{size}(X_\xi) \leqslant \kappa_0 \leqslant \kappa_1$ (cf. [10, Lemma I.10.40.]). Therefore, by an application of AB5 in the same way as in the end of the proof of Theorem 1.10 in [6, p. 137], there exists $\nu \in \lambda$ such that φ factors over A_ν. By our construction, φ has an extension to a morphism $Y_\xi \to A_{\nu+1} \subseteq A_\lambda$. By Corollary 3.11, A_λ is strictly injective. $\square$

4. Application to torsion classes in Grothendieck categories

Let $\mathscr{A}$ be a Grothendieck category and let $\mathscr{T} \subset \mathscr{A}$ be a full subcategory which is also an AB3 category and has the property that the inclusion functor $\mathscr{T} \to \mathscr{A}$ is exact and commutes with arbitrary coproducts. Assume also that every subobject (in $\mathscr{A}$) of an object of $\mathscr{T}$ belongs to $\mathscr{T}$. Let $\mathscr{D}_{\mathscr{T}}(\mathscr{A})$ be the full subcategory of all complexes whose cohomology objects belong to $\mathscr{T}$. This subcategory is stable under the shift functors $[k]$, and becomes a triangulated category using the class of all triangles formed by objects of $\mathscr{D}_{\mathscr{T}}(\mathscr{A})$ which are distinguished in $\mathscr{D}(\mathscr{A})$. It is easy to see that such a $\mathscr{T}$ automatically is the class of torsion objects in a torsion theory in the sense of [14, Section 4.7].

Theorem 4.1. *Under these assumptions, $\mathscr{D}_{\mathscr{T}}(\mathscr{A})$ satisfies the Brown representability theorem. This implies that $\mathscr{D}_{\mathscr{T}}(\mathscr{A}) \to \mathscr{D}(\mathscr{A})$ has a right adjoint.*

For the following considerations we fix a generator G of $\mathscr{A}$. The size of objects of $\mathscr{A}$ with respect to G will be the same as in the last section, and the size of objects of $\mathscr{T}$ will be their size in $\mathscr{A}$.

It is easy to see that $\mathscr{D}_{\mathscr{T}}(\mathscr{A})$ is stable under coproducts in $\mathscr{D}(\mathscr{A})$. Therefore, it has coproducts and the formulation of the theorem makes sense. By Proposition 3.5, it suffices to prove that $\mathscr{D}_{\mathscr{T}}(\mathscr{A})$ is strongly generated by a subset of its class of objects. This is established by the following proposition:

Proposition 4.2. *Let κ be the same as in Corollary 3.8. Then $\mathscr{D}_{\mathscr{T}}(\mathscr{A})$ is strongly generated by $\mathscr{D}_{\mathscr{T}}(\mathscr{A})_{\leqslant\kappa} = \mathscr{D}_{\mathscr{T}}(\mathscr{A}) \cap \mathscr{D}(\mathscr{A})_{\leqslant\kappa}$.*

Lemma 4.3. *Let κ be the same as above. If C^* is a cochain complex over $\mathscr{A}$ and if $\tilde{H}^0 \subset H^0(C^*)$ is a subobject of size $\leqslant \kappa$, then C^* has a subcomplex $\tilde{C}^*$ of size $\leqslant \kappa$ such that $H^*(\tilde{C}^*) \to H^*(C^*)$ is injective and such that the image of $H^0(\tilde{C}^*) \to H^0(C^*)$ is $\tilde{H}^0$.*

Proof. Let $\tilde{C}^i = 0$ for $i > 0$, and let $\tilde{C}^0 \subset Z^0$ be any subobject of size $\leqslant \kappa$ which epimorphically maps onto $\tilde{H}^0$. If subobjects $\tilde{C}^j \subset C^j$ of size $\leqslant \kappa$ forming a partial complex have been defined for $j > i$, let $\tilde{B}^{i+1}$ be the kernel of $Z^{i+1} \cap \tilde{C}^{i+1} \to H^{i+1}$ and let $\tilde{C}^i \subset C^i$ be any subobject of size $\leqslant \kappa$ such that $\tilde{C}^i \xrightarrow{d} C^{i+1}$ factors over an epimorphism $\tilde{C}^i \to \tilde{B}^{i-1}$. □

We are now able to prove Proposition 4.2 in two steps.

Step 1: Let C^* be a cochain complex over $\mathscr{A}$ with cohomology in $\mathscr{T}$ which is acyclic in positive dimensions. Then there exists a complex X^* which is a coproduct of objects of $\mathscr{D}_{\mathscr{T}}(\mathscr{A})_{\leqslant \kappa}$, such that X^i vanishes if $i > 0$, and a morphism $X^* \to C^*$ which induces an epimorphism on cohomology.

This is easily derived from the lemma.

Step 2: Let $\tilde{D} \subseteq \mathscr{D}_{\mathscr{T}}(\mathscr{A})$ be the smallest full triangulated subcategory which contains arbitrary coproducts of objects of $\mathscr{D}_{\mathscr{T}}(\mathscr{A})_{\leqslant \lambda}$ and is stable under countable coproducts. Then $\tilde{D} = \mathscr{D}_{\mathscr{T}}(\mathscr{A})$.

Let C be an arbitrary complex with cohomology in $\mathscr{T}$. Let $C_0 = C[1]$. By an iterated application of Step 1 and use of a cone construction, one constructs a sequence

$$C_0 \xrightarrow{i_0} C_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{k-1}} C_k \xrightarrow{i_k} C_{k+1} \xrightarrow{i_{k+1}} \cdots,$$

in which i_k defines the zero morphism on cohomology and $X_k = \operatorname{coker}(i_k)$ belongs to $\tilde{D}$. Let $\tilde{C}_k \to C$ be the homotopy fibre of $C_0 \to C_k$. Then the $\tilde{C}_k$ form an ascending sequence, and $\tilde{C}_{k+1}/\tilde{C}_k \cong X_k[-1]$. Since $\tilde{C}_0$ vanishes, this implies $\tilde{C}_k \in \tilde{D}$ by induction on k. Therefore, the countable colimit (which is a homotopy colimit in the sense of [2]) $\operatorname{colim}_k \tilde{C}_k$ belongs to $\tilde{D}$. But this colimit is quasi-isomorphic to C since $\operatorname{colim}_k C_k$ is easily seen to be acyclic.

The proof of Proposition 4.2 and of the theorem is complete. □

References

[1] H. Bass, Algebraic K-Theory, Benjamin, New York, 1968.

[2] M. Bökstedt, A. Neeman, Homotopy limits in triangulated categories, Compositio Mathematica 86 (1993) 209–234.

[3] J. Franke, Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence, Preprint, published on Grayson's K-theory server math.uiuc.edu, 1996.

[4] P.J. Freyd, A. Scedrov, Categories, Allegories, vol. 39, North-Holland Mathematical Library, Elsevier, Amsterdam, 1990.

[5] P. Gabriel, F. Ulmer, Lokal präsentierbare Kategorien, vol. 221, Lecture Notes in Mathematics, Springer, Berlin, 1971.

[6] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Mathematical Journal Second Series 9 (1957) 119–221.

[7] G. Harder, M. Goresky, R. MacPherson, Weighted cohomology, Inventiones Mathematicae 116 (1994) 139–213.

[8] A. Heller, On the representability of homotopy functors, Journal of London Mathematical Society 23 (2) (1981) 551–562.

[9] B. Keller, Deriving DG categories, Annales Scientifiques de l'Ecole Normale Superieure 4 (1994) 63–102.

[10] K. Kunen, Set theory, An Introduction to Independence Proofs, vol. 102, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam (London, New York, Tokyo), 1992.

[11] B. Mitchell, Rings with several objects, Advances in Mathematics 8 (1972) 1–161.
[12] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, Journal of American Mathematics Society 9 (1996) 205–236.
[13] A. Neeman, Triangulated Categories, Unpublished book manuscript, 1997.
[14] N. Popescu, Abelian Categories with Application to Rings and Modules, Academic Press, New York, 1973.
[15] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Mathematica 65 (1988) 121–154.