UNIQUENESS THEOREMS FOR CERTAIN TRIANGULATED CATEGORIES POSSESSING AN ADAMS SPECTRAL SEQUENCE

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Contents

It is a well-known and easy consequence of Serre's calculation of the rational homotopy groups of spheres that the homology functor defines an equivalence of the rational stable homotopy category to the category of graded rational vector spaces. However, it also became soon apparent that localisation at a prime number p does not simplify the picture to a similar extent. The reason is that from the point of view of a structure theory of the stable homotopy category of finite spectra, there is an infinite sequence of chromatic primes lying above each of the usual prime numbers. This was discovered by Ravenel in the late seventies, who formulated his picture in a sequence of conjectures motivated by earlier work of Quillen, Landweber, Novikov, Morava and Miller, Ravenel and Wilson. A breakthrough on this question has been achieved by Devinatz, Hopkins and Smith, which led to a proof of all Ravenel conjectures with the exception of the disproved telescope conjecture. We refer to [Rav93] and the references given there for all these questions.

The Ravenel conjectures make it clear that localisation at a chromatic prime (for instance, K-localisation) can simplify the task of giving an algebraic model for the stable homotopy category, and also that at least in the case of finite spectra no further simplification by other localisations is possible. The task of understanding the stable homotopy category at a chromatic prime therefore seems to be on the agenda. To my knowledge, the first work in this direction is due to Bousfield [Bou85], who classified the K-local spectra at an odd prime. He later extended his work to include the prime 2. The aim of this work is to give an extension of Bousfield's classification of spectra at an odd prime to a description of the localised stable homotopy category in terms of cochain complexes.

The construction of the equivalence of categories also uses, besides the K-local stable homotopy category itself, similar localisations of the category of $\boldsymbol{C}$-diagrams of spectra, where $\boldsymbol{C}$ is a finite poset. This makes it desirable to extend the equivalence between K-local spectra and cochain complexes to $\boldsymbol{C}$-diagram categories, but we can do this only for $\dim \boldsymbol{C} < 2p-4$. Therefore, our algebraic model can only be considered as an approximation to the topological picture, but not as a complete algebraisation comparable to Quillen's success in the rational unstable case. In the terminology of [Rav87], we have to give our model a place among the flat earth models, since its unrestricted validity (i.e., for $\dim \boldsymbol{C}$ arbitrary) would imply, among other futile things, that no higher order self-map of the sphere spectrum survives K-localisation at p. The reason why a flat earth model based on K-theory can give us information about the homotopy category is that the Adams spectral sequence based on K-theory is sparser than the classical Adams spectral sequence based on homology. I hope that the simplicity of our model, compared to what will possibly be involved in a full algebraisation of the theory, and the possibility that some of the methods can be useful in getting more realistic and more complicated models justifies its publication despite of its limited scope.

Our methods can also be applied to n-th part of the chromatic tower, where $n^2 + n < 2p-2$. In this case, it seems that no complete classification of the objects has been obtained before, but only a classification of the invertible spectra by B. Gross and M. Hopkins. The restriction under which they are able to obtain their classification is only slightly better than the one under which we hope to actually calculate the homotopy category. I think that the results obtained in that way are at least not far from being the best possible approximation of stable homotopy by derived categories. All approximations of greater precision probably have to use more complicated structures. Although the Ravenel conjectures (which certainly are an essential part of the picture) have been guessed from the experience in the stable case, it is not clear whether the stabilisation simplifies the task of an algebraisation of homotopy. It is possible that it obfuscates non-linear features of the problem which cannot be ignored even in the stable case.

We obtain our partial algebraisation of K-local stable homotopy by a combination of Bousfield's methods in [Bou85] with the idea underlying the construction of the realisation functor from the derived category of perverse sheaves to the derived category of usual sheaves in [BBD82]. Beilinson, Bernstein and Deligne use the filtered derived category. Since our construction is more complicated, we have to use homotopy categories of $\boldsymbol{C}$-diagrams for a poset $\boldsymbol{C}$. The axioms for such a system of categories are given in the first and longest chapter. They involve only the structures belonging to a system of diagram categories. The triangulated structure is reconstructed from these more basic data. This should be compared to another improvement of the axioms of triangulated categories due to Keller [Kel91]. Keller does not eliminate the triangulated structure. One the other side, he only needs the filtered derived categories for a finite number of two-step filtrations. This is less than the input needed by our axioms, but it also makes his systems unsuitable for the purposes of this paper, since our construction definitely needs diagram categories shaped by more complicated posets. However, his axioms and two of the axioms considered in our paper (namely, $\tilde{\mathfrak{C}}$-systems and $\tilde{\mathfrak{P}}$-systems) can be shown to be equivalent (cf. [GW95] and [Wil95]), and the same probably holds for the other two types of systems considered in this paper. I am indebted to Haynes Miller for pointing out a system of axioms similar to ours exists in a paper of Alex Heller [Hel88] on abstract homotopy theory. He imposes no linearity condition and assumes diagram categories shaped by an arbitrary small category as input. Apart from these differences, his axioms are similar to ours. He also has a universality result similar to our Theorem 4, but in his case the distinguished role is plaid by the homotopy category of simplicial sets of arbitrary size rather than of finite spectra.

The second section proves the abstract uniqueness theorem for categories with an Adams spectral sequence. In the third section this abstract result is applied to K-local stable homotopy. In the K-local case, the assumptions of our abstract uniqueness theorem have been verified by Bousfield [Bou85]. For the generalisation to higher chromatic primes, the necessary facts about the Adams-Novikov spectral sequence and the cohomological dimension of its E_2-term do not seem to be contained in the published literature (although slight modifications of them are). Therefore, these facts have to be proved before we can formulate our result about the structure of the n-th chromatic localisation of stable homotopy at an odd prime p for $n^2 + n < p$.

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1. Systems of triangulated diagram categories

1.1. Notations. For a category C, we denote by $\mathfrak{Ob}(C)$ the set of objects of C. The Nerve of C will be denoted by $N.C$. The dimension of this simplicial set (i. e., the supremum of the dimensions of its non-degenerate simplices) will be called the dimension of C. For a subcategory $D \subseteq C$ and $x \in C$, we will denote by $D \to x$ the following comma category: Objects are pairs (d,ϕ) where $d \in D$ and $d \xrightarrow{\phi} x$ is a morphism in C. Morphisms from (d,ϕ) to $(\tilde{d},\tilde{\phi})$ are morphisms $d \xrightarrow{\delta} \tilde{d}$ in D such that $\phi = \tilde{\phi}\delta$. The category $D \leftarrow x$ has pairs (d,ϕ) with $d \in D$ and $x \xrightarrow{\phi} d$ in C as objects. Morphisms are defined in the same way as for $D \to x$. If $M \subseteq \mathfrak{Ob}(C)$ is a subset, we will denote by $C - M$ the full subcategory of C with the set of objects $C - M$. If $M = \{x\}$ has just one element, we will also denote this subcategory by $C - x$. If $F\colon D \to C$ is a functor, the categories $F \Rightarrow x$ and $F \Leftarrow x$ have objects $(d \in D, F(d) \xrightarrow{\phi} x)$ and $(d \in D, x \xrightarrow{\phi} F(d))$. If F is the inclusion of a subcategory, this is the same as $D \to x$ and $D \leftarrow x$.

Let $F\colon \boldsymbol{C} \to \boldsymbol{D}$ be a functor between categories, and let $\mathcal{A}^{\boldsymbol{D}}$ be the category of functors from $\boldsymbol{D}$ to $\mathcal{A}$. Let LKan_F (resp. RKan_F) be the (in general, only partially defined) left (resp. right) Kan extension functor along F from $\mathcal{A}^{\boldsymbol{C}}$ to $\mathcal{A}^{\boldsymbol{D}}$. If these extensions are defined everywhere, they are the left (resp. right) adjoint to the pull-back functor $F^*\colon \mathcal{A}^{\boldsymbol{D}} \to \mathcal{A}^{\boldsymbol{C}}$. If the necessary limits (for instance, finite limits and colimits if $\boldsymbol{C}$ and $\boldsymbol{D}$ are finite) exist, then they exist and are given by

$$(\underset{F}{\mathrm{RKan}}\, A)_X = \lim_{F \Leftarrow X} p^* A, \tag{1}$$

(cf. for instance [Mac71, Theorem X.3.1]) where $A \in \mathcal{A}^{\boldsymbol{C}}$, the value of A at X is denoted A_X, and $p\colon (F \Leftarrow X) \to \boldsymbol{C}$ is the projection. There is a dual version for right Kan extensions. In the case of an abelian category A, we will denote the i-th left (resp. right) derived left (resp. right) Kan extension by LKan_{iF} (resp. $\mathrm{RKan}^i{}_F$). By (1), we have

$$(\underset{F}{\mathrm{RKan}}^i\, A)_X = \lim_{F \Leftarrow X}{}^i\, p^* A. \tag{2}$$

If $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G,\tilde{G}} \mathcal{C} \xrightarrow{H} \mathcal{D}$ are functors between categories and if $G \xrightarrow{\phi} \tilde{G}$ is a natural transformation, then $HG \xrightarrow{H(\phi)} H\tilde{G}$ and $GF \xrightarrow{\phi_F} \tilde{G}F$ denote the natural transformations derived from ϕ.

As usual, for a natural number n, $\underline{n}$ will denote the totally ordered set $\{0 < 1 \cdots < n\}$, $\underline{n} \xrightarrow{d_i} \underline{n+1}$ is the monotonic injection not containing i in its image, and $\underline{n} \xrightarrow{s_i} \underline{n-1}$ is the monotonic surjection satisfying $s_i(i) = s_i(i+1)$.

Throughout this paper, (k) refers to formula k in the same subsection, whereas (i.j.k) refers to formula k in subsection i.j. The other logical units are numbered in the same way, with the exceptions of definitions and theorems, which are numbered consecutively throughout the paper.

1.2. The axioms. Throughout this paper, we will use the term "poset" as an abbreviation of "finite partially ordered set". Every poset can be considered as a category in which $\mathrm{Hom}(X,Y)$ has precisely one element if $X \preceq Y$, and is empty otherwise. Our systems of filtered triangulated categories will attach a category to each poset $\boldsymbol{P}$. Let $\mathfrak{P}$ be the 2-category containing posets as objects. For any poset $\boldsymbol{C}$, let $\boldsymbol{C}^\star$ be $\boldsymbol{C}$ with a initial and final object $\star$ added. For x, y in $\boldsymbol{C}$, there is the unique homomorphism from x to y in $\boldsymbol{C}^\star$ which factorises over $\star$, which will be called the zero morphism. If $x \preceq y$, there is one more morphism from x to y in $\boldsymbol{C}^\star$, and there are no other morphisms. The composition is defined in the obvious way. Let horizontal morphisms $\boldsymbol{C} \to \boldsymbol{D}$ in $\mathfrak{P}$ be given by functors $\boldsymbol{C}^\star \to \boldsymbol{D}^\star$ mapping $\star$ to $\star$, and let bimorphisms be natural transformations between functors from $\boldsymbol{C}^\star$ to $\boldsymbol{D}^\star$.

It may sometimes be useful to consider homotopy categories of $\boldsymbol{C}$-diagrams for other categories $\boldsymbol{C}$. It will be assumed that $\boldsymbol{C}$ is finite and finite-dimensional (i. e., the simplicial set $N.(\boldsymbol{C})$ is finite-dimensional). Let $\boldsymbol{C}^\star$ be defined as above by adding a new initial and final object $\star$ to $\boldsymbol{C}$ and a new zero homomorphism factorising over $\star$ between any two objects of $\boldsymbol{C}$, such that a composition which contains a zero is zero, and the composition in $\boldsymbol{C}$ otherwise. $\mathfrak{C}$ be the 2-category of finite finite-dimensional categories $\boldsymbol{C}$, with functors $\boldsymbol{C}^\star \to \boldsymbol{D}^\star$ mapping $\star$ to $\star$ as morphisms and natural transformations as bimorphisms. Obviously, $\mathfrak{P} \subset \mathfrak{C}$.

Let $\mathfrak{K}$ be one of the 2-categories $\mathfrak{C}$ or $\mathfrak{P}$. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are objects of $\mathfrak{K}$, $\mathfrak{Hom}_{\mathfrak{K}}(\boldsymbol{C}, \boldsymbol{D})$ denotes the category of homomorphisms from $\boldsymbol{C}$ to $\boldsymbol{D}$. We define the sub-2-category $\tilde{\mathfrak{K}} \subseteq \mathfrak{K}$ which has the same objects as $\mathfrak{K}$ and for which $\mathfrak{Hom}_{\tilde{\mathfrak{K}}}(\boldsymbol{C}, \boldsymbol{D})$ is the full subcategory of $\mathfrak{Hom}_{\mathfrak{K}}(\boldsymbol{C}, \boldsymbol{D})$ whose objects are functors F from $\boldsymbol{C}^\star$ to $\boldsymbol{D}^\star$ with the following property: If $X \xrightarrow{a} Y$ is a morphism in $\boldsymbol{C}$ and if $F(X) \neq \star$ and $F(Y) \neq \star$, then $F(a) \neq \star$.

For set-theoretical reasons, it is sometimes necessary to consider small sub-2-categories of $\mathfrak{K}$ which are equivalent to $\mathfrak{K}$ as 2-categories. If U is an infinite class, let $\mathfrak{K}_U$ be the 2-category described as follows: An object of $\mathfrak{K}$ is an object of $\mathfrak{K}_U$ if its underlying set of objects is a subset of U. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are objects of $\mathfrak{K}_U$, then $\mathfrak{Hom}_{\mathfrak{K}_U}(\boldsymbol{C}, \boldsymbol{D}) = \mathfrak{Hom}_{\mathfrak{K}}(\boldsymbol{C}, \boldsymbol{D})$. Obviously, $\mathfrak{K}_U = \mathfrak{K}$ if U is the class of all sets.

Let $\mathfrak{K}$ be one of the 2-categories $\mathfrak{P}_U$, $\tilde{\mathfrak{P}}_U$, $\mathfrak{C}_U$, or $\tilde{\mathfrak{C}}_U$, where U is an infinite class. A $\mathfrak{K}$-system of triangulated diagram categories consists of the following data:

(1)
- For each $\boldsymbol{C} \in \mathfrak{K}$, a category $\mathcal{K}_{\boldsymbol{C}}$.
- For each functor $f\colon \boldsymbol{C} \to \tilde{\boldsymbol{C}}$, a functor $f^*\colon \mathcal{K}_{\tilde{\boldsymbol{C}}} \to \mathcal{K}_{\boldsymbol{C}}$.
- For composable functors $\boldsymbol{C} \xrightarrow{f} \boldsymbol{D} \xrightarrow{g} \boldsymbol{E}$, a natural isomorphism $\iota_{f,g}\colon f^* g^* \cong (gf)^*$.
- For each natural transformation $\phi\colon f \to g$, a natural transformation $\underline{\phi}\colon f^* \to g^*$.

These are all the data we require. In particular, the triangulated structure is not initially given but will be defined from these data. Of course, several axioms have to be satisfied. Before we explain them, let us briefly sketch the typical situation in which such data arise. One should think of $\mathcal{K}_{\boldsymbol{C}}$ as a homotopy category of diagrams (typically in some appropriate ("linear") closed model category, or (probably) in some DG-category in the sense of Bondal and Kapranov [BK91]) indexed by $\boldsymbol{C}$. Obviously, any functor $f\colon \mathcal{C}^\star \to \mathcal{D}^\star$ induces a functor f^* from $\mathcal{D}$-diagrams to $\mathcal{C}$-diagrams if we assume that in the $\boldsymbol{C}$-diagram objects which f sends to $\star$ are mapped to the zero object, and morphisms which f sends to the zero morphism become the zero morphism in the pull-back diagram. This functor should pass to the homotopy category of diagrams. It is also clear that in such a situation the natural transformations for the composition of two functors and for a natural transformation between two functors really exist. The reason for passing to $\boldsymbol{C}^\star$ when defining the morphisms is that this gives us a convenient way of prepending or appending zeros to a given diagram.

To formulate the axioms, we need some more notations. Let $\underline{n} = \{0 \prec \ldots \prec n\} \in \mathfrak{P}$. For each $X \in \boldsymbol{C}$, let $\underline{X}\colon \underline{0} \to \boldsymbol{C}$ be the functor sending 0 to X. For $A \in \mathcal{K}_{\boldsymbol{C}}$, let $A_X = \underline{X}^* A \in \mathcal{K}_{\underline{0}}$. Similarly, we will write $\alpha_X = \underline{X}^*\alpha$ for morphisms α in $\mathcal{K}_{\boldsymbol{C}}$. Any morphism $\phi\colon X \to Y$ in $\boldsymbol{C}$ defines a natural transformation $\underline{X} \to \underline{Y}$, hence $\underline{\phi}\colon A_X \to A_Y$.

The following axioms have to be satisfied:

Functoriality Axiom. The following two conditions hold:

- The maps $f \to f^*$ and $\phi \to \underline{\phi}$ define a functor from $\mathfrak{Hom}_{\mathfrak{K}}(\boldsymbol{C}, \boldsymbol{D})$ to the category of functors from $\mathcal{K}_{\boldsymbol{D}}$ to $\mathcal{K}_{\boldsymbol{C}}$.
- We have $\iota_{f,\mathrm{Id}} = \iota_{\mathrm{Id},f} = \mathrm{Id}$. Moreover, if $\boldsymbol{C} \xrightarrow{f} \boldsymbol{D} \xrightarrow{g} \boldsymbol{E} \xrightarrow{h} \boldsymbol{F}$ are morphisms in $\mathfrak{K}$, then
$$\iota_{f,hg} f^*(\iota_{g,h}) = \iota_{gf,h}(\iota_{f,g})_h.$$
If $\boldsymbol{D} \xrightarrow{\tilde{g}} \boldsymbol{E}$ is another morphism in $\mathfrak{K}$ and if $g \xrightarrow{\phi} \tilde{g}$ is a bimorphism, then
$$\iota_{f,\tilde{g}} f^*(\underline{\phi}) = \underline{\phi_f}\iota_{f,g}$$
$$\iota_{\tilde{g},h}\underline{\phi}_{h^*} = \underline{h(\phi)}\iota_{g,h}$$

The motivation of this axiom in the concrete case of a system of homotopy categories of diagrams should be clear.

Isomorphism Axiom. A morphism $\alpha\colon A \to B$ in $\mathcal{K}_{\boldsymbol{C}}$ is an isomorphism if and only if $\alpha_X\colon A_X \to B_X$ is an isomorphism in $\mathcal{K}_0$ for each $X \in \boldsymbol{C}$.

In other words, a morphism in the homotopy category of $\boldsymbol{C}$-diagrams is an isomorphism if and only if it induces isomorphisms on the vertices of the diagram.

Disjoint Union Axiom. If $\boldsymbol{C}$ is the disjoint union of its full subcategories $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, then the inclusions $i_{1;2}\colon \boldsymbol{C}_{1;2} \to \boldsymbol{C}$ define an equivalence of categories
$$\mathcal{K}_{\boldsymbol{C}} \cong \mathcal{K}_{\boldsymbol{C}_1} \times \mathcal{K}_{\boldsymbol{C}_2}.$$

This is motivated by the fact that a $\boldsymbol{C}$-diagram is completely determined by its restriction to the connected components of $\boldsymbol{C}$, and this determinacy should prevail after passing to the homotopy category.

Mapping Cylinder Axiom. Let $\mathfrak{Ar}(\boldsymbol{C})$ be the category of morphisms in $\boldsymbol{C}$ (i. e., the category $\boldsymbol{C}^{\underline{1}}$). The functors $\boldsymbol{C} \to \boldsymbol{C} \times \underline{1}$, $X \Rightarrow X \times 0$ and $X \Rightarrow X \times 1$, and the natural transformation between them given by $0 \prec 1$ define a functor

$$\mathfrak{a}_{\boldsymbol{C}} \colon \mathcal{K}_{\boldsymbol{C}\times\underline{1}} \to \mathfrak{Ar}(\mathcal{K}_{\boldsymbol{C}}).$$

This functor should be viewed as a functor passing from the homotopy category of morphisms between $\boldsymbol{C}$-Diagrams to the category of morphisms in the homotopy category of $\boldsymbol{C}$-Diagrams. The Mapping Cylinder Axiom requires that this functor is full and defines a bijection between the isomorphism classes of objects of $\mathcal{K}_{\boldsymbol{C}\times\underline{1}}$ and $\mathfrak{Ar}(\mathcal{K}_{\boldsymbol{C}})$.

In other words, every morphism in the homotopy category of $\boldsymbol{C}$-diagrams should come from a $\boldsymbol{C} \times \underline{1}$-diagram, and the functor from the homotopy category of $\boldsymbol{C} \times \underline{1}$-diagrams to morphisms in the homotopy category of $\boldsymbol{C}$-diagrams should be full.

Homotopy Kan Extension Axioms. The first homotopy Kan extension axiom requires that for any functor $f\colon \boldsymbol{C}^\star \to \boldsymbol{D}^\star$ in $\mathfrak{K}$, the functor $f^*\colon \mathcal{K}_{\boldsymbol{D}} \to \mathcal{K}_{\boldsymbol{C}}$ has a left adjoint $\operatorname{Ho}\operatorname{LKan}_f\colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{K}_{\boldsymbol{D}}$ and a right adjoint $\operatorname{Ho}\operatorname{RKan}_f\colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{K}_{\boldsymbol{D}}$.

The second homotopy Kan extension axiom for $\operatorname{Ho}\operatorname{LKan}$ applies to a functor f which has a right adjoint f_- and requires that for such f the morphism

$$\operatorname{Ho}\underset{f}{\operatorname{LKan}} \to f_-^*$$

given by the natural transformation $\mathrm{Id}_{\boldsymbol{C}} \to f_- f$ is an isomorphism. If f possesses a left adjoint f_+, we require that the map

$$f_+^* \to \operatorname{Ho}\underset{f}{\operatorname{RKan}}$$

given by the natural transformation $f_+ f \to \mathrm{Id}_{\boldsymbol{C}}$ is an isomorphism.

The motivation is that in the case of the homotopy categories of diagrams of cochain complexes or spectra (or simplicial sets, too) these functors exist, generalise the usual homotopy limit functors, and have the required properties.

In the special case where $f\colon \boldsymbol{C}^\star \to \underline{0}^\star$ comes from the unique functor $\boldsymbol{C} \to \underline{0}$, we shall write simply $\underrightarrow{\operatorname{Holim}}_{\boldsymbol{C}}$ for $\operatorname{Ho}\operatorname{LKan}_f$ and $\underleftarrow{\operatorname{Holim}}_{\boldsymbol{C}}$ for $\operatorname{Ho}\operatorname{RKan}_f$.

Linearity Axiom. This axiom is motivated by Goodwillie's calculus of functors [Goo90] and is the decisive condition which makes our categories triangulated (for instance, it excludes the homotopy categories of diagrams of simplicial sets from consideration). It asserts that a square is homotopy cartesian if and only if it is homotopy cocartesian. We first have to introduce the necessary notations. Let $\square \in \mathfrak{P}$ be the poset $\underline{1} \times \underline{1}$, possessing the following elements:

$$
\begin{tikzcd}
0 \times 0 \arrow[r] \arrow[d] & 1 \times 0 \arrow[d] \\
0 \times 1 \arrow[r] & 1 \times 1,
\end{tikzcd}
\tag{2}
$$

where $\to$ can be read as $\prec$. Let $\ulcorner \subset \square$ be the subposet obtained by removing the lower right corner 1×1, and let $\lrcorner \subset \square$ be the subposet containing all elements of $\square$ save for 0×0. Let $i_\ulcorner \colon \ulcorner \to \square$ and $i_\lrcorner \colon \lrcorner \to \square$ be the inclusions. An object A of $\mathcal{K}_\square$ is called homotopy cartesian if and only if the canonical morphism $A \to \operatorname{Ho}\operatorname{RKan}_{i_\lrcorner} i_\lrcorner^* A$ is an isomorphism. It is called homotopy cocartesian if and only if the canonical morphism $\operatorname{Ho}\operatorname{LKan}_{i_\ulcorner} i_\ulcorner^* A \to A$ is an isomorphism. As we already announced, the linearity axiom requires that an object of $\mathcal{K}_\square$ is homotopy cartesian if and only if it is homotopy cocartesian.

We first define the notion of a compatible system of triangulated $\mathfrak{P}$-diagram categories.

Definition 1. For $\mathcal{K} = \mathfrak{P}_U$, or $\mathfrak{K} = \tilde{\mathfrak{P}}_U$, the collection of data in (1) is called a system of triangulated $\mathfrak{K}$-diagram categories if the above axioms (the Functoriality Axiom, the Isomorphism Axiom, the Disjoint Union Axiom, the Mapping Cylinder Axiom, and the Linearity Axiom and the two Homotopy Kan Extension Axioms) are satisfied.

In the cases $\mathfrak{K} = \mathfrak{C}_U$ or $\mathfrak{K} = \tilde{\mathfrak{C}}_U$, I need another axiom about homotopy limits, which will be introduced below in Definition 3.

Let $\boldsymbol{C} \xrightarrow{f} \boldsymbol{D}$ be a functor between finite finite-dimensional categories. There is a unique functor $\boldsymbol{C}^\star \xrightarrow{f^\star} \boldsymbol{D}^\star$ which agrees with f on $\boldsymbol{C}$ and which sends $\star$ to $\star$. For the sake of simplicity, we will write f^* instead of $f^{\star *}$, $\operatorname{Ho}\operatorname{LKan}_f$ instead of $\operatorname{Ho}\operatorname{LKan}_{f^\star}$, and $\operatorname{Ho}\operatorname{RKan}_f$ instead of $\operatorname{Ho}\operatorname{RKan}_{f^\star}$. Obviously, we have $(fg)^\star = f^\star g^\star$. Moreover, if $\boldsymbol{C} \underset{R}{\overset{L}{\rightleftarrows}} \boldsymbol{D}$ are adjoint functors, then so are $\boldsymbol{C}^\star \underset{R^\star}{\overset{L^\star}{\rightleftarrows}} \boldsymbol{D}^\star$.

1.3. Examples. Although other examples could be given, the main examples which are needed for this paper are obtained from certain closed model categories in the sense of Quillen ([Qui67], [Qui69]).

1.3.1. *Diagram categories for a closed model category.* This is a generalisation of the appendix A to [BF78]. The following facts are probably well-known to the experts. Therefore, we make no claims to their originality. I understand that they will probably be contained in a work in progress by Dwyer and Kan, which will also be able to deal with infinite homotopy limits. If this preprint exists, it is well possible that this paragraph will be removed from the final version of our paper. Actually, it is quite surprising that the following facts never seem to have been published in a quotable form, more than 25 years after [Qui67].

Let $\mathcal{C}$ be a closed model category, and let $\boldsymbol{C}$ be a finite-dimensional finite category (for the moment, we do not assume that it posses initial or final objects). A morphism $F \xrightarrow{\phi} G$ in $\mathcal{C}^{\boldsymbol{C}}$ is called a weak equivalence (resp. a componentwise fibration or a componentwise cofibration) if, for each $X \in \boldsymbol{C}$, the map $F(X) \xrightarrow{\phi} G(X)$ is a weak equivalence (resp. a fibration or a cofibration) in $\mathcal{C}$. It is called a diagram fibration if it is a componentwise fibration and if, for each $X \in C$, the canonical morphism

$$F(X) \;\to\; G(X) \underset{\lim\limits_{C - X \leftarrow X} G(\cdot)}{\times} \lim_{C - X \leftarrow X} F(\cdot)$$

is a fibration in $\mathcal{C}$. A morphism $F \xrightarrow{\phi} G$ in $\mathcal{C}^{\boldsymbol{C}}$ is a diagram cofibration if it is a componentwise cofibration and if, for each $X \in \boldsymbol{C}$, the morphism

$$F(X) \underset{\operatorname*{colim}\limits_{C - X \to X} F(\cdot)}{\bigsqcup} \operatorname*{colim}_{C - X \to X} G(\cdot) \;\to\; G(X) \tag{1}$$

is a cofibration in $\mathcal{C}$.

Proposition 1. *Let $\boldsymbol{C}$ be a finite finite-dimensional category, and let $\mathcal{C}$ be a closed model category.*

a. *We equip $\mathcal{C}^{\boldsymbol{C}}$ with the family of weak equivalences defined above. Then the following two choices for the families of fibrations and cofibrations give $\mathcal{C}^{\boldsymbol{C}}$ the structure of a closed model category:*
   - *The closed model category $\mathcal{C}_f^{\boldsymbol{C}}$ having componentwise fibrations as fibrations and diagram cofibrations as cofibrations.*
   - *The closed model category $\mathcal{C}_c^{\boldsymbol{C}}$ having diagram fibrations as fibrations and componentwise cofibrations as cofibrations.*

b. *If a morphism $F \xrightarrow{\phi} G$ in $\mathcal{C}^{\boldsymbol{C}}$ is a diagram fibration (resp. a diagram fibration and a weak equivalence), then*

$$\lim_{\boldsymbol{C}} F \xrightarrow{\lim_{\boldsymbol{C}} \phi} \lim_{\boldsymbol{C}} G$$

is a fibration (resp. a trivial fibration) in $\mathcal{C}$. If ϕ is a diagram cofibration (resp. a diagram cofibration and a weak equivalence), then

$$\operatorname*{colim}_{\boldsymbol{C}} F \xrightarrow{\operatorname*{colim}_{\boldsymbol{C}} \phi} \operatorname*{colim}_{\boldsymbol{C}} G$$

is a cofibration (resp. a trivial cofibration) in $\mathcal{C}$.

Proof. We will prove both assertions simultaneously by induction on the dimension of $\boldsymbol{C}$, the case where this number is zero (i. e., all morphisms in $\boldsymbol{C}$ are identical) being trivial.

Within the set of categories of a given dimension d, we will also use induction on the number of objects of $\boldsymbol{C}$, starting with the impossible case that this number is zero.

Let both assertions be proved for all finite-dimensional finite categories of dimension $< d = \dim \boldsymbol{C}$, and also for all categories of dimension d with less objects than $\boldsymbol{C}$.

We prove that $\mathcal{C}_c^{\boldsymbol{C}}$ is a closed model category. We will use the axioms as they are formulated in [BF78] or [Qui69]. The verification of CM1–3 is trivial. To verify CM4, we have to find a lifting l for each diagram

$$\begin{tikzcd} \tilde{F} \arrow[r, "a"] \arrow[d, "i"'] & F \arrow[d, "p"] \\ \tilde{G} \arrow[r, "b"] \arrow[ur, dotted, "l"] & G \end{tikzcd} \tag{2}$$

when i is a componentwise cofibration, p is a diagram fibration, and at least one of p or i is a weak equivalence. Let

$$\alpha \to \alpha_1 \to \ldots \to \alpha_d$$

be a d-dimensional simplex in the nerve of $\boldsymbol{C}$. Then the induction assumption can be applied to $\boldsymbol{C}' = \boldsymbol{C} - \alpha$, since this category has at most the same dimension as and less objects than $\boldsymbol{C}$. Let a $'$ denote the restriction of functors to $\boldsymbol{C}'$. By the induction assumption, we find the dotted arrow $\tilde{G}' \xrightarrow{l'} F'$ for the restriction of (2) to $\boldsymbol{C}'$. We have to find

$$l_\alpha \colon \tilde{G}(\alpha) \to F(\alpha)$$

which is compatible with l' and makes (2) at α commutative. Let $\Phi = \lim_{C' \leftarrow \alpha} F(.) \in \mathcal{C}$ and $\Gamma = \lim_{C' \leftarrow \alpha} G(.)$. The conditions which l_α has to satisfy can be formulated as the commutativity of

$$\begin{tikzcd} \tilde{F}(\alpha) \arrow[r, "a"] \arrow[d, "i"'] & F(\alpha) \arrow[d] \\ \tilde{G}(\alpha) \arrow[r, "{(b,l')}"] \arrow[ur, dotted, "l_\alpha"] & G(\alpha) \underset{\Gamma}{\times} \Phi. \end{tikzcd} \tag{3}$$

The definition of a diagram fibration implies that the right vertical arrow in this diagram is a fibration. Therefore, the dotted arrow exists if i is a trivial cofibration. Otherwise, p has to be a weak equivalence, and the induction hypothesis applied to $C' \leftarrow x$ (whose dimension is strictly less than the dimension of $\boldsymbol{C}$) implies that $\Phi \to \Gamma$ is a trivial fibration in $\mathcal{C}$. Then the base change

$$G(\alpha) \underset{\Gamma}{\times} \Phi \to G(\alpha)$$

is a trivial fibration [Qui69, Corollary II.1.2], as is $F(\alpha) \to G(\alpha)$, and it follows that the right vertical arrow in (3) is a trivial fibration. Hence the dotted arrow exists in this case too, and the verification of CM4 for $\mathcal{C}_c^{\boldsymbol{C}}$ is complete.

To verify CM5, we have to factorise a morphism $F \xrightarrow{f} G$ in $\mathcal{C}_c^{\boldsymbol{C}}$ into a cofibration i and a fibration p, one of which has to be trivial. Let α and $'$ have the same meaning as in the proof of CM4. By the induction assumption, a factorisation of f' $F' \xrightarrow{i'} L' \xrightarrow{p'} G'$ with the desired properties exists. Here L' is not yet the restriction of some $\boldsymbol{C}$-diagram to $\boldsymbol{C}'$, but a $\boldsymbol{C}'$-diagram which was constructed by the induction assumption and which has to be extended to $\boldsymbol{C}$. Let

$$\begin{aligned}
\Phi &= \lim_{C' \leftarrow x} F(.) \\
\Lambda &= \lim_{C' \leftarrow x} L'(.) \\
\Gamma &= \lim_{C' \leftarrow x} G(.).
\end{aligned}$$

These objects, together with the obvious morphisms between them, form the solid part of the following diagram:

$$\begin{tikzcd}
 & G(\alpha) \arrow[r] & \Gamma \\
L(\alpha) \arrow[r, dotted, "q"] & \tilde{\Lambda} \arrow[u, "\tilde{\pi}"] \arrow[r] & \Lambda \arrow[u, "\pi"'] \\
 & F(\alpha) \arrow[u, "\tilde{\imath}"] \arrow[ul, dotted, "j"] \arrow[r] & \Phi \arrow[u, "\tilde{\imath}"']
\end{tikzcd}$$

Here $\tilde{\Lambda}$ is constructed in such a way that the upper square is cartesian. Let us first consider the case in which we want i to be a trivial cofibration. By an application of the induction hypothesis to $C' \leftarrow \alpha$, the projective limit π is a fibration. Hence the same applies to $\tilde{\pi}$. Choosing $L(\alpha)$ and the dotted arrows in such a way that q is a fibration and j is a trivial cofibration, we arrive at the desired factorisation. If we want p to be a trivial fibration, p' has already been constructed in that way. By the induction assumption and by [Qui69, Corollary II.1.2], its projective limit π and the base change $\tilde{\pi}$ are also trivial cofibrations. Choosing $L(\alpha)$ and the dotted arrows in such a way that j is a cofibration and q is a trivial fibration, we complete the verification of CM5. Therefore, $\mathcal{C}_c^{\boldsymbol{C}}$ is a closed model category. By duality, the same arguments apply to $\mathcal{C}_f^{\boldsymbol{C}}$.

It remains to prove the second part of the proposition for $\boldsymbol{C}$. By duality, it is sufficient to verify the assertion about right Kan extensions. The functor between closed model categories

$$\text{constant diagram}\colon \mathcal{C} \to \mathcal{C}_c^{\boldsymbol{C}}$$

preserves cofibrations and weak equivalences. By the characterisation of fibrations and trivial fibrations in terms of their lifting properties [Qui69, II.1.1.], its right adjoint $\lim_{\boldsymbol{C}}$ preserves fibrations and trivial fibrations. $\square$

Let $\operatorname{Ho}(\mathcal{C}^{\boldsymbol{C}})$ be the homotopy category obtained by inverting the weak equivalences. We will now assume that the initial and final objects of $\mathcal{C}$ coincide. We will just call them the zero object, and any morphism which factorises over it will be called the zero morphism. Then there is a canonical functor

$$\mathcal{C}^{\boldsymbol{C}} \to \mathcal{C}^{\boldsymbol{C}^\star}$$

which extends a $\boldsymbol{C}$-diagram to $\boldsymbol{C}^\star$ by sending the zero object and morphisms in $\boldsymbol{C}^\star$ to the zero object and morphisms in $\mathcal{C}$. Any functor $f\colon \boldsymbol{C}^\star \to \boldsymbol{D}^\star$ therefore defines a functor $f^*\colon \mathcal{C}^{\boldsymbol{D}} \to \mathcal{C}^{\boldsymbol{C}}$. It preserves weak equivalences, hence it defines a functor between homotopy categories denoted by the same letter.

Proposition 2. *The functor*

$$f^*\colon \operatorname{Ho}\Big(\mathcal{C}^{\boldsymbol{D}}\Big) \to \operatorname{Ho}\Big(\mathcal{C}^{\boldsymbol{C}}\Big)$$

has a left adjoint

$$\operatorname{Ho}\underset{f}{\operatorname{LKan}}\colon \operatorname{Ho}\Big(\mathcal{C}^{\boldsymbol{C}}\Big) \to \operatorname{Ho}\Big(\mathcal{C}^{\boldsymbol{D}}\Big)$$

and a right adjoint

$$\operatorname{Ho}\underset{f}{\operatorname{RKan}}\colon \operatorname{Ho}\Big(\mathcal{C}^{\boldsymbol{C}}\Big) \to \operatorname{Ho}\Big(\mathcal{C}^{\boldsymbol{D}}\Big).$$

If f has a right adjoint f_-, $\operatorname{Ho}\operatorname{LKan}_f \cong f_-^*$. If f has a left adjoint f_+, $\operatorname{Ho}\operatorname{RKan}_f \cong f_+^*$.

Proof. By duality, it suffices to consider $\operatorname{Ho}\operatorname{RKan}$. Let $\mathcal{C}_o^{\boldsymbol{C}^\star}$ be the full subcategory of $\mathcal{C}^{\boldsymbol{C}^\star}$ containing the functors which send $\star$ to the zero object in $\mathcal{C}$. It is equivalent to $\mathcal{C}^{\boldsymbol{C}}$. We first note that the right Kan extension along $f^\star$

$$\underset{f^\star}{\operatorname{RKan}}\colon \mathcal{C}_o^{\boldsymbol{C}^\star} \to \mathcal{C}_o^{\boldsymbol{D}^\star}$$

exists and is given by (1.1.1). If $f^\star$ has a left adjoint, the category over which the limit is taken in (1.1.1) has an initial object $f_+(X)$, hence $\operatorname{RKan}_f \cong f_+^*$ in this case.

An application of the following Lemma 1 to $\Phi = f^*$ and $\Gamma = \operatorname{RKan}_f$ proves the assertion. $\square$

Lemma 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be closed model categories, and let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be a functor which preserves cofibrations and weak equivalences and possesses a right adjoint Γ. Then the right derived functor*

$$R\Gamma\colon \mathrm{Ho}\mathcal{D} \to \mathrm{Ho}\mathcal{C}$$

exists and is right adjoint to the functor

$$\mathrm{Ho}\Phi = L\Phi\colon \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}.$$

Proof. We want to derive the assertion from [Qui67, Theorem I.4.3].

It follows easily from the characterisation of these classes in terms of lifting properties (cf. [Qui69, II.1.1.]) that Γ preserves the classes of fibrations and trivial fibrations in $\mathcal{D}$. We have to verify that Γ preserves weak equivalences between fibrant objects of $\boldsymbol{D}$. Since any weak equivalence factorises into a trivial cofibration followed by a trivial fibration, and since we already know that Γ preserves the latter class, it suffices to prove that Γ takes trivial cofibrations between fibrant objects of $\mathcal{D}$ to weak equivalences in $\mathcal{C}$.

We need the fact that Γ preserves path objects X^I of fibrant objects X of $\mathcal{D}$. Let $X \xrightarrow{s} X^I \xrightarrow{d_{0,1}} X$ be the constant path and beginning or end point morphisms. Then by the dual of [Qui67, lemma I.1.2] $d_{0,1}$ are trivial fibrations , which we know are preserved by Γ. Since $\Gamma(d_0)\Gamma(s)$ is the identity, it follows from the saturatedness of weak equivalences in $\mathcal{C}$ that $\Gamma(s)$ is a weak equivalence. Therefore, $\Gamma(X^I)$ is a path object and it follows in view of the dual of [Qui67, lemma I.1.1] that Γ preserves right homotopies between morphisms to X.

Now if f is a trivial cofibration between fibrant objects of $\mathcal{D}$, it has a inverse g up to right homotopy [Qui67, lemma I.1.7], and we have just seen that this implies that $\Gamma(g)$ and $\Gamma(f)$ are inverse up to right homotopies. It follows $\Gamma(f)$ is invertible in $\mathrm{Ho}\mathcal{C}$, hence it is a weak equivalence by [Qui67, Proposition I.5.1].

The verification of the assumptions of [Qui67, Theorem I.4.3] is now complete. □

As a consequence of Proposition 2, we can define notions of homotopy fibre products and coproducts in any closed model category $\mathcal{C}$. Therefore, we also have a notion of homotopy cartesian or cocartesian squares. In the case where $\mathcal{C}$ is proper, this notion coincides with the one in [BF78, Appendix A]. Since the homotopy category of a closed model category depends on the family of weak equivalences alone, it follows that our notion of homotopy cartesianness or cocartesianness also depends only on the family of weak equivalences (and, of course, the underlying category $\mathcal{C}$ itself).

1.3.2. *Linear Closed model categories.*

Definition 2. A closed model category $\mathcal{C}$ is called linear if the following conditions hold:

- The morphism from the initial to the final object of $\mathcal{C}$ is an isomorphism. In other words, the category is pointed.
- A commutative square in $\mathcal{C}$ is homotopy cartesian if and only if it is homotopy cocartesian.

By the remark made above, this condition depends only on the family of weak equivalences. We will now check that a closed model category with this property gives rise to a system of triangulated diagram categories. Let $\mathcal{K}_{\boldsymbol{C}} = \mathrm{Ho}\mathcal{C}^{\boldsymbol{C}}$ with the pull-back functors f^* introduced before the formulation of Proposition 2, and the obvious natural transformations between them. The Functorialtiy and Disjoint Union Axioms are trivial, the Homotopy Kan Extension Axioms follow from Proposition 2, and the Linearity Axiom follows from the definition of a linear closed model category. It remains to verify the mapping cylinder axiom. The fact that the functor

$$\mathcal{K}_{\boldsymbol{C}\times\underline{1}} \to \mathfrak{Ar}(\mathcal{K}_{\boldsymbol{C}})$$

is surjective on isomorphism classes of objects is clear from our definitions. To see that it is full, it suffices to show that every commutative square in the homotopy category of $\mathcal{C}$ comes from a commutative square in $\mathcal{C}$. Obviously, it comes from a square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd} \tag{4}$$

of fibrant and cofibrant objects of $\mathcal{C}$ which commutes up to homotopy. It follows easily from the axioms of a closed model category that we may assume γ to be a fibration. Let $i_{0,1}\colon A \to A \times I$ be a cylinder object for A and let $H\colon A \times I \to D$ be a homotopy between $Hi_0 = \delta\beta$ and $Hi_1 = \gamma\alpha$. Choosing a lifting $\tilde{H}\colon A \times I \to B$ of H with $\tilde{H}i_1 = \alpha$ and replacing α by $\tilde{H}i_0$, we can make (4) commutative in $\mathcal{C}$. This completes the verification of the axioms.

1.3.3. *Examples of linear closed model categories.* Our first example concerns closed model categories of cochain complexes. It is a generalisation of the unbounded derived category of an abelian category with sufficiently many K-injective cochain complexes [Spa88].

Let $\mathcal{A}$ be an abelian category together with an equivalence of categories $T\colon \mathcal{A} \to \mathcal{A}$, and let $N > 1$ be a natural number. A (T, N)-periodic complex C is a pair (C^*, α_C) consisting of a cochain complex in $\mathcal{A}$ together with an isomorphism of complexes $\alpha_C\colon C^* \cong C^*[N]$. For arbitrary cochain complexes E^*, F^* in $\mathcal{A}$, let $\mathfrak{Hom}^k(E^*, F^*)$ be the group of morphisms of graded $\mathcal{A}$-objects from E^* to $F^*[k]$. These groups form the cochain complex $\mathfrak{Hom}^\star(E^*, F^*)$ with the usual differential (see for instance [Spa88, 0.4(2)] If (C^*, α_C) and (D^*, α_D) are (T, N)-periodic complexes, let $\mathfrak{Hom}^\star(C, D) \subseteq \mathfrak{Hom}^*(C^*, D^*)$ be the subcomplex of those ϕ satisfying $\phi[N]\alpha_C = \alpha_D\phi$. Let $\mathcal{C}^{(T,N)}(\mathcal{A})$ be the category of (T, N)-periodic complexes, with homomorphisms $\mathrm{Hom}_{(T,N)}(C, D) = Z^0(\mathfrak{Hom}^*_{(T,N)}(C, D))$, the homomorphisms of cochain complexes ϕ satisfying $\phi[N]\alpha_C = \alpha_D\phi$. Let an object C of $\mathcal{C}^{(T,N)}(\mathcal{A})$ be called injective if each C^i is an injective object, acyclic if the cochain complex C^* is acyclic, K-injective if for each acyclic D, the cochain complex $\mathfrak{Hom}^*_{(T,N)}(D, C)$ is acyclic, and strictly injective if it is both injective and K-injective. A morphism in $\mathcal{C}^{(T,N)}(\mathcal{A})$ is called a quasi-isomorphism if it induces an isomorphism on cohomology. We say that there are sufficiently many strictly injective complexes if for every (T,N)-periodic chain complex C there is a strictly injective I and an embedding $C \to I$ which is a quasi-isomorphism. It is easy to see that this condition implies that $\mathcal{A}$ has sufficiently many injective objects. We say that there are sufficiently many K-injective objects if for every (T,N)-periodic complex C there exists a K-injective D and a quasi-isomorphism $C \to D$ (which can be assumed to be a componentwise split monomorphism). A (T,N)-periodic homotopy between two morphisms in $\mathcal{C}^{(T,N)}(\mathcal{A})$ is a homotopy between the corresponding morphisms of ordinary chain complexes which belongs to $\mathfrak{Hom}^{-1}_{(T,N)}(C,D)$. A morphism in $\mathcal{C}^{(T,N)}(\mathcal{A})$ is a homotopy equivalence if it is invertible up to (T,N)-periodic homotopy. It is a componentwise split monomorphism (resp. epimorphism) if each $C^i \to D^i$ is split monomorphism (resp. epimorphism).

Proposition 3.
- *If $\mathcal{A}$ has sufficiently many injective objects and finite cohomological dimension, then there are sufficiently many strictly injective (T,N)-periodic complexes.*
- *If there are sufficiently many strictly (resp. K-) injective (T,N)-periodic objects, then $\mathcal{C}^{(T,N)}(\mathcal{A})$ becomes a linear closed model category with monomorphisms (resp. componentwise split monomorphisms) as cofibrations, componentwise split epimorphisms with strictly (resp. K-) injective kernel as fibrations, and quasi-isomorphism as weak equivalences. Note that an epimorphism with injective kernel is automatically componentwise split.*
- *$\mathcal{C}^{(T,N)}(\mathcal{A})$ becomes a linear closed model category with componentwise split monomorphisms (resp. epimorphisms) as cofibrations (resp. fibrations) and homotopy equivalences as weak equivalences.*

Proof. At the price of replacing $\mathcal{A}$ by N copies of itself, we can assume for the sake of simplicity that $N = 1$.

For every injective object I of $\mathcal{A}$, the (T,N)-periodic complexes

$$V(I)^N = T^N I, \quad d = 0,$$

and

$$C(I)^N = T^N I \oplus T^{N-1} I, \quad d = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix},$$

with α defined in the tautological way, are easily seen to be strictly injective.

To prove the first assertion, we note that if $\mathcal{C}$ has finite injective dimension, then for every injective complex C with injective H^0, it follows easily that B^0 and Z^0 are also injective and that $C \cong V(H^0(C)) \oplus C(B^0(C))$ is strictly injective. Let D be the injective dimension. For every (T,N)-periodic cochain complex C, we may choose injective objects I and J and monomorphisms $C^0 \to I$ and $C^0/B^0 \to J$ defining a morphism $C \to K^{(0)} C(I) \oplus V(J)$ of (T,N)-periodic complexes which is injective and defines a monomorphism on cohomology. Iterating this procedure, we get a resolution $C \to K^{(0)} \to K^{(1)} \to \ldots \to K^{(D-1)}$, which is exact save for possibly at the last term and has the additional properties that each $K^{(i)}$ is strictly injective with injective cohomology and that $H^0(K^{(i)}) \to H^0(K^{(i+1)})$ is injective. The cokernel $K^{(D)}$ of $K^{(D-2)} \to K^{(D-1)}$ is therefore injective with injective cohomology, hence strictly injective by the above remark. Let K be the total complex of the double complex $K^{(0)} \to \ldots \to K^{(D)}$, it is strictly injective and the inclusion $C \to K$ is an embedding and a weak equivalence.

For the second and third assertions, the verification of the closed model axioms CM1–3 is trivial. Since the construction of the mapping cylinder of a morphism of cochain complexes carries over to the (T,N)-periodic case, every morphism of (T,N)-periodic complexes can be factorised as an componentwise split monomorphism followed by a componentwise split epimorphism which is a homotopy equivalence. This proves one half of CM5 for the closed model structure described in the third assertion, and the other half follows by duality. In particular, every morphism of (T,N)-periodic complexes can be factorised as a componentwise split monomorphism which is a weak equivalence followed by an epimorphism. In the case of the second assertion, it is therefore sufficient to verify the factorisation axiom CM5 for epimorphisms $C \xrightarrow{f} D$. Choosing a monomorphism $\left(\ker(f)\right)^0 \to I$ into an injective object, we obtain an injection $\ker(f) \xrightarrow{i} C(I)$ into an acyclic strictly injective complex (resp. the componentwise split monomorphism $\ker(f) \xrightarrow{i} C(\ker(f)^0)$ into a contractible and therefore K-injective complex). Let $C \xrightarrow{\tilde{i}} E$ be the push-out of f along i, then $\tilde{i}$ is a cofibration and $E \to D$ is a trivial fibration. Let $\ker(f) \xrightarrow{j} J$ be a quasi-isomorphic monomorphism into a strictly injective (T,N)-periodic complex J (resp. a componentwise split quasi-isomorphic monomorphism into a K-injective complex), and let $C \xrightarrow{\tilde{j}} F$ be the push-out. Then $\tilde{j}$ is a trivial cofibration and $F \to D$ is a fibration.

Let $A \xrightarrow{f} B \xrightarrow{p} D$ and $A \xrightarrow{i} C \xrightarrow{g} D$, $pf = gi$, be a commutative square with a fibration p and a cofibration i, one of which is a weak equivalence. To verify CM4 for this square, we note that in all cases the morphism of cochain complexes

$$\mathfrak{Hom}^*_{(T,N)}(C,B) \xrightarrow{(i,p)} H = \mathfrak{Hom}^*_{(T,N)}(A,B) \underset{\mathfrak{Hom}^*_{(T,N)}(A,D)}{\times} \mathfrak{Hom}^*_{(T,N)}(C,D). \tag{5}$$

is surjective. Indeed, this is easily seen to be the case if both i and p are componentwise split, or if $\ker(p)$ is injective and if i is a monomorphism, and this covers all the cases we need. If we can prove that (5) induces an isomorphism on cohomology, then it induces a surjection on Z^0, and the diagonal $C \to B$ for the square exists. But the kernel of (5) is

$$\mathfrak{Hom}^*_{(T,N)}(\mathrm{coker}(i), \ker(p)), \tag{6}$$

and it is acyclic if $\mathrm{coker}(i)$ is acyclic and $\ker(p)$ is K-injective, or if one of $\ker(p)$ or $\mathrm{coker}(i)$ is contractible, and again this covers all the cases we need.

To verify the linearity axiom, we first note that the usual facts about mapping cones generalise to the (T,N)-periodic case. In particular for $M \xrightarrow{f} N$ there are morphisms

$$\operatorname{cone}(f)[-1] \xrightarrow{e} M \xrightarrow{f} N \xrightarrow{g} \operatorname{cone}(f)$$

and a morphism $\ker(f) \to \operatorname{cone}(f)[-1]$ which is a quasi-isomorphism (resp. homotopy equivalence) if f is an epimorphism (resp. a componentwise split epimorphism) and a morphism $\operatorname{cone}(f) \to \operatorname{coker}(f)$ which is a quasi-isomorphism (resp. homotopy equivalence) if f is a (componentwise split) monomorphism. In particular, there are homotopy equivalences

$$\begin{aligned} \operatorname{cone}(g) &\to \operatorname{coker}(g) = M[1] \\ \operatorname{cone}(e) &\leftarrow \ker(e)[1] = N. \end{aligned} \tag{7}$$

Let $A_{01} \xleftarrow{\alpha} A_{00} \xrightarrow{\beta} A_{10}$ be an object of $\mathcal{C}^{(T,N)}(\mathcal{A})^{\ulcorner}$. We claim that for all three closed model structures we are considering, $\operatorname{Ho}\operatorname{LKan}_{i_\ulcorner} A$ is given by the commutative square

$$\begin{tikzcd} A_{00} \arrow[r] \arrow[d] & \operatorname{cone}(A_{00} \to A_{00} \oplus A_{01}) \arrow[d] \\ \operatorname{cone}(A_{00} \to A_{10} \oplus A_{00}) \arrow[r] & \operatorname{cone}(A_{00} \to A_{10} \oplus A_{01}.) \end{tikzcd} \tag{8}$$

Indeed, there is a natural homomorphism from $\operatorname{LKan}_{i_\ulcorner} A$ to (8) which is a quasi-isomorphism (resp. a homotopy equivalence) if $A_{00} \to A_{01} \oplus A_{10}$ is a (componentwise split) monomorphism.

In a similar way, one verifies that for an object A of $\mathcal{C}^{(T,N)}(\mathcal{A})^{\lrcorner}$ given by

$$A_{01} \to A_{11} \leftarrow A_{10},$$

we have

$$\Big(\operatorname{Ho}\operatorname*{RKan}_{i_\lrcorner} A\Big)_{00} \cong \operatorname{cone}(A_{01} \oplus A_{10} \to A_{11})[-1].$$

It follows that a commutative square of (T,N)-periodic complexes A is homotopy cartesian if and only if

$$A_{00} \to \operatorname{cone}(A_{01} \oplus A_{10} \to A_{11})[-1]$$

is a quasi-isomorphism (resp. a homotopy equivalence), and it is homotopy cocartesian if and only if

$$A_{11} \leftarrow \operatorname{cone}(A_{00} \to A_{10} \oplus A_{01})$$

is a quasi-isomorphism (resp. a homotopy equivalence). In view of (7), these two conditions are equivalent. □

Corollary 1. *If there are sufficiently many K-injective complexes, then the system of categories*

$$\mathcal{D}^{(T,N)}_{\boldsymbol{C}}(\mathcal{A}) = \mathrm{Ho}_{\mathrm{Quis}}\mathcal{C}^{(T,N)}(\mathcal{A}^{\boldsymbol{C}}), \tag{9}$$

where Quis *is the set of quasi-isomorphisms, is a system of triangulated diagram categories. In the general case, the system of categories*

$$\mathcal{K}^{(T,N)}_{\boldsymbol{C}}(\mathcal{A}) = \mathrm{Ho}_{\mathrm{He}}\mathcal{C}^{(T,N)}(\mathcal{A}^{\boldsymbol{C}}), \tag{10}$$

where He *is the set of homotopy equivalences, is a system of triangulated diagram categories. We will write $\mathcal{D}^{(T,N)}(\mathcal{A})$ and $\mathcal{K}^{(T,N)}(\mathcal{A})$ for $\mathcal{D}^{(T,N)}_{\underline{0}}(\mathcal{A})$ and $\mathcal{K}^{(T,N)}_{\underline{0}}(\mathcal{A})$.*

Remark 1. Let $\mathcal{B}$ be an abelian category with sufficiently many injective objects. Let $\mathcal{A}$ be the category of graded $\mathcal{B}$-objects which are bounded from below, and let T be the shift functor. Then every injective complex is K-injective, and we may apply the first example of the second assertion of above proposition (this example is also given in [Qui67]), and we obtain the system of categories $\mathcal{D}^+(\mathcal{B})$. If the cohomological dimension is finite, we can also take $\mathcal{A}$ equal to the category of bounded graded $\mathcal{B}$-objects, obtaining the bounded derived category. If there are enough strictly injective $\mathcal{B}$-complexes (for instance, if the homological dimension is finite or in the cases covered by [Spa88]), then we can take the category of all graded $\mathcal{B}$-objects as $\mathcal{A}$, obtaining $\mathcal{D}(\mathcal{A})$. For the application to stable homotopy, we will however be in a situation where $\mathcal{D}^{(T,N)}(\mathcal{A})$ has no t-structure.

In Corollary 1.5.1, we will see that (9) exists and forms a system of triangulated diagram paracategories (in the sense of Remark 1.4.3), which will usually (but not always) be actual categories. Note that while

$$\mathcal{D}^{(T,N)}_{\boldsymbol{C}}(\mathcal{A}) \cong \mathcal{D}^{(T,N)}(\mathcal{A}_{\boldsymbol{C}}),$$

it is not true that $\mathcal{K}^{(T,N)}_{\boldsymbol{C}}(\mathcal{A}) \cong \mathcal{D}^{(T,N)}(\mathcal{K}_{\boldsymbol{C}})$.

In [Adl96] third point has been generalized to pretriangulated DG-categories in the sense of Bondal and Kapranov [BK91].

Another example of a linear closed model category is the category of spectra, equipped with a closed model structure by [BF78]. That this category is linear easily follows from the fact [Ada74, III.3.10] that in the category of spectra the families of homotopy fibre and cofibre sequences coincide.

1.3.4. *The opposite system of categories.* If $\mathcal{K}$ is a system of triangulated diagram categories, then so is

$$\mathcal{K}^{\mathrm{op}}_{\boldsymbol{C}} = \left(\mathcal{K}_{\boldsymbol{C}^{\mathrm{op}}}\right)^{\mathrm{op}}. \tag{11}$$

1.4. Consequences of the axioms. Let $\mathfrak{K}$ be one of the 2-categories $\mathfrak{P}_U$, $\tilde{\mathfrak{P}}_U$, $\mathfrak{C}_U$, or $\tilde{\mathfrak{C}}_U$, where U is an infinite class. In the case $\mathfrak{K} = \mathfrak{P}_U$ or $\tilde{\mathfrak{P}}_U$, let $\mathcal{A}$ be a system of triangulated $\mathfrak{K}$-diagram categories. In the case $\mathfrak{K} = \mathfrak{C}_U$, or $\tilde{\mathfrak{C}}_U$, we have yet to impose another axiom, so we assume for the moment that all axioms formulated so far are satisfied.

1.4.1. *Properties of the homotopy Kan extension functors.* Let

$$\begin{tikzcd} \boldsymbol{C}^\star \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{p}"'] & \boldsymbol{D}^\star \arrow[d, "p"] \\ \boldsymbol{E}^\star \arrow[r, "f"] & \boldsymbol{F}^\star \end{tikzcd}$$

be a commutative diagram. From

$$\begin{aligned} \mathrm{Id} \in \mathrm{Hom}(\mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan}, \mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan}) &\cong \mathrm{Hom}(p^* \,\mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan}, \mathrm{Id}) \\ &\to \mathrm{Hom}(\tilde{f}^* p^* \,\mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan}, \tilde{f}^*) \\ &\cong \mathrm{Hom}(\tilde{p}^* f^* \,\mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan}, \tilde{f}^*) \\ &\cong \mathrm{Hom}(f^* \,\mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan}, \mathrm{Ho}\,\underset{\tilde{p}}{\mathrm{R}}\mathrm{Kan}\,\tilde{f}^*) \end{aligned}$$

we derive a base change morphism

$$f^* \,\mathrm{Ho}\,\underset{p}{\mathrm{R}}\mathrm{Kan} \to \mathrm{Ho}\,\underset{\tilde{p}}{\mathrm{R}}\mathrm{Kan}\,\tilde{f}^* \tag{1}$$

and similar morphisms

$$\mathrm{Ho}\,\underset{\tilde{f}}{\mathrm{L}}\mathrm{Kan}\,\tilde{p}^* \to p^* \,\mathrm{Ho}\,\underset{f}{\mathrm{L}}\mathrm{Kan} \tag{2}$$

$$p^* \,\mathrm{Ho}\,\underset{f}{\mathrm{R}}\mathrm{Kan} \to \mathrm{Ho}\,\underset{\tilde{f}}{\mathrm{R}}\mathrm{Kan}\,\tilde{p}^* \tag{3}$$

$$\mathrm{Ho}\,\underset{\tilde{p}}{\mathrm{L}}\mathrm{Kan}\,\tilde{f}^* \to f^* \,\mathrm{Ho}\,\underset{p}{\mathrm{L}}\mathrm{Kan}\,. \tag{4}$$

Proposition 1. a. *The base change morphism* (1) *is an isomorphism if and only if* (2) *is. Moreover, this is the case when f and $\tilde{f}$ have right adjoints f_- and $\tilde{f}_-$ satisfying $f_- p = \tilde{p}\tilde{f}_-$.*

b. *The base change morphism* (4) *is an isomorphism if and only if* (3) *is. Moreover, this is the case when f and $\tilde{f}$ have left adjoints f_+ and $\tilde{f}_+$ satisfying $f_+ p = \tilde{p}\tilde{f}_+$.*

Proof. Since (1) is adjoint to (2), the fact that one of them is a functorisomorphism implies that the other one also is an isomorphism. The similar relation between (4) and (3) follows by interchanging the roles of f and p. If f_- and $\tilde{f}_-$ exist and have the required properties, then it follows from the second homotopy Kan extension axiom that (2) is an isomorphism. The same applies to (3) if f_+ and $\tilde{f}_+$ exist and have the required properties. $\square$

Proposition 2. *Let $F\colon \boldsymbol{D} \to \boldsymbol{C}$ be a functor, let $X \in \boldsymbol{C}$, and let*

$$\begin{aligned} i_X &\colon (\boldsymbol{C} \to X) \to \boldsymbol{C} \\ j_X &\colon (F \Rightarrow X) \to \boldsymbol{D} \\ \tilde{F} &\colon (F \Rightarrow X) \to (\boldsymbol{C} \to X) \end{aligned}$$

be the canonical functors. Assume that $\boldsymbol{C}$ is a poset. Then for $A \in \mathcal{K}_{\boldsymbol{D}}$, we have an isomorphism

$$\begin{aligned} (\operatorname{Ho} \underset{F}{\mathrm{LKan}}\, A)_X &\cong \underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}\to X} i_X^* \operatorname{Ho} \underset{F}{\mathrm{LKan}}\, A \\ &\cong \underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}\to X} \operatorname{Ho} \underset{\tilde{F}}{\mathrm{LKan}}\, j_X^* A \\ &\cong \underrightarrow{\mathrm{Holim}}_{F\Rightarrow X} j_X^* A. \end{aligned} \tag{5}$$

A dual assertion holds for projective homotopy limits and right homotopy Kan extensions.

Proof. The first morphism in (5) is an isomorphism given by the second homotopy Kan extension axiom. The third of these morphisms is a tautological isomorphism. It remains to verify that the second morphism, which is a base change morphism, is an isomorphism.

Since $\boldsymbol{C}$ is a poset, $i_X^\star$ has a right adjoint i_{X-} which is a morphism in $\tilde{\mathfrak{P}}$ and sends every object of $\boldsymbol{C}$ to itself if it is $\preceq X$, and to $\star$ otherwise. Similarly, j_X has a right adjoint j_{X-} which sends $Y \in D$ to zero if $f(Y) \not\preceq X$, and to itself otherwise, with the expected behaviour on morphisms. It is straightforward to check $\tilde{F} j_{X-} = i_{X-} F$. The assertion therefore follows from Proposition 1. □

We are now ready to define systems of triangulated $\mathfrak{K}$-diagram categories, where $\mathfrak{K}$ is $\mathfrak{C}_U$ or $\tilde{\mathfrak{C}}_U$. In this case, we impose another condition which in the case $\mathfrak{K} = \mathfrak{P}$ was just derived from the other axioms.

Definition 3. For $\mathfrak{K} = \mathfrak{C}$, or $\mathfrak{K} = \tilde{\mathfrak{C}}$, the collection of data in (1.2.1) is called a system of triangulated $\mathfrak{C}$-diagram categories if the Functoriality Axiom, the Isomorphism Axiom, the Disjoint Union Axiom, the Mapping Cylinder Axiom, the Homotopy Kan extension axioms and the Linearity Axiom are satisfied and in addition the assertion of the last proposition is true in full generality, without the assumption that $\boldsymbol{C}$ is a poset.

In the case where F is the inclusion of a full subcategory $\boldsymbol{D} \subseteq \boldsymbol{C}$, then for every $X \in \mathfrak{Ob}(\mathfrak{D})$ the category $F \Rightarrow X$ has a final object (X, Id_X). Therefore:

Corollary 1. *Let $\boldsymbol{C} \xrightarrow{F} \mathfrak{D}$ be the inclusion of a full subcategory. Then the canonical morphisms in $\mathcal{K}_{\boldsymbol{D}}$*

$$\begin{gathered} A \longrightarrow F^* \operatorname{Ho} \underset{F}{\mathrm{LKan}}\, A \\ F^* \operatorname{Ho} \underset{F}{\mathrm{RKan}}\, A \longrightarrow A \end{gathered}$$

are isomorphisms for every object A of $\mathcal{K}_{\boldsymbol{D}}$.

Remark 1. In the case of a $\mathfrak{C}$-system, the assumption that the assertion of the last proposition is valid cannot be derived from the other axioms. Indeed, for every finite finite-dimensional category $\boldsymbol{C}$, let $\boldsymbol{C}^{(2)}$ be the category which has the same objects as $\boldsymbol{C}$ and

$$\operatorname{Hom}_{\boldsymbol{C}^{(2)}}(X,Y) = \operatorname{Hom}_{\boldsymbol{C}}(X,Y) \times \operatorname{Hom}_{\boldsymbol{C}}(X,Y)$$

as morphisms. If $\boldsymbol{C}^\star \xrightarrow{f} \boldsymbol{D}^\star$ is a functor, then let $\boldsymbol{C}^{(2)\star} \xrightarrow{f^{(2)}} \boldsymbol{D}^{(2)\star}$ be the functor preserving $\star$ which agrees with f on the set of objects and sends a morphism (a,b) in $\boldsymbol{C}^{(2)}$ to zero if f sends one of the two morphisms a and b in $\boldsymbol{C}$ to zero, and to $(f(a), f(b))$ otherwise. If a $\mathfrak{C}$-system $\mathcal{K}$ is given, let $\mathcal{K}^{(2)}$ be defined by $\mathcal{K}^{(2)}_{\boldsymbol{C}} = \mathcal{K}_{\boldsymbol{C}^{(2)}}$. The functor

$$\mathcal{K}^{(2)}_{\boldsymbol{D}} \xrightarrow{f^*} \mathcal{K}^{(2)}_{\boldsymbol{C}}$$

is given by

$$\mathcal{K}_{\boldsymbol{D}^{(2)}} \xrightarrow{f^{(2)*}} \mathcal{K}_{\boldsymbol{C}^{(2)}}.$$

It is easy to see that $\mathcal{K}^{(2)}$ satisfies the Functoriality Axiom, the Disjoint Union Axiom, the Mapping Cylinder Axiom, the two Homotopy Kan Extension Axioms, and the Linearity Axiom if they are satisfied for $\mathcal{K}$. On the other side, the assertion of Proposition 2 applied to the functor $\underline{0} \xrightarrow{i_X} \boldsymbol{C}$ asserts that the canonical morphism

$$\sum_{\alpha \in \operatorname{Hom}_{\boldsymbol{C}}(X,Y)} A \to \left(\operatorname{Ho}\operatorname{LKan}^{(\mathcal{K})}_{i_X} A\right)_Y \tag{6}$$

is an isomorphism. This gives

$$\sum_{(\alpha,\beta) \in \operatorname{Hom}_{\boldsymbol{C}}(X,Y) \times \operatorname{Hom}_{\boldsymbol{C}}(X,Y)} A \cong \left(\operatorname{Ho}\operatorname{LKan}^{(\mathcal{K}^{(2)})}_{i_X} A\right)_Y, \tag{7}$$

and this isomorphism identifies the counterpart of (6) for $\mathcal{K}^{(8)}$

$$\sum_{\alpha \in \operatorname{Hom}_{\boldsymbol{C}}(X,Y)} A \to \left(\operatorname{Ho}\operatorname{LKan}^{(\mathcal{K}^{(2)})}_{i_X} A\right)_Y \tag{8}$$

with the embedding of the summands corresponding to pairs of the form (α, α) in (7). Therefore, (8) can be an isomorphism only if A is the zero object. Thus, if $\mathcal{K}$ satisfies all the assumptions of Definition 3 and $\mathcal{K}_{\underline{0}}$ is not the zero category, then $\mathcal{K}^{(2)}$ violates Proposition 2 but satisfies all the other assumptions of Definition 3.

The last proposition can often be used to reduce assertions about the functors $\operatorname{Ho}\operatorname{LKan}_f$ and $\operatorname{Ho}\operatorname{RKan}_f$ to the similar assertions about $\underrightarrow{\operatorname{Holim}}_{\boldsymbol{C}}$ and $\underleftarrow{\operatorname{Holim}}_{\boldsymbol{C}}$. The first part of the following proposition is concerned with the question of replacing $\boldsymbol{C}$ by a smaller category.

Proposition 3. *a. Let $i\colon \boldsymbol{C}^\star \to \boldsymbol{D}^\star$ be some functor (typically the inclusion of a subcategory). If i has a left adjoint of the form $l^\star$, where $\boldsymbol{D} \xrightarrow{l} \boldsymbol{C}$, then $\underrightarrow{\operatorname{Holim}}_{\boldsymbol{C}} A \cong \underrightarrow{\operatorname{Holim}}_{\boldsymbol{D}} i^* A$. If i has a right adjoint of the form $r^\star$ for some functor $\boldsymbol{D} \xrightarrow{r} \boldsymbol{C}$, then $\underleftarrow{\operatorname{Holim}}_{\boldsymbol{D}} A \cong \underleftarrow{\operatorname{Holim}}_{\boldsymbol{C}} i^* A$.*

b. Let $\boldsymbol{D}$ be a prefibred category over $\boldsymbol{C}$, and let $f\colon \boldsymbol{D} \to \boldsymbol{C}$ be the projection, then for any $X \in \boldsymbol{C}$ and any $A \in \mathcal{K}_{\boldsymbol{D}}$ we have

$$\Big(\operatorname{Ho}\operatorname*{RKan}_f A\Big)_X \cong \underleftarrow{\operatorname{Holim}}_{f^{-1}(X)} j_X^* A,$$

where $j_X\colon f^{-1}(X) \to \boldsymbol{D}$ is the inclusion. Similarly, if $\boldsymbol{D}$ is a precofibred category over $\boldsymbol{C}$, then

$$\Big(\operatorname{Ho}\operatorname*{LKan}_f A\Big)_X \cong \underrightarrow{\operatorname{Holim}}_{f^{-1}(X)} j_X^* A.$$

Proof. For the first point, let l be a left adjoint of i, then $\underrightarrow{\operatorname{Holim}}_{\boldsymbol{C}} \cong \underrightarrow{\operatorname{Holim}}_{\boldsymbol{D}} \operatorname{Ho}\operatorname{LKan}_l \cong \underrightarrow{\operatorname{Holim}}_{\boldsymbol{D}} i^*$ by the second homotopy Kan extension axiom. The assertion about $\underleftarrow{\operatorname{Holim}}$ follows from the dual considerations.

For the second point, the definition of the condition that $\boldsymbol{D}$ is a prefibred category over $\boldsymbol{C}$ in [SGA1, Exp. VI, Definition 6.1], is equivalent to the assertion that the inclusion $\big(f^{-1}(X)\big) \to (f \Leftarrow X)$ has a right adjoint (cf. [Qui73, §1, after the formulation of Theorem A]). This allows us to apply the first part of the proposition. Again, the case of $\underrightarrow{\operatorname{Holim}}$ is dual. $\square$

1.4.2. *The extended linearity axiom.* Let an object of $\mathcal{K}_{\boldsymbol{C}\times\square}$ be called homotopy cartesian if

$$A \to \operatorname{Ho}\operatorname*{RKan}_{\mathrm{Id}_{\boldsymbol{C}}\times i_\lrcorner}(\mathrm{Id}_{\boldsymbol{C}} \times i_\lrcorner)^* A$$

is an isomorphism, and let cocartesianness be defined in the same way, replacing $\lrcorner$ by $\ulcorner$ and Ho RKan by Ho LKan and reversing the direction of the arrow. Since $\boldsymbol{C} \times \square$ is both fibrant and cofibrant over $\boldsymbol{C}$, it follows from Proposition 3.b. and the isomorphism axiom that an object A of $\mathcal{K}_{\boldsymbol{C}\times\square}$ is homotopy cartesian (resp. cocartesian) if and only if for every $X \in \boldsymbol{C}$, the object $j_X^* A$ of $\mathcal{K}_\square$ is homotopy cartesian (resp. cocartesian). By the linearity axiom, we arrive at the first part of the following proposition.

Proposition 4. *An object of $\mathcal{K}_{\boldsymbol{C}\times\square}$ is homotopy cartesian if and only if it is homotopy cocartesian. Consequently, for any $\boldsymbol{D} \in \mathfrak{K}$ the system $\mathcal{L}_{\boldsymbol{C}} = \mathcal{K}_{\boldsymbol{D}\times\boldsymbol{C}}$ is also a system of triangulated $\mathfrak{K}$-diagram categories.*

The second part of this proposition follows from the first part since all the other axioms obviously hold for the system $\mathcal{L}$.

From now on, we will use the term bicartesian for the equivalent properties of being homotopy cartesian and cocartesian.

Definition 4. A square in $\boldsymbol{C}$ is a functor $i\colon \square \to \boldsymbol{C}$ which is injective on the set of objects. Let $A \in \mathcal{K}_{\boldsymbol{C}}$, we say that A makes the square homotopy bicartesian if i^*A is homotopy bicartesian.

If in the following considerations (x) is the equation number of the definition of a poset by a commutative diagram, then by a visible square in (x) we will understand a square which actually becomes visible as an ordinary geometric square in (x).

Proposition 5. *Let i be a square in $\boldsymbol{C}$. If the functor $\ulcorner \to (\boldsymbol{C}-i(1\times 1)\to i(1\times 1))$ possesses a left adjoint and if $A = \operatorname{Ho}\operatorname{LKan}_f B$, where $f\colon \boldsymbol{D} \to \boldsymbol{C}$ is a functor not containing $i(1\times 1)$ in its image, then A makes i bicartesian. The same holds if the functor $\lrcorner \to (\boldsymbol{C}-i(0\times 0)\leftarrow i(0\times 0))$ possesses a right adjoint and if $A = \operatorname{Ho}\operatorname{RKan}_f B$, where $f\colon \boldsymbol{D}\to\boldsymbol{C}$ is a functor not containing $i(0\times 0)$ in its image.*

Proof. It suffices to prove the first assertion. Let $j\colon \boldsymbol{C} - i(1\times 1) \to \boldsymbol{C}$ be the inclusion. By our assumption on the image of f, f factorises as $\boldsymbol{D} \xrightarrow{\tilde{f}} \boldsymbol{C} - i(1\times 1) \xrightarrow{j} \boldsymbol{C}$. Then $A \cong \operatorname{Ho}\operatorname{LKan}_j \operatorname{Ho}\operatorname{LKan}_{\tilde{f}} B \cong \operatorname{Ho}\operatorname{LKan}_j j^*A$, where the second isomorphism follows from Corollary 1, hence

$$A_{i(1\times 1)} \cong \underrightarrow{\operatorname{Holim}}_{(\boldsymbol{C}-i(1\times 1)\to i(1\times 1))} h^*A \cong \underrightarrow{\operatorname{Holim}}_{\ulcorner} i\,|_{\ulcorner}{}^*A$$

by Proposition 2 and Proposition 3.a., where

$$(\boldsymbol{C} - i(1\times 1) \to i(1\times 1)) \xrightarrow{h} \boldsymbol{C}$$

is the canonical functor. $\square$

Proposition 6. a. *Let $f\colon \square^\star \to \underline{0}^\star$ be the functor sending 1×0 and 1×1 to 0, the nonzero morphism between them to the identity, and all the other objects to $\star$. Then for any object A of $\mathcal{K}_{\underline{0}}$, f^*A is bicartesian. The same applies to the functor g sending 0×0 and 0×1 to 0, and all the other objects of $\square$ to $\star$.*
As a consequence, if two adjacent vertices of a bicartesian square are contractible, the opposite side of the square is an isomorphism. (This should be viewed as a confirmation of our intuition that a morphism is an isomorphism if its cone or homotopy fibre is contractible).
b. *(Concatenation of squares and bicartesianness) Let $d_{0,1,2}\colon \underline{1} \to \underline{2}$ be the three monotonic injections, and let $A \in \mathcal{K}_{\underline{2}\times\underline{1}}$. Then if two of the three objects $(d_i \times \operatorname{Id}_{\underline{1}})^*A \in \mathcal{K}_\square$ are bicartesian, then so is the third one.*

Proof. For the assertion about squares in which two adjacent vertices are contractible, the $\operatorname{Ho}\operatorname{LKan}$-part of Proposition 5 can be applied to $f^* = \operatorname{Ho}\operatorname{LKan}_i$, where $i(0) = 1\times 0$. Similarly, the $\operatorname{Ho}\operatorname{RKan}$-part of Proposition 5 can be applied to g^*. The assertion about isomorphisms follows from the fact that a bicartesian square A can be reconstructed in a unique way from $i_\ulcorner^* A$ or $i_\lrcorner^* A$.

To prove the assertion about concatenation and bicartesianness, we can assume that $(d_0\times\operatorname{Id}_{\underline{1}})^*A$ or $(d_2\times\operatorname{Id}_{\underline{1}})^*A$ is bicartesian. In the second case, let $\boldsymbol{C} = \underline{2}\times\underline{1} - \{2\times 1\}$ and $\boldsymbol{D} = \boldsymbol{C} - \{1\times 1\}$, and let j and k be the inclusions of the subposets $\boldsymbol{C}$ and $\boldsymbol{D}$ into $\underline{2}\times\underline{1}$, and let $\boldsymbol{D} \xrightarrow{l} \boldsymbol{C}$ be the inclusion. Then by an application of Proposition 2, we have

$$(\operatorname{Ho}\operatorname*{LKan}_k k^*A)_{1\times 1} \cong (\operatorname{Ho}\operatorname*{LKan}_l l^*A)_{1\times 1} \cong \underrightarrow{\operatorname{Holim}}_{\ulcorner}(d_2\times\operatorname{Id}_{\underline{1}})^*A,$$

and by an additional application of Proposition 3.a.

$$(\mathrm{Ho}\,\underset{k}{\mathrm{LKan}}\,k^*A)_{2\times 1} \cong \underrightarrow{\mathrm{Holim}}_{\ulcorner}(d_1 \times \mathrm{Id}_{\underline{1}})^*A.$$

By our assumption on $(d_2 \times \mathrm{Id}_{\underline{1}})^*A$, it follows that $j^*A \cong \mathrm{Ho}\,\mathrm{LKan}_l\,k^*A$, and that $(d_1 \times \mathrm{Id}_{\underline{1}})^*A$ is bicartesian if and only if A is isomorphic to $\mathrm{Ho}\,\mathrm{LKan}_k\,k^*A \cong \mathrm{Ho}\,\mathrm{LKan}_j\,j^*A$. Since $(\mathrm{Ho}\,\mathrm{LKan}_j\,j^*A)_{2\times 1} \cong \underrightarrow{\mathrm{Holim}}_{\ulcorner}(d_0 \times \mathrm{Id}_{\underline{1}})^*A$, this is the case if and only if $(d_0 \times \mathrm{Id}_{\underline{1}})^*A$ is bicartesian.

We have proved that if $(d_2 \times \mathrm{Id}_{\underline{1}})^*A$ is bicartesian, either both or none of the remaining squares will be bicartesian. The same result with d_2 replaced by d_0 is proved in a similar way, using right instead of left homotopy Kan extensions. $\square$

1.4.3. *Preadditivity.*

Proposition 7. *$\mathcal{K}_\emptyset$ is a trivial category (having precisely one morphism between any pair of objects). For any $\boldsymbol{C}$, $\mathcal{K}_{\boldsymbol{C}}$ has an initial object which is also a final object, and finite coproducts exist and are isomorphic to the finite products. For every $\boldsymbol{C}^* \xrightarrow{f} \boldsymbol{D}^*$, the functors f^*, $\mathrm{Ho}\,\mathrm{LKan}_f$, and $\mathrm{Ho}\,\mathrm{RKan}_f$ preserve finite coproducts and products.*

Proof. By the disjoint union axiom, we have $\mathcal{K}_\emptyset \cong \mathcal{K}_\emptyset \times \mathcal{K}_\emptyset$, whence the first assertion.

Let 0 be any object of $\mathcal{K}_\emptyset$. For any $\boldsymbol{C}$, let $f\colon \boldsymbol{C}^\star \to \emptyset^\star$ be the unique functor. The object f^*0 of $\mathcal{K}_{\boldsymbol{C}}$ will also be denoted by 0. By the existence of homotopy Kan extensions for f, it is both an initial and a final object of $\mathcal{K}_{\boldsymbol{C}}$.

Let $\boldsymbol{C} \sqcup \boldsymbol{C}$ be the disjoint union of two copies of $\boldsymbol{C}$, and let $p\colon \boldsymbol{C} \sqcup \boldsymbol{C} \to \boldsymbol{C}$ be the functor which is the identity on each of the two copies of $\boldsymbol{C}$. By the disjoint union axiom, $\mathcal{K}_{\boldsymbol{C}\sqcup\boldsymbol{C}} \cong \mathcal{K}_{\boldsymbol{C}} \times \mathcal{K}_{\boldsymbol{C}}$. Hence, $\mathrm{Ho}\,\mathrm{LKan}_p$ provides the coproduct and $\mathrm{Ho}\,\mathrm{RKan}_p$ the product of two objects of $\mathcal{K}_{\boldsymbol{C}}$. Since it has both a left and a right adjoint, f^* preserves both products and coproducts. Since it has a right adjoint, $\mathrm{Ho}\,\mathrm{LKan}_f$ preserves coproducts and $\mathrm{Ho}\,\mathrm{RKan}_f$, having a left adjoint, preserves products.

It remains to prove that the coproduct and the product of two objects of $\mathcal{K}_{\boldsymbol{C}}$ coincide. By the isomorphism axiom, it suffices to do this for $\boldsymbol{C} = \underline{0}$. Let $q\colon \ulcorner^\star \to (\underline{0} \sqcup \underline{0})^\star$ be the functor sending 0×0 to $\star$ and 0×1 and 1×0 to the two copies of zero in $\underline{0} \sqcup \underline{0}$. It has a left adjoint $l^\star$, where $\underline{0} \sqcup \underline{0} \xrightarrow{l} \ulcorner$ sends the two copies of 0 to 0×1 and 1×0. By Proposition 3.a. we have

$$\underrightarrow{\mathrm{Holim}}_{\underline{0}\sqcup\underline{0}} \cong \underrightarrow{\mathrm{Holim}}_{\ulcorner}\,q^*. \tag{9}$$

Now consider the following poset $\boldsymbol{D}$:

$$\begin{tikzcd}
O_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & O_1 \arrow[d] \\
Y \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\
O_2 \arrow[r] & \tilde{X} \arrow[r] & O_3
\end{tikzcd}$$

Let $j\colon \ulcorner \to \boldsymbol{D}$ be the inclusion of the framed subposet. For any $(A, B) \in \mathcal{K}_{\underline{0}} \times \mathcal{K}_{\underline{0}} \cong \mathcal{K}_{\underline{0}\sqcup\underline{0}}$, let $S = \operatorname{Ho}\operatorname{LKan}_j q^*(A, B) \in \mathcal{K}_{\boldsymbol{D}}$. It follows from Proposition 5 that S makes all the visible squares in the above diagram bicartesian. In particular, by (9) S_P is coproduct of $S_X = A$ and $S_Y = B$, and by the dual version of (9) it is also a product of $S_{\tilde{X}}$ and $S_{\tilde{Y}}$. But it follows by applying Proposition 6.b. and Proposition 6.a. to the square formed by O_0, X, O_2, $\tilde{X}$ that $A \cong S_X \to S_{\tilde{X}}$ is an isomorphism. Also, $S_{\tilde{Y}} \cong B$. It follows that S_P also is a product of A and B. $\square$

Corollary 2. *Let $f\colon \boldsymbol{C}^\star \to \boldsymbol{D}^\star$ be a functor, and let $\boldsymbol{C}_1, \dots, \boldsymbol{C}_n$ be the connected components of $\boldsymbol{C}$. Let f_j be the restriction of f to $\boldsymbol{C}_j$ and let i_j be the inclusion of $\boldsymbol{C}_j$ into $\boldsymbol{C}$. Then*

$$\operatorname{Ho}\operatorname*{LKan}_f A \cong \bigoplus_{j=1}^n \operatorname{Ho}\operatorname*{LKan}_{f_j} i_j^* A, \qquad \operatorname{Ho}\operatorname*{RKan}_f A \cong \bigoplus_{j=1}^n \operatorname{Ho}\operatorname*{RKan}_{f_j} i_j^* A. \tag{10}$$

It follows from the last proposition that every object of $\mathcal{K}_{\boldsymbol{C}}$ has canonical (and unique) structures of a semigroup object and a semicogroup object and that these two canonical structures yield the same structure of a semigroup on $\operatorname{Hom}_{\mathcal{K}_{\boldsymbol{C}}}(A, B)$. This semigroup will be written additively, it will eventually turn out in Corollary 3 to be a group.

1.4.4. *The loop space and suspension functors.* Let $p_\ulcorner\colon \ulcorner^\star \to \underline{0}^\star$ be the projection sending 0×0 to 0, and all the other objects to $\star$. We define the suspension functor $\boldsymbol{\Sigma}\colon \mathcal{K}_{\underline{0}} \to \mathcal{K}_{\underline{0}}$ by

$$\boldsymbol{\Sigma} = \underrightarrow{\underline{\mathrm{Holim}}}_{\,\ulcorner} p_\ulcorner^*. \tag{11}$$

The loop space functor is defined by

$$\boldsymbol{\Omega} = \underleftarrow{\underline{\mathrm{Holim}}}_{\,\lrcorner} p_\lrcorner^*, \tag{12}$$

where $p_\lrcorner$ sends 1×1 to 0 and all other objects of $\lrcorner$ to $\star$.

We want to prove that these functors are mutually inverse equivalences of categories. To verify this, we consider for $A \in \mathcal{K}_{\underline{0}}$ the square $B = \mathrm{Ho}\,\mathrm{LKan}_{i_\ulcorner} p_\ulcorner^* A$. By Proposition 2, we have $B_{1\times 1} \cong \mathbf{\Sigma} A$, hence $p_\lrcorner^* \mathbf{\Sigma} A \cong i_\lrcorner^* B$. Since B is homotopy cocartesian,

$$\mathrm{Ho}\,\underset{i_\lrcorner}{\mathrm{RKan}}\, p_\lrcorner^* \mathbf{\Sigma} A \cong \mathrm{Ho}\,\underset{i_\lrcorner}{\mathrm{RKan}}\, i_\lrcorner^* B \cong B,$$

whence a canonical isomorphism $A \cong \mathbf{\Omega\Sigma} A$. The isomorphism $A \cong \mathbf{\Sigma\Omega} A$ is constructed in the same way.

We will put $A[i] = \mathbf{\Sigma}^i A$ for $i \geq 0$ and $A[i] = \mathbf{\Omega}^{-i} A$ for $i \leq 0$. By the above results, the functor $[-i]$ really is an inverse to $[i]$. Eventually, we shall show that this is the shift functor for a triangulated structure on $\mathcal{K}_{\underline{0}}$. When we verify the axiom about shifting distinguished triangles, it will be necessary to identify a sign change with a canonical automorphism of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$.

Let σ be the automorphism of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ defined by interchanging the vertices 1×0 and 0×1 of $\square$. More precisely, let $\boldsymbol{\sigma}$ be the automorphism of $\square$ which interchanges the vertices 1×0 and 0×1. We have an isomorphism

$$\begin{aligned} \sigma\colon \mathbf{\Sigma} A &= \underrightarrow{\mathrm{Holim}}_{\ulcorner} p_\ulcorner^* A \\ &= \underrightarrow{\mathrm{Holim}}_{\ulcorner} (p_\ulcorner \boldsymbol{\sigma})^* A \\ &\cong \underrightarrow{\mathrm{Holim}}_{\ulcorner} \boldsymbol{\sigma}^* p_\ulcorner^* A \qquad (13) \\ &\cong \underrightarrow{\mathrm{Holim}}_{\ulcorner} p_\ulcorner^* A \\ &= \mathbf{\Sigma} A. \qquad (14) \end{aligned}$$

A similar definition is made for $\mathbf{\Omega}$.

Proposition 8. *We have $\sigma + \mathrm{Id} = 0$ on both $\mathbf{\Sigma}$ and $\mathbf{\Omega}$. In particular, the semigroup and semicogroup objects $\mathbf{\Omega} X$ and $\mathbf{\Sigma} X$ are group and cogroup objects.*

Proof. It suffices to consider the case of $\mathbf{\Omega}$.

For each $n > 0$, let $\boldsymbol{C}_n$ be the poset having vertices X and $O_0, \ldots, O_n$ with $O_i \prec X$ and no other relations. For any object A of $\mathcal{K}_{\underline{0}}$, let $P_n A$ be the object $\underleftarrow{\mathrm{Holim}}_{\boldsymbol{C}_n} p_n^\star A$, where $p_n\colon \boldsymbol{C}_n \to \underline{0}$ sends X to 0 and O_i to $\star$. For any map $f\colon \underline{k} \to \underline{n}$, let $\underline{f}\colon \boldsymbol{C}_k \to \boldsymbol{C}_n$ be the map defined by $\underline{f}(X) = X$ and $\underline{f}(O_i) = O_{f(i)}$. Corresponding to $\underline{f}$, we have a morphism $\tilde{f}\colon P_n A \to P_k A$ by applying (1) to the projective homotopy limits over $\boldsymbol{C}_k$ and $\boldsymbol{C}_n$. These morphisms satisfy the obvious transitivity property.

Let $i_0 = 0 < i_1 < \ldots < i_N = n$, and let f_l be the unique monotonic bijection $\underline{i_l - i_{l-1}} \cong [i_{l-1}, i_l] = \{k | i_{l-1} \leq k \leq i_l\}$. We claim that

$$(\tilde{f}_1, \ldots, \tilde{f}_N)\colon P_n A \to \prod_{l=1}^{N} P_{i_l - i_{l-1}} A \qquad (15)$$

is an isomorphism. By an induction argument, it suffices to prove this for $N = 2$. Let $\boldsymbol{D}$ be the poset containing $\boldsymbol{C}_n$ and the additional vertices Y_1, Y_2 such that $O_0, \ldots, O_{i_1}$ are $\succ Y_1$

and $O_{i_1}, \dots, O_n$ are $\succ Y_2$. Let $j\colon \boldsymbol{C}_n \to \boldsymbol{D}$ be the inclusion and let $B = \operatorname{Ho}\operatorname{RKan}_j p_n^* A$. Then $P_n A \cong \underleftarrow{\operatorname{Holim}}_{\boldsymbol{D}} B$. By an application of Proposition 3.a. to the inclusion $k\colon \lrcorner \to \boldsymbol{D}$ defined by $k(1 \times 1) = O_{i_1}$, $k(0 \times 1) = Y_1$, $k(1 \times 0) = Y_2$, which has a right adjoint sending O_{i_1} and X to $O_{i_1}, O_0, \dots, O_{i_1 - 1}$ and Y_1 to Y_1, and $O_{i_1+1,\dots,i_n}$ and Y_2 to Y_2, we have $P_n A \cong \underleftarrow{\operatorname{Holim}}_{\lrcorner} k^* B$. But $B_{O_{i_1}} = 0$, while $B_{Y_1} = P_{i_1} A$ and $B_{Y_2} = P_{n-i_1} A$, proving that (15) is an isomorphism.

The isomorphism $\lrcorner \cong \boldsymbol{C}_1$ gives us an identification $P_1 \cong \boldsymbol{\Omega}$. The morphism $\tilde{d}_1\colon P_2 A \to P_1 A$ together with the inverse of the isomorphism $(\tilde{d}_2, \tilde{d}_0)A\colon P_2 A \cong P_1 A \times P_1 A$ gives us a morphism $a\colon \boldsymbol{\Omega} A \oplus \boldsymbol{\Omega} A \to \boldsymbol{\Omega} A$ (which can be viewed as the concatenation of loops map). We want to verify that this a satisfies the associativity law, i.e., that

$$a(x, a(y,z)) = a(a(x,y), z) \tag{16}$$

for three morphisms x, y, z of any object E of $\mathcal{K}_{\underline{0}}$ to $\boldsymbol{\Omega} A$. To do this, one notes that, by the transitivity of the morphisms $\tilde{f}$ between the P_k, both sides of (16) are equal to $\tilde{i}_{03} g$, where $i_{\alpha\beta}$ is the map from $\underline{1}$ to $\underline{3}$ sending 0 to α and 1 to β., and where $g\colon E \to P_3 A$ is determined uniquely by

$$\begin{aligned} \tilde{i}_{01} g &= x \\ \tilde{i}_{12} g &= y \\ \tilde{i}_{23} g &= z. \end{aligned}$$

There is an action of the symmetric group S_3 on P_2 for which a permutation π of $\{0;1;2\}$ acts by $\tilde{\pi}$. We use the identification $P_2 A \cong P_1 A \times P_1 A$ via d_2 and d_0 to study this action. Let σ_{ij} be the involution interchanging i and j. Thus, $\tilde{\sigma}_{01}$ on $P_1 A$ gives us the involution σ on $\boldsymbol{\Omega} A$. Using this, one sees that

$$\tilde{\sigma}_{02} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \tag{17}$$

on P_2. Since σ_{01} commutes with d_2, we have $\tilde{d}_2 \tilde{\sigma}_{01} = \sigma \tilde{d}_2$, hence

$$\tilde{\sigma}_{01} = \begin{pmatrix} \sigma & 0 \\ \alpha & \beta \end{pmatrix}. \tag{18}$$

This is an involution if and only if σ and β are involutions and

$$\alpha\sigma + \beta\alpha = 0. \tag{19}$$

Since $\tilde{d}_1 = \tilde{d}_0 \tilde{\sigma}_{01}$, we have

$$a(x,y) = \alpha(x) + \beta(y). \tag{20}$$

Conjugating (18) by (17), we have

$$\tilde{\sigma}_{12} = \begin{pmatrix} \sigma\beta\sigma & \sigma\alpha\sigma \\ 0 & \sigma \end{pmatrix}$$

Using this and $\tilde{d}_1 = \tilde{d}_2\tilde{\sigma}_{12}$, we see that $a(x,y) = \sigma\beta\sigma(x) + \sigma\alpha\sigma(y)$. Comparison with (20) gives us

$$\beta = \sigma\alpha\sigma \tag{21}$$

By (20), the associativity law for a implies that

$$\begin{aligned} a(x, a(y,z)) &= \alpha x + \beta(\alpha y + \beta z) \\ &= \alpha x + \beta\alpha y + z \end{aligned}$$

must be equal to

$$\begin{aligned} a(a(x,y),z) &= \alpha(\alpha x + \beta y) + \beta z \\ &= \alpha^2 x + \alpha\beta y + \beta z. \end{aligned}$$

We put $x = y = 0$ and conclude that $\beta = \mathrm{Id}$, which in view of (21) implies $\alpha = \mathrm{Id}$. Therefore (19) implies $\sigma + \mathrm{Id} = 0$. $\square$

Corollary 3. *The categories $\mathcal{K}_{\boldsymbol{C}}$ are additive categories, and the functors f^*, $\operatorname{Ho}\operatorname{LKan}_f$, and $\operatorname{Ho}\operatorname{RKan}_f$ are additive functors.*

Proof. By Proposition 7, it remains to show that $\operatorname{Hom}_{\mathcal{K}_{\boldsymbol{C}}}(A,B)$ is a group. In the case where B is isomorphic to $\boldsymbol{\Omega}C$ for some $C \in \mathfrak{Ob}\,\mathcal{K}_{\boldsymbol{C}}$, this follows from Proposition 8. But every B is isomorphic to $\boldsymbol{\Omega\Sigma}B$. $\square$

1.4.5. *Cone and Homotopy Fibre.* Let the cone functor $\mathfrak{Cone}\colon \mathcal{K}_{\underline{1}} \to \mathcal{K}_{\underline{2}}$ be given by $\mathfrak{Cone}(A) = i^* \operatorname{Ho}\operatorname{LKan}_{i_\ulcorner} p^*A$, where $p\colon \ulcorner^\star \to \underline{1}^\star$ sends 1×0 to $\star$, and 0×0 to 0 and 0×1 to 1, and where $i\colon \underline{2} \to \square$ is given by $i(0) = 0\times 0$, $i(1) = 0\times 1$, and $i(2) = 1 \times 1$. Similarly, let $\mathfrak{Hofi}(A) = i^* \operatorname{Ho}\operatorname{RKan}_{i_\lrcorner} q^*A$, where $q(1\times 0) = \star$, $q(0\times 1) = 0$, $q(1\times 1) = 1$.

Let $\mathrm{hofi}A = (\mathfrak{Hofi}\,A)_0$ and $\mathrm{cone}A = (\mathfrak{Cone}\,A)_2$. These objects should be thought of as the homotopy fibre and the cone of the morphism $A_0 \to A_1$, where $A \in \mathcal{K}_{\underline{1}}$ contains enough information about this morphism to define such objects up to unique isomorphism. $\mathfrak{Hofi}$ then is an object in the homotopy category of diagrams incarnating $\mathrm{hofi} \to A_0 \to A_1$. The following proposition confirms our expectation that the homotopy fibre of the morphism from A_1 to the cone of A is A_0.

Proposition 9. *The functors $d_0^*\,\mathfrak{Cone}$ and $d_2^*\,\mathfrak{Hofi}$ are mutually inverse self-equivalences of $\mathcal{K}_{\underline{1}}$.*

Proof. This is an application of the linearity axiom to the squares $\operatorname{Ho}\operatorname{LKan}_{i_\ulcorner} p^*A$ and $\operatorname{Ho}\operatorname{RKan}_{i_\lrcorner} q^*A$. $\square$

1.4.6. *Distinguished Triangles.* Let $\boldsymbol{D}$ be the following poset:

(22)
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & O_2 \arrow[d] \\
O_1 \arrow[r] & Z \arrow[r] & X_{\boldsymbol{\Sigma}}
\end{tikzcd}$$

and let $\boldsymbol{C}$ be the framed subposet. Let $f\colon \boldsymbol{C}^\star \to \underline{1}^\star$ be given by $f(O_i) = \star$, $f(X) = 0$, $f(Y) = 1$, sending the morphism $X \to Y$ in $\boldsymbol{C}$ to the morphism $0 \to 1$ in $\underline{1}$. Let i be the inclusion of $\boldsymbol{C}$ into $\boldsymbol{D}$. For $A \in \mathcal{K}_{\underline{1}}$, let $B = \operatorname{Ho}\operatorname{LKan}_i f^*A$. By Proposition 2 and Proposition 3.a., $B_Z \cong \operatorname{cone}(A)$ and $B_{X_{\boldsymbol{\Sigma}}} \cong \boldsymbol{\Sigma}A_0$. Therefore B gives us a morphism from $\operatorname{cone}(A)$ to $\boldsymbol{\Sigma}A_0$.

Definition 5. The triangle T_A is the triangle

$$A_0 \to A_1 \to \operatorname{cone}(A) \to \boldsymbol{\Sigma}A_0$$

in $\mathcal{K}_{\underline{0}}$. A triangle in $\mathcal{K}_{\underline{0}}$ is called distinguished if it is isomorphic to T_A for some $A \in \mathcal{K}_{\underline{1}}$.

Finally, an object B of $\mathcal{K}_{\boldsymbol{D}}$ is called distinguished if it is of the form $\operatorname{Ho}\operatorname{LKan}_i f^*A$ for some object A of $\mathcal{K}_{\underline{1}}$. This is the case if and only if $B_{O_1} = B_{O_2} = 0$ and B makes the two visible squares in (22) bicartesian. Such an object of $\mathcal{K}_{\boldsymbol{D}}$ gives rise to a distinguished triangle in $\mathcal{K}_{\underline{0}}$.

Remark 2. Although the dimension of the poset $\boldsymbol{D}$ is three, it is possible to characterise the triangulated structure in terms of $\mathcal{K}_{\boldsymbol{C}}$ with $\dim \boldsymbol{C} \leq 1$. For we have $\boldsymbol{\Sigma}A = \operatorname{cone}(A \to 0)$, and the distinguished triangles are the ones isomorphic to triangles of the form

$$A \to B \to \operatorname{cone}(A \to B) \to \operatorname{cone}(A \to 0).$$

Theorem 1. *The category $\mathcal{K}_{\underline{0}}$, equipped with this class of distinguished triangles, is a triangulated category. In a similar way, the categories $\mathcal{K}_{\boldsymbol{D}}$ can be made into triangulated categories, using the system of triangulated diagram categories $\mathcal{K}_{?\times\boldsymbol{D}}$ described in Proposition* 4.

Proof of Theorem 1: We will refer to the axioms of a triangulated category in the numeration in which they are given in [ΓM88] or [GM94]. The axiomTR1.a) follows from Proposition 6.a., which can be formulated as stating that the cone of the identity is 0. AxiomTR1.b) is clear since by our definition any triangle isomorphic to a distinguished one is distinguished. AxiomTR1.c) follows from the fact that the functor $\mathcal{K}_{\underline{1}} \to \mathcal{K}_{\underline{0}}^{\underline{1}}$ gives a surjection on the isomorphism classes of objects.

For axiom TR2, we define the shift of a triangle

$$T = (E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{w} E[1])$$

to be $ST = (F \xrightarrow{v} G \xrightarrow{w} E[1] \xrightarrow{-u[1]} F[1])$. We have to show that a triangle is distinguished if and only if so is its shift. The 'if'-part follows readily from the following lemma:

Lemma 1. *We have* $ST_A \cong T_{d_0^* \mathfrak{Cone}(A)}$.

The 'only if'-part follows from this and Proposition 9.

Proof of Lemma 1: Let $\boldsymbol{E} \supset \boldsymbol{D}$ be the following poset

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & O_2 \arrow[d] \\
O_1 \arrow[r] & Z \arrow[r] \arrow[d] & X_{\boldsymbol{\Sigma}} \arrow[d] \\
 & O_3 \arrow[r] & Y_{\boldsymbol{\Sigma}}
\end{tikzcd}$$

Let $j\colon \boldsymbol{D} \to \boldsymbol{E}$ be the inclusion and let $k\colon \boldsymbol{F} \to \boldsymbol{E}$ be the inclusion of the framed subposet. $\boldsymbol{F}$ is isomorphic to $\boldsymbol{D}$ by an isomorphism sending X to Y, Y to Z, Z to $X_{\boldsymbol{\Sigma}}$, $X_{\boldsymbol{\Sigma}}$ to $Y_{\boldsymbol{\Sigma}}$, O_1 to O_2 and O_2 to O_3. In particular, we can define the property of distinguishedness for an object of $\mathcal{K}_{\boldsymbol{F}}$, and we have a distinguished triangle associated to such an object. Let $\tilde{\boldsymbol{C}} \xrightarrow{\tilde{i}} \boldsymbol{E}$ be the subposet $\boldsymbol{C} \cup \{O_3\}$, and let $\tilde{\boldsymbol{C}}^\star \xrightarrow{\tilde{f}} \underline{1}^\star$ be defined by $\tilde{f}|_{\boldsymbol{C}} = f$ and $\tilde{f}(O_3) = \star$. Let $C = \operatorname{Ho}\operatorname{LKan}_{\tilde{i}} \tilde{f}^* A$. By Proposition 5, C makes all visible squares in the above diagram bicartesian. Since in addition $C_{O_i} = 0$ for $1 \le i \le 3$, j^*C and k^*C are distinguished objects of $\mathcal{K}_{\boldsymbol{D}}$ and $\mathcal{K}_{\boldsymbol{F}}$. In particular, there are canonical isomorphisms $C_{X_{\boldsymbol{\Sigma}}} \cong \boldsymbol{\Sigma} C_X$ and$C_{Y_{\boldsymbol{\Sigma}}} \cong \boldsymbol{\Sigma} C_Y$. Since for all $U \in \boldsymbol{D}$ we have $\boldsymbol{C} \to U = \tilde{\boldsymbol{C}} \to U$, the canonical morphism $B \longrightarrow j^*C$ is an isomorphism. Using this and Proposition 2 and Proposition 3.a., we see that the pull-back of C along the morphism from $\underline{1}$ to $\boldsymbol{E}$ sending 0 to Y and 1 to Z is $\mathfrak{Cone}(A)$. Therefore, the distinguished triangle belonging to k^*C is $T_{\mathfrak{Cone}(A)}$. The lemma follows if we can identify the triangle belonging to k^*C with ST_A. Only the identification of the morphism $C_{X_{\boldsymbol{\Sigma}}} \longrightarrow C_{Y_{\boldsymbol{\Sigma}}}$ is not completely trivial. The embeddings of $\ulcorner$ into $\boldsymbol{E}$ which give us the identifications of $C_{X_{\mathbf{\Sigma}}}$ and $C_{Y_{\mathbf{\Sigma}}}$ with $\mathbf{\Sigma} A_0$ and $\mathbf{\Sigma} A_1$ look as follows:

$$\begin{tikzcd}
1\times 0 \arrow[r] & O_1 \arrow[r] & O_3 & 0\times 1 \arrow[l] \\
0\times 0 \arrow[r] \arrow[u] \arrow[d] & X \arrow[r] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d] & 0\times 0 \arrow[l] \arrow[u] \arrow[d] \\
0\times 1 \arrow[r] & O_2 \arrow[r] & O_2 & 1\times 0 \arrow[l]
\end{tikzcd}$$

In this diagram, the first and the seventh column represent the elements of $\ulcorner$, and the second and the sixth column represent functors from $\ulcorner$ to the two subposets of $\boldsymbol{E}$ used to identify $C_{X_{\mathbf{\Sigma}}}$ and $C_{Y_{\mathbf{\Sigma}}}$. The horizontal arrows in the fourth column are morphisms in $\boldsymbol{E}$. Since the first and the seventh column of the diagram differ by the reflection of $\square$ along its main diagonal, it follows that the morphism $\mathbf{\Sigma} A_0 \cong C_{X_{\mathbf{\Sigma}}} \to C_{Y_{\mathbf{\Sigma}}} \cong \mathbf{\Sigma} A_1$ differs from the suspension of $A_0 \to A_1$ by σ, which by Proposition 8 is -1.

The proof of Lemma 1 is complete. **Q.E.D.**

This completes the verification of axiom TR2. Axiom TR3 follows from the fullness of the functor $\mathcal{K}_{\underline{1}} \to \mathcal{K}_{\underline{0}}{}^{\underline{1}}$ required by the mapping cylinder axiom.

It remains to verify the octahedron axiom TR4. This will be done by associating a full octahedron diagram in $\mathcal{K}_{\underline{0}}$ to each object of $\mathcal{K}_{\underline{2}}$ and verifying that each upper half of an octahedron diagram comes from such an object.

To associate an octahedron diagram to an object of $\mathcal{K}_{\underline{2}}$, we consider the following poset $\boldsymbol{O}$:

(23)
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & O_3 \arrow[d] & \\
O_1 \arrow[r] & Z' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & X_{\mathbf{\Sigma}} \arrow[r] \arrow[d] & O_4 \arrow[d] \\
& O_2 \arrow[r] & X' \arrow[r] & Y_{\mathbf{\Sigma}} \arrow[r] & Z'_{\mathbf{\Sigma}}
\end{tikzcd}$$

Let $l\colon \boldsymbol{P} \to \boldsymbol{O}$ be the inclusion of the framed subposet, and let $h\colon \boldsymbol{P}^{\star} \to \underline{2}^{\star}$ be defined by $h(X) = 0$, $h(Y) = 1$, $h(Z) = 2$, $h(O_i) = \star$, taking an arrow to the zero morphism only if its source or target are mapped to $\star$. For $D \in \mathcal{K}_{\underline{2}}$, let $E = \operatorname{Ho} \operatorname{LKan}_l h^* D$. By an application of Proposition 5, E makes all visible squares in (23) bicartesian. By Proposition 6.b., the same is true for all concatenations of visible squares. Applying this to the squares formed by X, O_1, O_3, and $X_{\mathbf{\Sigma}}$ (resp. by Y, O_2, O_3, and $Y_{\mathbf{\Sigma}}$ and by Z', O_2, O_4, and $Z'_{\mathbf{\Sigma}}$), we see that $E_{X_{\mathbf{\Sigma}}}$ can be identified with $\mathbf{\Sigma}D_0$ (resp. $E_{Y_{\mathbf{\Sigma}}}$ with $\mathbf{\Sigma}D_1$ and $E_{Z'_{\mathbf{\Sigma}}}$ with $\mathbf{\Sigma}E_{Z'}$). Therefore, we get all the necessary commutativities in the two halves of the octahedron diagram:

(24)
$$\begin{tikzcd}
E_{X'} \arrow[dr, "+"] \arrow[dd, "+"'] \arrow[rr, phantom, "\star"', yshift=-12pt] & & E_Z = D_2 \arrow[ll] \\
& E_Y = D_1 \arrow[ur] \arrow[dl] & \\
E_{Z'} \arrow[rr, "+"] \arrow[rr, phantom, "\star", yshift=12pt] & & E_X = D_0 \arrow[ul] \arrow[uu]
\end{tikzcd}$$

(25)
$$\begin{tikzcd}
E_{X'} \arrow[dd, "+"'] & & E_Z = D_2 \arrow[ll] \arrow[dl] \\
 \arrow[r, phantom, "\star"] & E_{Y'} \arrow[ul] \arrow[dr, "+"] \arrow[r, phantom, "\star"] & \\
E_{Z'} \arrow[rr, "+"] \arrow[ur] & & E_X = D_0, \arrow[uu]
\end{tikzcd}$$

including the commutativity of the two diagonal squares containing both E_Y and $E_{Y'}$. In these diagrams, arrows marked by a $+$ are of degree one, and the four triangles marked by $\star$ have to be distinguished instead of commutative. To check that they are really distinguished, we consider the following four embeddings $m_{1\ldots 4}\colon \boldsymbol{D} \to \boldsymbol{O}$:

- $m_1(X) = X$, $m_1(Y) = Y$, $m_1(O_2) = O_3$, $m_1(O_1) = O_1$, $m_1(Z) = Z'$, $m_1(X_{\mathbf{\Sigma}}) = X_{\mathbf{\Sigma}}$.
- $m_2(X) = Y$, $m_2(Y) = Z$, $m_2(O_2) = O_3$, $m_2(O_1) = O_2$, $m_2(Z) = X'$, $m_2(X_{\mathbf{\Sigma}}) = Y_{\mathbf{\Sigma}}$.
- $m_3(X) = X$, $m_3(Y) = Z$, $m_3(O_2) = O_3$, $m_3(O_1) = O_1$, $m_3(Z) = Y'$, $m_3(X_{\mathbf{\Sigma}}) = X_{\mathbf{\Sigma}}$.
- $m_4(X) = Z'$, $m_4(Y) = Y'$, $m_4(O_2) = O_4$, $m_4(O_1) = O_2$, $m_4(Z) = X'$, $m_4(X_{\mathbf{\Sigma}}) = Z'_{\mathbf{\Sigma}}$.

Of course, the arguments of each of the m_i are vertices of (22), while its values are vertices of (23). Each m_i takes visible squares and O-vertices in (22) to visible squares and O-vertices in (23), it follows that m_i^*E is distinguished in the sense of Definition 5. This proves the distinguishedness of the four triangles marked by $\star$.

It remains to prove that every upper cone

(26)
$$\begin{tikzcd}
A' \arrow[dr, "+"] \arrow[dd, "+"'] \arrow[rr, phantom, "\star"', yshift=-12pt] & & C \arrow[ll] \\
& B \arrow[ur] \arrow[dl] & \\
C' \arrow[rr, "+"] \arrow[rr, phantom, "\star", yshift=12pt] & & A \arrow[ul] \arrow[uu]
\end{tikzcd}$$

comes from an object of $\mathcal{K}_{\underline{2}}$. Let $M \in \mathcal{K}_{\underline{1}}$ such that $M_0 \to M_1$ is isomorphic to $A \to B$. The morphism $B \to C$ defines a morphism $M \to s_0^*C$ in $\mathcal{K}_{\underline{1}}$, which by the mapping cylinder axiom comes from an object $\tilde{D}$ of $\mathcal{K}_\square$ with $\tilde{D}_{0\times ?} \cong M$ and $\tilde{D}_{1\times ?} \cong s_0^* C$. Let $n\colon \underline{2} \to \square$ be defined by $n(0) = 0 \times 0$, $n(1) = 0 \times 1$, $n(2) = 1 \times 1$. Then $D = n^* \tilde{D}$ is an object of $\mathcal{K}_{\underline{2}}$ giving rise to $A \to B \to C$. We want to verify that with this choice of D, 1.4.6 is isomorphic to (24). Since isomorphisms between $A \cong E_X$, $B \cong E_Y$, and $C \cong E_Z$ are part of our construction of D, and since TR3 is already proved, there is a choice of isomorphisms $A' \cong E_{X'}$ and $C' \cong E_{Z'}$ such that we get isomorphisms between the two distinguished triangles in 1.4.6 and their counterparts in (24). It is clear that these isomorphisms intertwine between the long left vertical arrows in 1.4.6 and (24).

We have constructed an isomorphism between 1.4.6 and (24). Therefore, (25) gives us a lower half for 1.4.6, and we are through.

The proof of Theorem 1 is complete. **Q.E.D.**

1.4.7. *Some spectral sequences.* For any $\boldsymbol{C} \in \mathfrak{K}$ and $A \in \mathcal{K}_{\boldsymbol{C}}$, we have a functorial distinguished triangle

(27)
$$\begin{tikzcd} A \arrow[d] & & RA \arrow[ll] \\ PA = \bigoplus_{X \in \boldsymbol{C}} \operatorname{Ho} \operatorname{RKan}_{i_X} A_X \arrow[urr, "+"'] & & \end{tikzcd}$$

which can be defined in a functorial way as follows: Let $\tilde{\boldsymbol{C}}$ be the subcategory of $\boldsymbol{C} \times \underline{1}$ obtained by keeping all objects, but removing the non-identical morphisms between $X \times 1$ and $Y \times 1$. Let $p\colon \tilde{\boldsymbol{C}} \to \boldsymbol{C}$ be the projection and $i\colon \tilde{\boldsymbol{C}} \to \boldsymbol{C} \times \underline{1}$ be the inclusion. Let $B = \operatorname{Ho} \operatorname{RKan}_i p^* A \in \mathcal{K}_{\boldsymbol{C} \times \underline{1}}$, by (10) and Proposition 3, we have $(\mathrm{Id}_{\boldsymbol{C}} \times d_1)^* B = A$ and $(\mathrm{Id}_{\boldsymbol{C}} \times d_0)^* B = PA$. Thus, putting $RA = \mathrm{cone} B$, we get a functorial distinguished triangle (27).

Lemma 2. *We have $R^{\dim \boldsymbol{C}+1} = 0$. Thus, we get a resolution*

$$\begin{tikzcd} A \arrow[d] & RA \arrow[l] \arrow[d] & R^2A \arrow[l] \arrow[d] & \cdots \arrow[l] & R^{\dim \boldsymbol{C}-1}A \arrow[l] \arrow[d] & R^{\dim \boldsymbol{C}}A \arrow[l] \arrow[d, "\cong"] \\ PA \arrow[ur, "+"'] & PRA \arrow[ur, "+"'] & PR^2A \arrow[ur, "+"'] & \cdots & PR^{\dim \boldsymbol{C}-1}A \arrow[ur, "+"'] & PR^{\dim \boldsymbol{C}}A \end{tikzcd}$$
(28)

By duality, we have a similar resolution

$$
\begin{tikzcd}
A \arrow[r] & \tilde{R}A \arrow[r] \arrow[dl, "+"] & \tilde{R}^2 A \arrow[r] \arrow[dl, "+"] & \cdots \arrow[r] \arrow[dl, "+"] & \tilde{R}^{\dim \boldsymbol{C}-1}A \arrow[r] \arrow[dl, "+"] & \tilde{R}^{\dim \boldsymbol{C}}A \arrow[dl, "+"] \\
\tilde{P}A \arrow[u] & \tilde{P}\tilde{R}A \arrow[u] & \tilde{P}\tilde{R}^2A \arrow[u] & \cdots & \tilde{P}\tilde{R}^{\dim \boldsymbol{C}-1}A \arrow[u] & \tilde{P}\tilde{R}^{\dim \boldsymbol{C}}A \arrow[u, "\cong"']
\end{tikzcd}
\tag{29}
$$

with $\tilde{P}A = \bigoplus_{X\in\boldsymbol{C}} \mathrm{Ho}\,\mathrm{LKan}_{i_X} A_X$.

Proof. By Proposition 2, we have

$$(\mathrm{Ho}\,\underset{i_X}{\mathrm{RKan}}\, A)_Y = \bigoplus_{\mathrm{Hom}_{\boldsymbol{C}}(Y,X)} A, \tag{30}$$

hence

$$(PA)_X = \bigoplus_{\substack{Y\in\mathfrak{Ob}(\boldsymbol{C})\\ \mathrm{Hom}_{\boldsymbol{C}}(X,Y)}} A_Y. \tag{31}$$

Moreover, the morphism $A_X \longrightarrow (PA)_X$ defined by evaluating the downward arrow in (27) corresponds to the inclusion of the summand belonging to X. Let $\boldsymbol{C}^n$ denote the subset of all objects X of $\boldsymbol{C}$ for which there exists a non-degenerate n-simplex $X = X_0 \to X_1 \to \ldots \to X_n = X$ in $N.\boldsymbol{C}$, and let $\mathcal{K}^n_{\boldsymbol{C}}$ be the full subcategory of all $A \in \mathcal{K}_{\boldsymbol{C}}$ with $A_X = 0$ for $X \notin \boldsymbol{C}^n$. Then it follows easily from (31), and the exact triangle (27) that $R\mathcal{K}^n_{\boldsymbol{C}} \subseteq \mathcal{K}^{n+1}_{\boldsymbol{C}}$. By induction we have $R^{\dim \boldsymbol{C}+1}A \in \mathcal{K}^{\dim \boldsymbol{C}+1}_{\boldsymbol{C}}$. But $\boldsymbol{C}^{\dim \boldsymbol{C}+1} = \emptyset$, hence $\mathcal{K}^{\dim \boldsymbol{C}+1}_{\boldsymbol{C}} = 0$. □

Let $\mathrm{Sub}(\boldsymbol{C})$ denote the subdivision of $\boldsymbol{C}$, cf. [Gra76]. Its objects are morphisms f in $\boldsymbol{C}$, a morphism from f to g in $\mathrm{Sub}(\boldsymbol{C})$ being a factorisation $g = afb$. The dimensions of $\boldsymbol{C}$ and $\mathrm{Sub}(\boldsymbol{C})$ are equal.

Proposition 10. *Let A and B be objects of $\mathcal{K}_{\boldsymbol{C}}$, and let F be a covariant and G be a contravariant cohomological functor on $\mathcal{K}_{\underline{0}}$ with values in some abelian category. Then there are canonical spectral sequences*

$$E_2^{p,q} = \lim_{Y\to X\in\mathrm{Sub}(\boldsymbol{C})}{}^{p}\, \mathrm{Hom}^q_{\mathcal{K}_{\underline{0}}}(B_Y, A_X) \Rightarrow \mathrm{Hom}^{p+q}_{\mathcal{K}_{\boldsymbol{C}}}(B, A) \tag{32}$$

$$E_2^{p,q} = \lim_{\boldsymbol{C}}{}^{p}\, F^q(A_X) \Rightarrow F^{p+q}(\underleftarrow{\mathrm{Holim}}_{\boldsymbol{C}} A) \tag{33}$$

$$E_2^{p,q} = \operatorname*{colim}_{\boldsymbol{C}^{\mathrm{op}}}{}_{-p}\, G^q(A_X) \Rightarrow G^{p+q}(\underleftarrow{\mathrm{Holim}}_{\boldsymbol{C}} A) \tag{34}$$

$$E_2^{p,q} = \operatorname*{colim}_{\boldsymbol{C}}{}_{-p}\, F^q(A_X) \Rightarrow F^{p+q}(\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}} A) \tag{35}$$

$$E_2^{p,q} = \lim_{\boldsymbol{C}^{\mathrm{op}}}{}^{p}\, G^q(A_X) \Rightarrow G^{p+q}(\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}} A). \tag{36}$$

Proof. Let us first prove (32). Applying $\mathrm{Hom}^*_{\mathcal{K}_{\boldsymbol{C}}}(B,-)$ to the resolution, one gets a spectral sequence

$$E_1^{p,q} = \mathrm{Hom}^{p+q}_{\mathcal{K}_{\boldsymbol{C}}}(B, PR^pA) \Rightarrow \mathrm{Hom}^{p+q}_{\mathcal{K}_{\boldsymbol{C}}}(B, A). \tag{37}$$

We have to identify its E_2-term. For objects C, D of $\mathcal{K}_{\boldsymbol{C}}$, let $M^*(C,D)$ be the $\mathrm{Sub}(\boldsymbol{C})$-diagram of abelian groups given by

$$M^*(C,D)_{Y\to Z} = \mathrm{Hom}^*\mathcal{K}_{\underline{0}}(C_Y, D_Z).$$

By (31), the morphism $A_Z \to (PA)_Z$ derived from the downward arrow in (27) is a (split) monomorphism. Therefore, the morphism $(RA)_Z \to A_Z$ is the zero morphism. It follows that the homomorphism $M^*(B, R^{k+1}A) \to M^*(B, R^kA)$ is zero, therefore the complex $M^{*+k}(B, PR^kA)$ is a resolution of $M^*(B,A)$. Consequently, the E_2-term of (37) is canonically isomorphic to the initial term of (32) if we can prove

$$\lim_{\mathrm{Sub}(C)}{}^p M^*(B, PC) = \begin{cases} \bigoplus_{X\in\boldsymbol{C}} \mathrm{Hom}^*_{\mathcal{K}_{\underline{0}}}(B_X, C_X) = \mathrm{Hom}^*_{\mathcal{K}_{\boldsymbol{C}}}(B, PC) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases} \tag{38}$$

Note that the identity on the right hand side in the case $p = 0$ is a trivial consequence of the definition of $\mathrm{Ho}\,\mathrm{RKan}$ as an adjoint functor. The identification of the E_2-term follows from (38) by inserting $C = R^kA$ and applying the abstract de Rham theorem.

To verify (38), we note that by (31), we have

$$M^*(B, PC) \cong \bigoplus_{X\in\mathfrak{Ob}(\boldsymbol{C})} S_X, \tag{39}$$

where

$$S_X(Y \to Z) = \bigoplus_{\psi\in\mathrm{Hom}_{\boldsymbol{C}}(Z,X)} \mathrm{Hom}^*_{\mathcal{K}_{\underline{0}}}(B_Y, C_X).$$

Let $j_X\colon (\boldsymbol{C}\to X)^{\mathrm{op}} \to \mathrm{Sub}(C)$ be the obvious embedding. For every object $\boldsymbol{v} = (Y \xrightarrow{\phi} Z)$ of $\mathrm{Sub}(\boldsymbol{C})$, the connected components of $j_X \Leftarrow \boldsymbol{v}$ correspond to the morphisms $Z \xrightarrow{\psi} X$ in $\boldsymbol{C}$, and each connected component has an initial object $Y \xrightarrow{\psi\phi} X$. Let T_X be the $(\boldsymbol{C} \to X)^{\mathrm{op}}$-diagram of abelian groups given by $T_X(Y \to X) = \mathrm{Hom}^*_{\mathcal{K}_0}(B_Y, C_X)$. We conclude by (1.1.2) that $\mathrm{RKan}^l{}_{j_X} T_X$ vanishes for $l > 0$ while for $l = 0$ it is canonically isomorphic to S_X. Using this and the spectral sequence

$$E_2^{k,l} = \lim_{\mathrm{Sub}(\boldsymbol{C})}{}^k \mathop{\mathrm{RKan}^l}_{j_X} T_X \Rightarrow \lim_{(\boldsymbol{C}\to X)^{\mathrm{op}}}{}^{k+l} T_X,$$

we get

$$\lim_{\mathrm{Sub}(\boldsymbol{C})}{}^k S_X \cong \lim_{(\boldsymbol{C}\to X)^{\mathrm{op}}}{}^k T_X \cong \begin{cases} T_X(X \xrightarrow{\mathrm{Id}} X) = \mathrm{Hom}^*_{\mathcal{K}_{\underline{0}}}(B_X, C_X) & \text{if } k = 0 \\ 0 & \text{if } k > 0, \end{cases}$$

where the last identity holds since $X \xrightarrow{\mathrm{Id}} X$ is an initial object of $(\boldsymbol{C} \to X)^{\mathrm{op}}$. Combining this with (39), we get (38), completing the proof of (32).

To prove (35), we apply the cohomological functor F to the resolution (29) and identify the E_2-term of the resulting spectral sequence

$$E_1^{p,q} = F^{p+q}(\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}} \tilde{P}\tilde{R}^{-p} A) \Rightarrow F^{p+q}(\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}} A).$$

with the left hand side of (35). As in the proof of (32), one sees that the $\boldsymbol{C}$-diagrams $F((\tilde{P}\tilde{R}^{-q}A)_Y)_{Y\in\mathfrak{Ob}(\boldsymbol{C})}$ form a resolution of $F(A_Y)_{Y\in\mathfrak{Ob}(\boldsymbol{C})}$. But

$$F((\tilde{P}\tilde{R}^{-q}A)_Y)_{Y\in\mathfrak{Ob}(\boldsymbol{C})} = \bigoplus_{X\in\mathfrak{Ob}(\boldsymbol{C})} \underset{i_X}{\mathrm{LKan}}\, F(\tilde{R}^{-q}A_X),$$

with vanishing higher LKan_i. Therefore,

$$\underset{\boldsymbol{C}}{\mathrm{colim}}\left(F^*((\tilde{P}\tilde{R}^{-q}A)_X)_{X\in\mathfrak{Ob}(\boldsymbol{C})}\right) = F^*(\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}} \tilde{P}\tilde{R}^{-q}A),$$

without higher derived limits, proving the necessary formula for the E_2-term.

The spectral sequence (33) is constructed in the same way, using (28). The spectral sequences (36) and (34) are dual to the previous two cases. □

Remark 3. Throughout the preceding subsections, we have used the notion of a category in the usual sense, i.e., categories are assumed to have (small) Hom-sets but may have a proper class of objects. However, for the discussion of quotient categories in 1.5.2, it will be convenient to consider categories with proper Hom-classes. Let us call such mathematical structures paracategories. It is easy to see that this notion can be formulated in terms of the von Neumann-Bernays-Gödel axioms of set theory, and that all our preceding results still hold for paracategories. The spectral sequence (32) is a spectral sequence of paragroups (i. e., classes with an underlying group structure). There is no notion of a paracategory of paragroups, since a proper class cannot be element of a class of objects. However, it is still possible to define the notion of a diagram of paragroups (shaped by a small category), and of the limit of such a diagram. In particular, (37) holds. The same remark applies to the proof of the other spectral sequences in Proposition 10. If the E_2-term of a finitely convergent spectral sequence of paragroups is in fact a group, then its limit is also a group. Therefore, the spectral sequence (32) proves that if $\mathcal{K}_{\underline{0}}$ in a system of triangulated diagram paracategories is in fact a category, then all $\mathcal{K}_{\boldsymbol{C}}$ are categories in the usual sense.

When one wants to construct the category of fractions of a big category by a multiplicative class of morphisms as a paracategory, one has to form a quotient of a (possibly proper) class by an equivalence relation. We will assume in addition to the von Neumann-Bernays-Gödel axioms that such quotients exist. In other words, we assume that for a class X and an equivalence relation R on X there exists a class Y and a surjective map $X \xrightarrow{f} Y$ such that for $x, y \in X$ we have $f(x) = f(y)$ if and only if $(x, y) \in R$. This is the case if the class of all sets is well-ordered, a strengthened version of the axiom of choice whose consistency is proved in [Gö40].

1.5. Systems of Functors. Let $\mathfrak{K}$ be one of the 2-categories $\mathfrak{P}_U$, $\tilde{\mathfrak{P}}_U$, $\mathfrak{C}_U$, or $\tilde{\mathfrak{C}}_U$, where U is an infinite class.

1.5.1. *Systems of triangulated functors.*

Definition 6. Let $\mathcal{K}_{\boldsymbol{C}}$ and $\mathcal{L}_{\boldsymbol{C}}$ be systems of triangulated diagram categories. A compatible system of functors consists of a collection $F_{\boldsymbol{C}} \colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{L}_{\boldsymbol{C}}$ for every $\boldsymbol{C} \in \mathfrak{K}$ and of natural isomorphisms

$$f^* F_{\boldsymbol{D}} \cong F_{\boldsymbol{C}} f^* \tag{1}$$

for every functor $f \colon \boldsymbol{C} \to \boldsymbol{D}$. These natural transformations must be compatible with the ones in (1.2.1) in the sense that one has a morphism between 2-functors from $\mathfrak{K}$ to the 2-category of categories.

In order to avoid awkward notations, we will often just write F instead of $F_{\boldsymbol{C}}$.

The composition of two systems of functors is defined in the obvious way. A compatible system of natural transformations between two compatible systems of functors consists of a natural transformation $F_{\boldsymbol{C}} \to G_{\boldsymbol{C}}$ for every $\boldsymbol{C}$, which have to be compatible with (1) in the sense that they constitute a bimorphism between morphisms between bifunctors.

If such a system of functors is given, then we have natural transformations

$$\operatorname{Ho} \underset{f}{\mathrm{LKan}}\, F_{\boldsymbol{C}} \to F_{\boldsymbol{D}} \operatorname{Ho} \underset{f}{\mathrm{LKan}} \tag{2}$$

$$F_{\boldsymbol{D}} \operatorname{Ho} \underset{f}{\mathrm{RKan}} \to \operatorname{Ho} \underset{f}{\mathrm{RKan}}\, F_{\boldsymbol{C}} \tag{3}$$

defined by (1) and the universality property of the homotopy Kan extensions. It is natural to ask when these natural transformations are isomorphisms.

Theorem 2. • *The following assertions are equivalent:*

a. *For every f, (2) is an isomorphism.*
b. *The same condition, but applied only to $f \colon \boldsymbol{C} \to \boldsymbol{D}$, i. e., to those functors which do not map anything nontrivial to $\star$.*
c. *For every f, (3) is an isomorphism.*
d. *The same condition, but applied only to $f \colon \boldsymbol{C} \to \boldsymbol{D}$.*
e. *For every $\boldsymbol{C}$, $F_{\boldsymbol{C} \times \square}$ respects bicartesian squares.*
f. *For every $\boldsymbol{C}$, $F_{\boldsymbol{C}}$ can be made into a triangulated functor such that the isomorphisms* (1) *are triangulated, where f^* is given the structure of a triangulated functor defined by the fact Proposition* 1.4.5 *that it preserves bicartesian squares.*

- *If these equivalent conditions are satisfied, then the collection of triangulated structures on the functors $F_{\boldsymbol{C}}$ satisfying f is unique. More generally, if F and G satisfy these conditions and if $F_{\boldsymbol{C}} \xrightarrow{\phi_{\boldsymbol{C}}} G_{\boldsymbol{C}}$ is a compatible system of natural transformations, then for every $\boldsymbol{C}$ $\phi_{\boldsymbol{C}}$ is a triangulated functormorphism.*
- *These conditions are always satisfied in the case where $F_{\boldsymbol{C}}$ has a right adjoint for every $\boldsymbol{C}$ which is also a compatible system of functor, or in the case where it has left adjoints forming such a system. For instance, if $g\colon \boldsymbol{E}^\star \to \boldsymbol{D}^\star$ is a functor, these conditions are all satisfied for the functors*

$$\begin{aligned}(g \times \mathrm{Id}_{\boldsymbol{C}})^*\colon \mathcal{K}_{\boldsymbol{D}\times\boldsymbol{C}} &\to \mathcal{K}_{\boldsymbol{E}\times\boldsymbol{C}}\\ \operatorname{Ho}\operatorname*{LKan}_{g\times\mathrm{Id}_{\boldsymbol{C}}}\colon \mathcal{K}_{\boldsymbol{E}\times\boldsymbol{C}} &\to \mathcal{K}_{\boldsymbol{D}\times\boldsymbol{C}}\\ \operatorname{Ho}\operatorname*{RKan}_{g\times\mathrm{Id}_{\boldsymbol{C}}}\colon \mathcal{K}_{\boldsymbol{E}\times\boldsymbol{C}} &\to \mathcal{K}_{\boldsymbol{D}\times\boldsymbol{C}}.\end{aligned}$$

Proof. In the first assertion, the implications

$$\text{b or d} \Rightarrow \text{c or a} \Rightarrow \text{e} \Rightarrow \text{f}$$

are trivial. It is now sufficient to derive c and a from f. By (1.4.30), (3) is an isomorphism in the case of $\operatorname{Ho}\operatorname{RKan}_{i_X}$. Let $f\colon \boldsymbol{C}^\star \to \boldsymbol{D}^\star$ be a functor. Applying our previous remark to $\operatorname{Ho}\operatorname{RKan}_{i_X}$ and $\operatorname{Ho}\operatorname{RKan}_{i_{f(X)}}$ (which is to be interpreted as the zero functor if $f(X) = \star$), we see that for any object A of $\mathcal{K}_{\boldsymbol{C}}$, the morphism (3) applied to PA is an isomorphism, where PA is the same as in (1.4.27). The functor $\operatorname{Ho}\operatorname{RKan}_f$ is a triangulated functor since it commutes with other right homotopy Kan extension functors, in particular with homotopy fibres. From (1.4.28) we conclude therefore that (3) is an isomorphism for arbitrary A since it is an isomorphism for PR^kA. The case of $\operatorname{Ho}\operatorname{LKan}$ is similar, using the resolution (1.4.29).

The second assertion follows by applying [Kel91, Lemma 7.1.a)] to the epivalent tower $(\left(\mathcal{K}_{\boldsymbol{C}\times\underline{1}^n}\right)_{n\in\mathbb{Z}}$.

For the third point, let us first verify that for each of the functors considered at the end of the theorem, at least one of the six equivalent conditions is satisfied. If F has right adjoints forming a compatible system of functors, it clearly commutes with $\operatorname{Ho}\operatorname{LKan}$. If left adjoints with the same property exist, it commutes with $\operatorname{Ho}\operatorname{RKan}$. The functor g^* respects bicartesian squares by the remarks leading to Proposition 1.4.4. For $\operatorname{Ho}\operatorname{LKan}_{g\times\mathrm{Id}_{\boldsymbol{c}}}$ and $\operatorname{Ho}\operatorname{RKan}_{g\times\mathrm{Id}_{\boldsymbol{c}}}$, we have already verified that they are triangulated, by an argument which depends on the fact that they have an adjoint functor g^* which forms a compatible system of functors. □

Definition 7. A compatible system of triangulated functors $\mathcal{K} \to \mathcal{L}$ is a compatible system of functors satisfying the equivalent conditions of the above theorem. If the class of objects of the bicategory $\mathfrak{K}$ is a set and if for every $\mathfrak{C}$, $\mathfrak{K}_{\boldsymbol{C}}$ is a small category, then let $\mathfrak{Fun}_\triangle(\mathcal{K}, \mathcal{L})$

be the category which has systems of triangulated functors from $\mathcal{K}$ to $\mathcal{L}$ as objects and compatible systems of natural transformations as morphisms.

Proposition 1. *Let $F_{\boldsymbol{C}}\colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{L}_{\boldsymbol{C}}$ be a system of triangulated functors.*

- *If $F_{\underline{0}}$ is an equivalence of categories, then so is $F_{\boldsymbol{C}}$ for every $\boldsymbol{C} \in \mathfrak{K}$.*
- *Let $\phi_{\boldsymbol{C}}\colon F_{\boldsymbol{C}} \to G_{\boldsymbol{C}}$ be a compatible system of natural transformations from F to a second compatible system of triangulated functors. If $\phi_{\underline{0}}$ is an isomorphism, then so is $\phi_{\boldsymbol{C}}$ for every $\boldsymbol{C} \in \mathfrak{K}$.*

Proof. The second assertion immediately follows from (1.4.32). For the same reason, in the first assertion the functor $F_{\boldsymbol{C}}$ is full and faithful. Its essential image is therefore a full triangulated subcategory, which contains all objects of the form PA in (1.4.28) since $F_{\underline{0}}$ is essentially surjective. It follows that $F_{\boldsymbol{C}}$ is essentially surjective. □

1.5.2. *Thick Subcategories.* Recall that a thick subcategory of a triangulated category is a full subcategory closed under forming extensions (i.e, the third edge of a distinguished triangle) and direct summands. This is not literally Verdier's definition, but is equivalent to it by a theorem of Rickard [Ric89, Proposition 1.3.] (cf. also [Nee90, Criterion 1.3.]). One wants to have a notion of a factor category by a full subcategory. This factor category should be defined by inverting all morphisms whose cone is in the thick subcategory. This is possible, but the result of this procedure will in some cases be a paracategory (in the sense of Remark 1.4.3). It is necessary to assume that the quotient of a class by an equivalence relation always exists, cf. the remarks made at the end of Remark 1.4.3. We will say that the quotient category of a triangulated category by a thick subcategory exists if the quotient paracategory is in fact a category. There are examples of quotients of triangulated categories by thick subcategories which do not exist in the usual sense, but only as paracategories. For instance, [Fre66, Exercise 6A on p. 131] is an example of an abelian category whose derived category exists only as a paracategory.

Theorem 3. *Let $\mathcal{K}$ be a system of triangulated diagram categories and let $\mathcal{E}_{\underline{0}}$ be a full triangulated subcategory of $\mathcal{K}_{\underline{0}}$.*

- *Let $\mathcal{E}_{\boldsymbol{C}}$ be the full subcategory containing all objects A of $\mathcal{K}_{\boldsymbol{C}}$ with $A_X \in \mathcal{E}_{\underline{0}}$ for every X. Then $\mathcal{E}_{\boldsymbol{C}}$ is a system of triangulated diagram categories, and the functors*
$$\mathcal{E}_{\boldsymbol{C}} \to \mathcal{K}_{\boldsymbol{C}}$$
form a system of triangulated functors.
- *Let in addition $\mathcal{E}_{\underline{0}}$ be a thick subcategory. Then $\mathcal{E}_{\boldsymbol{C}}$ is a thick subcategory of $\mathcal{K}_{\boldsymbol{C}}$ for every $\boldsymbol{C}$. If $\mathcal{K}_{\underline{0}}/\mathcal{E}_{\underline{0}}$ exists, then the quotient categories $\mathcal{K}_{\boldsymbol{C}}/\mathcal{E}_{\boldsymbol{C}}$ exist and form a system of triangulated diagram categories, and the functors*
$$\mathcal{K}_{\boldsymbol{C}} \to (\mathcal{K}/\mathcal{E})_{\boldsymbol{C}}$$
form a system of triangulated functors.

Proof. To prove that $\mathcal{E}$ is a system of triangulated diagram categories, it is sufficient to verify that it is closed under taking homotopy Kan extensions. This is clear for Kan extensions along the embedding i_X, by (1.4.30) and its dual. By an application of (1.4.29) and (1.4.28), one derives the general case from this special case.

It remains to verify the second point. That $\mathcal{E}_{\boldsymbol{C}}$ is a thick subcategory if $\mathcal{E}_{\underline{0}}$ is a thick subcategory is clear. In general, we know that $\mathcal{K}_{\boldsymbol{C}}/\mathcal{E}_{\boldsymbol{C}}$ exists as a paracategory. Our arguments will focus on showing that this system of quotient categories is a system of triangulated diagram paracategories. Once this is established, it follows from Remark 1.4.3 that $\mathcal{K}_{\boldsymbol{C}}/\mathcal{E}_{\boldsymbol{C}}$ exists as a usual category if $\mathcal{K}_{\underline{0}}/\mathcal{E}_{\underline{0}}$ does.

Since the pull backs and homotopy Kan extensions preserve $\mathcal{E}$, they pass to the factorcategories. Using the description of the factorcategories by means of a calculus of fraction, one sees that the necessary adjointness relations remain valid.

For the factor category, the functoriality, isomorphism, disjoint union, and homotopy Kan extension axioms are now trivial. Since we know that passage to the factorcategory commutes with Kan extension, a square in $(\mathcal{K}/\mathcal{E})_{\square}$ is cartesian (resp. cocartesian) if and only if it is the image of a cartesian (resp. cocartesian) square in $\mathcal{K}_{\square}$, and these two conditions are equivalent. The essential surjectivity of the functor

$$(\mathcal{K}/\mathcal{E})_{\boldsymbol{C}\times\underline{1}} \to \mathfrak{Ar}\Big((\mathcal{K}/\mathcal{E})_{\boldsymbol{C}}\Big)$$

follows easily from the description of the localisation by a calculus of fractions. To prove its fullness, let S be the set of all morphisms in $\mathcal{K}_{\boldsymbol{C}}$ whose cone belongs to $\mathcal{E}_{\boldsymbol{C}}$ and let $\tilde{S}$ be the set of all morphisms in $\mathfrak{Ar}(\mathcal{K}_{\boldsymbol{C}})$ whose two components belong to S, then the standard proofs of axiom TR3 for the quotient category (for instance, the proof in [ΓM88, §4.2.6]) prove the fullness of the functor

$$\tilde{S}^{-1}\,\mathfrak{Ar}(\mathcal{K}_{\boldsymbol{C}}) \to \mathfrak{Ar}(S^{-1}\mathcal{K}_{\boldsymbol{C}}).$$

Therefore, the mapping cylinder axiom for $\mathcal{K}/\mathcal{E}$ follows from the mapping cylinder axiom for $\mathcal{K}$.

The verification of the axioms for a system of triangulated diagram paracategories is now complete. The inclusion and quotient functors mentioned at the end of the theorem are triangulated because we have already convinced ourselves that they commute with homotopy Kan Extensions. □

Corollary 1. *For any abelian category $\mathcal{A}$ and any self-equivalence T of $\mathcal{A}$, the homotopy categories* (1.3.9) *form a system of triangulated diagram paracategories. If they exist as usual categories, then this system is a system of triangulated diagram categories. This is the case if and only if $\mathcal{D}^{(T,N)}(\mathcal{A})$ exists as a usual category.*

Indeed, one can represent (1.3.9) as the quotient of (1.3.10) by the thick subcategory of acyclic complexes.

1.5.3. *Partial equivalences.* In the cases relevant to homotopy theory, we will only get a partial uniqueness result for systems of triangulated diagram categories with an Adams spectral sequence, giving a description for $\mathcal{K}_{\boldsymbol{C}}$ only when $\dim \boldsymbol{C}$ is not too big.

Definition 8. A compatible system of equivalences $F_{\boldsymbol{C}}\colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{L}_{\boldsymbol{C}}$ in dimension $\dim \boldsymbol{C} < L$ is a collection of equivalences of categories $F_{\boldsymbol{C}}$ for $\dim \boldsymbol{C} < L$, together with natural transformations (1) satisfying the same compatibility assumption as in (1). A compatible system of functor-isomorphisms between two compatible system of equivalences in dimension $< L$ is defined by restricting Definition 6 in the same way.

If F is a compatible system of equivalences in dimension $< L$, then $F_{\boldsymbol{C}}$ can be given a canonical structure of a triangulated functor if $\dim \boldsymbol{C} < L - 1$. This follows from the description of the suspension functor and the distinguished triangles in terms of the functor cone in Remark 1.4.2.

1.6. Strong Linearity.

1.6.1. *The distinguished role of stable homotopy.* The distinguished role of stable homotopy is expressed the following theorem:

Theorem 4. *Let $\mathcal{S}^{\mathrm{fin}}$ be the system of homotopy categories of $\mathfrak{K}$-diagrams of spectra with finitely many cells. By replacing it by an equivalent system of mall subcategories, we may assume that the categories $\mathcal{S}^{\mathrm{fin}}_{\boldsymbol{C}}$ are small. If the class of objects of $\mathfrak{K}$ is a set, then the evaluation of functors at the sphere spectrum defines an equivalence of categories*

$$\mathfrak{Fun}_{\triangle}(\mathcal{S}^{\mathrm{fin}}, \mathcal{K}) \cong \mathcal{K}_{\underline{0}}. \tag{1}$$

In the case where the class of objects of $\mathfrak{K}$ is a proper class, or if one does not make the assumption that the categories $\mathcal{S}^{\mathrm{fin}}_{\boldsymbol{C}}$ are small (such that the category $\mathfrak{Fun}_{\triangle}(\mathcal{S}^{\mathrm{fin}}, \mathcal{K})$ cannot be defined), it is still true that a compatible system of triangulated functors F from $\mathcal{S}^{\mathrm{fin}}_{\boldsymbol{C}}$ to $\mathcal{K}$ is determined uniquely up to unique compatible system of functor-isomorphisms by its value at the sphere spectrum, that every object of $\mathcal{K}_{\underline{0}}$ occurs in this way as $F_{\underline{0}}S^0$, and that a compatible system of functor-morphisms is uniquely determined by its value at S^0.

We will defer the somewhat technical proof of the theorem to the end of this subsection. That such a theorem holds should not be too surprising, since the stable homotopy category can be expressed through the combinatorics of the posets $\boldsymbol{C}$.

Corollary 1. *There is a unique way to define smash products*

$$\mathcal{S}^{\mathrm{fin}}_{\boldsymbol{C}} \times \mathcal{K}_{\boldsymbol{D}} \to \mathcal{K}_{\boldsymbol{C}\times\boldsymbol{D}}$$

which are systems of triangulated functors in both arguments, and such that the smash product by the sphere spectrum is the identity functor. For these smash products, there is a unique way to define an associativity law

$$P \wedge \Big(Q \wedge A\Big) \cong \Big(P \wedge Q\Big) \wedge A, \quad A \in \mathcal{K}_{\boldsymbol{E}}, P \in \mathcal{S}^{\mathrm{fin}}_{\boldsymbol{C}}, Q \in \mathcal{S}^{\mathrm{fin}}_{\boldsymbol{D}}, \tag{2}$$

which, for given P, Q, is a compatible system of natural transformations in A. This associativity law is automatically natural in P and Q as well and makes the pentagon axiom [DM82, 1.0.1.] *commutative. Moreover, if $\boldsymbol{C}$ is any poset and if $\boldsymbol{\pi_C}\colon \boldsymbol{C} \to \underline{0}$ is the unique functor, then*

$$\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}} \boldsymbol{\pi}_{\boldsymbol{C}}^* A \cong \Big(Q(|N.\boldsymbol{C}|_+)\Big) \wedge A, \tag{3}$$

where $\Big|N.\boldsymbol{C}\Big|_+$ is the classifying space of $\boldsymbol{C}$ with a point added and Q is the suspension spectrum, and

$$\underleftarrow{\mathrm{Holim}}_{\boldsymbol{C}} \boldsymbol{\pi}_{\boldsymbol{C}}^* A \cong \Big(DQ(|N.\boldsymbol{C}|_+)\Big) \wedge A, \tag{4}$$

where D is the Spanier-Whitehead duality. More over, if X is a finite spectrum, then the functor $DX\wedge ?$ is both left and right adjoint to the functor $X \wedge ?$, i. e.,

$$\mathrm{Hom}_{\mathcal{K}_{\underline{0}}}\Big(A, X \wedge B\Big) = \mathrm{Hom}_{\mathcal{K}_{\underline{0}}}(DX \wedge A, B) \tag{5}$$

Indeed, that a smash product which is a system of triangulated functors in the first argument exists and is unique is an immediate consequence of the theorem, applied to the system $\mathcal{K}_{\boldsymbol{D}\times ?}$. Its functorial properties in the second variable also follow readily from the theorem. The existence, uniqueness, and naturality of the associativity law follows by applying, for $A \in \mathcal{K}_{\boldsymbol{E}}$ and $Q \in \mathcal{S}_{\mathcal{D}}^{\mathrm{fin}}$, the theorem to the two compatible systems of triangulated functors from $\mathcal{S}_{\boldsymbol{C}}^{\mathrm{fin}}$ to $\mathcal{K}_{\boldsymbol{C}\times\boldsymbol{D}\times\boldsymbol{E}}$ given by $P \wedge (Q \wedge A)$ and $(P \wedge Q) \wedge A$, both of which send the sphere spectrum to $Q \wedge A$. The pentagon axiom is also derived that way. The relation between inductive homotopy limits and smash products will be proved in the course of the proof of the theorem.

1.6.2. *Strong linearity.*

Definition 9. A strongly linear structure for $\mathcal{K}$ consists of the following data: A system of functors

$$\overset{\mathrm{L}}{\otimes}\colon D^{b,\mathrm{fin}}(\mathbb{Z}^{\boldsymbol{C}}) \times \mathcal{K}_{\boldsymbol{D}} \to \mathcal{K}_{\boldsymbol{C}\times\boldsymbol{D}} \tag{6}$$

together with compatible isomorphisms

$$\alpha_{\boldsymbol{P}}\colon P \wedge A \cong \mathbb{C}(E) \overset{\mathrm{L}}{\otimes} A \tag{7}$$

for $P \in \mathcal{S}_{\boldsymbol{C}}^{\mathrm{fin}}$ and $A \in \mathcal{K}_{\boldsymbol{D}}$, where the functor $\mathbb{C}$

$$\mathcal{S}_{\boldsymbol{C}}^{\mathrm{fin}} \xrightarrow{\mathbb{C}} D^{b,\mathrm{fin}}(\mathbb{Z}^{\boldsymbol{C}}) \tag{8}$$

is obtained from the composition of the homology chain complex functor

finite spectra $\longrightarrow$ {$\boldsymbol{C}$-diagrams of chain complexes of finitely generated abelian groups} and the identification

$$\begin{aligned} \text{chain complexes} &\xrightarrow{I} \text{cochain complexes} \\ (A_k, d) &\longrightarrow (I(A)^k = A_{-k}, d), \end{aligned}$$

and a system of natural transformations

$$\beta_{K,L}\colon K \overset{\mathrm{L}}{\otimes} \left(L \overset{\mathrm{L}}{\otimes} A\right) \cong \left(K \overset{\mathrm{L}}{\otimes} L\right) \overset{\mathrm{L}}{\otimes} A \tag{9}$$

which is compatible with the associativity law (2) and satisfies the pentagon axiom [DM82, 1.0.1.]. An isomorphism between two strongly linear structures $\overset{\mathrm{L}}{\otimes}$ and $\overset{\mathrm{L}}{\boxtimes}$ is a collection of natural transformations:

$$K \overset{\mathrm{L}}{\otimes} A \cong K \overset{\mathrm{L}}{\boxtimes} A \tag{10}$$

which are compatible with the isomorphisms (7) and (9) for $\overset{\mathrm{L}}{\otimes}$ and $\overset{\mathrm{L}}{\boxtimes}$. If $\mathcal{K}$ and $\mathcal{L}$ are two strongly linear systems of triangulated diagram categories, then a strongly linear system of functors F betwixt them is a compatible system of triangulated functors, together with a system of natural transformations $F(K \overset{\mathrm{L}}{\otimes} A) \to K \overset{\mathrm{L}}{\otimes} F(A)$ which are compatible with the transformations (7) and (9).

Remark 1. Since every finitely generated chain complex is quasi-isomorphic to the homology chain complex of a finite spectrum, it follows that the natural transformations (9) and (10) are isomorphisms in the case where K and L is just a chain complex of abelian groups. By Proposition 1.5.1, they are isomorphisms in general.

For the same reasons, in the case of cochain complexes of abelian groups the natural transformations (9) and (10) are uniquely determined if they exist. This need no longer be the case for the derived tensor product by a diagram of abelian groups. The reason is that for arbitrary $\boldsymbol{C}$, there may be objects of $D(\mathbb{Z}^{\boldsymbol{C}})$ which can not be obtained from any $\boldsymbol{C}$-diagram of spectra. Since the endomorphism group of the (mod 2)-Moore spectrum is known to be $\mathbb{Z}/4\mathbb{Z}$, the diagram of abelian groups

$$\begin{tikzcd}
\mathbb{Z} \arrow[rrr] \arrow[dr, "2"] \arrow[ddd, "2"'] & & & 0 \arrow[dl] \arrow[ddd] \\
& \mathbb{Z} \arrow[r] \arrow[d, "2"'] & 0 \arrow[d] & \\
& \mathbb{Z} \arrow[r, "\mathrm{mod}\,2"] & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z} \arrow[ur, "2"'] \arrow[rrr, "\mathrm{mod}\,2"] & & & \mathbb{Z}/2\mathbb{Z} \arrow[ul] \arrow[uul]
\end{tikzcd}$$

regarded as an object of the derived category of diagrams of abelian groups, is a example of this behaviour. Therefore, it seems to be necessary to include (9) into the definition of a strongly linear structure rather than just requiring its existence. Also, the isomorphism class of $K \overset{\mathrm{L}}{\otimes} A$ may possibly depend on the strongly linear structure if K is a diagram of cochain complexes. Similarly, I cannot exclude the possibility that a strongly linear structure has automorphisms. There is no straightforward way to derive the pentagon diagram for tensor products by diagrams of cochain complexes from its counterpart for the smash product by diagrams of spectra.

Proposition 1. *If* $\mathrm{Hom}_{\mathcal{K}_0}(A, B)$ *is a* $\mathbb{Q}$*-vector space for all* A *and* B*, then there exists a strongly linear structure on* $\mathcal{K}$*, and it is unique up to unique isomomorphism. If* $\mathcal{L}$ *is another* $\mathbb{Q}$*-rational system of triangulated diagram categories, then any compatible system of triangulated functors from* $\mathcal{K}$ *to* $\mathcal{L}$ *can be made into a strongly linear system in a unique way.*

Proof. The proposition follows immediately if we prove the the homology chain complex functor from $\mathcal{S}^{\mathrm{fin}}_{\boldsymbol{C}} \otimes \mathbb{Q}$ to $D^b(\mathbb{Q}^{\boldsymbol{C}})$ is an equivalence of categories. But for $\boldsymbol{C} = \underline{0}$ this is a well known and straightforward consequence of Serre's calculation of the rational homotopy of spheres, and for arbitrary $\boldsymbol{C}$ it follows by Proposition 1.5.1. $\square$

Remark 2. In general there are, of course, several obstructions against the existence of strongly linear structures. For instance, by Corollary 1 $\mathrm{Hom}_{\mathcal{K}_{\boldsymbol{C}}}(X, Y)$ is a module over the stable homotopy of spheres $\pi_*(S^0)$, and the existence of a strongly linear structure implies that the elements of positive degree act trivially.

1.6.3. *Finite posets and finite simplicial sets.* For the proof of Theorem 4, We need the fact that the homotopy theory of posets (i. e., the category of posets with the homotopy equivalences inverted) is equivalent to the homotopy of finite simplicial sets.

Let $\boldsymbol{D}$ be any finite finite-dimensionalcategory, and let $\mathfrak{P}^{\boldsymbol{D}}$ be the category of functors from $\boldsymbol{D}$ to the category of posets. A morphism $F \to G$ in $\mathfrak{P}^{\boldsymbol{D}}$ is a homotopy equivalence if $F_X \to G_X$ is a homotopy equivalence for every X. Let η be the class of homotopy equivalences. The nerve functor to the diagram category of finite simplicial sets

$$N \colon \mathfrak{P}^{\boldsymbol{D}} \to \mathrm{sim\,sets}_{\mathrm{fin}}^{\boldsymbol{D}}$$

preserves homotopy equivalences, hence

$$N \colon h^{-1}\mathfrak{P}^{\boldsymbol{D}} \to \mathrm{Ho}(\mathrm{sim\,sets}_{\mathrm{fin}}^{\boldsymbol{D}}). \tag{11}$$

Proposition 2. *The functor* (11) *is an equivalence of categories.*

The proof uses work of Fritsch and Latch about inverses for the nerve functor. An alternative way would be to follow the indications at the beginning of [Qui73, §1].

Let Sd be Kan's subdivision functor for simplicial sets [Kan57]. It is easy to see that it preserves finite simplicial sets. Recall the natural weak equivalence [Kan57, Lemma 7.5]

$$\mathrm{Sd}X \longrightarrow X. \tag{12}$$

There is work of Fritsch and Latch [FL79, Theorems 3.1. and 3.4] showing that for a finite simplicial set X one has a finite poset $c\mathrm{Sd}^2X$ and a canonical weak equivalence

$$\mathrm{Sd}^2X \longrightarrow Nc\mathrm{Sd}^2,$$

which, together with (12), proves Proposition 2.

1.6.4. *Products by finite simplicial sets.* For each $\boldsymbol{C} \in \mathfrak{P}^{\boldsymbol{D}}$, viewed as a cofibred category $\overline{\boldsymbol{C}} \xrightarrow{p} \boldsymbol{C}$, denote by q the projection from $\overline{\boldsymbol{C}}$ to $\underline{0}$, and put

$$\boldsymbol{C} \frown A = \mathrm{Ho}\,\underset{p}{\mathrm{LKan}}\, q^*A \in \mathfrak{Ob}\,\mathcal{K}_{\boldsymbol{D}} \tag{13}$$

for $A \in \mathfrak{Ob}\,\mathcal{K}_{\underline{0}}$. It is clear that this is a functor from $\mathfrak{P}^{\boldsymbol{D}} \times \mathcal{K}_{\underline{0}}$ to $\mathcal{K}_{\boldsymbol{D}}$. It is an easy consequence of Proposition 1.4.3.b. that for any functor $f \colon \boldsymbol{E}_\star \to \boldsymbol{D}_\star$, (13) satisfies

$$f^*(\boldsymbol{C} \frown A) \cong (f^*\boldsymbol{C}) \frown A. \tag{14}$$

By Proposition 1.4.3.b. and (1.4.35), we have a homological spectral sequence

$$H_p(\boldsymbol{C}_X, F_q(A)) \Rightarrow F_{p+q}\big((\boldsymbol{C} \frown A)_X\big)$$

for every covariant homological functor F on $\mathcal{K}_{\underline{0}}$. This spectral sequence proves that (13) takes homotopy equivalences between diagrams of posets to isomorphisms. By Proposition 2, it can be obtained by the composition of the realisation functor with a product functor

$$\frown \colon \mathrm{Ho}(\mathrm{sim\,sets}^{\boldsymbol{D}}_{\mathrm{fin}}) \times \mathcal{K}_{\underline{0}} \to \mathcal{K}_{\boldsymbol{D}}. \tag{15}$$

For these functors, the analog of (14) holds. We want to prove that for any $\boldsymbol{D} \xrightarrow{f} \boldsymbol{E}$ and every $X \in \mathrm{sim\,sets}^{\boldsymbol{D}}_{\mathrm{fin}}$, the canonical homomorphism

$$\mathrm{Ho}\,\underset{f}{\mathrm{LKan}}(T \frown A) \to (\mathrm{Ho}\,\underset{f}{\mathrm{LKan}}\, T) \frown A \tag{16}$$

derived from the analog of (14) is an isomorphism. It suffices to prove this if T is the realisation of a diagram cofibrant diagram of ordered simplicial complexes M. Let N be the left Kan extension of M along f, since M is diagram cofibrant the realisation of N is the homotopy Kan extension of X. Let $\mathcal{N}M$ (resp. $\mathcal{N}N$) be the categories of simplices of M (resp. N), viewed as cofibred categories over $\boldsymbol{D}$ (resp. $\boldsymbol{E}$) such that the fibre at X is the poset of simplices of M_X (resp. N_X). Let $g\colon \mathcal{N}M \to \mathcal{N}N$, $p\colon \mathcal{N}M \to \boldsymbol{D}$, $q\colon \mathcal{N}N \to \boldsymbol{E}$, $r\colon \boldsymbol{D} \to \underline{0}$, $s\colon \boldsymbol{E} \to \underline{0}$ be the canonical functors. We have $T \frown A = \mathrm{Ho}\,\mathrm{LKan}_p(rp)^*A$, hence

$$\mathrm{Ho}\,\underset{f}{\mathrm{LKan}}(X \frown A) \cong \mathrm{Ho}\,\underset{q}{\mathrm{LKan}}\,\mathrm{Ho}\,\underset{g}{\mathrm{LKan}}(rp)^*A.$$

Since $(\mathrm{Ho}\,\mathrm{LKan}_f\, T) \frown A = \mathrm{Ho}\,\mathrm{LKan}_q(sq)^*A$, it suffices to prove that $\mathrm{Ho}\,\mathrm{LKan}_g(rp)^*A \cong (sq)^*A$. In view of the following lemma, this follows from (1.4.35) and Proposition 1.4.2.

Lemma 1. *In the situation described above, for every $\sigma \in \mathfrak{Ob}\,\mathcal{N}N$ the category $g \Rightarrow \sigma$ is contractible.*

Proof. Let $Y = q(\sigma)$, replacing $\boldsymbol{D}$ by $f \Rightarrow Y$ and $\boldsymbol{E}$ by $\{Y\}$, we may assume, without altering our assumption that M is diagram cofibrant, that $\boldsymbol{E}$ consists of a single point. Now σ is a simplex in the inductive limit of M, and $g \Rightarrow \sigma$ is the category of pairs (X, τ), where τ is a simplex of M_X which maps to a boundary simplex of σ in the inductive limit N. Associating to this pair the image of τ in N, we get a functor h from $g \Rightarrow \sigma$ to the poset $\hat{\sigma}$ of boundary simplices of σ. Since $\hat{\sigma}$ has a finial object σ and is thus contractible, it suffices to prove that h is a homotopy equivalence. But h makes $g \Rightarrow \sigma$ into a fibred category over $\hat{\sigma}$, and the fibre at $\vartheta \in \hat{\sigma}$ is the category $K_\vartheta(M)$ of pairs (X, τ), where the simplex τ of M_X maps to ϑ in the inductive limit. If we can prove that K_ϑ has an initial object, then it is contractible and it follows from [Qui73, Theorem A] that h is a homotopy equivalence, proving the lemma.

The existence of an initial object of $K_\tau(M)$ will be proved by induction on the dimension d of $\boldsymbol{D}$, starting from the trivial case where $d = 0$, and the number of d-dimensional simplices in $\boldsymbol{D}$, starting from the void case where this number is zero. Let X be a maximal object of $\boldsymbol{D}$ and $\tilde{\boldsymbol{D}} = \boldsymbol{D} - X$. Since M is diagram cofibrant, we have

$$\operatorname*{colim}_{\tilde{\boldsymbol{D}}} M \subseteq \operatorname*{colim}_{\boldsymbol{D}} M, \tag{17}$$

and it makes sense to speak of $K_\vartheta(M\,|_{\tilde{\boldsymbol{D}}})$. If no object of the form (X,τ) of $K_\tau(M)$ exists, then $K_\vartheta(M) = K_\vartheta(M\,|_{\tilde{\boldsymbol{D}}})$, and the induction assumption applies. If it exists, then τ is unique. If it does not belong to the subcomplex $\operatorname{colim}_{\tilde{\boldsymbol{D}}\to X} M \subseteq M_X$, then $K_\vartheta(M)$ has the unique object (x,τ). Otherwise, $K_\vartheta(M\,|_{\tilde{\boldsymbol{D}}})$ has an initial object (Y,φ), and $K_\tau(M\,|_{\tilde{D}\to X})$ has an initial object $(Z,\chi,Z \xrightarrow{\lambda} X)$. We have $(Z,\chi) \in K_\vartheta$ by (17), hence there exists a unique morphism $Y \xrightarrow{\mu} Z$ in $\boldsymbol{D}$ mapping φ to χ. But then $(Y,\varphi,\lambda\mu)$ is also an object of $K_\tau(M\,|_{\tilde{D}\to X})$, and since Z is initial we get $Y = Z$. This implies that (Y,φ) is also an initial object of $K_\vartheta(M)$. $\square$

The proof of our assertion that (16) is an isomorphism is complete.

1.6.5. *Smash products by finite spectra.* Let $\operatorname{sim\,sets}_{*,\mathrm{fin}}$ be the category of finite pointed simplicial sets. Every object X of this category gives rise to an object

$$i(X)\colon \text{base point} \to X,$$

whence a functor $i\colon \operatorname{sim\,sets}_{*,\mathrm{fin}}^{\boldsymbol{D}} \to \operatorname{sim\,sets}_{\mathrm{fin}}^{\boldsymbol{D}\times\underline{0}}$. It is easy to see that i passes to the homotopy category, where it has a left adjoint

$$\begin{aligned} C\colon \operatorname{Ho}\operatorname{sim\,sets}_{\mathrm{fin}}^{\boldsymbol{D}\times\underline{1}} &\to \operatorname{Ho}\operatorname{sim\,sets}_{*,\mathrm{fin}}^{\boldsymbol{D}} \\ C(T)_X &= \operatorname{cone}(T_{X\times 0} \to T_{X\times 1}). \end{aligned}$$

These functors identify $\operatorname{Ho}\operatorname{sim\,sets}_{*,\mathrm{fin}}^{\boldsymbol{D}}$ with the full subcategory of $\operatorname{Ho}\operatorname{sim\,sets}_{\mathrm{fin}}^{\boldsymbol{D}\times\underline{1}}$ consisting of all $\boldsymbol{D}\times\underline{1}$-diagrams T with $T_{X\times 0}$ contractible. Since there is a slight danger of confusing the homotopy Kan extensions in the categories of pointed and unpointed simplicial sets, we denote for the time being by $\operatorname{Ho}\operatorname{LKan}^*$ the homotopy Kan extension in the category of pointed spaces. From our considerations about i and Q, we conclude that it is related to its unpointed counterpart $\operatorname{Ho}\operatorname{LKan}$ by

$$\operatorname{Ho}\operatorname*{LKan}_f{}^* = C\Big(\operatorname{Ho}\operatorname*{LKan}_{f\times\underline{1}} i\Big). \tag{18}$$

For every $T \in \operatorname{Ho}\operatorname{sim\,sets}_{*,\mathrm{fin}}^{\boldsymbol{D}}$ and every $A \in \mathfrak{Ob}\,\mathcal{K}_{\underline{0}}$, $i(T) \frown A$ is an object in $\mathcal{K}_{\boldsymbol{D}\times\underline{1}}$, to which we can apply the cone functor $\operatorname{cone}\colon \mathcal{K}_{\boldsymbol{D}\times\underline{1}} \to \mathcal{K}_{\boldsymbol{D}}$. This defines the smash product

$$T \wedge A = \operatorname{cone}\Big(i(T) \frown A\Big),$$

a functor from $\operatorname{Ho}\operatorname{sim\,sets}_{*,\mathrm{fin}}^{\boldsymbol{D}}$ to $\mathcal{K}_{\boldsymbol{D}}$ with a compatible system of isomorphisms

$$(f^*T) \wedge A \cong f^*(T \wedge A) \tag{19}$$

for $f\colon \boldsymbol{D}_\star \to \boldsymbol{C}_\star$. We want to prove that the analog of (16) is an isomorphism

$$(\operatorname{Ho}\underset{f}{\mathrm{LKan}}^* T) \wedge A \cong \operatorname{Ho}\underset{f}{\mathrm{LKan}}(T \wedge A). \tag{20}$$

To prove this, we will need the fact

$$\operatorname{cone}(T \frown A) = \operatorname{cone}(iC(T) \frown A) \tag{21}$$

for $T \in \operatorname{Ho}\operatorname{sim}\operatorname{sets}_{\mathrm{fin}}^{\boldsymbol{D}\times\underline{1}}$. Indeed, let $\tilde{T}$ be the $\boldsymbol{D} \times \ulcorner$-diagram defined by $\tilde{T}_{X\times 0\times i} = T_{X\times i}$ for $i \in \{0; 1\}$ and $\tilde{T}_{X\times 1\times 0} = \{\text{point}\}$, and let p be the projection from $\boldsymbol{D} \times \ulcorner$ to $\boldsymbol{D}$. Then $C(T) = \operatorname{Ho}\mathrm{LKan}_p \tilde{T}$, hence

$$\Big(iC(T) \frown A\Big)_{X\times 1} = \operatorname{Ho}\underset{p}{\mathrm{LKan}}(\tilde{T} \frown A) = \operatorname{cone}(T \frown A)$$

by (16) and the definition of cone. Since $(iC(T) \frown A)_{X\times 0} = 0$, this implies (21).

We are now ready to prove (20). We have

$$\begin{aligned}
\operatorname{Ho}\underset{f}{\mathrm{LKan}}(T \wedge A) &= \operatorname{Ho}\underset{f}{\mathrm{LKan}}\operatorname{cone}\Big(i(X) \frown A\Big)\\
&= \operatorname{cone}\operatorname{Ho}\underset{f}{\mathrm{LKan}}\Big(i(X) \frown A\Big)\\
&= \operatorname{cone}\bigg(\Big(\operatorname{Ho}\underset{f\times\underline{1}}{\mathrm{LKan}}\, i(T)\Big) \frown A\bigg)\\
&= \operatorname{cone}\bigg(\Big(iC\operatorname{Ho}\underset{f\times\underline{1}}{\mathrm{LKan}}\, i(T)\Big) \frown A\bigg)\\
&= \operatorname{cone}\Big((i\operatorname{Ho}\underset{f}{\mathrm{LKan}}^* T) \frown A\Big)\\
&= (\operatorname{Ho}\underset{f}{\mathrm{LKan}}^* T) \wedge A,
\end{aligned}$$

where we have used the definition of the smash product in the first and the last line, the transitivity of left homotopy Kan extensions on the second, (16) on the third, (21) on the fourth, and (18) on the fifth line. The proof of (20) is complete.

From now till the end of the paper, we will give up the distinction between Ho LKan and Ho LKan*, since unpointed simplicial sets are no longer needed, so that the danger of a confusion is over. All homotopy Kan extensions will again be denoted by Ho LKan.

By (20) and the definition of the suspension functor in $\mathcal{K}$, we have $(\boldsymbol{\Sigma} T) \wedge A = \boldsymbol{\Sigma}(T \wedge A)$. Since $\boldsymbol{\Sigma}$ is an equivalence of categories on $\mathcal{K}$, there is a unique way to define smash products by finite spectra which are related to the smash products by pointed finite simplicial sets by $QT \wedge A = T \wedge A$, QT being the suspension spectrum. Since homotopy Kan extensions and pull-backs of diagrams commute with passing to the suspension spectrum, we also have (19) for $f\colon \boldsymbol{D}_\star \to \boldsymbol{E}_\star$ and (20) for $f\colon \boldsymbol{D} \to \mathcal{E}$. By Theorem 2, it follows that the smash product $P \wedge A$ is a compatible system of triangulated functors in its first argument. In particular, (20) holds in full generality and there is a similar compatibility with right homotopy Kan extensions. It is easy to see that $S^0 \wedge A \cong A$. Moreover, since our smash products have been defined by homotopy limits, we have a canonical isomorphism

$$P \wedge F(A) = F(P \wedge A) \tag{22}$$

for every compatible system of triangulated functors $F\colon \mathcal{K} \to \mathcal{L}$ and every diagram of finite spectra P and every $A \in \mathfrak{Ob}\,\mathcal{K}_{\underline{0}}$.

1.6.6. *Proof of Theorem* 4. Let the functor (1) be denoted by E. We construct an inverse I to E by putting $(IA)(P) = P \wedge A$ for $A \in \mathfrak{Ob}\,\mathcal{K}_{\underline{0}}$. By our previous assertion about the smash product, IA is a compatible system of triangulated functors from the stable homotopy category which satisfies $(IA)(S^0) \cong A$. In particular, $EI \cong \mathrm{Id}$. To prove $IE \cong \mathrm{Id}$, note that (22) defines a canonical isomorphism

$$F(P) \cong F(P \wedge S^0) \cong P \wedge F(S^0) = \big(IE(F)\big)(P)$$

for every spectrum P[1]. The proof of Theorem 4 is complete. It remains to prove its corollaries.

As we mentioned after formulating Corollary 1, most of its assertions are consequences of the theorem. It is clear that the smash products defined in the last paragraph satisfy the conditions by which the smash product was characterised in the corollary. This proves the formula (3).

For the proof of (4), we first note that by Theorem 4 there is a unique (up to unique natural transformations) compatible system contravariant triangulated functors D from the system $\mathcal{S}^{\mathrm{fin}}$ to itself such that $DS^0 = S^0$. We will first prove (4) for this D, and then identify it with the usual Spanier-Whitehead duality. Before we prove that we note that since D^2 is a compatible system of triangulated functors from $\mathcal{S}^{\mathrm{fin}}$ to itself preserving S^0, there is a unique compatible system of natural isomorphisms $D^2 \cong \mathrm{Id}$ which gives the identity when applied to S^0.

Let $[X, A]$ be the smash product of $A \in \mathfrak{Ob}\,\mathcal{K}_{\underline{0}} = \mathfrak{Ob}\,\mathcal{K}_{\underline{0}}^{\mathrm{op}}$ by the diagram of spectra X in the opposite system of triangulated diagram categories $\mathcal{K}^{\mathrm{op}}$. This is a compatible system of contravariant triangulated functors from the homotopy categories of diagrams of finite spectra to $\mathcal{K}$, and since passing to the opposite system of categories interchanges the two types of homotopy limits we know by (3) that (4) holds if its left hand side is replaced by $[Q(|N.\boldsymbol{C}|_+), A]$. On the other side, by Theorem 4 we have a canonical isomorphism $[X, A] \cong (DX) \wedge A$, proving (4).

The adjointness relation (5) immediately follows from (3) and (4). Since the usual Spanier-Whitehead duality satisfies this relation in the case $\mathcal{K} = \mathcal{S}^{\mathrm{fin}}$ [Ada74, III.5] and is characterised by it uniquely, we conclude that it is canonically isomorphic to $D\colon \mathcal{S}_{\underline{0}}^{\mathrm{fin\,op}} \to \mathcal{S}_{\underline{0}}^{\mathrm{fin}}$.

[1]Here the smash product between two finite spectra has to be defined by the method of the last paragraph. That it coincides with the usual smash product is a consequence of the uniqueness part of Theorem 4.

1.7. Comparison with other definitions of enhanced triangulated categories. We will consider other enhancements of the definition of triangulated diagram categories, in the order from little additional structure to much.

1.7.1. *Ordinary triangulated categories.* We have seen that there is a canonical way to put a triangulated structure on $\mathcal{K}_{\underline{0}}$.Since the topological application of our main uniqueness theorem about categories with an Adams spectral sequence gives examples of systems of triangulated diagram categories which are equivalent in low, but not in arbitrary dimension (cf. Remark 3.1.1), it is clear that the tower $\mathcal{K}_?$ can not usually be reconstructed from the triangulated category $\mathcal{K}_{\underline{0}}$.

1.7.2. *Neeman's categories of triangles.* In [Nee91], Neeman enhanced the original definition by considering categories of triangles for a triangulated category. It is easy to see that a Neeman structure on $\mathcal{K}_{\underline{0}}$ can be defined in terms of the pair $(\mathcal{K}_{\underline{0}}, \mathcal{K}_{\underline{1}})$ and the functors between them. For, by Remark 1.4.2, there is a functor from $\mathcal{K}_{\underline{1}}$ to what Neeman calls candidate triangles in $\mathcal{K}_{\underline{0}}$. Let $\mathcal{T}$ be the category obtained from $\mathcal{K}_{\underline{0}}$ by identifying two morphisms which give the same morphism of candidate triangles. Then it is easy to see that $\mathcal{T}$ is a Neeman structure for $\mathcal{K}_{\underline{0}}$. Since the topological application of our main uniqueness theorem about categories with an Adams spectral sequence gives examples of systems of triangulated diagram categories which are equivalent in low, but not in arbitrary dimension (cf. Remark 3.1.1), it is clear that the tower $\mathcal{K}_?$ can not usually be reconstructed from the pair $(\mathcal{K}_{\underline{0}}, \mathcal{T})$.

1.7.3. *Beilinson's f-categories.* Let $\mathcal{K}$ be a $\mathfrak{P}$-system of triangulated diagram categories, and let

$$F\mathcal{K} = \operatorname*{colim}_n \mathcal{K}_{[-n,n]}, \tag{1}$$

where $[-n,n]$ is the interval of integers between $-n$ and n and the transition functors are full immersions defined in the following way: Let for $m > n$ $p_{m,n}\colon [-m,-m] \to [-n,n]^\star$ be defined by

$$p_{m,n}(i) = \begin{cases} \star & \text{if } i < -n \\ i & \text{if } -n \leq i \leq n \\ n & \text{if } i > n, \end{cases}$$

and let the transition in (1) be made by $p^*_{m,n}$. Let $F(\leq 0)\mathcal{K}$ be the similar inductive limit over $\mathcal{K}_{[-n,0]}$, and let $F(\geq 0)\mathcal{K}$ be the limit over $\mathcal{K}_{[0,n]}$. There is an obvious way of defining a functor of a shift of the filtration by 1. Together, these data constitute a filtered triangulated category in the sense of [Bei87, Definition A.1.]. Thus, every system of triangulated diagram categories gives rise to a f-category in the sense of Beilinson. However, I think it is unlikely that it can be reconstructed from it.

1.7.4. *Keller's epivalent towers.* Keller [Kel91] considered only the categories $\mathcal{K}_{\underline{1}^n}$, and only functors between the various powers of $\underline{1}$ obtained by products of s_0, d_0, and d_1. He assumes that a triangulated structure on each of these categories is given, and imposes an axiom similar to our mapping cylinder axiom. It is easy to see that these axioms are satisfied by our categories $\mathcal{K}_{\underline{1}^n}$, hence one gets a Keller tower from a system of triangulated diagram categories.

The reconstruction of a system of triangulated diagram category from a Keller tower is much more involved. It has been carried out by Goertz and Wolff [GW95], who construct a $\tilde{\mathfrak{P}}$-system from a Keller tower. One intermediate step in this comparison is to drop the homotopy Kan extension and linearity axioms and to assume instead that we are given structures of triangulated categories on the $\mathcal{K}_{\boldsymbol{C}}$ and of triangulated functors on the f^* such that the two sorts of natural transformations in (1.2.1) are triangulated. They derive the remaining two axioms, and prove that the triangulated structures must be equivalent to the ones constructed here.

As the two definitions are equivalent, it is a matter of taste which of them one prefers. The fact that our definition contains more structures made it the method of choice for the proof of our abstract uniqueness theorem Theorem 5. Also, the distinguished role of stable homotopy is relatively straightforward to see for our definition, while it is rather non-obvious for Keller's, and in fact it seems to be so far unknown for Keller towers. On the other side, Keller's definition is more conservative, it is very similar to what one would get by an iteration of Beilinson's f-categories.

1.7.5. *Various choices for* $\mathfrak{K}$. Let $\mathfrak{L} \subset \mathcal{K}$ be two of the 2-categories of this paper. Obviously, the restriction of a $\mathfrak{K}$-system of triangulated diagram categories to $\mathfrak{L}$ is an $\mathfrak{L}$-system. We will say that $\mathfrak{K}$-systems and $\mathfrak{L}$-systems are equivalent if the following two assertions hold:

- For every $\mathfrak{L}$-system $\mathcal{K}$, there is a compatible system of equivalences of categories from $\mathcal{K}$ to the restriction to $\mathfrak{L}$ of a $\mathfrak{K}$-system of triangulated diagram categories.
- If $\mathcal{K}$ and $\mathcal{L}$ are $\mathfrak{K}$-systems, then the assertion which in the case where the necessary categories of functors are well-defined expresses the fact that

$$\mathfrak{Fun}_\triangle(\mathcal{K},\mathcal{L}) \longrightarrow \mathfrak{Fun}_\triangle(\mathcal{K}\,|_{\mathfrak{L}}\,, \mathcal{L}\,|_{\mathfrak{L}})$$

  is an equivalence of categories holds, i. e.:
  - Every compatible system of triangulated functors from $\mathcal{K}\,|_{\mathfrak{L}}$ to $\mathcal{L}\,|_{\mathfrak{L}}$ is the restriction to $\mathfrak{K}$ of a compatible system of triangulated functors from $\mathcal{K}$ to $\mathcal{L}$.
  - If F and G are compatible systems of triangulated functors from $\mathcal{K}$ to $\mathcal{L}$, then a compatible system of functormorphisms from F to G is uniquely determined by its restriction to $\mathfrak{L}$. Moreover, every compatible system of functormorphisms from $F\,|_{\mathfrak{L}}$ to $G\,|_{\mathfrak{L}}$ occurs as such a restriction.

It is very easy to see that if $\mathfrak{K}$ is one of $\mathfrak{P}$, $\tilde{\mathfrak{C}}$, $\mathfrak{C}$, $\tilde{\mathfrak{C}}$, and if $V \subseteq U$, then $\mathfrak{K}_U$-systems and $\mathcal{K}_V$-systems are equivalent. The equivalence of $\tilde{\mathfrak{C}}$-systems and $\tilde{\mathfrak{P}}$-systems has been proved by Willing [Wil95]. It seems likely that these types of systems are also equivalent to $\mathfrak{P}$-systems and to $\mathfrak{C}$-systems, but this comparison has not yet been worked out.

1.7.6. *Systems for infinite posets and arbitrary small categories.* There should be similar definition of $\mathfrak{P}_\infty$- or $\mathfrak{C}_\infty$-systems, where $\mathfrak{P}_\infty$ is the bicategory of (not necessarily finite) partially ordered sets and $\mathfrak{C}_\infty$ is the bicategory of small categories. I expect both types of systems to be equivalent (provided they are properly defined). The necessary modifications to the statements of our theorems probably all come from the fact that the infinite version of (1.4.28) gives us information only about functors preserving direct products. In Theorem 4, $\mathcal{S}_{\mathrm{fin}}$ is replaced by the category $\mathcal{S}$ of all spectra, but the theorem can be applied only to those functors $\mathcal{S} \to \mathcal{K}$ which preserve arbitrary direct sums. However, these remarks are only tentative conjectures.

2. The abstract uniqueness theorem

Let $\mathfrak{K}$ be one of the 2-categories $\mathfrak{P}_U$, $\tilde{\mathfrak{P}}_U$, $\mathfrak{C}_U$, or $\tilde{\mathfrak{C}}_U$, where U is an infinite class.

2.1. Adams spectral sequences for triangulated categories. The main result of our paper is a abstract uniqueness result for triangulated categories with an Adams spectral sequence which can be constructed by injective resolutions. This construction of the Adams spectral sequence has been pioneered by H. B. Brinkmann [Bri68], and plays a decisive role in the classification of K-local spectra by Bousfield [Bou85], who quoted it from R. M. F. Moss [Mos68]. We briefly recall this construction and prove some lemmas which will be needed later.

Definition 10. Let $\mathcal{A}$ be an abelian category with sufficiently many injective objects and with a shift functor [1] which is an equivalence of categories, let $\mathcal{D}$ be a triangulated category, and let $F\colon \mathcal{D} \to \mathcal{A}$ be a cohomological functor with a natural isomorphism $F(X[1]) \cong F(X)[1]$. Let I be an injective object of $\mathcal{A}$. We say that *the* (F, I)*-Eilenberg-MacLane object exists*, or that *the Eilenberg-MacLane object exists for* I, if the following two conditions hold:

- the functor $\mathfrak{E}_I$ from $\mathcal{D}$ to abelian groups defined by
$$\mathfrak{E}_I(X) = \mathrm{Hom}_{\mathcal{A}}\Big(F(X), I\Big)$$
is representable by an object E_I of $\mathcal{D}$.
- The canonical homomorphism $F(E_I) \to I$ defined by
$$\mathrm{Id} \in \mathrm{Hom}_{\mathcal{D}}(E_I, E_I) \cong \mathrm{Hom}_{\mathcal{A}}(F(E_I), I)$$
is an isomorphism.

In this case, we call E_I an I*-Eilenberg-MacLane object*.

We say that F *possesses an Adams spectral sequence by injective resolutions* if every object of $\mathcal{A}$ can be embedded into an injective object I for which the Eilenberg-MacLane object exists. A cohomological functor F is called *almost faithful* if $F(X) \cong 0$ implies $X \cong 0$.

Lemma 1. *Let F be an almost faithful cohomological functor possessing an Adams spectral sequence by injective resolutions, and let E be an object of $\mathcal{D}$ for which $I = F(E)$ is an injective object of $\mathcal{A}$. Then E is an I-Eilenberg-MacLane object. In other words, the canonical homomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(X, E) \longrightarrow \mathrm{Hom}_{\mathcal{A}}\Big(F(X), F(E)\Big)$$

is an isomorphism for every object X of $\mathcal{D}$.

Proof. By our assumption, we know that we have an embedding $I \xrightarrow{i} J$ into an injective object J for which the Eilenberg-MacLane object E_J exists. Corresponding to i we have $E \xrightarrow{\iota} E_J$. Let

$$E \xrightarrow{\iota} E_J \xrightarrow{\pi} \tilde{E} \xrightarrow{\delta} E[1]$$

be a distinguished triangle. There is a canonical isomorphism $F(\tilde{E}) \cong \mathrm{coker}(i)$. Since I is injective, its embedding into J splits and there is a section s of the projection $J \longrightarrow J/I$. Corresponding to s we have a morphism $\tilde{E} \xrightarrow{\sigma} E_J$, and the morphism

$$E \oplus \tilde{E} \xrightarrow{\iota+\sigma} E_J$$

becomes an isomorphism after applying F, hence it is itself an isomorphism since F is almost faithful. It follows that E is a direct summand of E_J, and this proves our claim. □

Let $\mathcal{A}$ have finite injective dimension, and let $F\colon \mathcal{D} \to \mathcal{A}$ be an almost faithful cohomological functor possessing an Adams spectral sequence by injective resolutions. For every object Y of $\mathcal{D}$ and every injective resolution $F(Y) \to I^0 \to \ldots \to I^k \to 0$, we get a resolution

(1)
$$\begin{tikzcd}
Y = Y^0 \arrow[d] & Y^1 \arrow[l] \arrow[d] & Y^2 \arrow[l] \arrow[d] & \cdots \arrow[l] & Y^{k-1} \arrow[l] \arrow[d] & Y^k \arrow[l] \arrow[d, "\cong"] \\
E_{I^0} \arrow[ur, "+"] & E_{I^1} \arrow[ur, "+"] & E_{I^2} \arrow[ur, "+"] & \cdots & E_{I^{k-1}} \arrow[ur, "+"] & E_{I^k}
\end{tikzcd}$$

in $\mathcal{D}$. That Y^k is an I^k-Eilenberg-MacLane object follows from Lemma 1. Applying $\mathrm{Hom}_{\mathcal{D}}(X, -)$, we get an Adams spectral sequence

(2)
$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{A}}\Big(F(X), F(Y)[q]\Big) \Rightarrow \mathrm{Hom}^{p+q}_{\mathcal{D}}(X, Y)$$

which can be shown to be independent of the choice of the resolution.

Remark 1. It is clear that it is absolutely necessary for $F(X) \cong 0$ to imply $X \cong 0$ if one wants to have an Adams spectral sequence (2). However, this condition is only sufficient if $\mathcal{A}$ has finite cohomological dimension. If the cohomological dimension is infinite, such that the diagram (1) becomes infinite, and if one assumes the existence of countable products, then in order to establish good convergence properties, one has to show that

$$\underleftarrow{\mathrm{Holim}}\, Y^k \cong 0,$$

where $\underleftarrow{\mathrm{Holim}}$ is the (unique up to possibly non-unique isomorphism) projective homotopy limit defined by Bökstedt and Neeman [BN93]. To conclude this vanishing in a straightforward way, one has to assume that F commutes with countable products. Of course, F also has to respect all infinite coproducts which exist if one wants the functor $\mathfrak{E}_I$ to be representable. In stable homotopy theory, Adams spectral sequences of infinite cohomological dimension usually come from a functor which does not respect infinite products. In this case, there is no straightforward way to prove the convergence of the Adams spectral sequence even if the faithfulness condition of the above definition is satisfied. Since there are other reasons for which our methods are severely restricted to the case of finite cohomological dimension, this problem, which is related to the difference between Bousfield localisation and convergence of the Adams spectral sequence discussed in [Bou79, §5,6] and also in [Rav84], does not have to bother us.

Definition 11. Let $K(\mathcal{A})$ be the Grothendieck Group of $\mathcal{A}$. The shift functor [1] makes this group into a module over $\mathbb{Z}[t, t^{-1}]$, and we put $\tilde{K}(\mathcal{A}, [1]) = K(\mathcal{A})/(t+1)K(\mathcal{A})$. The *value group* $\mathfrak{V}(F)$ of F is the set of all elements of $\tilde{K}(\mathcal{A}, [1])$ which can be realized as the equivalence class of $F(X)$ for some $X \in \mathfrak{Ob}(\mathcal{D})$. It is easy to see that it really is a subgroup.

Proposition 1. *Let $\mathcal{D} \xrightarrow{F} \mathcal{A}$ be an almost faithful cohomological functor possessing an Adams spectral sequence via injective resolutions. Assume that the cohomological dimension of $\mathcal{A}$ is finite. If I is an injective object of $\mathcal{A}$, then the I-Eilenberg-MacLane object exists if and only if the image of I in $\tilde{K}(\mathcal{A}, [1])$ belongs to $\mathfrak{V}(F)$. In particular, if $\tilde{K}(\mathcal{A}, [1]) = \{0\}$ (for instance because $\mathcal{A}$ has countable coproducts or products), then for every injective object I the Eilenberg-MacLane object exists.*

Proof. We first claim that for every injective I, $E_{I\oplus I[-1]}$ exists. Indeed, there is an injective object J such that $E_{I\oplus J}$ exists. Let e be the endomorphism of $E_{I\oplus J}$ corresponding to the projection of $I \oplus J$ to its second factor. Then by Lemma 1, a cone of e is an $I \oplus I[-1]$-Eilenberg-MacLane object.

Now we are ready to show that the existence of E_I depends only on the image of I in $\tilde{K}(\mathcal{A}, [1])$. Indeed, let E_J exist and let the images of I and J in $\tilde{K}(\mathcal{A}, [1])$ be the same. Then there exist injective objects K and L such that there is an isomorphism $I \oplus K \oplus K[-1] \xrightarrow{\phi} J \oplus L \oplus L[-1]$ in $\mathcal{A}$. We know that $E_{K\oplus K[-1]}$ and $E_{L\oplus L[-1]\oplus J}$ exist.

By Lemma 1, the cone of the morphism $E_{K\oplus K[-1]} \longrightarrow E_{L\oplus L[-1]\oplus J}$ defined by ϕ is an I-Eilenberg-MacLane object.

Let $\tilde{\mathfrak{V}}(F) \subset \tilde{K}(\mathcal{A},[1])$ be the subset containing the equivalence classes of all injective objects I for which E_I exists. Obviously, this is a subgroup, and we have $\tilde{V}(F) \subseteq \mathfrak{V}(F)$. By considering an Adams resolution (1), we see that the last inclusion actually is an equality. $\square$

So far, we only used ordinary triangulated categories. Now let $\mathcal{K}$ be a system of triangulated diagram categories.

Proposition 2. *If $F\colon \mathcal{K}_{\underline{0}} \to \mathcal{A}$ possesses an Adams spectral sequence via injective resolutions, then same is true for the functor $F_{\boldsymbol{C}}\colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{A}^{\boldsymbol{C}}$ defined by*

$$F_{\boldsymbol{C}}(A)_X = F(A_X), \quad A \in \mathfrak{Ob}(\mathcal{K}_{\boldsymbol{C}}).$$

Moreover, the value group of $F_{\boldsymbol{C}}$ is

$$\mathfrak{V}(F)^{\boldsymbol{C}} \subseteq \tilde{K}(\mathcal{A},[1])^{\boldsymbol{C}} \cong \tilde{K}(\mathcal{A}^{\boldsymbol{C}},[1]).$$

Proof. Since every object J of $\mathcal{A}^{\boldsymbol{C}}$ can be embedded into

$$\bigoplus_{X\in\mathfrak{Ob}(\boldsymbol{C})} \operatorname*{RKan}_{i_X} J_X,$$

it suffices to establish the existence of $E_{\mathrm{RKan}_{i_X} I}$ with the desired property for every injective object I of $\mathcal{A}$. But

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}^{\boldsymbol{C}}}(F_{\boldsymbol{C}}(A), \operatorname*{RKan}_{i_X} I) &= \mathrm{Hom}_{\mathcal{A}}(F(A_X), I)\\ &= \mathrm{Hom}_{\mathcal{K}_{\underline{0}}}(A_X, E_I)\\ &= \mathrm{Hom}_{\mathcal{K}_{\boldsymbol{C}}}(A, \mathrm{Ho}\operatorname*{RKan}_{i_X} E_I),\end{aligned}$$

hence $E_{\mathrm{RKan}_{i_X} I}$ exists and is given by $\mathrm{Ho}\,\mathrm{RKan}_{i_X} E_I$. But

$$F_{\boldsymbol{C}}\Big(\mathrm{Ho}\operatorname*{RKan}_{i_X} E_I\Big) \cong \operatorname*{RKan}_{i_X} F(E_I) \cong \operatorname*{RKan}_{i_X} I$$

by (1.4.30).

The assertion about value groups is clear. $\square$

There is another way in which the existence of an Adams spectral sequence via injective resolutions can be inherited by a category.

Proposition 3. *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{A}$ be an abelian category with a shift functor* $[1]$*, and let $\mathcal{D} \xrightarrow{F} \mathcal{A}$ be a functor possessing an Adams spectral sequence by injective resolutions. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre class which is stable under* $[1]$*, and assume that the quotient functor*

$$\mathcal{A} \xrightarrow{j^*} \mathcal{A}/\mathcal{B}$$

has a right adjoint j_* such that the canonical functormorphism $j^*j_* \longrightarrow \mathrm{Id}_{\mathcal{A}/\mathcal{B}}$ is an isomorphism. Let $\mathcal{D}_{\mathcal{B}} = \{X \in \mathcal{D} | F(X) \in \mathcal{B}\}$. Then the functor

$$\mathcal{D}/\mathcal{D}_{\mathcal{B}} \xrightarrow{j^*F} \mathcal{A}/\mathcal{B}$$

possesses an Adams spectral sequence by injective resolutions.

Proof. This is clear: Let I be an injective object of $\mathcal{A}/\mathcal{B}$, and let $J = j_*I$. Then J is an injective object of $\mathcal{A}$, and the image of a (F, J)-Eilenberg-MacLane object in $\mathcal{D}/\mathcal{D}_{\mathcal{B}}$ is a (j^*F, I)-Eilenberg-MacLane object. $\square$

2.2. The uniqueness theorem. The standard example of a triangulated category with an Adams spectral sequence is $\mathcal{D}^{([1],1)}(\mathcal{A})$ (cf. (1.3.9)) with the cohomological functor H^0. The uniqueness theorem proves that under certain conditions the other examples are equivalent to the standard example.

Definition 12. Let $\mathcal{A}$ be an abelian category with a self-equivalence [1]. A splitting of period N for $(\mathcal{A}, [1])$ is a Serre class $\mathcal{B} \subset \mathcal{A}$ which is preserved by $[N]$ and $[-N]$ and has the property that

$$\bigoplus_{0 \le i < N} \mathcal{B} \to \mathcal{A} \tag{1}$$

$$(B_i)_{0 \le i < N} \to \bigoplus_{0 \le i < N} B_i[i]$$

is an equivalence of categories. Let $s(\mathcal{A}, [1])$ be the supremum of all N for which there exists a splitting of period N.

Theorem 5. *Let $\mathcal{K}$ be a system of triangulated diagram categories and let $F\colon \mathcal{K}_{\underline{0}} \to \mathcal{A}$ be a cohomological functor possessing an Adams spectral sequence via injective resolutions. Let D be the injective dimension of $\mathcal{A}$ and assume that $L = s(\mathcal{A}, [1]) - D \ge 0$. Let*

$$\mathcal{D}^{([1],1)}_{\boldsymbol{C}}(\mathcal{A})_{\mathfrak{V}(F)^{\boldsymbol{C}}} \subseteq \mathcal{D}^{([1],1)}_{\boldsymbol{C}}(\mathcal{A}) \tag{2}$$

be the full subcategory containing all $([1], 1)$-quasiperiodic cochain complexes X for which the image of $H^0(X)$ in $\tilde{K}(\mathcal{A})^{\boldsymbol{C}}$ belongs to $\mathfrak{V}(F)^{\boldsymbol{C}}$. In the case where $\mathfrak{V}(F) = \tilde{K}(\mathcal{A}, [1])$ (for instance, when $\mathcal{A}$ has countable coproducts or products, which implies the vanishing of the K-group), it is equal to the full derived category.

- *There is a canonical way to define a compatible system of equivalences*

  $$\mathfrak{R}^F_{\boldsymbol{C}}\colon \mathcal{D}^{([1],1)}_{\boldsymbol{C}}(\mathcal{A})_{\mathfrak{V}(F)^{\boldsymbol{C}}} \to \mathcal{K}_{\boldsymbol{C}}$$

  in dimension $\dim \boldsymbol{C} < L$ (cf. Definition 8) together with a natural isomorphism of cohomological functors $\psi\colon F\mathfrak{R}^F_{\underline{0}} \cong H^0$.

- *Let $0 < k < L$ and let $G_{\boldsymbol{C}} \colon \mathcal{D}^{([1],1)}_{\boldsymbol{C}}(\mathcal{A}) \to \mathcal{K}_{\boldsymbol{C}}$ be a compatible system of equivalences in dimension $\leq \lambda$, together with a natural isomorphism*

$$\phi \colon FG_{\underline{0}} \cong H^0. \tag{3}$$

Then there is a canonical way to define compatible natural isomorphisms $\tau_{\boldsymbol{C}} \colon G_{\boldsymbol{C}} \cong \mathfrak{R}^F_{\boldsymbol{C}}$ in dimension $< k$ such that $\psi F(\tau_{\underline{0}}) = \phi$. If $\tilde{\tau}_{\boldsymbol{C}} \colon G_{\boldsymbol{C}} \cong \mathfrak{R}^F_{\boldsymbol{C}}$ is a compatible system of natural isomorphisms in dimension $\leq k$ with $\psi F(\tilde{\tau}_{\underline{0}}) = \phi$, then $\tau_{\boldsymbol{C}} = \tilde{\tau}_{\boldsymbol{C}}$ in dimension $\dim \boldsymbol{C} < \lambda$.

In view of Proposition 2.1.2 it will turn out to be sufficient to construct a realisation functor $\mathcal{D}^{([1],1)}(\mathcal{A}) \to \mathcal{K}_{\underline{0}}$. The construction of such a functor is a modification of the construction of the realisation functor in [BBD82, 3.1.9.]. To motivate the following constructions, let us briefly recall the construction of the realisation functor in that paper. Let an abelian category $\mathcal{X}$ and a t-structure (cf. [BBD82] or [ΓM88]) on the bounded derived category of $\mathcal{X}$, with heart $\mathcal{Y}$ be given. If C^* is a cochain complex of objects of $\mathcal{Y}$, then one considers the descending filtration

$$F^iC = (\ldots \to 0 \to C^i \to C^{i+1} \to C^{i+1} \to \ldots). \tag{4}$$

It is clear that the quotient F^i/F^{i+1} has a single cohomology object C^i. The idea in [BBD82, 3.1.9.] is to impose a similar condition on objects of the filtered derived category of $\mathcal{X}$, i. e., to consider the full subcategory of the filtered derived category of $\mathcal{X}$ consisting of all objects C with $(F^i/F^{i+1})C[i] \in \mathcal{Y}$. It turns out that this category is equivalent to the category of cochain complexes in $\mathcal{Y}$, and this can be used to construct a realisation functor. Let us now return to our original situation and consider a splitting $\mathcal{B}$ of $\mathcal{A}$ of period N. We want to use a similar idea to construct a realisation functor for $([N], N)$-periodic complexes in $\mathcal{B}$. Since the complexes are quasi-periodic, it is no longer possible to consider a filtration as in (4). Instead, we consider the more complicated diagram formed by the morphisms

$$G^i(C) \to B^i(C), \qquad G^i(C) \to B^{i-1}(C)[1], \tag{5}$$

where $G^i(C) = C^{i-1}/B^{i-1}$ for any $([N], N)$-periodic complex C. If the period N is bigger than the cohomological dimension of $\mathcal{A}$, then it turns out that one can construct a realisation functor by (5), imposing certain conditions on objects of $\mathcal{K}_{\boldsymbol{C}_N}$ for a certain poset $\boldsymbol{C}_N$. The reason for considering (5) instead of the more straightforward

$$B^i(C) \to Z^i(C), \qquad B^i(C) \to Z^{i-1}(C)[1] \tag{6}$$

is that the bound L in Theorem 5 obtained by (5) is by one better than the bound obtained by (6). If we had considered Adams spectral sequences obtained from projective resolutions, (6) would be the better way of constructing a realisation functor.

To realise the program we have outlined, let $\mathcal{B}$ be a splitting of period N for $(\mathcal{A}, [1])$. We may exclude the case $N = 1$, which is trivial since in this case the assumptions of Theorem 5 imply $D = 0$, hence all objects of $\mathcal{A}$ are injective. Let $\boldsymbol{C}_N$ be the following poset: The elements of $\boldsymbol{C}_N$ are $2N$ symbols β_i and γ_i for $i \in \mathbb{Z}/N\mathbb{Z}$. The relations are $\gamma_i \prec \beta_i$ and $\gamma_i \prec \beta_{i-1}$. We denote by $k_i \colon \underline{1} \to \boldsymbol{C}_N$ the map given by $k_i(0) = \gamma_{i+1}$ and $k_i(1) = \beta_i$. We will usually write γ_i, β_i and k_i even in a situation where we should actually write $\gamma_{i \bmod N}$ etc. Let $\mathcal{L}$ be the full subcategory of $\mathcal{K}_{\boldsymbol{C}_N}$ consisting of all objects A satisfying the following two conditions:

(7)
- The objects of $\mathcal{A}$ $G^i(A) = F(X_{\gamma_i})[i]$ and $B^i = F(X_{\beta_i})[i]$ are actually objects of $\mathcal{B}$.
- The morphism $G^i(A) \xrightarrow{\pi_i} B^i(A)$ defined by $\gamma_i \prec \beta_i$ is surjective.

Let us define a functor Q from $\mathcal{L}$ to the category of $([N], N)$-periodic chain complexes in $\mathcal{B}$. Our conditions imply that the objects of $\mathcal{A}$

$$C^i(A) = F(\mathrm{cone}(k_i^* A))[i] \tag{8}$$

are part of an exact sequence

$$0 \to B^i(A) \xrightarrow{\iota_i} C^i(A) \xrightarrow{\rho_i} G^{i+1}(A) \to 0,$$

which implies that they are actually objects of $\mathcal{B}$. Indeed, from the distinguished triangle $A_{\beta_i} \longrightarrow A_{\gamma_{i+1}} \longrightarrow \mathrm{cone}(k_i^* A) \longrightarrow A_{\beta_i}[1]$ we have an exact sequence in $\mathcal{A}$

$$G^{i+1}(A)[-1] \to B^i(A) \to C^i(A) \to G^{i+1}(A) \to B^i(A)[1]$$

in which the first and the last morphism vanish, since the second and the fourth object belong to $\mathcal{B} \subset \mathcal{A}$, while the first object belongs to $\mathcal{B}[-1]$ and the last object belongs to $\mathcal{B}[1]$. We define a differential $d \colon C^i(A) \to C^{i+1}(A)$ by the composition

$$C^i(A) \xrightarrow{\rho_i} G^i(A) \xrightarrow{\pi_i} B^{i+1}(A) \xrightarrow{\iota_{i+1}} C^{i+1}(A). \tag{9}$$

d really is a differential since d^2 factorises over the composition $\rho_{i+1}\iota_{i+1}$, which is zero. Since π and ρ are both surjective, the image of d is $B^*(A)$. We obtain the promised functor

$$Q \colon \mathcal{L} \to \mathcal{C}^{([N],N)}(\mathcal{B})$$

to the category of $([N], N)$-periodic cochain complexes in $\mathcal{B}$.

Let $\mathcal{M}$ be the full subcategory of objects A of $\mathcal{L}$ satisfying

$$\mathrm{inj}\dim\Big(B^i(A)\Big) < N-1, \qquad \mathrm{inj}\dim\Big(G^i(A)\Big) < N-1. \tag{10}$$

In the case where $N > D + 1$, this condition is automatically satisfied.

Proposition 1.
- *The restriction of the functor Q to $\mathcal{M}$ is a full and faithful functor*

$$\mathcal{M} \to \mathcal{C}^{([N],N)}(\mathcal{B}). \tag{11}$$

- *Let $A_0 \to A_1 \to A_2 \to A_0[1]$ be a distinguished triangle in $\mathcal{K}_{\boldsymbol{C}_N}$, and assume that A_2 and A_3 are objects of $\mathcal{L}$ and that the morphism $Q(A_2) \to Q(A_3)$ is an epimorphism of complexes which induces an epimorphism on cohomology. Then A_1 is an object of $\mathcal{L}$, and the sequence*
$$0 \to Q(A_1) \to Q(A_2) \to Q(A_3) \to 0$$
is exact.
- *The essential image of (11) is the full subcategory of those $([N], N)$-periodic complexes which belong to the subcategory (2) and satisfy the analogue of (10). In particular, if $N > D+1$, then Q is an equivalence of categories*
$$\mathcal{L} = \mathcal{M} \cong \mathcal{C}^{([N],N)}(\mathcal{B})_{\mathfrak{V}(F)^{\boldsymbol{C}}}.$$
If $N = D+1$, this is no longer the case but it is at least true that the essential image of Q contains all injective complexes.
- *We have a canonical isomorphism in $\mathcal{A}$*
$$F(\underrightarrow{\operatorname{Holim}}_{\boldsymbol{C}_N} A) \cong \sum_{i=0}^{N-1} H^i\Big(Q(A)\Big)[-i]. \tag{12}$$

Proof. The first assertion is the most difficult part. Let A and $\tilde{A}$ be objects of $\mathcal{M}$. It suffices to prove the injectivity of the map

(13)
$$M = \operatorname{Hom}_{\mathcal{K}_{\boldsymbol{C}_n}}(A, \tilde{A}) \xrightarrow{\alpha} N = \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}_{\mathcal{B}^{\underline{1}}}\Big((B^i(A) \to C^i(A)), (B^i(\tilde{A}) \to C^i(\tilde{A}))\Big).$$

and to show that its image consists of the homomorphisms of complexes. The group M is the limit of the spectral sequence (1.4.32), which in the given case amounts to an exact sequence

$$\begin{array}{ccccc}
\bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}^{-1}_{\mathcal{K}_{\underline{0}}}(A_{\beta_i}, \tilde{A}_{\beta_i}) \oplus & & \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}^{-1}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\beta_i}) \oplus & & \\
\oplus \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}^{-1}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\gamma_i}) & \to & \oplus \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}^{-1}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\beta_{i-1}}) & \to M \to & \\
& & \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}_{\mathcal{K}_{\underline{0}}}(A_{\beta_i}, \tilde{A}_{\beta_i}) \oplus & & \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\beta_i}) \oplus \\
& \to & \oplus \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}^{-1}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\gamma_i}) & \to & \oplus \bigoplus_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{Hom}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\beta_{i-1}})
\end{array} \tag{14}$$

To investigate the terms in this sequence, we use the Adams spectral sequence (2.1.2). For instance,
$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{A}}\Big(F(A_{\beta_i}), F(\tilde{A}_{\beta_i})[q]\Big) \Rightarrow \operatorname{Hom}^{p+q}_{\mathcal{K}_{\underline{0}}}(A_{\gamma_i}, \tilde{A}_{\gamma_i}).$$
Since up to the same shift both $F(?)$-arguments of the Ext in the initial term belong to $\mathcal{B}$, the splitting (1) shows that the initial term vanishes unless q is divisible by N. On the

other side, by our definition of $\mathcal{M}$, the injective dimension of $B^i(\tilde{A})$ is $\leq N-2$, hence the initial term vanishes unless $0 \leq p \leq N-2$. The Adams spectral sequence therefore degenerates, and we have $E_2^{p,q} = 0$ for $p+q = -1$ and also for $p+q = 0$ and $p \neq 0$. Thus, we find

$$\mathrm{Hom}^{-1}_{\mathcal{K}_{\underline{0}}}(A_{\beta_i}, \tilde{A}_{\beta_i}) = 0, \quad \mathrm{Hom}_{\mathcal{K}_{\underline{0}}}(A_{\beta_i}, \tilde{A}_{\beta_i}) = \mathrm{Hom}_{\mathcal{B}}(B^i(A), B^i(\tilde{A})).$$

Computing the other terms in (14) in the same way, we get

$$(15) \quad 0 \to \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), B^{i-1}(\tilde{A})) \to M$$
$$\to \begin{matrix} \bigoplus\limits_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(B^i(A), B^i(\tilde{A})) \oplus \\ \oplus \bigoplus\limits_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), G^i(\tilde{A})) \end{matrix} \to \begin{matrix} \bigoplus\limits_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{B}}(G^i(A), B^{i-1}(\tilde{A})) \oplus \\ \oplus \bigoplus\limits_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), B^i(\tilde{A})) \end{matrix}$$

On the other side, the right hand side of (13) sits in an exact sequence

$$(16) \quad 0 \to \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), B^{i-1}(\tilde{A})) \to N$$
$$\to \begin{matrix} \bigoplus\limits_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(B^i(A), B^i(\tilde{A})) \oplus \\ \oplus \bigoplus\limits_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), G^i(\tilde{A})) \end{matrix} \to \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{B}}(G^i(A), B^{i-1}(\tilde{A}))$$

By an investigation of the way in which the spectral sequence (1.4.32) was constructed, one sees that there is a homomorphism of exact sequences of groups from (15) to (16) given by $M \xrightarrow{\alpha} N$, the projection

$$\bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{B}}(G^i(A), B^{i-1}(\tilde{A})) \oplus \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), B^i(\tilde{A})) \to \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Ext}^1_{\mathcal{B}}(G^i(A), B^{i-1}(\tilde{A})),$$

and the identity on the other terms. We arrive at an exact sequence

$$0 \to M \xrightarrow{\alpha} N \to \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(G^i(A), B^i(\tilde{A})),$$

confirming our claim that α is injective, with image equal to the subgroup of all morphisms of cochain complexes. The proof of the first assertion is therefore complete.

To prove the second assertion, we note that the surjectivity of $H^i(Q(A_2)) \to H^i(Q(A_3))$ and of $C^i(A_2) \to C^i(A_3)$ also implies the surjectivity of $X^i(A_2) \to X^i(A_3)$ for $X \in \{B; G\}$, hence $X^i(A_1) = \ker(X^i(A_2) \to X^i(A_3))$ since F is a triangulated functor, and the validity of (7) for A_1 follows from its validity for A_2 and A_3.

For the proof of the third assertion, recall the complexes

$$C(I), V(I) \in \mathfrak{Ob}(\mathcal{C}^{([1],1)}(\mathcal{A})) = \mathfrak{Ob}(\mathcal{C}^{([N],N)}(\mathcal{B}))$$

for injective $I \in \mathcal{A}$ from the proof of Proposition 1.3.3. It is straightforward to see that they are in the essential image of Q if they belong to the subcategory (2). Since any injective complex with injective cohomology is a sum of two complexes of this form, it follows that injective complexes with injective cohomology are in the essential image of Q. We prove by induction on k that every complex C in the subcategory (2) with

$$\max(\operatorname{inj}\dim B^i(C), \operatorname{inj}\dim G^i(C)) \leq k \leq N-2 \tag{17}$$

is in the essential image of Q, starting from the case $k = 0$ which was just considered. Let the induction assertion be proved for injective dimension $\leq k-1$. As in the proof of Proposition 1.3.3, we find an embedding $C \xrightarrow{i} K$ into an injective complex $K = Q(A_2)$ with injective cohomology, such that i induces a monomorphism on cohomology. Let L be the cokernel of i. This is a complex satisfying (17) with k replaced by $k-1$, hence $L \cong Q(A_3)$ for some A_3 by the induction assumption. Since we know that Q is full, the morphism $K \to L$ comes from a morphism $A_2 \to A_3$. Extending it to a distinguished triangle to which the second assertion can be applied, we find $A_1 \in \mathfrak{Ob}\,\mathcal{M}$ with $Q(A_1) \cong C$. This completes the proof of our assertion that all complexes satisfying (10) are in the essential image of $\mathcal{M}$ by Q

In the case $D \leq N-2$, the condition (10) is void, hence in this case $\mathcal{L} = \mathcal{M} \cong \mathcal{C}^{([N],N)}(\mathcal{B})$. In the limiting case $D = N-1$, (10) is at least satisfied for quotients of injective objects, hence all injective complexes are in the essential image. The proof of the third assertion is complete.

For the proof of the last assertion, we note that by the conditions (7) the morphisms $F(A_{\gamma_i}) \to F(A_{\beta_{i-1}})$ are zero, while the morphisms $F(A_{\gamma_i}) \xrightarrow{\pi_i[-i]} F(A_{\beta_i})$ are surjective. It follows that

$$\operatorname*{colim}_{\boldsymbol{C}_n}{}_1\, F(A_?) = 0$$

and

$$\begin{aligned} \operatorname*{colim}_{\boldsymbol{C}_n} F(A_?) &\cong \sum_{i=0}^{N-1} \ker\Big(F(A_{\gamma_i}) \to F(A_{\beta_i})\Big) \\ &\cong \sum_{i=0}^{N-1} \ker\Big(G^i(A) \xrightarrow{\pi_i\rho_i} B^i(A)\Big)[-i] \\ &\cong \sum_{i=0}^{N-1} H^i(Q(A))[-i], \end{aligned}$$

where the isomorphism on the last line follows from the surjectivity of π_i and ρ_i and the injectivity of ι_i in the definition of the differential d (9). Therefore, the spectral sequence (1.4.35) degenerates to an isomorphism (12), completing the proof of the last assertion. $\square$

Let $\mathcal{N} \subset \mathcal{C}^{([N],N)}(\mathcal{B})_{\mathfrak{V}(F)^{\boldsymbol{C}}}$ be a full subcategory consisting of complexes satisfying (10) and containing all injective complexes, and let $Q^{-1} \colon \mathcal{N} \to Q^{-1}(\mathcal{N}) \subseteq \mathcal{M} \subset \mathcal{K}_{\boldsymbol{C}_N}$ be an inverse to the functor Q. Let

$$\mathfrak{R}^F_{\underline{0}} = \underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}_N} Q^{-1} \colon \mathcal{N} \to \mathcal{K}_{\underline{0}}.$$

The isomorphism (12) gives rise to an isomorphism

$$\psi\colon F\mathfrak{R}^F_{\underline{0}}(M) \cong \sum_{i=0}^{N-1} H^i(M)[-i], \quad M \in \mathfrak{Ob}(\mathcal{N}) \tag{18}$$

hence $\mathfrak{R}^F_{\underline{0}}$ preserves quasi-isomorphisms. By our assumption, $\mathcal{N}$ contains all injective complexes, hence the factorcategory of $\mathcal{N}$ by the quasi-isomorphisms is the derived category and (10) factorises over a functor from the derived category

$$\mathfrak{R}^F_{\underline{0}} \colon \mathcal{D}^{([N],N)}(\mathcal{A})_{\mathfrak{V}(F)} \to \mathcal{K}_{\underline{0}}, \tag{19}$$

which is denoted by the same letter and to which the natural isomorphism (12) also applies. It is clear that this functor is up to canonical isomorphism independent of the choice of $\mathcal{N}$.

Proposition 2.
- *If the sequence $M_0 \to M_1 \to M_2 \to M_0[1]$ is a distinguished triangle in $\mathcal{D}^{([N],N)}(\mathcal{A})_{\mathfrak{V}(F)}$ such that $H^*(M_0) \to H^*(M_1)$ is a monomorphism, then $\mathfrak{R}^F_{\underline{0}}$ maps this triangle into a distinguished one. In particular, it maps injective H^*-Adams resolutions of objects of $\mathcal{D}^{([N],N)}_{\mathfrak{V}(F)}$ to injective Adams resolutions of objects of $\mathcal{K}_{\underline{0}}$.*
- *The functor $\mathfrak{R}^F_{\underline{0}}$ is an equivalence of categories.*

Proof. The first point follows from the second point of Proposition 1. The first point implies that $\mathfrak{R}^F_{\underline{0}}$ respects the Adams spectral sequences on its source and target, therefore it is full and faithful. Since it is full, the first point proves by downward induction on i, starting from $i = k$, that the objects Y^i in (2.1.1) belong to the essential image. □

If $\dim \boldsymbol{C} < L$, then we can apply the last proposition to the functor $F_{\boldsymbol{C}}$ investigated in Proposition 2.1.2, obtaining the functor

$$\mathfrak{R}^F_{\boldsymbol{C}} = \mathfrak{R}^{F_{\boldsymbol{C}}}_{\underline{0}}. \tag{20}$$

It remains to show that the assertions of the theorem are true with this definition of $\mathfrak{R}^F$. We first have to construct a compatible system of natural isomorphisms

$$f^*\mathfrak{R}^F_{\boldsymbol{D}} \cong \mathfrak{R}^F_{\boldsymbol{C}} f^* \tag{21}$$

for any functor $f\colon \boldsymbol{C}^\star \to \boldsymbol{D}^\star$. Recall that for the definition of $\mathfrak{R}^F_{\boldsymbol{C}}$, we were free to choose a subcategory $\mathcal{N}_{\boldsymbol{C}}$ of $\mathcal{C}^{([N],N)}(\mathcal{B}^{\boldsymbol{C}})$ containing all injective complexes, such that the objects of $\mathcal{N}_{\boldsymbol{C}}$ satisfy (10). Up to canonical isomorphism, all choices give the same result. The definition of the compatible system of isomorphisms (21) becomes obvious if we choose $\mathcal{N}_{\boldsymbol{C}}$ in such a way that f^* maps $\mathcal{N}_{\boldsymbol{D}}$ to $\mathcal{N}_{\boldsymbol{C}}$. Such a choice is possible by defining $\mathcal{N}_{\boldsymbol{C}}$ as the category of all $\boldsymbol{C}$-diagrams of complexes A with the property that for every object X of $\boldsymbol{C}$, the complex A_X consists of injective objects of $\mathcal{B}$. The injective dimension of $G^i(A)$

or $B^i(A)$ is then $\le D + \dim \boldsymbol{C} - 1$, which in view of $\dim \boldsymbol{C} < L = N - D$ is sufficient to guarantee (10). This defines the structure of a compatible system of equivalences of categories in dimension $\dim \boldsymbol{C} < L$ on the functors

$$\mathcal{D}^{([1],1)}(\mathcal{A}) \cong \mathcal{D}^{([N],N)}(\mathcal{B}) \xrightarrow{\mathfrak{R}^F_{\boldsymbol{C}}} \mathcal{K}_{\boldsymbol{C}}.$$

It is clear that $\mathfrak{R}^F_{\boldsymbol{C}}$ is determined uniquely up to unique isomorphism by our construction. Moreover, if $\mathcal{J}$ is another system of triangulated diagram categories and if G is a cohomological functor on $\mathcal{J}_{\underline{0}}$ with values in $\mathcal{A}$ possessing an Adams spectral sequence via injective resolutions and if $\Lambda_{\boldsymbol{C}} \colon \mathcal{K}_{\boldsymbol{C}} \to \mathcal{L}_{\boldsymbol{C}}$ is a compatible system of equivalences in dimension $\dim \boldsymbol{C} \le k \le L$, then by our explicit construction of $\mathfrak{R}^F$ and $\mathfrak{R}^G$, every natural isomorphism of cohomological functors

$$\lambda \colon F \cong G\Lambda_{\underline{0}}$$

determines a unique isomorphism

$$\kappa_\lambda \colon \mathfrak{R}^G_{\boldsymbol{C}} \cong \Lambda_{\boldsymbol{C}} \mathfrak{R}^F_{\boldsymbol{C}}. \tag{22}$$

At this point, we want to check that neither $\mathfrak{R}^F$ nor the isomorphism (22) depend on the choice of the splitting $\mathcal{B}$. It is clear that $\mathcal{B}$ and its shifts $\mathcal{B}[i]$ give the same realisation functors, and that a refinement $\tilde{\mathcal{B}}$ of $\mathcal{B}$ (i.e, $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a splitting of $\mathcal{B}$ with respect to the shift functor $[N] \colon \mathcal{B} \to \mathcal{B}$) also defines the same realisation functors as $\mathcal{B}$ in the range of dimensions in which both realisation functors are defined. But if $\tilde{\mathcal{B}}$ is another splitting, of period $\tilde{N}$, then

$$\mathcal{A} = \bigoplus_{i=1}^{\gcd(N,\tilde{N})} \mathcal{A}_i,$$

where

$$\mathcal{A}_i = \bigoplus_{j=1}^{\operatorname{lcm}(N,\tilde{N})} \mathcal{B}[j] \cap \tilde{\mathcal{B}}[i+j].$$

Each $\mathcal{A}_i$ is $[1]$-invariant, and $\mathcal{B} \cap \mathcal{A}_i$ and $\tilde{\mathcal{B}} \cap \mathcal{A}_i$ have the common (up to shift by i) subsplitting $\mathcal{B} \cap \tilde{\mathcal{B}}[i]$. We conclude that the realisation functor is, up to canonical isomorphism preserving (22), independent of the choice of the splitting.

It remains to prove the second part of the theorem. This uniqueness property of $\mathfrak{R}^F$ will be derived from the easy fact that $\mathfrak{R}^{H^0} \cong \mathrm{Id}$ in the case of $\mathcal{D}^{([1],1)}(\mathcal{A})$, together with the isomorphism (22). If G is another compatible system of realisation functors with an isomorphism (3) defined in dimension $\le k \le L$, then we have an isomorphism

$$\mathrm{Id}_{\mathcal{D}^{([1],1)}_{\boldsymbol{C}}(\mathcal{A})_{\mathfrak{V}(F)}} \cong \mathfrak{R}^{H^0}_{\boldsymbol{C}} \xrightarrow{\kappa_\phi} G^{-1}\mathfrak{R}^F_{\boldsymbol{C}}, \tag{23}$$

defining the natural isomorphism $\tau_{\boldsymbol{C}}$. The remaining part of the theorem, namely the uniqueness property of $\tau_{\boldsymbol{C}}$, follows from the following consideration. If $\chi_{\boldsymbol{C}}$ is any compatible system of natural automorphisms of the identity functor on $\mathcal{K}_{\boldsymbol{C}}$ defined in dimensions $\dim \boldsymbol{C} \leq k \leq L$ and satisfying $F(\chi_{\underline{0}}) = \mathrm{Id}$, then we have $Q(\chi_{\boldsymbol{C}_N}) = \mathrm{Id}$, hence $\chi_{\boldsymbol{C}_N}$ is the identity on $\mathcal{M}$ by Proposition 1. Hence $\chi_{\underline{0}}$ must be the identity on the essential image of $\mathcal{M}$ by $\underrightarrow{\mathrm{Holim}}_{\boldsymbol{C}_N}$, which is all of $\mathcal{K}_{\underline{0}}$. Applying the same argument with $\mathcal{K}_?$ replaced by $\mathcal{K}_{\boldsymbol{C}\times ?}$, we get $\chi_{\boldsymbol{C}} = \mathrm{Id}$ if $\dim \boldsymbol{C} < k$. The proof of our main uniqueness theorem is now complete.

2.3. Comparison with the theory of t-Structures. Let $\mathcal{K}$ be a system of triangulated diagram categories, and let a t-structure on $\mathcal{K}_{\underline{0}}$ with heart $\mathcal{H}$ be given. We assume that $\mathcal{H}$ has sufficiently many injective objects and finite injective dimension, that the t-structure is non-degenerate (cf. [ΓM88] or [GM94]), and that the Ext-groups in $\mathcal{H}$ coincide with homomorphisms in $\mathcal{K}_{\underline{0}}$.

In this case, the functor ${}^tH^*$ possesses an Adams spectral sequence via injective resolutions in the graded abelian category of graded $\mathcal{H}$-objects whose grading is bounded from above and below. The graded abelian category admits splittings of arbitrarily big period, and we conclude

Proposition 1. *Under the assumptions made above, there is a compatible systems of equivalences $F_{\boldsymbol{C}}$ from $\mathcal{K}_{\boldsymbol{C}}$ to the bounded derived categories $\mathcal{D}^b(\mathcal{H}^{\boldsymbol{C}})$ with the property that $F_{\underline{0}}$ is t-exact (where the derived category is given its usual t-structure) and induces the identity on $\mathcal{H}$. These assumptions characterise F uniquely up to a unique compatible system of functor-isomorphisms.*

Remark 1. Our Theorem 5 also proves that if $\mathcal{H}$ is an abelian category with sufficiently many injective objects, then any Adams spectral sequence with values in the category of graded $\mathcal{H}$-objects with bounded grading comes from a t-structure with heart $\mathcal{H}$. The theorem also applies in the case where the heart has sufficiently many projective objects.

Remark 2. Similar theorems are well-known in the literature on triangulated categories and perverse sheaves, cf. [BBD82, Proposition 3.1.16], [Bei87, Appendix A], [Nee91, Theorem 5.1.], and [Kel91, Corollary 2.7]. Of course, Keller's result is (in view of the comparison result [GW95]) sharper than ours because Keller does not have to assume that the heart has sufficiently many injective or projective objects and finite cohomological dimension. Of course, the assumption about cohomological dimension is absolutely necessary for an application of our construction. The assumption about injective objects can probably be eliminated. Moreover, it seems that our method can in principle still be applied in the case of the unbounded derived category. We will not go into details here.

Remark 3. The theorem shows that for an abelian category of finite homological dimension and with sufficiently many injective objects, there is essentially only one way to find a system of triangulated diagram categories $\mathcal{K}$ such that $\mathcal{K}_{\underline{0}}$ is the category $\mathcal{D}^b(\mathcal{H})$. If $\mathcal{L}$ is a second system with the same property, then there is a compatible system of equivalences

$\mathcal{K} \cong \mathcal{L}$ which is the identity on $\mathcal{K}_{\underline{0}} = \mathcal{L}_{\underline{0}} = \mathcal{D}^b(\mathcal{H})$. Moreover, this equivalence is unique up to a unique compatible system of functor-isomorphisms, and a similar result is likely to hold for the unbounded derived category under suitable assumptions, and the assumption that there are sufficiently many injective objects can probably be dropped. Moreover, by using a construction similar to the one in [BBD82] or in Keller's paper, one sees that the assumption that the homological dimension is finite can be probably be dropped if one considers only the bounded derived category.

In the case of examples relevant to topology, it is no longer the case that the extension of a triangulated category to a system of triangulated diagram categories is unique. For instance, we shall see that there is more than one way to extend the stable homotopy category of K-local spectra at an odd prime to a system of triangulated diagram categories (cf. Remark 3.1.1).

3. Applications

The aim of this subsection is to apply the abstract uniqueness result to some of the chromatic quotients of the stable homotopy category. We start with the special case of K-local spectra at an odd prime, since in this case the assumptions of our uniqueness theorem have been verified by Bousfield. The generalisation of these results to higher chromatic primes is somewhat involved, although all the methods we need are in principle contained in the literature. We first show that the Adams-Novikov spectral sequence can be constructed by injective resolutions. Then we set up the chromatic spectral sequence combine it with the result of Morava [Mor85] to understand the cohomological dimension of the Adams-Novikov E_2-term. We arrive at an algebraic description of the stable homotopy category of spectra which are localised in chromatic dimension $\leq n$ at an odd prime p, where $n^2 + n < 2p - 2$.

3.1. Bousfield's classification of K-local spectra at an odd prime. Fix an odd prime p. As in [Bou85], we denote by $\mathcal{A}$ the category of $\mathbb{Z}_{(p)}$-modules M equipped with Adams operations ψ^k for rational numbers k prime to p, such that the following conditions hold:

- $\psi^k \psi^l = \psi^{kl}$.
- We have a decomposition

$$M \otimes \mathbb{Q} = \coprod_{j=-\infty}^{\infty} M_j, \tag{1}$$

  where ψ^k acts on M_j by multiplication by k^j.
- Every element of M is contained in a ψ-invariant finitely generated submodule N such that for every m there is a $n(m)$ such that the action of ψ^k on $N/p^m N$ depends only on k mod $p^{n(m)}$.

Let T be the self equivalence of $\mathcal{A}$ defined by letting TM the same $\mathbb{Z}_{(p)}$-module M, but with the action of the Adams operations twisted:

$$\psi^k_{TM} = k\psi_M.$$

Let $\tilde{\mathcal{A}}$ be the category of graded $\mathcal{A}$-objects L with an isomorphism $L[2] \xrightarrow{b} TL$, morphisms being the graded morphism compatible with the Bott periodicity b.

Let $\mathfrak{K}$ be the K-theory spectrum and let $\mathcal{S}^{\mathfrak{K}}$ be the system of triangulated diagram categories obtained by localising the stable homotopy category $\mathcal{S}$ with respect to the thick subcategory of spectra A satisfying $\mathfrak{K}_*A = 0$. It was verified by Bousfield [Bou85, Theorem 8.2.] that $\mathfrak{K}_{-*}$ is a cohomological functor on $\mathcal{S}^{\mathfrak{K}}_{\underline{0}}$ with values in $\tilde{\mathcal{A}}$ which has an Adams spectral sequence via injective resolutions.

Let $\mathcal{B} \subset \mathcal{A}$ be the full subcategory of all M satisfying the following two conditions:

- In (1), the summands M_j vanish unless j is divisible by $p-1$.
- Every element of M is contained in a ψ-invariant finitely generated submodule N such that for every m there is a $n(m)$ such that the action of ψ^k on N/p^mN depends only on the image of k in the p-primary component of $\mathbb{Z}/p^{n(m)}\mathbb{Z}$.

Let $\tilde{\mathcal{B}}$ be the full subcategory of $\tilde{\mathcal{A}}$ consisting of all objects L concentrated in even degree and satisfying $L_0 \in \mathfrak{Ob}\,\mathcal{B}$. This is a splitting of period $2p-2$ for $\tilde{\mathcal{A}}$. Since the injective dimension of $\mathcal{B}$ (and hence also of $\mathcal{A}$ and $\tilde{\mathcal{A}}$) is 2 [Bou85, Proposition 7.7], we arrive at the following theorem:

Theorem 6. *Let p be an odd prime, then the K-local stable homotopy category of $\boldsymbol{C}$-diagrams of spectra is, in dimension $\dim \boldsymbol{C} < 2p-4$, equivalent to the derived category of quasi-periodic cochain complexes*

$$\mathcal{D}^{(T,2)}(\mathcal{A}^{\boldsymbol{C}}) \cong \mathcal{D}^{T^{p-1},2p-2}(\mathcal{B}^{\boldsymbol{C}}). \tag{2}$$

This isomorphism identifies K_0 with the zeroth cohomology functor and has the uniqueness properties described in the second part of Theorem 5.

Remark 1. By Remark 1.6.2, and since there are classes in the stable homotopy of spheres which survive K-localisation, the system of homotopy categories of $\boldsymbol{C}$-diagrams of K-local spectra has no strongly linear structure. On the other side, (2) clearly has a strongly linear structure. Therefore, our equivalence cannot be extended to arbitrary dimensions of $\boldsymbol{C}$.

A similar argument was also used by A. Neeman [Nee92, Remark 4.8] in the discussion of his triangulated lifting of the stable homotopy functor which, however, is far from being an equivalence of categories.

3.2. The Adams-Novikov spectral sequence via injective resolutions. We now start to prepare for a generalisation of the last subsection to higher chromatic primes. We first have to verify that the Adams-Novikov spectral sequence can be set up by injective resolutions.

Let E be any ring spectrum, and let $E_*X = \pi_*(X \wedge E)$ be the homology theory defined by E. We will write E_* for E_*S^0. Then it is known (cf. [Rav86, §2.2]) that the pair (E_*, E_*E) is a cogroupoid object in the category of graded rings. We will consider the ring E_*E as an E_*-bimodule with the left multiplication given by the cotarget morphism $E_* \xrightarrow{t} E_*E$ and the right multiplication given by the cosource map.

By an (E_*, E_*E)-comodule we understand a graded E_*-module M_* together with a comultiplication $\Delta\colon E_*E \bigotimes_{E_*} M_*$ satisfying $i \otimes \Delta = \mathrm{Id}_M$, where i is the coidentity morphism, and

$$(c \otimes \mathrm{Id}_M)\Delta = (\mathrm{Id}_{E_*} \otimes \Delta)\Delta,$$

where $c\colon E_*E \to E_*E \bigotimes_{E_*} E_*E$ is the cocomposition map. By [Rav86, Proposition 2.2.8.], the homology theory E_*X takes values in the abelian category of (E_*, E_*E)-comodules.

Proposition 1. *Let E be a ring spectrum which is the inductive limit of finite subspectra ε_α such that $E_*\varepsilon_\alpha$ is a projective E_*-module and such that the canonical morphism*

$$E^*(\varepsilon_\alpha) \longrightarrow \mathrm{Hom}_{E_*}(E_*\varepsilon_\alpha, E_*) \tag{1}$$

sending $\varepsilon_\alpha \xrightarrow{\lambda} E$ to the homomorphism

$$E_*\varepsilon_\alpha \xrightarrow{E_*\lambda} E_*E \xrightarrow{i} E_*,$$

where i is the coidentity of the cocatgory ring (E_*, E_*E), is an isomorphism. Then the homological functor E_* has an Adams spectral sequence via injective resolutions.

Proof. The forgetful functor from the category of (E_*, E_*E)-comodules to the category of E_*-modules has a right adjoint which sends M to $M \bigotimes_{E_*} E_*E$. Therefore, it suffices to show that for every injective E_*-module J the functor

$$X \longrightarrow \mathrm{Hom}_{E_*}(E_*X, J) = \mathrm{Hom}_{(E_*,E_*E)}(E_*X, E_*E \bigotimes_{E_*} J) \tag{2}$$

is representable by a spectrum $\mathfrak{E}_J$ for which the morphism $E_*\mathfrak{E}_J \longrightarrow E_*E \bigotimes_{E_*} J$ is an isomorphism.

The representability of (2) follows from the Brown representability theorem. We have

$$\pi_*\mathfrak{E}_J \cong J. \tag{3}$$

Let us prove that $\mathfrak{E}_J$ possesses the structure of a E-module spectrum (in the sense of [Ada74, §III.13]) in such a way that (3) becomes an isomorphism of E_*-modules. Indeed, from

$$\operatorname{Hom}_{\mathcal{S}}(X, \mathfrak{E}_J) \cong \operatorname{Hom}_{E_*}(E_*X, J) \xrightarrow{i} \operatorname{Hom}_{E_*}(E_*E \bigotimes_{E_*} E_*X, J) \cong \operatorname{Hom}_{E_*}(E_*(X \wedge E), J) \cong \operatorname{Hom}_{\mathcal{S}}(X \wedge E, \mathfrak{E}_J),$$

where i is defined by the coidentity morphism of the cocategory object (E_*, E_*E) and we have used [Rav86, Lemma 2.2.7], we obtain a natural transformation

$$\operatorname{Hom}_{\mathcal{S}}(X, \mathfrak{E}_J) \longrightarrow \operatorname{Hom}_{\mathcal{S}}(X \wedge E, \mathfrak{E}_J)$$

which is compatible with the structure of E as a ring spectrum. Applying it to $X = \mathfrak{E}_J$, we get the desired structure of a E-module up to homotopy on $\mathfrak{E}_J$.

The proof will be complete if we show that the multiplication map

$$\pi_*\mathfrak{E}_J \bigotimes_{E_*} E_*X \longrightarrow \pi_*(\mathfrak{E}_J \wedge X) \tag{4}$$

is an isomorphism in the case $X = E$. That (4) is an isomorphism in the case $X = \varepsilon_\alpha$ follows from

$$\begin{aligned} \pi_*(\mathfrak{E}_J \wedge \varepsilon_\alpha) &\cong \operatorname{Hom}_{\mathcal{S}}(D\varepsilon_\alpha, \mathfrak{E}_J) \\ &\cong \operatorname{Hom}_{E_*}(E_*D\varepsilon_\alpha, J) \\ &\cong \operatorname{Hom}_{E_*}(\operatorname{Hom}_{E_*}(E_*\varepsilon_\alpha, E_*), J) \\ &\cong E_*\varepsilon_\alpha \bigotimes_{E_*} J. \end{aligned}$$

The first line uses Spanier-Whitehead duality, the second line is (2), the third line is our assumption (1), and the fourth line follows from our assumption that $E_*\varepsilon_\alpha$ is a projective E_*-module. We have seen that (4) is an isomorphism in the case $X = \varepsilon_\alpha$, hence it is also an isomorphism in the case $X = E$, since E is the inductive limit of ε_α. $\square$

Corollary 1. *Let* $\mathbf{MU}$ *be the complex bordism spectrum. The functor* $\mathbf{MU}_*$ *with values in the category of* $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$*-comodules possesses an Adams spectral sequence by injective resolutions.*

Proof. The assumption of the last proposition can be verified by taking E_α to be the Thom constructions on the universal bundles over finite-dimensional Graßmannians. For these varieties, the necessary calculations are easily made using either the complex orientation or the Atiyah-Hirzebruch spectral sequence (cf. [Ada74, Proof of Proposition III.13.4]) $\square$

The author hopes that he can prove the convergence of the spectral sequence thusly obtained under conditions similar to the ones which are known to guarantee the convergence of the usual Adams spectral sequence. We will not do this in the present paper. In the application to a generalisation of Bousfield's result, it is anyway necessary to pass to a chromatic localisation of the category of $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-comodules. The investigation of the cohomological dimension of these chromatic localisations is the main aim of the next two subsections.

3.3. The abstract chromatic spectral sequence. Let $\mathfrak{G} = (\mathfrak{O}, \mathfrak{M})$ be a groupoid object in the category of preschemes, with $\mathfrak{O}$ as the prescheme of objects and $\mathfrak{M}$ as the prescheme of morphisms. We make the convention that in a fibre product $\mathfrak{M} \times_{\mathfrak{O}} \cdot$, $\mathfrak{M}$ is considered as an $\mathfrak{O}$-prescheme via the target morphism $\mathfrak{t}$, while a fibre product $\cdot \times_{\mathfrak{O}} \mathfrak{M}$ is defined by the source morphism $\mathfrak{s}$. The composition morphism of the groupoid object $\mathfrak{G}$ is a morphism $\mathfrak{c}\colon \mathfrak{M} \times_{\mathfrak{O}} \mathfrak{M} \to \mathfrak{M}$, and the identity morphism is a morphism $\mathfrak{i}\colon \mathfrak{O} \to \mathfrak{M}$. A linear $\mathfrak{G}$-representation is a quasi-coherent $\mathcal{O}_{\mathfrak{O}}$-module M, together with an isomorphism

$$\phi_M\colon \mathfrak{s}^* M \cong \mathfrak{t}^* M \tag{1}$$

such that

$$\mathfrak{c}^*(\phi_M) = p_2^*(\phi_M) p_1^*(\phi_M),$$

where $p_i\colon \mathfrak{M} \times_{\mathfrak{O}} \mathfrak{M} \to \mathfrak{M}$ is the projections to the i-th factor and $\mathfrak{c}$ is the composition morphism. Let $\mathcal{M}(\mathfrak{G})$ denote the category of linear $\mathfrak{G}$-representations. Throughout this subsection, we assume in addition that $\mathfrak{s}$ (and hence $\mathfrak{t}$ as well) is flat. Then $\mathcal{M}(\mathfrak{G})$ is an abelian category, and the forgetful functor to the category of quasi-coherent $\mathcal{O}_{\mathfrak{O}}$-modules is exact and faithful.

A morphism

$$\mathfrak{G} \xrightarrow{f} \tilde{\mathfrak{G}}$$

of groupoid preschemes is a pair $(f_{\mathfrak{O}}, f_{\mathfrak{M}})$ of morphisms of preschemes $\mathfrak{O} \xrightarrow{f_{\mathfrak{O}}} \tilde{\mathfrak{O}}$ and $\mathfrak{M} \xrightarrow{f_{\mathfrak{M}}} \tilde{\mathfrak{M}}$ making the obvious diagrams commutative. If f is a morphism of groupoid schemes and if (M, ϕ_M) is an object of $\mathcal{M}(\tilde{\mathfrak{G}})$, then we denote the object $(f_{\mathfrak{O}}^*(M), f_{\mathfrak{M}}^*(\phi_{\mathfrak{M}}))$ by $f^*\big((M, \phi_M)\big)$.

If S is any prescheme, then $\mathfrak{G}(S)$ denotes the groupoid $(\mathfrak{O}(S), \mathfrak{M}(S))$.

3.3.1. *Homological algebra in $\mathcal{M}(\mathfrak{G})$.*

Proposition 1. *$\mathcal{M}(\mathfrak{G})$ is a Grothendieck category. In particular, it has sufficiently many injective objects and all limits.*

Proof. Using the fact that $\mathfrak{s}^*$ commutes with direct sums of sheaves, it is easy to see that $\mathcal{M}(\mathfrak{G})$ has arbitrary direct sums, which coincide with the direct sums in the category of (quasi-coherent) sheaves. Since the category of sheaves is known to be AB5, the same is true for $\mathcal{M}(\mathfrak{G})$. It remains to prove the existence of a generating set.

For the purpose of this proof, let us call the size of a module M over a ring R the minimum of the cardinalities of the subsets of M generating M as an R-module. The size of a quasi-coherent sheaf $\mathcal{M}$ on a prescheme X is the supremum of the sizes of the $\mathcal{O}_X(U)$-modules $\mathcal{M}(U)$, taken over all affine open subsets U of X or (which amounts to the same result) over the elements of some covering of X by affine open subsets.

For a cardinality $\aleph$, it is easy to construct a set $Q_\aleph(\mathfrak{G})$ of representatives for the isomorphism classes of objects of $\mathcal{M}(\mathfrak{G})$ of size $\leq \aleph$.

Let $\mathfrak{O}$ be covered by affine open subsets U_i, $i \in I$. There exist an index set J and a map $J \xrightarrow{\iota} I \times I$ and a family V_j, $j \in J$, of affine open subsets of $\mathfrak{M}$ such that

$$U_i \bigtimes_{\mathfrak{O}} \mathfrak{M} \bigtimes_{\mathfrak{O}} U_k = \bigcup_{\substack{j \in J \\ \iota(j)=(i,k)}} V_j.$$

Let $\aleph$ be an infinite cardinality which is at least as large as

$$\sup_{i \in I} \iota^{-1}\mathrm{card}(\{i\} \times I).$$

We claim that $Q_\aleph(\mathfrak{G})$ is a generating set for the category $\mathcal{M}(\mathfrak{G})$. This claim immediately follows from the following lemma:

Lemma 1. *Let M be an object of $\mathcal{M}(\mathfrak{G})$, $i_o \in I$ and $s \in M(U_{i_o})$. Then there exists a subobject $N \subset M$ of size $\leq \aleph$ such that $s \in N(U_{i_o})$.*

Therefore, the proof of the proposition is reduced to the proof of the lemma. □

Remark 1. According to [TT90, Appendix B.2.], the result of this proposition seems to new even in the case of quasi-coherent sheaves on a prescheme which fails to be quasi-compact and quasi-separated.

Proof of Lemma 1: Let $U_i = \mathrm{Spec}(R_i)$, $V_j = \mathrm{Spec}(S_j)$. Giving the object M of $\mathcal{M}(\mathfrak{G})$ is equivalent to giving R_i-modules $M_i = M(U_i)$ and isomorphisms

$$S_j \bigotimes_{R_k} M_k \xrightarrow{\phi_j} M_i \bigotimes_{R_i} S_j$$

for $j \in J$, where $\iota(j) = (i,k)$. Of course, the ϕ_j have to satisfy various compatibilities. A collection of R_i-submodules $N_i \subseteq M_i$ defines a subobject of M if and only if

$$\phi_j(S_j \bigotimes_{R_k} N_k) \subseteq N_i \bigotimes_{R_i} S_j \tag{2}$$

holds for $j \in J$ and $\iota(j) = (i,k)$. (The compatibility conditions among the ϕ_j and the fact that $\mathfrak{G}$ is a groupoid imply that this inclusion amounts to an equality if it is satisfied for all j.)

We inductively construct an increasing sequence of submodules $N_i^{(l)} \subseteq M_i$ of size $\leq \aleph$ as follows: Let $N_i^{(0)}$ be the submodule generated by s if $i = i_o$ and zero otherwise. Let the $N_i^{(l)}$ be defined. We will define the submodule $N_i^{(l+1)} \supseteq N_i^{(l)}$ in such a way that

$$\phi_j(S_j \bigotimes_{R_k} N_k^{(l)}) \subseteq N_i^{(l+1)} \bigotimes_{R_i} S_j \tag{3}$$

holds for all $j \in I$ and $\iota(j) = (i,k)$. By the induction assumption, the size of $N_k^{(l)}$ is at most $\aleph$. Therefore, for each $j \in J$, we have to add at most $\aleph$ generators to $N_i^{(l)}$ in order to achieve (3). By our definition of $\aleph$, for each i the cardinality of all j for which this has to be done is also at most $\aleph$. Therefore, by the rules of arithmetic for infinite cardinals, we can construct $N_i^{(l+1)}$ of size $\leq \aleph$ such that (3) holds.

Let N_i be the inductive limit of the $N_i^{(l)}$ as l tends to infinity. By (3), (2) is satisfied. Obviously, the size of N_i is at most $\aleph$, and we have $s \in N_{i_o}$.

The proof of Lemma 1 is complete. **Q.E.D.**

Recall that an open subprescheme $U \subset \mathfrak{O}$ is called invariant if $\mathfrak{s}^{-1}(U) = \mathfrak{t}^{-1}(U)$. In this case, we denote by $\mathfrak{G}_U$ the groupoid prescheme $(U, \mathfrak{s}^{-1}(U))$. Let $Z = \mathfrak{O} - U$ and let $\mathcal{M}_Z(\mathfrak{G})$ be the torsion class of all objects of $\mathcal{M}(\mathfrak{G})$ which are supported in Z.

Proposition 2. *Let $j\colon U \to \mathfrak{O}$ denote the embedding. Then the restriction functor*

$$j^*\colon \mathcal{M}(\mathfrak{G}) \longrightarrow \mathcal{M}(\mathfrak{G}_U)$$

factorises over the functor p from $\mathcal{M}(\mathfrak{G})$ to the quotient category $\mathcal{M}(\mathfrak{G})/\mathcal{M}_Z(\mathfrak{G})$ and a uniquely defined faithful functor

$$j^\sharp\colon \mathcal{M}(\mathfrak{G})/\mathcal{M}_Z(\mathfrak{G}) \longrightarrow \mathcal{M}(\mathfrak{G}_U).$$

Moreover, j^* has a right adjoint j_* and p has a right adjoint L, the localisation with respect to U, and for an object M of $\mathcal{M}(\mathfrak{G})/\mathcal{M}_Z(\mathfrak{G})$ the canonical morphism

$$L(M) \longrightarrow j_* j^\sharp M \tag{4}$$

is a monomorphism. If the morphism j is quasi-compact, then $j^\sharp$ is an equivalence of categories

$$\mathcal{M}(\mathfrak{G})\Big/\mathcal{M}_Z(\mathfrak{G}) \cong \mathcal{M}(\mathfrak{G}_U),$$

and (4) *is an isomorphism. Moreover, in this case the functor j_* coincides with the usual direct image of sheaves on U.*

Proof. The existence and faithfulness of $j^\sharp$ follow from the exactness of j^* and the fact that the objects killed by j^* are precisely the objects of the torsion class $\mathcal{M}_Z(\mathfrak{G})$. The existence of right adjoints to j^* and p follows from the fact that these functors are exact and commute with arbitrary sums and from the special adjoint functor theorem (cf. [Fre66] or [Mac71]). That (4) is a monomorphism follows from the adjunction relation and the fact that $j^\sharp$ is faithful.

In the case where j is quasi-compact, it is known ([EGAIII, 1.4.10], [EGAIII, Proposition 1.4.15] and [EGAIV, 1.7.21]) that the direct image functor from the category of sheaves on U to the category of sheaves on X respects the classes of quasi-coherent sheaves and commutes with base-change by the flat morphisms $\mathfrak{s}$ and $\mathfrak{t}$ if it is applied to a quasi-coherent sheaf. This proves the remaining assertions of the proposition. $\square$

Proposition 3. *The inclusion $\mathcal{M}_Z(\mathfrak{G}) \longrightarrow \mathcal{M}(\mathfrak{G})$ has a right adjoint $\mathcal{H}^0_Z$ respecting injective objects. If in addition the embedding $U = \mathfrak{O} - Z \xrightarrow{j} \mathfrak{O}$ is a quasi-compact morphism, then $\mathcal{H}^0_Z(M)$ can be constructed as the subsheaf of M whose sections are the sections of M supported in Z, and*

$$\mathcal{H}^0_Z(M) = \ker(M \longrightarrow j_*j^*M)$$

holds for an arbitrary object M of $\mathcal{M}(\mathfrak{G})$.

Proof. This is clear. □

Proposition 4. *In the situation of the last proposition, assume that j is quasi-compact and that every injective object of $\mathcal{M}_Z(\mathfrak{G})$ is also injective in $\mathcal{M}(\mathfrak{G})$. Then for every injective object I of $\mathcal{M}(\mathfrak{G})$, the objects j^*I of $\mathcal{M}(\mathfrak{G}_U)$ and j_*j^*I and $\mathcal{H}^0_Z I$ of $\mathcal{M}(\mathfrak{G})$ are injective, and we have a (non-canonically) split short exact sequence*

$$0 \longrightarrow \mathcal{H}^0_Z I \longrightarrow I \longrightarrow j_*j^*I \longrightarrow 0. \tag{5}$$

Proof. Clearly, $\mathcal{H}^0_Z I$ is injective in $\mathcal{M}_Z(\mathfrak{G})$, hence also in $\mathcal{M}(\mathfrak{G})$ by our assumption. Therefore, the monomorphism $\mathcal{H}^0_Z I \longrightarrow I$ splits. Let $j^*I \longrightarrow J$ be an embedding into an injective object of $\mathcal{M}(\mathfrak{G}_U)$. Then we have a monomorphism

$$I \longrightarrow \tilde{I} = \mathcal{H}^0_Z I \oplus j_*J.$$

Since j is quasi-compact, j_* is the usual direct image of sheaves, hence $j^*j_*J = J$, and it follows that the morphism $\tilde{I} \longrightarrow j_*j^*\tilde{I}$ is an epimorphism. Since I is injective, it is a direct summand of $\tilde{I}$. In particular, the morphism $I \longrightarrow j_*j^*I$ is an epimorphism. Moreover, j^*I is injective since it is a direct summand of the injective object $j^*\tilde{I} = J$. □

3.3.2. *The abstract chromatic spectral sequence.* Let $\mathfrak{O} = Z^0 \supseteq Z^1 \supseteq \ldots$ be a descending sequence of invariant closed subsets of $\mathfrak{O}$. Let $Z^\infty = \bigcap_{i=0}^\infty Z^i$. Let $U_k = \mathfrak{O} - Z^{k+1}$ and $j_k \colon U_k \longrightarrow \mathfrak{O}$ be the immersion. Let $R\mathcal{H}_{Z^k}(M)$ be the derived functor of $\mathcal{H}^0_{Z^k}$, viewed as an element of the derived category of $\mathcal{M}(\mathfrak{G})$. Let $\mathcal{H}^i_{Z^k}(M)$ be the i-th cohomology object of $R\mathcal{H}_{Z^k}(M)$.

Theorem 7. *Assume that for all finite k, every injective object of $\mathcal{M}_{Z^k}(\mathfrak{G}_{U_\infty})$ is also injective in $\mathcal{M}(\mathfrak{G}_{U_\infty})$ and that the immersions j_k for $k < \infty$ are quasi-compact morphisms. Then for linear $\mathfrak{G}$-representations M, N there is a spectral sequence of a filtered complex*

$$E_1^{p,q} = \mathrm{Hom}^{p+q}_{\mathcal{D}^+(\mathcal{M}(\mathfrak{G}))}(M, Rj_{p*}R\mathcal{H}_{Z^p \cap U_p} j_p^* N) \tag{6}$$

with cohomology of the total complex equal to

$$\mathrm{Ext}^{p+q}_{\mathcal{M}(\mathfrak{G})}(M, Rj_{\infty *} j_\infty^* N).$$

The initial term of (6) *can also be written as*

$$E_1^{p,q} = \mathrm{Hom}^{p+q}_{\mathcal{D}^+\left(\mathcal{M}(\mathfrak{G})/\mathcal{M}_{Z^{p+1}}(\mathfrak{G})\right)}(M, j_p^* R\mathcal{H}_{Z^p} N). \tag{7}$$

Proof. Let I^* be an injective resolution of $j_\infty^* N$. We have a canonical filtration by the subsheaves of section with support in $Z^q - Z^\infty$:

$$I^k \supseteq \mathcal{H}^0_{Z^1-Z^\infty}(I^k) \supseteq \ldots \supseteq \mathcal{H}^0_{Z^q-Z^\infty}(I^k) \ldots . \tag{8}$$

Let j_p^∞ be the embedding $U_p \longrightarrow U_\infty$. Applying Proposition 4 to $Z^p - Z^\infty \subset U_\infty$, we see that the p-th quotient of the filtration (8) is canonically isomorphic to $j_{p*}^\infty \mathcal{H}^0_{Z^p\cap U_p} j_p^{\infty *} I^k$ and also that it is an injective object of $\mathcal{M}(\mathfrak{G}_{U_\infty})$.

We conclude that $\mathrm{Gr}^q I^*$ is an injective complex representing $Rj_{q*}^\infty R\mathcal{H}_{Z^q\cap U_q} j_q^* N$. We have

$$\mathrm{Hom}^{p+q}_{\mathcal{D}^+(\mathcal{M}(\mathfrak{G}))}(M, Rj_{q*}R\mathcal{H}_{Z^q\cap U_q} j_q^* N) \cong \mathrm{Hom}^{p+q}_{\mathcal{D}^+(\mathcal{M}(\mathfrak{G}_{U_\infty}))}(j_\infty^* M, Rj_{q*}^\infty R\mathcal{H}_{Z^q\cap U_q} j_q^* N).$$

Therefore, the initial term of the spectral sequence of the filtered complex

$$\mathrm{Hom}_{\mathcal{M}(\mathfrak{G})}(M, I^*)$$

has the form described in (6). The description of the limit of (6) follows from the fact that by the sheaf axioms we have

$$\lim_{q\to\infty} j_{q*}^\infty j_q^{\infty *} I^k \cong j_\infty^* I^k$$

in $\mathcal{M}(\mathfrak{G}_{U_\infty})$ and from the adjointness relation between j_∞^* and $Rj_{\infty *}$.

The reformulation of the initial term as (7) follows from the relation between $\mathcal{M}(\mathfrak{G}_{U_k})$ and $\mathcal{M}(\mathfrak{G})/\mathcal{M}_{Z^{k+1}}(\mathfrak{G})$ described in Proposition 2. $\square$

Corollary 1. *If the assumptions of Theorem 7 hold, then for $k < \infty$ and all objects M, N of $\mathcal{M}(\mathfrak{G}_{U_k})$, we have a finitely convergent spectral sequence*

(9)
$$E_1^{m,n} = \begin{cases} \mathrm{Hom}^{m+n}_{\mathcal{D}^+(\mathcal{M}(\mathfrak{G}_{U_m}))}(j_m^* M, R\mathcal{H}_{Z^m\cap U_m} j_m^* N) & \text{if } 0 \le m \le k \\ 0 & \text{otherwise} \end{cases} \Rightarrow \mathrm{Ext}^{m+n}_{\mathcal{M}(\mathfrak{G}_{U_k})}(M, N).$$

Proof. To get (9), one replaces $\mathfrak{G}$ by $\mathfrak{G}_{U_k}$ in (6). The spectral sequence is finitely convergent because in this case the filtration of the complex is finite. $\square$

In our application to the generalisation of Bousfield's work, Corollary 1 is the only consequence of Theorem 7 we need. In particular, only finitely many chromatic levels are involved. If this is no longer the case, there is the question of giving a condition under which the limit of (6) can is in fact isomorphic to $\mathrm{Ext}^*_{\mathcal{M}(\mathfrak{G})}(M, N)$ (which usually will be the initial term of some Adams-Novikov spectral sequence). This question is answered by the following corollary:

Corollary 2. *If the assumptions of Theorem 7 hold, then for every $N \in \mathfrak{Ob}(\mathcal{M}(\mathfrak{G}))$, there exists an isomorphism*

$$Rj_{\infty *} j_\infty^* N \longrightarrow \underleftarrow{\mathrm{Holim}}\, Rj_{q*} j_q^* N$$

projecting to the canonical morphisms

$$Rj_{\infty *} j_\infty^* N \longrightarrow Rj_{q*} j_q^* N.$$

Here the homotopy limit is understood in the sense of [BN93]. *If, moreover, for every $i \geq 0$ there is an $k(i)$ such that*

$$\mathcal{H}^i_{Z^m} N = 0 \quad \textit{if } m > k(i), \tag{10}$$

then the canonical homomorphism

$$N \longrightarrow Rj_{\infty *} j^*_\infty N$$

is an isomorphism, and (6) *finitely converges to*

$$\mathrm{Ext}^{p+q}_{\mathcal{M}(\mathfrak{G})}(M, N).$$

Proof. This is an easy consequence of the proof of Theorem 7. The spectral sequence converges finitely because the filtered complex is cohomologically finite if (10) holds (cf. [HS71, Theorem VIII.3.5]). □

3.3.3. *The case of a Noetherian scheme of objects.* The Corollary 1 is what we need because it allows us, under certain circumstances, to relate the cohomological dimensions of $\mathcal{M}(\mathfrak{G}_{U_k})$ and of the categories $\mathcal{M}_{Z^l}(\mathfrak{G}_U)$ for $l < k$. In the application to complex bordism, the last category is understood because of the work of Morava. In order to get these things done, we need two more technical facts.

Of course, we need a criterion which allows us to make sure that Theorem 7 can be applied.

Proposition 5. *Assume that the prescheme of objects $\mathfrak{O}$ underlying $\mathfrak{G}$ is Noetherian, and assume that the source morphism $\mathfrak{s}$ (and hence also $\mathfrak{t}$) is quasi-compact. If $Z \subseteq \mathfrak{O}$ is an invariant closed subset, then every injective object of $\mathcal{M}_Z(\mathfrak{G})$ is also injective in $\mathcal{M}(\mathfrak{G})$.*

For a prescheme X, let $\mathfrak{Qc}(X)$ be the category of quasi-coherent sheaves on X, and let $\mathfrak{Qc}_Z(X)$ be its full subcategory of all objects supported in the closed subset Z.

In order to prove this proposition, we need a preparation.

Proposition 6. *The forgetful functor from $\mathcal{M}(\mathfrak{G})$ to the category of all quasi-coherent $\mathcal{O}_{\mathfrak{O}}$-modules has a right adjoint R. In the case where $\mathfrak{s}$ (and hence also $\mathfrak{t}$) is quasi-compact, the $\mathcal{O}_{\mathfrak{O}}$-module underlying RM is given by $\mathfrak{s}_*\mathfrak{t}^*M$.*

Proof. The existence of R follows from the special adjoint functor theorem (cf. [Fre66] or [Mac71]). Let $\mathfrak{M} \times_{\mathfrak{O}} \mathfrak{M} \xrightarrow{p_{1,2}} \mathfrak{M}$ be the projections to the two factors, and recall that $\mathfrak{M} \times_{\mathfrak{O}} \mathfrak{M} \xrightarrow{\mathfrak{c}} \mathfrak{M}$ is the composition morphism. If the source and target morphisms are quasi-compact, then we have an isomorphism

$$\begin{aligned} \mathfrak{s}^*\mathfrak{s}_*\mathfrak{t}^*M &\longrightarrow p_{1*}\mathfrak{c}^*\mathfrak{t}^*M \\ &\longrightarrow p_{1*}p_2^*\mathfrak{t}^*M \\ &\longrightarrow \mathfrak{t}^*\mathfrak{s}_*\mathfrak{t}^*M, \end{aligned} \tag{11}$$

where the second line uses the equality $\mathfrak{tc} = \mathfrak{t}p_2$ and the first and third line are obtained by applying the base change result [EGAIII, Proposition 1.4.15] and [EGAIV, 1.7.21] to the two Cartesian squares

$$\begin{tikzcd} \mathfrak{M}\times\mathfrak{M} \arrow[r, "p_1"] \arrow[d, "\mathfrak{c}"] & \mathfrak{M} \arrow[d, "\mathfrak{s}"] \\ \mathfrak{M} \arrow[r, "\mathfrak{s}"] & \mathfrak{O} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{M}\times\mathfrak{M} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathfrak{M} \arrow[d, "\mathfrak{t}"] \\ \mathfrak{M} \arrow[r, "\mathfrak{s}"] & \mathfrak{M}. \end{tikzcd}$$

It is easy to see that (11) defines the structure of an object of $\mathcal{M}(\mathfrak{G})$ on $\mathfrak{s}_*\mathfrak{t}^*M$. Also, let an object L of $\mathcal{M}(\mathfrak{G})$ and a morphism $L \xrightarrow{f} M$ of $\mathcal{O}_{\mathfrak{O}}$-modules be given. Applying the adjointness between $\mathfrak{s}_*$ and $\mathfrak{s}^*$ to the composition

$$\mathfrak{s}^*L \xrightarrow{\phi_L} \mathfrak{t}^*L \xrightarrow{\mathfrak{t}^*(f)} \mathfrak{t}^*M,$$

we get a canonical homomorphism $L \longrightarrow \mathfrak{s}_*\mathfrak{t}^*M$, which is easily seen to be a morphism in $\mathcal{M}(\mathfrak{G})$. One easily checks that this morphism is universal, proving that $\mathfrak{s}_*\mathfrak{t}^*$ is right adjoint to the forgetful functor. $\square$

Proof of Proposition 5: It is sufficient to construct, for every object M of $\mathcal{M}_Z(\mathfrak{G})$, an embedding $M \longrightarrow I$, where I is and injective object of $\mathcal{M}(\mathfrak{G})$ with support in Z. In the case where M is injective in $\mathcal{M}_Z(\mathfrak{G})$, this embedding splits, and it follows that M is also injective in $\mathcal{M}(\mathfrak{G})$.

To construct the desired embedding, we first find an embedding $M \xrightarrow{j} J$, where J is an injective object of $\mathfrak{Qc}(\mathfrak{O})$ with support in Z. This is possible: By [Har66, Theorem II.7.18], the injective hull J of M in the category of all $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module. But this injective hull is isomorphic to i_*K, where i is the embedding of the topological space Z into X and K is an injective hull of i^*M in the category of all $i^*\mathcal{O}_{\mathfrak{O}}$-modules. In particular, J is supported in Z.

Having chosen J, we put $I = RJ$, where R was constructed in Proposition 6. The embedding $M \xrightarrow{i} I$ in $\mathcal{M}(\mathfrak{G})$ is the one derived from the embedding $M \xrightarrow{j} J$ in $\mathfrak{Qc}(\mathfrak{O})$ and the adjointness property of R. That i really is monomorphism follows from the relation $j = pi$, where $RI \xrightarrow{p} I$ is the adjunction morphism in $\mathfrak{Qc}(\mathfrak{O})$, and the fact that j is an embedding. That J is supported in Z follows from the explicit description of R in Proposition 6.

The proof of Proposition 5 is complete. **Q.E.D.**

3.3.4. *A base change result.* Of course, Proposition 5 cannot be directly applied to the category of $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-comodules since $\mathbf{MU}_*$ is not Noetherian. However, it will turn out that at a finite chromatic level we can reduce to a Noetherian situation by applying an abstract result which we explain next. Let $X \xrightarrow{f} \mathfrak{O}$ be a morphism of preschemes. Assume that the two projections

$$X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} X \to X \tag{12}$$

are flat. Then $(X, X \times_{\mathfrak{O}} \mathfrak{M} \times_{\mathfrak{O}} X)$ has the structure of a groupoid prescheme with flat cosource and cotarget map, which we denote by $\mathfrak{G}_X$. Obviously this is compatible with the notation which we had previously introduced in the case where $X = U$ is an open subprescheme of $\mathfrak{O}$. If M is an object of $\mathcal{M}(\mathfrak{G})$, then the quasi-coherent sheaf f^*M has a canonical structure of an object of $\mathcal{M}(\mathfrak{G}_X)$. For a prescheme S, the groupoid $\mathfrak{G}_X(S)$ has the set $X(S)$ as its set of objects, and a morphism from x to $\tilde{x}$ in $\mathfrak{G}_X(S)$ is a morphism from $f(x)$ to $f(\tilde{x})$ in $\mathfrak{G}(S)$. The morphism of groupoid schemes $\mathfrak{G}_X \to \mathfrak{G}$ which on objects is given by f and on morphisms by the projection

$$X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} X \to \mathfrak{M}$$

will also be denoted by f. We are looking for a result stating that f^* is an equivalence of categories if $\mathfrak{G}_X \to \mathfrak{G}$ is what one might call an equivalence of groupoid preschemes. One way to define the notion of an equivalence of groupoid preschemes is to assume the existence of an inverse up to natural isomorphism. This property was called "natural similarity" and used in [Mor85, Proposition 1.2.3]. Our aim is to prove a similar result, which is more suitable for our considerations.

Proposition 7. *Assume that the morphism*

$$\mathfrak{M} \underset{\mathfrak{O}}{\times} X \xrightarrow{p} \mathfrak{O} \tag{13}$$

induced by $\mathfrak{M} \xrightarrow{\mathfrak{s}} \mathfrak{O}$ *is faithfully flat and quasi-compact. Then* (12) *is flat, and the functor*

$$\mathcal{M}(\mathfrak{G}) \xrightarrow{f^*} \mathcal{M}(\mathfrak{G}_X)$$

is an equivalence of categories. Moreover, if $Z \subset \mathfrak{O}$ *is a* $\mathfrak{G}$*-invariant closed subset, then the restriction of this functor*

$$\mathcal{M}_Z(\mathfrak{G}) \xrightarrow{f^*} \mathcal{M}_{f^{-1}(Z)}(\mathfrak{G}_X)$$

is also an equivalence of categories, and we have a canonical isomorphism

$$f^*\mathcal{H}^p_Z M \cong \mathcal{H}^p_{f^{-1}(Z)} f^*M.$$

As the reader will have guessed, this essentially amounts to an application of flat descent. Let $\tilde{\mathfrak{G}} \xrightarrow{f} \mathfrak{G}$ is a morphism of groupoid preschemes. The fibre product $\tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}}$ in the category of groupoid preschemes exists and has $\tilde{\mathfrak{O}} \times_{\mathfrak{O}} \tilde{\mathfrak{O}}$ as its prescheme of objects and $\tilde{\mathfrak{M}} \times_{\mathfrak{M}} \tilde{\mathfrak{M}}$ as its prescheme of morphisms, and its structure morphisms are defined from the structure morphisms for $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ in the obvious way. Let $\tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}} \xrightarrow{\mathrm{pr}_{1,2}} \tilde{\mathfrak{G}}$ be the projections to the two factors. The category $\mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$ of descent data for f has as objects the pairs $(M, \mathfrak{d}_M)$, where M is an object of $\mathcal{M}(\tilde{\mathfrak{G}})$ and $\mathfrak{d}$ is an isomorphism $\mathrm{pr}_1^* M \cong \mathrm{pr}_2^* M$ such that the isomorphism of $\mathcal{O}_{\tilde{\mathfrak{M}} \times_{\mathfrak{M}} \tilde{\mathfrak{M}}}$-modules underlying $\mathfrak{d}$ is a descent datum for quasi-coherent sheaves in the sense of [SGA1, Exp. VIII.1]. If N is an object of $\mathcal{M}(\mathfrak{G})$, then the object $f^* N$ of $\mathcal{M}(\tilde{\mathfrak{G}})$, together with the isomorphism

$$\mathrm{pr}_1^* f^* N \cong (f\mathrm{pr}_1)^* N = (f\mathrm{pr}_2)^* N \cong \mathrm{pr}_2^* f^* N$$

becomes an object of $\mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$ which we denote by $f^+ N$. The following fact follows readily from the general theory of flat descent for quasi-coherent modules, cf. [SGA1, Exp. VIII.1, 1.2. et 1.3.].

Proposition 8. *Assume that the morphisms $\tilde{\mathfrak{O}} \xrightarrow{g_{\mathfrak{O}}} \mathfrak{O}$ and $\tilde{\mathfrak{M}} \to \mathfrak{M}$ are quasi-compact and faithfully flat. Then the functor*

$$\mathcal{M}(\mathfrak{G}) \xrightarrow{g^+} \mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$$

is an equivalence of categories.

Proof. If an object of $\mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$ is given, then applying flat descent to $\tilde{\mathfrak{G}} \xrightarrow{f_{\mathfrak{G}}} \mathfrak{G}$ shows how to descent the underlying quasi-coherent sheaf to $\mathfrak{G}$, and descent for $\tilde{\mathfrak{M}} \xrightarrow{f_{\mathfrak{M}}} \mathfrak{M}$ shows how to define the structure of an object of $\mathcal{M}(\mathfrak{G})$ on this descented sheaf. Similarly, descent for $f_{\mathfrak{G}}$ shows that every morphism in $\mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$ can be descented to a morphism between quasi-coherent sheaves on $\mathfrak{G}$, and the faithful flatness of $f_{\mathfrak{M}}$ shows that this descent is a morphism in $\mathcal{M}(\mathfrak{G})$. □

Remark 2. There is no obvious way to see that the structure morphisms $\mathfrak{s}$ and $\mathfrak{t}$ for $\tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}}$ are flat if the similar fact holds for $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$. From the factorisation of $\mathfrak{s}_{\tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}}}$

$$\tilde{\mathfrak{M}} \times_{\mathfrak{M}} \tilde{\mathfrak{M}} \xrightarrow{(\mathfrak{s}_{\tilde{\mathfrak{G}}}, f_{\mathfrak{M}})} \tilde{\mathfrak{M}} \times_{\mathfrak{M}} (\tilde{\mathfrak{O}} \times_{\mathfrak{O}} \mathfrak{M}) \cong \tilde{\mathfrak{M}} \times_{\mathfrak{O}} \tilde{\mathfrak{O}} \to \tilde{\mathfrak{O}} \times_{\mathfrak{O}} \tilde{\mathfrak{O}}$$

one sees that is is flat if $\tilde{\mathfrak{M}} \xrightarrow{(\mathfrak{s}_{\tilde{\mathfrak{G}}}, f_{\mathfrak{M}})} \tilde{\mathfrak{O}} \times_{\mathfrak{O}} \mathfrak{M}$ is flat. For our subsequent application, this will be the case. However, it is not necessary for our purposes to know that $\mathcal{M}(\tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}})$ is an abelian category.

For a groupoid prescheme $\mathfrak{H}$, there is a groupoid prescheme $\mathfrak{Ar}(\mathfrak{H})$ such that $\mathfrak{Ar}(\mathfrak{H})(S) = \mathfrak{Ar}(\mathfrak{H}(S))$ holds for any prescheme S, where $\mathfrak{Ar}$ was defined in the formulation of the mapping cylinder axiom. We have the two morphisms of groupoid prescheme $\mathfrak{Ar}(H) \xrightarrow{\sigma,\tau} \mathfrak{H}$

which to a morphism in $\mathfrak{H}$ associates its source and target. There is a morphism of groupoid preschemes

$$\mathfrak{Ar}(\mathfrak{H}) \underset{\mathfrak{H}}{\times} \mathfrak{Ar}(\mathfrak{H}) \xrightarrow{\chi} \mathfrak{Ar}(\mathfrak{H}),$$

where the first $\mathfrak{Ar}(\mathfrak{H})$ is made into an $\mathfrak{H}$-prescheme via τ and the second $\mathfrak{Ar}(\mathfrak{H})$ via σ.

Lemma 2. *Let $\mathfrak{H} \xrightarrow{f} \mathfrak{G}$ be a morphism of groupoid preschemes which is part of a diagram*

$$\begin{tikzcd} \tilde{\mathfrak{G}} \arrow[r, "h"] \arrow[d, "g"'] & \mathfrak{H} \arrow[dl, "f"] \\ \mathfrak{G} & \end{tikzcd} \tag{14}$$

in which g satisfies the assumptions of Proposition 8. Assume that (14) commutes up to natural isomorphism, i. e., that there exist a morphism from the prescheme of objects of $\tilde{\mathfrak{G}}$ to the prescheme of morphisms of $\mathfrak{G}$ which for every S defines a natural transformation between the two functors $\tilde{\mathfrak{G}}(S) \xrightarrow{g,fh} \mathfrak{G}(S)$. Also, assume that there is a morphism

$$\tilde{\mathfrak{G}} \underset{\mathfrak{G}}{\times} \tilde{\mathfrak{G}} \xrightarrow{a} \mathfrak{Ar}(\mathfrak{H})$$

satisfying the following two conditions:

- *We have $h\mathrm{pr}_1 = \mathfrak{s}a$ and $h\mathrm{pr}_2 = \mathfrak{t}a$, where $\tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}} \xrightarrow{\mathrm{pr}_{1,2}} \tilde{\mathfrak{G}}$ are the two projections.*
- *We have $a\mathrm{pr}_{1,3} = \mathfrak{c}(a\mathrm{pr}_{1,2}, a\mathrm{pr}_{2,3})$, where*

$$\tilde{\mathfrak{G}} \underset{\mathfrak{G}}{\times} \tilde{\mathfrak{G}} \underset{\mathfrak{G}}{\times} \tilde{\mathfrak{G}} \xrightarrow{\mathrm{pr}_{i,j}} \tilde{\mathfrak{G}} \underset{\mathfrak{G}}{\times} \tilde{\mathfrak{G}}$$

  is the projection to the subproduct formed from the i-th and the j-th factor.

Finally, assume that the morphism of object preschemes underlying h is faithfully flat. Then f^* is an equivalence of categories.

Proof. Let $\boldsymbol{M} = (M, \phi_M)$ be an object of $\mathcal{M}(\mathfrak{H})$. Its structure morphism ϕ_M defines an isomorphism $\mathfrak{s}^*\boldsymbol{M} \cong \mathfrak{t}^*\boldsymbol{M}$ in $\mathcal{M}(\mathfrak{Ar}(\mathfrak{H}))$. By our two conditions on a, the pull-back of this isomorphism with respect to a defines the structure of an object of $\mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$ on $h^*\boldsymbol{M}$. Let us denote the resulting functor

$$\mathcal{M}(\mathfrak{H}) \longrightarrow \mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G})$$

by $(h, a)^\sharp$.

The following diagram of functors commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathcal{M}(\mathfrak{H}) \arrow[r, "(h,a)^\sharp"] & \mathfrak{Desc}(\tilde{\mathfrak{G}}/\mathfrak{G}) \\
\mathcal{M}(\mathfrak{G}) \arrow[u, "f^*"] \arrow[ur, "g^+"'] &
\end{tikzcd}$$

By Proposition 8, we know that g^+ is an equivalence of categories. It follows that $(h,a)^\sharp$ is full and essentially surjective. But our assumption about the faithful flatness of the morphism of preschemes of objects underlying h implies that $(h,a)^\sharp$ is faithful. If follows that $(h,a)^\sharp$ is an equivalence of categories. But then f^* is an equivalence of categories. $\square$

Proof of Proposition 7: The flatness of (12) is obvious. We will derive Proposition 7 by applying the Lemma 2 to the morphism $\mathfrak{G}_X = \mathfrak{H} \xrightarrow{f} \mathfrak{G}$. Let $\tilde{\mathfrak{G}}$ be the following groupoid prescheme: Objects of $\tilde{\mathfrak{G}}(S)$ are triples (x,m,o), where $x \in X(S)$, $o \in \mathfrak{O}(S)$, and m is a morphism in $\mathfrak{G}(S)$ from o to $f(x) \in \mathfrak{O}(S)$. The image of (o,m,x) by g is o and its image by h is x. A morphism from (o,m,x) to $(\tilde{o},\tilde{m},\tilde{x})$ in $\tilde{\mathfrak{G}}(S)$ is just a morphism n from o to $\tilde{o}$ in $\mathfrak{G}(S)$, its image by g is n and its image by h is the morphism from x to $\tilde{x}$ in $\mathfrak{G}_X(S)$ given by $\tilde{m}nm^{-1}$. It is clear from these definitions that the diagram (14) commutes up to canonical isomorphism. The morphism h is faithfully flat because $\mathfrak{M}$ is a faithfully flat $\mathfrak{O}$-prescheme. That g satisfies the Cartesianness assumption of Proposition 8 is clear, and the assumption about the quasi-compactness and faithful flatness of the underlying morphism between the preschemes of objects is precisely our assumption about (13). The morphism $\mathfrak{A} = \tilde{\mathfrak{G}} \times_{\mathfrak{G}} \tilde{\mathfrak{G}} \xrightarrow{a} \mathfrak{Ar}(\mathfrak{H})$ sends an object $\big((o,m,x),(o,m',x')\big)$ of $\mathfrak{A}(S)$ to the morphism $x \xrightarrow{m'm^{-1}} x'$ in $\mathfrak{H}(S)$, and the morphism ν from this object of $\mathfrak{A}(S)$ to $\big((\tilde{o},\tilde{m},\tilde{x}),(\tilde{o},\tilde{m}',\tilde{x}')\big)$, which is simply given by a morphism $o \xrightarrow{n} \tilde{o}$ in $\mathfrak{G}(S)$, to the commutative diagram

$$\begin{tikzcd}
x \arrow[r, "m'm^{-1}"] \arrow[d, "\tilde{m}nm^{-1}"'] & x' \arrow[d, "\tilde{m}'nm'^{-1}"] \\
\tilde{x} \arrow[r, "\tilde{m}'\tilde{m}^{-1}"] & \tilde{x}'
\end{tikzcd}$$

in $\mathfrak{H}(S)$. It is easy to verify that this construction really gives a morphism of groupoid preschemes, and that the assumptions of Lemma 2 about a are valid.

The proof of Proposition 7 is complete. **Q.E.D.**

In order to verify the assumption of Proposition 7 in the situation in which we will apply it to bordism comodules, it will be convenient to verify the flatness of (13) in infinitesimal neighborhoods of $Z^k - Z^{k+1}$ by applying the deformation theory of formal group laws. If $\mathbf{MU}_*$ was a Noetherian ring, it would be a standard fact of algebraic geometry that it is sufficient to verify the flatness in infinitesimal neighborhoods of the strata. That we can still apply the same principle although $\mathbf{MU}_*$ is not Noetherian will be derived from the next proposition.

Proposition 9. *Let $\mathfrak{G}$ be an arbitrary groupoid prescheme. Assume that we are given a filtering index set I with a final object o and a projective system of preschemes $\mathfrak{M}_\iota$ indexed by $\iota \in I$ with $\mathfrak{M}_o = \mathfrak{O}$ and such that for $\iota \prec \kappa$ the morphism $\mathfrak{M}_\iota \longrightarrow \mathfrak{M}_\kappa$ is affine, faithfully flat and of finite type, and assume that we are given an isomorphism $\mathfrak{M} \cong \lim_{\iota\in I} \mathfrak{M}_\iota$ (where the inverse limit exists by* [EGAIV, Proposition 8.2.3]*) such that the projection $\mathfrak{M} \longrightarrow \mathfrak{M}_o = \mathfrak{O}$ is $\mathfrak{t}$. Assume also that $\mathfrak{O}$ is the filtering projective limit of Noetherian preschemes $\mathfrak{O}_\iota$, $\iota \in J$, with transition morphisms which are affine, faithfully flat and of finite type, such that J has a final element o for which the projection $\mathfrak{O} \xrightarrow{\pi_o} \mathfrak{O}_o$ is invariant (i.e, $\mathfrak{s}\pi_o = \mathfrak{t}\pi_o$), where*

$$\mathfrak{O} \xrightarrow{\pi_j} \mathfrak{O}_j$$

are the projections identifying $\mathfrak{O}$ with the inverse limit of the $\mathfrak{O}_j$. Assume that a filtration

$$\mathfrak{O} = Z^0 \supset \ldots \supset Z^k \supseteq \ldots$$

of $\mathfrak{O}$ by invariant closed subsets is given. View Z^k as a reduced subprescheme of $\mathfrak{O}$ defined by a quasi-coherent sheaf of ideals $\mathcal{I}_k$. Assume that $\mathcal{I}_k$ is coherent, and let $Z^{k(l)}$ be the l-th infinitesimal neighborhood of Z^k, i.e., the closed subprescheme defined by $\mathcal{I}_k^l$. Let a morphism $X \xrightarrow{f} U_\infty$ be given, and assume that X is Noetherian. Assume also that for all k and l, the projection to the first factor

$$\left(Z^{k(l)} - Z^{k+1}\right) \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} X \to Z^{k(l)} - Z^{k+1} \tag{15}$$

is flat. Then (13) *is flat.*

Proof. It suffices to show that at every $m \in \mathfrak{M} \times_{\mathfrak{O}} X$ and for every sufficiently big i, $\mathfrak{M} \times_{\mathfrak{O}} X \xrightarrow{\pi_j p} \mathfrak{O}_i$ is flat. We have $p(m) \in Z^k - Z^{k+1}$ for some $k < \infty$. By [EGAIV, Proposition 8.6.3], if i is sufficiently big then there exists a reduced closed subprescheme $\tilde{Z}^k \subset \mathfrak{O}_i$ such that $Z^k = \pi_i^{-1}(\tilde{Z}^k)$. Let $\mathfrak{M} \xrightarrow{\rho_j} \mathfrak{M}_j$ be the projections identifying $\mathfrak{M}$ with the inverse limit of the $\mathfrak{M}_j$. Since $\mathfrak{O}_i$ is an $\mathfrak{O}_o$-prescheme of finite type and since $s\pi_o = \mathfrak{t}\pi_o$, by [EGAIV, Proposition 8.13.1] there exist a j and a morphism $\mathfrak{M}_j \xrightarrow{\tilde{\mathfrak{s}}} \mathfrak{O}_i$ such that $\pi_i\mathfrak{s} = \tilde{\mathfrak{s}}\rho_j$. We have a commutative diagram

$$\begin{tikzcd}
\mathfrak{M} \underset{\mathfrak{O}}{\times} X \arrow[r, "\tilde{\rho}=\rho_j\times \mathrm{Id}_X"] \arrow[d, "p"'] & \mathfrak{M}_j \underset{\mathfrak{O}}{\times} X \arrow[d, "q=\tilde{\mathfrak{s}}\mathrm{pr}_1"] \\
\mathfrak{O} \arrow[r, "\pi_i"] & \mathfrak{O}_i,
\end{tikzcd}$$

where $\mathfrak{M}_j \times_{\mathfrak{O}} X \xrightarrow{\mathrm{pr}_1} \mathfrak{M}_j$ is the projection to the first factor, and the fibre product is taken with respect to the projection $\mathfrak{M}_j \to \mathfrak{M}_o = \mathfrak{O}$. Since $\tilde{\rho}$ is flat, it is sufficient to show that q is flat at $\tilde{\rho}(m)$. We claim that for every l, the morphism

$$\tilde{Z}^{k(l)} \times_{\mathfrak{O}_i} \mathfrak{M}_j \times_{\mathfrak{O}} X \to \mathfrak{O}_i \tag{16}$$

is flat at $\tilde{\rho}(m)$. This follows from the fact that this morphism is part of a commutative diagram

$$\begin{tikzcd}
Z^k(l) \times_{\mathfrak{O}} \mathfrak{M} \times_{\mathfrak{O}} X = \tilde{Z}^{k(l)} \times_{\mathfrak{O}_i} \mathfrak{M} \times_{\mathfrak{O}} X \arrow[r, "\tilde{\rho}"] \arrow[d, "(15)"] & \tilde{Z}^{k(l)} \times_{\mathfrak{O}_i} \mathfrak{M}_j \times_{\mathfrak{O}} X \arrow[d, "(16)"] \\
Z^{k(l)} \arrow[r, "\text{restriction of } \pi_i"] & \tilde{Z}^{k(l)}
\end{tikzcd}$$

in which, by our assumptions, the bottom horizontal morphism is flat everywhere and the left vertical morphism is flat at m since $p(m)$ does not belong to Z^{k+1}. Since $\tilde{\rho}$ is flat, it follows that the right vertical morphism is flat at $\tilde{\rho}(m)$, which is our claim about (16). From this claim and from [EGAIV, Théorème 11.5.1] (cf. also [Mat86, Theorem 22.3]), it follows that q is flat at $\tilde{\rho}(m)$. Since this holds for every sufficiently big i, p is flat at m. $\square$

3.3.5. *Investigation of the functor $\mathcal{H}^i_Z$.* In order to apply Theorem 7 and its corollaries, it will be useful to have a condition which guarantees that the sheaf underlying the derived functor $\mathcal{H}^i_Z$ in the category $\mathcal{M}(\mathfrak{G})$ coincides with ${}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^i_Z$, the derived functor of $\mathcal{H}^0_Z$ in the category of $\mathcal{O}_{\mathfrak{O}}$-modules, and ${}^{\mathfrak{Qc}}\mathcal{H}^i_Z$, the derived functor of $\mathcal{H}^0_Z$ in the category $\mathfrak{Qc}(\mathfrak{O})$.

Lemma 3. *Let us assume that the open immersion $\mathfrak{O} - Z \longrightarrow \mathfrak{O}$ is quasi-compact. Then if M is a quasi-coherent $\mathcal{O}_{\mathfrak{O}}$-module, ${}^{\mathcal{O}-M}\mathcal{H}^i_Z$ is also quasi-coherent. Moreover, there are canonical base-change isomorphisms*

$$\begin{aligned}
\mathfrak{s}^{*\mathcal{O}-M}\mathcal{H}^i_Z M &\cong {}^{\mathcal{O}-M}\mathcal{H}^i_{\mathfrak{s}^{-1}(Z)}\Big(\mathfrak{s}^* M\Big) \\
\mathfrak{t}^{*\mathcal{O}-M}\mathcal{H}^i_Z M &\cong {}^{\mathcal{O}-M}\mathcal{H}^i_{\mathfrak{t}^{-1}(Z)}\Big(\mathfrak{t}^* M\Big).
\end{aligned} \tag{17}$$

Proof. The first assertion is [Har67, Proposition 2.1.], and the second a consequence of the exact sequence [Har67, Proposition 1.1.b] and a combination of [EGAIII, Proposition 1.4.15] and [EGAIV, 1.7.21]. $\square$

The isomorphisms (17), together with the structure of an object of $\mathcal{M}(\mathfrak{G})$ on M, define the structure of an object of $\mathcal{M}(\mathfrak{G})$ on ${}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^i_Z M$. Let us denote this object of $\mathcal{M}(\mathfrak{G})$ by $\tilde{\mathcal{H}}^i_Z M$.

Proposition 10. *Assume that $Z \subset \mathfrak{D}$ is an invariant closed subset and that the inclusion $\mathfrak{D} - Z \longrightarrow \mathfrak{D}$ is quasi-compact. Then there is a unique natural transformation of cohomological δ-functors on $\mathcal{M}(\mathfrak{G})$:*

$$\mathcal{H}^i_Z M \longrightarrow \tilde{\mathcal{H}}^i_Z M.$$

This natural transformation is an isomorphism if moreover the following two conditions are satisfied:

- *$\mathfrak{s}$ is affine.*
- *For every injective object A of $\mathfrak{Qc}(\mathfrak{D})$ and every $j > 0$, ${}^{\mathcal{O}-M}\mathcal{H}^j A$ vanishes. In other words, the canonical natural transformation*

$$ {}^{\mathfrak{Qc}}\mathcal{H}^j_Z \longrightarrow {}^{\mathcal{O}-M}\mathcal{H}^j_Z$$

  on functors from $\mathfrak{Qc}(\mathfrak{D})$ to itself is an isomorphism.

Proof. The existence of the natural transformation follows from the universal property of a derived functor. To prove that it is an isomorphism, it is sufficient to show that every object M of $\mathcal{M}(\mathfrak{G})$ can be embedded into an object I of $\mathcal{M}(\mathfrak{G})$ with ${}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^j_Z(I) = 0$ for $j > 0$. To achieve this, embed M into an injective object J of $\mathfrak{Qc}(X)$ and put $I = RJ$, where R is the right adjoint to the forgetful functor $\mathcal{M}(\mathfrak{G}) \longrightarrow \mathfrak{Qc}(\mathfrak{D})$ (cf. Proposition 6). The composition of the embedding $M \longrightarrow I$ with the adjunction morphism $I \longrightarrow RI$ is a monomorphism in $\mathcal{M}(\mathfrak{G})$. Therefore, it is sufficient to show that RI is ${}^{\mathcal{O}-\mathrm{M}}\mathcal{H}$-acyclic. By the explicit formula for R in Proposition 6, this amounts to

$$ {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^j\Big(\mathfrak{s}_*\mathfrak{t}^* J\Big) = 0 \quad \text{if } j > 0. \tag{18}$$

There are obvious Leray-type spectral sequences

$$\begin{aligned} E_2^{p,q} &= {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^p_Z R^q \mathfrak{s}_* X \Rightarrow R^{p+q}\Big(\mathcal{H}^0_Z \mathfrak{s}_*\Big) X \\ E_2^{p,q} &= R^p \mathfrak{s}_* {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^q_{\mathfrak{s}^{-1}(Z)} X \Rightarrow R^{p+q}\Big(\mathfrak{s}_* \mathcal{H}^0_{\mathfrak{s}^{-1}(Z)}\Big) X, \end{aligned} \tag{19}$$

where the derived functor R^{p+q} is taken on the category of $\mathcal{O}_{\mathfrak{M}}$-modules. Since $\mathfrak{s}_*\mathcal{H}^0_{\mathfrak{s}^{-1}(Z)} \cong \mathcal{H}^0_Z \mathfrak{s}_*$, the derived functors to which these spectral sequence converge are canonically isomorphic. Using this and applying [EGAIII, Proposition 1.4.14] to $f = \mathrm{Id}_{\mathfrak{M}}$ and $g = \mathfrak{s}$, and by [Har67, Proposition 2.1], we see that the spectral sequence, applied to $X = \mathfrak{t}^* J$, amounts to an isomorphism

$$ {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^p_Z \mathfrak{s}_* \mathfrak{t}^* J \cong \mathfrak{s}_* {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^p_{\mathfrak{s}^{-1}(Z)} \mathfrak{t}^* J.$$

But

$$\begin{aligned} {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^p_{\mathfrak{s}^{-1}(Z)}\mathfrak{t}^* J &= {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^p_{\mathfrak{t}^{-1}(Z)}\mathfrak{t}^* J \\ &\cong \mathfrak{t}^{*\,\mathcal{O}-\mathrm{M}}\mathcal{H}^p_Z J \\ &\cong \mathfrak{t}^{*\,\mathfrak{Qc}}\mathcal{H}^p_Z J \\ &= 0. \end{aligned}$$

The first of these line is the invariance of Z, the second is (17), the third is a consequence of our second assumptions, and the fourth line follows from the injectivity of I. The proof of (18) is complete. $\square$

We will also need a condition which can be used to verify the second assumption of Proposition 10.

Proposition 11. *Let X be an affine prescheme, Z a closed subset of X which can be defined by finitely many elements of a Noetherian subring of $\mathcal{O}_X(X)$ over which $\mathcal{O}_X(X)$ is flat. Then the natural transformation ${}^{\mathfrak{Qc}}\mathcal{H}^j_Z \longrightarrow {}^{\mathcal{O}-M}\mathcal{H}^j_Z$ is an isomorphism.*

Proof. This follows from a combination of [Har67, Theorem 2.3, Lemma 2.4, and Proposition 2.6] with the fact that the second of the two equivalent conditions of [Har67, Lemma 2.4] is invariant under flat base change. $\square$

3.4. Application of Morava's result. Let p be a prime number and $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at p. Let $\mathfrak{G} = (\mathfrak{O}, \mathfrak{M})$ be the affine groupoid scheme representing the following functor from the category of affine schemes to the category of groupoids: For every affine scheme $X = \operatorname{Spec}(R)$, objects of $\mathfrak{G}(X)$ are formal group laws $F(S,T) = S + T = \sum_{a+b\geq 2} f_{a,b}S^aT^b$ in one variable with coefficients in R. Morphisms from F to G are formal power series $\phi(S) = \sum_{a=1}^{\infty} \phi_a S^a$ satisfying $\phi(F(S,T)) = G(\phi(S), \phi(T))$, where $\phi_i \in R$ for all i, and ϕ_1 is a unit in R. Moreover, let $\tilde{\mathfrak{G}} \subset \mathfrak{G}$ be the subgroupoid scheme with the same underlying scheme of objects, but in which $\phi_1 = 1$ is required for morphisms. We have

$$\tilde{\mathfrak{G}} = \ker(\mathfrak{G} \xrightarrow{\delta} \mathfrak{G}_{\mathrm{m}}), \tag{1}$$

where $\mathfrak{G}_{\mathrm{m}} = \operatorname{Spec}\mathbb{Z}_{(p)}[T, T^{-1}]$ is the multiplicative group over $\mathbb{Z}_{(p)}$ and δ, the differential at the origin, sends every morphism $\phi \in \mathfrak{M}(X)$ to ϕ_1. There also is a morphism of schemes $\mathfrak{G}_{\mathrm{m}} \times_{\mathbb{Z}_{(p)}} \mathfrak{O} \xrightarrow{\gamma} \mathfrak{M}$ which sends every unit $\rho \in R$ to the morphism $\phi(T) = \rho T$ in $\mathfrak{M}(X)$.

It is a section for δ, satisfies $\mathfrak{s}\gamma = \mathrm{Id}_{\mathfrak{O}}$ and makes the diagram

$$\begin{tikzcd}
\mathfrak{G}_{\mathrm{m}} \times \mathfrak{G}_{\mathrm{m}} \times \mathfrak{O} \arrow[r, "(\mathrm{Id}\cdot p_1, \gamma\cdot(p_2,p_3))"] \arrow[d, "(\mu\cdot(p_1,p_2), Id_{\mathfrak{O}}\cdot p_3)"] & \mathfrak{G}_{\mathrm{m}} \times \mathfrak{M} \arrow[r, "(\mu\cdot\mathfrak{t}, \mathrm{Id}_{\mathfrak{M}})"] & \mathfrak{M} \times \mathfrak{M} \arrow[d, "\mathfrak{c}"] \\
\mathfrak{G}_{\mathrm{m}} \times \mathfrak{O} \arrow[rr, "\gamma"] & & \mathfrak{M}
\end{tikzcd}$$

commutative, where μ is the multiplication on $\mathfrak{G}_{\mathrm{m}}$, $\mathfrak{c}$ is the composition law of $\mathfrak{G}$, and all products are in the category of $\mathbb{Z}_{(p)}$-schemes.

By well-known results of Lazard, Milnor, Quillen, and Landweber-Novikov, $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ exist and we have an isomorphism of groupoid schemes

$$\tilde{\mathfrak{G}} = (\operatorname{Spec} \mathbf{MU}_*, \operatorname{Spec} \mathbf{MU}_*\mathbf{MU}).$$

Therefore, $\mathcal{M}(\tilde{\mathfrak{G}})$ is equivalent to the category of ungraded comodules over the cogroupoid ring $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$. In particular, every object F of $\mathcal{M}(\mathfrak{G})$ defines an object of $\mathcal{M}(\tilde{\mathfrak{G}})$ by restriction, and therefore the structure of an ungraded $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-comodule on the $\mathbf{MU}_*$-module $F(\mathfrak{O})$. Since F is an object of $\mathcal{M}(\mathfrak{G})$, every $f \in F(\mathfrak{O})$ has a unique finite decomposition $f = \sum_{k=-\infty}^{\infty} f_k$ such that for the pull back with respect to the composition $\mathfrak{G}_{\mathrm{m}} \times_{\mathbb{Z}_{(p)}} \mathfrak{O} \xrightarrow{\gamma} \mathfrak{M} \xrightarrow{\mathfrak{s}} \mathfrak{O}$ we have

$$(\mu\gamma)^* f = \sum_{k=-\infty}^{\infty} T^{-k} f_k,$$

where T is the parameter of $\mathfrak{G}_{\mathrm{m}}$. There is a grading on $F(\mathfrak{O})$ defined by

$$F(\mathfrak{O})_l = \Big\{ f \in F(\mathfrak{O}) \;\Big|\; f_k = 0 \text{ unless } l = 2k \Big\}.$$

It is easy to see that the grading on $\mathbf{MU}_* = \mathcal{O}_{\mathfrak{O}}(\mathfrak{O})$ coincides with the usual one, that $F(\mathfrak{O})$ with this grading becomes a graded $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-comodule, and that we get an equivalence of categories between $\mathcal{M}(\mathfrak{G})$ and the category of graded $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-comodules concentrated in even dimension.

For $0 \le k \le \infty$, let $Z^k \subseteq \mathfrak{O}$ be the closed subset corresponding to formal group laws of height $\ge k$ on a $\mathbb{F}_p$-algebra, and let $U_k = \mathfrak{O} - Z^k$. The reduced closed subscheme Z^k corresponds to the ideal $I_k = \langle p, V_1, \ldots, V_{k-1} \rangle$. Let $\mathcal{I}_k$ be the quasi-coherent sheaf on $\mathfrak{O}$ defined by I_k.

Our aim is to apply the results of the last subsection to the groupoid scheme $\mathfrak{G}$. We first have to convince ourselves that the assumption of Proposition 3.3.4 holds. We will do this by applying Proposition 3.3.5. However, as I said before, we first have to reduce to a Noetherian situation. This will be done using the result of the following subsubsection.

3.4.1. *Base change to Noetherian* $\mathbf{MU}_*$*-algebras satisfying the Landweber condition.* Let X be a Noetherian prescheme and let $X \xrightarrow{f} U_\infty \subset \mathfrak{O}$ be a morphism. We say that f satisfies the assumptions of the Landweber exact functor theorem if for every $x \in X$, every k and every $i > 0$, we have

$$\operatorname{Tor}_i^{\mathcal{O}_{\mathfrak{O},f(x)}}\Big(\mathcal{O}_{X,x}, \mathcal{O}_{\mathfrak{O},f(x)} \Big/ \mathcal{I}_{k,f(x)}\mathcal{O}_{\mathfrak{O},f(x)}\Big) = \{0\}.$$

Another formulation of this condition is that $p, V_1, \dots, V_k$ is a regular sequence of $\mathcal{O}_{X,x}$ (cf. [Lan76, Theorem 2.6]). By the Hauptidealsatz, this condition implies that $f^{-1}(Z^{k+1})$ is either empty or of codimension 1 in $f^{-1}(Z^k)$. In particular, if a Noetherian $\mathfrak{O}$-prescheme X satisfies the assumptions of the exact functor theorem, then there exists an m such that $f(X) \subseteq U_m$.

Theorem 8. *Let X be a Noetherian $\mathfrak{O}$-prescheme. The following conditions are equivalent:*

- *For every k and l,* (3.3.15) *is flat.*
- (3.3.13) *is flat.*
- *X satisfies the assumptions of the Landweber exact functor theorem.*

Corollary 1. *Let $X \xrightarrow{f} \mathfrak{O}$ be a Noetherian $\mathfrak{O}$-prescheme which satisfies the assumptions of the exact functor theorem. Let m be the smallest number with $f(X) \subseteq U_m$. Then*

$$\mathcal{M}(\mathfrak{G}_{U_m}) \xrightarrow{f^*} \mathcal{M}(\mathfrak{G}_X)$$

and

$$\mathcal{M}_{Z^k}(\mathfrak{G}_{U_m}) \xrightarrow{f^*} \mathcal{M}_{f^{-1}(Z^k)}(\mathfrak{G}_X), \quad 0 \le k \le m$$

are equivalences of categories, and we have a canonical isomorphism

$$f^*\mathcal{H}^p_{Z^k}M \cong \mathcal{H}^p_{f^{-1}(Z^k)}f^*M.$$

Indeed, a point of $\mathfrak{O}$ is in the image of (3.3.13) if and only if the formal group law it parameterizes is, over some field extension, isomorphic to the formal group law parametrized by a point of X. By a result of Lazard [Rav86, Theorem A2.2.11], it follows that the image (3.3.13) is either disjoint to or contains all of $U_k \cap Z^k$. By the Landweber condition, $f^{-1}(Z^{k-1})$ is either empty or strictly larger than $f^{-1}(Z^k)$. Therefore, if the image of (3.3.13) intersects $U_k \cap Z^k$, then it contains all of U^k. This implies that the image of (3.3.13) it must be all of U_m. It follows that (3.3.13) becomes quasi-compact and faithfully flat if its image is replaced by U_m. Therefore, that the corollary is a consequence of the theorem follows from Proposition 3.3.7. We will now prove Theorem 8.

Let us first assume that (3.3.15) is flat. We want to apply Proposition 3.3.9 to verify that (3.3.13) is flat. Indeed, $\mathbf{MU}_*\mathbf{MU}$ is a polynomial ring in infinitely many variables

$\mathbf{MU}_*[X_1,\dots]$, where $\deg(X_i) = 2i$ (cf. [Rav86, A2.1.10]), such that the cotarget homomorphism sends every element of $\mathbf{MU}_*$ to the constant polynomial, and we put

$$\mathfrak{M}_i = \operatorname{Spec}\Big(\mathbf{MU}_*[X_1,\dots,X_i] \otimes \mathbb{Z}_{(p)}\Big).$$

Also, $\mathbf{MU}_*$ is a polynomial ring in infinitely many variables $\mathbf{MU}_*[z_1,\dots]$ (cf. [Rav86, A2.1.10]), and we put

$$\mathcal{O}_i = \operatorname{Spec}\Big(\mathbb{Z}_{(p)}[z_1,\dots,z_i]\Big).$$

It is easy to see that the assumptions of Proposition 3.3.9 are valid.

Let us now assume that (3.3.13) is flat. We want to verify the Landweber exact functor condition. Let $E = \mathfrak{M} \times_{\mathfrak{O}} X$. The projection from E to the second factor is always faithfully flat, therefore the Landweber condition for $X \xrightarrow{f} \mathfrak{O}$ are equivalent to the Landweber conditions for $E \xrightarrow{g} \mathfrak{O}$, where g is the composition

$$E = \mathfrak{M} \underset{\mathfrak{O}}{\times} X \to X \xrightarrow{f} \mathfrak{O}.$$

Also, since the Landweber conditions are trivially valid for $\mathfrak{O} \xrightarrow{\mathrm{Id}} \mathfrak{O}$, they are valid for the morphism $E \xrightarrow{h} \mathfrak{O}$ defined as the composition

$$E = \mathfrak{M} \underset{\mathfrak{O}}{\times} X \to \mathfrak{M} \xrightarrow{\mathfrak{s}} \mathfrak{O}, \tag{2}$$

since we are assuming that h is flat. But since $\mathcal{I}_k$ is an invariant sheaf of ideals on $\mathfrak{O}$, we have $g^{-1}\mathcal{I}_k = h^{-1}\mathcal{I}_k$, and this sheaf of ideals contains the section $g^*V_{k+1} - h^*V_{k+1}$ since V_{k+1} is known to be invariant modulo $\mathcal{I}_k$. Therefore, the Landweber conditions for g and h are equivalent, and we conclude that f satisfies the Landweber conditions.

To complete the final step in the proof of Theorem 8, we need a result of Lubin and Tate [LT66] which we now formulate. It is well-known (cf. [Rav86, Appendix A]) that two formal groups laws of height m over an algebraically closed field are isomorphic, and that their endomorphism rings are isomorphic to the maximal order in a division algebra of invariant $\frac{1}{n}$ with center $\mathbb{Q}_p$. Moreover, there is a formal group law $F_{m,o}$ of height m over $\mathbb{F}_p$ such that all endomorphisms of $F_{m,o}$ over $\overline{\mathbb{F}}_p$ are already defined over $\mathbb{F}_{p^m}$.Let $W_m = \operatorname{Spec}(\mathbb{Z}_p[\![V_1,\dots,V_{m-1}]\!]) = \operatorname{Spec}(R_m)$. The result of Lubin and Tate asserts that there is a formal group law F_m on W_m which is a universal deformation of $F_{m,o}$, i.e., which classifies all formal group laws on Artinian local rings with residue field containing $\mathbb{F}_p$ and whose image in the residue field is $F_{m,o}$. Let E_m be the ring of endomorphisms of $F_{m,o}$ over $\overline{\mathbb{F}}_p$. It is the maximal order in a central division algebra D_m of dimension m^2 and invariant $\frac{1}{m}$ over $\mathbb{Z}_p$ (cf. [Rav86, Theorem A2.2.17]). Let

$$\tilde{W}_m = W_m \underset{\operatorname{Spec}\mathbb{Z}_p}{\times} \operatorname{Spec}\mathbb{W}(\mathbb{F}_{p^m}) = \operatorname{Spec}(\tilde{R}_m),$$

where $\mathbb{W}(\mathbb{F}_{p^m})$ is the Witt ring and

$$\tilde{R}_m = R_m \bigotimes_{\mathbb{Z}_p} \mathbb{W}(\mathbb{F}_{p^m}) = \mathbb{W}(\mathbb{F}_{p^m})[\![V_1, \ldots, V_{m-1}]\!],$$

and let $\tilde{F}_m$ be the pull-back of F_m to $\tilde{W}_m$, i. e., the same formal group law as F_m, but with coefficients regarded as elements of the larger ring $\tilde{R}_m$. Let $d \in E_m^\times$. By the universal property of F_m, there is a unique action $\underline{d}$ of d on the scheme $\tilde{W}_m$ such that there exists an isomorphism

$$\underline{d}^* \tilde{F}_m \xrightarrow{\tilde{d}} \tilde{F}_m$$

whose evaluation at the closed point is d. Moreover, $\tilde{d}$ is determined uniquely by this condition.

The Frobenius endomorphism of $F_{m,o}$

$$\varphi_m(T) = T^p$$

defines a uniformizing element $\phi_m \in E_m$, and ϕ_m^m is a central element $\boldsymbol{\pi}$ of E_m by our assumptions about the field of definition of the endomorphisms of $F_{m,o}$. We define $\underline{d}$ and $\tilde{d}$ for arbitrary $d \in D^\times$ as follows: $\underline{\varphi}_m$ is the automorphism of $\tilde{W}_m$ over W_m defined by the Frobenius automorphism of $\mathbb{W}(\mathbb{F}_{p^m})$, and $\tilde{\varphi}_m$ is the identity automorphism of F_m, regarded as an isomorphism between $\tilde{F}_m$ and $\underline{\varphi}_m^* \tilde{F}_m$. An arbitrary $d \in D^\times$ can be represented as $d = r\varphi_m^k$, we put $\underline{d} = \underline{r}\underline{\varphi}_m^k$ and $\tilde{d} = \tilde{r}\tilde{\varphi}_m^k$. It is easy to see that $\underline{de} = \underline{de}$ and $\widetilde{de} = \tilde{d}\tilde{e}$ hold in full generality. Moreover, $\underline{p}$ and $\tilde{p}$ are the identity. Therefore, $\underline{d}$ and $\tilde{d}$ depend only on the image of d in the factorgroup

$$Q_m = D_m^\times \Big/ \Big\{ \boldsymbol{\pi}^k \;\Big|\; k \in \mathbb{Z} \Big\}.$$

For any $\mathfrak{O}$-prescheme Y, $\underline{d}$ and $\tilde{d}$ determine an action of Q_m on the Y-prescheme

$$Y \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m.$$

The following result is a consequence of the work of Lubin and Tate:

Proposition 1. *Let $Y \xrightarrow{g} U_m$ be any morphism and assume that the ideal $g^{-1}\mathcal{I}_m$ is nilpotent on Y. Then the projection*

$$I = Y \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \xrightarrow{\pi} Y \tag{3}$$

is an pro-étale Q_m-principal homogeneous space. More precisely, for every open normal subgroup $K \subset Q_m$, the quotient prescheme $Y \times_{\mathfrak{O}} \tilde{W}_m/K$ exists, is étale over Y (in the sense of [SGA1, Exp. IX, Definition 1.1]*) and is a principal homogeneous space for Q_m/K, and the natural morphism*

$$Y \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \to \lim_K Y \underset{\mathfrak{O}}{\times} \tilde{W}_m/K$$

is an isomorphism.

Proof. Let us first assume that $g^{-1}\mathcal{I}_m$ is not only nilpotent, but is even zero. Then for the formal group law on Y, we have $p = V_1 = \ldots = V_{m-1} = 0$, while V_m is a unit. Let G be the formal group law classified by g, let $G^{(k)}$ be the finite closed subgroupscheme of G defined by the p^k-th power of the generator, let $F_o^{(k)}$ be defined in the same way, and let I_k be the prescheme of isomorphisms from $G^{(k)}$ to $F_o^{(k)}$. The same considerations which were used after formula A2.2.12 in the proof of [Rav86, A2.2.11] prove that $I_{k+1} \longrightarrow I_k$ is an Artin-Schreier covering of degree p^m for $k \geq 0$ and that $I_0 \longrightarrow Y$ is an unramified Kummer covering of degree $p^m - 1$. Since (3) is the limit of the preschemes I_k, this proves our assertion in the special case $g^{-1}\mathcal{I}_m = 0$.

Now we deal with the general case, in which $g^{-1}\mathcal{I}_m$ is only supposed to be nilpotent. Let Y_o be the closed subscheme of Y defined by this sheaf of ideals. For every Y-scheme X, let X_o be the preimage of Y_o in X. We claim that

$$\mathrm{Hom}_{Y\text{-schemes}}(X_o, I_o) \longrightarrow \mathrm{Hom}_{Y\text{-schemes}}(X, I) \tag{4}$$

is a bijection. This is a consequence of the following claim, which will follow from the work of Lubin and Tate:

Lemma 1. *Let X be a prescheme, F a formal group law and $\mathcal{I}$ a nilpotent sheaf of ideals on X containing p. Let $X_o = V(\mathcal{I})$ and let F_o be the reduction of F modulo $\mathcal{I}$. Let a morphism $X_o \xrightarrow{f_o} \tilde{W}_m$ mapping every point of X_o to the closed point of $\tilde{W}_m$ and an isomorphism $F_o \xrightarrow{\varphi_o} f_o^*\tilde{F}_m$ be given. Then there exists a unique pair (f, φ), where $X \xrightarrow{f} \tilde{W}_m$ is an extension of f_o and $F \xrightarrow{\phi} f^*\tilde{F}_m$ is an isomorphism whose reduction modulo $\mathcal{I}$ is φ_o.*

We have already seen that I_o is a pro-étale Y_o-prescheme. By [SGA1, Exp. IX, Proposition 1.7], there is a unique way to extend I_o to a pro-étale Y-prescheme $\tilde{I}$. By the bijection (4), the isomorphism $\tilde{I}_o \xrightarrow{\alpha} I_o$ extends in a unique way to a morphism $\tilde{I} \xrightarrow{\beta} I$. The inverse isomorphism $I_o \xrightarrow{\alpha^{-1}} \tilde{I}_o$ extends by [SGA1, Exp. IX, Proposition 1.7] in a unique way to a morphism $I \xrightarrow{\beta} \tilde{I}$. We have $\alpha\beta = \mathrm{Id}_I$ since (4) is injective and $\beta\alpha = \mathrm{Id}_{\tilde{I}}$ by [SGA1, Exp. IX, Proposition 1.7]. Therefore, $I \cong \tilde{I}$, and the result follows.

It remains to convince ourselves that Lemma 1 really is a consequence of the work of Lubin and Tate. Here the only difficulty is that [LT66, Theorem 3.1] (of which Lemma 1 essentially is a reformulation) is only formulated in the Noetherian case. However, the proof given there works in full generality. Also, one could use [Mes72, Theorem V.1.6. and Corollary II.4.5] (using the method of the proof of [Mes72, Proposition IV.1.10] to translate the deformation result from the language of universal extensions to the language used by Lubin and Tate). □

We will now prove by induction on k that for every Noetherian $\mathfrak{O}$-prescheme satisfying the assumptions of the Landweber exact functor theorem, the morphism (3.3.15) is flat. We assume that this assertion holds for all $k < m$ and prove it for $k = m$.

It follows from the induction assumption, from Proposition 1, and from the part of Theorem 8 which we already proved that the projection

$$\mathfrak{O} \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \to \mathfrak{O} \tag{5}$$

is flat. Indeed, since $\tilde{W}_m$ satisfies the Landweber condition, the induction assumption proves that (3.3.15) with X replaced by $\tilde{W}_m$ is flat for all $k < m$. The same also holds for $k = m$ by Proposition 1 and for $k > m$ since then the left hand side of (3.3.15) is empty. By the part Theorem 8 which we already proved we conclude that (5) is flat.

Let X satisfy the assumptions of the exact functor theorem. We claim that the projection

$$X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \longrightarrow \tilde{W}_m \tag{6}$$

is flat. Since $p, \dots, V_{m-1}$ is a regular parameter sequence for the unique closed point of $\tilde{W}_m$, this amounts to the assertion that the pull-back of this sequence with respect to (6) is again regular. By the argument which we used after (2), this is equivalent to the assertion that the pull-pack of $p, \dots, V_{m-1}$ with respect to the composition

$$X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \xrightarrow{p} X \xrightarrow{f} \mathfrak{O}$$

is regular, which follows since p, which is the base change of (5) to X, is flat and since f satisfies the Landweber conditions.

We are now ready to prove the flatness of (3.3.15) for $k = m$. Let $S = Z^{k(l)} - Z^{k+1}$. We consider the following commutative diagram:

$$\begin{tikzcd}
S \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} X \arrow[dd, "\alpha"'] & S \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \arrow[l, "\beta"'] \arrow[d, "\gamma"] \\
 & (S \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m) \underset{\tilde{W}_m}{\times} (X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m) \arrow[d, "\delta"] \\
S & S \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m \arrow[l, "\varepsilon"']
\end{tikzcd}$$

In this diagram, γ is defined by

$$\gamma\Big((\sigma, \mu_1, \xi, \mu_2, \upsilon)\Big) = \Big((\sigma, \mu_1\mu_2, \upsilon), (\xi, \mu_1, \upsilon)\Big),$$

where σ, μ_i, ξ, υ are points of S, $\mathfrak{M}$, X, and $\tilde{W}_m$ with values in some prescheme A, and the composition $\mu_1\mu_2$ is defined using the groupoid law of $\mathfrak{G}$. It is easy to see that γ is an isomorphism. We have to prove the flatness of α. Since β and ε are faithfully flat by Proposition 1, it is sufficient to prove that δ is flat. But δ is obtained from the flat morphism (6) by base-change.

Corollary 2. *Let $\tilde{W}_m \xrightarrow{\tilde{f}_m} \mathfrak{O}$ be the morphism classifying the formal group law $\tilde{F}_m$. Then*

$$\mathcal{M}(\mathcal{G}_{U_m}) \xrightarrow{\tilde{f}_m^*} \mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$$

and

$$\mathcal{M}_{Z^k}(\mathcal{G}_{U_m}) \xrightarrow{\tilde{f}_m^*} \mathcal{M}_{\tilde{f}_m^{-1}(Z^k)}(\mathfrak{G}_{\tilde{W}_m}), \quad 0 \leq k \leq m,$$

are equivalences of categories, and we have a canonical isomorphism

$$\tilde{f}_m^* \mathcal{H}^p_{Z^k} M \cong \mathcal{H}^p_{\tilde{f}_m^{-1}(Z^k)} \tilde{f}_m^* M.$$

Indeed, $\tilde{f}_m$ satisfies the assumptions of the exact functor theorem.

It seems that this corollary is not covered by [Mor85], although the second equivalence of categories with $k = m$ is part of the main result of that paper.

3.4.2. *The chromatic spectral sequence.*

Corollary 3. *The groupoid scheme $\mathfrak{G}$ and the filtration $\mathfrak{O} = Z^0 \supset Z^1 \supset \ldots \supset Z^k \supset \ldots$ satisfy the assumptions of Theorem 7. In particular, there is a chromatic spectral sequence* (3.3.6) *converging in the sense explained in* [Ada74, Theorem III.8.2] *to the limit described in* (3.3.6) *and Corollary* 3.3.2. *Moreover, the cohomological dimension of the category $\mathcal{M}(\mathfrak{G}_{U_n})$ can be investigated by means of Corollary* 3.3.1.

Indeed, what we have to prove is that every injective object of $\mathcal{M}_{Z^k}(\mathfrak{G}_{U_m})$ is injective in $\mathcal{M}(\mathfrak{G}_{U_m})$. By Corollary 2, we may replace U_m by $\tilde{W}_m$ and Z^k by $\tilde{f}_m^{-1}(Z^k)$ and derive the result from Proposition 3.3.5.

Although we will not need it in our application to the generalisation of Bousfield's description of K-local spectra, it may still be worthwhile to give a criterion which guarantees that the limit of (3.3.6) is $\mathrm{Ext}^*_{\mathcal{M}(\mathfrak{G})}(M, N)$.

Corollary 4. *Assume that N is an object of $\mathcal{M}(\mathfrak{G})$ whose underlying $\mathbf{MU}_*$-module is a finitely presented $\mathbf{MU}_*$-module (for instance, because N arises as the even or odd degree part of the complex bordism of a finite spectrum). Then the canonical morphism in $\mathcal{D}\big(\mathcal{M}(\mathfrak{G})\big)$*

$$N \longrightarrow \underleftarrow{\mathrm{Holim}}\, Rj_{q*} j_q^* N$$

is an isomorphism. In particular, in this case the limit of the chromatic spectral sequence (3.3.6) *is* $\mathrm{Ext}^*_{\mathcal{M}(\mathfrak{G})}(M, N)$.

Proof. It is sufficient to verify the vanishing condition (3.3.10). By Proposition 3.3.10 and Proposition 3.3.11, we have an isomorphism in $\mathfrak{Qc}(\mathfrak{O})$

$$\mathcal{H}^i_{Z^m}(N) \cong {}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^i_{Z^m}(N).$$

Therefore, it is sufficient to verify condition (3.3.10) for finitely presented $\mathcal{O}_{\mathfrak{O}}$-modules N and with $\mathcal{H}^i_{Z^m}$ replaced by ${}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^i_{Z^m}$. Since $\mathcal{O}_{\mathfrak{O}}(\mathfrak{O})$ is the union an ascending sequence of polynomial rings over which it is flat, every finitely presented $\mathcal{O}_{\mathfrak{O}}$-module has a finite free resolution. This reduces our assertion to the case $\mathcal{N} = \mathcal{O}_{\mathfrak{O}}$. But in this case, it follows from [Har67, Theorem 2.3] and the fact that $(p, V_1, \dots, V_{m-1})$ is a regular sequence that

$${}^{\mathcal{O}-\mathrm{M}}\mathcal{H}^i_{Z^m}(\mathcal{O}_{\mathfrak{O}}) = 0 \qquad \text{unless } i = m,$$

proving (3.3.10). □

3.4.3. *The injective dimension of $\mathcal{M}(\mathfrak{G}_{U_k})$.*

Theorem 9. *Let $k < p-1$, then the injective dimension of $\mathcal{M}(\mathfrak{G}_{U_k})$ is k^2+k.*

By Corollary 3, we can try to prove the theorem by an application of Corollary 3.3.1. In order to prepare an application of the chromatic spectral sequence, we use Morava's result to give an explicit description of $\mathcal{M}_{Z^m}(\mathfrak{G}_{U_m})$ in terms of torsion modules over a certain ring with an action of a profinite group.

Recall from the formulation of Proposition 1 the action $\underline{d}$ of $d \in Q_m$ on $\tilde{W}_m = \mathrm{Spec}(\tilde{R}_m)$. The isomorphism of formal group laws $\tilde{d}$ is classified by a morphism

$$\tilde{W}_m \to \tilde{W}_m \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m$$

whose compositions with the projections to the first and second factor $\tilde{W}_m$ are $\underline{d}$ and $\mathrm{Id}_{\tilde{W}_m}$. For every object object M of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$, evaluating (3.3.1) along this morphism gives an automorphism ψ^d of the group $M(\tilde{W}_m)$. The following identities are satisfied for $m \in M(\tilde{W}_m)$, $f \in \tilde{R}_m$, and $d, e \in Q_m$:

$$\begin{aligned} \psi^d\big(\psi^e(m)\big) &= \psi^{de}(m) \\ \psi^d(fm) &= \psi^d(f)\cdot\psi^d(m), \end{aligned} \tag{7}$$

where we have put

$$\psi^d(f) = (\underline{d}^* f).$$

Recall that I_m was the invariant ideal of $\mathbf{MU}_*$ classifying height m formal group laws in characteristic p. In the following considerations, we will denote

$$I_m\tilde{R}_m = \langle p, V_1, \dots, V_{m-1}\rangle_{\tilde{R}_m}$$

by the same letter $I-M$. The following fact is a slight reformulation of results of Morava [Mor85]:

Proposition 2. *Let $S_m \subset \tilde{W}_m$ be the closed point. The functor which to the object M of $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ associates the group $M(\tilde{W}_m)$ with its additional structures is an equivalence of categories between $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ and the following category: Objects are $\tilde{R}_m$-modules M, together with an action of the profinite group Q_m, such that the relations* (7) *hold and such that every element of M is annihilated by some power of I_m and invariant under some open subgroup of Q_m.*

Proof. For every n, let

$$\tilde{W}_m^{(n)} = \operatorname{Spec}\Big(\tilde{R}_m \big/ I_m^n\Big) \subset \tilde{W}_m$$

and

$$Q_m \hat{\times} \tilde{W}_m^{(n)} = \lim_U \Big(\Big(Q_m \big/ U\Big) \times \tilde{W}_m^n\Big),$$

where the inverse limit over all open subgroups $U \subseteq Q_m$ exists by [EGAIV, 8.2.3]. Recall the action $\tilde{d}$ of $d \in Q_m$ on the formal group law F_m over $\tilde{W}_m$. Since modulo $\langle p, V_1, \dots, V_m \rangle^n$ every coefficient of the power series representing $\tilde{d}$ depends only on d modulo an open subgroup of Q_m, there is a morphism of schemes

$$Q_m \hat{\times} \tilde{W}_m^{(n)} \to \tilde{W}_m^{(n)} \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} \tilde{W}_m^{(n)}$$

whose pull-back by the morphism $\tilde{W}_M^{(n)} \to Q_m \hat{\times} \tilde{W}_m^{(n)}$ defined by $d \in Q_m$ corresponds to $\tilde{d}$. By Proposition 1, this morphism an isomorphism. It follows readily from this observation that the proposition is true if one restricts its formulation to objects annihilated by I_m^n. But an object of any of the two categories occurring in the formulation of the proposition is in a canonical way the colimit over n of subobjects satisfying this condition. $\square$

As an immediate consequence of this result, we have:

Corollary 5. *The class of all objects M of $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ for which $M(\tilde{W}_m)$ is a finitely generated $\tilde{R}_m$-module is a generating class.*

Now we want to study Ext-groups in $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ using the description of this category by Proposition 2. Our result is a slight generalisation of [Mor85, Proposition 2.1.4], which is essentially obtained by putting $M = \mathcal{O}_{\tilde{W}_m}$.

Proposition 3. *Let M be an object of $\mathcal{M}(\mathcal{G}_{\tilde{W}_m})$ such that $M(\tilde{W}_m)$ is a finitely generated $\tilde{R}_m$-module, and let N be any object of $\mathcal{M}(\mathcal{G}_{\tilde{W}_m})$. Assume that one of the objects M and N is supported in S_m. Then the action of Q_m on*

$$\operatorname{Ext}^*_{\tilde{R}_m}\Big(M(\tilde{W}_m), N(\tilde{W}_m)\Big) \tag{8}$$

defined by

$$\psi^d(\varepsilon) = \psi^d \varepsilon \psi^{d^{-1}}$$

for an extension class ε is continuous for the discrete topology on the Ext*-module, and we have a spectral sequence*

$$E_2^{a,b} = H^a\Big(Q_m, \operatorname{Ext}^b_{\tilde{R}_m}\big(M(\tilde{W}_m), N(\tilde{W}_m)\big)\Big) \Rightarrow \operatorname{Ext}^{a+b}_{\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})}(M,N), \tag{9}$$

where the cohomology is the cohomology of the profinite *group Q_m (i. e., using locally constant cochains).*

Proof. If M is finitely generated and supported in S_m, for an object L of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$ the continuity of the action of Q_m on the discrete group

$$\operatorname{Hom}_{\tilde{R}_m}\big(M(\tilde{W}_m), L(\tilde{W}_m)\big) \tag{10}$$

is clear. If M is finitely generated and L is supported in S_m, then we still have

$$\operatorname{Hom}_{\tilde{R}_m}\big(M(\tilde{W}_m), L(\tilde{W}_m)\big) = \operatorname*{colim}_k \operatorname{Hom}_{\tilde{R}_m}\big(M(\tilde{W}_m)/I_m^k M(\tilde{W}_m), L(\tilde{W}_m)\big). \tag{11}$$

If N is an object of $\mathcal{M}_{S^m}(\mathfrak{G}_{\tilde{W}_m})$, it has an injective resolution in this category. We shall see later that this resolution also defines an injective resolution of $N(\tilde{W}_m)$ in the category of all $\tilde{R}_m$-modules. By (11), this implies

$$\operatorname{Ext}^b_{\tilde{R}_m}\big(M(\tilde{W}_m), N(\tilde{W}_m)\big) = \operatorname*{colim}_k \operatorname{Ext}^b_{\tilde{R}_m}\big(M(\tilde{W}_m)/I_m^k M(\tilde{W}_m), N(\tilde{W}_m)\big). \tag{12}$$

In the case where M is supported in S^m, we have

$$\operatorname{Hom}_{\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})}(M,L) = \operatorname{Hom}_{\tilde{R}_m}\big(M(\tilde{W}_m), L(\tilde{W}_m)\big)^{Q_m}$$

by Proposition 2 since the left hand side is equal to $\operatorname{Hom}_{\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})}(M, \mathcal{H}^0_{S^m}L)$ and since $(\mathcal{H}^0_{\mathcal{S}_m}L)(\tilde{W}_m)$ is the submodule of I_m-torsion elements of $L(\tilde{W}_m)$. If L is supported in S^m, a similar fact holds by (11). Therefore, and (in the case where we do not assume that M is supported in S^m) by the arguments which we used to derive (12) from (11), (9) follows from the general Grothendieck spectral sequence for the derived functor of a composition once we prove the following two facts:

- For every injective object I of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$, and for every I_m-torsion R_m-module M, we have
$$\operatorname{Ext}^b_{\tilde{R}_m}\big(M, I(\tilde{W}_m)\big) = 0$$
if $a > 0$. If in addition I is supported in S^m, then $I(\tilde{W}_m)$ is an injective $\tilde{R}_m$-module.
- For the same I and for M subject to the assumptions of the proposition,
$$H^a\Big(Q_m, \operatorname{Hom}_{\tilde{R}_m}\big(M(\tilde{W}_m), I(\tilde{W}_m)\big)\Big)$$
vanishes if $a > 0$.

We claim that for the proof of both assertions it is possible to assume that I is I_m-torsion. Indeed, if M is supported in S^m we have

$$\operatorname{Ext}^b_{\tilde{R}_m}(M, I) = \operatorname{Ext}^b_{\tilde{R}_m}(M, \mathcal{H}^0_{S^m} I).$$

But $\mathcal{H}^0_{S^m} I$ is an injective object of $\mathcal{M}_{S^m}(\mathfrak{G}_{\tilde{W}_m})$, by Corollary 3. In both cases this allows us to assume that I is an injective object of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$ with support in S^m.

We denote the right adjoint of the forgetful functor from $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ to the category of I_m-torsion $\tilde{R}_m$-modules by $\mathfrak{R}$. It is given by

$$\Big(\mathfrak{R}X\Big)(\tilde{W}_m) = \mathcal{R}(X) = \Big\{\alpha\colon Q_m \longrightarrow X \;\Big|\; \alpha \text{ is locally constant}\Big\}, \tag{13}$$

where the actions of $\tilde{R}_m$ and Q_m on the right hand side of (13) are given by

$$\begin{aligned}(f \cdot \alpha)(d) &= \psi^d(f) \cdot \alpha(d)\\ \Big(\psi^e(\alpha)\Big)(d) &= \alpha(de),\end{aligned}$$

where $f \in \tilde{R}_m$ and $d, e \in Q_m$, and α belongs to the right hand side of (13).

For any finitely generated $\tilde{R}_m$-Module M, it follows that we have an isomorphism of $\tilde{R}_m$-modules with Q_m-action

$$\operatorname{Hom}_{\tilde{R}_m}\Big(M, \mathcal{R}(X)\Big) \cong \mathcal{R}\Big(\operatorname{Hom}_{\tilde{R}_m}(M, X)\Big). \tag{14}$$

If X is an injective $\tilde{R}_m$-module, the right hand side of (14) is exact in M, therefore the left hand side is also exact in M for finitely generated $\tilde{R}_m$-modules M. Since $\tilde{R}_m$ is Noetherian, this implies the injectivity of $\mathcal{R}(X)$ as an $\tilde{R}_m$-module. For arbitrary X, $H^a(Q_m, \mathcal{R}(X))$ vanishes if $a > 0$. The same therefore holds for the cohomology of Q_m with coefficients in the left hand side of (14). We have verified the two points which we had to prove for arbitrary injective objects I of $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ in the special case $I = \mathfrak{R}X$, where X is an injective I_m-torsion $\tilde{R}_m$-module. But an arbitrary injective object I can always be embedded as a direct summand into an injective object of this form. $\square$

Corollary 6. *Let m be not divisible by $p-1$, and let A and B be objects of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$, then*

$$\operatorname{Hom}^r_{\mathcal{D}\left(\mathcal{M}(\mathfrak{G}(U_m))\right)}\Big(A, R\mathcal{H}_{S^m}(B)\Big)$$

vanishes for $r > m^2 + m$.

Proof. Let $N^* = R\mathcal{H}_{S^m}(B)$. We first show that for every object M of $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ such that $M(\tilde{W}_m)$ is a finitely generated $\tilde{R}_m$-module, we have

$$\operatorname{Hom}^k_{\mathcal{D}\left(\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})\right)}(M, N^*) = 0 \quad \text{if } k > m^2 + d. \tag{15}$$

Once this is proved, the corollary easily follows from the following result:

Lemma 2. *Let $\mathcal{A}$ be an AB5 category and let $\mathcal{U}$ be a generating class for $\mathcal{A}$ such that any quotient object of an element of $\mathcal{U}$ also is in $\mathcal{U}$.*

- *Let $K \xrightarrow{\pi} L$ be a morphism in $\mathcal{A}$ such that*
$$\mathrm{Hom}_{\mathcal{A}}(U, K) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(U, L)$$
is an epimorphism for every $U \in \mathcal{U}$. Assume also that K is injective. Then π is a split epimorphism.
- *Let $\mathcal{A}$ have sufficiently many injective objects (for instance, because it is a Grothendieck category) and let X be an object of $\mathcal{D}^+(\mathcal{A})$ with the property that for all elements U of $\mathcal{U}$, $\mathrm{Hom}^i_{\mathcal{D}^+(\mathcal{A})}(U, X)$ vanishes unless $i \leq e$. Then X can be represented by an injective complex which is concentrated in dimensions $\leq e$.*

To apply the second point of the lemma to the proof of Corollary 6, one takes $\mathcal{M} = \mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$, $X^* = N^*$, and $\mathcal{U}$ the class of all objects U of $\mathcal{A}$ for which $U(\tilde{W}_n)$ is a finitely generated $\tilde{R}_m$-module. By Corollary 5, this is a generating class of $\mathcal{M}_{S^m}(\mathfrak{G}_{\tilde{W}_m})$. The remaining assumption of Lemma 2 is (15). It follows that N^* can be represented by an injective complex I^* in $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ which is concentrated in dimensions between 0 and $d + m^2$. But every injective object of $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ also is injective in $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$, by Corollary 3.

To prove (15) for $M(\tilde{W}_m)$ finitely generated, we note that

$$\mathrm{Hom}^k_{\mathcal{D}\left(\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})\right)}(M, R\mathcal{H}_{S^m} B) \cong \mathrm{Ext}^k_{\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})}(M, B) \tag{16}$$

since every injective object of $\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m})$ is injective in $\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})$ and since $\mathcal{H}^0_{S_m}$ is right adjoint to the inclusion functor

$$\mathcal{M}_{S_m}(\mathfrak{G}_{\tilde{W}_m}) \longrightarrow \mathcal{M}(\mathfrak{G}_{\tilde{W}_m}).$$

But the right hand side of (16) is the limit of the spectral sequence (9). Let

$$X^b = \mathrm{Ext}^b_{\tilde{R}_m}\Big(M(\tilde{W}_m), B(\tilde{W}_m)\Big),$$

the spectral sequence has the form

$$E_2^{a,b} = H^a(Q_m, X^b) \Rightarrow \mathrm{Ext}^{a+b}_{\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})}(M, B).$$

By Serre's result about the injective dimension of regular local Noetherian rings, we have $X^b = 0$ if $b > m$. Therefore, it is sufficient to show that $E_2^{a,b} = 0$ if $a > m^2$. We have $E_m^\times \subset Q_m$, and it is known that the cohomological dimension of the profinite group $E_m^\times$ is m^2 and the strict cohomological dimension is $m^2 + 1$ (cf.[Mor85, 2.2.0–2] or [Rav86, Theorem 6.2.10]). Therefore, $Y^{a,b} = H^a(E_m^\times, X^b) = 0$ for $a > m^2$, since X^b is p-torsion. By Hochschild-Serre, it is sufficient to show that $H^r(Q_m/E_m^\times, Y^{a,b}) = 0$ for $r > 0$. But $Q_m/E_m^\times$

is cyclic of order m, generated by image of the Frobenius endomorphism φ_m of $F_{m,o}$, and the action of the image of φ_m in $Q_m/E_m^\times$ is σ-linear:

$$\varphi_m(wy) = F(w)\varphi_m(y), \quad y \in Y^{a,b},\ w \in W(\mathbb{F}_{p^m}).$$

It is well-known that this implies that the higher cohomology of $Y^{a,b}$ vanishes: For instance, by flat descent it follows that Q_m/E_m- and $W(\mathbb{F}_p)$-modules

$$Y^{a,b} = W(\mathbb{F}_{p^m}) \bigotimes_{\mathbb{Z}_p} \tilde{Y}^{a,b}$$

for $\mathbb{Z}_p$-modules $\tilde{Y}^{a,b}$, such that φ_m acts as the Frobenius automorphism of the Witt ring and as the identity on the second factor of the tensor product. Since $W(\mathbb{F}_{p^m})/\mathbb{Z}_p$ is unramified, the normal basis theorem holds for the rings of integers in unramified extensions, $W(\mathbb{F}_{p^m})$ as a Galois module is induced from the zero subgroup of Q_m/E_m^*, hence the same holds for $Y^{a,b}$, and $Y^{a,b}$ has no higher cohomology. $\square$

Proof of Lemma 2: Both parts of the lemma are probably well-known, but we prove them here because we did not find a reference for it.

For the first point, let Π be the poset of all pairs (Λ, σ), where $\Lambda \xrightarrow{i_\Lambda} L$ is a subobject of L and $\Lambda \xrightarrow{\sigma} K$ satisfies $\pi\sigma = i_\Lambda$. The partial ordering is defined by $(\Lambda, \sigma) \preceq (\tilde{\Lambda}, \tilde{\sigma})$ if $\Lambda \subseteq \tilde{\Lambda}$ and $\tilde{\sigma}\,|_\Lambda = \sigma$. Π is not empty because it contains the zero subobject, and every totally ordered subset of Π has an upper bound because in an AB5 category the colimit of a totally ordered family of subobjects of L is a subobject of L. By Zorn's lemma, it follows that Π has a maximal element $(\tilde{L}, \sigma)$. Assume that $\tilde{L} \subset L$. Then there exists a $\tilde{L} \subset \overline{L} \subseteq L$ such that $\overline{L}/\tilde{L} \in \mathcal{U}$. By the injectivity of K there exists an extension of $\tilde{\sigma}$ to some morphism $\overline{L} \xrightarrow{\tau} K$. The morphism

$$\overline{L} \xrightarrow{\mathrm{Id}_{\overline{L}} - \pi\tau} L$$

annihilates $\tilde{L}$. Since $\overline{L}/\tilde{L} \in \mathcal{U}$, by our assumption on π this implies the existence of a morphism $\overline{L}/\tilde{L} \xrightarrow{\alpha} K$ such that $\mathrm{Id}_{\overline{L}} - \pi\tau = \pi\alpha$. Let $\overline{\sigma} = \tau + \alpha$. Then $(\overline{L}, \overline{\sigma}) \in \Pi$ and $(\tilde{L}, \tilde{\sigma}) \preceq (\overline{L}, \overline{\sigma})$, contradicting the maximality of $(\tilde{L}, \tilde{\sigma})$. We conclude $\tilde{L} = L$, and $\tilde{\sigma}$ is the desired left inverse of π.

For the second point we may assume that X^* is given by an injective complex I^* which is bounded from below. By our assumption, for $k > e$ and $U \in \mathcal{U}$ the mapping

$$\mathrm{Hom}_{\mathcal{A}}(U, I^{k-1}) \xrightarrow{d} \mathrm{Hom}_{\mathcal{A}}(U, Z^k)$$

must be surjective. By the first point, this implies that $I^{k-1} \xrightarrow{d} Z^k$ is a split epimorphism. In other words, $Z^k = B^k$ and there exists a section $B^k \xrightarrow{\tilde{\sigma}} I^{k-1}$. Extending $\tilde{\sigma}$ to $I^k \xrightarrow{\sigma} I^{k-1}$, we see that $Z^k = B^k$ splits off I^k and also that Z^{k-1} splits off I^{k-1}. Therefore, we may replace I^* by the complex $\{0\} \longrightarrow I^0 \longrightarrow \ldots \longrightarrow I^{e-1} \longrightarrow Z^e \longrightarrow \{0\}$.

The proof of Lemma 2 is complete. **Q.E.D.**

Let now M and N be objects of $\mathcal{M}(\mathfrak{G}_{U_k})$, with $k < p-1$. We have already verified that we can apply Corollary 3.3.1 to get a spectral sequence

$$E_1^{m,n} = \begin{cases} \mathrm{Hom}^{m+n}_{\mathcal{D}^+(\mathcal{M}(\mathfrak{G}_{U_m}))}(j_m^* M, R\mathcal{H}_{Z^m \cap U_m} j_m^* N) & \text{if } 0 \leq m \leq k \\ 0 & \text{otherwise} \end{cases} \Rightarrow \mathrm{Ext}^{m+n}_{\mathcal{M}(\mathfrak{G}_{U_k})}(M, N).$$

By Corollary 2, the initial term is isomorphic to

$$\mathrm{Hom}^{m+n}_{\mathcal{D}^+\left(\mathcal{M}(\mathfrak{G}_{\tilde{W}_m})\right)}(\tilde{f}_m^* M, R\mathcal{H}_{S^m} \tilde{f}_m^* N),$$

which by Corollary 6 vanishes for $m+n > m^2+m$, in particular for $m+n > k^2+k$. This proves one half of Theorem 9.

It remains to prove that the cohomological dimension is also exactly k^2+k. Let χ be the unique character of Q_k with values in $\mathbb{F}_p^\times$ such that $H^{k^2}(Q_k, \mathbb{F}_{p,\chi}) \cong \mathbb{F}_p$ (for the existence of such a character, cf. [Mor85, § 2.2]). Let M be the object of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_k})$ whose underlying quasi-coherent sheaf is $\mathcal{O}_{\tilde{W}_k}$, and whose structure morphism is the tautological isomorphism $\mathfrak{s}^*\mathcal{O}_{\mathfrak{D}} \cong \mathcal{O}_{\mathfrak{M}} \cong \mathfrak{t}^*\mathcal{O}_{\mathfrak{D}}$. Let N be the object of $\mathcal{M}(\mathfrak{G}_{\tilde{W}_k})$ described by the correspondence of Proposition 2 as follows: The underlying $\tilde{R}_k$-module is $\mathbb{F}_{p^k}$, with I_k acting trivially, and ψ^d is $\chi(d)^{-1}\mathrm{Frob}^{v(d)}$. It is well known from commutative algebra that $\mathrm{Ext}^k_{\tilde{R}_k}(\mathbb{F}_{p^k}, \tilde{R}_k)$ is a one-dimensional $\mathbb{F}_{q^k}$-vector space M, and that this is the only non-vanishing Ext-group. Moreover, the action of Q_k on this space is given by χ and the Frobenius automorphism. It follows from (8) that $\mathrm{Ext}^{k^2+k}_{\mathcal{M}(\mathfrak{G}_{\tilde{W}_k})}(N, M) \cong \mathbb{F}_p$. By Corollary 2, there exist objects A and B of $\mathcal{M}_{U_k \cap Z^k}(\mathfrak{G}_{U_k})$ such that $\tilde{f}_k^* A \cong N$ and $\tilde{f}_k^* B \cong M$. We have

$$\mathrm{Ext}^{k^2+k}_{\mathcal{M}(\mathfrak{G}_{U_k})}(A, B) \cong \mathbb{F}_p.$$

The proof of Theorem 9 is complete.

Remark 1. The same methods can be used to prove the following result: Let $\tilde{k} \leq k$. If there exists no number l which is a multiple of $p-1$ and satisfies $\tilde{k} \leq l \leq k$, then the cohomological dimension of $\mathcal{M}_{Z^{\tilde{k}}}(\mathfrak{G}_{U_k})$ is k^2+k. Otherwise, the cohomological dimension is infinite.

3.5. Generalisation of Bousfield's result to higher chromatic primes. Let an odd prime number p be fixed and let $\mathfrak{G}$ be the same as in the last subsection. Let $\mathcal{M}(\mathfrak{G}) \xrightarrow{T} \mathcal{M}(\mathfrak{G})$ be the following self-equivalence of categories: For $(M, \phi_M) \in \mathfrak{Ob}(\mathcal{M}(\mathfrak{G}))$, let $T\big((M,\delta)\big)$ be the object (M, ϕ_{TM}) of $\mathcal{M}(G)$ with the same underlying $\mathcal{O}_{\mathfrak{D}}$-module M and the isomorphism

$$\phi_{TM} \colon \mathfrak{s}^* M \cong \mathfrak{t}^* M$$

given by

$$\phi_{TM} = \delta \cdot \phi_M,$$

where $\mathfrak{G} \xrightarrow{\delta} \mathfrak{G}_{\mathrm{m}}$ was defined after (3.4.1). We claim that $(\mathcal{M}(\mathfrak{G}), T)$ has a splitting of period $p-1$. This will follow readily from the following fact:

Lemma 1. *Let F be a formal group law over flat $\mathbb{Z}_{(p)}$-algebra A, and let $\zeta \in A$ be a $p-1$-th root of unity. Then there exists a unique automorphism $F \xrightarrow{[\zeta]} F$ such that the differential of $[\zeta]$ at the origin is ζ.*

Proof. Obviously, the assertion depends only on the isomorphism class of A. In the case where A is a $\mathbb{Q}$-algebra, F is isomorphic to the additive formal group law. If F is the additive formal group law, then all automorphisms of F are given by multiplication by a certain element of A. The existence and uniqueness of $[\zeta]$ are clear in that case. This proves the uniqueness of $[\zeta]$ in general, since A is assumed to be a flat $\mathbb{Z}_{(p)}$-algebra.

In general, by a theorem of Cartier [Rav86, A2.1.18], F is isomorphic to a p-typical formal group law. Thus, it remains to prove the existence of $[\zeta]$ for p-typical F. We claim that for F p-typical,

$$[\zeta](T) = \zeta T$$

is an automorphism of F. Again, we may replace A by $A \otimes \mathbb{Q}$ since A was assumed to be without torsion. Then an isomorphism $\log_F$ from F to the additive formal group law exists, and the p-typicalness of F is equivalent to the assertion that $\log_F$ has the form

$$\log_F(T) = \sum_{k=0}^{\infty} w_k T^{p^k}.$$

Then $\log_F(\zeta T) = \zeta \log_F(T)$, and this implies that $[\zeta]$ really is an automorphism of F. $\square$

If X is a flat $\mathfrak{O}$-scheme and if $\zeta \in \mathcal{O}_X(X)$ is a $p-1$-th root of unity, then the endomorphism $[\zeta]$ of the pull-back to X of the universal formal group law on $\mathfrak{O}$ is classified by a morphism

$$X \xrightarrow{\sigma_\zeta} X \underset{\mathfrak{O}}{\times} \mathfrak{M}_{\mathfrak{D}} \subseteq X \underset{\mathfrak{O}}{\times} \mathfrak{M} \underset{\mathfrak{O}}{\times} X,$$

where

$$\mathfrak{M}_\Delta = \mathfrak{O} \underset{\mathfrak{O}\times\mathfrak{O}}{\times} \mathfrak{M}.$$

In this formula for $\mathfrak{M}_\Delta$, $\mathfrak{O}$ is made into an $\mathfrak{O}\times\mathfrak{O}$-scheme by the diagonal embedding, and $\mathfrak{M}$ by the pair of morphisms $(\mathfrak{s}, \mathfrak{t})$. In other words, $\mathfrak{M}_\Delta$ is the scheme of endomorphisms of the objects parametrized by $\mathfrak{O}$. If (M, ϕ_M) is an arbitrary object of $\mathcal{M}(\mathfrak{G}_X)$, then the pull-back $\sigma_\zeta^*(\phi_M)$ of the isomorphism $\mathfrak{s}_X^* M \xrightarrow{\phi_M} \mathfrak{t}_X^* M$ with respect to σ_ζ is an automorphism S_ζ of M. It is clear that S_ζ is a natural automorphism of the identity functor of $\mathcal{M}(\mathfrak{G}_X)$. Moreover, if $Y \xrightarrow{f} X$ is a flat morphism, then the automorphism $S_{f^*\zeta}$ and $f^*(S_\zeta)$ of the functor $\mathcal{M}(\mathfrak{G}_X) \longrightarrow \mathcal{M}(\mathfrak{G}_Y)$ coincide. If η is another $p-1$-th root of unity on X, then we have

$$\sigma_{\zeta\eta} = \tilde{\mathfrak{c}}(\sigma_\zeta, \sigma_\eta),$$

where

$$\Big(\mathfrak{M}_\Delta \underset{\mathfrak{O}}{\times} X\Big) \underset{X}{\times} \Big(\mathfrak{M}_\Delta \underset{\mathfrak{O}}{\times} X\Big) \xrightarrow{\tilde{\mathfrak{c}}} \mathfrak{M}_\Delta \underset{\mathfrak{O}}{\times} X$$

is the morphism obtained by base-changing the restriction of $\mathfrak{c}$ to $\mathfrak{M}_\Delta$ to X. Consequently, we have $S_{\zeta\eta} = S_\zeta S_\eta$. In particular, $S_\zeta^{p-1} = S_{\zeta^{p-1}} = \mathrm{Id}$.

We apply these remarks to the $\mathfrak{O}$-scheme

$$X = \mathfrak{O} \underset{\mathbb{Z}_{(p)}}{\times} \mu\!\!\mu_{p-1} \xrightarrow{p} \mathfrak{O}$$

where $\mu\!\!\mu_{p-1}$ is the scheme of $p-1$-th roots of unity. Let ζ be the universal $(p-1)$-th root of unity. If M is an object of $\mathcal{M}(\mathfrak{G})$ and if $m \in M(\mathfrak{O})$, then p^*m is a section of p^*M to which we may apply S_ζ. Since $\mathcal{O}_{\mathfrak{O}\times\mu\!\!\mu_{p-1}}$ is a free $\mathcal{O}_{\mathfrak{O}}$-module with base $1, \zeta, \dots, \zeta^{p-2}$, we have

$$S_\zeta\Big(p^*m\Big) = \sum_{k\in\mathbb{Z}/(p-1)\mathbb{Z}} \zeta^k p^*\Big(\pi_k(m)\Big)$$

with $\pi_k(m) \in M(\mathfrak{O})$. We claim that the $\pi_k(m)$ are idempotents and that $\pi_k\pi_l = 0$ if $k \neq l$. Indeed, let

$$\tilde{X} = \mathfrak{O} \underset{\mathbb{Z}_{(p)}}{\times} \mu\!\!\mu_{p-1} \underset{\mathbb{Z}_{(p)}}{\times} \mu\!\!\mu_{p-1} \xrightarrow{r} \mathfrak{O},$$

let η and ϑ be the two copies of the universal $(p-1)$-th root of unity on $\mu\!\!\mu_{p-1} \times \mu\!\!\mu_{p-1}$, and let

$$\tilde{X} \xrightarrow{q_{1,2,3}} X$$

be defined by $pq_i = r$ and $q_1^*(\zeta) = \eta$, $q_2^*(\zeta) = \vartheta$, and $q_3^*(\zeta) = \eta\vartheta$. The remarks about the functor-automorphisms S_ζ made in the last paragraph imply

$$\begin{aligned}
\sum_{i\in\mathbb{Z}/(p-1)\mathbb{Z}} \sum_{j\in\mathbb{Z}/(p-1)\mathbb{Z}} \eta^l\vartheta^m r^*\pi_i(\pi_j(m)\Big)) &= \sum_{j\in\mathbb{Z}/(p-1)\mathbb{Z}} \vartheta^j q_1^*(\zeta^i p^*(\pi_i(\pi_j(m)))) \\
&= \sum_{j\in\mathbb{Z}/(p-1)\mathbb{Z}} \vartheta^j q_1^*(S_\zeta(p^*(\pi_j(m)))) \\
&= \sum_{j\in\mathbb{Z}/(p-1)\mathbb{Z}} \vartheta^j S_\eta(r^*(\pi_j(m))) \\
&= S_\eta\Big(\sum_{j\in\mathbb{Z}/(p-1)\mathbb{Z}} \vartheta^j r^*(\pi_j(m))\Big) \\
&= S_\eta\Big(S_\vartheta(m)\Big) \\
&= S_{\eta\vartheta}(r^*(m)) \\
&= \sum_{k\in\mathbb{Z}/(p-1)\mathbb{Z}} \eta^k\vartheta^k r^*(\pi_k(m)),
\end{aligned}$$

from which identity our assertions about π_k ensue. If we denote the image of M under the idempotent π_k by M_k, then M is the direct sum of its subobjects M_k, and $\mathcal{M}(\mathfrak{G})$ splits as

$$\mathcal{M}(\mathfrak{G}) \cong \bigoplus_{k \in \mathbb{Z}/(p-1)\mathbb{Z}} \mathcal{M}(\mathfrak{G})_k,$$

where

$$\mathcal{M}(\mathfrak{G})_k = \Big\{ M \in \mathfrak{Ob}(\mathcal{M}(\mathfrak{G})) \Bigm| M_k = M \Big\}.$$

Since $\delta(\sigma_\zeta) = \zeta$, it is easy to see from the above definition of T that $T(\mathcal{M}(\mathfrak{G})_k) = \mathcal{M}(\mathfrak{G})_{k+1}$. Therefore, $\mathcal{M}(\mathfrak{G})_0$ is a splitting of period $p-1$ for $(\mathcal{M}(\mathfrak{G}), T)$.

Remark 1. It is clear from the proof of Lemma 1 and from Quillen's relation between $\mathbf{MU}_*$ and $\mathbf{BP}$ that this splitting corresponds to the splitting of the category of even degree $(\mathbf{BP}_*, \mathbf{BP}_*\mathbf{BP})$-comodules by those comodules which are concentrated in degree $2p-2$.

Let $\widetilde{\mathcal{M}(\mathfrak{G})}$ be the category of graded objects M of $\mathcal{M}(\mathfrak{G})$, together with an isomorphism $M[2] \cong TM$. In other words, $\widetilde{\mathcal{M}(\mathfrak{G})}$ is obtained from $\mathcal{M}(\mathfrak{G})$ and T the same way that $\tilde{\mathcal{A}}$ was obtained from $\mathcal{A}$ and T in our consideration of K-local spectra. Let $\widetilde{\mathcal{M}(\mathfrak{G}_{U_k})}$ be obtained in the same way from $\mathcal{M}(\mathfrak{G}_{U_k})$ and T. Let $\mathcal{S}^{(p)}_{\boldsymbol{C}}$ be the localisation of the stable homotopy category of $\boldsymbol{C}$-diagrams of spectra at p (the odd prime number which we kept fixed from the beginning of this subsection). The complex bordism functor, localised at p, defines a cohomological functor

$$\mathcal{S}^{(p)}_{\underline{0}} \to \widetilde{\mathcal{M}(\mathfrak{G})},$$

which we denote by $\mathfrak{MU}$. We have $\mathfrak{MU}\Big(S[1]\Big) \cong \mathfrak{MU}\Big(S\Big)[1]$. Let the functor

$$\widetilde{\mathcal{M}(\mathfrak{G})} \to \widetilde{\mathcal{M}(\mathfrak{G}_{U_k})}$$

obtained from

$$\mathcal{M}(\mathfrak{G}) \xrightarrow{j_k^*} \mathcal{M}(\mathfrak{G}_{U_k})$$

be also denoted by j_k^*, and let

$$\mathfrak{MU}_k(S) = j_k^*(S).$$

Let $\mathcal{S}^{(p),k}_{\boldsymbol{C}}$ be the thick subcategory of all objects S of $\mathcal{S}^{(p)}_{\boldsymbol{C}}$ with $\mathfrak{MU}_k(S_X) = 0$ for all $X \in \boldsymbol{C}$. Since $\widetilde{\mathcal{M}(\mathfrak{G}_{U_k})}$ is a quotient category of the category of all $(\mathbf{MU}_*, \mathbf{MU}_*\mathbf{MU})$-comodules, Corollary 3.2.1 and Proposition 2.1.3 imply that $\mathfrak{MU}_k$ has an Adams spectral sequence by injective resolutions. We have verified all assumptions of Theorem 5 and can formulate the result of an application of this theorem as follows:

Theorem 10. *Let p be an odd prime, and let $k > 0$ be an integer such that $L = 2p - 2 - k^2 - k$ is positive. Then $\mathcal{S}_{\boldsymbol{C}}^{(p)} \big/ \mathcal{S}_{\boldsymbol{C}}^{(p),k}$ is, in dimension $\dim \boldsymbol{C} < L$, equivalent to the derived category of quasi-periodic cochain complexes*

$$\mathcal{D}^{([1],1)}(\widetilde{\mathcal{M}(\mathfrak{G}_{U_k})}^{\boldsymbol{C}}) \cong \mathcal{D}^{(T,2)}(\mathcal{M}(\mathfrak{G}_{U_k})^{\boldsymbol{C}}). \tag{1}$$

This isomorphism identifies the localisation of the even degree part of the complex bordism functor with the zeroth cohomology functor and has the uniqueness properties described in the second part of Theorem 5.

References

[Ada74] J. F. Adams. *Stable Homotopy and Generalised Homology*, volume 10 of *Chicago Lectures in Mathematics*. The University of Chicago Press, 1974.

[Adl96] Jürgen Adleff. Konstruktionen für Homotopielimites in Frobeniuskategorien und prätriangulierten DG-Kategorien. Diplomarbeit, Bonn, 1996.

[BBD82] A. A. Beilinson, J. N. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers*, volume 100 of *Astérisque*, pages 7–171. 1982.

[Bei87] A. A. Beilinson. On the derived category of perverse sheaves. In Yu. I. Manin, editor, *K-theory, Arithmetic and Geometry*, volume 1289 of *Lecture Notes in Mathematics*, pages 27–41. Springer, 1987.

[BF78] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric Applications of Homotopy Theory II*, volume 658 of *Lecture Notes in Mathematics*, pages 80–131. Springer, 1978.

[BK91] A. I. Bondal and M. M. Kapranov. Enhanced triangulated categories. *Math. USSR Sbornik*, 70:93–107, 1991.

[BN93] Marcel Bökstedt and Amnon Neeman. Homotopy limits in triangulated categories. *Compositio Mathematica*, 86:209–234, 1993.

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18:257–281, 1979.

[Bou85] A. K. Bousfield. On the homotopy theory of K-local spectra at an odd prime. *Amer. J. Math.*, 107(4):895–932, August 1985.

[Bri68] Hans-Berndt Brinkmann. Relative homological algebra and the Adams spectral sequence. *Arch. Math. (Basel)*, XIX(2):137–155, 1968.

[DM82] Pierre Deligne and James S. Milne. Tannakian categories. In Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-Yen Shih, editors, *Hodge Cycles, Motives, and Shimura Varieties*, volume 900 of *Lecture Notes in Mathematics*, pages 101–228. Springer, 1982.

[EGAI–IV] A. Grothendieck and J. Dieudonné. Eléments de Géométrie Algébrique I-IV. *Inst. Hautes Études Sci. Publ. Math.*, 4 (EGAI), 8 (EGAII), 11, 17 (EGAIII), 20, 24, 28, 32 (EGAIV), 1960–67.

[FL79] Rudolf Fritsch and Dana May Latch. Homotopy inverses for nerve. *Bull. Amer. Math. Soc. (N.S.)*, 1(1):258–262, January 1979.

[Fre66] P. Freyd. *Abelian Categories. An Introduction to the Theory of Functors*. Harper & Row, 1966.

[ГМ88] С. И. Гельфанд and Ю. И. Манин. Методы гомологической алгебры. Мир, 1988. In Russian. English Translation to appear soon.

[GM94] S. I. Gel'fand and Yu. I. Manin. Homological algebra. In *Algebra V*, volume 38 of *Encyclopedia of Mathematical Sciences*. Springer, 1994.

[Goo90] Thomas G. Goodwillie. Calculus I. The first derivative of pseudoisotopy theory. *K-Theory*, 4:1–27, 1990.

[Gö40] Kurt Gödel. *The Consistency the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory*, volume 3 of *Annals of Mathematics Studies*. Princeton University Press, 1940.

[Gra76] Daniel Grayson. Higher algebraic K-theory II. After D. Quillen. In Micheal R. Stein, editor, *Algebraic K-Theory. Evanston 1976*, volume 551 of *Lecture Notes in Mathematics*, pages 217–240, 1976.

[GW95] Patricia Goertz and Jürgen Wolff. Systeme triangulierter Diagrammkategorien und epivalente Co-Türme triangulierter Kategorien. Diplomarbeit, Bonn, 1995.

[Har66] Robin Hartshorne. *Residues and Duality*, volume 20 of *Lecture Notes in Mathematics*. Springer, 1966. Lecture Notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64.

[Har67] Robin Hartshorne. *Local cohomology*, volume 41 of *Lecture Notes in Mathematics*. Springer, 1967. A seminar given by A. Grothedieck, Harvard University, Fall 1961.

[Hel88] Alex Heller. Homotopy theories, January 1988.

[HS71] Peter John Hilton and Urs Stammbach. *A Course in Homological Algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer, 1971.

[Kan57] Daniel M. Kan. On c.s.s. complexes. *Amer. J. Math.*, 79:449–476, 1957.

[Kel91] Bernhard Keller. Derived categories and universal problems. *Comm. Algebra*, 19(3):699–747, 1991.

[Lan76] Peter S. Landweber. Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$. *Amer. J. Math.*, 98(3):591–617, 1976.

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal lie groups. *Bull. Soc. Math. France*, 94:49–59, 1966.

[Mac71] Saunders MacLane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, 1971.

[Mat86] Hideuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[Mes72] William Messing. *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes.*, volume 264 of *Lecture Notes in Mathematics*. Springer, 1972.

[Mor85] Jack Morava. Noetherian localisation of cobordism comodules. *Ann. of Math. (2)*, 121:1–39, 1985.

[Mos68] R. M. F. Moss. On the composition pairing of Adams spectral sequences. *Proc. London Math. Soc.*, 18:179–192, 1968.

[Nee90] Amnon Neeman. The derived category of an exact category. *J. Algebra*, 135:388–394, 1990.

[Nee91] Amnon Neeman. Some new axioms for triangulated categories. *J. Algebra*, 139:221–255, 1991.

[Nee92] Amnon Neeman. Stable homotopy as a triangulated functor. *Invent. Math.*, 109:17–40, 1992.

[Qui67] Daniel G. Quillen. *Homotopical Algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer, 1967.

[Qui69] Daniel G. Quillen. Rational homotopy theory. *Annals of Mathematics*, 90:205–295, 1969.

[Qui73] Daniel G. Quillen. Higher algebraic K-theory I. In *Higher K-Theories*, volume 341 of *Lecture Notes in Mathematics*, pages 85–147. Springer, 1973.

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106:351–414, 1984.

[Rav86] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., 1986.

[Rav87] Douglas C. Ravenel. Localization and periodicity in homotopy theory. In E. Rees and J. D. S. Jones, editors, *Homotopy Theory*, volume 117 of *London Math. Soc. Lecture Notes Series*, pages 174–194. Cambridge University Press, 1987. Proceedings of the Durham Symposium 1985.

[Rav93] Douglas C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, 1993.

[Ric89] Jeremy Rickard. Derived categories and weak equivalences. *J. Pure Appl. Algebra*, 61:303–317, 1989.

[SGA1–7] A. Grothendieck and Collaborators. *Séminaire de Géométrie Algébrique 1–7*, volume 224 (SGA1), 151, 152, 153 (SGA3), 269, 270, 305 (SGA4), 569 (SGA4$\frac{1}{2}$), 225 (SGA6), 288, 340 (SGA7) of *Lecture Notes in Mathematics*. Springer, 1971–1977.

[Spa88] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Mathematica*, 65:121–154, 1988.

[TT90] Robert W. Thomason and Thomas F. Trobaugh. Higher algebraic K-theory of schemes and derived categories, 1990.

[Wil95] Jan Willing. Zur Axiomatik verfeinerter triangulierter Diagrammkategorien. Diplomarbeit, Bonn, 1995.